OPTIMAL CONTROL PROBLEMS WITH SWITCHING POINTS

by

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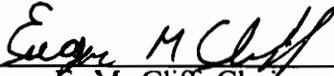
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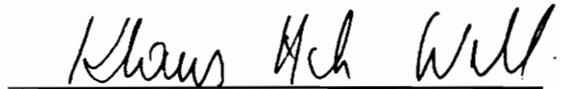
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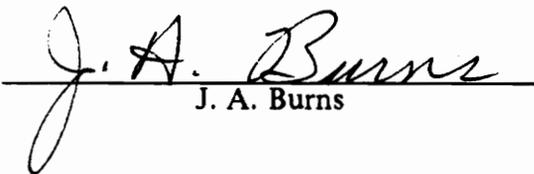
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(ABSTRACT)

The main idea of this thesis is to give an overview of the problems and difficulties that arise in solving optimal control problems with switching points. A brief discussion of existing optimality conditions is given and a numerical approach for solving the multi-point boundary value problems associated with the first-order necessary conditions of optimal control is presented. Two real-life aerospace optimization problems are treated explicitly. These are altitude maximization for a sounding rocket (Goddard Problem) in presence of a dynamic pressure limit, and range maximization for a supersonic aircraft flying in the vertical plane, also in presence of a dynamic pressure limit. In the second problem singular control appears along arcs with active dynamic pressure limit, which, in the context of optimal control, represents a first-order state inequality constraint. An extension of the Generalized Legendre-Clebsch Condition to the case of singular control along state/control constrained arcs is presented and is applied to the aircraft range maximization problem stated above. A contribution to the field of Jacobi Necessary Conditions is made by giving a new proof for the non-optimality of conjugate paths in the Accessory Minimum Problem. Because of its simple and explicit character the new

proof may provide the basis for an extension of Jacobi's Necessary Condition to the case of trajectories with interior point constraints. Finally, the result that touch points cannot occur for first-order state inequality constraints is extended to the case of vector valued control functions.

Acknowledgements

I am deeply indepted to Dr. Cliff under whom I could continue my studies when my dear teacher Dr. Kelley passed away. How lucky I was having these bright and sincere individuals as advisors and friends.

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Last not least I want to thank my wife Yuping for all the love and support she gave

me, especially when she got job offers and I did not!

Dedication

To my parents and the loving memory of Dr. Kelley

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Chapter 1

A "Road Map" for this Thesis

1.1 Overview

A particularly simple case of optimal control problems is given as follows:

$$\min_{u} \ \Phi(x(t_f), t_f) \tag{1.1}$$

subject to

$$\dot{x}(t) = f(x(t), u(t), t) \tag{1.2}$$

$$x(t_0) = x_0; \ t_0, \ x_0 \ \text{fixed} \tag{1.3}$$

$$\Psi(x(t_f), t_f) = 0. \tag{1.4}$$

Equations (1.2), (1.3), (1.4) describe the evolution of the underlying dynamical system, its initial states, and the target set to which the states have to be driven at final time. The "driving force" for the dynamical system is given by the control vector function of time $u(t)$ appearing in the right-hand side of the state equations (1.2). Control $u(t)$ can be chosen freely, but the aim is, and this is the essence of optimal control, to find control $u(t)$ such that the cost criterion Φ in (1.1) is minimized.

Many engineering problems of very different nature can be identified as optimal control problems. In aerospace engineering applications equation (1.2) may typically descibe the dynamics of helicopters, aircrafts, launch vehicles, or space probes. A common cost criterion (1.1) is $\Phi(x(t_f), t_f) = t_f$ (minimum time problem: state x of the dynamical system (1.2) has to be driven from initial state (1.3) to target set (1.4) such that final time t_f is minimized) or $\Phi(x(t_f), t_f) = -m(t_f)$ (minimum fuel problem: state x of the dynamical system (1.2) has to be driven from initial state (1.3) to target set (1.4) such that final mass $m(t_f)$ is maximized. In this case, of course, mass m has to be a component of the state vector).

Another area for the application of optimal control is the chemical industry. Here the dynamical system (1.2) may describe processes in a chemical plant. Typically, the control components are energy input and/or input of catalyzers. Common objectives are usually to minimize quantities such as the total energy consumption, the use of certain chemicals, or the total output of undesired toxic byproducts.

More exotic areas for the application of optimal control are, for example, industrial engineering and economics. Here the objectives are to organize a work force or a money market such that some measure of productivity or profit is maximized. In these areas it is usually very difficult to model the underlying dynamical system with sufficient precision and consequently results are often of academic interest only.

Chapter 2 of this thesis gives an introduction to boundary value problems and shows how they are associated with optimal control problems. Special emphasis is put on existence and uniqueness properties, as well as numerical well-posedness. A robust, easy-to-use FORTRAN code is introduced, that has been used to generate all numerical results that are presented in this thesis. Chapter 3 deals with the mathematical aspects of optimal control. First, the standard optimal control problem (1.1), (1.2), (1.3), (1.4) is restated in Section 3.2 in mathematically more precise form (Definition 3.2.1). In a remark following this definition it is explained how certain engineering requirements make it necessary to allow minimization over all control vector functions of time $u \in (L_2[t_0, t_f])^m$, where m is the dimension of u and L_2 denotes the well-known Hilbert space of all square integrable functions on $[t_0, t_f]$ in conjunction with the natural norm induced by the scalar product $< u_1, u_2 > = \sqrt{\int_{t_0}^{t_f} u_1 u_2 \, dt}$. In further remarks it is shown which types of seemingly more general optimal control problems are covered by the simple problem formulation in Def-

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inition 3.2.1 and which are not. Section 3.3 states the first-order necessary conditions that a solution to problem (3.1), (3.2), (3.3), (3.4) has to satisfy. While Section 3.4 introduces the concept of singular control, Sections 3.5, 3.6, 3.7 cover important extensions of the standard problem, namely interior point constraints, control constraints, and state constraints, respectively.

Closer examination shows that optimality conditions stated in Sections 3.3, 3.4, 3.5, 3.6, 3.7 lead to two-point boundary value problems in the simplest case and to multi-point boundary value problems in general. These boundary value problems (BVPs) have to be solved numerically. As this is a quite non-trivial task a whole Chapter (Chapter 2) is devoted to the discussion of problems and difficulties that may arise in solving BVPs. An approach is introduced, namely the well-known Shooting Method, that reduces BVPs to (usually highly non-linear) zero finding problems. Using simple examples it is indicated that existence and uniqueness of a solution to a given BVP cannot be guaranteed. Important for the numerical treatment of zero finding problems, the concept of condition number is discussed, which provides a measure for how well a problem is suited for treatment on a digital computer with finite word length. Furthermore, some important statements are made about the numerical evaluation of Jacobi matrices. A robust, but simple and easy-to-use software package that has evolved out of the needs described in Section 2.4 is described in Section 2.5. All numerical results stated in this thesis have been obtained with this software.

In Chapter 4 an extension of the famous Goddard Problem is treated as a first example for the application of optimal control to real-life aerospace problems. Here the problem is to find the thrust history for a vertically ascending rocket such that maximum altitude is

attained. The extensions beyond the classical Goddard Problem are a dynamic pressure constraint, which, in the context of optimal control, represents a first-order state inequality constraint, and an isoperimetric constraint. Despite the simple structure of the problem (only one control variable; this control variable appears only linearly in the equations of motion), solutions are found to be of considerable complexity. Theoretically, four different control logics are possible, namely zero thrust, full thrust, singular thrust, and arcs of active dynamic pressure limit. Upon varying the maximum dynamic pressure limit and the prescribed final time, nine different switching structures are obtained. All four theoretically possible control logics are found to be active in various sequences. The complexity of the solutions provides numerous opportunities for improving or deteriorating the numerical solvability by formulating different zero finding problems associated with the same BVP.

Chapter 5 deals with range optimization for a high performance fighter aircraft flying in the vertical plane. The controls are load factor n and throttle η, with η only appearing linearly in the equations of motion. Constraints are explicit limits on the absolute values of throttle and load factor (control constraints) and an upper bound on the dynamic pressure (first-order state inequality constraint). Theoretically, eleven different control logics are possible. The explicit derivation of these control logics along with the higher-order convexity tests ("Kelley Condition", or "Generalized Legendre-Clebch Condition") for the singular control cases is presented in Appendix A.

In the case of active state or control constraints the theoretical background for these higher-order convexity tests is not available in the literature. In Chapter 6 a generalized definition of singular control is presented. Based on a work by Goh [11] an extension

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of the Generalized Legendre-Clebsch Condition to singular control along state/control constrained arcs is derived (Theorem 6.6.2). The results obtained here are applied in Section A.5.

In Chapter 7 a theoretical result on the existence of touch points for first-order state inequality constraints is presented. The well-known first-order necessary conditions associated with the assumed switching structure

(i) unconstrained arc

(ii) state contraint active at a single point

(iii) unconstrained arc

are used to derive new concise conditions. In most cases of practical interest these conditions exclude the existence of touch points. For practical applications results of this type are invaluable, as they can significantly reduce the time consuming and frustrating search for the correct switching structure.

In Chapter 8 finally the problem of conjugate point testing is addressed. In 1965 Breakwell & Ho [3] showed that the existence of a conjugate point for a linear quadratic optimal control problem with zero initial states and homogeneous final conditions implies that the trivial solution (identically zero) cannot furnish a relative minimum. Through the concept of the Accessory Minimum Problem this implies for general non-linear optimal control problems that a solution candidate with a conjugate point cannot be optimal. In Chapter 8 the proof given by Breakwell & Ho is modified such that the trajectory that furnishes negative cost to the AMP is constructed explicitly. The explicit character of the proof makes it possible to extend results immediately to the case where the reference solution has discontinuities at fixed points in time. For the future it is hoped that a Jacobi

testing procedure can also be developed for trajectories with corners of more general type. For covenience, a derivation of the AMP for problems with interior point constraints of the type described in Section 3.5 is presented in Appendix B. As a useful byproduct this derivation also yields the first-order necessary conditions associated with such trajectories.

Chapter 2

Boundary Value Problems in Optimal Control

Chapter Overview

The concepts of two-point boundary value problems and multi-point boundary value problems are introduced. It is shown how these problems arise from applying the first-order necessary conditions of optomal control. Existence and uniqueness questions are addressed as well as questions of numerical well-posedness. A robust, easy-to-use software package for solving boundary value problems is introduced.

2.1 Introduction

In practical applications it is common that about 80 % of the total time spent on solving an optimal control problem is spent on the numerical treatment of boundary value problems (BVPs). On a simple example it is demonstrated that existence and uniqueness of the solution of a BVP can not be guaranteed. In fact, for the practically important non-linear case, there are hardly any theoretical results that can answer these questions a priori without actually trying to solve the BVP by running "numerical experiments". Practically, this is a very unpleasant feature. Even if a given optimal control problem is known to have a solution, the switching structure, i.e. the sequence of different control logics that actually solves the problem is not known in advance and has to be found by a numerical trial and error approach. Depending on the assumed switching structure different BVPs are obtained. If a numerical solution of such a BVP can be found, then fine. But if the numerical search for a solution fails it can not immediately be concluded that a solution to the BVP does not exist. Hence one is stuck with the question whether to continue the search for a solution of the assumed structure (which may not exist) or

whether to give up the present effort and start searching for a solution with a different structure. Naturally, in this situation it is very important that the applied zero-finding software actually does find a solution if there is a solution. In this context the condition number of a zero-finding problem plays an important role.

2.2 Theoretical Background for Boundary Value Problems (BVPs)

A two-point BVP is a problem of the following form:

Definition 2.2.1 (Two-Point Boundary Value Problem (TPBVP)) *Given is an ordinary differential equation (ODE) of the form*

$$\dot{z}(t) = f(z(t)) \tag{2.1}$$

on the time interval $[0,1]$*, where* $z(t) \in \mathbf{R}^n$ *and*

$$f : \begin{cases} \mathbf{R}^n & \to & \mathbf{R}^n \\ z & \mapsto & f(z) \end{cases} \in C^1.$$

Find a solution of this ODE such that n *given conditions*

$$h\left(z(0), z(1)\right) = 0$$

are satisfied, where

$$h : \begin{cases} \mathbf{R}^{2n} & \to & \mathbf{R}^n \\ (z(0), z(1)) & \mapsto & h\left(z(0), z(1)\right) \end{cases} \in C^1.$$

The more general case of a multi-point boundary value problem can be stated as follows:

Definition 2.2.2 (Multi-Point Boundary Value Problem (MPBVP)) *Given is a piecewise defined ODE of the form*

$$\dot{z}(t) = f_i(z(t)) \quad on \ [t_{i-1}, t_i], \ i = 1, ..., k+1 \qquad (2.2)$$

with $t_0 = 0$, $t_{k+1} = 1$; $t_{i+1} > t_i \ \forall i$; $z(t) \in \mathbf{R}^n$ and

$$f_i : \begin{cases} \mathbf{R}^n & \to & \mathbf{R}^n \\ z & \mapsto & f(z) \end{cases} \in C^1.$$

Find a solution of this ODE such that $n + k$ given conditions

$$h\left(z(0), z(t_1), ..., z(t_k), z(1), t_1, ..., t_k\right) = 0$$

are satisfied, where

$$h : \begin{cases} \mathbf{R}^{(k+2)n+k} & \to & \mathbf{R}^{n+k} \\ (z(0), z(t_1), ..., z(t_k), z(1), t_1, ..., t_k) & \mapsto & h\left(z(0), z(t_1), ..., z(t_k), z(1), t_1, ..., t_k\right) \end{cases} \in C^1.$$

Figures 2.1 and 2.2 give a schematic representation of two-point and multi-point BVPs, respectively.

Let us now have a closer look at the TPBVP. If we pick an arbitrary set of initial states $z_0 \in \mathbf{R}^n$, then the solution to the initial value problem $\dot{z} = f(z)$, $z(0) = z_0$, if it exists, is determined uniquely as long as the right-hand side $f(z)$ of the differential equations is Lipshitz bounded. Furthermore, if $f \in C^1$ then the final states $z(1)$ vary smoothly with the initial states z_0 (see Lee & Markus [24]). Now it is clear that a TPBVP reduces

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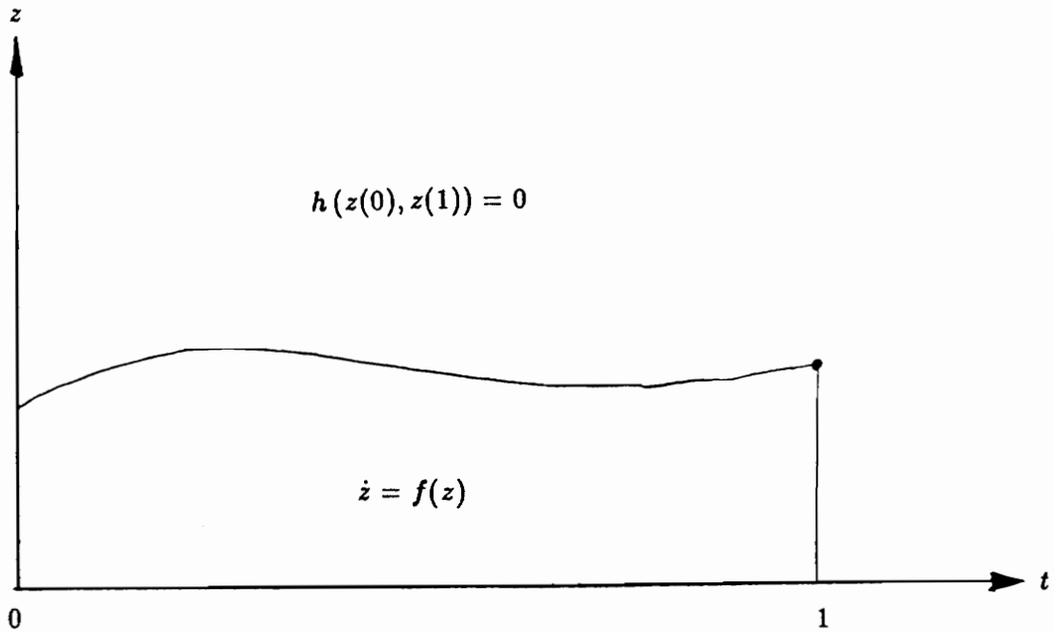
Figure 2.1: Two-Point Boundary Value Problem

to the problem of picking the right initial states $z(0) \in \mathbf{R}^n$ such that the n conditions $h\left(z(0), z(1)\right) = 0$ are satisfied.

For MPBVPs the extensions are only of technical nature. In the case of the problem stated in Definition 2.2.2 the independent parameters that can be chosen freely are initial states $z(0)$ and the location of the switching times $t_1, ..., t_k$. For both, two-point and multi-point BVPs this yields consistent zero-finding problems (i.e. number of conditions is equal to number of independent variables). Because of the smooth dependence of conditions $h = 0$ on the free variables, a Newton Method can be applied to solve these zero-finding problems. It is interesting to note that the zero-finding problem associated with a given BVP is not determined uniquely. In case of the MPBVP given in Definition 2.2.2 it is clear that instead of using $z(0), t_1, ..., t_k$ as independent parameters to represent

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$$h\left(z(0), z(t_1), ..., z(t_k), z(1), t_1, ..., t_k\right) = 0$$

$\dot{z} = f_1(z)$ $\dot{z} = f_2(z)$ $\dot{z} = f_k(z)$

0   t_1 t_2 t_k 1 t
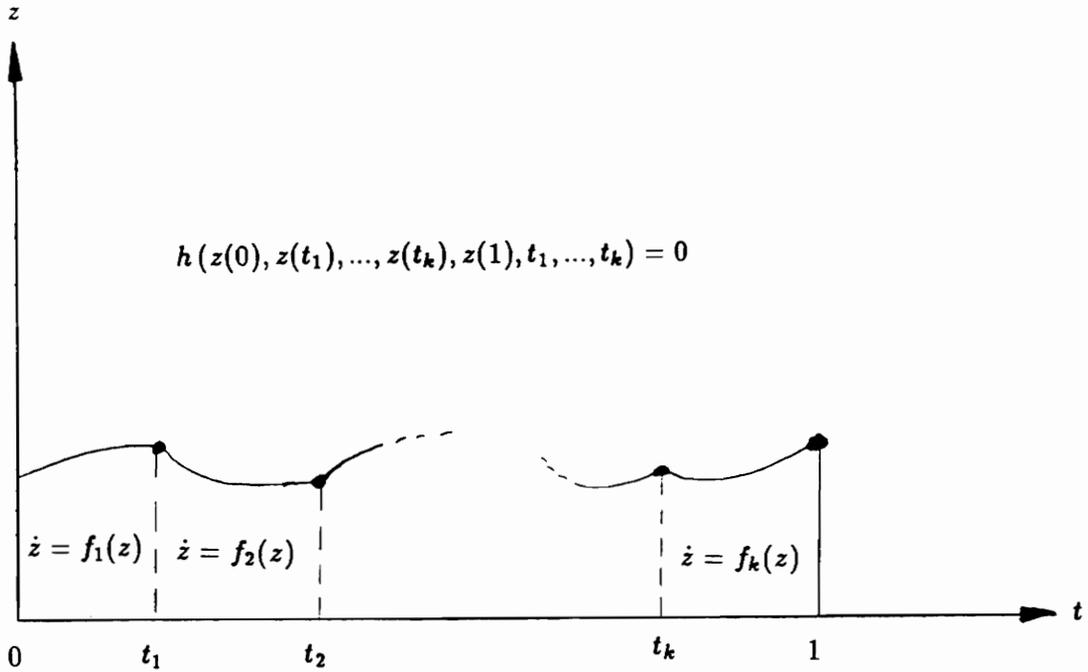
Figure 2.2: Multi-Point Boundary Value Problem

the solution one could also choose $z(0), (t_1 - t_0),, ..., (t_{k+1} - t_k)$. Or, in order to make things more complicated, one could choose some point t_i as a starting point and obtain the trajectory by integrating backward and forward. In this case the independent variables can be chosen as $z(t_i), t_1, ..., t_k$. Mathematically it does not make any difference how a trajectory is represented. However, numerically one may benefit considerably from a change in parameters. Before discussing these points in more detail a simple example is given in the next Section to answer questions about existence and uniqueness of solutions to BVPs.
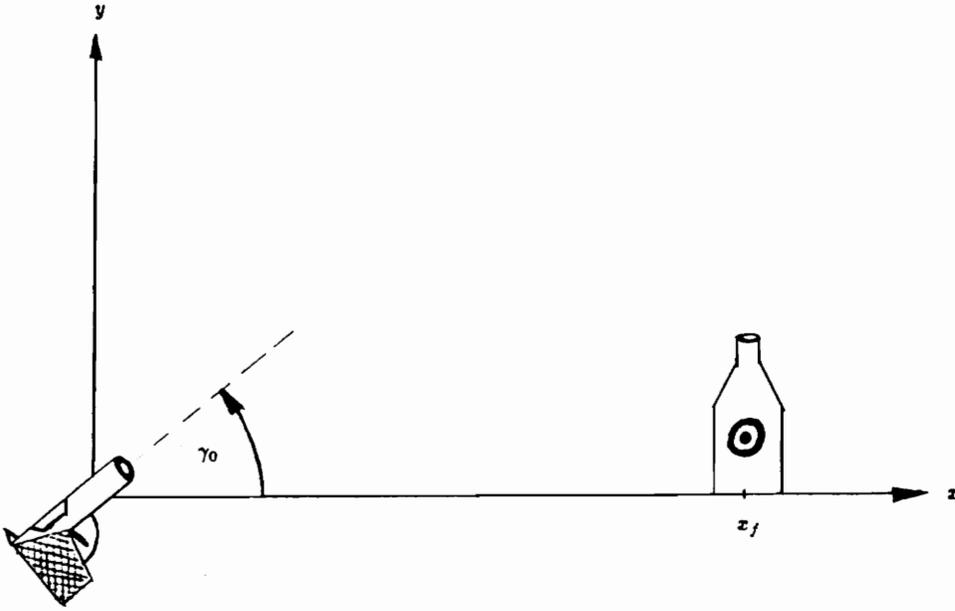
Figure 2.3: A Physical Example for a Boundary Value Problem

2.3 Existence and Uniqueness of Solutions to BVPs

Let us consider the simple physical example of shooting at an (empty) beer bottle with a
gun (see Figure 2.3). It is easily verified that (neglecting atmospheric drag) this problem
can be described by the following BVP

$$
\begin{array}{llll}
\frac{dx}{d\tau} & = & (v\cos\gamma_0)\Delta t & x(0) = 0 \\[2mm]
\frac{dy}{d\tau} & = & (v\sin\gamma_0 - g\tau)\Delta t & y(0) = 0 \\[2mm]
\frac{d\gamma_0}{d\tau} & = & 0 & x(1) = x_f \geq 0 \\[2mm]
\frac{d\Delta t}{d\tau} & = & 0 & y(1) = 0
\end{array}
\tag{2.3}
$$

Here g is the gravitational acceleration, v is the initial velocity of the bullet, γ_0 is its
initial flight path angle, and Δt is the total time that it takes the bullet to travel from
starting point $(x, y) = (0, 0)$ to the target point $(x, y) = (x_f, 0)$. In order to normalize

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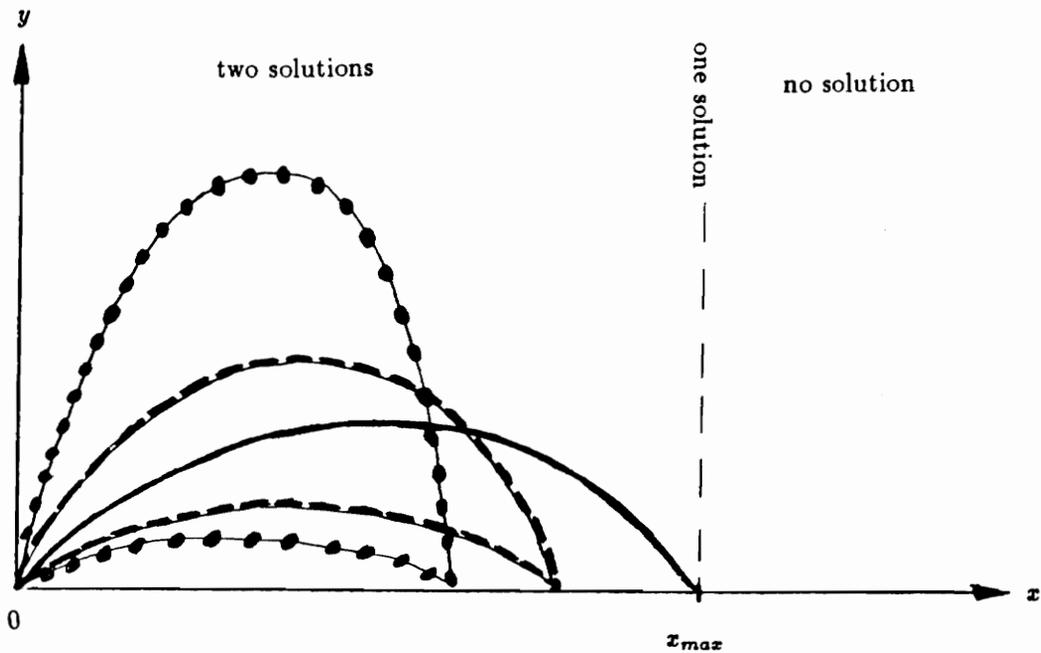
Figure 2.4: Structure of Solution for Example Problem

the time interval to $[0,1]$ the independent variable, time t, is replaced by some scaled

time-like variable τ. From physics it is immediately clear that, since v_0 is fixed, there is

an $x_{max} > 0$ such that (see Figure 2.4)

problem 2.3 has no solution if $x_f > x_{max}$,

problem 2.3 has exactly one solution if $x_f = x_{max}$,

problem 2.3 has exactly two solutions if $x_f < x_{max}$.

Hence we see that even the numerical value of the prescribed boundary conditions

may have a strong influence on existence and uniqueness of the solution. In fact, for

the practically important case of non-linear BVPs there are hardly any theorems that

can give a priori answers to existence and uniqueness questions. In our simple example

only physical intuition can lead to an immediate answer without actually solving the BVP

numerically. It should be noted that physical intuition usually cannot be applied to BVPs associated with optimal control problems. Even if the behavior of the plant is well-known and understood, the evolution of the costates is usually quite unpredictable and one has to rely exclusively on information obtained from running numerical experiments.

2.4   Numerical Well-Posedness

It is clear that different zero-finding problems can be formulated in association with the same given BVP. Instead of integrating forward one could integrate the trajectory backward, or one could even start the integration in the interior of the time interval and obtain the trajectory through successive forward and backward integration. Analytically, the associated zero-finding problems are usually somewhat equivalent, but numerically an intelligent choice of the parameters to describe a trajectory and the conditions to determine these parameters can make all the difference between not getting a result at all and getting a result pretty easily. Smooth zero-finding problems are usually solved by some kind of Newton Method, which, in each iteration, solves a linear system of equations. Hence it is natural to first investigate the scalar equation

$$ax + b = 0; \ a \in \mathbf{R}, \ b \in \mathbf{R}; \ a, b \ \text{given.} \tag{2.4}$$

A digital computer with finite word length will first add round-off errors to the input variables a and b. Then these round-off errors will propagate according to the type of operation that has to be performed on a and b to compute x, and finally another round-off error is added to the result x. From these considerations it becomes clear that the problem is best suited for numerical treatment if a is of order unity, i.e. $|a| = 1$. If $|a|$

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is very small ($|a| << 1$) then even small perturbations in a (caused by round-off errors) imply large changes in the solution x. On the other hand, if $|a|$ is very large ($|a| >> 1$) then the magnitude of the solution is very small and even small perturbations in form of round-off errors added to the solution x may corrupt the relative precision of the result considerably. The generalization of this concept from scalar to vector equations is quite straightforward and one comes to the conclusion that $\| A \|$ close to unity is desirable for problems of the form

$$Ax + b = 0; \quad A \in \mathbf{R}^{n,n}, \, b \in \mathbf{R}^n. \tag{2.5}$$

However, careful examination of the non-scalar case shows that there is another quantity, namely the so-called condition of A defined by $\text{cond}(A) = \| A \| \, \| A^{-1} \|$ which is of fundamental importance for the numerical solution of linear systems of equations. We have the following theorem (see for example Stoer & Bulirsch [43]):

Theorem 2.4.1 *Let $x \in \mathbf{R}^n$ be a solution to the linear system of equations $Ax = b$ with $A \in \mathbf{R}^{n,n}$, $b \in \mathbf{R}^n$ given. If $x + \delta x$ is a solution to $(A + \delta A)(x + \delta x) = b + \delta b$ for given $\delta A \in \mathbf{R}^{n,n}$, $\delta b \in \mathbf{R}^n$, then we have*

$$\frac{\| \delta x \|}{\| x \|} \left(1 - \| A^{-1} \| \, \| A \| \, \frac{\| \delta A \|}{\| A \|} \right) \leq \| A^{-1} \| \, \| A \| \left(\frac{\| \delta b \|}{\| b \|} + \frac{\| \delta A \|}{\| A \|} \right). \tag{2.6}$$

This is true for any norm on $\mathbf{R}^n$ (the space in which x and b live). The matrix norm is understood as the natural norm induced by the norm chosen for $\mathbf{R}^n$, i.e. $\| A \| = \sup_{\|x\|=1} \frac{\|Ax\|}{\|x\|}$.

PROOF: Assume for given A, b we have x such that $Ax = b$. Then, for given δA, δb we have

$$(A + \delta A)(x + \delta x) = b + \delta b$$

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$$Ax + A\delta x + \delta Ax + \delta A\delta x = b + \delta b$$

$$A\delta x + \delta A\delta x = \delta b - \delta Ax$$

$$\| A\delta x + \delta A\delta x \| = \| \delta b - \delta Ax \|$$

$$\| A\delta x \| - \| \delta A\delta x \| \leq \| \delta b \| + \| \delta A \| \| x \|$$

Also

$$\| \delta x \| \;=\; \| A^{-1}A\delta x \|$$

$$\leq\; \| A^{-1} \| \| A\delta x \|$$

$$\Rightarrow\; \| A\delta x \| \geq \frac{\| \delta x \|}{\| A^{-1} \|}.$$

Using this we get

$$\frac{\| \delta x \|}{\| A^{-1} \|} - \| \delta A \| \| \delta x \| \leq \| \delta b \| + \| \delta A \| \| x \|$$

$$\| \delta x \| \left(\frac{1}{\| A^{-1} \|} - \| \delta A \| \right) \leq \| \delta b \| + \| \delta A \| \| x \|$$

$$\frac{\| \delta x \|}{\| x \|} \left(1 - \| \delta A \| \| A^{-1} \| \right) \leq \| \delta b \| \frac{\| A^{-1} \|}{\| x \|} + \| A^{-1} \| \| \delta A \|$$

$$\frac{\| \delta x \|}{\| x \|} \left(1 - \frac{\| \delta A \| \| A^{-1} \| \| A \|}{\| A \|} \right) \leq \| \delta b \| \frac{\| A^{-1} \| \| Ax \|}{\| x \| \| b \|} + \| A^{-1} \| \| \delta A \|$$

$$\frac{\| \delta x \|}{\| x \|} \left(1 - \| A^{-1} \| \| A \| \frac{\| \delta A \|}{\| A \|} \right) \leq \| A^{-1} \| \| A \| \frac{\| \delta b \|}{\| b \|} + \| A^{-1} \| \| A \| \frac{\| \delta A \|}{\| A \|}$$

$$\frac{\| \delta x \|}{\| x \|} \left(1 - \| A^{-1} \| \| A \| \frac{\| \delta A \|}{\| A \|} \right) \leq \| A^{-1} \| \| A \| \left(\frac{\| \delta b \|}{\| b \|} + \frac{\| \delta A \|}{\| A \|} \right).$$

q.e.d.

Essentially this theorem states that for small perturbations $\delta b \in \mathbf{R}^n$, $\delta A \in \mathbf{R}^{n,n}$,

$$\mathrm{cond}(A) := \| A^{-1} \| \| A \|$$

is an amplification factor by which relative errors in the data A, b may influence the relative precision of the result x even if all operations are performed with total precision. This property can be best demonstrated by the simple example problem of finding the intersection point between two straight lines in the horizontal plane. Let the two lines be given by the equations $n_1^T x = b_1$ and $n_2^T x = b_2$ where $n_1 \in \mathbf{R}^2$, $n_2 \in \mathbf{R}^2$, $b_1 \in \mathbf{R}$, $b_2 \in \mathbf{R}$. Then the intersection point $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbf{R}^2$ is determined by the linear 2×2 system of equations $\begin{bmatrix} n_1^T \\ n_2^T \end{bmatrix} x = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$. It is clear that without loss of generality the normal vectors n_1, n_2 can be assumed normalized, i.e. $\| n_1 \| = \| n_2 \| = 1$, so that matrix $\begin{bmatrix} n_1^T \\ n_2^T \end{bmatrix}$ has norm close to unity. Then $\text{cond}(A)$ is large if and only if matrix $\begin{bmatrix} n_1^T \\ n_2^T \end{bmatrix}$ has a singular value close to zero, i.e. if n_1 and n_2 are close to parallel. Applying Theorem 2.4.1, this implies that it is numerically difficult to calculate the intersection point between two straight lines in $\mathbf{R}^2$ if and only if these two lines are close to parallel. This result is of no surprise. Of course we expect difficulties in calculating the intersection point between two nearly parallel lines and it is immediately clear that the obtained solution may be worthless if the normal directions n_1, n_2 are corrupted even with only very small errors.

To summarize this point we note that for a given BVP the associated zero-finding problem

$$F(x) = 0,$$

$$F : \begin{cases} \mathbf{R}^p & \to & \mathbf{R}^p \\ \\ x & \mapsto & F(x) \end{cases}$$

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should be formulated such that both, norm and condition of the Jacobian matrix $J = \frac{\partial F}{\partial x}$ evaluated at the solution point x^* are "as close as possible" to unity. To be more specific, a norm or condition in the order of 10^{10} is still pretty much o.k. on a computer with double precision accuracy (16 decimal digits mantissa). When norm and/or condition go close to $\frac{1}{\epsilon}$, where ϵ = machine precision, results in each Newton iteration step are governed mainly by round-off errors and are becoming useless. Both, norm and condition of the Jacobian associated with the problem $F(x) = 0$ can be influenced also by scaling parameters x and/or conditions F. Even though the pure Newton Method is invariant under scaling, the (usually applied) Relaxed Newton Method is not and scaling has an effect beyond the amplification of round-off errors (see for example Stoer & Bulirsch [43] or Ortega & Rheinboldt [35]).

There are numerous other criteria to examine when setting up a numerically well-posed zero-finding problem associated with a given BVP. In general, BVPs are full of surprises and it is probably not possible to give a reasonably complete list of what might go wrong. The following two practical examples are given in order to demonstrate somewhat typical problems that may occur and to show how these problems can be solved.

For the first example let us have a look at the rocket ascent problem treated in Chapter 4. As described in Section 4.2 the rocket model involves only a single control, namely thrust T. This control appears only linearly in the equations of motion and is subject to fixed bounds $0 \leq T \leq T_{max}$. For final time t_f prescribed between (roughly) 0.13 and 0.15 it is found that the solution is of the structure

full thrust − singular thrust − full thrust − zero thrust.

Figure 2.5 gives a schematic representation of the BVP associated with this switching

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structure and indicates two different zero-finding problems that may be associated with this BVP, designated type 1 and type 2. The first zero-finding problem is obtained by integrating from initial time 0 to switching time t_1, then to switching time t_2, and so on until final time t_f is reached. The set of parameters that makes this procedure unique is given by the initial states and costates $r(0)$, $v(0)$, $m(0)$, $\lambda_r(0)$, $\lambda_v(0)$, $\lambda_m(0)$, and the lengths of the integration intervals Δt_1, Δt_2, Δt_3, Δt_4. (Of course parameters $r(0)$, $v(0)$, $m(0)$ can be eliminated directly by using the conditions numbered 1), 2), 3) in Figure 2.5, but this does not change the nature and the severity of the problem that is described below). If the rocket ascent problem is solved successively for different values of prescribed final time t_f, it turns out that if the prescribed final time t_f approaches 0.15 from below, the length of the second full thrust arc $[t_2, t_3]$ goes to zero and ultimately vanishes completely, so that switching structure

full thrust $-$ singular thrust $-$ full thrust $-$ zero thrust

blends into switching structure

full thrust $-$ singular thrust $-$ zero thrust

for $t_f > 0.15$. In this process the numerical solution of the zero-finding problem of type 1 fails if Δt_3, the length of the second full thrust arc becomes less than 10^{-3}. Switching times t_2 and t_3 are basically determined by the condition that the switching function be zero. For the numerical calculation of the Jacobian J associated with the zero-finding problem type 1, small perturbations have to be applied in the initial states $r(0)$, $v(0)$, $m(0)$, $\lambda_r(0)$, $\lambda_v(0)$, $\lambda_m(0)$ as well as the lengths of the arcs Δt_1, Δt_2, Δt_3, Δt_4. As usual, partial derivatives are then approximated by quotients of finite differences. This procedure is based on the assumption that finite differences and partial derivatives don't differ

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Figure 2.5: Schematic Representation of Boundary Value Problem
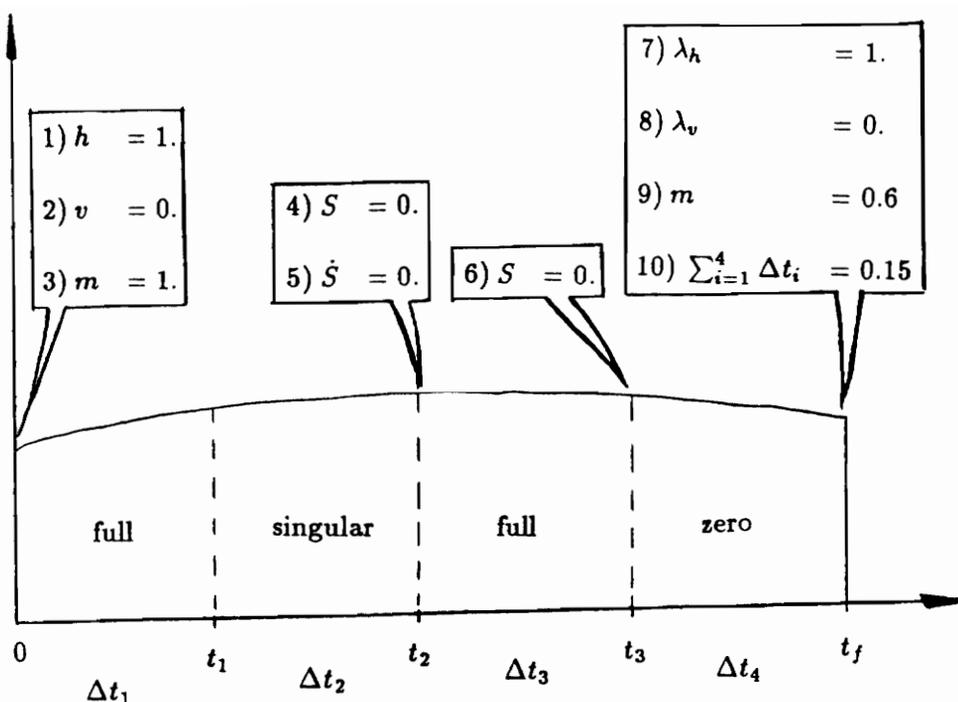
zero-finding problem type 1

10 parameters: $h(0)$, $v(0)$, $m(0)$, $\lambda_h(0)$, $\lambda_v(0)$, $\lambda_m(0)$, Δt_1, Δt_2, Δt_3, Δt_4

10 conditions: 1), 2), 3), 4), 5), 6), 7), 8), 9), 10)

zero-finding problem type 2

8 parameters: $h(t_2)$, $v(t_2)$, $m(t_2)$, $\lambda_v(t_2)$, Δt_1, Δt_2, Δt_3, Δt_4

8 conditions: 1), 2), 3), 6), 7), 8), 9), 10)

too much. Of course this is true as long as the numerical offset for the finite difference calculations are "small enough". But what is small enough may vary considerably from problem to problem. In our example where the length of the second full thrust arc $[t_2, t_3]$ is of order 10^{-3} the magnitude of switching function S is at most in the order 10^{-6} on $[t_2, t_3]$ (as $S = \dot{S} = 0$ at t_2 and $S = 0$ at t_3) and a perturbation applied at initial time can be called small only if it yields a perturbation in S at times t_2, t_3 that is "small" compared to 10^{-6}, the order of magnitude of S on $[t_2, t_3]$. But the effect of perturbations at initial time that are small enough to satisfy this criterion are likely to be dominated by round-off errors that build up during the integration. Hence the Jacobian matrix can not be calculated with sufficient precision and the Newton Method must fail.

It is easy to visualize that the problem above can be avoided if the integration of the trajectory is started at switching time t_2 rather than at initial time $t_0 = 0$ (zero-finding problem of type 2 in Figure 2.5). In this case perturbations in the parameters $(r(t_2),$ $v(t_2)$, $m(t_2)$, $\lambda_r(t_2)$, $\lambda_v(t_2)$, $\lambda_m(t_2)$, Δt_1, Δt_2, Δt_3, $\Delta t_4)$ have a direct effect on conditions $S|_{t_2} = 0$, $\dot{S}|_{t_2} = 0$, $S|_{t_3} = 0$ and can be chosen reasonably large. In the zero-finding problem of type 1 perturbations applied at initial time are going through an amplification phase along the integration on time interval $[0, t_2]$ before effects on the above conditions are measured.

Another example for how different zero-finding problems associated with the same BVP can have completely different numerical solvability qualities is provided by the aircraft range optimization problem treated in Chapter 5. As described in Sections 5.2 and 5.4 the aircraft model involves two controls, namely throttle δ and load factor n. Throttle δ appears only linearly in the equations of motion and both controls, δ and n, are subject

to fixed bounds $0 \leq \delta \leq 1$, $-n_{max} \leq n \leq +n_{max}$. Additionally, a dynamic pressure limit which, in the context of optimal control, represents a first-order state inequality constraint has to be satisfied by the trajectories. Several different switching structures are found to solve the problem upon varying the prescribed value of n_{max} while keeping the prescribed initial and final states fixed. A schematic representation of one of the switching structures (denoted by (S6)) is given in Figure 5.4. The switching structure consists of five arcs. With respect to the dynamic pressure limit $C_0 := q - q_{max} \leq 0$ the switching structure is

not active − active − active − active − not active

(and this is the only property of switching structure (S6) that is important for this Section). The most obvious zero-finding problem associated with this BVP is to search numerical values of the 14 parameters $E(0)$, $h(0)$, $\gamma(0)$, $x(0)$, $\lambda_E(0)$, $\lambda_h(0)$, $\lambda_\gamma(0)$, $\lambda_x(0)$, l_0, Δt_1, Δt_2, Δt_3, Δt_4, Δt_5 such that the 14 conditions numbered 1), ..., 14) in Figure 5.4 are satisfied. But it turns out that it is numerically nearly impossible to solve the problem this way. Small perturbations in initial states, initial costates, or the lenth of the time interval Δt_1 lead to (usually slightly bigger) perturbations in the states at time t_1, the beginning of the active-state-constraint phase. Along state constrained arcs (intervals $[t_1, t_2]$, $[t_2, t_3]$, $[t_3, t_4]$) the differential equations are such that the dynamic pressure is identically constant, so that $q - q_{max} \equiv \Delta q$ on $[t_1, t_4]$, where $\Delta q = (q - q_{max})|_{t_1}$. Loosely speaking, in order to stay on a higher dynamic pressure limit over some extended period of time either the thrust has to be increased or the load factor has to be decreased. In fact, even small positive increments Δq make it impossible for the aircraft to stay on the dynamic pressure limit all along time interval $[t_1, t_4]$. Numerically it turns out that load factor n is reduced along $[t_1, t_4]$ until $n = 0$ is reached. After that point either the calcu-

lation of n stops with a negative square root or the equation $\frac{d}{dt}(q - q_{max}) = 0$ that has to be satisfied along arcs with active dynamic pressure limit is no longer fulfilled. This means that the calculation of the Jacobian is either impossible or leads to inconsistent results (depending on how the software to integrate the trajectory is set up). Again it is easy to see that all these difficulties can be avoided if the integration is started at a time where the dynamic pressure constraint is active, i.e. somewhere along the time interval $[t_1, t_4]$. The most preferrable starting point for the integration is switching time t_2 (or t_3). At this time conditions 6), 8), 9) in Figure 2.5 can be used to reduce the free parameters to the set $h(t_2), \gamma(t_2), x(t_2), \lambda_E(t_2), \lambda_x(t_2), l_0, \Delta t_1, \Delta t_2, \Delta t_3, \Delta t_4, \Delta t_5$. The conditions left to satisfy are given by equations 1), 2), 3), 4), 5), 7), 10), 11), 12), 13), 14). In this setup the characteristic properties $S \equiv 0$ along the singular arc $[t_2, t_3]$, as well as $q - q_{max} \equiv 0$ along the state constrained arcs $[t_1, t_2]$, $[t_2, t_3]$, $[t_3, t_4]$ are always satisfied irrespective of the perturbations that are applied in computing the Jacobian matrix.

2.5 A Robust, East-To-Use Software Package

In this Section we describe the software package for solving multi-point BVPs that has been used to generate all numerical results that are presented in this thesis. This package has evolved out of the needs that arise in the practical work with optimal control problems. The heart of the software is subroutine ZSCNT, an IMSL 9.2 implementation of Newton's Method for solving non-linear zero-finding problems (see IMSL 9.2 User's Manual [15]). The main program, into which the user has to make only a few entries, like the number of equations to solve, is also provided in some standard form. This main program calls
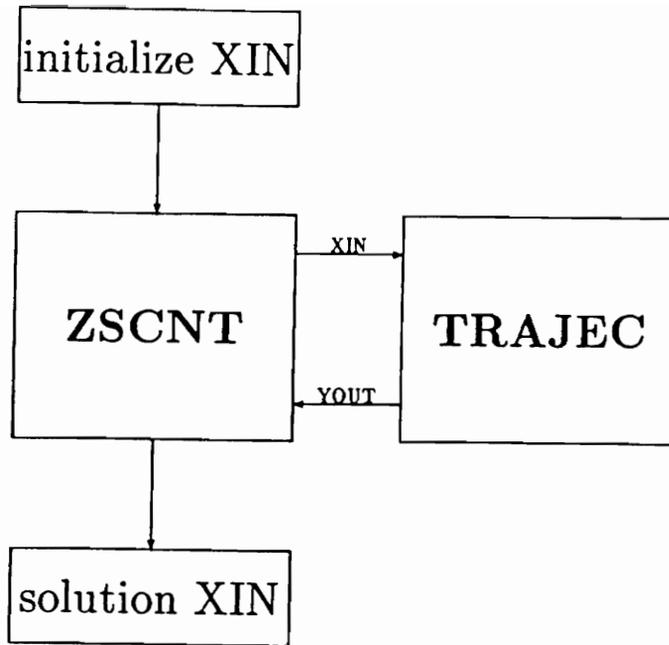
Figure 2.6: Flow Chart for Boundary Value Problem Solver

the Newton Method and the Newton Method in return calls subroutine TRAJEC in an iterative way (see Figure 2.6). For given parameter vector x (= input XIN in TRAJEC) TRAJEC computes the value of the non-linear vector-valued function $F(x)$ (= output YOUT in TRAJEC) that has to be made zero. This subroutine has to be provided completely by the user.

It is not claimed that this software package incorporates any new numerical theories. The advantages of this software come purely from its user friendliness. In practice it is usually necessary to write some program that can integrate the trajectory for a given initial state/costate vector. This should always be done in order to test the right-hand side of the differential equations (e.g. for constancy of the Hamiltonian) but may also be required to find a starting trajectory for homotopy runs. Once such a program is

written it is trivial to transform it into a subroutine and to provide the input/output interface as required in TRAJEC. Clearly, if the same subroutine that is used for software test runs can be immediately embedded into the BVP solver, then an obvious source for programming errors is eliminated.

Another advantage of the described program architecture (see Figure 2.6) is that the user has complete control over the parameters used to characterize the solution of a given BVP. All parameters have to be selected by the user and all conditions (to determine these parameters) have to be programmed explicitly in TRAJEC. Also, no automatic scaling is provided by the software package. The aim is to give the user as much control as possible. By "playing around" with scaling factors and by trying different zero-finding problems for a given BVP the user is able to get a good feeling for the BVP. In case of troubles it is usually easy to clearly identify where the difficulties are coming from. In programs with a lot of automatic features this may be very difficult. Besides, the problems and difficulties arising in the practical work with optimal control problems are so different in nature that it is probably impossible to make automatic procedures fool-proof. The software package described here leaves it up to the user to do a good or bad job in fine-tuning the numerical procedure. For the experienced user this feature is quite welcome. For the unexperienced user it provides the opportunity of learning something and of growing with the software.

Chapter 3

Existing Optimality Conditions

A standard type optimal control problem is defined. Existing optimality conditions are stated for this standard problem. Extensions to the practically important cases of singular control, interior point constraints, control constraints, and state constraints are treated. Finally, the existence of solutions and Jacobi Testing is addressed.

3.1 Introduction

In this Chapter it is intended to give an overview of existing optimality conditions for a reasonably large class of optimal control problems. In the first Section of this Chapter a very simple reference optimal control problem is defined. Section 2 states necessary conditions for optimality and points out a few general difficulties associated with optimal control problems. In Sections 3, 4, and 5 a few complications, such as control constraints, interior point constraints, and state constraints are introduced to the simple reference problem. Finally, Section 6 addresses the question of existence of a solution. Throughout this Chapter proofs are avoided in favor of giving only the basic idea. For details numerous references are provided in this Chapter.

3.2 Standard Optimal Control Problem

In this Section we consider the following simple optimization problem

Definition 3.2.1 (Standard Problem)

$$\min_{u \in (L_2[t_0, t_{\underline{\gtrless}}])^m} \Phi(x(t_f), t_f) \tag{3.1}$$

subject to the equations of motion

$$\dot{x}(t) = f(x(t), u(t)) \tag{3.2}$$

the initial conditions

$$x(t_0) = x_0; \ t_0, \ x_0 \ \text{fixed} \tag{3.3}$$

and the boundary conditions

$$\Psi(x(t_f), t_f) = 0 \tag{3.4}$$

where $u(t) \in \mathbf{R}^m$; $x(t) \in \mathbf{R}^n$; $\Phi : \mathbf{R}^{n+1} \to \mathbf{R}$; $f : \mathbf{R}^{n+m} \to \mathbf{R}^n$; $\Psi : \mathbf{R}^{n+1} \to \mathbf{R}^k$, $k \leq n$.

Remarks:

(i) It is important to allow minimization of the cost function (3.1) over a reasonably large class of control functions u. For engineering purposes we want this class of control functions to include at least the set of all piecewise continuous functions. Moreover, we want the metric defined on the principle set of functions to be such that two continuous functions f and g have distance zero ("are the same") if and only if $f(t) = g(t) \ \forall t \in [t_0, t_f]$. Besides these engineering requirements an important issue from the mathematical point of view is completeness of the set of control functions with respect to the selected metric. (Note that for instance, the minimization problem of finding the smallest number in the set $X = \{x \in \mathbf{R} | 0 < x \leq 1\}$ does not have a solution because the set X is not complete. To avoid difficulties of this kind we want the set of admissible controls to be complete). Loosely speaking, the set $L_2[t_0, t_f]$ in conjunction with the norm $\| f \|_2 = \int_{t_0}^{t_f} f^2 \, dt$ for $f \in L_2[t_0, t_f]$ can be defined as the completion of the set of all piecewise continuous functions on $[t_0, t_f]$ (completion w.r.t. the metric on L_2 as implied by the norm $\| \cdot \|_2$), and hence satisfies all desired qualifications. For all practical purposes it is possible to

view $PWC[t_0, t_1]$ (the set of all piecewise continuous functions on $[t_0, t_f]$) as the set of allowed control functions. This makes things easier to visualize. When this engineering approach leads to difficulties it will be necessary to recall that we are really minimizing over all control functions in $L_2[t_0, t_f]$.

(ii) Frequently, optimal control problems are stated in terms of a cost function

$$J = \varphi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x(t), u(t))\, dt$$

Problems of this form are not more general than the one stated in Definition 3.2.1. By introducing the additional state y as solution of

$$\dot{y} = L(x, u), \quad y(t_0) = 0$$

the above cost function takes the form

$$J = \varphi(x(t_f), t_f) + y(t_f)$$

and hence is of the general form given in Definition 3.2.1.

(iii) Frequently, optimal control problems are stated with an explicit time dependence of the right-hand side of the state equations

$$\dot{x}(t) = f(x(t), u(t), t).$$

This explicit time dependence can be transformed away by introducing the additional state y as solution of the initial value problem

$$\dot{y} = 1$$

$$y(t_0) = t_0.$$

Then, obviously, the right-hand side of the state equations is only a function of states (x, y) and controls u,

$$\dot{x} = f(x, u, y)$$

(iv) It is customary to consider initial conditions only of the simple form (3.3). It is not possible to obtain such simple initial conditions by applying some transformation on the general problem with initial/boundary conditions of the form, say

$$\overline{\Psi}(x(t_0), x(t_f), t_0, t_f) = 0 \tag{3.5}$$

Hence, considering only initial/boundary conditions of the form (3.3), (3.4) poses a nontrivial restriction on the generality of the standard optimal control problem. On the other hand, optimality conditions (and methods of deriving them) associated with the more general initial/boundary conditions (3.5) are a quite straightforward generalization of the optimality conditions (and methods of deriving them) associated with the simple conditions (3.3), (3.4).

3.3 Optimality Conditions for the Standard Problem

Assume the control function $u^*(t)$, $t \in [t_0, t_f]$, furnishes a solution to the standard problem given in Definition 3.2.1. Then necessary conditions for optimality are obtained from the following formalism (the theory also guarantees that these optimality conditions have a solution if the optimal control problem has a solution):

define the variational-Hamiltonian

$$H(x, \lambda, u) := \lambda^T f(x, u) \tag{3.6}$$

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and define the Lagrange multiplier vector λ as solution of the final value problem

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \tag{3.7}$$

$$\lambda(t_f) = \frac{\partial \Phi}{\partial x(t_f)} + \nu^T \frac{\partial \Psi}{\partial x(t_f)}. \tag{3.8}$$

Then, at (almost) every instant of time, the optimal control u^* is such that the Hamiltonian (3.6) is minimized (Minimum Principle):

$$u^*(t) = \arg \min_{u \in \mathbf{R}^m} H(x(t), \lambda(t), u). \tag{3.9}$$

An optimality condition associated with final time t_f is given by

$$H|_{t_f} + \frac{\partial \Phi}{\partial t_f} + \nu^T \frac{\partial \Psi}{\partial t_f} = 0. \tag{3.10}$$

In an engineering approach these conditions can be obtained from setting the first variation of the augmented cost function equal zero (see Appendix B). A more rigorous, yet very geometrical and illustrative approach is presented in Leitmann [25]. Leitmann extends trajectories into an $n + 1$-dimensional space (n state dimension, 1 cost dimension). Then he introduces limiting surfaces defined as the set of all points in the $n+1$-dimensional space associated with trajectories that connect arbitrary, admissible initial and final states, with the cost at initial time chosen such that each trajectory ends up with the same fixed final cost. Finally, the Lagrange multipliers introduced above are identified as (part of) the normal vector on these limiting surfaces.

Another illustrative, yet mathematically very clean approach is given in Lee & Markus [24]. For the minimum time problem Lee & Markus investigate the evolution of the set of attainability $K(t)$. This is the set of all possible states that can be reached within time t, starting at a fixed point x_0 at time t_0. Optimality conditions are derived from the

requirement that for all times $t \geq t_0$ the optimal state $x^*(t)$ has to be a boundary point of the set of attainability $K(t)$.

Finally, it should be mentioned that the optimal control formalism can be derived also in very abstract functional analysis approach (see Neustadt [33]). Most theorems and proofs still have some geometrical interpretation, but often this is hard to see because one is generally working in infinite-dimensional spaces. In some cases the results obtained from the functional analysis approach lead to a significant strengthening of theorems derived in a classical way (e.g. in [31] supplementary optimality conditions on multiplier μ associated with state inequality constraints can be derived by applying results obtained from the general multiplier theory).

It should be noted also that at each instant of time where the Hamiltonian is a sufficiently smooth function of control vector u, the Minimum Principle (3.9) implies

$$\frac{\partial H}{\partial u} = 0 \tag{3.11}$$

$$\frac{\partial^2 H}{\partial u^2} \geq 0. \tag{3.12}$$

Condition (3.12) is called the Legendre-Clebsch Condition or Convexity Condition.

3.4 Singular Control

As stated in the previous Section, at each instant of time t, the optimal control u^* has to satisfy (Minimum Principle)

$$u^*(t) = \arg \min_{u \in \mathbf{R}^m} H(x(t), \lambda(t), u) \tag{3.13}$$

It is possible that this condition does not determine u^* uniquely. Typically, and for practical applications most important, this can happen if some component u_j of control

vector u appears only linearly in the right-hand side $f(x, u)$ of the state equations (3.2) such that $\frac{\partial f(x,u)}{\partial u_j}$ is independent of all controls u and is a function of states x only. If, at some instant of time, states x and costates λ are such that $S(x, \lambda) := \frac{\partial H}{\partial u_j}$ is zero, i.e.

$$S = 0 \qquad (3.14)$$

then the Hamiltonian is independent of u_j at that instant of time and the Minimum Principle (3.13) does not furnish any information on how to choose u_j. It is clear that pointwise occurrence of this situation can be ignored. This is true as we can choose any arbitrary control value u_j at an isolated point in time without changing the evolution of states. If we are willing to think in terms of L_2-functions (rather than piecewise continuous functions) as admissible control functions then we can generalize this statement by saying that arbitrary control values on a set of measure zero along the time axis do not have any effect on the evolution of the states. Hence, we only have to investigate the case where S is zero on a set of positive measure along the time axis. After replacing all control components u_i, $i \neq j$, by functions of states x and costates λ as determined by the Minimum Principle (3.13), S is an absolutely continuous function and hence S being zero on a set of measure greater than zero implies that there is a non-trivial time interval $[t_1, t_2] \subset [t_0, t_f]$, $t_2 > t_1$, such that $S \equiv 0$ on $[t_1, t_2]$. This is called a singular control case (note that the Hessian matrix $\frac{\partial^2 H}{\partial u^2}$ is singular on $[t_1, t_2]$). In this case control component u_j is determined implicitly by condition (3.14). Explicit information on u_j can be obtained by differentiating identity (3.14) until the undetermined control u_j appears explicitly. In [19] it is shown that S has to be differentiated an even number of times, say $2q$ times for some $q \in \mathbb{N}$, until control u_j appears explicitly. Then q is called the order of the singular

control.

Hence, on an arc $[t_1, t_2]$, $t_2 > t_1$, with singular control u_j, $j \in \{1, ..., m\}$ of order q we have the following necessary conditions:

$$\{u_i | i \neq j\} = \arg \min_{u \in \mathbb{R}^m} H \tag{3.15}$$

$$S(x, \lambda)|_{t_1} = 0$$

$$\frac{dS}{dt}(x, \lambda)\Big|_{t_1} = 0$$

$$\vdots \tag{3.16}$$

$$\frac{d^{2q-1}S}{dt^{2q-1}}(x, \lambda)\Big|_{t_1} = 0$$

$$\frac{d^{2q}S}{dt^{2q}}(x, \lambda, u_j) \equiv 0 \text{ on } [t_1, t_2] \tag{3.17}$$

where

$$S = \frac{\partial H}{\partial u_j} \tag{3.18}$$

with all controls u_i, $i \neq j$ expressed in terms of x, λ as obtained from (3.15). A more general definition of singular control which applies also for non-linearly appearing controls, as well as for arcs with active control and/or state constraints is given in Section 6.4.

In the 1960's singular control was found to play an important role in numerous engineering problems of great practical interest (e.g. the atmospheric ascent of the Saturn V rocket). This prompted intense research in supplementary optimality conditions for singular control. Note that along singular arcs the classical Legendre-Clebsch condition

$$\frac{\partial^2 H}{\partial u^2} \geq 0 \tag{3.19}$$

is satisfied for the singular control component u_j only in the weak form (i.e. with equality). In 1964 H. J. Kelley [17] was the first to state second order necessary conditions for this

type of control. In the following years many authors e.g. Kelley & Kopp & Moyer [18], [19], Robbins [36], Goh [11], Krener [20] have extended Kelley's idea to derive what is now known as the Generalized Legendre-Clebsch condition. In compact form it can be stated as

$$(-1)^q \frac{\partial}{\partial u_j} \left[\frac{d^{2q}}{dt^{2q}} \left(\frac{\partial H}{\partial u_j} \right) \right] \geq 0 \tag{3.20}$$

In Chapter 6, Goh's Necessary Condition, which implies (3.20), is extended to the case of singular control along state/control constrained arcs.

3.5 Interior Point Conditions

Let us introduce the additional condition

$$N \left(x(t_1^-), x(t_1^+), t_1 \right) = 0, \quad N : \mathbf{R}^{2n+1} \rightarrow \mathbf{R}^p \tag{3.21}$$

to the standard optimal control problem stated in Definition 3.2.1. With conditions of this form it is possible to formulate explicit conditions on the states and/or prescribe discontinuous jumps in states at points in the interior of the trajectory. A typical example is the staging of a rocket, where the mass changes discontinuously at staging time, say, t_1. The optimality conditions associated with constraint (3.21) can be easily derived by investigating the first variation of the augmented cost function as demonstrated in Appendix B. For convenience the results are restated below (superscripts $-$, $+$ denote evaluation just before t_1 and just after t_1, respectively):

$$N \left(x(t_1^-), x(t_1^+), t_1 \right) = 0, \quad N : \mathbf{R}^{2n+1} \rightarrow \mathbf{R}^p \tag{3.22}$$

$$\left(l^T \frac{\partial N}{\partial x_1^+} + \lambda^{+T} \right) dx_1^+ + \left(l^T \frac{\partial N}{\partial x_1^-} - \lambda^{+T} \right) dx_1^- + \left(l^T \frac{\partial N}{\partial t_1} - H^+ + H^- \right) dt_1 = 0 \tag{3.23}$$

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for all $[dx_1{}^+, dx_1{}^-, dt_1] \in \mathbf{R}^{n+n+1}$ with

$$\frac{\partial N}{\partial x_1{}^+} dx_1{}^+ + \frac{\partial N}{\partial x_1{}^-} dx_1{}^- + \frac{\partial N}{\partial t_1} dt_1 = 0 \tag{3.24}$$

The constant multiplier vector $l \in \mathbf{R}^p$ compensates for the p degrees of freedom lost through condition $N = 0$.

In the practically important case where all states are continuous at t_1 and where time t_1 is free, i.e.

$$N(x(t_1)) = 0$$

the optimality conditions simplify to

$$N(x(t_1)) = 0 \tag{3.25}$$

$$\lambda^{+T} = \lambda^{-T} - l^T \frac{\partial N}{\partial x_1} \tag{3.26}$$

$$H^+ - H^- = 0. \tag{3.27}$$

We will make use of this result later in Section 3.7 when we are dealing with state constraints.

3.6 Control Constraints

In the standard optimal control problem stated in Definition 3.2.1 the range of control vector function $u \in (L_2[t_0, t_f])^m$ is assumed to be all of $\mathbf{R}^m$. In this Section we consider control constraints of the general form

$$g(x, u) \leq 0 \tag{3.28}$$

with $g : \mathbf{R}^{n+m} \to \mathbf{R}^p$, $1 \le p \le m$, and

$$\text{rank } \frac{\partial g}{\partial u} = p. \tag{3.29}$$

Assume u^* furnishes a solution to the optimal control problem (3.1), (3.2), (3.3), (3.4), (3.28) and assume control constraint (3.28) is active on some subinterval $[t_1, t_2]$ of $[t_0, t_f]$. (i.e. $g(x, u^*) = 0$ on $[t_1, t_2]$), and control constraint (3.28) is non-active on $[t_0, t_1) \cup (t_2, t_f]$ (i.e. $g(x, u^*) < 0$ on $[t_0, t_1) \cup (t_2, t_f]$). Then necessary conditions for optimality of u^* are obtained from the following formalism: define the Hamiltonian

$$H(x, \lambda, u) = \lambda^T f(x, u) \tag{3.30}$$

and define the Lagrange multiplier vector λ as solution of the final value problem

$$\dot{\lambda} = -\frac{\partial H}{\partial x} - \mu \frac{\partial g}{\partial x} \tag{3.31}$$

$$\lambda(t_f) = \frac{\partial \Phi}{\partial x(t_f)} + \nu^T \frac{\partial \Psi}{\partial x(t_f)}. \tag{3.32}$$

Then, at (almost) every instant of time, the optimal control u^* is such that the Hamiltonian (3.6) is minimized subject to all active control constraints (Minimum Principle):

$$u^*(t) = \arg \min_{u \in \mathbf{R}^m} H(x(t), \lambda(t), u) \quad \text{on } [t_0, t_1) \cup (t_2, t_f] \tag{3.33}$$

$$u^*(t) = \arg \min_{u \in \mathbf{R}^m,\, g(x(t), u) = 0} H(x(t), \lambda(t), u) \quad \text{on } [t_1, t_2]. \tag{3.34}$$

The multiplier vector function of time $\mu : \mathbf{R} \to \mathbf{R}^p$ satisfies

$$\begin{cases} \mu = 0 & \text{on intervals where } g(x, u) < 0 \\ \frac{\partial H}{\partial u} + \mu^T \frac{\partial g}{\partial u} = 0 & \text{on intervals where } g(x, u) = 0. \end{cases} \tag{3.35}$$

The "switching times" t_1, t_2 are determined from the condition that the Hamiltonian be continuous

$$H(t_1{}^+) = H(t_1{}^-) \tag{3.36}$$

$$H(t_2{}^+) = H(t_2{}^-). \tag{3.37}$$

An optimality condition associated with final time t_f is given by

$$H|_{t_f} + \frac{\partial \Phi}{\partial t_f} + \nu^T \frac{\partial \Psi}{\partial t_f} = 0. \tag{3.38}$$

Again, the easiest way to obtain these results is to analyze the total variation of the augmented cost function. For the case of state constraints instead of control constraints this is demonstrated in Jacobson, Lele, Speyer [24]. (Mathematically, state constraints can be viewed as control constraints in conjunction with additional interior point constraints, see next Section). An alternative way of derivation which is even easier, but sometimes very powerful, is given as follows:

First note that (as long as we don't have "chattering control", see [24]) every instant of time $t \in [t_0, t_f]$ belongs to a time interval $[\tau_1, \tau_2]$ of length > 0 which is either completely constrained (i.e. $g(x, u) \equiv 0$ on $[\tau_1, \tau_2]$) or completely unconstrained (i.e. $g(x, u) < 0$ on $[\tau_1, \tau_2]$). By the Principle of Optimality (see [24]) the control on unconstrained arcs is determined from the usual unconstrained optimality conditions (3.6), (3.7), (3.8), (3.9), (3.10), which is equivalent to conditions (3.30), (3.31), (3.32), (3.33), (3.35), (3.38) with $\mu = 0$. On arcs with active control constraints, condition $g(x, u) = 0$, $g : \mathbf{R}^{n+m} \rightarrow \mathbf{R}^p$ can be viewed as specifying p controls w ($u = \begin{bmatrix} v \\ w \end{bmatrix}$, $v \in \mathbf{R}^{m-p}$, $w \in \mathbf{R}^p$) in terms of states x and the remaining $m - p$ controls v. The existence of a solution $w = h(x, v)$ of $g(x, u) = g(x, v, w) = 0$ is guaranteed at least locally by assumption (3.29) (see Implicit Function Theorem [2]) even if an explicit solution of $g(x, v, w) = 0$ is not possible. Upon substituting $w = h(x, v)$ the constrained optimal control problem is transformed into an unconstrained optimal control problem. For the evolution of costate vector λ we get from

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(3.7) (chain rule)

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} - \frac{\partial H}{\partial w}\frac{\partial h}{\partial x}.$$

From differentiation of the identity $g(x, v, h(x, v)) \equiv 0$ w.r.t. x we obtain

$$\frac{\partial h}{\partial x} = -\left(\frac{\partial g}{\partial w}\right)^{-1}\left(\frac{\partial g}{\partial x}\right),$$

so that the adjoint differential equation can be written as

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} - \underbrace{\frac{\partial H}{\partial w}\left(-\frac{\partial g}{\partial w}\right)^{-1}}_{=:\mu^T}\frac{\partial g}{\partial x}.$$

Similarly, the constrained minimization problem (3.34) written in the form

$$u^* = [v^*, w^*] = \arg\min_{u \in \mathbf{R}^r, \, g(x,u)=0} H \tag{3.39}$$

yields

$$g(x; v, w) = 0 \tag{3.40}$$

$$\frac{\partial H}{\partial w} + \underbrace{\frac{\partial H}{\partial w}\left(-\frac{\partial g}{\partial w}\right)^{-1}}_{=:\mu^T}\frac{\partial g}{\partial w} = 0 \tag{3.41}$$

and we see that multiplier μ in (3.31), (3.35) is exactly the multiplier obtained from applying the Kuhn-Tucker conditions to the finite dimensional minimization problem (3.39). This basic concept is the general idea used in Chapter 6 to derive an extension of Goh's Necessary Condition for singular control along arcs with active state and/or control constraints.

3.7 State Constraints

In the previous two Sections we introduced control constraints (3.28) and interior point constraints (3.21) to the standard optimal control problem stated in Definition 3.2.1. In

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this Section we consider state constraints. These are of the general form

$$h(x) \leq 0 \qquad\qquad (3.42)$$

with $h : \mathbf{R}^n \to \mathbf{R}^p$, $1 \leq p \leq m$. As in the previous Section, the Principle of Optimality (see [24]) implies that along time intervals where the state constraint is non-active the trajectory evolves as if there were no state constraints at all, i.e. equations (3.2), (3.6), (3.7), (3.9) are valid. Along time intervals, say $[t_1, t_2]$, where the state constraint is active we note that

$$h(x) \equiv 0 \text{ on } [t_1, t_2]$$

is equivalent into

$$h^{(0)}(x) = 0 \quad \text{at } t = t_1$$

$$\vdots \qquad\qquad (3.43)$$

$$h^{(q-1)}(x) = 0 \quad \text{at } t = t_1$$

$$h^{(q)}(x, u) \equiv 0 \text{ on } [t_1, t_2]. \qquad\qquad (3.44)$$

Here superscript (i) denotes i-th total time derivative and q is the smallest integer $i \in \mathbf{N}$ such that u appears explicitly in $h^{(i)}$. Hence we see that an active state constraint $h(x) \equiv 0$ on some interval $[t_1, t_2]$ is equivalent into control constraint (3.44) along $[t_1, t_2]$ and interior point constraints (3.43) at time $t = t_1$. The regularity condition (3.29) for control constraints translates immediately into rank $\frac{\partial h^{(q)}}{\partial u} = p$. Optimality conditions implied by (3.43), (3.44) are given in the previous two Sections. For convenience these results are restated below:

Optimality conditions for entering the state constrained arc:

$$h^{(0)}(x) = 0 \quad \text{at } t = t_1$$

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$$\vdots$$

$$h^{(q-1)}(x) = 0 \quad \text{at } t = t_1$$

$$H^+ - H^- = 0 \text{ at } t = t_1$$

$$\lambda^+ = \lambda^- - l_0 \frac{\partial h^{(0)}}{\partial x} - \cdots - l_{q-1} \frac{\partial h^{(q-1)}}{\partial x} \quad \text{at } t = t_1.$$

Optimality conditions in the interior of the state constrained arc:

$$u^*(t) = \arg \min_{u \in \mathbb{R}^m, g(x(t),u)=0} H(x(t), \lambda(t), u)$$

$$\dot{x} = f(x, u)$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} - \mu \frac{\partial h^{(q)}}{\partial x}.$$

The multiplier μ is determined by (3.35) (with g replaced by h), and hence, as noted earlier, is exactly the multiplier obtained from applying the Kuhn-Tucker conditions applied on the finite-dimensional minimization problem (3.34) (with g replaced by h). From this observation we find immediately the supplementary optimality condition

$$\mu \geq 0.$$

Even stronger supplementary optimality conditions on multiplier μ are given by

$$(-1)^i \frac{d^i \mu}{dt^i} > 0 \quad \text{for } i = 0, 1, ..., q - 1. \tag{3.45}$$

This result is obtained from linking state constraint $g(x) \equiv 0$ directly to the cost function rather than splitting $g(x) \equiv 0$ up into an interior point constraint and a control constraint as shown in (3.43), (3.44) and linking both parts separately. The theoretical background

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for this procedure is provided by a generalized multiplier theory (see [33]). Result (3.45) is then basically obtained through simple transformations involving integration by parts (see [31]).

Optimality conditions for leaving the state constrained are:

$$H^+ - H^- = 0 \text{ at } t = t_2.$$

In contrary to control constraints we have to consider also the possibility of a state constraint being active only at an isolated point in time, a so-called touch point. (Note that a control constraint being active at an isolated point in time, say t_1, is equivalent to associating a fixed numerical value with the control function $u(t)$ at time t_1. This is not sensible as point evaluation for L_2-functions is not well-defined. Hence non-trivial touch-points can never exist for control constraints). In case of a touch point, say t_1, the only active constraints are the interior point constraints (3.43), while before t_1 and after t_1 (at least in some neighborhood) the trajectory evolves like a free trajectory. Hence for a touch point t_1 associated with a q-th order state inequality constraint we get

$$h^{(0)}(x) = 0 \quad \text{at } t = t_1$$

$$\vdots$$

$$h^{(q-1)}(x) = 0 \quad \text{at } t = t_1$$

$$H^+ - H^- = 0 \text{ at } t = t_1$$

$$\lambda^+ = \lambda^- - l_0 \frac{\partial h^{(0)}}{\partial x} - \cdots - l_{q-1} \frac{\partial h^{(q-1)}}{\partial x}.$$

In [31] it has been shown for the case of scalar control u that touch points are not possible for state constraints of order $q = 1$, except if very special conditions are satisfied. In

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Chapter 7 this result is generalized to the case of controls of arbitrary dimensions $m \in \mathbf{N}$.

3.8 Existence of a Solution and Jacobi's Condition

In the formulation of the Minimum Principle it is always assumed that the optimal control problem under consideration does have a solution. It is possible then to conclude that the optimal solution must satisfy the optimality conditions stated in the previous Sections. In general it is not possible to reverse this process, i.e. an extremal which satisfies the necessary conditions stated in the previous Sections need not furnish a solution to the optimal control problem. In Lee & Markus [24] fairly general existence theorems are stated. One of these theorems is given as follows:

Theorem 3.8.1 *Consider the non-linear process in* $\mathbf{R}^n$

$$\dot{x} = f(x, t, u) \tag{3.46}$$

The data are as follows:

1) The initial and target set $X_0(t)$ *and* $X_1(t)$ *are nonempty compact sets varying continuously in* $\mathbf{R}^n$ *for all* t *in the basic prescribed compact interval* $\tau_0 \le t \le \tau_1$.

2) The control constraint set $\Omega(x, t)$ *is a nonempty compact set varying continuously in* $\mathbf{R}^m$ *for* $(x, t) \in \mathbf{R}^n \times [\tau_0 \le t \le \tau_1]$.

3) The state constraints are (possibly vacuous) $h^1(x) \le 0$, ..., $h^r(x) \le 0$, *a finite or infinite family of constraints, where* h^1, ..., h^r *are real continuous functions on* $\mathbf{R}^n$.

4) The family $\mathbf{F}$ *of admissible controllers consists of all measurable functions* $u(t)$ *on various time intervals* $t_0 \le t \le t_1$ *in* $[\tau_0 \le t \le \tau_1]$ *such that each* $u(t)$ *has a response* $x(t)$ *on* $t_0 \le t \le t_1$ *steering* $x(t_0) \in X(t_0)$ *to* $x(t_1) \in X(t_1)$ *and* $u(t) \in \Omega(x(t), t)$, $h^1(x(t)) \le 0$,

..., $h^r(x(t)) \leq 0$.

5) The cost for each $u \in \mathbf{F}$ is

$$C(u) = g\left(x(t_1)\right) + \int_{t_0}^{t_1} f^0(x(t), t, u(t))\, dt + \max_{t_0 \leq t \leq t_1} \gamma(x(t))$$

where $f^0 \in C^1$ in $\mathbf{R}^{n+1+m}$, and $g(x)$ and $\gamma(x)$ are continuous in $\mathbf{R}^n$.

Assume

(a) The family $\mathbf{F}$ of admissible controllers is not empty.

(b) There is a uniform bound

$$|x(t)| \leq b \text{ on } t_0 \leq t \leq t_1$$

for all responses $x(t)$ to controllers $u \in \mathbf{F}$.

(4) The extended velocity set

$$\hat{V}(x, t) = \{f^0(x, t, u),\ f(x, t, u)|\ u \in \Omega(x, t)\}$$

is convex in $\mathbf{R}^{n+1}$ for each fixed (x, t).

Then there exists an optimal controller $u^*(t)$ on $t_0^* \leq t \leq T_1^*$ in $\mathbf{F}$ minimizing $C(u)$.

Conditions on states x and controls u are stated such that a priori the existence of a lower bound on cost function $C(u)$ is guaranteed. Let $c_0 \in \mathbf{R}$ be the greatest lower bound on cost function $C(u)$. Then, together with the assumption that the set of admissible controls is not empty, it is concluded that there is a sequence of admissible control functions such that the associated costs are monotonically decreasing and have a cluster point at c_0. Finally it is shown that this sequence of control functions has a weakly convergent subsequence and that an admissible control function is in this weak limit.

For practical purposes theorems of the form 3.8.1 may be inadequate because of the strong assumptions necessary. An alternative line of thinking is presented in Bryson & Ho [5]. Without knowing whether the original optimal control problem has a solution it is proposed to generate a solution candidate by solving the first order necessary conditions. About this solution candidate one considers so-called weak perturbations. (these are perturbations δx in states x with the property that for $\delta x \to 0$ also $\delta \dot{x} \to 0$). Restricting ourselves to weak perturbations (rather than all possible perturbations available in L_2) it is possible to expand the augmented cost function into a Taylor series about the reference solution. By construction the first order term of this expansion is zero and the leading term is of second order. Minimization of this second order term is equivalent to solving a linear quadratic optimal control problem, the so-called Accessory Minimum Problem (AMP).Controls and states in the AMP are exactly the (first order approximations of) perturbations in controls and states about the reference solution for the original problem. Now it is clear that

i) the reference solution for the original problem is non-optimal if there is a solution to the AMP which furnishes negative cost.

ii) the reference solution for the original problem furnishes at least a weak local minimum if all non-trivial control functions (control functions which are not identically zero) yield cost greater than zero.

Because of these considerations the study of linear quadratic optimal control problems gains tremendous theoretical importance. Explicit results on this matter are presented for instance in Bryson & Ho [5], Chapter 6.3, where sufficient conditions are stated that allow one to transform the second variation of the cost function into a perfect square.

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By construction these conditions are sufficient conditions for the reference solution under consideration to furnish a weak local minimum to the original problem. Still, if a reference solution is shown to furnish a weak local minimum, it is not clear whether it also furnishes a strong local minimum. In fact it is not even guaranteed that the original problem under consideration does have a solution.

Another interesting treatment of the AMP is given in a paper by Breakwell & Ho [3]. In this paper it is shown that the existence of a conjugate point along the reference solution (for the definition of conjugate points see [3], [5], or Chapter 8 of this thesis) implies the existence of a trajectory which furnishes negative cost to the AMP. Hence, by virtue of i) above, the existence of a conjugate point implies non-optimality of the original solution candidate. The proof given in [3] is not constructive. That means, a trajectory that furnishes negative cost to the AMP is shown to exist, but it is not given explicitly. In Chapter 8 of this thesis a modification of the proof in [3] is given which explicitly constructs a trajectory that furnishes negative cost. Because of its explicit character, it is possible to extend the new proof to the case where the coefficient functions in the AMP have any finite number of corners. The results obtained so far apply immediately for corners in the AMP that are induced by an explicit non-smooth time dependence of the state equations for the original problem. In the future it is planned to extend the new approach also to corners in the AMP induced by interior point conditions as discussed in Section 3.5.

Chapter 4

Example: Vertical Rocket-Powered Ascent Study

Chapter Overview

The Goddard Problem is that of maximizing the final altitude for a vertically ascending, rocket-powered vehicle under the influence of an inverse square gravitational field and atmospheric drag. The present example is concerned with the effects of two additional constraints: a dynamic pressure limit q_{max}, and a specified final time t_f. Nine different switching structures involving zero-thrust arcs, full-thrust arcs, singular-thrust arcs, and state constrained arcs are obtained for prescribed values of q_{max} between 0 and ∞ and the final time t_f between $t_{f,min}$ and t_f^*. Here $t_{f,min}$ is the minimum possible time within which all the fuel can be burned, and t_f^* is the optimal final time. For all points in the above defined domain of the q_{max}, t_f-plane the associated optimal switching structure is clearly identified. Finally, a comparison between the optimal solutions and a simple intuitive feedback law is given.

4.1 Introduction

The problem of maximizing the final altitude for a vertically ascending rocket was first formulated by Goddard [10] in 1919. Numerous authors such as Hamel [14] in 1927, Tsien and Evans [44] in 1951, Miele and Cavoti [32] in 1958, Leitmann [26], [27], [28], [29], [30] in 1956-1963, and Garfinkel [9] in 1963 have analyzed the problem using various mathematical methods and assumptions on the equations of motion. An extensive study of the problem under realistic assumptions on the equations of motion has become possible only with the development of the theory of optimal control in conjunction with powerful digital computers. Recently, Tsiotras and Kelley [45], [46] have studied the effect of a

final time specification and of drag modelling.

In the present treatment a dynamic pressure constraint, which in the context of optimal control represents a first-order state inequality constraint, is introduced to the problem. The effect of this constraint as well as the effect of restrictions on the final time are investigated for their effect on the switching structure and the maximum attainable altitude.

4.2 Problem Formulation

The problem is to maximize the final altitude for a rocket ascending vertically under the influence of atmospheric drag and an inverse square gravitational field. The thrust magnitude T is the only control and is subject to fixed bounds $0 \leq T \leq T_{max}$ (control constraints) and a dynamic pressure limit $q \leq q_{max}$ (state constraint). The following assumptions are made: point-mass model, Newtonian central gravitational field, one-dimensional trajectory, air density varies exponentially with altitude, constant drag coefficient, and constant exhaust velocity.

In non-dimensionalized form the problem is given as follows:

$$\min \ - r(t_f) \tag{4.1}$$

subject to the equations of motion

$$
\begin{aligned}
\dot{r} &= v \\
\dot{v} &= \frac{T - D}{m} - \frac{1}{r^2} \\
\dot{m} &= -\frac{T}{c}
\end{aligned}
\tag{4.2}
$$

the control constraints

$$T \in [0, T_{max}] \tag{4.3}$$

the boundary conditions

$$
\begin{array}{ll}
a) \ r(0) = 1 & d) \ r(t_f) \ \text{to be maximized} \\
b) \ v(0) = 0 & e) \ v(t_f) \ \text{free} \\
c) \ m(0) = 1 & f) \ m(t_f) = m_f
\end{array}
\tag{4.4}
$$

and the state inequality constraint

$$P_0(r, v, m) = v - v_{max} \leq 0; \quad v_{max}(r) := \sqrt{\frac{q_{max}}{b}} e^{\beta(1-r)}. \tag{4.5}$$

The final time t_f may be fixed or free. The dynamic pressure limit q_{max} is prescribed with values between $q_{max} = 0$ (trivial case, rocket is allowed only hovering with maximum velocity zero) and $q_{max} = +\infty$ (dynamic pressure limit can be ignored). Radial distance r, velocity v, and mass m are the states; thrust T is the only control. Drag D is given by

$$D = q\, C_D$$

where

$$q = v^2 b e^{\beta(1-r)}$$

is the dynamic pressure times cross section area [A]. Note that constraint (4.5) can be identified as a dynamic pressure limit $q - q_{max} \leq 0$.

The variables in the system description (4.1) - (4.5) have been non-dimensionalized with initial radius $[r_0]$ as the length-scale, initial mass $[m_0]$ as the mass-scale and time-scale given by $t_0 = \sqrt{\frac{r_0}{g}}$, where g is the value of Earth's gravitational acceleration at the initial radius.

4.3 Minimum Principle

Problem (4.1) - (4.5) is solved by applying the Pontryagin Minimum Principle. Assuming that a solution of (4.1) - (4.5) exists, the Minimum Principle states that at every point in time the control is such that the variational-Hamiltonian

$$
\begin{aligned}
H(r, v, m, \lambda_r, \lambda_v, \lambda_m) &= \lambda_r \dot{r} + \lambda_v \dot{v} + \lambda_m \dot{m} \\
&= \lambda_r v + \lambda_v \left(\frac{T - D}{m} - \frac{1}{r^2} \right) - \lambda_m \frac{T}{c}
\end{aligned}
\tag{4.6}
$$

is minimzed subject to all control constraints:

$$
T = \arg \min_{T \in U} H; \quad U = \{ T \in R | \, T \text{ admissible} \}.
\tag{4.7}
$$

On 'unconstrained arcs' (i.e. on time intervals where (4.5) is satisfied with strict inequality)

$$
U = \{ T \in R | \, 0 \leq T \leq T_{max} \}.
\tag{4.8}
$$

On 'constrained arcs' (i.e. on time intervals, say $[\tau_1, \tau_2]$ where (4.5) is satisfied with strict equality)

$$
P_0 \equiv 0 \text{ on } [\tau_1, \tau_2]
\tag{4.9}
$$

is equivalent to

$$
P_0 = 0 \text{ at } t = \tau_1
\tag{4.10}
$$

$$
P_1 \equiv 0 \text{ on } t \in (\tau_1, \tau_2)
\tag{4.11}
$$

where

$$
P_1 := \frac{dP_0}{dt} = \frac{T - D}{m} - \frac{1}{r^2} - v'_{max}(r) v
\tag{4.12}
$$

so that the set of admissible controls is

$$U = \{T \in R \mid 0 \le T \le T_{max}, P_1 = 0\}. \tag{4.13}$$

This set 'usually' consists of a single point, but may be empty. The evolution of the Lagrange multipliers λ_x, $x \in \{r, v, m\}$ is governed by

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} \tag{4.14}$$

on unconstrained arcs. On constrained arcs the implicit dependence of the control on states via (4.11) implies

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} - \mu \frac{\partial P_1}{\partial x} \tag{4.15}$$

(see Bryson, Denham & Dreyfus [4] or Sections 3.6, 3.7 of this thesis). Supplementary optimality conditions are given by

$$\mu \ge 0 \tag{4.16}$$

$$\dot{\mu} \le 0 \tag{4.17}$$

where μ is the Valentine multiplier defined in (3.35), i.e. $\mu = -\frac{\partial H}{\partial T} / \frac{\partial P_1}{\partial T} = -\lambda_v + \lambda_m \frac{m}{c}$ and (4.16), (4.17) is obtained from (3.45) in Section 3.7.

4.4   Hamiltonian and Adjoint Equations

Explicitly the Hamiltonian (4.6) takes the form

$$H = \lambda_r v + \lambda_v \left(\frac{T - D}{m} - \frac{1}{r^2} \right) - \lambda_m \frac{T}{c} \tag{4.18}$$

and the adjoint equations are

$$\dot{\lambda}_r = \frac{\lambda_v}{m} \frac{\partial D}{\partial r} - \lambda_v \frac{2}{r^3} - \mu \left(-\frac{1}{m} \frac{\partial D}{\partial r} + \frac{2}{r^3} - v''_{max}(r) v \right)$$

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$$\dot{\lambda}_v = \frac{\lambda_v}{m}\frac{\partial D}{\partial v} - \lambda_r - \mu\left(-\frac{1}{m}\frac{\partial D}{\partial v} - v'_{max}(r)\right) \tag{4.19}$$

$$\dot{\lambda}_m = \frac{\lambda_v}{m^2}(T - D) - \mu\left(-\frac{T - D}{m^2}\right)$$

where

$$\mu = 0 \qquad \text{on unconstrained arcs} \tag{4.20}$$

$$\frac{\partial H}{\partial T} + \mu\frac{\partial P_1}{\partial T} = 0 \qquad \text{on constrained arcs.} \tag{4.21}$$

Here $'$ denotes differentiation w.r.t. radial distance r.

4.5 Control Logic

With the switching function S defined by

$$S := \frac{\partial H}{\partial T} = \frac{\lambda_v}{m} - \frac{\lambda_m}{c} \tag{4.22}$$

the Minimum Principle (4.7), (4.8) implies

$$T = \begin{cases} 0 & \text{if } S > 0 \\ T_{max} & \text{if } S < 0 \\ T_{sing} & \text{if } S \equiv 0. \end{cases} \tag{4.23}$$

On 'singular arcs' $S = 0$, $\dot{S} = 0$, $\ddot{S} = 0$ imply

$$\lambda_v - \lambda_m\frac{m}{c} = 0, \tag{4.24}$$

$$\lambda_r - \frac{\lambda_m}{c}\left(\frac{\partial D}{\partial v} + \frac{D}{c}\right) = 0, \tag{4.25}$$

and

$$T_{sing} = \frac{A_1 + A_2}{A_3}, \tag{4.26}$$

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where

$$A_1 = -\frac{D}{mc}\left(\frac{\partial D}{\partial v} + \frac{D}{c}\right) + \frac{\partial^2 D}{\partial v \partial r}v - \frac{\partial^2 D}{\partial v^2}\left(\frac{D}{m} + \frac{1}{r^2}\right),$$

$$A_2 = \frac{1}{c}\left(\frac{\partial D}{\partial r}v - \frac{\partial D}{\partial v}\left(\frac{D}{m} + \frac{1}{r^2}\right)\right) + \left(\frac{\partial D}{\partial r} + \frac{2m}{r^3}\right),$$

and

$$A_3 = \frac{1}{mc}\left(\frac{\partial D}{\partial v} + \frac{D}{c}\right) + \frac{\partial^2 D}{\partial v^2}\frac{1}{m} + \frac{\partial D}{\partial v}\frac{1}{mc},$$

respectively. The singular arc is of first order (see Bryson & Ho [5]). On constrained arcs (4.7), (4.13) imply

$$\mu = -\lambda_v + \lambda_m \frac{m}{c} \qquad \left(\text{from } \frac{\partial H}{\partial T} + \mu\frac{\partial P_1}{\partial T}\right) \qquad (4.27)$$

$$T = D + \frac{m}{r^2} + mv'_{max}(r)v \qquad (\text{ from } P_1 = 0). \qquad (4.28)$$

Along singular arcs the Generalized Legendre-Clesch condition (see Kelley, Kopp & Moyer [18])

$$(-1)^q \frac{\partial}{\partial T}\left(\frac{d^{2q}}{dt^{2q}}\left(\frac{\partial H}{\partial T}\right)\right) \geq 0, \quad q = 1 \qquad (4.29)$$

yields

$$-\frac{\lambda_m}{mc}\left(\frac{1}{mc}\left(\frac{\partial D}{\partial v} + \frac{D}{c}\right) + \frac{\partial^2 D}{\partial v^2}\frac{1}{m} + \frac{\partial D}{\partial v}\frac{1}{mc}\right) \geq 0 \qquad (4.30)$$

and is checked numerically.

4.6 Transversality and Corner Conditions

All transversality and corner conditions are given such that the first variation δJ of the cost function (4.1) $J = -r(t_f)$ is zero. For the boundary conditions (4.4) this yields

$$\lambda_r(t_f) = -1 \qquad (4.31)$$

$$\lambda_v(t_f) = 0. \qquad (4.32)$$

In case of free final time t_f, the associated optimality condition is

$$H(t_f) = 0. \qquad (4.33)$$

The Hamiltonian H is continuous throughout the time interval $[0, t_f]$, including across corners. At switching points between minimum and maximum thrust T this implies

$$S = 0, \qquad (4.34)$$

where S is the switching function given in (4.22). At the beginning of singular arcs (4.24) and (4.25) have to be satisfied. At the beginning, say t_1, of the constrained arc conditions are

$$
\begin{aligned}
a) \quad & P_0(t_1) = 0 \\[2mm]
b) \quad & S^-(T^+ - T^-) = 0 \\[2mm]
c) \quad & \lambda_r{}^+ = \lambda_r{}^- - l_0 \frac{\partial P_1}{\partial r} \\[2mm]
d) \quad & \lambda_v{}^+ = \lambda_v{}^- - l_0 \frac{\partial P_1}{\partial v} \\[2mm]
e) \quad & \lambda_m{}^+ = \lambda_m{}^- - l_0 \frac{\partial P_1}{\partial m}
\end{aligned}
\qquad (4.35)
$$

where superscripts $+$, $-$ denote denote evaluation at times $t_1 + \epsilon$, $t_1 - \epsilon$, $\epsilon > 0$, $\epsilon \to 0$, respectively. The end, say t_2, of the constrained arc is determined by

$$S^-(T^+ - T^-) = 0 \text{ at } t = t_2. \qquad (4.36)$$

Conditions (4.35b) and (4.36) are equivalent to the continuity of the of the Hamiltonian at t_1, t_2, respectively. Note that two solutions, namely $S^- = 0$ and $T^+ - T^- = 0$ are possible. The jump in the multipliers (4.35c), (4.35d), (4.35e) is implied by the interior point condition (4.35a).

4.7 Switching Structures

Problem (4.1) - (4.5) is solved for a range of prescribed values $q_{max} \geq 0$ and $t_f > 0$. For explicit calculations the numerical values $T_{max} = 3.5$, $m_f = 0.6$, $b = 6200$, $\beta = 500$, $c = 0.5$, $C_D = 0.05$ are used. In dimensional form this implies the exhaust velocity $3.95 \ 10^3 \ \frac{m}{s}$ and from the given value for b one determines that $\frac{m_0}{A} = 629.6 \ \frac{kg}{m^2}$ (A=cross section area). These values are adopted from [47] and correspond to a Soviet surface-to-air missile SA-2. The dimensional value of maximum dynamic pressure is recovered by multiplying q_{max} by $6174 \ \frac{N}{m^2}$.

As noted above, the switching structure, that is, the sequence of different control logics that actually solves the problem is not known in advance. For a given problem it has to be found by 'numerical experiments'. Assuming a certain switching structure the state equations (4.2), costate equations (4.19), along with boundary conditions (4.4a), (4.4b), (4.4c), (4.4f), transversality conditions (4.31), (4.32) and corner conditions implied by the assumed switching structure yield a multipoint boundary value problem (see Figure (4.6) for an example case). For $q_{max} > 0$ and t_f ranging between the minimum possible flight time $t_{f,min}(q_{max})$ within which all the fuel can be burned, and the optimal flight time $t_f{}^*(q_{max})$, the following different switching structures are found to solve the problem:

(S1) full - zero

(S2) full - singular - zero

(S3) full - singular - full - zero

(S4) full - constrained - zero

(S5) full - constrained - singular - zero

(S6) full - constrained - singular - full - zero

(S7) full - constrained - full - zero

(S8) full - constrained - full - singular - zero

(S9) full - constrained - full - singular - full - zero

For switching structures (S4), (S5), (S6) the continuity of the Hamiltonian at the end of the constrained arc, say at time t_2, imposed by condition (4.36) is satisfied through $S^- = 0$ (or equivalently $\mu^- = 0$). For switching structures (S7), (S8), (S9) condition (4.36) is satisfied through $T^+ - T^- = 0$. The domains in the t_f, q_{max}-plane where the above switching structures solve problem (4.1) - (4.4) are shown in Figure (4.1). The time histories of thrust T, the switching function S, and the dynamic pressure q for selected trajectories in the free time case are given in Figures (4.2), (4.3), (4.4), .

4.8 Numerical Procedure for Solving Multipoint Boundary Value Problems (MPBVPs)

For switching structure (S5) [full - constrained - singular - zero] the associated MPBVP is indicated in Figure (4.6). It is clear that the trajectory can be obtained by simple forward integration once all parameters h_0, v_0, m_0, λ_{h0}, λ_{v0}, λ_{m0}, l_0, Δt_1, Δt_2, Δt_3, Δt_4 are known. These 11 parameters are determined by the 11 conditions (1) - (10) and (11a) for fixed final time, (11b) for free final time. (The jump conditions for the Lagrange multipliers can be considered directly during the integration). The smoothness of the right-hand side of the differential equations on each subarc implies smooth dependence of all conditions on the parameters, so that a Newton Method can be applied to solve this

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root finding problem. As noted in Chapter 2 the routine ZSCNT of the IMSL subroutine library (version 9.2) is used for this purpose. The software package presented in Section 2.5 proved to be suited very well for the challenges that one meets in solving multipoint BVPs associated with optimal control problems with switching points. As shown in Section 2.4 caution has to be applied when setting up the zero-finding problem associated with a given BVP. This zero-finding problem is not determined uniquely and different zero-finding problems associated with the same BVP may have very different numerical solvability properties.

4.9 A Simple Feedback Strategy

An intuitive feedback law to solve problem (4.1) - (4.5) is given as follows: choose T as large as possible subject to the constraints $T \in [0, T_{max}]$ and $P_0 = q - q_{max} \leq 0$. That means the thrust is always set $T = T_{max}$ and is reduced only along arcs where the dynamic pressure limit $P_0 = 0$ is active. The optimal final time t_f^* is obtained from $v(t_f^*) = 0$. The structure of these feedback solutions turns out to be of the following form:

(FB0) constrained   if $q_{max} = 0$

(FB1) full - constrained - zero   if $q_{max} \in (0, 8.303)$

(FB2) full - constrained - full - zero if $q_{max} \in (8.303, 21.334)$

(FB2) full - zero   if $q_{max} > 21.334$

By comparison with the optimal switching structures given in Figure (4.1) it is found that these feedback strategies actually yield the optimal solution for $q_{max} < 8.303$. For $q_{max} > 8.303$ the loss in final altitude increases until $q_{max,free}$, the maximum attainable dynamic pressure with thrusters burning on full throttle, is reached. For $q_{max} > q_{max,free}$

the loss in performance remains constant. These results are shown in Figure (4.5). It is observed that the loss in final altitude never exceeds 2.5 %.

4.10   Conclusion

The effect of a dynamic pressure constraint on the vertical ascent of a sounding rocket has been studied. Trajectories leading to maximum possible altitudes have been obtained for arbitrarily prescribed limits q_{max} on the dynamic pressure q and for final times t_f ranging between the minimum possible value within which all the fuel can be burned and the optimal final time t_f^*. Nine different switching structures have been obtained and the regions in the t_f, q_{max}-plane where they furnish the optimal solutions have been clearly identified. Finally the optimal solutions in the free-time case have been compared with a simple intuitive feedback strategy.
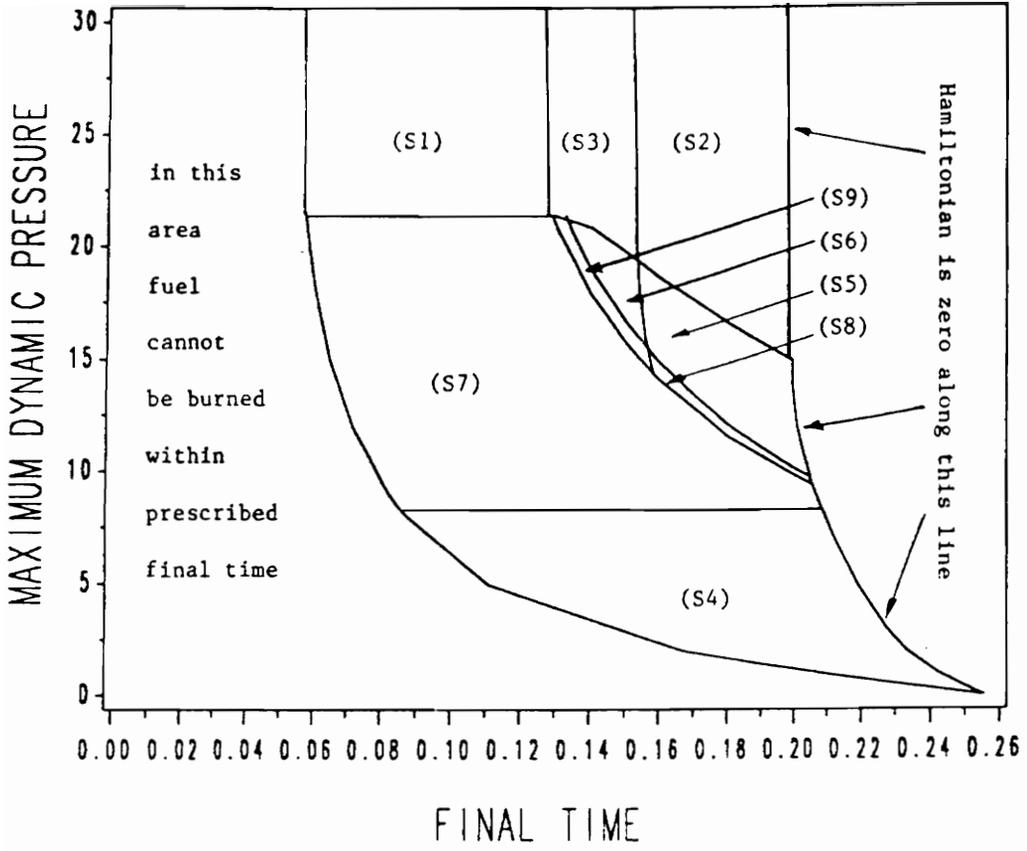
Figure 4.1: Optimal Switching Structures in the t_f, q_{max}-Plane

The plot shows THRUST (vertical axis, from -0.5 to 4.0) versus TIME (horizontal axis, from 0.00 to 0.22). Legend labels:

t_f free, q_{max} free (full-sing.-zero)
t_f free, q_{max} = 10. (full-constr.-sing.-zero)
t_f free, q_{max} = 9.5 (full-constr.-full-sing.-zero)
t_f free, q_{max} = 9.0 (full-constr.-full-zero)
t_f free, q_{max} = 7.0 (full-constr.-zero)
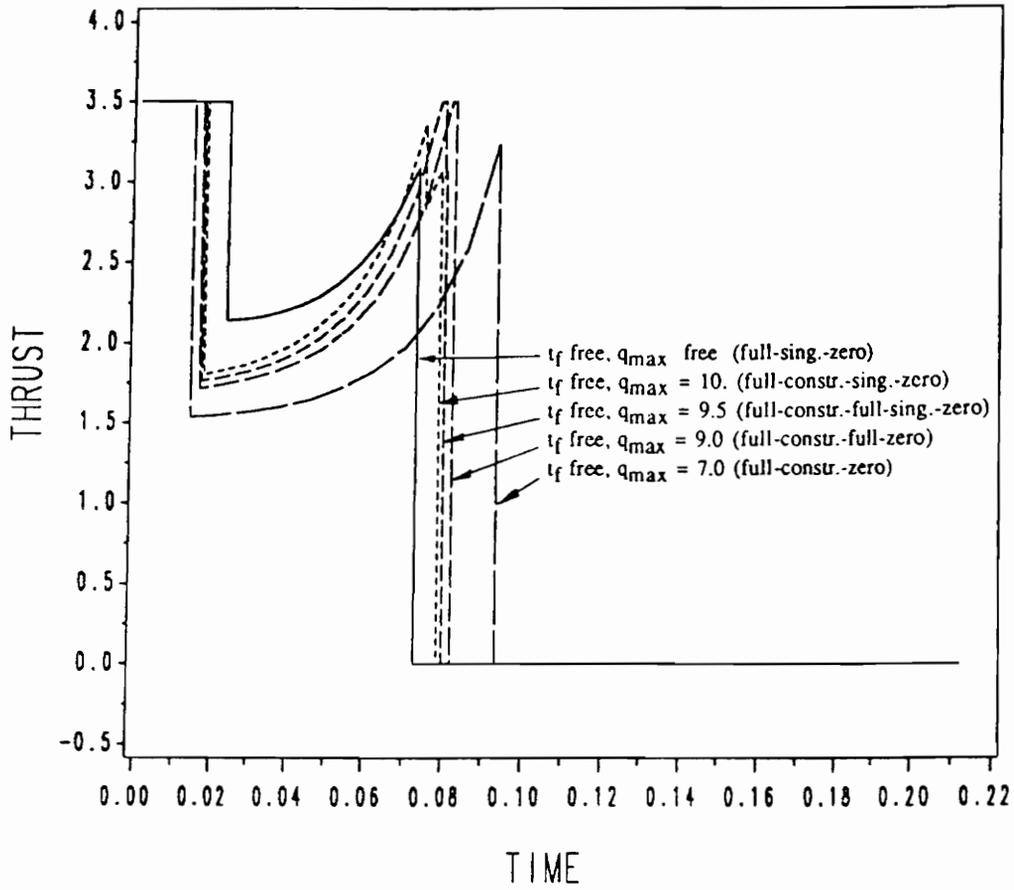
Figure 4.2: Thrust vs. Time for Selected Trajectories in the Free-Time Case

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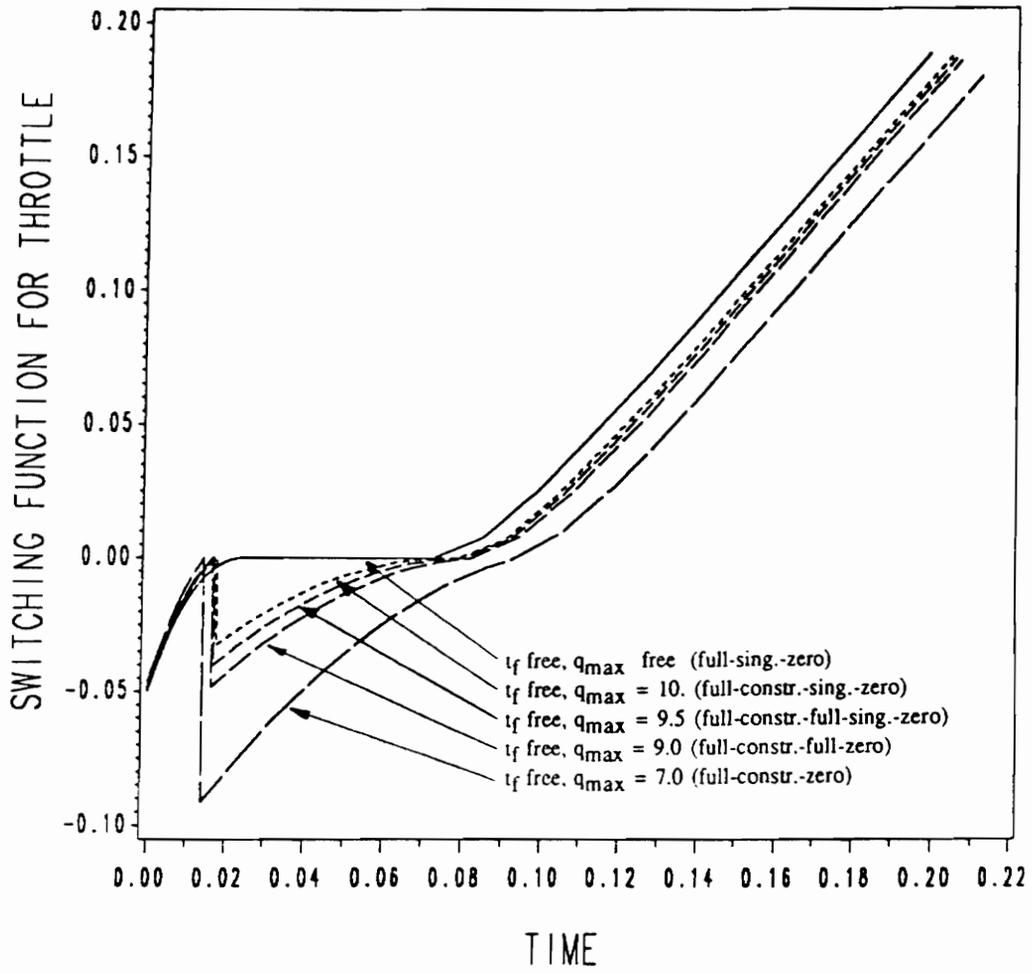
Figure 4.3: Switching Function vs. Time for Selected Trajectories in the Free-Time Case

The figure shows "SWITCHING FUNCTION FOR THROTTLE" on the vertical axis (ranging from -0.10 to 0.20) versus "TIME" on the horizontal axis (ranging from 0.00 to 0.22). The legend reads:

t_f free, q_{max} free (full-sing.-zero)
t_f free, q_{max} = 10. (full-constr.-sing.-zero)
t_f free, q_{max} = 9.5 (full-constr.-full-sing.-zero)
t_f free, q_{max} = 9.0 (full-constr.-full-zero)
t_f free, q_{max} = 7.0 (full-constr.-zero)
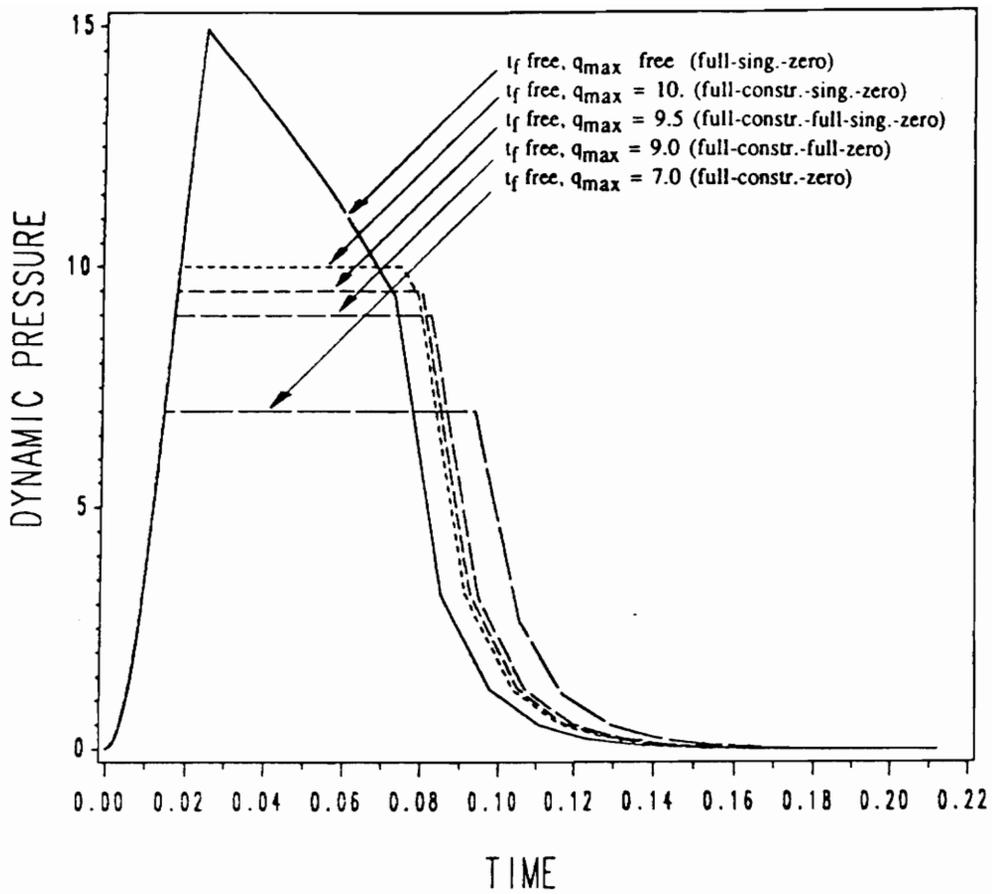
Figure 4.4: Dynamic Pressure q vs. Time for Selected Trajectories in the Free-Time Case

The legend within the figure reads:

t_f free, q_{max} free (full-sing.-zero)
t_f free, q_{max} = 10. (full-constr.-sing.-zero)
t_f free, q_{max} = 9.5 (full-constr.-full-sing.-zero)
t_f free, q_{max} = 9.0 (full-constr.-full-zero)
t_f free, q_{max} = 7.0 (full-constr.-zero)
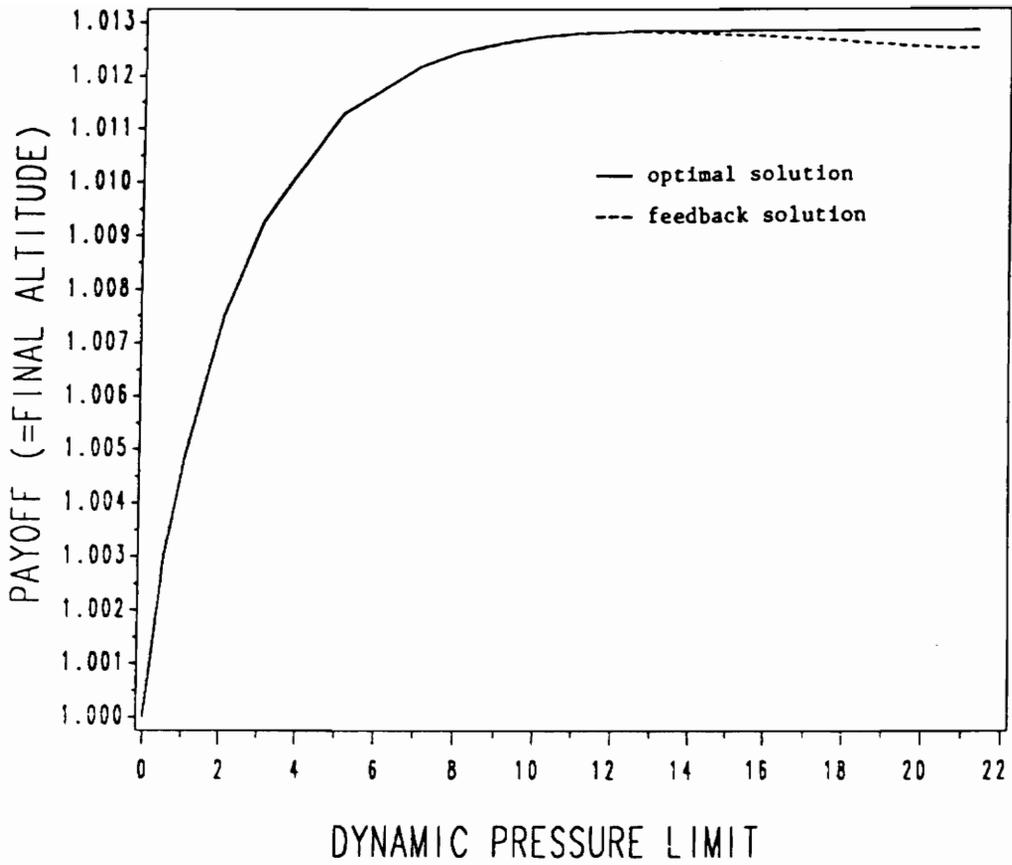
Figure 4.5: Payoff vs. Dynamic Pressure Limit Comparing the Optimal Solution and the Feedback Solution

The figure contains the following labeled boxes:

4) $v - v_{max} = 0$

5) $H^+ - H^- = 0$

$\lambda_r^+ = \lambda_r^- - l_0 \cdot \dfrac{\partial(v - v_{max})}{\partial r}$

$\lambda_v^+ = \lambda_v^- - l_0 \dfrac{\partial(v - v_{max})}{\partial v}$

$\lambda_m^+ = \lambda_m^- - l_0 \dfrac{\partial(v - v_{max})}{\partial m}$

1) $r = 1.$
2) $v = 0.$
3) $m = 1.$

6) $S = 0$
7) $S = 0$

8) $\lambda_r = 1.$
9) $\lambda_v = 0.$
10) $m = 0.6$
11a) $\Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4 = \ldots$
11b) $H(t_4) = 0.$

| full | constrained | singular | zero |

$0 \quad \Delta t_1 \quad t_1 \quad \Delta t_2 \quad t_2 \quad \Delta t_3 \quad t_3 \quad \Delta t_4 \quad t_4$
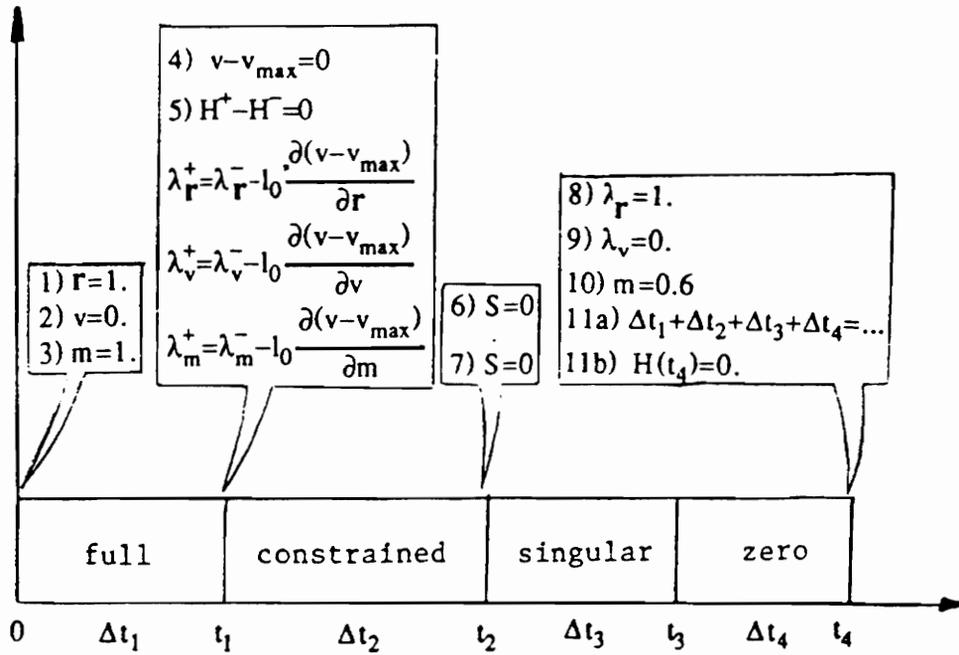
11 parameters:
$r(0),\ v(0),\ m(0),\ \lambda_r(0),\ \lambda_v(0),\ \lambda_m(0),\ l_0,\ \Delta t_1,\ \Delta t_2,\ \Delta t_3,\ \Delta t_4$

11 coditions: (see above)

Figure 4.6: Schematic Representation of the Boundary Value Problem Associated with the Switching Structure Full $-$ Singular $-$ Full $-$ Zero

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Chapter 5

Example: Range Optimization

for a Supersonic Aircraft

Chapter Overview

Range optimal trajectories for an aircraft flying in the vertical plane are obtained from Pontryagin's Minimum Principle. Control variables are load factor n which appears non-linearly in the equations of motion and throttle setting η, which appears only linearly in the equations of motion. Both controls are subject to the fixed bounds, namely $0 \leq \eta \leq 1$ and $|n| \leq n_{max}$. Additionally, a dynamic pressure limit is imposed, which represents a first-order state-inequality constraint. For fixed flight time, initial coordinates, and final coordinates of the trajectory the effect of the load factor limit $|n| \leq n_{max}$ is studied. Upon varying n_{max}, six different switching structures are obtained. All trajectories involve singular control along arcs with active dynamic pressure limit. The explicit derivation of possible control logics is presented in Appendix A. This includes the application of the higher-order convexity test (Generalized Legendre-Clebsch Condition) for singular control logics as presented in Chapter 6.

5.1   Introduction

Great efforts are being undertaken to develop real-time, near-optimal feedback algorithms either for enhancement of aircraft performance by optimizing specified maneuvers or as autonomous guidance schemes for short and medium range air-to-air missiles ([6],[12],[42], [13],[41]). Open-loop control logics obtained by state-of-the-art optimization techniques are an important tool in testing the accuracy and finding the limits of such feedback laws. In a recent study ([21],[22]) open-loop optimal control solutions in conjunction with perturbation techniques have been used directly to develop feedback algorithms. In

this context minimum time intercept trajectories, or, often equivalently, maximum range trajectories for fixed flight time play an important role in modern air combat scenarios. In the present example Pontryagin's Minimum Principle is applied to determine range-optimal trajectories for an aircraft flying in the vertical plane. State variables are energy E, altitude h, and fligh-path angle γ; control variables are load factor n and throttle setting η. Control η appears only linearly in the equations of motion and is subject to fixed bounds $0 \leq \eta \leq 1$. Additionally, a dynamic pressure limit is imposed on the trajectory, which, in the context of optimal control represents a first-order state inequality constraint. For fixed flight time, and fixed initial and final states of the trajectory the effect of a load factor limit $|n| \leq n_{max}$ is studied. Six different switching structures, involving singular control on state constrained arcs are encountered if n_{max} is varied between $n_{max} = \infty$ and $n_{max} = 5$.

Numerical tests, which are not presented here show that the obtained switching structures solve the range optimization problem for a large class of prescribed initial states, final states, and flight times. Yet, for 'large' flight times these switching structures do not furnish the optimal solution.

5.2 Aircraft Model

The equations of motion of an aircraft flying in the vertical plane are

$$\dot{E} = (\eta T - D)\frac{v}{W} \tag{5.1}$$

$$\dot{h} = v \sin \gamma \tag{5.2}$$

$$\dot{\gamma} = \frac{g}{v} \left(\frac{L}{W} - \cos \gamma \right) \tag{5.3}$$

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$$\dot{x} = v \, \cos \gamma. \tag{5.4}$$

The specific energy E, replacing velocity v, the altitude h, the flight-path angle γ, and the range x are the state variables. Load factor n and the power setting η are the control variables. Weight W and gravitational acceleration g are assumed to be constant. Velocity v is a short notation for $v = \sqrt{2g(E-h)}$. The air density ρ in $[kg/m^3]$ is given by

$$\rho(h) = \frac{1.225}{g} e^y$$

$$y = -1.0228055 - 0.12122693 \, 10^{-3}h + r$$

$$r = 1.0228055 \, e^{-z}$$

$$z = \quad -3.48643241 \, 10^{-5}h + 3.50991865 \, 10^{-9}h^2 +$$

$$-8.33000535 \, 10^{-14}h^3 + 1.15219733 \, 10^{-18}h^4$$

The speed of sound in $[m/s]$ is given by

$$a(h) = 20.0468\sqrt{\theta}$$

where the temperature θ is given by

$$\theta = 292.1 - 8.87743 \, 10^{-3}h + 0.193315 \, 10^{-6}h^2 + 3.72 \, 10^{-12}h^3.$$

In these expressions h is altitude in $[m]$. The Mach number is given by $M = \frac{v}{a(h)}$. The lift L, the drag D, and the maximum thrust T are given as functions of h, M, and n

$$q = \frac{1}{2}\rho(h)v^2 S$$

$$L = W \, n$$

$$D = q \left(C_{D0}(M) + K(M) \frac{W^2}{q^2}n^2 \right)$$

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$$C_{D0} = \frac{a_4 M^4 + a_3 M^3 + a_2 M^2 + a_1 M + a_0}{b_4 M^4 + b_3 M^3 + b_2 M^2 + b_1 M + b_0}$$

$$K = \frac{c_4 M^4 + c_3 M^3 + c_2 M^2 + c_1 M + c_0}{d_5 M^5 + d_4 M^4 + d_3 M^3 + d_2 M^2 + d_1 M + d_0}$$

$$T(h, M) = e_5(M) h^5 + e_4(M) h^4 + e_3(M) h^3 + e_2(M) h^2 + e_1(M) h + e_0(M)$$

where for i = 0, 1, ..., 5

$$e_i(M) = f_{i5} M^5 + f_{i4} M^4 + f_{i3} M^3 + f_{i2} M^2 + f_{i1} M + f_{i0}.$$

The numerical values of the constants a_i, b_i, c_i, d_i, f_{ij} are given in Tables 5.1, 5.2, 5.3, and represent a high performance fighter-interceptor.

5.3   Problem Formulation

The problem under consideration is that of finding control functions $\eta(t)$ and $n(t)$ that steer an aircraft from prescribed initial states energy E_0, altitude h_0, and flight-path angle γ_0 to prescribed final states energy E_f, altitude h_f, and flight-path angle γ_f in prescribed flight time $(t_f - t_0)$ (without loss of generality $t_0 = 0$) such that the downrange x is maximized. Along the optimal trajectory a set of state and control constraints has to be satisfied. Explicitly the problem can be stated in Mayer form as follows:

$$min \ - x(t_f) \tag{5.5}$$

subject to the state equations (5.1), (5.2), (5.3), (5.4), the control constraints

$$- \eta \leq 0 \tag{5.6}$$

$$\eta - 1 \leq 0 \tag{5.7}$$

$$- n - n_{max} \leq 0 \tag{5.8}$$

$$+ n - n_{max} \leq 0 \tag{5.9}$$

the state constraint

$$C_0(E, h, \gamma, x) := v - v_{max}(h) \leq 0 \tag{5.10}$$

and the boundary conditions

$$
\begin{array}{ll}
a) \ E(0) = 38029.207 \ [m] & e) \ E(t_f) = 9000 \ [m] \\
b) \ h(0) = 12119.324 \ [m] & f) \ h(t_f) = 942.292 \ [m] \\
c) \ \gamma(0) = 0 \ [Rad] & g) \ \gamma(t_f) = -0.2 \ [Rad] \\
d) \ x(0) = 0 \ [m] & h) \ x(t_f) \ to \ be \ optimized
\end{array}
\tag{5.11}
$$

and the final time t_f prescribed, e.g.

$$t_f = 60 \ [s]. \tag{5.12}$$

Here n_{max} is a specified constant denoting the maximum allowed absolute value of the load factor $n = \frac{L}{W}$. In state constraint (5.10) $v_{max}(h)$ is a specified function of altitude h. With $v_{max}(h)$ chosen appropriately this covers the important case of a dynamic pressure constraint. Boundary conditions (5.11a), (5.11b), (5.11c) refer to the so-called dash-point or high speed point which is defined by

$$(E_{dash}, h_{dash}) = arg \ \sup_{E,h} v$$

subject to the quasi steady flight constraint

$$\eta = 1, \ \gamma = 0, \ \dot\gamma = 0, \ \dot E = 0.$$

The boundary conditions (5.11 e), (5.11 f), (5.11 g) are picked more or less arbitrarily. The only important features are that $h(t_f) < h(0)$ and $v(t_f) = \sqrt{2g(E(t_f) - h(t_f))} <$

$v_{max}(h(t_f))$, i.e. in the altitude-velocity chart the prescribed final point of the trajectory is located to the left of the state constraint (5.10).

5.4   Relaxed Problem Formulation

Among the conditions sufficient to ensure existence of a solution to an optimal control problem is the convexity of the Hodograph. Given a state equation $\dot{x} = f(x, u)$, $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$ with admissible controls $u \in U \subset \mathbf{R}^m$, the Hodograph at some fixed state $x_0 \in \mathbf{R}^n$ is defined as the set $S = \{\dot{x} \in \mathbf{R}^n \,|\, \dot{x} = f(x_0, u), \ u \in U\}$ of possible state rates. For state equations (5.1), (5.2), (5.3), (5.4) with controls $[\eta, n] \in \mathbf{R}^2$ subject to the constraints (5.6), (5.7), (5.8), (5.9), the Hodograph is clearly non-convex as indicated in Figure 5.1. For aircraft models with quadratic drag polar, as the one used in this paper, this deficiency can be overcome by rewriting state equation (5.1) as

$$\dot{E} = [\delta(T - D + D_{max}) - D_{max}]\frac{v}{W} \tag{5.13}$$

with $D_{max}(M, h) := D(M, h, n_{max})$ or any function $D_{max}(M,h)$ with $D_{max}(M,h) \geq D(M,h,n_{max})$. The new control δ replaces the old control η and is subject to the constraints

$$-\delta \leq 0 \tag{5.14}$$

$$\delta - 1 \leq 0. \tag{5.15}$$

Now the relaxed control problem is given by

$$min - x(t_f)$$

subject to the state equations (5.13), (5.2), (5.3), (5.4), control constraints (5.14), (5.15), (5.8), (5.9), state constraint (5.10), boundary conditions (5.11a), (5.11b), (5.11c), (5.11d), (5.11e), (5.11f), (5.11g), and the final time t_f prescribed as in (5.12). The Hodograph associated with the new system dynamics is obviously convex as indicated in Figure 5.2. Note that state inequality constraint (5.10) being active on some time interval $[\tau_1, \tau_2]$ (i.e. $C_0(E, h, \gamma, x) \equiv 0$ on $[\tau_1, \tau_2]$) is equivalent to

$$C_0(E, h, \gamma, x) = 0 \text{ at } t = \tau_1 \tag{5.16}$$

$$C_1(E, h, \gamma, x; n, \delta) \equiv 0 \text{ on } [\tau_1, \tau_2], \tag{5.17}$$

where

$$C_1(E, h, \gamma, x; n, \delta) := \frac{d}{dt} C_0(E, h, \gamma, x)$$

$$= (\delta(T - D + D_{max}) - D_{max}) \frac{g}{W} - v \sin \gamma \left(v'_{max} + \frac{g}{v} \right). \tag{5.18}$$

5.5 Minimum Principle

The relaxed optimization problem as stated above is solved by applying the Pontryagin Minimum principle. It states that at every point in time the controls have to be chosen such that the variational Hamiltonian

$$
\begin{aligned}
H(E, h, \gamma, x, \lambda_E, \lambda_h, \lambda_\gamma, \lambda_x, \delta, n) &= \lambda_E \dot{E} + \lambda_h \dot{h} + \lambda_\gamma \dot{\gamma} + \lambda_x \dot{x} \\
&= \lambda_E (\delta (T - D + D_{max}) - D_{max}) \frac{v}{W} \\
&\quad + \lambda_h v \sin \gamma \\
&\quad + \lambda_\gamma \frac{g}{v} (n - \cos \gamma) \\
&\quad + \lambda_x v \cos \gamma
\end{aligned}
\tag{5.19}
$$

is minimized subject to all control constraints. Let the vector valued function $g : \mathbf{R}^6 \mapsto \mathbf{R}^4$ be defined by

$$g_1(E, h, \gamma, x; \delta, n) = -\delta$$

$$g_2(E, h, \gamma, x; \delta, n) = \delta - 1$$

$$g_3(E, h, \gamma, x; \delta, n) = -n - n_{max}$$

$$g_4(E, h, \gamma, x; \delta, n) = +n - n_{max}$$

(5.20)

so that inequalities (5.14), (5.15), (5.8), (5.9) can be written concisely as $g \leq 0$. Then Lagrange multipliers $\lambda_E, \lambda_h, \lambda_\gamma, \lambda_x$ are solutions of the adjoint equations

$$\dot{\lambda}_E = -\frac{\partial H}{\partial E} - \sigma^T \frac{\partial g}{\partial E} - \mu \frac{\partial C_1}{\partial E}$$

$$\dot{\lambda}_h = -\frac{\partial H}{\partial h} - \sigma^T \frac{\partial g}{\partial h} - \mu \frac{\partial C_1}{\partial h}$$

$$\dot{\lambda}_\gamma = -\frac{\partial H}{\partial \gamma} - \sigma^T \frac{\partial g}{\partial \gamma} - \mu \frac{\partial C_1}{\partial \gamma}$$

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} - \sigma^T \frac{\partial g}{\partial x} - \mu \frac{\partial C_1}{\partial x}$$

(5.21)

where σ_i, $i = 1, ..., 4$ and μ are multipliers associated with constraints $g_i \leq 0$, $i = 1, ..., 4$ and $C_1 \leq 0$ (C_1 as given in (5.18)), respectively. On time intervals where constraint (5.10) is not active (i.e. $C_0(E, h, \gamma, x) < 0$), multiplier μ is identically zero:

$$\mu = 0 \text{ if } C_0(t) < 0.$$

(5.22)

On these intervals multipliers σ_i, $i = 1, ..., 4$ are determined at each instant of time from the Kuhn-Tucker conditions applied to the finite dimensional parameter optimization problem

$$(\delta, n) = arg \min_{g \leq 0} H.$$

(5.23)

At times where state constraint (5.10) is active (i.e. $C_0(E, h, \gamma, x) = 0$), multipliers σ, μ are determined from the Kuhn-Tucker conditions applied to the finite dimensional

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parameter optimization problem

$$(\delta,\, n) = arg \min_{g \leq 0,\, C_1 \leq 0} H. \tag{5.24}$$

In both cases, as a consequence of the Kuhn-Tucker conditions, components of multiplier vector σ are zero along intervals where the associated constraint is not active:

$$\sigma_i = 0 \text{ if } g_i > 0, \quad i = 1, \ldots, 4, \tag{5.25}$$

and components of multiplier vector σ are non-negative along intervals where the associated constraints are active, i.e.

$$\sigma_i \geq 0 \text{ if } g_i = 0, \quad i = 1, \ldots, 4. \tag{5.26}$$

5.6 Possible Control Logics

At each instant of time controls n, δ are determined from the Minimum Principle given by equations (5.23) in case $C_1 < 0$ and (5.24) in case $C_1 = 0$, respectively. Since the Hamiltonian H and the constraint functions g and C_1 are smooth functions of their arguments δ and n, the Kuhn-Tucker conditions imply that at the solution point $[\delta^*, n^*]$ the following conditions have to be satisfied at each instant of time:

$$\frac{\partial}{\partial \delta}(H + \sigma^T g + \mu C_1) = 0 \tag{5.27}$$

$$\frac{\partial}{\partial n}(H + \sigma^T g + \mu C_1) = 0 \tag{5.28}$$

$$g \leq 0 \tag{5.29}$$

$$C_1 \leq 0 \tag{5.30}$$

$$\sigma \geq 0 \tag{5.31}$$

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$$\mu \geq 0 \tag{5.32}$$

$$[\Delta\delta,\ \Delta n] \begin{bmatrix} \dfrac{\partial^2(H+\sigma^T g+\mu C_1)}{\partial\delta^2} & \dfrac{\partial^2(H+\sigma^T g+\mu C_1)}{\partial\delta\partial n} \\[2ex] \dfrac{\partial^2(H+\sigma^T g+\mu C_1)}{\partial n\partial\delta} & \dfrac{\partial^2(H+\sigma^T g+\mu C_1)}{\partial n^2} \end{bmatrix} \begin{bmatrix} \Delta\delta \\[2ex] \Delta n \end{bmatrix} \geq 0 \tag{5.33}$$

for all $(\Delta\delta,\ \Delta n) \in \mathbf{R}^2$ satisfying $\frac{\partial h}{\partial\delta}\Delta\delta + \frac{\partial h}{\partial n}\Delta n = 0$. Here vector function h contains exactly the active components of the inequality constraints $g \leq 0$ and $C_1 \leq 0$. The set of active control constraints and the character of the solution of (5.27) - (5.33) depends greatly on the direction of the multiplier vector $[\lambda_E,\ \lambda_h,\ \lambda_\gamma,\ \lambda_x]^T$, and through state constraint (5.10) also on the state itself. The explicit analysis for solving this finite dimensional constrained minimization problem as well as application of the Generalized Legendre-Clebsch Condition are given in Appendix A. Below the results are summarized. Define

$$n_0 = \frac{\lambda_\gamma}{\lambda_E}\frac{gq}{2\,v^2 W K} \tag{5.34}$$

$$n_1 = \frac{\lambda_E}{\lambda_x}\frac{v}{g}\cos^2\gamma\frac{\partial\left(D_{max}\frac{v}{W}\right)}{\partial h} + 2\cos\gamma \tag{5.35}$$

$$n_2 = \frac{q}{W}\sqrt{\frac{T - \frac{v\,W}{g}\sin\gamma\left(v'_{max} + \frac{q}{v}\right) - q\,C_{D0}}{q\,K}} \tag{5.36}$$

$$\delta_1 = \frac{D_{max} + \frac{v\,W}{g}\sin\gamma\left(v'_{max} + \frac{q}{v}\right)}{T - D|_{n=n_{max}} + D_{max}} \tag{5.37}$$

$$\delta_2 = \frac{D_{max} + \frac{v\,W}{g}\sin\gamma\left(v'_{max} + \frac{q}{v}\right)}{T - D|_{n=0} + D_{max}}. \tag{5.38}$$

Here n_0 is obtained from $\frac{\partial H}{\partial n} = 0$, n_1 solves $\ddot{\lambda}_\gamma \equiv 0$ in the singular case 'constraint (5.10) active, $\lambda_\gamma \equiv 0$, $\lambda_x \neq 0$' (case 6b below), and n_2 is implied by $C_1 = 0$ with $\delta = 1$. The expressions δ_1, δ_2 stem from $C_1 = 0$ with $n = n_{max}$ and $n = 0$, respectively. Then the different possible control logics are as follows (a derivation of the results stated here is given in Appendix A:

case 1: constraint (5.10) not active, $\lambda_E < 0$, $n_0 \in [-n_{max}, +n_{max}]$:

$$\delta = 1$$

$$n = n_0$$

$$\sigma_1 = 0$$

$$\sigma_2 = -\lambda_E \left(T - D + D_{max}\right) \frac{v}{W} \qquad (5.39)$$

$$\sigma_3 = 0$$

$$\sigma_4 = 0$$

$$\mu = 0$$

case 2: constraint (5.10) not active, $\lambda_E < 0$, $n_0 < -n_{max}$:

$$\delta = 1$$

$$n = -n_{max}$$

$$\sigma_1 = 0$$

$$\sigma_2 = -\lambda_E \left(T - D + D_{max}\right) \frac{v}{W} \qquad (5.40)$$

$$\sigma_3 = \frac{\partial H}{\partial n}$$

$$\sigma_4 = 0$$

$$\mu = 0$$

case 3: constraint (5.10) not active, $\lambda_E < 0$, $n_0 > n_{max}$:

$$\delta = 1$$

$$n = n_{max}$$

$$\sigma_1 = 0$$

$$\sigma_2 = -\lambda_E \left(T - D + D_{max} \right) \frac{v}{W} \qquad (5.41)$$

$$\sigma_3 = 0$$

$$\sigma_4 = -\frac{\partial H}{\partial n}$$

$$\mu = 0$$

case 4: constraint (5.10) not active, $\lambda_E > 0$, $\lambda_\gamma > 0$:

$$\delta = 0$$

$$n = -n_{max}$$

$$\sigma_1 = \lambda_E \left(T - D + D_{max} \right) \frac{v}{W}$$

$$\sigma_2 = 0 \qquad (5.42)$$

$$\sigma_3 = \frac{\partial H}{\partial n}$$

$$\sigma_4 = 0$$

$$\mu = 0$$

case 5: constraint (5.10) not active, $\lambda_E > 0$, $\lambda_\gamma < 0$:

$$\delta = 0$$

$$n = n_{max}$$

$$\sigma_1 = \lambda_E \left(T - D + D_{max}\right) \frac{v}{W}$$

$$\sigma_2 = 0 \qquad\qquad (5.43)$$

$$\sigma_3 = 0$$

$$\sigma_4 = -\frac{\partial H}{\partial n}$$

$$\mu = 0$$

case 6: constraint (5.10) not active, $\lambda_E > 0$, $\lambda_\gamma = 0$:

In this case the controls are not determined uniquely by the Minimum Principle. Pointwise occurrence of this situation can be ignored. Assuming that $\lambda_\gamma \equiv 0$ on some non-zero time interval yields control n after differentiating twice (singular control of first order). Two cases have to be distinguished, namely $\lambda_x = 0$, and $\lambda_x \neq 0$:

case 6a: constraint (5.10) not active, $\lambda_E > 0$, $\lambda_\gamma = 0$, and $\lambda_x = 0$:

Then necessarily $\lambda_h \neq 0$ and

$$\delta = 0$$

$$\lambda_\gamma = 0 \qquad \text{(from } \lambda_\gamma = 0\text{)}$$

$$\cos \gamma = 0 \qquad \text{(from } \dot{\lambda}_\gamma = 0\text{)}$$

$$n = 0 \qquad \text{(from } \ddot{\lambda}_\gamma = 0\text{)}$$

$$\sigma_1 = \lambda_E \left(T - D + D_{max} \right) \tfrac{v}{W} \qquad\qquad (5.44)$$

$$\sigma_2 = 0$$

$$\sigma_3 = 0$$

$$\sigma_4 = 0$$

$$\mu = 0$$

The Generalized Legendre-Clebsch Condition (see Chapter 6) implies

$$\sin \gamma > 0 \quad \text{if} \quad \lambda_h < 0$$

$$\sin \gamma < 0 \quad \text{if} \quad \lambda_h > 0$$

case 6b: constraint (5.10) not active, $\lambda_E > 0$, $\lambda_\gamma = 0$, and $\lambda_x \neq 0$:

$$\delta = 0$$

$$\lambda_\gamma = 0 \qquad \text{(from } \lambda_\gamma = 0\text{)}$$

$$\lambda_h - \lambda_x \tan\gamma = 0 \qquad \text{(from } \dot{\lambda}_\gamma = 0\text{)}$$

$$n = n_1 \qquad \text{(from } \ddot{\lambda}_\gamma = 0\text{)}$$

$$\sigma_1 = \lambda_E \left(T - D + D_{max}\right) \frac{v}{W} \qquad (5.45)$$

$$\sigma_2 = 0$$

$$\sigma_3 = 0$$

$$\sigma_4 = 0$$

$$\mu = 0$$

The Generalized Legendre-Clebsch Condition implies $\lambda_x < 0$.

The case $\lambda_E \equiv 0$, $\lambda_\gamma \equiv 0$ can be excluded. The case $\lambda_E \equiv 0$, $\lambda_\gamma \neq 0$ leads to first order singular control in throttle δ which is rejected as non-optimal by the Generalized Legendre-Clebsch Condition.

case 7: constraint (5.10) active, $\lambda_E + \mu\frac{g}{v} < 0$, $\lambda_\gamma < 0$:

$$\delta = 1$$

$$n = n_2$$

$$\sigma_1 = 0$$

$$\sigma_2 = -\left(\lambda_E + \mu\frac{g}{v}\right)(T - D + D_{max})\frac{v}{W} \qquad (5.46)$$

$$\sigma_3 = 0$$

$$\sigma_4 = 0$$

$$\mu = \frac{\lambda_\gamma}{2\,v\,W\,K\,n} - \lambda_E \frac{v}{g}$$

case 8: constraint (5.10) active, $\lambda_E + \mu\frac{g}{v} < 0$, $\lambda_\gamma > 0$:

$$\delta = 1$$

$$n = -n_2$$

$$\sigma_1 = 0$$

$$\sigma_2 = -\left(\lambda_E + \mu\frac{g}{v}\right)(T - D + D_{max})\frac{v}{W} \qquad (5.47)$$

$$\sigma_3 = 0$$

$$\sigma_4 = 0$$

$$\mu = \frac{\lambda_\gamma}{2\,v\,W\,K\,n} - \lambda_E \frac{v}{g}$$

The case 'constraint (5.10) active, $\lambda_E + \mu\frac{g}{v} > 0$, $\lambda_\gamma \neq 0$' can be excluded. The case 'constraint (5.10) active, $\lambda_E + \mu\frac{g}{v} = 0$, $\lambda_\gamma = 0$' is treated later.

case 9: constraint (5.10) active, $\lambda_E + \mu\frac{g}{v} = 0$, $\lambda_\gamma < 0$:

$$\delta = \delta_1 \text{ (from } C_1 = 0 \text{ with } n = n_{max})$$

$$n = n_{max}$$

$$\sigma_1 = 0$$

$$\sigma_2 = 0 \tag{5.48}$$

$$\sigma_3 = 0$$

$$\sigma_4 = -\lambda_\gamma \frac{g}{v}$$

$$\mu = -\lambda_E \frac{v}{g}$$

case 10: constraint (5.10) active, $\lambda_E + \mu\frac{g}{v} = 0$, $\lambda_\gamma > 0$:

$$\delta = \delta_1 \text{ (from } C_1 = 0 \text{ with } n = n_{max})$$

$$n = -n_{max}$$

$$\sigma_1 = 0$$

$$\sigma_2 = 0 \tag{5.49}$$

$$\sigma_3 = \lambda_\gamma \frac{g}{v}$$

$$\sigma_4 = 0$$

$$\mu = -\lambda_E \frac{v}{g}$$

case 11: constraint (5.10) active, $\lambda_E + \mu_v^g = 0$, $\lambda_\gamma = 0$:

In this case the controls are not determined uniquely by the Minimium Principle. Point-wise occurrance of this situation can be ignored. Assuming that $\lambda_\gamma \equiv 0$ on some non-zero time interval yields control n after differentiating twice (singular control of first order). Two cases have to be distinguished, namely $\lambda_x = 0$ and $\lambda_x \neq 0$.

case 11a: constraint (5.10) active, $\lambda_E + \mu_v^g = 0$, $\lambda_\gamma = 0$, $\lambda_x = 0$, $\cos \gamma \neq 0$:

$$\delta = \frac{D_{max} + \frac{vW}{g}\sin\gamma(v'_{max} + \frac{g}{v})}{T - D|_{n=\cos\gamma} + D_{max}}$$

$$\lambda_\gamma = 0$$

$$\lambda_h + \lambda_E\left(1 + \frac{v'_{max}v}{g}\right) = 0$$

$$\sin\gamma = 0$$

$$n = \cos\gamma$$

$$\sigma_1 = 0$$

$$\sigma_2 = 0$$

$$\sigma_3 = 0$$

$$\sigma_4 = 0$$

$$\mu = -\lambda_E\frac{v}{g}$$

(5.50)

case 11b: constraint (5.10) active, $\lambda_E + \mu \frac{g}{v} = 0$, $\lambda_\gamma = 0$, $\lambda_x = 0$, $\cos \gamma = 0$:

$$\delta = \delta_2 \qquad \text{(from } C_1 = 0 \text{ with } n \text{ as below)}$$

$$\lambda_\gamma = 0 \qquad \text{(from } \lambda_\gamma = 0\text{)}$$

$$\cos \gamma = 0 \qquad \text{(from } \dot{\lambda}_\gamma = 0\text{)}$$

$$n = 0 \qquad \text{(from } \ddot{\lambda}_\gamma = 0\text{)}$$

$$\sigma_1 = 0 \qquad\qquad\qquad\qquad\qquad\qquad (5.51)$$

$$\sigma_2 = 0$$

$$\sigma_3 = 0$$

$$\sigma_4 = 0$$

$$\mu = -\lambda_E \frac{v}{g}$$

The Generalized Legendre-Clebsch Condition implies

$$\gamma = +90^0 \text{ if } \lambda_E \left(\frac{v'_{max} v^2}{g} \right) + \lambda_h \, v < 0$$

$$\gamma = -90^0 \text{ if } \lambda_E \left(\frac{v'_{max} v^2}{g} \right) + \lambda_h \, v > 0$$

case 11c: constraint (5.10) active, $\lambda_E + \mu \frac{g}{v} = 0$, $\lambda_\gamma = 0$, and $\lambda_x \neq 0$:

Then necessarily $\cos \gamma \neq 0$ and

$$\delta = \frac{D_{max} + \frac{vW}{g} \sin \gamma \left(v'_{max} + \frac{g}{v} \right)}{T - D|_{n \text{ as below}} + D_{max}}$$

$$\lambda_\gamma = 0 \qquad\qquad\qquad\qquad \text{(from } \lambda_\gamma = 0)$$

$$\lambda_h - \lambda_x \tan \gamma + \lambda_E \left(1 + \frac{v'_{max} v}{g} \right) = 0 \quad \text{(from } \dot{\lambda}_\gamma = 0)$$

$$n = \left(\frac{W^2}{q^2} \right) \left(1 - \frac{v'_{max} v}{g} \right) \cos \gamma \qquad\qquad \text{(from } \ddot{\lambda}_\gamma = 0)$$

$$\sigma_1 = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (5.52)$$

$$\sigma_2 = 0$$

$$\sigma_3 = 0$$

$$\sigma_4 = 0$$

$$\mu = -\lambda_E \frac{v}{g}$$

The Generalized Legendre-Clebsch Condition implies $\lambda_x < 0$.

case 12: constraint (5.10) active, $\lambda_E + \mu\frac{g}{v} \neq 0$, $\lambda_\gamma = 0$:

Then necessarily $\cos\gamma \neq 0$ and

$$\delta = 0$$

$$\lambda_\gamma = 0$$

$$D_{max}\frac{g}{W} + v\sin\gamma\left(v'_{max} + \frac{g}{v}\right) = 0$$

$$n = \cos\gamma - \frac{\left[\left(\frac{\partial D_{max}}{\partial E}\left(\frac{v'_{max}v}{g}+1\right)+\frac{\partial D_{max}}{\partial h}\right)\frac{g}{W}+v'_{max}{}^2+v''_{max}v\right]v\sin\gamma}{\frac{g}{v}(v'_{max}v+g)\cos\gamma}$$

$$\sigma_1 = \left(\lambda_E + \mu\frac{g}{v}\right)(T - D + D_{max})\frac{v}{W} \tag{5.53}$$

$$\sigma_2 = 0$$

$$\sigma_3 = 0$$

$$\sigma_4 = 0$$

$$\mu = \frac{\lambda_h - \lambda_x\tan\gamma}{v'_{max}+\frac{g}{v}}$$

5.7   Transversality and Corner Conditions

All transversality and corner conditions are given such that the first variation of the cost function (5.5) $J = -x(t_f)$ is zero. With boundary conditions (5.11) this yields

$$\lambda_x = -1. \tag{5.54}$$

In case of final time t_f to be minimized (i.e. cost function (5.5) $J = -x(t_f)$ being replaced by $J = t_f$), the associated boundary condition is

$$H(t_f) = 1. \tag{5.55}$$

The Hamiltonian H is continuous throughout the time interval $[0,\ t_f]$. At any corner point, say at time t_c, this yields an optimality condition on the switching time t_c, namely

$$H(t_c{}^+) - H(t_c{}^-) = 0. \tag{5.56}$$

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Here and below superscripts $+$, $-$ denote evaluation just right, and just left of the time under consideration, respectively. At the beginning, say t_1, of a state constrained arc additional conditions are

$$C_0(t_1) = 0 \tag{5.57}$$

$$\lambda_E{}^+ = \lambda_E{}^- \quad - \quad l_0 \frac{\partial C_0}{\partial E}$$

$$\lambda_h{}^+ = \lambda_h{}^- \quad - \quad l_0 \frac{\partial C_0}{\partial h}$$

$$\lambda_\gamma{}^+ = \lambda_\gamma{}^- \quad - \quad l_0 \frac{\partial C_0}{\partial \gamma} \tag{5.58}$$

$$\lambda_x{}^+ = \lambda_x{}^- \quad - \quad l_0 \frac{\partial C_0}{\partial x}$$

where l_0 is a constant multiplier. The end, say t_2, of a constrained arc is determined by the continuity of the Hamiltonian. The jump in multipliers (5.58) is implied by the interior point condition (5.57).

5.8 Supplementary Optimality Conditions

Along constrained arcs we have the sign conditions

$$\sigma_i \geq 0 \text{ on arcs where } g_i = 0, \; i = 1, ..., 4 \tag{5.59}$$

$$\left.\begin{array}{l} a) \, \mu \geq 0 \\[2mm] b) \, \dot{\mu} \leq 0 \end{array}\right\} \text{ on arcs where } C_0 = 0. \tag{5.60}$$

Along singular arcs an additional optimality condition is the Generalized Legendre-Clebsch condition (see Chapter 6). This condition is already considered in the possible control logics stated in Section 5.6. The explicit analysis is given in Appendix A.

5.9 Switching Structures

Problem (5.5) subject to the equations of motion (5.13), (5.2), (5.3), (5.4) and boundary conditions (5.11) is solved for fixed final time (5.12). In a first step only control constraints (5.14), (5.15) and the state constraint (5.10) are enforced, while load factor limits (5.8), (5.9) are neglected. The associated switching structure turns out to be

(S1) 1 - 7 - 11b - 7 - 1

where any number i in the above sequence refers to case i of the possible control logics listed in Section 5.6). The load factor $n = \frac{L}{W}$ increases rapidly near the final time t_f and reaches a maximum value of approximately $n_{max} = 56.5886$. Mathematically this is perfectly reasonable, as will be explained heuristically in the next Section. To make the solution meaningful from an engineering point of view lower values of n_{max} have to be enforced. Starting with switching structure (S1) this is done by reducing the load factor limit (5.9) in steps

$$|n| \leq n_{max}, \quad n_{max} = 56, 55, ..., 5.$$

In the process we come across the following switching structures.

$$
\begin{array}{lll}
(S1) & 1 - 7 - 11b - 7 - 1 & for\ n_{max} \in [56.6, \quad \infty] \\
(S2) & 1 - 7 - 11b - 7 - 1 - 3 & for\ n_{max} \in [32.7, \quad 56.5] \\
(S3) & 1 - 7 - 11b - 7 - 1 - 3 - 5 & for\ n_{max} \in [22.7, \quad 32.6] \\
(S4) & 1 - 7 - 11b - 9 - 7 - 1 - 3 - 5 & for\ n_{max} \in [20.8, \quad 22.6] \\
(S5) & 1 - 7 - 11b - 9 - 3 - 5 & for\ n_{max} \in [18.2, \quad 20.7] \\
(S6) & 1 - 7 - 11b - 9 - 5 & for\ n_{max} \in [5.0, \quad 18.1]
\end{array}
\tag{5.61}
$$

5.10 Large Load Factors Near Final Time

Figure 5.3 shows the time history of the load factor along the solution without the bounds (5.8), (5.9), (i.e. $n_{max} = \infty$). It may be helpful to provide some explanation for why the peak-value occurs at the final time and how the peak-value depends on the boundary conditions. To this end, suppose the conditions (5.11e), (5.11f), (5.11g) are replaced with

$$E(t_f) = E_f = \frac{v_{max}(h_f)^2}{2\,g} + h_f, \qquad (5.62)$$

$$h(t_f) = h_f = 9000, \qquad (5.63)$$

$$\lambda_\gamma(t_f) = 0, \qquad (5.64)$$

i.e. the prescribed final state lies on the dynamic pressure limit and the final flight path angle is free. Then numerical calculations show that the switching structure associated with the solution of this problem is given by switching structure (S1) of the previous section with the last two arcs deleted. Now, if boundary condition (5.62) is replaced by

$$E(t_f) = E_f - \Delta E, \qquad (5.65)$$

for some $\Delta E > 0$, then, if the load factor is unbounded, the 'optimal maneuver' for the aircraft would be to fly exactly as in the solution of the previous case (i.e. boundary conditions (5.62), (5.63), (5.64) until $E(t_f) = E_f$ is reached and to impulsively apply a high load factor $n \to \infty$ on an infinitesimal time interval $[t_f, t_f + \delta t_f]$, $\delta t_f \to 0$, such that the energy drops instantaneously to the prescribed value $E_f - \Delta E$. By noting that, in the dynamical equations, the load factor appears linearly in the $\dot\gamma$-equation and quadratically in the $\dot E$-equation, we expect that along this infinitesimal arc $\delta E \sim n^2 \delta t$, while $\delta\gamma \sim n\delta t$.

Hence, with n and δt_f such that $\delta E = -\Delta E$ we expect $\delta\gamma \to 0$ for $n \to \infty$, and the flight-path angle does not change along this arc.

If the final flight-path angle is prescribed at a value different from the natural one, i.e. (5.64)) is replaced by

$$\gamma(t_f) = \gamma_{free} + \Delta\gamma, \ \Delta\gamma \neq 0, \tag{5.66}$$

then the dissipation of energy turns into a gradual process extending over a non-zero time interval and the load factor remains finite. Paradoxically, non-zero $\Delta\gamma$ results in a smaller peak value load-factor than does $\Delta\gamma = 0$.

5.11 Numerical Procedures

The switching structure, that is, the sequence of different control logics that actually solves a problem is not known in advance. For a given problem it has to be found by 'numerical experiments'. Assuming a certain switching structure the state and costate equations along with the boundary conditions, transversality conditions, and corner conditions implied by the assumed switching structure yield a multi-point boundary value problem (MPBVP). As an example case a schematic representation of the MPBVP associated with switching structure (S6) is given in Figure 5.4. By inspection it is clear that the trajectory can be determined by simple forward integration if all parameters $E(0), h(0), \gamma(0), x(0), \lambda_E(0), \lambda_h(0), \lambda_\gamma(0), \lambda_x(0), l_0, \Delta t_1, \Delta t_2, \Delta t_3, \Delta t_4, \Delta t_5$ are known. Basically, the numerical problem is to determine these 14 parameters such that all 14 conditions (numbered 1, .., 14 in Figure 5.4) are satisfied. As noted in Chapter 2 the routine ZSCNT of the IMSL subroutine library (version 9.2) is used for this purpose. In practice,

forward integration causes the associated Newton Problem to be very badly conditioned. A remedy is to consider t_3 as new 'initial point' and generate trajectories by successively integrating backward and forward, starting at switching time t_3, respectively. In an obvious way this generates a new set of parameters $E(t_3)$, $h(t_3)$, $\gamma(t_3)$, $x(t_3)$, $\lambda_E(t_3)$, $\lambda_h(t_3)$, $\lambda_\gamma(t_3)$, $\lambda_x(t_3)$, l_0, Δt_1, Δt_2, Δt_3, Δt_4, Δt_5 along with the conditions numbered 1,..., 14 in Figure 5.4. Noting that conditions 6, 8, 10 in Figure 5.4 can equivalently be enforced at time t_3, three unknowns, say $E(t_3)$, $\lambda_h(t_3)$, $\lambda_\gamma(t_3)$ can be expressed in terms of the remaining twelve parameters $h(t_3)$, $\gamma(t_3)$, $x(t_3)$, $\lambda_E(t_3)$, $\lambda_x(t_3)$, l_0, Δt_1, Δt_2, Δt_3, Δt_4, Δt_5. While it is only of minor importance that this reduces the number of parameters and conditions, it is very significant that this substitution ensures that

(i) the characteristics of the singular arc, i.e. conditions 8, 9 in Figure 5.4, are satisfied along $[t_2, t_3]$

(ii) the dynamic pressure limit $v - v_{max}(h) = 0$ is satisfied along $[t_1, t_4]$.

Note that both points hold true even before the root finding process converges. A more detailed analysis of numerical problems and difficulties associated with optimal control problems is given in Chapter 2. The software package used to generate the numerical results stated in this thesis is presented in Section 2.5. Switching structure (S1) is in some sense the simplest of the switching structures (S1), ..., (S6) because no control constraints on load factor n are active. Furthermore, in numerical experiments no other switching structures could be found for trajectories involving an arc of active dynamic pressure limit (and no active load factor limit). It is clear that switching structure (S1) can hardly be found in an ad hoc method by just making an intelligent guess and getting the rest done by a computer. In practice, the first attempts were to generate solutions with active dynamic

pressure limit that do not involve singular control arcs (of type 11b). When this failed
trajectories were generated by just integrating along an arc of type 11b (singular control
along active dynamic pressure limit). The next problem was how to leave the constrained
arc and enter the free arc. Numerous different switching structures were tried out. In
this process only switching structure (S1) was found to lead to a consistent BVP. A first
guess for a trajectory was obtained by pure backward and forward integration starting at
some point in the interior of the singular arc of type 11b. Starting with this guess solution
(S1) could be found after a number of homotopy steps in which the prescribed initial and
final values of states E, h, γ were varied. Introducing the load factor limit $n \leq n_{max}$ was
comparatively easy. The necessary switching structures could immediately be guessed by
analyzing the time history of load factor n. It took the author more than one year to find
switching structure (S1). The other switching structures (S2), ..., (S6) were obtained the
same day. What a day!

5.12 Results

As a general trend it is observed that all trajectories consist of mainly 3 phases.

Phase 1: full thrust flight off the dynamic pressure limit (type 1) until dynamic pressure
limit is reached

Phase 2: rapid descent with dynamic pressure limit active and singular control power
(type 11b) until close to prescribed final altitude.

Phase 3: rapid pitch up maneuver off the dynamic pressure limit with load factor on its
upper limit and thrust first full, then zero.

Below, the lengths of each arc in seconds are given for selected solutions with switching structures (S1), ..., (S6) (compare (5.61))

$(S1)$ $27.622 - 0.525 - 30.614 - 0.796 - 0.443$ with n_{max} free

$(S2)$ $27.618 - 0.525 - 30.686 - 0.536 - 0.104 - 0.530$ with $n_{max} = 34$

$(S3)$ $27.491 - 0.521 - 30.539 - 0.451 - 0.015 - 0.515 - 0.567$ with $n_{max} = 23$

$(S4)$ $27.449 - 0.520 - 30.467 - 0.420 - 0.045 - 0.002 - 0.506 - 0.591$ with $n_{max} = 21$

$(S5)$ $27.423 - 0.519 - 30.412 - 0.651 - 0.340 - 0.654$ with $n_{max} = 20$

$(S6)$ $26.856 - 0.504 - 29.231 - 2.31 - 1.098$ with $n_{max} = 10$.

For the case of $n_{max} = 10$ (switching structure S6) time histories for throttle η, load factor n, Lagrange multiplier λ_γ, and switching function $S = \lambda_E \frac{v}{W} + \mu \frac{g}{W}$ are given in Figures 5.5, 5.6, 5.7, and 5.8, respectively. Figure 5.9 shows the altitude-velocity chart for this solution.

All switching structures found seem to be of some general nature in the sense that the same switching structures arise if initial or final coordinates of the trajectory are moderately changed. In this context trajectories starting at ground level with speed around take-off velocity have been calculated for prescribed flight times over 200 seconds and final conditions as in (5.11). For "long flight times" (over 62 seconds for initial and final conditions as given in (5.11)) the obtained switching structures S1, ..., S6 do not solve the problem (thrust over saturates at the beginning of the singular thrust arc). The correct switching structure for "long flight times" has not yet been found.

5.13 Conclusions

Range optimal trajectories for an aircraft flying in the vertical plane have been synthezised in the presence of a dynamic pressure limit (state inequality constraint) and a load factor limit (control inequality constraint). Six different switching structures are obtained with singular control along state constrained arcs always playing an important role. For large flight times the control over saturates at the beginning of a singular arc. For this case the correct switching structure has not yet been found.

Table 5.1: Coefficients for C_{D0}-Model

i	a_i	b_i
0	$-2.61059846050 \ 10^{-2}$	$+7.29821847445 \ 10^{-1}$
1	$+8.57043966269 \ 10^{-2}$	$-3.25219000620 \ 10^{0}$
2	$+1.07863115049 \ 10^{-1}$	$+5.72789877344 \ 10^{0}$
3	$-6.44772018636 \ 10^{-2}$	$-4.57116286752 \ 10^{0}$
4	$+1.64933626507 \ 10^{-2}$	$+1.37368651246 \ 10^{0}$

Table 5.2: Coefficients for K-Model

i	c_i	d_i
0	$+1.23001735612 \ 10^{0}$	$+1.42392902737 \ 10^{+1}$
1	$-2.97244144190 \ 10^{0}$	$-3.24759126471 \ 10^{+1}$
2	$+2.78009092756 \ 10^{0}$	$+2.96838743792 \ 10^{+1}$
3	$-1.16227834301 \ 10^{0}$	$-1.33316812491 \ 10^{+1}$
4	$+1.81868987624 \ 10^{-1}$	$+2.87165882405 \ 10^{+1}$
5		$-2.27239723756 \ 10^{-1}$

Table 5.3: Coefficients for Thrust Model

f_{ij}	$j = 0$	$j = 1$	$j = 2$
$i = 0$	$+0.11969995703 \, 10^6$	$-0.35217318620 \, 10^6$	$+0.60452159152 \, 10^6$
$i = 1$	$-0.14644656421 \, 10^5$	$+0.51808811078 \, 10^5$	$-0.95597112936 \, 10^5$
$i = 2$	$-0.45534597613 \, 10^3$	$+0.23143969006 \, 10^4$	$-0.38860323817 \, 10^4$
$i = 3$	$+0.49544694509 \, 10^3$	$-0.22482310455 \, 10^4$	$+0.39771922607 \, 10^4$
$i = 4$	$-0.46253181596 \, 10^2$	$+0.20894683419 \, 10^3$	$-0.36835984294 \, 10^3$
$i = 5$	$+0.12000480258 \, 10^1$	$-0.53807416658 \, 10^1$	$+0.94529288471 \, 10^1$

f_{ij}	$j = 3$	$j = 4$	$j = 5$
$i = 0$	$-0.43042985701 \, 10^6$	$+0.13656937908 \, 10^6$	$-0.16647992124 \, 10^5$
$i = 1$	$+0.83271826575 \, 10^5$	$-0.32867923740 \, 10^5$	$+0.49102536402 \, 10^4$
$i = 2$	$+0.12357128390 \, 10^4$	$+0.55572727442 \, 10^3$	$-0.23591380327 \, 10^3$
$i = 3$	$-0.30734191752 \, 10^4$	$+0.10635494768 \, 10^4$	$-0.13626703723 \, 10^3$
$i = 4$	$+0.29388870979 \, 10^3$	$-0.10784916936 \, 10^3$	$+0.14880019422 \, 10^2$
$i = 5$	$-0.76204728620 \, 10^1$	$+0.28552696781 \, 10^1$	$-0.40379767869 \, 10^0$
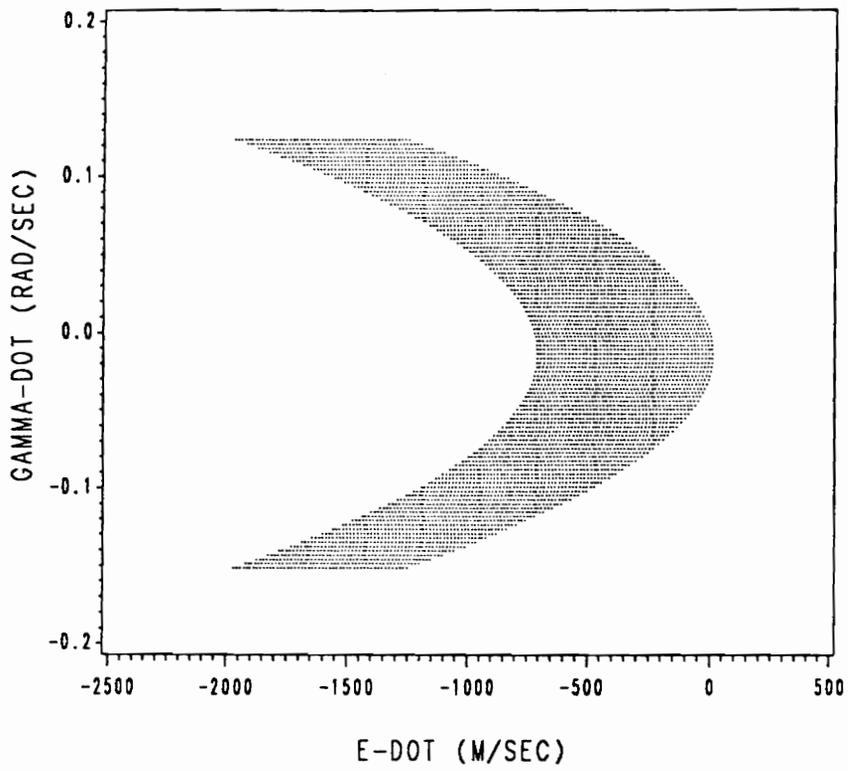
Figure 5.1: Hodograph for Unrelaxed Problem

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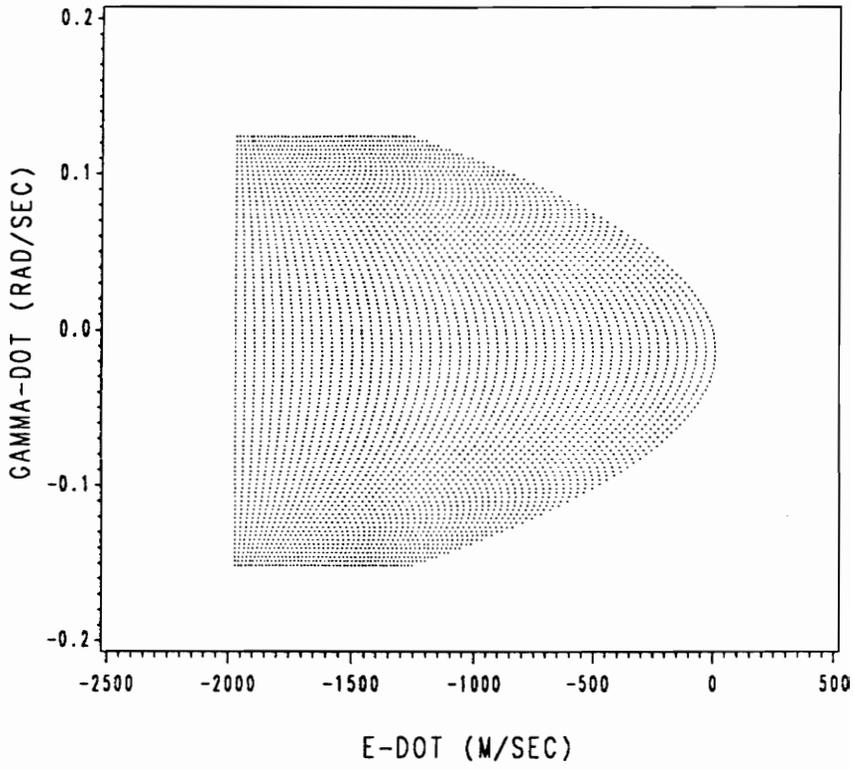
Figure 5.2: Hodograph for Relaxed Problem

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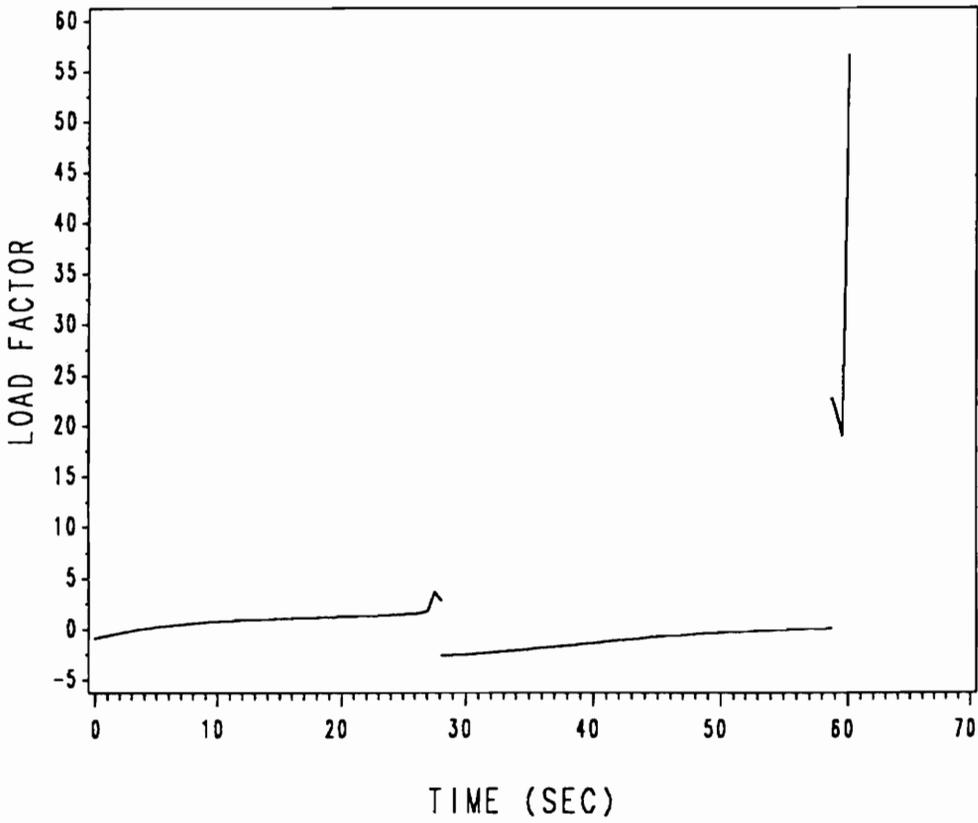
Figure 5.3: Load Factor n vs. Time t for $n_{max} = \infty$ (switching structure (S1))

$$\begin{array}{l} 1)\ E = 38029.207 \\ 2)\ h = 12119.324 \\ 3)\ \gamma = 0. \\ 4)\ z = 0. \\ 5)\ \lambda_z = -1. \end{array}$$

$$\begin{array}{l} 6)\ v - v_{max}(h) = 0 \\ 7)\ n^+ - n^- = 0 \\[4pt] \lambda_E^+ = \lambda_E^- - l_0 \frac{\partial(v - v_{max})}{\partial E} \\ \lambda_h^+ = \lambda_h^- - l_0 \frac{\partial(v - v_{max})}{\partial h} \\ \lambda_\gamma^+ = \lambda_\gamma^- - l_0 \frac{\partial(v - v_{max})}{\partial \gamma} \end{array}$$

$$\begin{array}{l} 8)\ \lambda_\gamma = 0 \\ 9)\ \lambda_h + \lambda_E \left(1 + \frac{v_{max} v'_{max}}{g}\right) - \lambda_z \tan\gamma = 0 \end{array}$$

$$10)\ \mu = 0$$

$$\begin{array}{l} 11)\ E = 9000. \\ 12)\ h = 942.292 \\ 13)\ \gamma = -0.2 \\ 14)\ t_5 - t_0 = 60. \end{array}$$

$\underline{case\ 1}$:	$\underline{case\ 7}$:	$\underline{case\ 11b}$:	$\underline{case\ 7}$:	$\underline{case\ 5}$:
$C_0 < 0$	$C_0 \equiv 0$	$C_0 \equiv 0$	$C_0 \equiv 0$	$C_0 < 0$
$\delta = 1$	δ from $C_1 = 0$	δ from $C_1 = 0$	δ from $C_1 = 0$	$\delta = 0$
n from	n as large	n from	n as large	n from
$\frac{\partial H}{\partial n} = 0$	as possible	$\ddot{S} = 0$	as possible	$n = n_{max}$

$t_0 \quad \Delta t_1 \quad t_1 \quad \Delta t_2 \quad t_2 \quad \Delta t_3 \quad t_3 \quad \Delta t_4 \quad t_4 \quad \Delta t_5 \quad t_5$
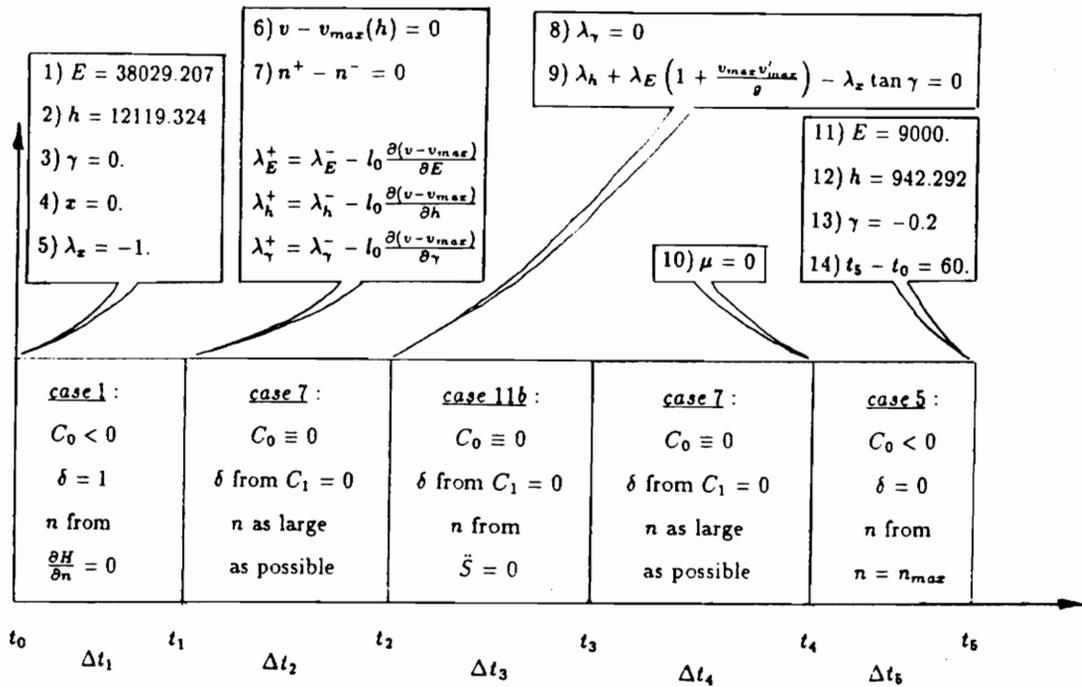
14 parameters: $E(0),\ h(0),\ \gamma(0),\ z(0),\ \lambda_E(0),\ \lambda_h(0),\ \lambda_\gamma(0),\ \lambda_z(0),\ l_0,\ \Delta t_1,\ \Delta t_2,\ \Delta t_3,\ \Delta t_4,\ \Delta t_5$

14 conditions: (numbered 1, ..., 14 above)

Figure 5.4: Schematic Representation of the Boundary Value Problem Associated with Switching Structure (S6)
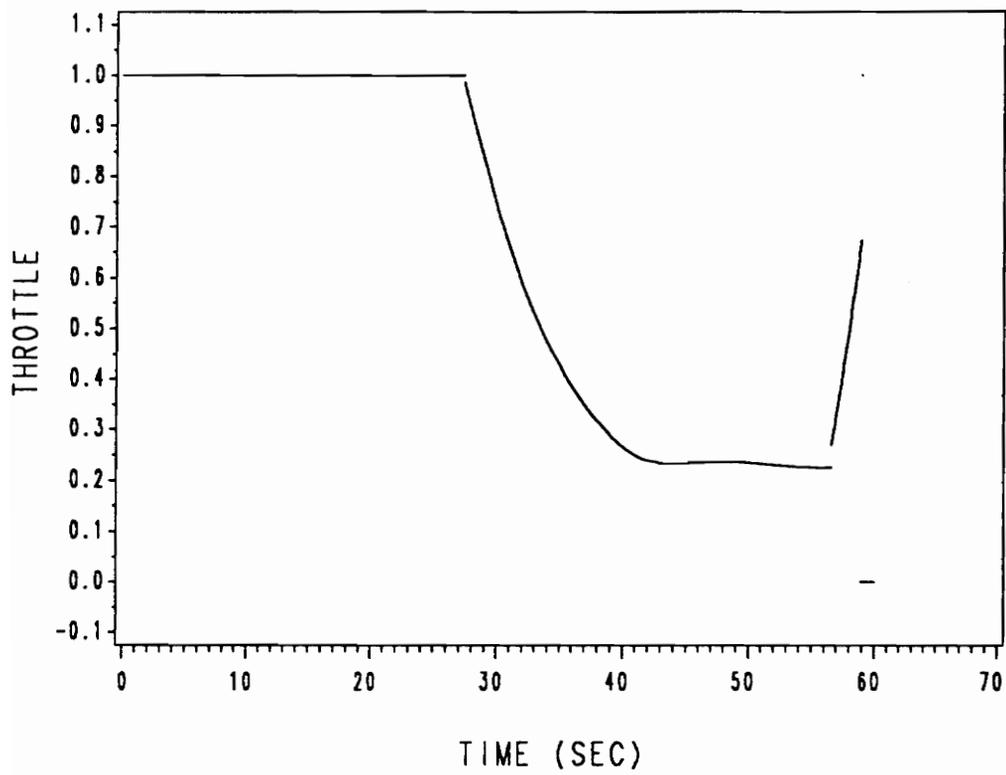
Figure 5.5: Throttle η vs. Time t for $n_{max} = 10$ (switching structure (S6))
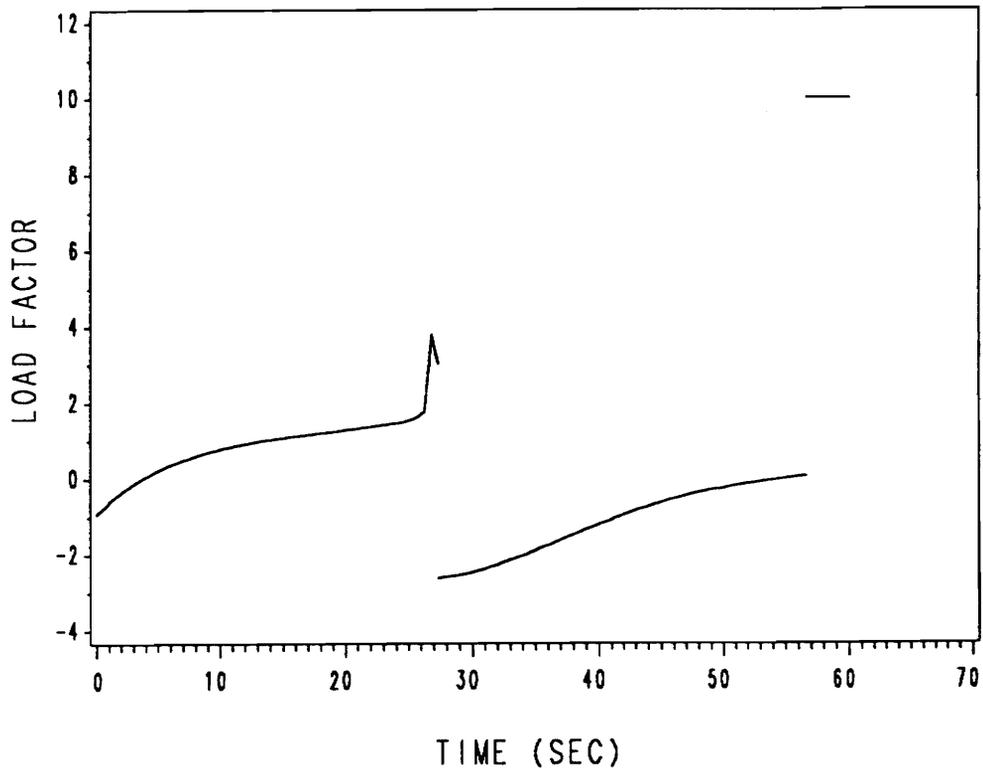
Figure 5.6: Load Factor n vs. Time t for $n_{max} = 10$ (switching structure (S6))
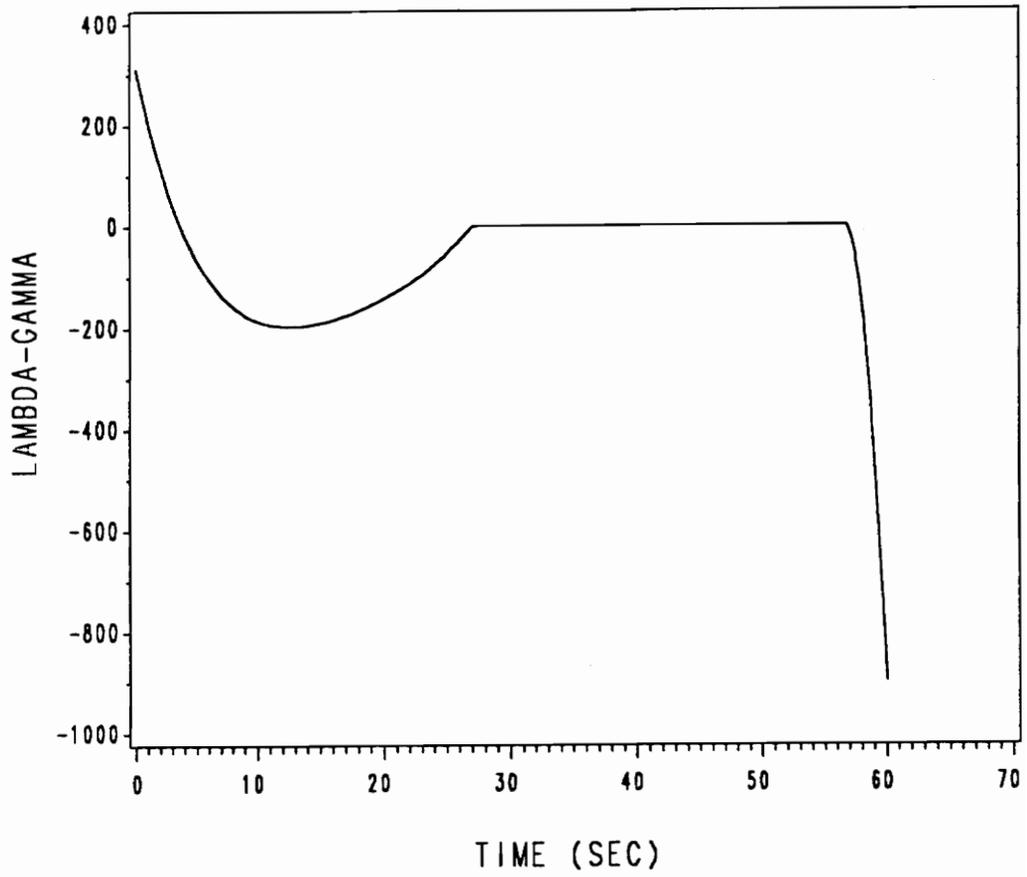
Figure 5.7: Costate λ_γ vs. Time t for $n_{max} = 10$ (switching structure (S6))
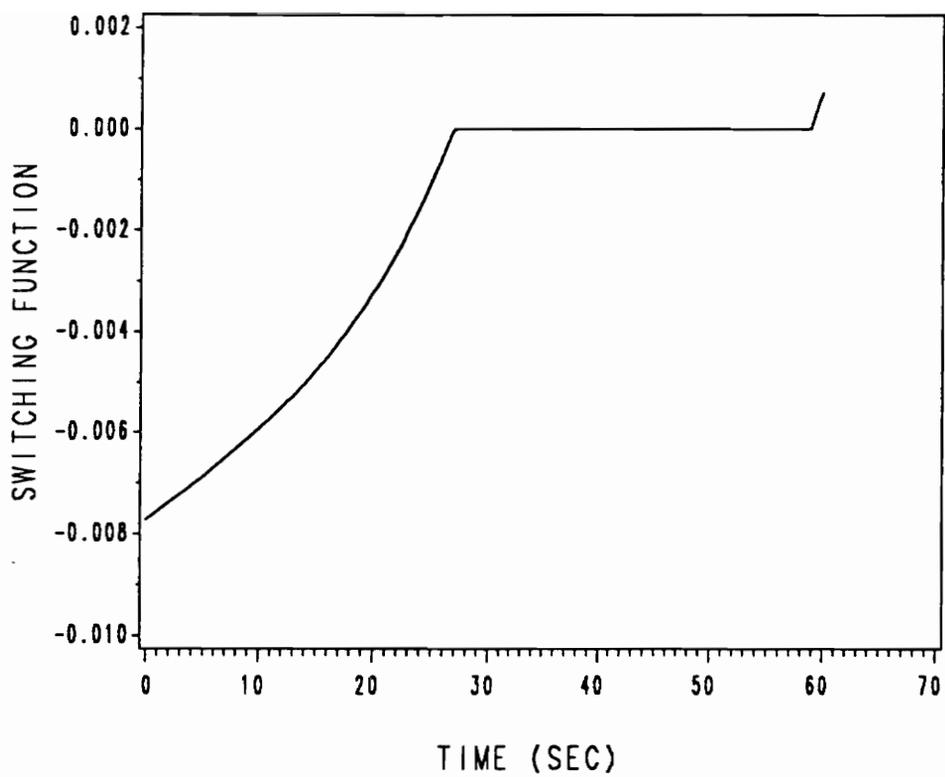
Figure 5.8: Switching Function S vs. Time t for $n_{max} = 10$ (switching structure (S6))
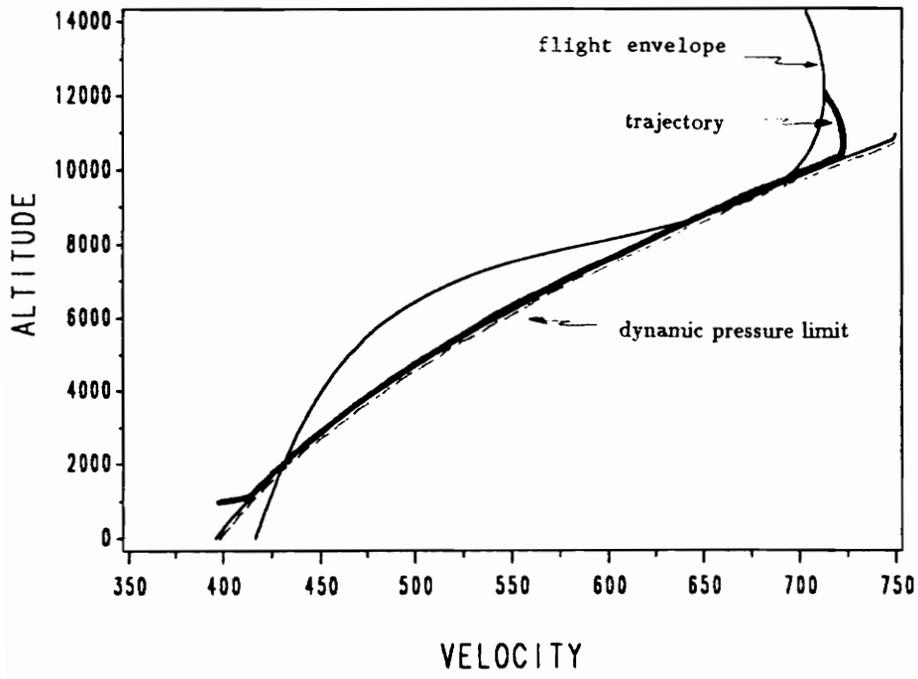
Figure 5.9: Altitude-Velocity Chart for a Typical Extremal

Chapter 6

The Generalized

Legendre-Clebsch Condition on

Constrained Arcs

Chapter Overview

An extension of the Generalized Legendre-Clebsch Condition is obtained for problems with singular control along arcs with active state or control constraints. This is achieved by first transforming the Accessory Minimum Problem associated with constrained singular arcs into an unconstrained singular, linear quadratic problem. In a second step (theorems and proofs are largely based on Goh's work [11]) necessary conditions are derived for such singular linear quadratic problems to yield non-negative cost.

6.1   Introduction

In the 1960's singular control arcs were found to play an important role in many optimal control problems of practical interest. H.J. Kelley, in 1964, was the first to formulate second-order, necessary conditions for this type of control (see [17]). In the following years many authors such as Kelley, Kopp & Moyer [18] and Goh [11] extended Kelley's idea to what is now known largely as the Generalized Legendre-Clebsch Condition. To the author's knowledge, singular control in the presence of active state or control constraints has not been treated in the literature.

In Chapter 5 range optimal aircraft trajectories subject to a dynamic pressure limit are synthesized. The appearance of singular control along arcs with active dynamic pressure limit has prompted the research that lead to the results presented in this Chapter.

6.2 Problem Formulation

Let us consider the following optimal control problem stated in Mayer form:

$$\min_{u \in (PWC[t_0, t_f])^m} \Phi(x(t_f), t_f) \tag{6.1}$$

subject to the conditions

$$\dot{x}(t) = f(x(t), u(t)) \ \forall \ t \in [t_0, t_f] \tag{6.2}$$

$$x(t_0) = x_0, \ x_0 \in \mathbf{R}^n \ and \ t_0 \in \mathbf{R} \ fixed \tag{6.3}$$

$$\Psi(x(t_f), t_f) = 0 \tag{6.4}$$

$$c(x(t), u(t)) = 0 \ \forall \ t \in [t_0, t_f] \tag{6.5}$$

$$h(x(t), u(t)) \leq 0 \ \forall \ t \in [t_0, t_f]. \tag{6.6}$$

Here $t \in \mathbf{R}$, $x(t) \in \mathbf{R}^n$, and $u(t) \in \mathbf{R}^m$ are time, state vector and control vector, respectively. The functions $\Phi : \mathbf{R}^{n+1} \to \mathbf{R}$, $f : \mathbf{R}^{n+m} \to \mathbf{R}^n$, $\Psi : \mathbf{R}^{n+1} \to \mathbf{R}^s$, $s \leq n$, $c : \mathbf{R}^{n+m} \to \mathbf{R}^{k_1}$, and $h : \mathbf{R}^{n+m} \to \mathbf{R}^{k_2}$ are assumed to be sufficiently smooth w.r.t. their arguments of whatever order is required in this Chapter. $(PWC[t_0, t_f])^m$ denotes the set of all piecewise continuous functions defined on the interval $[t_0, t_f]$ into $\mathbf{R}^m$. Conditions (6.2), (6.3), (6.4), represent the differential equations of the underlying dynamical system, the initial conditions, and the boundary conditions, respectively. Components of vector functions $c(x, u)$ and $h(x, u)$, in which control u appears explicitly are called control constraints; components of $c(x, u)$ and $h(x, u)$ which are independent of u are called state constraints. As time marches from t_0 to t_f the type of a given component of $c(x, u)$ or $h(x, u)$ may change back and forth between state constraint and control constraint. The

present Chapter does not address switching conditions that have to be satisfied at junction points between arcs. We are only concerned with optimality conditions that have to be satisfied along the interior of a given arc. By appropriately choosing the boundaries τ_1, τ_2 of an arc $[\tau_1, \tau_2]$ it is clear that each component of $c(x, u)$ and $h(x, u)$ can be considered being of the same type (state constraint or control constraint) throughout the time interval under consideration.

6.3   Minimum Principle

Let us assume that a solution to problem (6.1) - (6.6) exists, that $k_1 = 0$ (no equality constraints), and that along the optimal solution conditions (6.6) are all non-active, i.e. are satisfied with strict inequality. Then (see Bryson & Ho [5], Lee & Markus [24], Neustadt [33]) there is a constant multiplier vector $\nu \in \mathbf{R}^s$ and a time varying multiplier vector $\lambda(t) \in \mathbf{R}^n$ which is non-zero for all times $t \in [t_0, t_f]$ such that

$$H(x, \lambda, u) := \lambda^T f(x, u) \tag{6.7}$$

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} \tag{6.8}$$

$$\lambda(t_f)^T = \frac{\partial \Phi}{\partial x(t_f)} + \nu^T \frac{\partial \Psi}{\partial x(t_f)} \tag{6.9}$$

$$H(x(t_f), \lambda(t_f), u(t_f)) = -\frac{\partial \Phi}{\partial t_f} - \nu^T \frac{\partial \Psi}{\partial t_f}. \tag{6.10}$$

At each instant of time the optimal control u^* satisfies (Minimum Principle)

$$u^* = arg \min_{u \in \mathbf{R}^m} H. \tag{6.11}$$

By virtue of the assumed smoothness of f(x,u) equation (6.11) implies

$$\frac{\partial H}{\partial u} = 0 \tag{6.12}$$

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$$\delta u^T \frac{\partial^2 H}{\partial u^2} \, \delta u \geq 0 \; \forall \, \delta u \in \mathbf{R}^m. \tag{6.13}$$

6.4   Singular Control and Goh's Necessary Condition

A necessary condition for optimality directly implied by the Pontryagin Minimum Principle (6.11) is given by

$$\left. \frac{\partial H}{\partial u} \right|_{u^*} = 0. \tag{6.14}$$

In the regular case the second derivative matrix $\frac{\partial^2 H}{\partial u^2}$ has full rank

$$\operatorname{rank} \left(\frac{\partial^2 H}{\partial u^2} \right) \bigg|_{u^*} = m \tag{6.15}$$

and all components of the optimal control u^* are determined explicitly through (6.14), possibly in conjunction with the convexity condition (6.13). The singular case, $\operatorname{rank} \left(\frac{\partial^2 H}{\partial u^2} \right) \big|_{u^*} <$ m, occurs typically if some control component, say u_j, appears only linearly in the Hamiltonian. Then the associated component of the gradient $\frac{\partial H}{\partial u}$ is a function of x and λ only, and can not be influenced by the choice of controls. Assuming that control u_j is in the interior of its allowed domain, i.e. no control constraint is active on control u_j, condition (6.14) implies that $\frac{\partial H}{\partial u_j} =: S(x, \lambda) = 0$ has to be satisfied. Implicitly, this condition determines the control component u_j through its derivatives $\frac{d}{dt} S(x, \lambda) = 0$, $\frac{d^2}{dt^2} S(x, \lambda) = 0$, ... and so on.

In this Chapter we use the following more general definition of singular control.

Definition 6.4.1 *An arc $[\tau_1, \tau_2]$ is called singular of degree m^* if there is a smooth function $S : \mathbf{R}^{2n} \to \mathbf{R}^{m^*}$ of x and λ such that $\forall t \in [\tau_1, \tau_2]$ the optimal control u^* is determined*

by

$$\frac{\partial H}{\partial u} = 0 \tag{6.16}$$

$$\frac{\partial^2 H}{\partial u^2} \geq 0 \tag{6.17}$$

$$S(x, \lambda) = 0 \tag{6.18}$$

and

$$rank\left(\frac{\partial^2 H}{\partial u^2}\right) = m - m^* \tag{6.19}$$

is satisfied along the solution of (6.16), (6.17), (6.18).

Note that S does not depend explicitly on u so that differentiation of identity (6.18) w.r.t. time t is well-defined and can be used to obtain additional conditions on control u. Assuming that the classical Legendre-Clebsch condition (6.13), (6.17) $\frac{\partial^2 H}{\partial u^2} \geq 0$ is satisfied, we find

$$\frac{\partial^2 H}{\partial u^2} = \begin{bmatrix} R_1 & 0_{m-m^*,m^*} \\ & \\ 0_{m^*,m-m^*} & 0_{m^*,m^*} \end{bmatrix}, \quad R_1 \in \mathbf{R}^{m-m^*,m-m^*} \tag{6.20}$$

possibly after a permutation of the components u_i, $i = 1, ..., m$ of control vector u. Along an extremal $x^*(t)$, $\lambda^*(t)$, $u^*(t)$, t_f^*, ν^* of (6.1), (6.2), (6.3), (6.4) the second variation of the augmented cost functional

$$J = \Phi(x(t_f), t_f) + \nu^T \Psi(x(t_f), t_f) + \int_{t_0}^{t_f} \lambda^T(f(x, u) - \dot{x}) \, dt \tag{6.21}$$

is given by (see results obtained in Appendix B)

$$J_2 = \gamma + \int_{t_0}^{t_f} \omega \, dt \tag{6.22}$$

where

$$\gamma = dx_f^T \left(\frac{\partial^2 \Phi}{\partial x(t_f)^2} + \nu^T \frac{\partial^2 \Psi}{\partial x(t_f)^2}\right) dx_f + 2 \, dx_f^T \left(\frac{\partial^2 \Phi}{\partial x(t_f)\partial t_f} + \nu^T \frac{\partial^2 \Psi}{\partial x(t_f)\partial t_f}\right) dt_f +$$

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$$+ dt_f{}^T \left(\frac{\partial^2 \Phi}{\partial t_f{}^2} + \nu^T \frac{\partial^2 \Psi}{\partial t_f{}^2} \right) dt_f \tag{6.23}$$

$$\omega = \delta x^T \frac{\partial^2 H}{\partial x^2} \delta x + 2\, \delta x^T \frac{\partial^2 H}{\partial x \partial u} \delta u + \delta u^T \frac{\partial^2 H}{\partial u^2} \delta u. \tag{6.24}$$

Here the variation $\delta x(t)$ of state x is the solution of

$$\dot{\delta x} = \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial u} \delta u, \quad \delta x(t_0) = 0 \tag{6.25}$$

and the variation dx_f of the final state is given by

$$dx_f = \delta x(t_f^*) + f(x^*(t_f^*), u^*(t_f^*))\, dt_f. \tag{6.26}$$

The variation δu of control u is arbitrary, and all matrices are evaluated along the solution candidate x^*, λ^*, u^*. Applying Theorem 6.8.2 of this thesis to the second variation immediately yields the following result.

Theorem 6.4.2 *Let the optimal solution u^* corresponding to the solution of (6.1), (6.2), (6.3), (6.4) be singular of degree m^* on some arc $[\tau_1, \tau_2]$, i.e. equations (6.16), (6.17), (6.18), (6.19) hold for all times $t \in [\tau_1, \tau_2]$. Then a necessary condition for u^* to be optimal on $[\tau_1, \tau_2]$, are the conditions*

(i) the $(m - m^*) \times (m - m^*)$ matrix function of time $Q_2 B_2$ is identically symmetric, i.e.

$$Q_2 B_2 = B_2{}^T Q_2{}^T \quad \forall t \in [t_0, t_f] \tag{6.27}$$

(ii) if $Q_2 B_2$ is identically symmetric, then

$$R_4 := \begin{bmatrix} R_1 & R_2{}^T \\ R_2 & R_3 \end{bmatrix} \geq 0 \quad \forall t \in [t_0,\, t_f] \tag{6.28}$$

where

$$R_2 := B_2{}^T Q_1{}^T - Q_2 B_1 \tag{6.29}$$

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$$R_3 := B_2^T P_1 B_2 - \frac{d}{dt}(Q_2 B_2) - B_3^T Q_2^T \qquad (6.30)$$

$$B_3 := A B_2 - \dot{B}_2 \qquad (6.31)$$

Here matrices $R_1 \in \mathbf{R}^{m-m^*, m-m^*}, P_1 \in \mathbf{R}^{n,n}, Q_1 \in \mathbf{R}^{m-m^*, n}, Q_2 \in \mathbf{R}^{m^*, n}, A \in \mathbf{R}^{n,n}, B_1 \in$

$\mathbf{R}^{n, m-m^*}, B_2 \in \mathbf{R}^{n, m^*}$, are defined by (6.20) and

$$P_1 := \frac{\partial^2 H}{\partial x^2}, \qquad (6.32)$$

$$\begin{bmatrix} Q_1 \\ \\ Q_2 \end{bmatrix} := \frac{\partial^2 H}{\partial u \partial x}, \qquad (6.33)$$

$$A := \frac{\partial f}{\partial x}, \qquad (6.34)$$

$$\begin{bmatrix} B_1, & B_2 \end{bmatrix} := \frac{\partial f}{\partial u}, \qquad (6.35)$$

respectively, and are evaluated along the extremal x^*, λ^*, u^*.

6.5 Constrained Arcs

A constrained arc is a non-zero time interval, say $[\tau_1, \tau_2]$, along which a fixed set of components of constraints (6.5), (6.6) is satisfied with strict equality. Let this set of constraints be given by $d(x, u) = 0$, $d : \mathbf{R}^{n+m} \to \mathbf{R}^{k_1 + k_2'}$, $k_1 + k_2' > 0$, where the first k_1 components of vector function d are all constraints (6.5) and the last k_2' components of d are those components of (6.6) that are 'active'. For components i of $d(x, u)$ which represent state constraints (i.e. $d(x, u)$ is independent of control u), the order of the state constraint is defined as the smallest integer $p_i \in \mathbf{N}$ such that $\frac{d^{p_i}}{d t^{p_i}} d_i(x)$ contains control u explicitly, i.e. $\frac{\partial}{\partial u} \frac{d^{p_i}}{d t^{p_i}} d_i(x) \neq 0$ on $[\tau_1, \tau_2]$. (If p_i changes on $[\tau_1, \tau_2]$ then consider a new

interval $[\tau_1', \tau_2'] \subset [\tau_1, \tau_2]$ small enough such that p_i remains constant along $[\tau_1', \tau_2']$). Then

$$d_i(x) \equiv 0 \ on \ t \in [\tau_1, \tau_2] \qquad (6.36)$$

is obviously equivalent to

$$\begin{cases} d_i^{(0)} = 0 \ at \ t = \tau_1 \\ \vdots \\ d_i^{(p_i-1)} = 0 \ at \ t = \tau_1 \\ d_i^{(p_i)} \equiv 0 \ on \ t \in [\tau_1, \tau_2]. \end{cases} \qquad (6.37)$$

Here superscript (j) denotes the j-th total time derivative. By virtue of this equivalence the effect of all constraints $d(x, u)$ in the interior of any constrained arc is completely characterized by a vector valued function $g : \mathbf{R}^{n+m} \to \mathbf{R}^p$, $p = k_1 + k_2'$, with

$$g(x(t), u(t)) \equiv 0 \ on \ t \in (\tau_1, \tau_2), \qquad (6.38)$$

as long as some set of "initial conditions"

$$\zeta(x(t)) = 0 \ at \ t = \tau_1 \qquad (6.39)$$

is satisfied. Now assume

$$rank \left(\frac{\partial g(x, u)}{\partial u} \right) = p \ \forall \, t \in [\tau_1, \tau_2]. \qquad (6.40)$$

Then, without loss of generality, control vector u can be separated as

$$u^T = [v^T, w^T], \ v \in \mathbf{R}^{m-p}, \ w \in \mathbf{R}^p \qquad (6.41)$$

such that

$$rank \left(\frac{\partial g(x, v, w)}{\partial w} \right) = p \qquad (6.42)$$

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$(i.e.\ \frac{\partial g(x,v,w)}{\partial w}$ is non-singular) and the Implicit Functon Theorem [2] implies the existence of a smooth function

$$w = W(x, v) \tag{6.43}$$

such that

$$g(x, v, W(x, v)) \equiv 0 \tag{6.44}$$

is an identity in x and v.

6.6 Reduction of Constrained Arcs to Unconstrained Arcs

By the Principle of Optimality (see Lee & Markus [24]) every subarc of an optimal trajectory is an optimal trajectory between its end points. Hence, along a constrained arc, say $t \in [\tau_1, \tau_2]$, the formalism of (6.7) and (6.8) and the Minimum Principle (6.11) can still be applied after substituting (6.41) and (6.43) into the equations of motion (6.2). With the Hamiltonian (6.7) written in the form

$$H := \lambda^T f(x, v, w) \tag{6.45}$$

and with

$$\frac{d\cdot}{dx} := \left[\begin{array}{ccc} \frac{d\cdot}{dx_1}, & \cdots, & \frac{d\cdot}{dx_n} \end{array} \right]$$

$$= \left[\begin{array}{ccc} \frac{\partial\cdot}{\partial x_1}, & \cdots, & \frac{\partial\cdot}{\partial x_n} \end{array} \right] + \left[\begin{array}{ccc} \frac{\partial\cdot}{\partial w_1}, & \cdots, & \frac{\partial\cdot}{\partial w_p} \end{array} \right] \left[\begin{array}{ccc} \frac{\partial W_1}{\partial x_1} & \cdots & \frac{\partial W_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial W_p}{\partial x_1} & \cdots, & \frac{\partial W_p}{\partial x_n} \end{array} \right] \tag{6.46}$$

$$\frac{d\cdot}{dv} := \left[\begin{array}{ccc} \frac{d\cdot}{dv_1}, & \cdots, & \frac{d\cdot}{dv_{m-p}} \end{array} \right]$$

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$$
= \left[\frac{\partial \cdot}{\partial v_1}, \quad \cdots, \quad \frac{\partial \cdot}{\partial v_{m-p}} \right] + \left[\frac{\partial \cdot}{\partial w_1}, \quad \cdots, \quad \frac{\partial \cdot}{\partial w_p} \right]
\begin{bmatrix}
\frac{\partial W_1}{\partial v_1} & \cdots & \frac{\partial W_1}{\partial v_{m-p}} \\
\vdots & \ddots & \vdots \\
\frac{\partial W_p}{\partial v_1} & \cdots, & \frac{\partial W_p}{\partial v_{m-p}}
\end{bmatrix}
\tag{6.47}
$$

this yields on $[\tau_1, \tau_2]$

$$
\dot{\lambda}^T = -\frac{dH(x, \lambda, v, W(x, v)))}{dx}
\tag{6.48}
$$

$$
v^* = arg \min_{v \in \mathbf{R}^{m-p}} H(x, v, W(x, v)).
\tag{6.49}
$$

Again with the assumed smoothness of all participating functions (6.49) implies

$$
\frac{dH(x, v, W(x, v))}{dv} = 0
\tag{6.50}
$$

$$
\delta v^T \frac{d^2 H(x, v, W(x, v))}{dv^2} \delta v \geq 0 \quad \forall \delta v \in \mathbf{R}^{m-p}
\tag{6.51}
$$

In complete analogy to the unconstrained case we have

Definition 6.6.1 *An arc $[\tau_1, \tau_2]$ with constraints (6.38), (6.39) is called singular of degree m^* if there is a smooth function $S : \mathbf{R}^{2n} \to \mathbf{R}^{m^*}$ of x and λ such that along $[\tau_1, \tau_2]$ the optimal control v^* is determined by*

$$
\frac{dH}{dv} = 0
\tag{6.52}
$$

$$
\frac{d^2 H}{dv^2} \geq 0
\tag{6.53}
$$

$$
S(x, \lambda) = 0
\tag{6.54}
$$

and

$$
rank \left(\frac{d^2 H}{dv^2} \right) = m - p - m^*
\tag{6.55}
$$

is satisfied along the solution of (6.52), (6.53) and (6.54). The differential operator $\frac{d}{dv}$ is defined by equations (6.40) through (6.47).

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As before, note that S does not depend explicitly on any controls so that differentiation of identity (6.54) w.r.t. time t is well-defined and can be used to get additional conditions on control v. As in the unconstrained case, (6.55) along with the Legendre-Clebsch condition (6.53) implies the existence of a matrix R_1 such that

$$\frac{d^2 H}{dv^2} = \begin{bmatrix} R_1 & 0_{m-p-m^*,m} \\ 0_{m,m-p-m^*} & 0_{m^*,m^*} \end{bmatrix}, \quad R_1 \in \mathbf{R}^{m-p-m^*,m-p-m^*}, \tag{6.56}$$

possibly after rearranging controls v. For the second variation to be non-negative on $[\tau_1, \tau_2]$ Theorem 6.8.2 yields the necessary condition

Theorem 6.6.2 *Let the interval $[\tau_1, \tau_2]$ be a constrained singular arc of a solution to problem (6.1) - (6.6), i.e. with Definitions (6.41) and (6.45), (6.46), (6.47) equations (6.42), (6.43), (6.44), and (6.48), (6.52), (6.53), (6.54), (6.55) hold true on $[\tau_1, \tau_2]$, respectively. Then a necessary condition for v^* to be optimal on $[\tau_1, \tau_2]$ are the conditions (i) the $(m-p-m^*) \times (m-p-m^*)$ matrix function of time $Q_2 B_2$ is identically symmetric, i.e.*

$$Q_2 B_2 = B_2^T Q_2^T \quad \forall t \in [t_0, t_f] \tag{6.57}$$

(ii) if $Q_2 B_2$ is identically symmetric, then

$$R_4 := \begin{bmatrix} R_1 & R_2^T \\ R_2 & R_3 \end{bmatrix} \geq 0 \quad \forall t \in [t_0, t_f] \tag{6.58}$$

where

$$R_2 := B_2^T Q_1^T - Q_2 B_1 \tag{6.59}$$

$$R_3 := B_2^T P_1 B_2 - \frac{d}{dt}(Q_2 B_2) - B_3^T Q_2^T \tag{6.60}$$

$$B_3 := A B_2 - \dot{B}_2 \tag{6.61}$$

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Here matrices $R_1 \in \mathbf{R}^{m-p-m^*,m-p-m^*}, P_1 \in \mathbf{R}^{n,n}, Q_1 \in \mathbf{R}^{m-p-m^*,n}, Q_2 \in \mathbf{R}^{m^*,n}, A \in$

$\mathbf{R}^{n,n}, B_1 \in \mathbf{R}^{n,m-p-m^*}, B_2 \in \mathbf{R}^{n,m^*}$, are defined by (6.56) and

$$P_1 := \frac{d^2 H}{dx^2}, \tag{6.62}$$

$$\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} := \frac{d^2 H}{dv\, dx}, \tag{6.63}$$

$$A := \frac{df}{dx}, \tag{6.64}$$

$$\begin{bmatrix} B_1, & B_2 \end{bmatrix} := \frac{df}{dv}, \tag{6.65}$$

respectively, and are evaluated along the extremal x^*, λ^*, v^*.

6.7 Express Unknown Quantities in Terms of Known Quantities

Equations (6.48), (6.51) - (6.55) seem to require the explicit knowledge of the functional dependence $w = W(x, v)$. This may not be available, as it may not be possible to solve (6.38) analytically for w. Differentiation of identity (6.44) w.r.t. x and v yields

$$\begin{bmatrix} \frac{\partial W_1}{\partial x_1} & \cdots & \frac{\partial W_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial W_p}{\partial x_1} & \cdots & \frac{\partial W_p}{\partial x_n} \end{bmatrix} = - \begin{bmatrix} \frac{\partial g_1}{\partial w_1} & \cdots & \frac{\partial g_1}{\partial w_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial w_1} & \cdots & \frac{\partial g_p}{\partial w_p} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial x_1} & \cdots & \frac{\partial g_p}{\partial x_n} \end{bmatrix} \tag{6.66}$$

$$\begin{bmatrix} \frac{\partial W_1}{\partial v_1} & \cdots & \frac{\partial W_1}{\partial v_{m-p}} \\ \vdots & \ddots & \vdots \\ \frac{\partial W_p}{\partial v_1} & \cdots & \frac{\partial W_p}{\partial v_{m-p}} \end{bmatrix} = - \begin{bmatrix} \frac{\partial g_1}{\partial w_1} & \cdots & \frac{\partial g_1}{\partial w_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial w_1} & \cdots & \frac{\partial g_p}{\partial w_p} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial g_1}{\partial v_1} & \cdots & \frac{\partial g_1}{\partial v_{m-p}} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial v_1} & \cdots & \frac{\partial g_p}{\partial v_{m-p}} \end{bmatrix} \tag{6.67}$$

respectively, and the differential operators $\frac{d\cdot}{dx}$, $\frac{d\cdot}{dv}$ defined by (6.46), (6.47) take the form

$$\frac{d\cdot}{dx} := \left[\frac{d\cdot}{dx_1}, \quad \cdots, \quad \frac{d\cdot}{dx_n} \right] = \left[\frac{\partial\cdot}{\partial x_1}, \quad \cdots, \quad \frac{\partial\cdot}{\partial x_n} \right]$$

$$- \left[\frac{\partial\cdot}{\partial w_1}, \quad \cdots, \quad \frac{\partial\cdot}{\partial w_p} \right] \begin{bmatrix} \frac{\partial g_1}{\partial w_1} & \cdots & \frac{\partial g_1}{\partial w_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial w_1} & \cdots & \frac{\partial g_p}{\partial w_p} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial x_1} & \cdots & \frac{\partial g_p}{\partial x_n} \end{bmatrix} \qquad (6.68)$$

$$\frac{d\cdot}{dv} := \left[\frac{d\cdot}{dv_1}, \quad \cdots, \quad \frac{d\cdot}{dv_{m-p}} \right] = \left[\frac{\partial\cdot}{\partial v_1}, \quad \cdots, \quad \frac{\partial\cdot}{\partial v_{m-p}} \right]$$

$$- \left[\frac{\partial\cdot}{\partial w_1}, \quad \cdots, \quad \frac{\partial\cdot}{\partial w_p} \right] \begin{bmatrix} \frac{\partial g_1}{\partial w_1} & \cdots & \frac{\partial g_1}{\partial w_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial w_1} & \cdots & \frac{\partial g_p}{\partial w_p} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial g_1}{\partial v_1} & \cdots & \frac{\partial g_1}{\partial v_{m-p}} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial v_1} & \cdots & \frac{\partial g_p}{\partial v_{m-p}} \end{bmatrix}. \qquad (6.69)$$

Also define

$$\begin{bmatrix} \mu_1 \\ \vdots \\ \mu_p \end{bmatrix} = - \begin{bmatrix} \frac{\partial g_1}{\partial w_1} & \cdots & \frac{\partial g_1}{\partial w_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_p}{\partial w_1} & \cdots & \frac{\partial g_p}{\partial w_p} \end{bmatrix}^{-T} \begin{bmatrix} \frac{\partial H}{\partial w_1} \\ \vdots \\ \frac{\partial H}{\partial w_p} \end{bmatrix}. \qquad (6.70)$$

Then equation (6.48) can be written as

$$\dot\lambda^T = -\frac{\partial H}{\partial x} - \mu^T \frac{\partial g}{\partial x} \qquad (6.71)$$

and equations (6.50) and (6.70) can be restated together as

$$\frac{\partial H}{\partial u} + \mu^T \frac{\partial g}{\partial u} = 0. \qquad (6.72)$$

The non-singularity of matrix $\frac{\partial g}{\partial w}$ in equations (6.66), (6.67), (6.68), (6.69), (6.70) is guaranteed by assumption (6.40), (refK-e 5.7). It is clear that also higher order derivatives can be treated in this way by successively applying the differential operators $\frac{d}{dx}$ and $\frac{d}{dv}$

stated in (6.68) and (6.69), respectively. Hence it is possible to test the optimality conditions stated in Theorem 6.6.2 without explicit knowledge of the functional dependence of $w = W(x, v)$ defined by (6.44). Stating general expressions would be lengthy and unnecessarily confusing without providing further insight. In practice it is recommended to perform all necessary operations step by step, simplifying expressions in every stage as far as possible. The classical Legendre-Clebsch condition (6.51) can be restated as

$$\delta v^T \left[I, \ \left(\frac{\partial W}{\partial v}\right)^T \right] \begin{bmatrix} \frac{\partial^2 H}{\partial v^2} + \mu^T \frac{\partial^2 g}{\partial v^2} & \frac{\partial^2 H}{\partial v \partial w} + \mu^T \frac{\partial^2 g}{\partial v \partial w} \\ \frac{\partial^2 H}{\partial w \partial v} + \mu^T \frac{\partial^2 g}{\partial w \partial v} & \frac{\partial^2 H}{\partial w^2} + \mu^T \frac{\partial^2 g}{\partial w^2} \end{bmatrix} \begin{bmatrix} I \\ \frac{\partial W}{\partial v} \end{bmatrix} \delta v \geq 0 \ \forall \ \delta v \in \mathbf{R}^{m-p}.$$

$$(6.73)$$

With Definition (6.41) and assumption (6.42)

$$\frac{\partial g}{\partial u} \delta u = 0$$

is equivalent to

$$\delta w = - \left(\frac{\partial g}{\partial w}\right)^{-1} \frac{\partial g}{\partial v} \delta v,$$

or with (6.67)

$$\delta w = \frac{\partial W}{\partial v} \delta v.$$

Hence

$$\left\{ \delta v^T \left[I, \left(\frac{\partial W}{\partial v}\right)^T \right] \Big| \ \delta v \in \mathbf{R}^{m-p} \right\} = \left\{ \left[\delta v^T, \delta w^T \right] \Big| \ \delta v \in \mathbf{R}^{m-p}, \ \delta w = \frac{\partial W}{\partial v} \delta v \right\}$$

$$= \left\{ \left[\delta u^T \in \mathbf{R}^m \right] \Big| \ \frac{\partial g}{\partial u} \delta u = 0 \right\}$$

and condition (6.73) yields the well-known result

$$\delta u^T \left[\frac{\partial^2 H}{\partial u^2} + \mu^T \frac{\partial^2 g}{\partial u^2} \right] \delta u \geq 0 \ \forall \ \delta u \in \mathbf{R}^m \ \text{satisfying} \ \frac{\partial g}{\partial u} \delta u = 0. \qquad (6.74)$$

6.8 Singular Linear-Quadratic Optimal Control Problem

Definition 6.8.1 *Let problem (*) be defined as*

$$\min_{u_1 \in (L_2[t_0,t_f])^{m^*},\, u_2 \in (L_2[t_0,t_f])^{m-m^*}} J_2 \tag{6.75}$$

$$J_2 := \gamma(x(t_f), t_f) + \int_{t_0}^{t_f} \omega \, dt$$

$$\omega := x^T P_1 x + 2 \begin{bmatrix} u_1^T, u_2^T \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} x + \begin{bmatrix} u_1^T, u_2^T \end{bmatrix} \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

subject to

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \quad x(t_0) = 0 \tag{6.76}$$

$$\Psi(x(t_f), t_f) = 0. \tag{6.77}$$

Here the matrix functions of time

$$R_1 \in \mathbf{R}^{m^*, m^*}, \quad R_1 \geq 0$$

$$Q_1 \in \mathbf{R}^{m^*, n}, Q_2 \in \mathbf{R}^{m-m^*, n}$$

$$B_1 \in \mathbf{R}^{n, m^*}, B_2 \in \mathbf{R}^{n, m-m^*}$$

$$P_1 \in \mathbf{R}^{n,n}, \quad P_1 = P_1^T$$

$$A \in \mathbf{R}^{n,n},$$

are assumed continuous and B_2, Q_2 are assumed continuously differentiable. The functions $\gamma : \mathbf{R}^{n+1} \to \mathbf{R}$, $(x(t_f), t_f) \mapsto \gamma(x(t_f), t_f)$, $\Psi : \mathbf{R}^{n+1} \to \mathbf{R}^s$, $s \leq n$, $(x(t_f), t_f) \mapsto \Psi(x(t_f), t_f)$ are assumed smooth and satisfy $\gamma(0,0) = 0$, $\Psi(0,0) = 0$. For all $k \in \mathbf{N}$ $(L_2[t_0, t_f])^k$ denotes the Hilbert space of all quadratically integrable functions $f(t)$ from the interval $[t_0, t_f]$ into $\mathbf{R}^k$.

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Theorem 6.8.2 *A necessary condition for* $\left[x(t)^T, u_1(t)^T, u_2(t)^T\right] = [0, 0, 0]$ $\forall t \in [t_0, t_f]$

to be an optimal solution of problem (*) is that

(i) the $(m - m^*) \times (m - m^*)$ *matrix function of time* $Q_2 B_2$ *is identically symmetric, i.e.*

$$Q_2 B_2 = B_2{}^T Q_2{}^T \quad \forall t \in [t_0, t_f] \tag{6.78}$$

(ii) if $Q_2 B_2$ *is identically symmetric, then*

$$R_4 := \begin{bmatrix} R_1 & R_2{}^T \\ R_2 & R_3 \end{bmatrix} \geq 0 \quad \forall t \in [t_0, t_f] \tag{6.79}$$

where

$$R_2 := B_2{}^T Q_1{}^T - Q_2 B_1 \tag{6.80}$$

$$R_3 := B_2{}^T P_1 B_2 - \frac{d}{dt}(Q_2 B_2) - B_3{}^T Q_2{}^T \tag{6.81}$$

$$B_3 := A B_2 - \dot{B}_2 \tag{6.82}$$

Proof: The proof is adopted from Goh [11] and, for convenience, is restated here in slightly modified form.

Introducing the matrices

$$R := \begin{bmatrix} 0 & 0 & 0 \\ 0 & R_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbf{R}^{n+m^*+(m-m^*),n+m^*+(m-m^*)} \tag{6.83}$$

$$Q := \begin{bmatrix} 0 & 0 & 0 \\ Q_1 & 0 & 0 \\ Q_2 & 0 & 0 \end{bmatrix} \in \mathbf{R}^{n+m^*+(m-m^*),n+m^*+(m-m^*)} \tag{6.84}$$

$$P := \begin{bmatrix} P_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbf{R}^{n+m^*+(m-m^*),n+m^*+(m-m^*)} \qquad (6.85)$$

and defining the new state vector $\eta \in \mathbf{R}^{n+m^*+(m-m^*)}$ by

$$\eta(t)^T := \begin{bmatrix} \eta_1(t)^T, & \eta_2(t)^T, & \eta_3(t)^T \end{bmatrix}$$

$$= \begin{bmatrix} x(t)^T, & \int_{t_0}^t u_1(\tau)d\tau^T, & \int_{t_0}^t u_2(\tau)d\tau^T \end{bmatrix} \qquad (6.86)$$

problem (*) can be restated in the form

$$\min_{[u_1^T, u_2^T] \in (L_2[t_0,t_f])^{m^*+(m-m^*)}} J_2 \qquad (6.87)$$

$$J_2 = \gamma(\eta_1(t_f), t_f) + \int_{t_0}^{t_f} \eta^T P \eta + 2\dot{\eta}^T Q \eta + \dot{\eta}^T R \dot{\eta} \, dt \qquad (6.88)$$

subject to the state equations

$$\begin{aligned} \dot{\eta}_1 &= A\eta_1 + B_1 u_1 + B_2 u_2 \\ \dot{\eta}_2 &= u_1 \\ \dot{\eta}_3 &= u_2 \end{aligned} \qquad (6.89)$$

the initial conditions

$$\begin{aligned} \eta_1(t_0) &= x_0 \\ \eta_2(t_0) &= 0 \\ \eta_3(t_0) &= 0 \end{aligned} \qquad (6.90)$$

and the boundary conditions

$$\Psi(\eta_1(t_f), t_f) = 0. \qquad (6.91)$$

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Now define $\bar{\eta}$ by the regular transformation

$$\bar{\eta} = V\eta \qquad (6.92)$$

where

$$V := \begin{bmatrix} I_n & 0 & -B_2 \\ 0 & I_{m^\bullet} & 0 \\ 0 & 0 & I_{m-m^\bullet} \end{bmatrix} \in \mathbf{R}^{n+m^\bullet+(m-m^\bullet),n+m^\bullet+(m-m^\bullet)}. \qquad (6.93)$$

Then we have

$$\bar{\eta}(t)^T := \begin{bmatrix} \bar{\eta}_1^T, & \bar{\eta}_2^T, & \bar{\eta}_3^T \end{bmatrix}$$

$$:= \begin{bmatrix} (x(t) - B_2 \int_{t_0}^t u_2(\tau)d\tau)^T, & \int_{t_0}^t u_1(\tau)d\tau^T, & \int_{t_0}^t u_2(\tau)d\tau^T \end{bmatrix} \qquad (6.94)$$

and problem (*) can be rewritten as

$$\min_{[u_1^T,\, u_2^T] \in (L_2[t_0,t_f])^{m^\bullet+(m-m^\bullet)}} J_2 \qquad (6.95)$$

$$J_2 = \gamma(\bar{\eta}_1(t_f) + B_2(t_f)\bar{\eta}_3(t_f), t_f) + \int_{t_0}^{t_f} \bar{\eta}^T P \bar{\eta} + 2\dot{\bar{\eta}}^T Q \bar{\eta} + \dot{\bar{\eta}}^T R \dot{\bar{\eta}}\, dt \qquad (6.96)$$

subject to the state equations

$$\dot{\bar{\eta}}_1 = A\bar{\eta}_1 + B_1 u_1 + (AB_2 - \dot{B}_2)\bar{\eta}_3$$

$$\dot{\bar{\eta}}_2 = u_1 \qquad (6.97)$$

$$\dot{\bar{\eta}}_3 = u_2$$

the initial conditions

$$\bar{\eta}_1(t_0) = x_0$$

$$\bar{\eta}_2(t_0) = 0 \qquad (6.98)$$

$$\bar{\eta}_3(t_0) = 0$$

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and the boundary conditions

$$\Psi(\bar{\eta}_1(t_f) + B_2(t_f)\bar{\eta}_3(t_f), t_f) = 0. \tag{6.99}$$

Here the old matrices R, Q, P given by (6.83) - (6.85) are replaced by

$$R := \begin{bmatrix} 0 & 0 & 0 \\ 0 & R_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.100}$$

$$Q := \begin{bmatrix} 0 & 0 & 0 \\ Q_1 & 0 & Q_1 B_2 \\ Q_2 & 0 & Q_2 B_2 \end{bmatrix} \tag{6.101}$$

$$P := \begin{bmatrix} P_1 & 0 & P_1 B_2 \\ 0 & 0 & 0 \\ B_2{}^T P_1 & 0 & B_2{}^T P_1 B_2 \end{bmatrix}. \tag{6.102}$$

In the so-transformed problem $\dot{\bar{\eta}}_3(t) = u_2(t)$ appears only in the bilinear forms $2\dot{\bar{\eta}}_3^T Q_2 \bar{\eta}_1$

and $2\dot{\bar{\eta}}_3^T Q_2 B_2 \bar{\eta}_3$. Integrating by parts, employing state equations (6.97) and assuming that

$Q_2 B_2$ is identically symmetric it follows that

$$\int_{t_0}^{t_f} 2\dot{\bar{\eta}}_3^T Q_2 \bar{\eta}_1 dt = 2\bar{\eta}_3^T Q_2 \bar{\eta}_1 \Big|_{t_0}^{t_f}$$

$$- \int_{t_0}^{t_f} 2\bar{\eta}_3^T \dot{Q}_2 \bar{\eta}_1 + 2\bar{\eta}_3^T Q_2 B_1 \bar{\eta}_2 + 2\bar{\eta}_3^T Q_2 A \bar{\eta}_1 + 2\bar{\eta}_3^T Q_2(A B_2 - \dot{B}_2)\bar{\eta}_3 \, dt \tag{6.103}$$

and

$$\int_{t_0}^{t_f} 2\dot{\bar{\eta}}_3^T Q_2 B_2 \bar{\eta}_3 dt = \bar{\eta}_3^T Q_2 B_2 \bar{\eta}_3 \Big|_{t_0}^{t_f} - \int_{t_0}^{t_f} \bar{\eta}_3^T \frac{d}{dt}(Q_2 B_2)\bar{\eta}_3 \, dt. \tag{6.104}$$

Hence problem (*) takes the form

$$\min_{[u_1^T, u_2^T] \in (L_2[t_0, t_f])^{m^* + (m-m^*)}} J_2 \tag{6.105}$$

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$$J_2 = 2\,\bar{\eta}_3^T Q_2 \bar{\eta}_1 \Big|_{t_0}^{t_f} + \bar{\eta}_3^T Q_2 B_2 \bar{\eta}_3 \Big|_{t_0}^{t_f} +$$

$$+ \gamma(\bar{\eta}_1(t_f) + B_2(t_f)\bar{\eta}_3(t_f), t_f) + \int_{t_0}^{t_f} \bar{\eta}^T P \bar{\eta} + 2\dot{\bar{\eta}}^T Q \bar{\eta} + \dot{\bar{\eta}}^T R \dot{\bar{\eta}}\, dt \qquad (6.106)$$

subject to the state equations

$$\dot{\bar{\eta}}_1 = A\bar{\eta}_1 + B_1 u_1 + (AB_2 - \dot{B}_2)\bar{\eta}_3$$

$$\dot{\bar{\eta}}_2 = u_1 \qquad\qquad\qquad\qquad\qquad (6.107)$$

$$\dot{\bar{\eta}}_3 = u_2$$

the initial conditions

$$\bar{\eta}_1(t_0) = x_0$$

$$\bar{\eta}_2(t_0) = 0 \qquad\qquad\qquad (6.108)$$

$$\bar{\eta}_3(t_0) = 0$$

and the boundary conditions

$$\Psi(\bar{\eta}_1(t_f) + B_2(t_f)\bar{\eta}_3(t_f), t_f) = 0. \qquad (6.109)$$

Here the old matrices R, Q, P given by (6.100) - (6.102) replaced by

$$R := \begin{bmatrix} 0 & 0 & 0 \\ 0 & R_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (6.110)$$

$$Q := \begin{bmatrix} 0 & 0 & 0 \\ Q_1 & 0 & Q_1 B_2 + B_1^T Q_2^T \\ Q_2 & 0 & Q_2 B_2 \end{bmatrix} \qquad (6.111)$$

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$$P := \begin{bmatrix} P_1 & 0 & P_1 B_2 \\ 0 & 0 & 0 \\ B_2{}^T P_1 + 2\dot{Q}_2 + 2Q_2 A & 0 & B_2{}^T P_1 B_2 + \frac{d}{dt}(Q_2 B_2) + 2Q_2(AB_2 - \dot{B}_2) \end{bmatrix}. \tag{6.112}$$

Again defining a new state vector by

$$\left[\bar{\bar{\eta}}(t)^T, \ v(t)^T \right]$$

$$:= \left[\bar{\bar{\eta}}_1(t)^T, \ \bar{\bar{\eta}}_2(t)^T, \ \bar{\bar{\eta}}_3(t)^T, \ v(t)^T \right]$$

$$:= \left[\bar{\eta}_1^T, \ \bar{\eta}_2^T, \ \left(\int_{t_0}^{t_f} \bar{\eta}_3(\tau) d\tau \right)^T \left(\int_{t_0}^{t_f} u_2(\tau) d\tau \right)^T \right]$$

$$= \left[\left(x(t) - B_2 \int_{t_0}^{t} u_2(\tau) d\tau \right)^T, \ \left(\int_{t_0}^{t} u_1(\tau) d\tau \right)^T, \ \left(\int_{t_0}^{t} \int_{t_0}^{\tau} u_2(\sigma) d\sigma d\tau \right)^T, \ \left(\int_{t_0}^{t} u_2(\tau) d\tau \right)^T \right]$$

$$\tag{6.113}$$

problem (*) finally takes the form

$$\min_{[u_1^T, \ u_2^T] \in (L_2[t_0, t_f])^{m^* + (m - m^*)}} J_2 \tag{6.114}$$

$$J_2 = v(t_f)^T Q_2 \bar{\bar{\eta}}_1(t_f) + v(t_f)^T Q_2 B_2 v(t_f) +$$

$$+ \gamma(\bar{\bar{\eta}}_1(t_f) + B_2(t_f) v(t_f), t_f) + \int_{t_0}^{t_f} \left(\bar{\bar{\eta}}^T P \bar{\bar{\eta}} + 2 \dot{\bar{\bar{\eta}}} Q \bar{\bar{\eta}} + \dot{\bar{\bar{\eta}}}^T R \dot{\bar{\bar{\eta}}} \right) dt \tag{6.115}$$

subject to the state equations

$$\dot{\bar{\bar{\eta}}}_1 = A\bar{\bar{\eta}}_1 + B_1 u_1 + (AB_2 - \dot{B}_2)v$$

$$\dot{\bar{\bar{\eta}}}_2 = u_1 \tag{6.116}$$

$$\dot{\bar{\bar{\eta}}}_3 = v$$

$$\dot{v} = u_2$$

the initial conditions

$$\bar{\bar{\eta}}_1(t_0) = x_0$$

$$\bar{\bar{\eta}}_2(t_0) = 0$$

$$\bar{\bar{\eta}}_3(t_0) = 0 \qquad (6.117)$$

$$v(t_0) = 0$$

and the boundary conditions

$$\Psi(\bar{\bar{\eta}}_1(t_f) + B_2(t_f)v(t_f), t_f) = 0. \qquad (6.118)$$

Here the matrices P, Q, R are as follows

$$P := \begin{bmatrix} P_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (6.119)$$

$$Q := \begin{bmatrix} 0 & 0 & 0 \\ Q_1 & 0 & 0 \\ Q_3 & 0 & 0 \end{bmatrix} \qquad (6.120)$$

$$R := \begin{bmatrix} 0 & 0 & 0 \\ 0 & R_1 & R_2^T \\ 0 & R_2 & R_3 \end{bmatrix} \qquad (6.121)$$

where

$$R_2 := B_2^T Q_1^T - Q_2 B_1 \qquad (6.122)$$

$$R_3 := B_2^T P_1 B_2 - \frac{d}{dt}(Q_2 B_2) - Q_2 B_3 - B_3^T Q_2^T \qquad (6.123)$$

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$$Q_3 := B_2^T P_1 - Q_2 A - \dot{Q}_2 \tag{6.124}$$

$$B_3 := AB_2 - \dot{B}_2. \tag{6.125}$$

In this problem control u_2 is "free of charge". Note that u_2 does not appear anywhere in the cost function J_2 in (6.115). Furthermore, in the right-hand side of the state equations u_2 appears only in the expression $\dot{v} = u_2$. Hence, on every interval $[t_1, t_1 + \epsilon] \subset [t_0, t_f]$, $\epsilon > 0$, state $v(t)$ can be changed from any $c_1 \in \mathbf{R}^{m-m^*}$ to any $c_2 \in \mathbf{R}^{m-m^*}$ (i.e. $v(t_1) = c_1$, $v(t_1 + \epsilon) = c_2$) and the penalty in cost is at most of the order ϵ. This implies that the status of v can be raised to the status of a control. Let $c^* \in \mathbf{R}^{m-m^*}$ be the optimal value of $v(t_f)$ consistent with the boundary conditions (6.118) (the existence of a solution to problem (*) defined in Definition 6.8.1 implies the existence of such a c^*), then problem (*) can be rewritten as follows:

$$\min_{[u_1^T, v^T] \in (L_2[t_0, t_f])^{m^* + (m-m^*)}} J_2 \tag{6.126}$$

$$J_2 = c^{*T} Q_2 \bar{\bar{\eta}}_1(t_f) + \gamma(\bar{\bar{\eta}}_1(t_f) + B_2(t_f)c^*, t_f) + \int_{t_0}^{t_f} \left(\bar{\bar{\eta}}^T P \bar{\bar{\eta}} + 2\dot{\bar{\bar{\eta}}} Q \bar{\bar{\eta}} + \dot{\bar{\bar{\eta}}}^T R \dot{\bar{\bar{\eta}}} \right) dt \tag{6.127}$$

subject to the state equations

$$\dot{\bar{\bar{\eta}}}_1 = A\bar{\bar{\eta}}_1 + B_1 u_1 + (AB_2 - \dot{B}_2)v$$

$$\dot{\bar{\bar{\eta}}}_2 = u_1 \tag{6.128}$$

$$\dot{\bar{\bar{\eta}}}_3 = v$$

the initial conditions

$$\bar{\bar{\eta}}_1(t_0) = x_0$$

$$\bar{\bar{\eta}}_2(t_0) = 0 \tag{6.129}$$

$$\bar{\bar{\eta}}_3(t_0) = 0$$

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and the boundary conditions

$$\Psi(\bar{\bar{\eta}}_1(t_f) + B_2(t_f)c^*, t_f) = 0. \tag{6.130}$$

Here matrices P, Q, R are given by (6.119) - (6.121). Now the classical Legendre-Clebsch condition implies that necessarily

$$\begin{bmatrix} R_1 & R_2^T \\ R_2 & R_3 \end{bmatrix} \geq 0. \tag{6.131}$$

This proves (ii) under the assumption that (i) holds true, and it is left to show that indeed

(i) $Q_2 B_2 = B_2^T Q_2^T \quad \forall t \in [t_0, t_f]$

Assume this is not the case. Define $B_2^* \in \mathbf{R}^{n, m^{**}}$, $B_2^{**} \in \mathbf{R}^{n, n-m^*-m^{**}}$, $Q_2^* \in \mathbf{R}^{m^{**}, n}$, $Q_2^{**} \in \mathbf{R}^{m-m^*-m^{**}, n}$, by

$$\begin{bmatrix} Q_2^* \\ Q_2^{**} \end{bmatrix} := Q_2 \tag{6.132}$$

$$[B_2^*, B_2^{**}] := B_2 \tag{6.133}$$

where $m^{**} \geq 0$ is the smallest possible integer such that the $(m-m^*-m^{**})\times(m-m^*-m^{**})$ order submatrix $Q_2^{**} B_2^{**}$ of

$$Q_2 B_2 = \begin{bmatrix} Q_2^* B_2^* & Q_2^* B_2^{**} \\ Q_2^{**} B_2^* & Q_2^{**} B_2^{**} \end{bmatrix} \tag{6.134}$$

is identically symmetric. Then by construction the nonsymmetry of $Q_2 B_2$ implies

$$Q_2^{**} B_2^* - B_2^{**T} Q_2^{*T} \neq 0 \text{ for some } t \in [t_0, t_f] \tag{6.135}$$

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Now applying (ii) of the Theorem on problem (6.87) - (6.91), with matrices (6.83) - (6.86) replaced by

$$
R := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & R_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbf{R}^{n+m^*+m^{**}+(m-m^*-m^{**}),n+m^*+m^{**}+(m-m^*-m^{**})} \tag{6.136}
$$

$$
Q := \begin{bmatrix} 0 & 0 & 0 & 0 \\ Q_1 & 0 & 0 & 0 \\ Q_2^* & 0 & 0 & 0 \\ Q_2^{**} & 0 & 0 & 0 \end{bmatrix} \in \mathbf{R}^{n+m^*+m^{**}+(m-m^*-m^{**}),n+m^*+m^{**}+(m-m^*-m^{**})} \tag{6.137}
$$

$$
P := \begin{bmatrix} P_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbf{R}^{n+m^*+m^{**}+(m-m^*-m^{**}),n+m^*+m^{**}+(m-m^*-m^{**})} \tag{6.138}
$$

$$
\Phi := \begin{bmatrix} I_n, & -B_1, & -B_2^*, & -B_2^{**} \end{bmatrix} \in \mathbf{R}^{n,n+m^*+m^{**}+(m-m^*-m^{**})} \tag{6.139}
$$

$$
\Theta := \begin{bmatrix} -A, & 0, & 0, & 0 \end{bmatrix} \in \mathbf{R}^{n,n+m^*+m^{**}+(m-m^*-m^{**})} \tag{6.140}
$$

$$
\eta(t)^T := \begin{bmatrix} x(t)^T, & \left(\int_{t_0}^t u_1(\tau)d\tau\right)^T, & \left(\int_{t_0}^t u_2^*(\tau)d\tau\right)^T, & \left(\int_{t_0}^t u_2^{**}(\tau)d\tau\right)^T \end{bmatrix}
$$

$$
\in \mathbf{R}^{1,n+m^*+m^{**}+(m-m^*-m^{**})} \tag{6.141}
$$

implies that

$$
R_4 := \begin{bmatrix} R_1 & 0_{m^\bullet, m^{\bullet\bullet}} & R_2^{*T} \\ 0_{m^{\bullet\bullet}, m^\bullet} & 0_{m^{\bullet\bullet}, m^{\bullet\bullet}} & R_2^{**T} \\ R_2^* & R_2^{**} & R_3 \end{bmatrix} \geq 0 \quad \forall t \in [t_0, t_f] \tag{6.142}
$$

where

$$
[R_2^*, R_2^{**}] = B_2^{**T} \left[Q_1^{T}, Q_2^{*T} \right] - Q_2^{**} [B_1, B_2^*]
$$

$$
= \left[B_2^{**T} Q_1^{T} - Q_2^{**} B_1, \; B_2^{**T} Q_2^{*T} - Q_2^{**} B_2^* \right].
$$

Hence

$$
R_2^* = B_2^{**T} Q_1^T - Q_2^{**} B_1 \tag{6.143}
$$

and

$$
R_2^{**} = B_2^{**T} Q_2^{*T} - Q_2^{**} B_2^*. \tag{6.144}
$$

Now the positive semidefiniteness of R_4 implies that all 2×2 order submatrices of the

form

$$
\begin{vmatrix} 0 & (R_2^{**})_{pq} \\ (R_2^{**})_{pq} & (R_3)_{qq} \end{vmatrix} \geq 0 \quad \forall t \in [t_0, t_f]. \tag{6.145}
$$

This implies $-(R_2^{**})_{pq}^2 \geq 0$ $\forall t \in [t_0, t_f]$ so that necessarily

$$
R_2^{**} \equiv 0 \quad \forall t \in [t_0, t_f]. \tag{6.146}
$$

But now (6.144) and (6.146) contradict the assumption (6.135). Hence assumption (6.135)

is wrong and $Q_2 B_2$ is identically symmetric.

q.e.d.

6.9 Example

Examples for the application of the Generalized Legendre-Clebsch Condition to singular control along state/control constrained arcs can be found in Appendix A. In Section A.5 the results obtained in this Chapter are applied to the singular control cases in the aircraft range optimization problem treated in Chapter 5.

Chapter 7

Touch Points for First Order

State Inequality Constraints

Chapter 7

Touch Points for First Order

State Inequality Constraints

Chapter Overview

The appearance of touch points in state constrained optimal control problems with general vector-valued control is studied. Under the assumption that the Hamiltonian is regular, touch points for first-order state inequalities are shown to exist only under very special conditions, stated in Corollary 7.3.4. For state inequality constraints of arbitrary order the control is shown to be continuous across a touch point.

7.1   Introduction

In optimal control problems state inequality constraints can turn active in two different ways, namely in form of constrained arcs and in form of touch points. In practice, the form in which a constraint becomes active (i.e. the switching structure) has to be found by numerical experiments and involves some kind of "smart guessing". This time consuming process can be cut down considerably if certain switching structures can be excluded a priori. In this Chapter the existing first-order necessary conditions associated with interior point constraints are used to derive concise new statements. Conditions similar to the ones derived here have been derived in [31] for the case of a scalar control. In this Chapter no restriction on the dimension of the control vector is imposed.

7.2 Problem Formulation and Existing Optimality Conditions

Definition 7.2.1 (Reference Problem) *Let the reference problem be given as follows:*

$$\min_{u(t)\in\mathbf{R}^m} \Phi(x(t_f),t_f) \tag{7.1}$$

$$\dot{x} = f(x,u,t) \tag{7.2}$$

$$x(t_0) = x_0 \tag{7.3}$$

$$\Psi(x(t_f),t_f) = 0 \tag{7.4}$$

$$L(x) \leq 0 \tag{7.5}$$

where $x(t) \in \mathbf{R}^n$; $u(t) \in \mathbf{R}^m$; $\Phi : \mathbf{R}^{n+1} \mapsto \mathbf{R}^1$; $f : \mathbf{R}^{n+m+1} \mapsto \mathbf{R}^n$; $\Psi : \mathbf{R}^{n+1} \mapsto \mathbf{R}^k$, $k \leq n$; $L : \mathbf{R}^n \mapsto \mathbf{R}^1$ *is a q-th order state constraint, i.e.*

$$\frac{\partial}{\partial u}\left(\frac{d^i L(x)}{dt^i}\right) \begin{cases} = 0 & for \quad i = 0,1,...,q-1 \\[2mm] \neq 0 & for \quad i = q. \end{cases}$$

Definition 7.2.2 (Touch Point) *Let* $x^*(t)$, $u^*(t)$ *be an extremal associated with the reference problem stated in Definition 7.2.1. Time* $t_1 \in (t_0,t_f)$ *is called a touch point if there is some* $\epsilon > 0$ *such that*

$$\begin{aligned} L(x(t)) &= 0 \quad at \quad t = t_1 \\ L(x(t)) &< 0 \quad on \quad t \in (t_1 - \epsilon, t_1) \cup (t_1, t_1 + \epsilon) \end{aligned} \tag{7.6}$$

(see Figure 7.1).
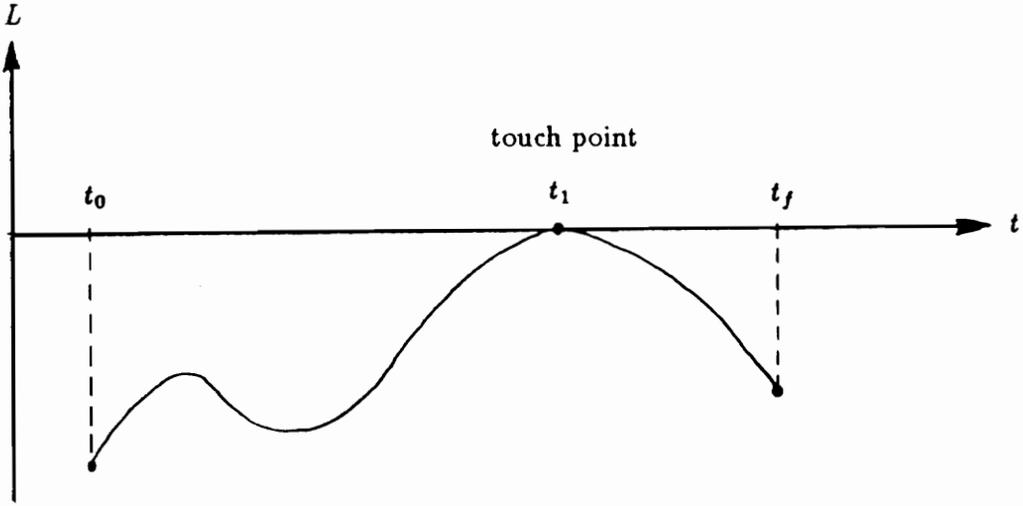
Figure 7.1: Touch Point

Definition 7.2.3 (Notation) *In this Chapter, let superscripts $(+)$, $(-)$ denote evaluation just left and just right of touch point t_1, respectively. Explicitly, for any function of time $f(t)$ we define*

$$f^+ = \lim_{\epsilon>0, \epsilon \Rightarrow 0} f(t_1 + \epsilon),$$

$$f^- = \lim_{\epsilon>0, \epsilon \Rightarrow 0} f(t_1 - \epsilon).$$

Lemma 7.2.4 *Assume the optimal solution to the reference problem given in Definition 7.2.1 has a touch point at some time $t_1 \in (t_0, t_f)$. Then the following necessary conditions for optimality are obtained from the Minimum Principle (for notation see Def-*

inition 7.2.3):

$$
\left.
\begin{aligned}
L^{(0)}(x) &= 0 \\
L^{(1)}(x) &= 0 \\
&\vdots \\
L^{(q-1)}(x) &= 0
\end{aligned}
\right\} \quad at\ t = t_1
\tag{7.7}
$$

$$
\left.
\begin{aligned}
L^{(q)}(x, u) &> 0 \quad on\ t \in (t_1 - \epsilon, t_1) \\
L^{(q)}(x, u) &< 0 \quad on\ t \in (t_1, t_1 + \epsilon)
\end{aligned}
\right\} \quad for\ some\ \epsilon > 0
\tag{7.8}
$$

$$
(\lambda^+) = (\lambda^-) - l_0 \frac{\partial L^{(0)}}{\partial x} - ... - l_{q-1} \frac{\partial L^{(q-1)}}{\partial x}
\tag{7.9}
$$

with

$$
l_0 \geq 0,\ l_1 \geq 0,\ ...\ l_{q-1} \geq 0
\tag{7.10}
$$

$$
H^+ - H^- = 0
\tag{7.11}
$$

$$
u^* = \arg \min_{u \in U} H \quad a.e..
\tag{7.12}
$$

Here and in the remainder of this Chapter superscript (i) denotes i-th total derivative w.r.t. time.

PROOF: Just apply the well-known first order necessary conditions (see Bryson & Ho [5]).

<div align="right">q.e.d.</div>

7.3 Concise Statements Implied by First-Order Necessary Conditions

Definition 7.3.1 (Regular Hamiltonian) *The Hamiltonian is called <u>regular</u> if $u^* = \arg\min_{u \in U} H$ is unique (i.e. if the Hodograph is strictly convex).*

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Lemma 7.3.2 *At a touch point $t_1 \in (t_0, t_f)$ in the solution to the reference problem stated in Definition 7.2.1 the following conditions are implied by the conditions given in Lemma 7.2.4 (for notation see Definition 7.2.3):*

$$\lambda^- f^+ = \lambda^- f^- = \lambda^+ f^+ = \lambda^+ f^- \tag{7.13}$$

$$l_{q-1}(L^q)^- = 0 \tag{7.14}$$

$$l_{q-1}(L^q)^+ = 0. \tag{7.15}$$

PROOF:

Equation (7.11) $H^+ - H^- = 0$ can be written as

$$\lambda^+ f^+ - \lambda^- f^- = 0.$$

With (7.9) this implies

$$\left(\lambda^- - l_0 \frac{\partial L^{(0)}}{\partial x} - \dots - l_{q-1} \frac{\partial L^{(q-1)}}{\partial x} \right) f^+ - \lambda^- f^- = 0$$

$$\lambda^- f^+ - l_0 \frac{\partial L^{(0)}}{\partial x} f^+ - \dots - l_{q-1} \frac{\partial L^{(q-1)}}{\partial x} f^+ - \lambda^- f^- = 0$$

$$\lambda^- f^+ - l_0 \left(L^{(1)} \right)^+ - \dots - l_{q-1} \left(L^{(q)} \right)^+ - \lambda^- f^- = 0.$$

Conditions (7.7) imply $\left(L^{(1)} \right)^+ = \dots = \left(L^{(q-1)} \right)^+ = 0$ so that

$$\lambda^- f^+ - l_{q-1} \left(L^{(q)} \right)^+ - \lambda^- f^- = 0$$

$$\lambda^- f^+ - \lambda^- f^- = l_{q-1} \left(L^{(q)} \right)^+.$$

Now recall that

$$(L^{(q)})^+ \leq 0 \qquad \text{(from (7.8)}$$

$$l_{q-1} \geq 0 \qquad \text{(from (7.10)}$$

$$\lambda^- f^+ - \lambda^- f^- \geq 0 \quad \text{(from (7.12)}.$$

Then

$$\underbrace{\lambda^- f^+ - \lambda^- f^-}_{\geq 0} = \underbrace{l_{q-1}}_{\geq 0} \underbrace{(L^{(q)})^+}_{\leq 0}$$

implies

$$\lambda^- f^+ - \lambda^- f^- = 0 \qquad\qquad (7.16)$$

$$l_{q-1}(L^{(q)})^+ = 0. \qquad\qquad (7.17)$$

Again starting with equation (7.11) $H^+ - H^- = 0$ we can write

$$\lambda^+ f^+ - \lambda^- f^- = 0.$$

Now using (7.9) to substitute for λ^- yields

$$\lambda^+ f^+ - \left(\lambda^+ + l_0 \frac{\partial L^{(0)}}{\partial x} + ... + l_{q-1} \frac{\partial L^{(q-1)}}{\partial x} \right) f^- = 0$$

$$\lambda^+ f^+ - \lambda^+ f^- - l_0 \frac{\partial L^{(0)}}{\partial x} f^- - ... - l_{q-1} \frac{\partial L^{(q-1)}}{\partial x} f^- = 0$$

$$\lambda^+ f^+ - \lambda^+ f^- - l_0 \left(L^{(1)} \right)^- - ... - l_{q-1} \left(L^{(q)} \right)^- = 0.$$

Again using conditions (7.7) we find $\left(L^{(1)} \right)^- = ... = \left(L^{(q-1)} \right)^- = 0$ so that

$$\lambda^+ f^+ - \lambda^+ f^- - l_{q-1} \left(L^{(q)} \right)^- = 0$$

$$\lambda^+ f^+ - \lambda^+ f^- = l_{q-1} \left(L^{(q)} \right)^-$$

Now recall that

$$(L^{(q)})^- \geq 0 \qquad \text{(from (7.8))}$$

$$l_{q-1} \geq 0 \qquad \text{(from (7.10))}$$

$$\lambda^+ f^+ - \lambda^+ f^- \leq 0 \quad \text{(from (7.12))}.$$

Then

$$\underbrace{\lambda^+ f^+ - \lambda^+ f^-}_{\leq 0} = \underbrace{l_{q-1}}_{\geq 0} \underbrace{(L^{(q)})^-}_{\geq 0}$$

implies

$$\lambda^+ f^+ - \lambda^+ f^- = 0 \tag{7.18}$$

$$l_{q-1}(L^{(q)})^- = 0. \tag{7.19}$$

Observing that equations (7.16), (7.18) in conjunction with the continuity of the Hamiltionian (7.11) imply (7.13) completes the proof.

<div align="right">q.e.d.</div>

Corollary 7.3.3 *Let t_1 be a touch point. If $H(t_1{}^+)$ or $H(t_1{}^-)$ is regular then the control is continuous across t_1.*

PROOF:

Without loss of generality assume $H(t_1{}^-)$ is regular. Condition (7.13) of Lemma 7.3.2 implies

$$\lambda^- f^+ = \lambda^- f^-.$$

By definition of regularity of the Hamiltonian (see Definition 7.3.1) this immediately implies

$$u^+ = u^-. \tag{7.20}$$

<div align="right">q.e.d.</div>
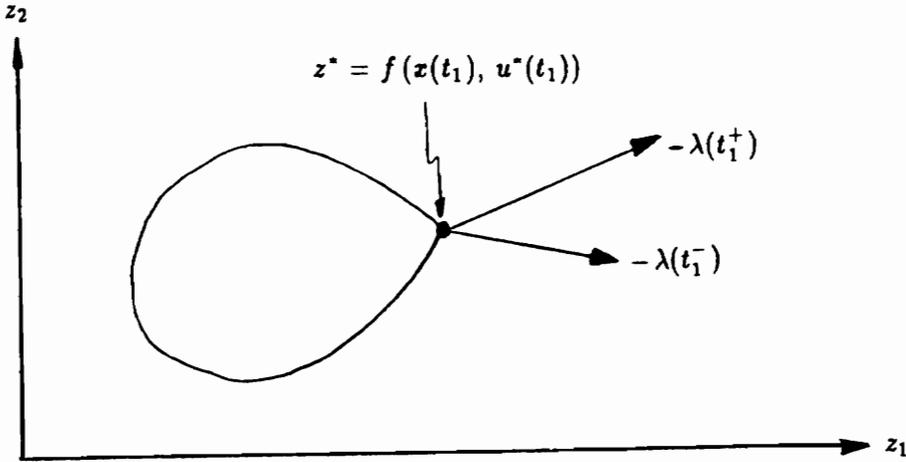
Figure 7.2: Optimal Control Located in a Corner Point of the Hodograph

Corollary 7.3.4 *Consider the reference problem stated in Definition 7.2.1 with state inequality constraint $L(x) \leq 0$ of order $q = 1$, i.e. $\frac{\partial}{\partial u}\frac{dL}{dt} \neq 0$, (meaning that $\frac{\partial}{\partial u}\frac{dL}{dt}$ is not the zero function). Let $t_1 \in (t_0, t_f)$ be a touch point (see Definition 7.2.2) and assume that either $H(t_1{}^+)$ or $H(t_1{}^-)$ is regular (see Definition 7.3.1). Then only the following three cases are possible:*

a) (trivial case) $l_0 = 0$, i.e. the touch point t_1 is a natural touch point.

b) $l_0 > 0$ and the Hodograph $\{z \in \mathbf{R}^n | z = f(x(t_1), u)$ for some $u \in \mathbf{R}^m\}$ has a corner point at time t_1 and $u^*(t_1)$ is located in that corner point (see Figure 7.2).

c) $l_0 > 0$ and $\left.\frac{\partial}{\partial u}\frac{dL}{dt}\right|_{u=u^*,\, t=t_1} = 0.$

PROOF:

a) Assume $l_0 = 0$, trivial.

b, c) Assume $l_0 \neq 0$. Then by (7.10) necessarily $l_0 > 0$. First let us split up all n-vectors

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$x, f, \frac{\partial L^{(0)}}{\partial x}$ into two components

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

$$\frac{\partial L^{(0)}}{\partial x} = \begin{bmatrix} \frac{\partial L^{(0)}}{\partial x_1}, & \frac{\partial L^{(0)}}{\partial x_2} \end{bmatrix}.$$

Here subscript 1 denotes the set of all components i of the original n-vector for which the associated state rate $f_i(x(t_1), u)$ depends explicitly on control u, and subscript 2 denotes the set of all components j of the original n-vector for which the associated state rate $f_j(x(t_1), u) = f_j(x(t_1))$ does not depend explicitly on control u.

Now, from $l_0 \neq 0$, we find $\lambda(t_1{}^+) \neq \lambda(t_1{}^-)$. This gives rise to the two possible cases

i) there is no $c \in \mathbf{R}$ such that $\lambda_1(t_1{}^+) = c\,\lambda_1(t_1{}^-)$ (case b))

ii) there is a $c \in \mathbf{R}$ such that $\lambda_1(t_1{}^+) = c\,\lambda_1(t_1{}^-)$ (case c))

case b):

In this case the vectors $\lambda_1(t_1{}^+)$ and $\lambda_1(t_1{}^-)$ have different directions. From (7.12) and (7.20) it follows that $\lambda_1(t_1{}^+)$ and $\lambda_1(t_1{}^-)$ are outward normal vectors associated with two different hyperplanes through the same point $z_1 = f_1(x, u^*)$ in the Hodograph (more precisely: in the projection of the Hodograph into the "1-plane"). This immediately implies that the Hodograph has a corner at time t_1 and that the optimal control $u^*(t_1)$ is located in such a corner.

case c):

Assume $\exists\, c \in \mathbf{R}$ such that $\lambda_1(t_1{}^+) = c\,\lambda_1(t_1{}^-)$. First note that necessarily $c \neq 1$.

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Otherwise (if $c = 1$) (7.9) in conjunction with $\frac{\partial L^{(0)}}{\partial x_1} \neq 0$ yields $l_0 = 0$ which contradicts the assumption. Hence $\lambda_1(t_1{}^+) = c\,\lambda_1(t_1{}^-)$ is equivalent into

$$\lambda_1(t_1{}^-) = \frac{\lambda_1(t_1{}^-) - \lambda_1(t_1{}^+)}{1 - c}$$

and with (7.9) this implies

$$\lambda_1(t_1{}^-)^T = \frac{l_0}{1 - c}\frac{\partial L^{(0)}}{\partial x_1}. \tag{7.21}$$

Now assume $u^*(t_1)$ is not located in a corner point of the Hodograph. Then the Minimum Principle (7.12) implies that along the optimal trajectory

$$\left.\frac{\partial H}{\partial u}\right|_{t_1-} = 0$$

or explicitly

$$\left.\left(\lambda_1{}^T\frac{\partial f_1}{\partial u}\right)\right|_{t_1-} = 0. \tag{7.22}$$

Inserting (7.21) in (7.22) yields

$$\underbrace{\frac{l_0}{1 - c}}_{\neq 0}\left.\left[\frac{\partial L^{(0)}}{\partial x_1}\left(\frac{\partial f}{\partial u}\right)_1\right]\right|_{t_1-} = 0.$$

Because of the continuity of control u across switching time t_1 (see Corollary 7.3.3) this implies $\left.\frac{\partial}{\partial u}\frac{dL^{(0)}}{dt}\right|_{u=u^*,\,t=t_1} = 0.$

<div align="right">q.e.d.</div>

Remarks:

a) Note that the conditions stated in Corollary 7.3.4 b), c) are only necessary for the existence of a touch-point, not sufficient.

b) Also note that these conditions (except for condition $l_0 > 0$) can be tested a priori without solving a boundary value problem. In this test, additionally conditions

$$L(x(t))|_{x=x^*,\,t=t_1} = 0$$

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$$\frac{dL}{dt}(x(t), u(t))\bigg|_{x=x^*, u=u^*, t=t_1} = 0$$

(see equations (7.7) and (7.14), (7.15)) can be used.

Chapter 8

Non-Optimality of the Accessory Minimum Problem in Presence of a Conjugate Point

A new proof for the non-optimality of the solution to a linear quadratic optimal control problem in presence of a conjugate point is presented. Contrary to an earlier proof by Breakwell & Ho [3] the new proof also covers the case where the coefficient functions of the linear quadratic problem have corners.

8.1 Introduction

Given an optimal control problem, it is possible to expand the cost function into a Taylor series about a solution candidate which is obtained by solving the first-order necessary conditions. The leading term of the expansion is of second order and can be identified as the cost associated with a linear quadratic optimal control problem, the so-called Accessory Minimum Problem (AMP). Necessary for the reference solution to furnish minimum cost is that the minimum cost associated with the AMP is non-negative. In this Chapter a proof of Jacobi's No-Conjugate-Point Condition is presented. In a constructive way it is shown that a conjugate point in the interior of the trajectory implies the existence of a solution to the AMP with negative cost. The general approach is adopted from [3]. The proof given in this Chapter applies also for linear quadratic optimal control problems with corners in the coefficient functions. It is hoped that it will be possible with the new approach to derive a Jacobi Necessary Condition for optimal control problems with interior point conditions. For convenience, and to provide a starting point for future work, the AMP for optimal control problems with interior point constraints is given in Appendix B.

8.2 The No-Conjugate-Point Condition for Linear Quadratic Optimal Control Problems with Corners

Lemma 8.2.1 *Let $A(t)$ be a Lipschitz continuous matrix function of time. Then the transition matrix $\Phi(t)$ associated with $A(t)$ is non-singular for all times, i.e.*

$$\det \Phi(t, t_0) \neq 0$$

where

$$\frac{d\, \Phi(t, t_0)}{dt} = A(t)\, \Phi(t, t_0)$$

$$\Phi(t, t_0) = I.$$

Proof:

(See for example Coddington & Levinson [7]). Assume $\det \Phi(t, t_0) = 0$ for some $t_1 \in \mathbf{R}$. Then $\exists x_0 \neq 0$ such that $\Phi(t_1, t_0) x_0 = 0$. But then the final value problem

$$\dot{x} = A(t)\, x$$

$$x(t_1) = 0$$

has at least two solutions, namely $x(t) \equiv 0$, and another solution with $x(t_0) = x_0 \neq 0$. This contradicts the assumed Lipschitz boundedness of $A(t)$.

<div align="right">q.e.d.</div>

Lemma 8.2.2 *Let $A(t)$ be a continuous Lipschitz bounded matrix function of time with one point, say t_1, of discontinuity. Then the transition matrix $\Phi(t, t_0)$ associated with*

A(t) is non-singular for all times, i.e.

$$det \ \Phi(t,t_0) \neq 0$$

where

$$\frac{d \ \Phi(t,t_0)}{dt} = A(t) \ \Phi(t,t_0)$$

$$\Phi(t,t_0) = I.$$

Proof:

Without loss of generality assume $t_1 > t_0$. According to Lemma 8.2.1 det $\Phi(t,t_0) \neq 0$

$\forall t \in [t_0,t_1]$. Assume det $\Phi(t_2,t_0) = 0$ for some $t_2 > t_1$. Then $\exists x_0 \neq 0$ such that

$\Phi(t_2,t_0) \ x_0 = 0$. As det $\Phi(t_1,t_0) \neq 0$ we have $\Phi(t_1,t_0) \ x_0 \neq 0$. But now we have two

solutions of the final value problem

$$\dot{x} = A(t) \ x$$

$$x(t_2) = 0$$

namely one solution given by $x(t) \equiv 0$ and another solution with the property $x(t_1) =$

$\Phi(t_1,t_0)x_0$. This contradicts the assumed Lipschitz boundedness of $A(t)$ on the interval

$[t_1,t_2]$.

<div align="right">q.e.d.</div>

Remark 8.2.3 *It is clear that Lemma 8.2.2 holds true also for continuous, Lipschitz*

bounded matrix functions of time $A(t)$ with any finite number of discontinuities.

Lemma 8.2.4 *Consider the problem*

$$\dot{x} = A(t)x + B(t)u$$

$$x(t_0) = x_0$$

with A, B Lipschitz continuous with at most finitely many discontinuities. Let $\Phi(t, t_0)$ be the transition matrix, i.e.

$$\frac{d\, \Phi(t, t_0)}{dt} = A(t)\, \Phi(t, t_0)$$

$$\Phi(t, t_0) = I$$

and assume that the Controllability-Grammian

$$K := \int_{t_0}^{t_f} \Phi(t, t_0)^{-1} B(t) B(t)^T \Phi(t, t_0)^{-T} dt$$

is non-singular. Then the control

$$u^*(t) = B(t)^T \Phi(t, t_0)^{-T} K^{-1} \Phi(t_f, t_0)^{-1} (x_f - \Phi(t_f, t_0)x_0)$$

takes the state from $x(t_0) = x_0$ to $x(t_f) = x_f$.

Remark: Note that by Remark 8.2.3 control u^* is well-defined.

Proof:

From the variation of parameters formula we get

$x(t) - \Phi(t, t_0)x_0 =$

$= \int_{t_0}^{t} \Phi(t, \tau) B(\tau) u^*(\tau) d\tau$

$= \int_{t_0}^{t} \Phi(t, \tau) B(\tau) B(\tau)^T \Phi(\tau, t_0)^{-T} K^{-1} \Phi(t_f, t_0)^{-1} (x_f - \Phi(t_f, t_0)x_0)\, d\tau$

$= \int_{t_0}^{t} \Phi(t, t_0) \Phi(\tau, t_0)^{-1} B(\tau) B(\tau)^T \Phi(\tau, t_0)^{-T} d\tau\; K^{-1} \Phi(t_f, t_0)^{-1} (x_f - \Phi(t_f, t_0)x_0)$

$= \Phi(t, t_0) \int_{t_0}^{t} \Phi(\tau, t_0)^{-1} B(\tau) B(\tau)^T \Phi(\tau, t_0)^{-T} d\tau\; K^{-1} \Phi(t_f, t_0)^{-1} (x_f - \Phi(t_f, t_0)x_0)$

Hence we have at time t_f

$$x(t_f) - \Phi(t_f, t_0)x_0 =$$

$$= \Phi(t_f, t_0) K K^{-1} \Phi(t_f, t_0)^{-1}(x_f - \Phi(t_f, t_0)x_0)$$

$$= \Phi(t_f, t_0) \Phi(t_f, t_0)^{-1}(x_f - \Phi(t_f, t_0)x_0)$$

$$= x_f - \Phi(t_f, t_0)x_0.$$

Hence

$$x(t_f) = x_f$$

q.e.d.

Definition 8.2.5 (LQP) *Let the linear quadratic optimal control problem (LQP) be defined by*

$$min \; \frac{1}{2}x(t_f)^T S x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} x(t)^T Q(t)x(t) + u(t)^T R(t)u(t) \; dt$$

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$x(t_0) = 0$$

$$T x(t_f) = 0$$

$$t_0, t_f \; \text{fixed.}$$

Here $S \in \mathbf{R}^{n,n}$, $T \in \mathbf{R}^{n,s}$, $s < n$ are fixed matrices; $A \in \mathbf{R}^{n,n}$, $B \in \mathbf{R}^{n,m}$, $Q \in \mathbf{R}^{n,n}$, $R \in \mathbf{R}^{m,m}$ are time-varying, Lipschitz continuous matrix functions of time with at most finitely many points of discontinuity. Also

$$S^T = S$$

$$Q^T = Q \quad \forall t$$

$$R^T = R \quad \forall t$$

$$R > 0 \quad \forall t$$

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Lemma 8.2.6 *Necessary conditions for a solution of LQP (see Definition 8.2.5) are given*

by

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

$$x(t_0) = 0$$

$$Tx(t_f) = 0$$

$$\lambda(t_f) = Sx(t_f) + T^T\nu$$

$$\lambda^+ - \lambda^- = 0 \text{ at any point of discontinuity of } A, B, Q, R$$

$$u(t) = -R^{-1}B^T\lambda.$$

Proof:

Hamiltonian:

$$H = \lambda^T(Ax + Bu) + \tfrac{1}{2}x^TQx + \tfrac{1}{2}u^TRu$$

adjoints:

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x}$$

$$= -\lambda^T A - x^T Q$$

controls:

$$\frac{\partial H}{\partial u} = 0 \quad \Rightarrow \quad \lambda^T B + u^T R = 0$$

$$\Rightarrow \quad Ru + B^T\lambda = 0$$

$$\Rightarrow \quad u = -R^{-1}B^T\lambda$$

state and adjoint equations together:

$$\dot{x} = Ax \quad -BR^{-1}B^T\lambda$$

$$\dot{\lambda} = -Qx \quad -A^T\lambda$$

$$\lambda(t_f) = \left[\frac{\partial}{\partial x_f} \left(\frac{1}{2} x_f^T S x_f + \nu^T T x_f \right) \right]^T$$

$$= \left[x_f^T S + \nu^T T \right]^T$$

$$= S x_f + T^T \nu$$

conditions at points of discontinuous right-hand side:

Let time t_1 be a point of discontinuous right-hand side. The first-order optimality conditions at t_1 are given by

$$\left(\mu^T \frac{\partial N}{\partial t_1} + H_1^- - H_1^+ \right) dt_1 + \left(\mu^T \frac{\partial N}{\partial x_1^-} - \lambda_1^- \right) dx_1^- + \left(\mu^T \frac{\partial N}{\partial x_1^+} - \lambda_1^+ \right) dx_1^+ = 0.$$

Here $N = N(x_1^+, x_1^-, t_1) = 0$ are conditions imposed at t_1. In our case the conditions $N = 0$ are empty. As t_1 is fixed $dt_1 = 0$. The continuity of x implies $dx_1^- = dx_1^+ =: dx_1$. Hence we get $-\lambda_1^- dx_1 + \lambda_1^+ dx_1 = 0$ and as dx_1 is free this finally implies $\lambda_1^- - \lambda_1^+ = 0$.

q.e.d.

Definition 8.2.7 ("conjugate point") *Time $t_1 \in (t_0, t_f)$ is called a conjugate point for problem LQP stated in Definition 8.2.5 if there is a non-trivial solution to the boundary value problem (BVP) implied by the stationarity conditions for LQP (this BVP is given in Lemma 8.2.6) such that*

$$x(t) \equiv 0 \quad \forall \, t \leq t_1$$

$$x(t) \neq 0 \quad \forall \, t_1 < t < t_1 + \epsilon, \text{ some } \epsilon > 0$$

(see Figure 8.1).

Lemma 8.2.8 *Let $cost_i$ be the cost for going from t_1 to t_f along path i, $i \in \{1,2\}$, i.e.*

$$cost_i = \frac{1}{2} x_f^T S x_f + \int_{path_i} \left(\frac{1}{2} x^T Q x + \frac{1}{2} u^t R u \right) dt$$

Then $cost_1 = cost_2$.

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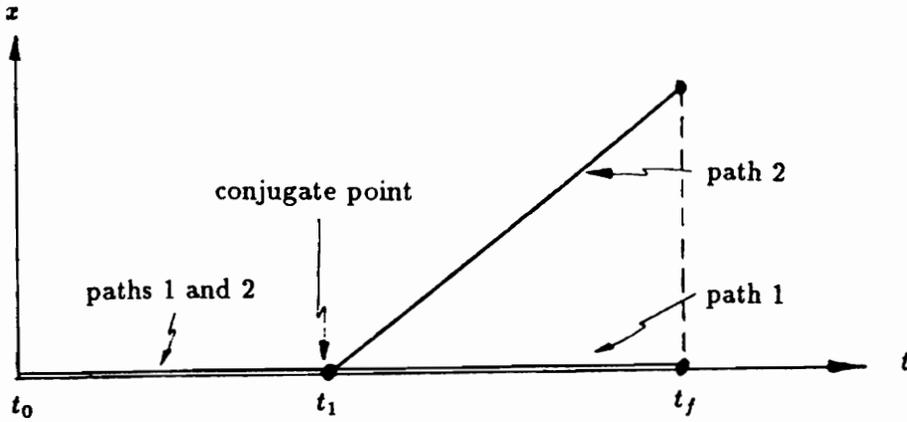
Figure 8.1: Conjugate Path

Proof:

Trivially, $cost_1 = 0$, as $x(t) \equiv 0$ and $u(t) \equiv 0$ along path 1. Along path 2 we have

$$0 = \int_{t_1}^{t_f} \lambda^T(\dot{x} - Ax + BR^{-1}B^T\lambda)\, dt$$

$$= \int_{t_1}^{t_f} \lambda^T\dot{x} + \dot{\lambda}^Tx - \dot{\lambda}^Tx - \lambda^T(Ax - BR^{-1}B^T\lambda)\, dt$$

$$= \lambda^Tx\,\big|_{t_0}^{t_f} - \int_{t_1}^{t_f} \dot{\lambda}^Tx + \lambda^T(Ax - BR^{-1}B^T\lambda)\, dt$$

$$= (Sx_f + T^T\nu)^Tx_f - \lambda_1^Tx_1 - \int_{t_1}^{t_f}(-Qx - A^T\lambda)^Tx + \lambda^T(Ax - BR^{-1}B^T\lambda)\, dt$$

$$= x_f^T Sx_f + \nu^t Tx_f + \int_{t_1}^{t_f} x^TQx + \lambda^TAx - \lambda^TAx + \lambda^T BR^{-1}B^T\lambda\, dt$$

$$= x_f Sx_f + \int_{t_1}^{t_f} x^TQx + \lambda^T BR^{-1}B^T\lambda\, dt$$

$$= x_f Sx_f + \int_{t_1}^{t_f} x^TQx + u^T Ru\, dt$$

$$= 2\, cost_2.$$

The same result holds true also if some or all matrices A, B, Q, R have at most finitely many corners or discontinuities. Let the times of discontinuity be given by $t_{c_1}, ..., t_{c_n}$. Then

$$0 = \int_{t_1}^{t_f} \lambda^T (\dot{x} - Ax + BR^{-1}B^T\lambda)\, dt$$

$$= \int_{t_1}^{t_f} \lambda^T \dot{x} + \dot{\lambda}^T x - \dot{\lambda}^T x - \lambda^T (Ax - BR^{-1}B^T\lambda)\, dt$$

$$= \lambda^T x \Big|_{t_1}^{t_{c_1}^-} + \lambda^T x \Big|_{t_{c_1}^+}^{t_{c_2}^-} + ... + \lambda^T x \Big|_{t_{c_n}^+}^{t_f} - \int_{t_1}^{t_f} \dot{\lambda}^T x + \lambda^T (Ax - BR^{-1}B^T\lambda)\, dt$$

$$= \lambda^T x \Big|_{t_1}^{t_f} - \int_{t_1}^{t_f} (-Qx - A^T\lambda)^T x + \lambda^T (Ax - BR^{-1}B^T\lambda)\, dt$$

$$= x_f{}^T S x_f + \nu^T T x_f + \int_{t_1}^{t_f} x^T Q x + \lambda^T A x - \lambda^T A x + \lambda^T B R^{-1} B^T \lambda\, dt$$

$$= x_f S x_f + \int_{t_1}^{t_f} x^T Q x + \lambda^T B R^{-1} B^T \lambda\, dt$$

$$= x_f S x_f + \int_{t_1}^{t_f} x^T Q x + u^T R u\, dt$$

$$= 2\, cost_2. \hspace{6cm} \text{q.e.d.}$$

Theorem 8.2.9 *Assume problem LQP stated in Definition 8.2.5 has a conjugate point, say, at time $t_1 \in (t_0, t_f)$. Furthermore assume that $B(t_1^-) \neq 0$ and $B(t_1^+) \neq 0$. Finally assume that there is some non-zero vector $c \in \mathbf{R}^m$ such that $R(t_1+)^{-1}B(t_1+)^T\lambda(t_1+) = c$ (i.e. $\dot{x}(t_1^+) = c$) holds along the conjugate path x, λ, u. Then there is a control $\bar{u}(t)$ which yields negative cost for problem LQP and hence the trivial solution $x^*(t) \equiv 0$, $u^*(t) \equiv 0$ (which yields cost $= 0$) is not optimal.*

Proof:

Let

path 1: $A \rightarrow B \rightarrow D$

path 2: $A \rightarrow B \rightarrow C$

path 3: $A \rightarrow E \rightarrow F \rightarrow C$

In Lemmma 8.2.8 we have seen that the costs associated with path 1 and path 2 are the same. Hence, to show that path 1 is not optimal it suffices to show that there is a control u such that

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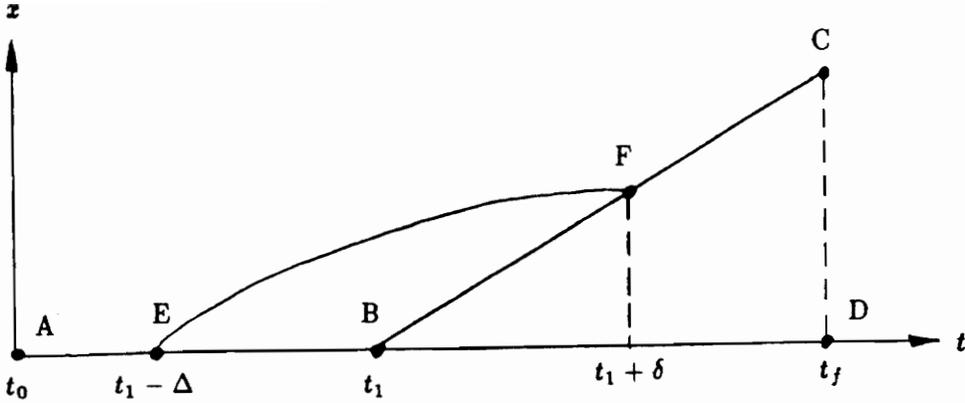
Figure 8.2: Construction of an Extremal with Negative Cost

$(**) := \int_{E \to F} x^T Q x + u^T R u \, dt < \int_{E \to B \to F} x^T Q x + u^T R u \, dt =: (*)$

as then $cost_3 < cost_2 = cost_1$.

1. case

Assume all matrix functions of time A, B, Q, R are continuous at time t_1. Then choose

δ, Δ small enough such that A, B, Q, R are continuous on $t \in [t_1 - \Delta, t_1 + \delta]$.

estimate (*):

The continuity of all matrices A, B, Q, R on $[t_1 - \Delta, t_1 + \delta]$ in conjunction with the

differential equations

$\dot{x} = Ax \quad -BR^{-1}B^T\lambda$

$\dot{\lambda} = -Qx \quad -A^T\lambda$

and the "initial conditions"

$$x(t_1^+) = 0$$

$$R^{-1}(t_1^+)B^T(t_1^+)\lambda(t_1^+) = c$$

imply that $\| u(t) \| = \mathcal{O}(1)$, $\| x(t) \| = \mathcal{O}(\delta)$ along $B \to F$ if only δ is chosen sufficiently small. Hence we have

$$\int_{B \to F} x^T Q x + u^T R u \, dt = \mathcal{O}(\delta^2) + \mathcal{O}(\delta)$$

. If necessary, reducing δ further until the $\mathcal{O}(\delta)$-term dominates the $\mathcal{O}(\delta^2)$-term, we have

$$\int_{B \to F} x^T Q x + u^T R u \, dt = \mathcal{O}(\delta)$$

and hence $(*) = \mathcal{O}(\delta)$.

estimate $(**)$:

Define $\alpha \in \mathbf{R}^n$ by $\alpha := x(t_1 + \delta)$. As shown above we have $\alpha = \mathcal{O}(\delta)$. Let $\Phi(t, t_1)$ be the transition matrix for $A(t)$, i.e.

$$\frac{d\, \Phi(t, t_1)}{dt} = A(t)\Phi(t, t_1)$$

$$\Phi(t_1, t_1) = I$$

and consider the control defined by

$$\overline{u}(t) := B(t)^T \Phi(t, t_1 - \Delta)^{-T} K^{-1} \Phi(t_1 + \delta, t_1 - \Delta)^{-1} \alpha$$

$$K := \int_{t_1 - \Delta}^{t_1 + \delta} \Phi(t, t_1 - \Delta)^{-1} B(t) B(t)^T \Phi(t, t_1 - \Delta)^{-T} \, dt.$$

Because of the controllability assumption $B(t_1^-) \neq 0$, $B(t_1^+) \neq 0$, K is non-singular and hence $\overline{u}(t)$ is well-defined and takes the state $\overline{x}$ from $\overline{x}(t_1 - \Delta) = 0$ to $\overline{x}(t_1 + \delta) = \alpha$. Furthermore, if Δ, δ are chosen sufficiently small, a lower bound for $\| K \|$ can be given by $\| K \| = \mathcal{O}(\Delta + \delta)$ Hence an upper bound for $\| K^{-1} \|$ is $\| K^{-1} \| = \mathcal{O}(\frac{1}{\Delta + \delta})$ and consequently, along $E \to F$

$$\| \overline{u}(t) \| = \mathcal{O}\left(\frac{1}{\Delta + \delta}\right) \mathcal{O}(\delta)$$

$$= \mathcal{O}\left(\frac{\delta}{\Delta + \delta}\right)$$

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and for the state $\bar{x}(t)$ the variation of parameters formula

$$\bar{x}(t) = \bar{x}(t_1 - \Delta) + \int_{t_1 - \Delta}^{t} \Phi(t, \tau)\bar{u}(\tau)\,d\tau$$

yields

$$\| \bar{x}(t) \| \; = \mathcal{O}\left(\tfrac{\delta}{\Delta+\delta}\right)\mathcal{O}(\Delta + \delta)$$

$$= \mathcal{O}(\delta).$$

Hence the cost $\int_{t_1 - \Delta}^{t_1 + \delta} x^T Q x + u^T R u \, dt$ associated with $\bar{u}(t)$ is

$(**) = \int_{t_1 - \Delta}^{t_1 + \delta} \bar{x}^T Q \bar{x} + \bar{u}^T R \bar{u} \, dt$

$= (t_1 + \delta - (t_1 - \Delta))\left[\mathcal{O}(\bar{x}^T Q \bar{x}) + \mathcal{O}(\bar{u}^T R \bar{u})\right]$

$= \mathcal{O}(\Delta + \delta)\left[\mathcal{O}(\delta^2) + \mathcal{O}\left(\tfrac{\delta^2}{(\delta+\Delta)^2}\right)\right]$

$= \mathcal{O}(\delta^3 + \Delta\delta^2) + \mathcal{O}\left(\tfrac{\delta^2}{(\Delta+\delta)}\right)$

Hence, if we first choose Δ, δ as small as necessary to perform the above estimations, then

fix Δ and further reduce δ, we find $(**) = \mathcal{O}(\delta^2)$. Hence we found that $(**) = \mathcal{O}(\delta^2)$,

while $(*) = \mathcal{O}(\delta)$.

As for Δ, δ "small enough" always $u^T R u$ is the dominating term both in $(*)$ and in $(**)$

we find $(*) > 0$ and $(**) > 0$ so that $(**) < (*)$ for Δ, δ sufficiently small and hence path

$A \to B \to C$ is not optimal.

2. case:

At least one of the matrix functions of time A, B, Q, R has a discontinuity at time t_1. Then

choose Δ, δ small enough such that A, B, Q, R are continuous on $[t_1 - \Delta, t_1) \cup (t_1, t_1 + \delta]$.

compute $(*)$:

As before (in case 1) $(*) = \mathcal{O}(\delta)$. (Note that continuity properties were required only on

$(t_1, t_1 + \delta]$ in the calculation of $(*)$ in case 1).

compute $(**)$:

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As before (in case 1) $let \alpha := x(t_2) = \mathcal{O}(\delta)$ at point F. Again let $\Phi(t, t_1)$ denote the transition matrix for matrix $A(t)$, i.e.

$$\frac{d\,\Phi(t, t_1)}{dt} = A(t)\Phi(t, t_1)$$

$$\Phi(t_1, t_1) = I$$

and consider the control as before

$$\overline{u}(t) := B(t)^T \Phi(t, t_1 - \delta)^{-T} K^{-1} \Phi(t_1 - \delta, t_1 - \Delta)^{-1} \alpha$$

$$K := \int_{t_1 - \Delta}^{t_1 + \delta} \Phi(t, t_1 - \Delta)^{-1} B(t) B(t)^T \Phi(t, t_1 - \Delta)^{-T}\, dt.$$

Here too, the controllability asumption $B(t_1^-) \neq 0$, $B(t_1^+) \neq 0$ implies that $\overline{u}(t)$ is well-defined. As before $\overline{u}(t)$ takes state x from $x(t - \Delta) = 0$ to $x(t + \delta) = \alpha$. The controllability assumption also guarantees a lower bound on $\| K \|$, namely $\| K \| = \mathcal{O}(\delta)$. Hence $\| K^{-1} \| = \mathcal{O}(\frac{1}{\Delta + \delta})$.

Hence $K = \mathcal{O}(\Delta + \delta)$ and $K^{-1} = \mathcal{O}\left(\frac{1}{\Delta + \delta}\right)$.

Now, in the same way as in case 1 it can be shown that $(*) = \mathcal{O}(\delta)$, while $(**) = \mathcal{O}\left(\frac{\delta^2}{\Delta + \delta}\right)$ Hence, by first choosing Δ "small" and then for fixed Δ varying δ it can be shown that $(**) < (*)$ and hence path 2 is not optimal.

<div align="right">q.e.d.</div>

Appendix A

Explicit Calculations to Derive the Possible Control Logics Stated in Section 5.6

A.1 Introduction

This Chapter contains the explicit calculations to determine the possible control logics in the aircraft range optimization problem treated in Chapter 5. Several singular control logics are found to be possible. In Section A.5 the supplementary optimality conditions derived in Chapter 6 are applied to these singular control cases. For convenience the finite dimensional minimization problem obtained by applying the Minimum Principle on the original optimal control problem is restated below.

A.2 Finite Dimensional Minimization Problem Implied by Minimum Principle

Cost Function

$$\text{min! } \phi\left(E(t_f), h(t_f), \gamma(t_f), x(t_f), t_f\right) \text{ with } \begin{cases} \text{a) } \phi = -x(t_f) \\ \text{or} \\ \text{b) } \phi = t_f. \end{cases} \qquad (A.1)$$

State Equations

$$\begin{aligned} \dot{E} &= \left[\delta(T(E,h) - D(E,h,n) + D_{max}(E,h)) - D_{max}(E,h)\right] \frac{v}{W} \\ \dot{h} &= v \, \sin\gamma \\ \dot{\gamma} &= \frac{g}{v}(n - \cos\gamma) \\ \dot{x} &= v \, \cos\gamma. \end{aligned} \qquad (A.2)$$

Controls

$$\delta \qquad \text{throttle}$$

$$n \qquad \text{load factor}$$

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Control Constraints

$$g_1 = \quad -\delta \quad \leq 0 \tag{A.3}$$

$$g_2 = \quad \delta - 1 \quad \leq 0 \tag{A.4}$$

$$g_3 = \quad -n - n_{max} \quad \leq 0 \tag{A.5}$$

$$g_4 = \quad n - n_{max} \quad \leq 0 \tag{A.6}$$

State Constraints

$$C_0 := v - v_{max}(h) \leq 0 \tag{A.7}$$

On arcs of active state constraint differentiation yields

$$C_1 \quad := \quad \frac{dC_0}{dt}$$
$$= \quad (\delta(T - D + D_{max}) - D_{max})\frac{g}{W} - v \sin\gamma \left(v'_{max} + \frac{g}{v}\right) = 0 \tag{A.8}$$

Hamiltonian

$$H = \quad +\lambda_E \left[\delta(T - D + D_{max}) - D_{max}\right]\frac{v}{W}$$

$$+\lambda_h v \sin\gamma$$

$$+\lambda_\gamma \frac{g}{v}(n - \cos\gamma) \tag{A.9}$$

$$+\lambda_x v \cos\gamma$$

Minimum Principle

At every instant of time the control is determined by the condition that the Hamiltonian (A.9) be minimized subject to all active control and state constraints. At each instant of time this leads to a finite-dimensional constrained or unconstrained parameter optimization problem. The following cases have to be distinguished.

A.3 State Constraint Not Active

The Kuhn-Tucker conditions associated with the finite-dimensional optimization problem

$$(\delta, n) = \arg \min_{g \leq 0} H$$

can be stated as follows

$$\text{I} \quad \frac{\partial(H + \sigma^T g)}{\partial \delta} = 0$$

$$\text{II} \quad \frac{\partial(H + \sigma^T g)}{\partial n} = 0$$

$$\text{III} \quad g_1 \leq 0 \text{ and } \sigma_1 \begin{cases} = 0 & \text{if} \quad g_1 < 0 \\[2mm] \geq 0 & \text{if} \quad g_1 = 0 \end{cases}$$

$$\text{IV} \quad g_2 \leq 0 \text{ and } \sigma_2 \begin{cases} = 0 & \text{if} \quad g_2 < 0 \\[2mm] \geq 0 & \text{if} \quad g_2 = 0 \end{cases}$$

$$\text{V} \quad g_3 \leq 0 \text{ and } \sigma_3 \begin{cases} = 0 & \text{if} \quad g_3 < 0 \\[2mm] \geq 0 & \text{if} \quad g_3 = 0 \end{cases} \qquad \text{(A.10)}$$

$$\text{VI} \quad g_4 \leq 0 \text{ and } \sigma_4 \begin{cases} = 0 & \text{if} \quad g_4 < 0 \\[2mm] \geq 0 & \text{if} \quad g_4 = 0 \end{cases}$$

$$\text{VII} \quad [\Delta\delta, \ \Delta n] \begin{bmatrix} \frac{\partial^2(H+\sigma^T g)}{\partial \delta^2} & \frac{\partial^2(H+\sigma^T g)}{\partial \delta \partial n} \\[2mm] \frac{\partial^2(H+\sigma^T g)}{\partial n \partial \delta} & \frac{\partial^2(H+\sigma^T g)}{\partial n^2} \end{bmatrix} \begin{bmatrix} \Delta\delta \\[2mm] \Delta n \end{bmatrix} \geq 0$$

for all $[\Delta\delta, \ \Delta n] \in \mathbf{R}^2$ satisfying $\dfrac{\partial h}{\partial \delta} \Delta\delta + \dfrac{\partial h}{\partial n} \Delta n = 0$

where vector function h contains exactly the active components

of the inequality constraints $g \leq 0$.

Explicitly this implies

$$\text{I} \quad \lambda_E(T - D + D_{max})\frac{v}{W} - \sigma_1 + \sigma_2 = 0$$

$$\text{II} \quad -\lambda_E\delta\frac{2KvW}{q}n + \lambda_\gamma\frac{g}{v} - \sigma_3 + \sigma_4 = 0$$

$$\text{III} \quad -\delta \leq 0 \text{ and } \sigma_1 \begin{cases} = 0 & \text{if } g_1 < 0 \\ \geq 0 & \text{if } g_1 = 0 \end{cases}$$

$$\text{IV} \quad \delta - 1 \leq 0 \text{ and } \sigma_2 \begin{cases} = 0 & \text{if } g_2 < 0 \\ \geq 0 & \text{if } g_2 = 0 \end{cases}$$

$$\text{V} \quad -n - n_{max} \leq 0 \text{ and } \sigma_3 \begin{cases} = 0 & \text{if } g_3 < 0 \\ \geq 0 & \text{if } g_3 = 0 \end{cases} \qquad (\text{A.11})$$

$$\text{VI} \quad +n - n_{max} \leq 0 \text{ and } \sigma_4 \begin{cases} = 0 & \text{if } g_4 < 0 \\ \geq 0 & \text{if } g_4 = 0 \end{cases}$$

$$\text{VII} \quad [\Delta\delta, \; \Delta n] \begin{bmatrix} 0 & -n \\ -n & -\delta \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta n \end{bmatrix} \lambda_E\frac{2KvW}{q} \geq 0$$

for all $[\Delta\delta, \; \Delta n] \in \mathbf{R}^2$ satisfying $\frac{\partial h}{\partial \delta}\Delta\delta + \frac{\partial h}{\partial n}\Delta n = 0$

where vector function h contains exactly the active components

of the inequality constraints $g \leq 0$.

To solve this problem several cases have to be distinguished.

A.3.1 $\lambda_E < 0$, $\lambda_\gamma \neq 0$

With $\lambda_E < 0$ equation (A.11-I) can be satisfied only if $\sigma_2 > 0$, i.e. constraint (A.4) is active. Hence

$$
\begin{aligned}
&\text{I} \quad && \lambda_E(T - D + D_{max})\frac{v}{W} + \sigma_2 = 0 \\[2mm]
&\text{II} \quad && -\lambda_E\frac{2KvW}{q}n + \lambda_\gamma\frac{g}{v} - \sigma_3 + \sigma_4 = 0 \\[2mm]
&\text{III} \quad && \sigma_1 = 0 \\[2mm]
&\text{IV} \quad && \delta - 1 = 0 \\[2mm]
&\text{V} \quad && -n - n_{max} \leq 0 \text{ and } \sigma_3 \begin{cases} = 0 & \text{if } g_3 < 0 \\[2mm] \geq 0 & \text{if } g_3 = 0 \end{cases} \\[4mm]
&\text{VI} \quad && +n - n_{max} \leq 0 \text{ and } \sigma_4 \begin{cases} = 0 & \text{if } g_4 < 0 \\[2mm] \geq 0 & \text{if } g_4 = 0 \end{cases} \\[4mm]
&\text{VII} \quad && [\Delta\delta,\ \Delta n]\begin{bmatrix} 0 & -n \\ -n & -\delta \end{bmatrix}\begin{bmatrix} \Delta\delta \\ \Delta n \end{bmatrix}\lambda_E\frac{2KvW}{q} \geq 0
\end{aligned}
\tag{A.12}
$$

for all $[\Delta\delta,\ \Delta n] \in \mathbf{R}^2$ satisfying $\dfrac{\partial h}{\partial \delta}\Delta\delta + \dfrac{\partial h}{\partial n}\Delta n = 0$

where vector function h contains exactly the active components

of the inequality constraints $g \leq 0$.

Define

$$
n_0 := \frac{\lambda_\gamma}{\lambda_E}\frac{gq}{2v^2KW}
\tag{A.13}
$$

(This is the solution of (A.12-II) with $\sigma_3 = \sigma_4 = 0$). Simple analysis involving equations (A.12-II), (A.12-V), (A.12-VI) shows

costraints (A.5), (A.6) are non-active   if $-n_{max} \leq n_0 \leq n_{max}$

costraint (A.5) is active   if $n_0 < -n_{max}$

costraint (A.6) is active   if $n_0 > +n_{max}$

Explicitly, we get

$$
\text{I} \quad \sigma_2 = -\lambda_E(T - D + D_{max})\frac{v}{W}
$$

$$
\text{II} \quad
\begin{cases}
n = n_0 & \text{if } -n_{max} \leq n_0 \leq +n_{max} \\[2mm]
\sigma_3 = -\lambda_E \frac{2KvW}{q}n + \lambda_\gamma \frac{g}{v} & \text{if } n_0 < -n_{max} \\[2mm]
\sigma_4 = +\lambda_E \frac{2KvW}{q}n - \lambda_\gamma \frac{g}{v} & \text{if } n_0 > +n_{max}
\end{cases}
$$

$$
\text{III} \quad \sigma_1 = 0
$$

$$
\text{IV} \quad \delta = 1
$$

$$
\text{V} \quad
\begin{cases}
\sigma_3 = 0 & \text{if } -n_{max} \leq n_0 \leq +n_{max} \\[2mm]
n = -n_{max} & \text{if } n_0 < -n_{max} \\[2mm]
\sigma_3 = 0 & \text{if } n_0 > +n_{max}
\end{cases}
$$

$$
\text{VI} \quad
\begin{cases}
\sigma_4 = 0 & \text{if } -n_{max} \leq n_0 \leq +n_{max} \\[2mm]
\sigma_4 = 0 & \text{if } n_0 < -n_{max} \\[2mm]
n = +n_{max} & \text{if } n_0 > +n_{max}
\end{cases}
$$

$$
\text{VII} \quad \text{satisfied in all 3 cases without further restrictions}
$$

(A.14)

A.3.2 $\lambda_E > 0,\ \lambda_\gamma \neq 0$

With $\lambda_E > 0$ equation (A.11-I) can be satisfied only if $\sigma_1 > 0$, i.e. constraint (A.3) is active. Hence

$$
\begin{array}{ll}
\text{I} & \lambda_E(T - D + D_{max})\dfrac{v}{W} - \sigma_1 = 0 \\[3mm]
\text{II} & -\lambda_E\delta\dfrac{2KvW}{q}n + \lambda_\gamma\dfrac{g}{v} - \sigma_3 + \sigma_4 = 0 \\[3mm]
\text{III} & \delta = 0 \\[3mm]
\text{IV} & \sigma_2 = 0 \\[3mm]
\text{V} & -n - n_{max} \leq 0 \text{ and } \sigma_3 \begin{cases} = 0 & \text{if } g_3 < 0 \\[2mm] \geq 0 & \text{if } g_3 = 0 \end{cases} \\[6mm]
\text{VI} & +n - n_{max} \leq 0 \text{ and } \sigma_4 \begin{cases} = 0 & \text{if } g_4 < 0 \\[2mm] \geq 0 & \text{if } g_4 = 0 \end{cases} \\[6mm]
\text{VII} & [\Delta\delta,\ \Delta n]\begin{bmatrix} 0 & -n \\ -n & -\delta \end{bmatrix}\begin{bmatrix} \Delta\delta \\ \Delta n \end{bmatrix}\lambda_E\dfrac{2KvW}{q} \geq 0
\end{array}
\tag{A.15}
$$

for all $[\Delta\delta,\ \Delta n] \in \mathbf{R}^2$ satisfying $\dfrac{\partial h}{\partial \delta}\Delta\delta + \dfrac{\partial h}{\partial n}\Delta n = 0$

where vector function h contains exactly the active components

of the inequality constraints $g \leq 0$.

By inserting $\delta = 0$ into equation (A.15-II) we see that constraint (A.5) is active if $\lambda_\gamma > 0$ and constraint (A.6) is active if $\lambda_\gamma < 0$. We get

$$
\begin{array}{ll}
\text{I} & \sigma_1 = +\lambda_E(T - D + D_{max})\dfrac{v}{W} \\[4mm]
\text{II} & \begin{cases} \sigma_3 = +\lambda_\gamma\dfrac{g}{v} & \text{if } \lambda_\gamma > 0 \\[3mm] \sigma_4 = -\lambda_\gamma\dfrac{g}{v} & \text{if } \lambda_\gamma < 0 \end{cases}
\end{array}
$$

III $\quad \delta = 0$

IV $\quad \sigma_2 = 0$

V $\quad \begin{cases} n = -n_{max} & \text{if } \lambda_\gamma > 0 \\[2mm] \sigma_3 = 0 & \text{if } \lambda_\gamma < 0 \end{cases}$ $\qquad\qquad$ (A.16)

VI $\quad \begin{cases} \sigma_4 = 0 & \text{if } \lambda_\gamma > 0 \\[2mm] n = +n_{max} & \text{if } \lambda_\gamma < 0 \end{cases}$

VII $\quad$ satisfied in both cases without further restrictions

A.3.3 $\quad \lambda_E \equiv 0,\ \lambda_\gamma \neq 0$

With $\lambda_E \equiv 0$ equation (A.11-I) can be satisfied only if $\sigma_1 = 0$ and $\sigma_2 = 0$, i.e. constraints (A.3), (A.4) are both non-active. Equation (A.11-II) implies that constraint (A.5) is active if $\lambda_\gamma > 0$, constraint (A.6) is active if $\lambda_\gamma < 0$. We get

I $\quad \lambda_E = 0$

II $\quad \begin{cases} \sigma_3 = +\lambda_\gamma \frac{q}{v} & \text{if } \lambda_\gamma > 0 \\[2mm] \sigma_4 = -\lambda_\gamma \frac{q}{v} & \text{if } \lambda_\gamma < 0 \end{cases}$

III $\quad \sigma_1 = 0$

IV $\quad \sigma_2 = 0$

V $\quad \begin{cases} n = -n_{max} & \text{if } \lambda_\gamma > 0 \\[2mm] \sigma_3 = 0 & \text{if } \lambda_\gamma < 0 \end{cases}$ $\qquad\qquad$ (A.17)

VI $\quad \begin{cases} \sigma_4 = 0 & \text{if } \lambda_\gamma > 0 \\[2mm] n = +n_{max} & \text{if } \lambda_\gamma < 0 \end{cases}$

VII $\quad$ in both cases trivially satisfied with strict equality

Obviously, we have not yet obtained explicit information on control δ. Control δ has to be determined from the condition that $\frac{\partial H}{\partial \delta}$ and all its derivatives are identically zero (singular control). With

$$S = \frac{\partial H}{\partial \delta} = \lambda_E \left(T - D + D_{max}\right) \frac{v}{W} \tag{A.18}$$

we have

$$S = 0 \Rightarrow \lambda_E = 0 \tag{A.19}$$

Below, further information is obtained by differentiation. Whenever control n appears explicitly in this process it is formally replaced by $\pm n_{max}$. In accordance with conditions (A.17-V), (A.17-VI) it is understood that $n = +n_{max}$ if $\lambda_\gamma < 0$ and $n = -n_{max}$ if $\lambda_\gamma > 0$. We get

$$
\begin{aligned}
\dot{S} &= \dot{\lambda}_E \left(T - D + D_{max}\right) \frac{v}{W} \\
&= \left[-\lambda_h \frac{g}{v} \sin\gamma + \lambda_\gamma \frac{g^2}{v^3} (\pm n_{max} - \cos\gamma) - \lambda_x \frac{g}{v} \cos\gamma\right] \left(T - D + D_{max}\right) \frac{v}{W} \\
&= \left[-\lambda_h \sin\gamma + \lambda_\gamma \frac{g}{v^2} (\pm n_{max} - \cos\gamma) - \lambda_x \cos\gamma\right] \left(T - D + D_{max}\right) \frac{g}{W}
\end{aligned}
$$

As $\left(T - D + D_{max}\right) \frac{g}{W} \neq 0$ we have

$$\dot{S} = 0 \Rightarrow -\lambda_h \sin\gamma + \lambda_\gamma \frac{g}{v^2} (\pm n_{max} - \cos\gamma) - \lambda_x \cos\gamma = 0 \tag{A.20}$$

Further differentiation yields (using conditions (A.19), (A.20))

$$
\begin{aligned}
\ddot{S} &= \left[-\dot{\lambda}_h \sin\gamma - \lambda_h \cos\gamma \dot{\gamma} + \dot{\lambda}_\gamma \frac{g}{v^2}(\pm n_{max} - \cos\gamma)+ \right. \\
&\quad \left. + \lambda_\gamma \left(-\frac{2g^2}{v^4}(\pm n_{max} - \cos\gamma)(\dot{E} - \dot{h}) + \frac{g}{v^2}\sin\gamma \dot{\gamma}\right) \right. \\
&\quad \left. + \lambda_x \sin\gamma \dot{\gamma}\right] \left(T - D + D_{max}\right) \frac{g}{W}
\end{aligned}
$$

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Here we know explicitly

$$
\begin{aligned}
\dot{\lambda}_h &= -\frac{\partial H}{\partial h} \\
&= \underbrace{-\lambda_E[...]}_{=0} + \lambda_h \frac{g}{v} \sin\gamma - \lambda_\gamma \frac{g}{v^2}\frac{g}{v}(\pm n_{max} - \cos\gamma) + \lambda_x \frac{g}{v}\cos\gamma \\
&= -\frac{g}{v}\underbrace{\left[-\lambda_h \sin\gamma + \lambda_\gamma \frac{g}{v^2}(\pm n_{max} - \cos\gamma) - \lambda_x \cos\gamma\right]}_{=0\ (from\ \dot{S}=0)}
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\lambda}_\gamma &= -\frac{\partial H}{\partial \gamma} \\
&= \underbrace{-\lambda_E[...]}_{=0} - \lambda_h v \cos\gamma - \lambda_\gamma \frac{g}{v}\sin\gamma + \lambda_x v \sin\gamma
\end{aligned}
$$

Using this and

$$
\dot{\gamma} = \frac{g}{v}(\pm n_{max} - \cos\gamma)
$$

yields

$$
\begin{aligned}
\ddot{S} &= \left[-\lambda_h \cos\gamma + \left(-\lambda_h \cos\gamma - \lambda_\gamma \frac{g}{v^2}\sin\gamma + \lambda_x \sin\gamma\right) + \right. \\
&\quad \left. -\lambda_\gamma \frac{2g}{v^3}(\dot{E} - \dot{h}) + \lambda_\gamma \frac{g}{v^2}\sin\gamma + \lambda_x \sin\gamma\right](T - D + D_{max})\frac{g}{W}\dot{\gamma} \\
&= \left[-\lambda_h \cos\gamma + \lambda_x \sin\gamma - \lambda_\gamma \frac{g}{v^3}(\dot{E} - \dot{h})\right] 2(T - D + Dmax)\frac{g}{W}\dot{\gamma}
\end{aligned}
$$

For all practical purposes $n_{max} > 1$ so that $\dot{\gamma} \neq 0$. As has been stated earlier also the factor $(T - D + D_{max})\frac{g}{W} > 0$. Hence we find

$$
\ddot{S} = 0 \Rightarrow -\lambda_h \cos\gamma + \lambda_x \sin\gamma - \lambda_\gamma \frac{g}{v^3}(\dot{E} - \dot{h}) = 0
$$

Using

$$
\dot{h} = v \sin\gamma
$$

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and

$$\dot{E} = [\delta(T - D(n = \pm n_{max}) + D_{max}) - D_{max}]\frac{v}{W}$$

$$= [\delta T - D_{max}]\frac{v}{W}$$

finally yields

$$\ddot{S} = 0 \Rightarrow \delta = \frac{D_{max} + W\left(\sin\gamma + \frac{v^2}{\lambda_\gamma g}(-\lambda_h\cos\gamma + \lambda_x\sin\gamma)\right)}{T} \tag{A.21}$$

A.3.4 $\quad \lambda_E \equiv 0,\ \lambda_\gamma \equiv 0$

Differentiation leads to

$$\left.\begin{array}{l} \dot{\lambda}_E \equiv 0 \\[2ex] \dot{\lambda}_\gamma \equiv 0 \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} -\lambda_h\frac{g}{v}\sin\gamma - \lambda_x\frac{g}{v}\cos\gamma \equiv 0 \\[2ex] -\lambda_h v\cos\gamma + \lambda_x v\sin\gamma \equiv 0 \end{array}\right\} \Rightarrow$$

$$\begin{pmatrix} -\sin\gamma & -\cos\gamma \\[2ex] -\cos\gamma & +\sin\gamma \end{pmatrix} \begin{pmatrix} \lambda_h \\[2ex] \lambda_x \end{pmatrix} \equiv \begin{pmatrix} 0 \\[2ex] 0 \end{pmatrix} \Rightarrow$$

$$\lambda_h \equiv 0, \quad \lambda_x \equiv 0$$

Together with the initial assumption $\lambda_E \equiv \lambda_\gamma \equiv 0$ we get

$$\lambda_E \equiv \lambda_h \equiv \lambda_\gamma \equiv \lambda_x \equiv 0$$

Hence case A.3.4 can be excluded.

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A.3.5 $\lambda_E < 0$, $\lambda_\gamma = 0$

With $\lambda_E < 0$ equation (A.11-I) can be satisfied only if constraint (A.4) is active. Equation (A.11-II) implies that both constraints (A.5) and (A.6) have to be non-active. We get

$$\text{I} \qquad \lambda_E(T - D + D_{max})\frac{v}{W} + \sigma_2 = 0$$

$$\text{II} \qquad -\lambda_E\delta\frac{2KvW}{q}n = 0$$

$$\text{III} \qquad \sigma_1 = 0$$

$$\text{IV} \qquad \delta = 1$$

$$\text{V} \qquad \sigma_3 = 0 \qquad\qquad\qquad\qquad\qquad \text{(A.22)}$$

$$\text{VI} \qquad \sigma_4 = 0$$

$$\text{VII} \qquad \text{satisfied without implying further restrictions}$$

Hence

$$\text{I} \qquad \sigma_2 = -\lambda_E(T - D + D_{max})\frac{v}{W}$$

$$\text{II} \qquad n = 0$$

$$\text{III} \qquad \sigma_1 = 0$$

$$\text{IV} \qquad \delta = 1$$

$$\text{V} \qquad \sigma_3 = 0 \qquad\qquad\qquad\qquad\qquad \text{(A.23)}$$

$$\text{VI} \qquad \sigma_4 = 0$$

$$\text{VII} \qquad \text{satisfied without implying further restrictions}$$

This is no singular control. Obviously, case A.3.5 is just an extension of case A.3.1 from $\lambda_E < 0$, $\lambda_\gamma \neq 0$ to $\lambda_E < 0$ (and no restrictions on λ_γ).

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A.3.6 $\lambda_E > 0$, $\lambda_\gamma \equiv 0$

With $\lambda_E > 0$ equation (A.11-I) can be satisfied only if constraint (A.3) is active. Using $\lambda_\gamma = 0$ and $\delta = 0$, equation (A.11-II) implies that constraints (A.5) and (A.6) must be both non-active. We get

$$
\begin{aligned}
\text{I} \quad & \lambda_E(T - D + D_{max})\frac{v}{W} - \sigma_1 = 0 \\[2mm]
\text{II} \quad & \lambda_\gamma \frac{g}{v} = 0 \\[2mm]
\text{III} \quad & \delta = 0 \\[2mm]
\text{IV} \quad & \sigma_2 = 0 \\[2mm]
\text{V} \quad & \sigma_3 = 0 \\[2mm]
\text{VI} \quad & \sigma_4 = 0 \\[2mm]
\text{VII} \quad & [\Delta\delta, \ \Delta n]
\begin{bmatrix} 0 & -n \\ -n & -\delta \end{bmatrix}
\begin{bmatrix} \Delta\delta \\ \Delta n \end{bmatrix}
\lambda_E \frac{2KvW}{q} \geq 0 \\
& \text{for all } [\Delta\delta, \ \Delta n] \in \mathbf{R}^2 \text{ satisfying } \Delta\delta = 0
\end{aligned}
\tag{A.24}
$$

Hence

$$
\begin{aligned}
\text{I} \quad & \sigma_1 = \lambda_E(T - D + D_{max})\frac{v}{W} \\[2mm]
\text{II} \quad & \lambda_\gamma = 0 \\[2mm]
\text{III} \quad & \delta = 0 \\[2mm]
\text{IV} \quad & \sigma_2 = 0 \\[2mm]
\text{V} \quad & \sigma_3 = 0 \\[2mm]
\text{VI} \quad & \sigma_4 = 0 \\[2mm]
\text{VII} \quad & \text{trivially satisfied with strict equality}
\end{aligned}
\tag{A.25}
$$

Obviously, we do not yet get any explicit information on control n. This information has to be obtained by differentiation of identity (A.25-II). We get

$$\dot{\lambda}_\gamma \equiv 0 \Rightarrow -\lambda_h v \cos \gamma + \lambda_x v \sin \gamma \equiv 0 \qquad (A.26)$$

Before differentiating further, several cases have to be distinguished:

Case 1: $\lambda_E > 0$, $\lambda_\gamma \equiv 0$, $\lambda_x \equiv 0$, $\lambda_h \equiv 0$

Then (A.26) is satisfied. From $\lambda_h \equiv 0$ we get by differentiation

$$\dot{\lambda}_h = +\lambda_E \frac{\partial \left(D_{max} \frac{v}{W} \right)}{\partial h} = 0$$

For the present aircraft model $\frac{\partial \left(D_{max} \frac{v}{W} \right)}{\partial h} > 0$ so that case 1 can be excluded.

Case 2: $\lambda_E > 0$, $\lambda_\gamma \equiv 0$, $\lambda_x \equiv 0$, $\lambda_h \neq 0$

Then (A.26) yields

$$\cos \gamma \equiv 0$$

Differentiation gives (using $\sin \gamma \neq 0$, as $\cos \gamma \equiv 0$)

$$\dot{\gamma} \equiv 0$$

With $\cos \gamma \equiv 0$ this implies

$$n \equiv 0$$

Comprehension case 2:

$$\lambda_E > 0, \ \lambda_\gamma \equiv 0, \ \lambda_x \equiv 0, \ \lambda_h \neq 0$$

$$\delta = 0$$

$S = \lambda_\gamma$ takes over the role of a switching function with

$$\lambda_\gamma = 0 \qquad (S = 0)$$

$$\cos \gamma = 0 \quad (\dot{S} = 0)$$

$$n = 0 \qquad (\ddot{S} = 0)$$

Case 3: $\lambda_E > 0$, $\lambda_\gamma \equiv 0$, $\lambda_x \neq 0$

Then necessarily $\cos \gamma \neq 0$. Otherwise (A.26) can never be satisfied. With $\cos \gamma \neq 0$ (A.26) implies

$$\lambda_h \equiv \lambda_x \tan \gamma.$$

Differentiation gives (using (A.25-II), (A.25-III) and the result above)

$$
\begin{aligned}
0 &\equiv \frac{d}{dt}(\lambda_h - \lambda_x \tan \gamma) \\
&\equiv \dot{\lambda}_h - \lambda_x \frac{\dot{\gamma}}{\cos^2 \gamma} \\
&\equiv \lambda_E \frac{\partial \left(D_{max}\frac{v}{W}\right)}{\partial h} + \lambda_h \frac{g}{v}\sin \gamma + \lambda_x \frac{g}{v}\cos \gamma - \lambda_x \frac{\frac{g}{v}(n - \cos \gamma)}{\cos^2 \gamma} \\
&\equiv \lambda_E \frac{\partial \left(D_{max}\frac{v}{W}\right)}{\partial h} + \lambda_x \tan \gamma \frac{g}{v}\sin \gamma + \lambda_x \frac{g}{v}\cos \gamma - \lambda_x \frac{\frac{g}{v}(n - \cos \gamma)}{\cos^2 \gamma} \\
&\equiv \lambda_E \frac{\partial \left(D_{max}\frac{v}{W}\right)}{\partial h} + \lambda_x \frac{g}{v}\frac{1}{\cos \gamma} - \lambda_x \frac{\frac{g}{v}(n - \cos \gamma)}{\cos^2 \gamma}
\end{aligned}
$$

Solving for load factor n yields

$$n \equiv \frac{\lambda_E}{\lambda_x}\frac{v}{g}\cos^2 \gamma \frac{\partial \left(D_{max}\frac{v}{W}\right)}{\partial h} + 2 \cos \gamma$$

Comprehension case 3:

$$\lambda_E > 0, \ \lambda_\gamma \equiv 0, \ \lambda_x \neq 0$$

$$\delta = 0$$

$$\lambda_\gamma = 0$$

$$\lambda_h \;=\; \lambda_x \tan \gamma$$

$$n \;=\; \frac{\lambda_E}{\lambda_x} \frac{v}{g} \cos^2 \gamma \frac{\partial \left(D_{max} \frac{v}{W} \right)}{\partial h} + 2 \cos \gamma.$$

A.4 State Constraint Active

The Kuhn-Tucker conditions associated with the finite-dimensional optimization problem

$$(\delta, n) \;=\; \arg \min_{C_1 = 0,\, g \le 0} H$$

can be stated as follows

$$\text{I} \qquad \frac{\partial (H + \mu C_1 + \sigma^T g)}{\partial \delta} = 0$$

$$\text{II} \qquad \frac{\partial (H + \mu C_1 + \sigma^T g)}{\partial n} = 0$$

$$\text{III} \qquad C_1 = 0$$

$$\text{IV} \qquad g_1 \le 0 \text{ and } \sigma_1 \begin{cases} = 0 & \text{if } g_1 < 0 \\[2ex] \ge 0 & \text{if } g_1 = 0 \end{cases}$$

$$\text{V} \qquad g_2 \le 0 \text{ and } \sigma_2 \begin{cases} = 0 & \text{if } g_2 < 0 \\[2ex] \ge 0 & \text{if } g_2 = 0 \end{cases}$$

$$\text{VI} \qquad g_3 \le 0 \text{ and } \sigma_3 \begin{cases} = 0 & \text{if } g_3 < 0 \\[2ex] \ge 0 & \text{if } g_3 = 0 \end{cases} \qquad\qquad \text{(A.27)}$$

$$\text{VII} \qquad g_4 \le 0 \text{ and } \sigma_4 \begin{cases} = 0 & \text{if } g_4 < 0 \\[2ex] \ge 0 & \text{if } g_4 = 0 \end{cases}$$

$$\text{VIII} \qquad [\Delta\delta, \; \Delta n] \begin{bmatrix} \frac{\partial^2 (H + \mu C_1 + \sigma^T g)}{\partial \delta^2} & \frac{\partial^2 (H + \mu C_1 + \sigma^T g)}{\partial \delta \partial n} \\[2ex] \frac{\partial^2 (H + \mu C_1 + \sigma^T g)}{\partial n \partial \delta} & \frac{\partial^2 (H + \mu C_1 + \sigma^T g)}{\partial n^2} \end{bmatrix} \begin{bmatrix} \Delta\delta \\[2ex] \Delta n \end{bmatrix} \ge 0$$

for all $[\Delta\delta, \; \Delta n] \in \mathbf{R}^2$ satisfying $\dfrac{\partial h}{\partial \delta}\Delta\delta + \dfrac{\partial h}{\partial n}\Delta n = 0$

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where vector function h consists exactly of equality constraint $C_1 = 0$

and the active components of the inequality constraints $g \leq 0$.

With conditions $g \leq 0$, $C_1 = 0$, and Hamiltonian H given by equations (A.3), (A.4), (A.5), (A.6), (A.8), and (A.9) this implies explicitely

$$
\text{I} \quad \left(\lambda_E + \mu \frac{g}{v}\right)(T - D + D_{max})\frac{v}{W} - \sigma_1 + \sigma_2 = 0
$$

$$
\text{II} \quad -\delta\left(\lambda_E + \mu\frac{g}{v}\right)\frac{2KvW}{q}n + \lambda_\gamma\frac{g}{v} - \sigma_3 + \sigma_4 = 0
$$

$$
\text{III} \quad (\delta(T - D + D_{max}) - D_{max})\frac{g}{W} - v\sin\gamma\left(v'_{max} + \frac{g}{v}\right) = 0
$$

$$
\text{IV} \quad -\delta \leq 0 \text{ and } \sigma_1 \begin{cases} = 0 & \text{if } g_1 < 0 \\ \geq 0 & \text{if } g_1 = 0 \end{cases}
$$

$$
\text{V} \quad \delta - 1 \leq 0 \text{ and } \sigma_2 \begin{cases} = 0 & \text{if } g_2 < 0 \\ \geq 0 & \text{if } g_2 = 0 \end{cases}
$$

$$
\text{VI} \quad -n - n_{max} \leq 0 \text{ and } \sigma_3 \begin{cases} = 0 & \text{if } g_3 < 0 \\ \geq 0 & \text{if } g_3 = 0 \end{cases}
$$

(A.28)

$$
\text{VII} \quad +n - n_{max} \leq 0 \text{ and } \sigma_4 \begin{cases} = 0 & \text{if } g_4 < 0 \\ \geq 0 & \text{if } g_4 = 0 \end{cases}
$$

$$
\text{VIII} \quad [\Delta\delta, \Delta n] \begin{bmatrix} 0 & -n \\ -n & -\delta \end{bmatrix} \begin{bmatrix} \Delta\delta \\ \Delta n \end{bmatrix} \left(\lambda_E + \mu\frac{g}{v}\right)\frac{2KvW}{q} \geq 0
$$

for all $[\Delta\delta, \Delta n] \in \mathbf{R}^2$ satisfying $\frac{\partial h}{\partial \delta}\Delta\delta + \frac{\partial h}{\partial n}\Delta n = 0$

where vector function h consists exactly of equality constraint $C_1 = 0$

and the active components of the inequality constraints $g \leq 0$.

To solve this problem several cases have to be distinguished.

A.4.1 $\lambda_\gamma \neq 0$, $\left(\lambda_E + \mu \frac{g}{v}\right) \neq 0$

In this case constraint (A.3) or (A.4) has to be active, otherwise condition (A.28-I) would imply $\left(\lambda_E + \mu \frac{g}{v}\right) = 0$. Furthermore, with constraints (A.3), (A.8) active or constraints (A.4), (A.8) active it is clear that constraints (A.5), (A.6) have to be non-active because otherwise the two controls were overdetermined. Hence $\sigma_3 = \sigma_4 = 0$. In equation (A.28-II) this implies $\delta \neq 0$. Hence the active constraints are exactly constraints (A.4) and (A.8). We get

$$
\begin{array}{ll}
\text{I} & \left(\lambda_E + \mu\frac{g}{v}\right)(T - D + D_{max})\frac{v}{W} + \sigma_2 = 0 \\[2mm]
\text{II} & -\delta\left(\lambda_E + \mu\frac{g}{v}\right)\frac{2KvW}{q}n + \lambda_\gamma\frac{g}{v} = 0 \\[2mm]
\text{III} & (\delta(T - D + D_{max}) - D_{max})\frac{g}{W} - v\sin\gamma\left(v'_{max} + \frac{g}{v}\right) = 0 \\[2mm]
\text{IV} & \sigma_1 = 0 \\[2mm]
\text{V} & \delta - 1 = 0 \\[2mm]
\text{VI} & \sigma_3 = 0 \\[2mm]
\text{VII} & \sigma_4 = 0 \\[2mm]
\text{VIII} & [\Delta\delta, \ \Delta n] \begin{bmatrix} 0 & -n \\[2mm] -n & -\delta \end{bmatrix} \begin{bmatrix} \Delta\delta \\[2mm] \Delta n \end{bmatrix} \left(\lambda_E + \mu\frac{g}{v}\right)\frac{2KvW}{q} \geq 0 \\[2mm]
& \text{for all } [\Delta\delta, \ \Delta n] \in \mathbf{R}^2 \text{ satisfying} \\[2mm]
& \dfrac{\partial C_1}{\partial\delta}\Delta\delta + \dfrac{\partial C_1}{\partial n}\Delta n = 0 \\[2mm]
& \dfrac{\partial g_2}{\partial\delta}\Delta\delta + \dfrac{\partial g_2}{\partial n}\Delta n = 0
\end{array}
\tag{A.29}
$$

Hence

$$
\text{I} \qquad \sigma_2 = -\left(\lambda_E + \mu\frac{g}{v}\right)(T - D + D_{max})\frac{v}{W}
$$

$$\text{II} \qquad \left(\lambda_E + \mu\frac{g}{v}\right) = \frac{\lambda_\gamma}{n}\frac{gq}{2Kv^2W}$$

$$\text{III} \qquad |n| = \frac{q}{W}\sqrt{\frac{T - \frac{vW}{g}\sin\gamma\,(v'_{max} + \frac{g}{v}) - qC_{D0}}{qK}}$$

$$\text{IV} \qquad \sigma_1 = 0$$

$$\text{V} \qquad \delta = 1$$

$$\text{VI} \qquad \sigma_3 = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(A.30)}$$

$$\text{VII} \qquad \sigma_4 = 0$$

$$\text{VIII} \qquad [\Delta\delta,\ \Delta n]\begin{bmatrix} 0 & -n \\ -n & -\delta \end{bmatrix}\begin{bmatrix} \Delta\delta \\ \Delta n \end{bmatrix}\left(\lambda_E + \mu\frac{g}{v}\right)\frac{2KvW}{q} \geq 0$$

for all $[\Delta\delta,\ \Delta n] \in \mathbf{R}^2$ satisfying

$$\Delta\delta = 0$$

$$\Delta n = 0$$

With $\sigma_2 > 0$ equation (A.30-I) implies $\left(\lambda_E + \mu\frac{g}{v}\right) < 0$. In equation (A.30-II) this yields the sign condition

$$n \begin{cases} > 0 & \text{if } \lambda_\gamma < 0 \\[2ex] < 0 & \text{if } \lambda_\gamma > 0 \end{cases}$$

Explicitly, we get

$$\text{I} \qquad \sigma_2 = -\left(\lambda_E + \mu\frac{g}{v}\right)(T - D + D_{max})\frac{v}{W}$$

$$\text{II} \qquad \mu = -\lambda_E\frac{v}{g} + \frac{\lambda_\gamma}{n}\frac{g}{2KvW}$$

$$\text{III} \qquad n = \begin{cases} +\frac{q}{W}\sqrt{\dfrac{T - \frac{vW}{g}\sin\gamma\left(v'_{max} + \frac{g}{v}\right) - qC_{D0}}{qK}} & \text{if } \lambda_\gamma < 0 \\[4ex] -\frac{q}{W}\sqrt{\dfrac{T - \frac{vW}{g}\sin\gamma\left(v'_{max} + \frac{g}{v}\right) - qC_{D0}}{qK}} & \text{if } \lambda_\gamma > 0 \end{cases}$$

$$\text{IV} \qquad \sigma_1 = 0$$

$$\text{V} \qquad \delta = 1$$

$$\text{VI} \qquad \sigma_3 = 0 \tag{A.31}$$

$$\text{VII} \qquad \sigma_4 = 0$$

VIII   satisfied without implying further restrictions

Note that (A.31-I) in conjunction with condition $\sigma_2 > 0$ implies $\lambda_E + \mu\frac{g}{v} < 0$, i.e. the case $\lambda_\gamma \neq 0$, $(\lambda_E + \mu\frac{g}{v}) > 0$ can be excluded.

A.4.2 $\lambda_\gamma \neq 0$, $\left(\lambda_E + \mu\frac{g}{v}\right) = 0$

Here condition (A.28-I) immediately implies $\sigma_1 = \sigma_2 = 0$, i.e. constraints (A.3), (A.4) are both non-active. Condition (A.28-II) implies that constraint (A.5) is active if $\lambda_\gamma > 0$, constraint (A.6) is active if $\lambda_\gamma < 0$. We get

$$\text{I} \qquad \left(\lambda_E + \mu\frac{g}{v}\right) = 0$$

$$\text{II} \qquad \begin{cases} \sigma_3 = +\lambda_\gamma\frac{g}{v} & \text{if } \lambda_\gamma > 0 \\[2mm] \sigma_4 = -\lambda_\gamma\frac{g}{v} & \text{if } \lambda_\gamma < 0 \end{cases}$$

$$\text{III} \qquad (\delta(T - D + D_{max}) - D_{max})\frac{g}{W} - v\sin\gamma\left(v'_{max} + \frac{g}{v}\right) = 0$$

$$\text{IV} \qquad \sigma_1 = 0$$

$$\text{V} \qquad \sigma_2 = 0$$

$$\text{VI} \qquad \begin{cases} n = -n_{max} & \text{if } \lambda_\gamma > 0 \\[2mm] \sigma_3 = 0 & \text{if } \lambda_\gamma < 0 \end{cases} \tag{A.32}$$

$$\text{VII} \qquad \begin{cases} \sigma_4 = 0 & \text{if } \lambda_\gamma > 0 \\[2mm] n = +n_{max} & \text{if } \lambda_\gamma < 0 \end{cases}$$

$$\text{VIII} \qquad [\Delta\delta, \Delta n]\begin{bmatrix} 0 & -n \\ -n & -\delta \end{bmatrix}\begin{bmatrix} \Delta\delta \\ \Delta n \end{bmatrix}\left(\lambda_E + \mu\frac{g}{v}\right)\frac{2KvW}{q} \geq 0$$

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for all $[\Delta\delta, \Delta n] \in \mathbf{R}^2$ satisfying $\dfrac{\partial h}{\partial \delta}\Delta\delta + \dfrac{\partial h}{\partial n}\Delta n = 0$

where vector function h consists exactly of equality constraint $C_1 = 0$

and the active components of the inequality constraints $g \leq 0$.

Hence

$$
\begin{aligned}
&\text{I} && \left(\lambda_E + \mu\frac{g}{v}\right) = 0 \\[2ex]
&\text{II} && \begin{cases} \sigma_3 = +\lambda_\gamma\frac{g}{v} & \text{if } \lambda_\gamma > 0 \\[1ex] \sigma_4 = -\lambda_\gamma\frac{g}{v} & \text{if } \lambda_\gamma < 0 \end{cases} \\[2ex]
&\text{III} && \delta = \frac{D_{max} + \frac{W}{g}v\sin\gamma\,(v'_{max} + \frac{g}{v})}{T - D + D_{max}} \\[2ex]
&\text{IV} && \sigma_1 = 0 \\[2ex]
&\text{V} && \sigma_2 = 0 \\[2ex]
&\text{VI} && \begin{cases} n = -n_{max} & \text{if } \lambda_\gamma > 0 \\[1ex] \sigma_3 = 0 & \text{if } \lambda_\gamma < 0 \end{cases} \\[2ex]
&\text{VII} && \begin{cases} \sigma_4 = 0 & \text{if } \lambda_\gamma > 0 \\[1ex] n = +n_{max} & \text{if } \lambda_\gamma < 0 \end{cases} \\[2ex]
&\text{VIII} && \text{satisfied with strict equality}
\end{aligned}
\tag{A.33}
$$

A.4.3 $\lambda_\gamma \equiv 0$, $\lambda_E + \mu\frac{g}{v} \equiv 0$

Equations (A.28-I), (A.28-II) now immediately imply $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = 0$, i.e. constraints (A.3), (A.4), (A.5), (A.6) are all non-active. We get

$$
\begin{aligned}
&\text{I} && \left(\lambda_E + \mu\frac{g}{v}\right)(T - D + D_{max})\frac{v}{W} = 0 \\[2ex]
&\text{II} && \lambda_\gamma\frac{g}{v} = 0
\end{aligned}
$$

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III $\quad (\delta(T - D + D_{max}) - D_{max}) \dfrac{g}{W} - v \sin \gamma \left(v'_{max} + \dfrac{g}{v} \right) = 0$

IV $\quad \sigma_1 = 0$

V $\quad \sigma_2 = 0$

VI $\quad \sigma_3 = 0$ $\hspace{6cm}$ (A.34)

VII $\quad \sigma_4 = 0$

VIII $\quad [\Delta \delta, \ \Delta n] \begin{bmatrix} 0 & -n \\ -n & -\delta \end{bmatrix} \begin{bmatrix} \Delta \delta \\ \Delta n \end{bmatrix} \left(\lambda_E + \mu \dfrac{g}{v} \right) \dfrac{2 K v W}{q} \geq 0$

$\quad\quad$ for all $[\Delta \delta, \ \Delta n] \in \mathbf{R}^2$ satisfying $\dfrac{\partial C_1}{\partial \delta} \Delta \delta + \dfrac{\partial C_1}{\partial n} \Delta n = 0.$

This implies

I $\quad \mu = -\lambda_E \dfrac{v}{g}$

II $\quad \lambda_\gamma = 0$

III $\quad \delta = \dfrac{D_{max} + \frac{W}{g} v \sin \gamma \left(v'_{max} + \frac{g}{v} \right)}{T - D + D_{max}}$

IV $\quad \sigma_1 = 0$

V $\quad \sigma_2 = 0$

VI $\quad \sigma_3 = 0$ $\hspace{6cm}$ (A.35)

VII $\quad \sigma_4 = 0$

VIII $\quad$ satisfied with strict equality

Obviously, we do not yet have an expression for control n. This information has to be obtained by differentiation of the "switching function" $S = \lambda_\gamma$. Before proceeding with differentiation we state all adjoint differential equations.

$$\dot{\lambda}_E \ = \ -\dfrac{\partial (H + \mu C_1)}{\partial E}$$

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$$= -\lambda_E \left[\left(\delta \left(\frac{\partial T}{\partial E} - \frac{\partial D}{\partial E} + \frac{\partial D_{max}}{\partial E} \right) - \frac{\partial D_{max}}{\partial E} \right) \frac{v}{W} + (\delta(T - D + D_{max}) - D_{max}) \frac{g}{vW} \right]$$

$$-\lambda_h \frac{g}{v} \sin \gamma$$

$$-\lambda_x \frac{g}{v} \cos \gamma \qquad \qquad (A.36)$$

$$-\mu \left[\left(\delta \left(\frac{\partial T}{\partial E} - \frac{\partial D}{\partial E} + \frac{\partial D_{max}}{\partial E} \right) - \frac{\partial D_{max}}{\partial E} \right) \frac{g}{W} - \frac{g}{v} v'_{max} \sin \gamma \right]$$

Condition (A.35-II) $\lambda_\gamma = 0$ is used here already. If we insert (A.35-I) $\mu = -\lambda_E \frac{v}{g}$ then the

terms

$$-\lambda_E \left(\delta \left(\frac{\partial T}{\partial E} - \frac{\partial D}{\partial E} + \frac{\partial D_{max}}{\partial E} \right) - \frac{\partial D_{max}}{\partial E} \right) \frac{v}{W}$$

$$-\mu \left(\delta \left(\frac{\partial T}{\partial E} - \frac{\partial D}{\partial E} + \frac{\partial D_{max}}{\partial E} \right) - \frac{\partial D_{max}}{\partial E} \right) \frac{g}{W}$$

cancel out. Furthermore, by use of (A.34-III) (which is equivalent to (A.35-III)) we can

replace

$(\delta(T - D + D_{max}) - D_{max}) \frac{g}{vW}$ by $\sin \gamma \, (v'_{max} + \frac{g}{v})$.

Then we get from (A.36)

$$\dot{\lambda}_E = -\lambda_E \left(v'_{max} + \frac{g}{v} \right) \sin \gamma$$

$$-\lambda_h \frac{g}{v} \sin \gamma$$

$$-\lambda_x \frac{g}{v} \cos \gamma$$

$$-\lambda_E v'_{max} \sin \gamma$$

and hence

$$\dot{\lambda}_E = -\lambda_E \left(2v'_{max} + \frac{g}{v} \right) \sin \gamma$$

$$-\lambda_h \frac{g}{v} \sin \gamma \qquad \qquad (A.37)$$

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$$-\lambda_x \frac{g}{v} \cos \gamma$$

Similarly

$$\dot{\lambda}_h = -\frac{\partial(H + \mu C_1)}{\partial h}$$

$$= -\lambda_E \left[\left(\delta\left(\frac{\partial T}{\partial h} - \frac{\partial D}{\partial h} + \frac{\partial D_{max}}{\partial h}\right) - \frac{\partial D_{max}}{\partial h}\right)\frac{v}{W} - (\delta(T - D + D_{max}) - D_{max})\frac{g}{vW}\right]$$

$$+\lambda_h \frac{g}{v} \sin \gamma$$

$$+\lambda_x \frac{g}{v} \cos \gamma$$

$$-\mu\left[\left(\delta\left(\frac{\partial T}{\partial h} - \frac{\partial D}{\partial h} + \frac{\partial D_{max}}{\partial h}\right) - \frac{\partial D_{max}}{\partial h}\right)\frac{g}{W} - \left(v''_{max}v - v'_{max}\frac{g}{v}\right)\sin \gamma\right]$$

$$= \lambda_E \left(v'_{max} + \frac{g}{v}\right)\sin \gamma$$

$$+\lambda_h \frac{g}{v} \sin \gamma$$

$$+\lambda_x \frac{g}{v} \cos \gamma$$

$$+\lambda_E \left(v'_{max} - \frac{v''_{max}v^2}{g}\right)\sin \gamma$$

$$= \lambda_E \left(2v'_{max} + \frac{g}{v} - \frac{v''_{max}v^2}{g}\right)\sin \gamma$$

$$+\lambda_h \frac{g}{v} \sin \gamma \tag{A.38}$$

$$+\lambda_x \frac{g}{v} \cos \gamma$$

and

$$\dot{\lambda}_\gamma = -\frac{\partial(H + \mu C_1)}{\partial \gamma}$$

$$= -\lambda_h v \cos\gamma$$

$$+\lambda_x v \sin\gamma$$

$$+\mu \left(v'_{max} + \frac{g}{v} \right) v \cos\gamma$$

$$= -\lambda_E \frac{v}{g} \left(v'_{max} v + g \right) \cos\gamma$$

$$-\lambda_h v \cos\gamma \tag{A.39}$$

$$+\lambda_x v \sin\gamma$$

$$\dot{\lambda}_x = -\frac{\partial(H + \mu C_1)}{\partial x}$$

$$= 0 \tag{A.40}$$

Now condition (A.35-II) $S = \lambda_\gamma = 0$ implies by differentiation

$$\lambda_x \sin\gamma - \left(\lambda_h + \lambda_E \left(1 + \frac{v'_{max} v}{g} \right) \right) \cos\gamma = 0 \tag{A.41}$$

Before differentiating equation (A.41) further it is convenient to distinguish the following four cases:

$$\text{case 1:} \quad \lambda_x = 0, \quad \cos\gamma \neq 0$$

$$\text{case 2:} \quad \lambda_x = 0, \quad \cos\gamma = 0$$

$$\text{case 3:} \quad \lambda_x \neq 0, \quad \cos\gamma = 0$$

$$\text{case 4:} \quad \lambda_x \neq 0, \quad \cos\gamma \neq 0$$

In the course of the following calculations we will also make use of the fact that constraint

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(A.7) is active, i.e. $v = v_{max}(h)$.

case 1: $\lambda_x = 0$, $\cos \gamma \neq 0$

Then equation (A.41) reduces to

$$\lambda_h + \lambda_E \left(1 + \frac{v'_{max}v}{g}\right) = 0 \qquad (A.42)$$

Differentiation gives

$$\dot{\lambda}_h + \dot{\lambda}_E \left(1 + \frac{v'_{max}v}{g}\right) + \lambda_E \frac{v''_{max}v + v'_{max}{}^2}{g} v \sin \gamma = 0$$

Using (A.37) and (A.38) we find

$$\left[\lambda_E \left(2v'_{max} + \frac{g}{v} - \frac{v''_{max}v^2}{g}\right) \sin \gamma + \lambda_h \frac{g}{v} \sin \gamma\right]$$
$$+ \left[-\lambda_E \left(2v'_{max} + \frac{g}{v}\right) \sin \gamma - \lambda_h \frac{g}{v} \sin \gamma\right] \left(1 + \frac{v'_{max}v}{g}\right)$$
$$+ \lambda_E \frac{v''_{max}v + v'_{max}{}^2}{g} v \sin \gamma \quad = 0$$

$\Rightarrow$

$$\left[-\lambda_E \left(2v'_{max} + \frac{g}{v}\right) \sin \gamma - \lambda_h \frac{g}{v} \sin \gamma\right] \frac{v'_{max}v}{g}$$
$$+ \lambda_E \frac{v'_{max}{}^2}{g} v \sin \gamma \quad = 0$$

$\Rightarrow$

$$\lambda_E \left(-\frac{2v'_{max}{}^2 v}{g} - v'_{max} + \frac{v'_{max}{}^2 v}{g}\right) \sin \gamma - \lambda_h v'_{max} \sin \gamma \quad = 0$$

$\Rightarrow$

$$\left(\lambda_E \left(1 + \frac{v'_{max}v}{g}\right) + \lambda_h\right) (-v'_{max} \sin \gamma) \quad = 0$$

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If $\sin \gamma \neq 0$, then this equation implies

$$\lambda_E \left(1 + \frac{v'_{max} v}{g}\right) + \lambda_h = 0 \qquad (A.43)$$

Obviously, this condition is satisfied already due to condition (A.42). Further differentiation gives no new information. This case refers to a singular arc of infinite order and can be excluded. Hence necessarily

$$\sin \gamma = 0.$$

Differentiation yields

$$\cos \gamma \, \dot{\gamma} = 0.$$

With the assumption $\cos \gamma \neq 0$ this implies

$$\dot{\gamma} = 0$$

so that finally

$$n = \cos \gamma.$$

case 2: $\lambda_x = 0$, $\cos \gamma = 0$

Then equation (A.41) is satisfied already. Further information can be obtained from differentiating $\cos \gamma = 0$. We get

$$\begin{aligned}
0 &= \frac{d}{dt} \cos \gamma \\
&= -\dot{\gamma} \sin \gamma
\end{aligned}$$

As $\sin \gamma \neq 0$, this yields

$$\dot{\gamma} = 0$$

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With $\cos \gamma = 0$ and $\dot{\gamma} = 0$ we get from (A.2)

$$n = 0 \qquad\qquad (A.44)$$

case 3: $\lambda_x \neq 0$, $\cos \gamma = 0$

Then equation (A.41) leads to

$$\lambda_x(\pm 1) = 0$$

which contradicts the assumption. Hence this case can be excluded.

case 4: $\lambda_x \neq 0$, $\cos \gamma \neq 0$

To obtain further information we have to differentiate equation (A.41). As this involves $\dot{\lambda}_E$, $\dot{\lambda}_h$ we first state the associated differential equations, simplified by use of equation (A.41). From (A.41) we get

$$\lambda_h = \lambda_x \tan \gamma - \lambda_E \left(1 + \frac{v'_{max} v}{g}\right) \qquad\qquad (A.45)$$

Inserting this into (A.37) gives

$$
\begin{aligned}
\dot{\lambda}_E &= -\lambda_E \left(2v'_{max} + \frac{g}{v}\right) \sin \gamma \\
&\quad - \left(\lambda_x \tan \gamma - \lambda_E \left(1 + \frac{v'_{max} v}{g}\right)\right) \frac{g}{v} \sin \gamma \\
&\quad - \lambda_x \frac{g}{v} \cos \gamma \\
&= -\lambda_E v'_{max} \sin \gamma - \lambda_x \frac{g}{v} (\tan \gamma \sin \gamma + \cos \gamma) \\
&= -\lambda_E v'_{max} \sin \gamma - \lambda_x \frac{g}{v} \frac{1}{\cos \gamma} \qquad\qquad (A.46)
\end{aligned}
$$

Inserting (A.45) in (A.38) gives

$$\dot{\lambda}_h = \lambda_E \left(2v'_{max} + \frac{g}{v} - \frac{v''_{max} v^2}{g}\right) \sin \gamma$$

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$$+ \left(\lambda_x \tan \gamma - \lambda_E \left(1 + \frac{v'_{max} v}{g} \right) \right) \frac{g}{v} \sin \gamma$$

$$+ \lambda_x \frac{g}{v} \cos \gamma$$

$$= \; + \lambda_E \left(v'_{max} - \frac{v''_{max} v^2}{g} \right) \sin \gamma$$

$$+ \lambda_x \frac{g}{v} \left(\tan \gamma \sin \gamma + \cos \gamma \right)$$

$$= \; + \lambda_E \left(v'_{max} - \frac{v''_{max} v^2}{g} \right) \sin \gamma + \lambda_x \frac{g}{v} \frac{1}{\cos \gamma} \tag{A.47}$$

Before differentiation we write (A.45) as

$$\lambda_x \tan \gamma - \lambda_h - \lambda_E \left(1 + \frac{v'_{max} v}{g} \right) = 0$$

Differentiation gives

$$\dot{\lambda}_x \tan \gamma + \lambda_x \frac{\dot{\gamma}}{\cos^2 \gamma} - \dot{\lambda}_h - \dot{\lambda}_E \left(1 + \frac{v'_{max} v}{g} \right) - \lambda_E \frac{v''_{max} v \dot{h} + v'_{max} \frac{g}{v}(\dot{E} - \dot{h})}{g} = 0$$

Now insert $\dot{h}$, $\dot{\lambda}_E$, $\dot{\lambda}_h$, $\dot{\lambda}_x$ as given in (A.2), (A.46), (A.47), (A.40) and use

$$\dot{E} - \dot{h} = v \sin \gamma \left(v'_{max} + \frac{g}{v} \right) \frac{v}{g} - v \sin \gamma$$

$$= \frac{v'_{max} v^2}{g} \sin \gamma$$

Then we get

$$\left[\lambda_x \frac{\dot{\gamma}}{\cos^2 \gamma} - \left(+\lambda_E v'_{max} \sin \gamma - \lambda_E \frac{v''_{max} v^2}{g} \sin \gamma + \lambda_x \frac{g}{v} \frac{1}{\cos \gamma} \right) \right.$$

$$+ \left(\lambda_E v'_{max} \sin \gamma + \lambda_x \frac{g}{v} \frac{1}{\cos \gamma} \right) \left(1 + \frac{v'_{max} v}{g} \right) +$$

$$\left. - \lambda_E \left(v''_{max} v v_{max} \sin \gamma + v'_{max} \frac{g}{v} \frac{v'_{max} v^2}{g} \sin \gamma \right) \frac{1}{g} \right] = 0$$

$$\Rightarrow$$

$$\lambda_E \left(-v'_{max} \sin \gamma + \frac{v''_{max} v^2}{g} \sin \gamma + v'_{max} \sin \gamma \right.$$

$$
+\frac{v_{max}'^2 v}{g} \sin\gamma - \frac{v_{max}'' v v_{max}}{g} \sin\gamma - \frac{v_{max}'^2 v}{g} \sin\gamma \Bigg)
$$

$$
+\lambda_x \left(\frac{\dot\gamma}{\cos^2\gamma} - \frac{g}{v}\frac{1}{\cos\gamma} + \frac{g}{v}\frac{1}{\cos\gamma} + v_{max}'\frac{1}{\cos\gamma} \right) = 0
$$

$\Rightarrow$

$$
\lambda_E \left(\frac{v_{max}'' v^2}{g} \sin\gamma - \frac{v_{max}'' v v_{max}}{g} \sin\gamma \right)
$$

$$
+\lambda_x \frac{\dot\gamma + v_{max}'\cos\gamma}{\cos^2\gamma} = 0
$$

As we on an arc with active state constraint $v - v_{max} = 0$ the coefficient of λ_E is zero and we have

$$
\lambda_x(\dot\gamma + v_{max}'\cos\gamma) = 0
$$

or explicitly

$$
\lambda_x \left(\frac{g}{v}(n - \cos\gamma) + v_{max}'\cos\gamma \right) = 0
$$

As λ_x is assumed not-equal zero this finally implies

$$
n = \left(1 - \frac{v_{max}'v}{g} \right) \cos\gamma. \tag{A.48}
$$

A.4.4 $\lambda_\gamma \equiv 0,\ \lambda_E + \mu\frac{g}{v} \neq 0$

Then condition (A.28-I) can be satisfied only if either constraint (A.3) or constraint (A.4) is active. Then again, with already two constraints active, namely constraints (A.3) and (A.8), or constraints (A.4) and (A.8), no further constraint can be active so that $\sigma_3 = \sigma_4 = 0$. But now condition (A.28-II) becomes $\delta \left(\lambda_E + \mu\frac{g}{v} \right) \frac{2KvW}{q}n = 0$. This equation can be satisfied only through $\delta = 0$ or through $n = 0$. The case $n = 0$ can be eliminated quickly, as both cases, $\delta = 1\ \&\ n = 0$, and, $\delta = 0\ \&\ n = 0$, are inconsistent with constraint (A.8).

Hence the active constraints are exactly (A.3) and (A.8). Note that condition (A.28-I) in conjunction with condition $\sigma_1 \geq 0$ now implies that only the case $\lambda_E + \mu\frac{g}{v} > 0$ is possible. We get

$$
\text{I} \quad \left(\lambda_E + \mu\frac{g}{v}\right)(T - D + D_{max})\frac{v}{W} - \sigma_1 = 0
$$

$$
\text{II} \quad \lambda_\gamma = 0
$$

$$
\text{III} \quad -D_{max}\frac{g}{W} - v\sin\gamma\left(v'_{max} + \frac{g}{v}\right) = 0
$$

$$
\text{IV} \quad \delta = 0
$$

$$
\text{V} \quad \sigma_2 = 0
$$

$$
\text{VI} \quad \sigma_3 = 0 \tag{A.49}
$$

$$
\text{VII} \quad \sigma_4 = 0
$$

$$
\text{VIII} \quad [\Delta\delta, \Delta n]\begin{bmatrix} 0 & -n \\ -n & -\delta \end{bmatrix}\begin{bmatrix} \Delta\delta \\ \Delta n \end{bmatrix}\left(\lambda_E + \mu\frac{g}{v}\right)\frac{2KvW}{q} \geq 0
$$

for all $[\Delta\delta, \Delta n] \in \mathbf{R}^2$ satisfying

$$
\frac{\partial g_1}{\partial \delta}\Delta\delta + \frac{\partial g_1}{\partial n}\Delta n = 0
$$

$$
\frac{\partial C_1}{\partial \delta}\Delta\delta + \frac{\partial C_1}{\partial n}\Delta n = 0.
$$

In presence of condition (A.49-IV) $\delta = 0$, the conditions

$$
\begin{aligned}
\frac{\partial g_1}{\partial \delta}\Delta\delta + \frac{\partial g_1}{\partial n}\Delta n &= 0 \\
\frac{\partial C_1}{\partial \delta}\Delta\delta + \frac{\partial C_1}{\partial n}\Delta n &= 0
\end{aligned}
$$

on the perturbation vector $[\Delta\delta, \Delta n] \in \mathbf{R}^2$ in (A.49-VIII) yield explicitly

$$
-\Delta\delta = 0
$$

$$
(T - D + D_{max})\frac{g}{W}\Delta\delta = 0
$$

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and are both satisfied if and only if

$$\Delta\delta = 0.$$

Inserting $\Delta\delta = 0$ in (A.49-VIII) and using (A.49-IV) $\delta = 0$ shows that (A.49-VIII) is always satisfied with strict equality. Hence we have

$$
\begin{array}{rl}
\text{I} & \sigma_1 = \left(\lambda_E + \mu\dfrac{g}{v}\right)(T - D + D_{max})\dfrac{v}{W} \\[2ex]
\text{II} & \lambda_\gamma = 0 \\[2ex]
\text{III} & D_{max}\dfrac{g}{W} + (v'_{max}v + g)\sin\gamma = 0 \\[2ex]
\text{IV} & \delta = 0 \\[2ex]
\text{V} & \sigma_2 = 0 \\[2ex]
\text{VI} & \sigma_3 = 0 \\[2ex]
\text{VII} & \sigma_4 = 0 \\[2ex]
\text{VIII} & \text{satisfied with strict equality.}
\end{array}
\tag{A.50}
$$

Now, control n and multiplier μ have to be determined from equations (A.50-II) and (A.50-III) by successive differentiation. Using (A.50-II) and (A.50-IV), differentiation of (A.50-II) yields

$$
\begin{aligned}
0 &= \dot{\lambda}_\gamma \\
&= -\frac{\partial(H + \mu C_1)}{\partial\gamma} \\
&= -\lambda_h v\cos\gamma + \lambda_x v\sin\gamma + \mu\left(v'_{max}v + g\right)\cos\gamma
\end{aligned}
\tag{A.51}
$$

To satisfy this equation, two different cases have to be distinguished, namely

case 1: $\cos\gamma = 0$,

2. case: $\cos \gamma \neq 0$.

We get

case 1: $\cos \gamma = 0$

Equation (A.50-III) involves only states and no costates. Hence, control n or multiplier μ can enter derivatives of equation (A.50-III) only through the right-hand sides of the state equations (A.2). This immediately implies that multiplier μ can never appear in any derivative of equation (A.50-III). In conjunction with (A.50-IV) $\delta = 0$, it also implies that control n can appear in derivatives of equation (A.50-III) only through terms of the form $\cos \gamma \, \dot{\gamma}$. But with the assumption $\cos \gamma = 0$ this implies that also control n can never appear explicitly in any derivative of (A.50-III). Hence equation (A.50-III) either leads to a contradiction after a finite number of differentiations, or it leads to a situation where after some finite number of differentiations all further derivatives of (A.50-III) are satisfied automatically. In either case we find that case 1 can be excluded.

case 2: $\cos \gamma \neq 0$

As $v'_{max}(h) > 0 \ \forall \, h \geq 0$ it is clear that $\left(v'_{max} + \frac{g}{v}\right) \neq 0$ and hence equation (A.51) can be used to determine multiplier μ. Explicitly, we get

$$\mu = \frac{\lambda_h v \cos \gamma - \lambda_x v \sin \gamma}{(v'_{max} v + g) \cos \gamma}$$

and hence

$$\mu = \frac{\lambda_h - \lambda_x \tan \gamma}{v'_{max} + \frac{g}{v}}. \tag{A.52}$$

Now control n has to be determined from equation (A.50-III) by successive differentiation. In the following calculations we will make use of the fact that constraint (A.7) is active,

i.e. $v = v_{max}(h)$. We get

$$0 = \left(\frac{\partial D_{max}}{\partial E} \dot{E} + \frac{\partial D_{max}}{\partial h} \dot{h} \right) \frac{g}{W}$$
$$+ \left(v'_{max} v'_{max} \dot{h} + v''_{max} v \dot{h} \right) \sin \gamma +$$
$$+ (v'_{max} v + g) \cos \gamma \, \dot{\gamma}$$

With the state rates

$$\dot{E} = -D_{max} \frac{v}{W} = \left(\frac{v'_{max} v^2}{g} + v \right) \sin \gamma$$

$$\dot{h} = v \sin \gamma$$

$$\dot{\gamma} = \frac{g}{v}(n - \cos \gamma)$$

this yields

$$0 = \left(\frac{\partial D_{max}}{\partial E} \left(\frac{v'_{max} v^2}{g} + v \right) \sin \gamma + \frac{\partial D_{max}}{\partial h} v \sin \gamma \right) \frac{g}{W} +$$
$$+ \left((v'_{max})^2 v \sin \gamma + v''_{max} v^2 \sin \gamma \right) \sin \gamma +$$
$$+ (v'_{max} v + g) \cos \gamma \, (n - \cos \gamma) \frac{g}{v}.$$

As $(v'_{max} v + g) \neq 0$ and $\cos \gamma \neq 0$, this equation determines control n. We get

$$n = \cos \gamma - \frac{\left[\left(\frac{\partial D_{max}}{\partial E} \left(\frac{v'_{max} v}{g} + 1 \right) + \frac{\partial D_{max}}{\partial h} \right) \frac{g}{W} + v'_{max}{}^2 + v''_{max} v \right] v \sin \gamma}{\frac{g}{v}(v'_{max} v + g) \cos \gamma}. \tag{A.53}$$

A.5 Generalized Legendre-Clebsch Condition in the Singular Control Cases

A.5.1 Singular Control Case A.3.3: State Constraint Not Active, $\lambda_E \equiv 0$, $\lambda_\gamma \neq 0$

Here we have singular control in presence of an active control constraint, namely constraint (A.5) $g_3 = -n - n_{max} = 0$ if $\lambda_\gamma > 0$ or constraint (A.6) $g_4 = n - n_{max} = 0$ if $\lambda_\gamma < 0$. In either case (as $\frac{\partial g_3}{\partial n} \neq 0$, $\frac{\partial g_4}{\partial n} \neq 0$) control n can be regarded as the control that is determined by the constraint. From (6.41), (6.42) we find that the symbols u, v, w, m, p used in Chapter 6 have the following meaning in the case presently under consideration:

$$u^T = [\delta, n]$$

$$v = \delta$$

$$w = n$$

$$m = 2$$

$$p = 1.$$

From (6.69) we find that the differential operator $\frac{d}{d\delta}$ takes the following form

$$\frac{d \cdot}{d\delta} = \frac{\partial \cdot}{\partial \delta} - \frac{\partial \cdot}{\partial n} \left(\frac{\partial g_i}{\partial n} \right)^{-1} \frac{\partial g_i}{\partial \delta}, \quad i = 3, 4$$

$$= \frac{\partial \cdot}{\partial \delta}$$

and from (6.68) we find for all states $y \in \{E, h, \gamma, x\}$

$$\frac{d \cdot}{dy} = \frac{\partial \cdot}{\partial y} - \frac{\partial \cdot}{\partial n} \left(\frac{\partial g_i}{\partial n} \right)^{-1} \frac{\partial g_i}{\partial y}, \quad i = 3, 4$$

$$= \frac{\partial \cdot}{\partial y}$$

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Explicitly, this implies

$$\frac{dH}{d\delta} = \frac{\partial H}{\partial \delta}$$

$$= \lambda_E (T - D + D_{max}) \frac{v}{W}$$

and, by applying Definition 6.6.1, we find that control δ is singular if and only if $\lambda_E = 0$ and the degree m^* of the singularity is given by $m^* = 1$. Applying Theorem 6.6.2 we find

$$R_1 \in R^{0,0} \Rightarrow \text{non-existent}$$

$$Q_1 \in R^{0,4} \Rightarrow \text{non-existent}$$

$$B_1 \in R^{4,0} \Rightarrow \text{non-existent}$$

and

$$\left. \begin{array}{c} Q_2 \in R^{1,4} \\ \\ B_2 \in R^{4,1} \end{array} \right\} \Rightarrow Q_2 B_2 \in R^{1,1} \Rightarrow Q_2 B_2 \equiv (Q_2 B_2)^T$$

so that condition i) of Theorem 6.6.2 is always satisfied. Furthermore

$$\left. \begin{array}{c} \left. \begin{array}{c} B_2 \in R^{4,1} \\ \\ Q_1 \in R^{0,4} \end{array} \right\} \Rightarrow B_2^T Q_1^T \in R^{0,0} \\ \\ \left. \begin{array}{c} Q_2 \in R^{1,4} \\ \\ B_1 \in R^{4,0} \end{array} \right\} \Rightarrow Q_2 B_2 \in R^{0,0} \end{array} \right\} \Rightarrow R_2 \in R^{0,0} \text{ non-existent}$$

$$Q_2 = \left[\frac{d^2 H}{d\delta dE}, \frac{d^2 H}{d\delta dh}, \frac{d^2 H}{d\delta d\gamma}, \frac{d^2 H}{d\delta dx} \right]$$

$$= [0, 0, 0, 0]$$

$$B_2 = \left[\frac{d\dot{E}}{d\delta}, \frac{d\dot{h}}{d\delta}, \frac{d\dot{\gamma}}{d\delta}, \frac{d\dot{x}}{d\delta} \right]^T$$

$$= \left[(T - D + D_{max}) \frac{v}{W}, 0, 0, 0 \right]^T$$

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so that

$$R_3 = B_2^T P_1 B_2 - \underbrace{\frac{d}{dt}(Q_2 B_2)}_{=0} - \underbrace{\left(AB_2 - \dot{B}_2\right)^T Q_2^T}_{=0}$$

$$= \left((T - D + D_{max})\frac{v}{W}\right)^2 \frac{d^2 H}{dE^2}$$

$$= \left((T - D + D_{max})\frac{v}{W}\right)^2 \left[-\lambda_h \sin\gamma \frac{g^2}{v^3} + \lambda_\gamma(n - \cos\gamma)\frac{3g^3}{v^5} - \lambda_x \cos\gamma \frac{g^2}{v^3}\right]$$

$$= \left((T - D + D_{max})\frac{v}{W}\right)^2 \frac{g^2}{v^3} \left[-\lambda_h \sin\gamma + \lambda_\gamma(n - \cos\gamma)\frac{3g}{v^2} - \lambda_x \cos\gamma\right]$$

$$= \left((T - D + D_{max})\frac{v}{W}\right)^2 \frac{g^2}{v^3} \left[\lambda_\gamma(n - \cos\gamma)\frac{3g}{v^2} - \lambda_\gamma\frac{g}{v^2}(n - \cos\gamma)\right]$$

$$= \underbrace{\left((T - D + D_{max})\frac{v}{W}\right)^2 \frac{g^3}{v^5}}_{>0} 2\lambda_\gamma(n - \cos\gamma).$$

For all practically important cases $n_{max} > 1$. Hence the control logic

$$\begin{cases} n = +n_{max} & \text{if} \quad \lambda_\gamma < 0 \\ n = -n_{max} & \text{if} \quad \lambda_\gamma > 0 \end{cases}$$

(see A.3.3) implies that always $R_3 < 0$. Consequently,

$$R = \begin{bmatrix} R_1 & R_2^T \\ R_2 & R_3 \end{bmatrix} \geq 0$$

is always violated and the singular control given in case A.3.3 can be rejected as non-optimal.

A.5.2 Singular Control Case A.3.6: State Constraint Not Active, $\lambda_E > 0$, $\lambda_\gamma \equiv 0$

Here we have singular control in presence of the active control constraint (A.3) $g_1 = -\delta = 0$. As $\frac{\partial g_1}{\partial \delta} \neq 0$ control δ can be regarded as the control that is determined by the

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constraint. From (6.41), (6.42) we find that the symbols u, v, w, m, p used in Chapter 6 have the following meaning in the case presently under consideration:

$$u^T = [n, \delta]$$

$$v = n$$

$$w = \delta$$

$$m = 2$$

$$p = 1.$$

From (6.69) we find that the differential operator $\frac{d}{dn}$ takes the following form

$$
\begin{aligned}
\frac{d\cdot}{dn} &= \frac{\partial\cdot}{\partial n} - \frac{\partial\cdot}{\partial\delta}\left(\frac{\partial g_1}{\partial\delta}\right)^{-1}\frac{\partial g_1}{\partial n} \\
&= \frac{\partial\cdot}{\partial n}
\end{aligned}
$$

and from (6.68) we find for all states $y \in \{E, h, \gamma, x\}$

$$
\begin{aligned}
\frac{d\cdot}{dy} &= \frac{\partial\cdot}{\partial y} - \frac{\partial\cdot}{\partial\delta}\left(\frac{\partial g_1}{\partial\delta}\right)^{-1}\frac{\partial g_1}{\partial y} \\
&= \frac{\partial\cdot}{\partial y}
\end{aligned}
$$

Explicitly, this implies

$$
\begin{aligned}
\frac{dH}{dn} &= \frac{\partial H}{\partial n} \\
&= \lambda_\gamma\frac{g}{v}
\end{aligned}
$$

and, by applying Definition 6.6.1, we find that control n is singular if and only if $\lambda_\gamma = 0$ and the degree m^* of the singularity is given by $m^* = 1$. Applying Theorem 6.6.2 we find

$$R_1 \in R^{0,0} \quad \Rightarrow \quad \text{non-existent}$$

$$Q_1 \in R^{0,4} \quad \Rightarrow \quad \text{non-existent}$$

$$B_1 \in R^{4,0} \quad \Rightarrow \quad \text{non-existent}$$

and

$$\left.\begin{array}{c} Q_2 \in R^{1,4} \\[2mm] B_2 \in R^{4,1} \end{array}\right\} \Rightarrow Q_2 B_2 \in R^{1,1} \Rightarrow Q_2 B_2 \equiv (Q_2 B_2)^T$$

so that condition i) of Theorem 6.6.2 is always satisfied. Furthermore

$$\left.\begin{array}{c} \left.\begin{array}{c} B_2 \in R^{4,1} \\[2mm] Q_1 \in R^{0,4} \end{array}\right\} \Rightarrow B_2{}^T Q_1{}^T \in R^{0,0} \\[6mm] \left.\begin{array}{c} Q_2 \in R^{1,4} \\[2mm] B_1 \in R^{4,0} \end{array}\right\} \Rightarrow Q_2 B_2 \in R^{0,0} \end{array}\right\} \Rightarrow R_2 \in R^{0,0} \text{ non-existent}$$

$$\begin{aligned} Q_2 &= \left[\frac{d^2 H}{dndE}, \frac{d^2 H}{dndh}, \frac{d^2 H}{dnd\gamma}, \frac{d^2 H}{dndx}\right] \\[3mm] &= [0,0,0,0] \end{aligned}$$

$$\begin{aligned} B_2 &= \left[\frac{d\dot{E}}{dn}, \frac{d\dot{h}}{dn}, \frac{d\dot{\gamma}}{dn}, \frac{d\dot{x}}{dn}\right]^T \\[3mm] &= \left[0, 0, \frac{g}{v}, 0\right]^T \end{aligned}$$

so that

$$\begin{aligned} R_3 &= B_2{}^T P_1 B_2 - \underbrace{\frac{d}{dt}(Q_2 B_2)}_{=0} - \underbrace{\left(AB_2 - \dot{B}_2\right)^T Q_2^T}_{=0} \\[3mm] &= \left(\frac{g}{v}\right)^2 \frac{d^2 H}{d\gamma^2} \\[3mm] &= \left(\frac{g}{v}\right)^2 \left(-\lambda_h v \sin\gamma - \lambda_x v \cos\gamma\right). \end{aligned}$$

Hence

$$
R = \begin{bmatrix} R_1 & R_2{}^T \\ R_2 & R_3 \end{bmatrix}
$$

$$
= R_3
$$

$$
= \left(\frac{g}{v}\right)^2 (-\lambda_h v \sin\gamma - \lambda_x v \cos\gamma).
$$

In A.3.6 case 2, the condition $R \geq 0$ implies

$$
R = \left(\frac{g}{v}\right)^2 \left(-\lambda_h v \sin\gamma - \underbrace{\lambda_x v \cos\gamma}_{=0}\right)
$$

$$
= -\frac{g^2}{v}\lambda_h \sin\gamma \geq 0
$$

so that necessarily

$$
\begin{cases} \sin\gamma > 0 & \text{if} \quad \lambda_h < 0 \\ \sin\gamma < 0 & \text{if} \quad \lambda_h > 0. \end{cases}
$$

Together with the condition $\cos\gamma = 0$ implied by $\dot{S} = 0$ this finally yields

$$
\begin{cases} \gamma = +90^0 & \text{if} \quad \lambda_h < 0 \\ \gamma < -90^0 & \text{if} \quad \lambda_h > 0. \end{cases}
$$

In A.3.6 case 3, the condition $R \geq 0$ implies

$$
R = \left(\frac{g}{v}\right)^2 \left(-\underbrace{\lambda_h}_{=\lambda_x \tan\gamma} v \sin\gamma - \lambda_x v \cos\gamma\right)
$$

$$
= -\frac{g^2}{v}(\lambda_x \tan\gamma \sin\gamma + \lambda_x \cos\gamma)
$$

$$
= -\frac{g^2}{v}\frac{\lambda_x}{\cos\gamma} \geq 0.
$$

Hence, for $-90^0 < \gamma < +90^0$ necessarily $\lambda_x \leq 0$. Together with the assumption $\lambda_x \neq 0$ this finally implies $\lambda_x < 0$.

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A.5.3 Singular Control Case A.4.3: State Constraint Active, $\lambda_\gamma \equiv 0$

Here we have singular control in presence of the active control constraint (A.8) (due to state constraint (A.7)). As $\frac{\partial C_1}{\partial \delta} \neq 0$ control δ can be regarded as the control that is determined by the constraint. From (6.41), (6.42) we find that the symbols u, v, w, m, p used in Chapter 6 have the following meaning in the case presently under consideration:

$$u^T = [n, \delta]$$

$$v = n$$

$$w = \delta$$

$$m = 2$$

$$p = 1.$$

From (6.69) we find that the differential operator $\frac{d}{dn}$ takes the following form

$$
\begin{aligned}
\frac{d \cdot}{dn} &= \frac{\partial \cdot}{\partial n} - \frac{\partial \cdot}{\partial \delta} \left(\frac{\partial C_1}{\partial \delta} \right)^{-1} \frac{\partial C_1}{\partial n} \\
&= \frac{\partial \cdot}{\partial n} - \frac{\partial \cdot}{\partial \delta} \frac{-\delta \frac{\partial D}{\partial n} \frac{g}{W}}{(T - D + D_{max})\frac{g}{W}} \\
&= \frac{\partial \cdot}{\partial n} + \frac{\delta \frac{\partial D}{\partial n}}{T - D + D_{max}} \frac{\partial \cdot}{\partial \delta}
\end{aligned}
$$

and from (6.68) we find for all states $y \in \{E, h, \gamma, x\}$

$$
\begin{aligned}
\frac{d \cdot}{dy} &= \frac{\partial \cdot}{\partial y} - \frac{\partial \cdot}{\partial \delta} \left(\frac{\partial C_1}{\partial \delta} \right)^{-1} \frac{\partial C_1}{\partial y} \\
&= \frac{\partial \cdot}{\partial y} + \frac{\frac{\partial C_1}{\partial y}}{(T - D + D_{max})\frac{g}{W}}
\end{aligned}
$$

Explicitly, this implies

$$\frac{dH}{dn} = \frac{\partial H}{\partial n} + \frac{\delta \frac{\partial D}{\partial n}}{T - D + D_{max}} \frac{\partial H}{\partial \delta}$$

$$= -\lambda_E \delta \frac{\partial D}{\partial n} \frac{v}{W} + \lambda_\gamma \frac{g}{v} +$$

$$+ \frac{\delta \frac{\partial D}{\partial n}}{T - D + D_{max}} \lambda_E (T - D + D_{max}) \frac{v}{W}$$

$$= \lambda_\gamma \frac{g}{v}$$

and, by applying Definition 6.6.1, we find that control n is singular if and only if $\lambda_\gamma = 0$ and the degree m^* of the singularity is given by $m^* = 1$. Applying Theorem 6.6.2 we find

$$R_1 \in R^{0,0} \quad \Rightarrow \quad \text{non-existent}$$

$$Q_1 \in R^{0,4} \quad \Rightarrow \quad \text{non-existent}$$

$$B_1 \in R^{4,0} \quad \Rightarrow \quad \text{non-existent}$$

and

$$\left.\begin{array}{c} Q_2 \in R^{1,4} \\[2mm] B_2 \in R^{4,1} \end{array}\right\} \Rightarrow Q_2 B_2 \in R^{1,1} \Rightarrow Q_2 B_2 \equiv (Q_2 B_2)^T$$

so that condition i) of Theorem 6.6.2 is always satisfied. Furthermore

$$\left.\begin{array}{l} \left.\begin{array}{l} B_2 \in R^{4,1} \\[2mm] Q_1 \in R^{0,4} \end{array}\right\} \Rightarrow B_2^T Q_1^T \in R^{0,0} \\[6mm] \left.\begin{array}{l} Q_2 \in R^{1,4} \\[2mm] B_1 \in R^{4,0} \end{array}\right\} \Rightarrow Q_2 B_2 \in R^{0,0} \end{array}\right\} \Rightarrow R_2 \in R^{0,0} \text{ non-existent}$$

$$Q_2 = \left[\frac{d^2 H}{dndE}, \frac{d^2 H}{dndh}, \frac{d^2 H}{dnd\gamma}, \frac{d^2 H}{dndx}\right]$$

$$= [0, 0, 0, 0]$$

$$B_2 = \left[\frac{d\dot{E}}{dn}, \frac{d\dot{h}}{dn}, \frac{d\dot{\gamma}}{dn}, \frac{d\dot{x}}{dn}\right]^T$$

$$= \left[0, 0, \frac{g}{v}, 0\right]^T$$

so that

$$\begin{aligned}
R_3 &= B_2^T P_1 B_2 - \underbrace{\frac{d}{dt}(Q_2 B_2)}_{=0} - \underbrace{\left(A B_2 - \dot{B}_2\right)^T Q_2^T}_{=0} \\
&= \left(\frac{g}{v}\right)^2 \frac{d^2 H}{d\gamma^2} \\
&= \left(\frac{g}{v}\right)^2 \left[-\lambda_E \sin\gamma \left(\frac{v'_{max} v^2}{g} + v \right) - \lambda_h v \sin\gamma - \lambda_x v \cos\gamma \right].
\end{aligned}$$

Hence

$$\begin{aligned}
R &= \begin{bmatrix} R_1 & R_2^T \\ R_2 & R_3 \end{bmatrix} \\
&= R_3 \\
&= \left(\frac{g}{v}\right)^2 \left[-\lambda_E \sin\gamma \left(\frac{v'_{max} v^2}{g} + v \right) - \lambda_h v \sin\gamma - \lambda_x v \cos\gamma \right].
\end{aligned}$$

In A.4.3 case 2, the condition $R \geq 0$ is always satisfied with strict equality (because of $\lambda_x = 0$ and (A.43)). In A.4.3 case 2, the condition $R \geq 0$ implies

$$\begin{aligned}
R &= \left(\frac{g}{v}\right)^2 \left[-\lambda_E \sin\gamma \left(\frac{v'_{max} v^2}{g} + v \right) - \lambda_h v \sin\gamma \right] \\
&= -\frac{g^2}{v} \left(\lambda_E \left(\frac{v'_{max} v}{g} + 1 \right) + \lambda_h \right) \sin\gamma,
\end{aligned}$$

so that necessarily

$$\gamma = \begin{cases} +90^0 & \text{if} \quad \lambda_E \left(\frac{v'_{max} v}{g} + 1 \right) + \lambda_h < 0 \\ -90^0 & \text{if} \quad \lambda_E \left(\frac{v'_{max} v}{g} + 1 \right) + \lambda_h > 0. \end{cases}$$

If $\lambda_E \left(\frac{v'_{max} v}{g} + 1 \right) + \lambda_h = 0$ then both, $\gamma = +90^0$, and $\gamma = -90^0$ are compatible with the Generalized Legendre-Clebsch Condition. Explicit calculation shows that the latter case is indeed possible as along arcs of control logic A.4.3 case 2 all derivatives of $z :=$ $\lambda_E \left(\frac{v'_{max} v}{g} + 1 \right) + \lambda_h$ are zero automatically if only $z = 0$ is satisfied at a single point.

In A.4.3 case 4, using (A.45) $\lambda_h = \lambda_x \tan\gamma - \lambda_E \left(1 + \frac{v'_{max}v}{g}\right)$, condition $R \geq 0$ implies

$$
\begin{aligned}
R &= \left(\frac{g}{v}\right)^2 \left[-\lambda_E \sin\gamma \left(\frac{v'_{max}v^2}{g} + v \right) - \lambda_x v \cos\gamma + \right. \\
&\quad \left. - \left(\lambda_x \tan\gamma - \lambda_E \left(\frac{v'_{max}v}{g} + 1 \right)\right) v \sin\gamma \right] \\
&= \left(\frac{g}{v}\right)^2 \left[-\lambda_E \sin\gamma \left(\frac{v'_{max}v^2}{g} + v \right) - \lambda_x v \cos\gamma + \right. \\
&\quad \left. \lambda_x v \sin\gamma \, \tan\gamma + \lambda_E \left(\frac{v'_{max}v^2}{g} + v \right) \sin\gamma \right] \\
&= -\left(\frac{g}{v}\right)^2 \lambda_x v \left[\cos\gamma + \sin\gamma \, \tan\gamma\right] \\
&= -\left(\frac{g}{v}\right)^2 \lambda_x v \frac{1}{\cos\gamma}
\end{aligned}
$$

Hence, for $-90^0 < \gamma < +90^0$ necessarily $\lambda_x \leq 0$. Together with the assumption $\lambda_x \neq 0$ this finally implies $\lambda_x < 0$.

Appendix B

Accessory Minimum Problem for

Extremals with Corners

B.1 Introduction

In the following Section the Accessory Minimum Problem (AMP) is derived for optimal control problems in presence of an interior point constraint. Along these calculations also the first-order necessary conditions stated in Section 3.5 are obtained. For future research it is planned to apply the preliminary results on the Jacobi Necessary Condition stated in Chapter 8 to the AMP derived in this Chapter. The aim is to derive a Jacobi Necessary Condition for optimal control problems with interior point constraints.

B.2 Derivation of the Acessory Minimum Problem (AMP)

Consider the non-linear optimal control problem

$$\min \ ! \ \phi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x, u, t) \, dt$$

$$\dot{x} = f(x, u, t)$$

$$x(t_0) = x_0$$

$$\psi(x(t_f), t_f) = 0$$

$$N(x(t_1^+), x(t_1^-), t_1) = 0 \tag{B.1}$$

$$t_0 \text{ fixed}$$

$$t_1 \text{ free}$$

$$t_f \text{ free.}$$

Let $*$ and $^-$ denote quantities associated with a reference solution and an associated perturbed solution, respectively. Furthermore let J denote the augmented cost function

$$
\begin{aligned}
J := \quad & \phi(x(t_f), t_f)+ \\
& +\nu^T \psi(x(t_f), t_f)+ \\
& +\mu^T N(x(t_1{}^+), x(t_1{}^-), t_1)+ \\
& + \int_{t_0}^{t_1} L + \lambda^T(f - \dot{x})\, dt + \int_{t_1}^{t_f} L + \lambda^T(f - \dot{x})\, dt
\end{aligned}
\tag{B.2}
$$

and define

$$
\Psi(x(t_f), t_f, \nu) := \phi(x(t_f), t_f) + \nu^T \psi(x(t_f), t_f)
$$

$$
H(x, u, \lambda, t) = L(x, u, t) + \lambda^T f(x, u, t)
$$

$$
M(x(t_1{}^+), x(t_1{}^-), t_1, \mu) = \mu^T N(x(t_1{}^+), x(t_1{}^-), t_1).
$$

In the following we give an expansion of $\overline{J} - J^*$ about the reference solution $*$. Then we state necessary conditions that have to hold along the reference solution in order that all first order terms in the expansion of $\overline{J} - J^*$ (the "first variation") be zero. Finally the problem of minimizing the remaining second order terms is identified as a linear quadratic optimal control problem.

Throughout this Section the following nomenclature is valid:

superscript $+$ denotes evaluation at time $t_1{}^{+*}$

superscript $-$ denotes evaluation at time $t_1{}^{-*}$

subscript 0 denotes evaluation at time t_0

subscript f denotes evaluation at time $t_f{}^*$

other subscripts denote partial derivatives.

Let

$$\delta x(t) = \overline{x}(t) - x^*(t)$$

$$\delta u(t) = \overline{u}(t) - u^*(t)$$

$$\delta t_1 = \overline{t_1} - (t_1)^*$$

$$\delta t_f = \overline{t_f} - (t_f)^*,$$

then we have

$$
\begin{aligned}
\overline{J} - J^* =\ & \Psi_x dx_f + \Psi_t dt_f + (M_{x_1+}dx_1{}^+ + M_{x_1-}dx_1{}^- + M_t dt_1) + \\
& + \int_{t_0}^{t_1} H_x \delta x + H_u \delta u - \lambda^T \delta \dot{x}\, dt + \int_{t_1}^{t_f} H_x \delta x + H_u \delta u - \lambda^T \delta \dot{x}\, dt + \\
& + \frac{1}{2}[dx_f, dt_f]
\begin{bmatrix} \Psi_{xx} & \Psi_{xt} \\ \Psi_{tx} & \Psi_{tt} \end{bmatrix}
\begin{bmatrix} dx_f \\ dt_f \end{bmatrix} + \\
& + \frac{1}{2}[dx_1{}^+, dx_1{}^-, dt_1]
\begin{bmatrix} M_{x_1+x_1+} & M_{x_1+x_1-} & M_{x_1+t} \\ M_{x_1-x_1+} & M_{x_1-x_1-} & M_{x_1-t} \\ M_{tx_1+} & M_{tx_1-} & M_{tt} \end{bmatrix}
\begin{bmatrix} dx_1{}^- \\ dx_1{}^+ \\ dt_1 \end{bmatrix} + \\
& + \frac{1}{2} \int_{t_0}^{t_1} [\delta x, \delta u]
\begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix}
\begin{bmatrix} \delta x \\ \delta u \end{bmatrix} dt + \\
& + \frac{1}{2} \int_{t_1}^{t_f} [\delta x, \delta u]
\begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix}
\begin{bmatrix} \delta x \\ \delta u \end{bmatrix} dt + \\
& + \mathcal{O}^3.
\end{aligned}
$$

Now use

$$\int_{t_0}^{t_1} -\lambda^T \delta \dot{x}\, dt = - \left(\lambda^T \delta x\right)\Big|_{t_0}^{t_1{}^-} + \int_{t_0}^{t_1} \dot{\lambda}^T \delta x\, dt$$

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$$= \ -(\lambda^T \delta x)^- + \underbrace{(\lambda^T \delta x)_0}_{=0} + \int_{t_0}^{t_1} \dot{\lambda}^T \delta x \, dt$$

$$= \ -\lambda^{-T} \left(dx_1{}^- - \dot{x}^- dt_1 - \frac{\ddot{x}^-}{2} dt_1{}^2 - \delta\dot{x}^- dt_1 \right) + \mathcal{O}^3 +$$

$$+ \int_{t_0}^{t_1} \dot{\lambda}^T \delta x \, dt$$

and similarly

$$\int_{t_1}^{t_f} -\lambda^T \delta\dot{x} \, dt \ = \ - \left(\lambda^T \delta x \right) \Big|_{t_1+}^{t_f} + \int_{t_1}^{t_f} \dot{\lambda}^T \delta x \, dt$$

$$= \ - \left(\lambda^T \delta x \right)_f + \left(\lambda^T \delta x \right)^+ + \int_{t_1}^{t_f} \dot{\lambda}^T \delta x \, dt$$

$$= \ -\lambda_f{}^T \left(dx_f - \dot{x}_f dt_f - \frac{\ddot{x}_f}{2} dt_f{}^2 - \delta\dot{x}_f dt_f \right) +$$

$$-\lambda^{+T} \left(dx_1{}^+ - \dot{x}^+ dt_1 - \frac{\ddot{x}^+}{2} dt_1{}^2 - \delta\dot{x}^+ dt_1 \right) + \mathcal{O}^3 +$$

$$+ \int_{t_1}^{t_f} \dot{\lambda}^T \delta x \, dt.$$

Then

$$\overline{J} - J^* \ = \ \Psi_x dx_f + \Psi_t dt_f + (M_{x_1+} dx_1{}^+ + M_{x_1-} dx_1{}^- + M_{t_1} dt_1) +$$

$$+ \int_{t_0}^{t_1} \left(H_x + \dot{\lambda}^T \right) \delta x + H_u \delta u \, dt + \int_{t_1}^{t_f} \left(H_x + \dot{\lambda}^T \right) \delta x + H_u \delta u \, dt +$$

$$-\lambda_f{}^T \left(dx_f - \dot{x}_f dt_f - \frac{\ddot{x}_f}{2} dt_f{}^2 - \delta\dot{x}_f dt_f \right) +$$

$$+\lambda^{+T} \left(dx_1{}^+ - \dot{x}^+ dt_1 - \frac{\ddot{x}^+}{2} dt_1{}^2 - \delta\dot{x}^+ dt_1 \right) +$$

$$-\lambda^{-T} \left(dx_1{}^- - \dot{x}^- dt_1 - \frac{\ddot{x}^-}{2} dt_1{}^2 - \delta\dot{x}^- dt_1 \right) +$$

$$+\frac{1}{2}[dx_f, dt_f]\begin{bmatrix} \Psi_{xx} & \Psi_{xt} \\ \Psi_{tx} & \Psi_{tt} \end{bmatrix}\begin{bmatrix} dx_f \\ dt_f \end{bmatrix} +$$

$$+\frac{1}{2}[dx_1{}^+, dx_1{}^-, dt_1]\begin{bmatrix} M_{x_1{}^+x_1{}^+} & M_{x_1{}^+x_1{}^-} & M_{x_1{}^+t_1} \\ M_{x_1{}^-x_1{}^+} & M_{x_1{}^-x_1{}^-} & M_{x_1{}^-t_1} \\ M_{t_1x_1{}^+} & M_{t_1x_1{}^-} & M_{t_1t_1} \end{bmatrix}\begin{bmatrix} dx_1{}^- \\ dx_1{}^+ \\ dt_1 \end{bmatrix} +$$

$$+\frac{1}{2}\int_{t_0}^{t_1}[\delta x, \delta u]\begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix}\begin{bmatrix} \delta x \\ \delta u \end{bmatrix} dt +$$

$$+\frac{1}{2}\int_{t_1}^{t_f}[\delta x, \delta u]\begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix}\begin{bmatrix} \delta x \\ \delta u \end{bmatrix} dt +$$

$$+\mathcal{O}^3.$$

Rearranging yields

$$\begin{aligned}
\overline{J} - J^* =\ & \left(\Psi_x - \lambda_f{}^T\right) dx_f + (\Psi_t + H_f) dt_f + \\
& + \left(M_{x_1{}^+} + \lambda^{+T}\right) dx_1{}^+ + \\
& + \left(M_{x_1{}^-} - \lambda^{-T}\right) dx_1{}^- + \\
& + (M_{t_1} - H^+ + H^-) dt_1 + \\
& + \int_{t_0}^{t_1} \left(H_x + \dot{\lambda}^T\right) \delta x + (H_u)\,\delta u\, dt + \\
& + \int_{t_1}^{t_f} \left(H_x + \dot{\lambda}^T\right) \delta x + (H_u)\,\delta u\, dt + \\
& + \lambda_f{}^T \left(\frac{\ddot{x}_f}{2} dt_f{}^2 + \dot{\delta x}_f dt_f\right) + \\
& - \lambda^{+T} \left(\frac{\ddot{x}^+}{2} dt_1{}^2 + \dot{\delta x}^+ dt_1\right) + \\
& + \lambda^{-T} \left(\frac{\ddot{x}^-}{2} dt_1{}^2 + \dot{\delta x}^- dt_1\right) +
\end{aligned}$$

$$+\frac{1}{2}[dx_f, dt_f] \begin{bmatrix} \Psi_{xx} & \Psi_{xt} \\ \Psi_{tx} & \Psi_{tt} \end{bmatrix} \begin{bmatrix} dx_f \\ dt_f \end{bmatrix} +$$

$$+\frac{1}{2}[dx_1{}^+, dx_1{}^-, dt_1] \begin{bmatrix} M_{x_1+x_1+} & M_{x_1+x_1-} & M_{x_1+t_1} \\ M_{x_1-x_1+} & M_{x_1-x_1-} & M_{x_1-t_1} \\ M_{t_1x_1+} & M_{t_1x_1-} & M_{t_1t_1} \end{bmatrix} \begin{bmatrix} dx_1{}^- \\ dx_1{}^+ \\ dt_1 \end{bmatrix} +$$

$$+\frac{1}{2}\int_{t_0}^{t_1}[\delta x, \delta u] \begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} dt +$$

$$+\frac{1}{2}\int_{t_1}^{t_f}[\delta x, \delta u] \begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta u \end{bmatrix} dt +$$

$$+\mathcal{O}^3.$$

Now set

$$H_x + \dot{\lambda}^T = 0$$

$$H_u = 0$$

$$\Psi_x - \lambda_f{}^T = 0$$

$$\Psi_t + H_f = 0$$

$$\left(\mu^T N_{x_1+} + \lambda^{+T}\right) dx_1{}^+ + \left(\mu^T N_{x_1-} - \lambda^{-T}\right) dx_1{}^- + \left(\mu^T N_t - H^+ + H^-\right) dt_1 = 0. \tag{B.3}$$

These conditions eliminate all first order terms in $\overline{J} - J^*$. We wish to express the remaining second order expression for $\overline{J} - J^*$ completely in terms of δx rather than dx. As all dx-terms appear at least quadratically it is clear that $\overline{J} - J^*$ remains correct up to second

order if we make the first order approximations

$$
\begin{aligned}
dx_f &= \delta x_f + \dot{x}_f dt_f + \mathcal{O}^2 \\
dx^+ &= \delta x^+ + \dot{x}^+ dt_1 + \mathcal{O}^2 \\
dx^- &= \delta x^- + \dot{x}^- dt_1 + \mathcal{O}^2.
\end{aligned}
\tag{B.4}
$$

Then we get

$$
\begin{aligned}
\overline{J} - J^* &= +\lambda_f^T \left(\frac{\ddot{x}_f}{2} dt_f{}^2 + \delta \dot{x}_f dt_f \right) + \\
&\quad -\lambda^{+T} \left(\frac{\ddot{x}^+}{2} dt_1{}^2 + \delta \dot{x}^+ dt_1 \right) + \\
&\quad +\lambda^{-T} \left(\frac{\ddot{x}^-}{2} dt_1{}^2 + \delta \dot{x}^- dt_1 \right) + \\
&\quad +\frac{1}{2} \begin{bmatrix} \delta x_f + \dot{x}_f dt_f \\ \\ dt_f \end{bmatrix}^T \begin{bmatrix} \Psi_{xx} & \Psi_{xt} \\ \\ \Psi_{tx} & \Psi_{tt} \end{bmatrix} \begin{bmatrix} \delta x_f + \dot{x}_f dt_f \\ \\ dt_f \end{bmatrix} + \\
&\quad +\frac{1}{2} \begin{bmatrix} \delta x^+ + \dot{x}^+ dt_1 \\ \delta x^- + \dot{x}^- dt_1 \\ dt_1 \end{bmatrix}^T \begin{bmatrix} M_{x_1^+ x_1^+} & M_{x_1^+ x_1^-} & M_{x_1^+ t_1} \\ M_{x_1^- x_1^+} & M_{x_1^- x_1^-} & M_{x_1^- t_1} \\ M_{t_1 x_1^+} & M_{t_1 x_1^-} & M_{t_1 t_1} \end{bmatrix} \begin{bmatrix} \delta x^+ + \dot{x}^+ dt_1 \\ \delta x^- + \dot{x}^- dt_1 \\ dt_1 \end{bmatrix} + \\
&\quad +\frac{1}{2} \int_{t_0}^{t_1} [\delta x, \delta u] \begin{bmatrix} H_{xx} & H_{xu} \\ \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} \delta x \\ \\ \delta u \end{bmatrix} dt + \\
&\quad +\frac{1}{2} \int_{t_1}^{t_f} [\delta x, \delta u] \begin{bmatrix} H_{xx} & H_{xu} \\ \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} \delta x \\ \\ \delta u \end{bmatrix} dt + \\
&\quad +\mathcal{O}^3.
\end{aligned}
$$

The terms $\lambda^T \frac{\ddot{x}}{2}$ and $\lambda^T \delta \dot{x}$, can be written as follows:

$$
\begin{aligned}
\lambda^T \frac{\ddot{x}}{2} &= \frac{d}{dt} \left(\lambda^T \frac{\dot{x}}{2} \right) - \dot{\lambda}^T \frac{\dot{x}}{2} \\
&= \frac{1}{2} \frac{dH}{dt} - \frac{1}{2} \dot{\lambda}^T \dot{x}
\end{aligned}
$$

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$$\lambda^T \delta \dot{x} = \frac{d}{dt}\left(\lambda^T \delta x\right) - \dot{\lambda}^T \delta x$$

$$= \dot{\lambda}^T \delta x + \lambda^T \delta \dot{x} - \dot{\lambda}^T \delta x$$

$$= \underbrace{-H_x \delta x + \lambda^T f_x \delta x}_{=0} + \underbrace{\lambda^T f_u \delta u}_{=0} - \dot{\lambda}^T \delta x$$

$$= -\dot{\lambda}^T \delta x.$$

Hence

$$
\begin{aligned}
\overline{J} - J^* = &+ \frac{1}{2}\left[\left(\frac{dH}{dt}\right)_f - \dot{\lambda}_f^T \dot{x}_f\right] dt_f{}^2 - \dot{\lambda}_f^T \delta x_f dt_f + \\
&- \frac{1}{2}\left[\left(\frac{dH}{dt}\right)^+ - \dot{\lambda}^{+T} \dot{x}^+\right] dt_1{}^2 + \dot{\lambda}^{+T} \delta x^+ dt_1 + \\
&+ \frac{1}{2}\left[\left(\frac{dH}{dt}\right)^- - \dot{\lambda}^{-T} \dot{x}^-\right] dt_1{}^2 - \dot{\lambda}^{-T} \delta x^- dt_1 + \\
&+ \frac{1}{2}\begin{bmatrix} \delta x_f + \dot{x}_f dt_f \\[1ex] dt_f \end{bmatrix}^T \begin{bmatrix} \Psi_{xx} & \Psi_{xt} \\[1ex] \Psi_{tx} & \Psi_{tt} \end{bmatrix} \begin{bmatrix} \delta x_f + \dot{x}_f dt_f \\[1ex] dt_f \end{bmatrix} + \\
&+ \frac{1}{2}\begin{bmatrix} \delta x^+ + \dot{x}^+ dt_1 \\[1ex] \delta x^- + \dot{x}^- dt_1 \\[1ex] dt_1 \end{bmatrix}^T \begin{bmatrix} M_{x_1^+ x_1^+} & M_{x_1^+ x_1^-} & M_{x_1^+ t_1} \\[1ex] M_{x_1^- x_1^+} & M_{x_1^- x_1^-} & M_{x_1^- t_1} \\[1ex] M_{t_1 x_1^+} & M_{t_1 x_1^-} & M_{t_1 t_1} \end{bmatrix} \begin{bmatrix} \delta x^+ + \dot{x}^+ dt_1 \\[1ex] \delta x^- + \dot{x}^- dt_1 \\[1ex] dt_1 \end{bmatrix} + \\
&+ \frac{1}{2}\int_{t_0}^{t_1} [\delta x, \delta u] \begin{bmatrix} H_{xx} & H_{xu} \\[1ex] H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} \delta x \\[1ex] \delta u \end{bmatrix} dt + \\
&+ \frac{1}{2}\int_{t_1}^{t_f} [\delta x, \delta u] \begin{bmatrix} H_{xx} & H_{xu} \\[1ex] H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} \delta x \\[1ex] \delta u \end{bmatrix} dt + \\
&+ \mathcal{O}^3.
\end{aligned}
$$

Now use

$$\frac{1}{2}\left[\left(\frac{dH}{dt}\right)_f - \dot{\lambda}_f^T \dot{x}_f\right] dt_f{}^2 - \dot{\lambda}_f^T \delta x_f dt_f +$$

$$+\frac{1}{2}\begin{bmatrix} \delta x_f + \dot{x}_f dt_f \\ \\ dt_f \end{bmatrix}^T \begin{bmatrix} \Psi_{xx} & \Psi_{xt} \\ \\ \Psi_{tx} & \Psi_{tt} \end{bmatrix} \begin{bmatrix} \delta x_f + \dot{x}_f dt_f \\ \\ dt_f \end{bmatrix}$$

$$= \frac{1}{2}\left[\left(\frac{dH}{dt}\right)_f - \dot{\lambda}_f^T \dot{x}_f\right] dt_f^2 - \dot{\lambda}_f^T \delta x_f dt_f +$$

$$+\frac{1}{2}\begin{bmatrix} \delta x_f + \dot{x}_f dt_f \\ \\ dt_f \end{bmatrix}^T \begin{bmatrix} \Psi_{xx} \delta x_f + \frac{d\Psi_x}{dt} dt_f \\ \\ \Psi_{tx}\delta x_f + \frac{d\Psi_t}{dt} dt_f \end{bmatrix}$$

$$= \frac{1}{2}\left[\left(\frac{dH}{dt}\right)_f - \dot{\lambda}_f^T \dot{x}_f\right] dt_f^2 - \dot{\lambda}_f^T \delta x_f dt_f +$$

$$+\frac{1}{2}\left[(\delta x_f + \dot{x}_f dt_f)^T \left(\Psi_{xx}\delta x_f + \frac{d\Psi_x}{dt} dt_f\right) + dt_f \left(\Psi_{tx}\delta x_f + \frac{d\Psi_t}{dt} dt_f\right)\right]$$

$$= \frac{1}{2}\left[\left(\frac{dH}{dt}\right)_f - \dot{\lambda}_f^T \dot{x}_f\right] dt_f^2 - \dot{\lambda}_f^T \delta x_f dt_f +$$

$$+\frac{1}{2}\left[\delta x_f^T \Psi_{xx}\delta x_f + \dot{x}_f^T \Psi_{xx}\delta x_f dt_f + \delta x_f^T \frac{d\Psi_x}{dt} dt_f +\right.$$

$$\left.+\dot{x}_f^T \frac{d\Psi_x}{dt} dt_f^2 + \Psi_{tx}\delta x_f dt_f + \frac{d\Psi_t}{dt} dt_f^2\right]$$

$$= \frac{1}{2}\left[\left(\frac{dH}{dt}\right)_f - \dot{\lambda}_f^T \dot{x}_f\right] dt_f^2 - \dot{\lambda}_f^T \delta x_f dt_f +$$

$$+\frac{1}{2}\left[\delta x_f^T \Psi_{xx}\delta x_f + 2\frac{d\Psi_x}{dt}\delta x_f dt_f + \left(\dot{x}_f^T \frac{d\Psi_x}{dt} + \frac{d\Psi}{dt}\right) dt_f^2\right]$$

$$= \frac{1}{2}\delta x_f^T \Psi_{xx}\delta x_f +$$

$$+\left(\frac{d\Psi_x}{dt} - \dot{\lambda}^T\right)_f \delta x_f dt_f +$$

$$+\frac{1}{2}\left(\frac{dH}{dt} - \dot{\lambda}^T \dot{x} + \dot{x}^T \frac{d\Psi_x}{dt} + \frac{d\Psi}{dt}\right)_f dt_f^2$$

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$$
= \frac{1}{2}[\delta x_f, dt_f] \begin{bmatrix} \Psi_{xx} & \left(\frac{d\Psi_x}{dt} - \dot{\lambda}\right)_f \\ \left(\frac{d\Psi_x}{dt} - \dot{\lambda}\right)_f & \left(\frac{dH}{dt} - \dot{\lambda}^T\dot{x} + \dot{x}^T\frac{d\Psi_x}{dt} + \frac{d\Psi}{dt}\right)_f \end{bmatrix} \begin{bmatrix} \delta x_f \\ dt_f \end{bmatrix}
$$

and similarly

$$
-\frac{1}{2}\left[\left(\frac{dH}{dt}\right)^+ - \dot{\lambda}^{+T}\dot{x}^+\right] dt_1{}^2 + \dot{\lambda}^{+T}\delta x^+ dt_1 +
$$

$$
+\frac{1}{2}\left[\left(\frac{dH}{dt}\right)^- - \dot{\lambda}^{-T}\dot{x}^-\right] dt_1{}^2 - \dot{\lambda}^{-T}\delta x^- dt_1 +
$$

$$
+\frac{1}{2}\begin{bmatrix} \delta x^+ + \dot{x}^+ dt_1 \\ \delta x^- + \dot{x}^- dt_1 \\ dt_1 \end{bmatrix}^T \begin{bmatrix} M_{x_{1+}x_{1+}} & M_{x_{1+}x_{1-}} & M_{x_{1+}t} \\ M_{x_{1-}x_{1+}} & M_{x_{1-}x_{1-}} & M_{x_{1-}t} \\ M_{tx_{1+}} & M_{tx_{1-}} & M_{tt} \end{bmatrix} \begin{bmatrix} \delta x^+ + \dot{x}^+ dt_1 \\ \delta x^- + \dot{x}^- dt_1 \\ dt_1 \end{bmatrix}
$$

$$
= -\frac{1}{2}\left[\left(\frac{dH}{dt}\right)^+ - \dot{\lambda}^{+T}\dot{x}^+\right] dt_1{}^2 + \dot{\lambda}^{+T}\delta x^+ dt_1 +
$$

$$
+\frac{1}{2}\left[\left(\frac{dH}{dt}\right)^- - \dot{\lambda}^{-T}\dot{x}^-\right] dt_1{}^2 - \dot{\lambda}^{-T}\delta x^- dt_1 +
$$

$$
+\frac{1}{2}\begin{bmatrix} \delta x^+ + \dot{x}^+ dt_1 \\ \delta x^- + \dot{x}^- dt_1 \\ dt_1 \end{bmatrix}^T \begin{bmatrix} M_{x_{1+}x_{1+}}\delta x^+ + M_{x_{1+}x_{1-}}\delta x^- + \frac{dM_{x_{1+}}}{dt}dt_1 \\ M_{x_{1-}x_{1+}}\delta x^+ + M_{x_{1-}x_{1-}}\delta x^- + \frac{dM_{x_{1-}}}{dt}dt_1 \\ M_{tx_{1+}}\delta x^+ + M_{tx_{1-}}\delta x^- + \frac{dM_t}{dt}dt_1 \end{bmatrix}
$$

$$
= -\frac{1}{2}\left[\left(\frac{dH}{dt}\right)^+ - \dot{\lambda}^{+T}\dot{x}^+\right] dt_1{}^2 + \dot{\lambda}^{+T}\delta x^+ dt_1 +
$$

$$
+\frac{1}{2}\left[\left(\frac{dH}{dt}\right)^- - \dot{\lambda}^{-T}\dot{x}^-\right] dt_1{}^2 - \dot{\lambda}^{-T}\delta x^- dt_1 +
$$

$$
+\frac{1}{2}\bigg[\delta x^{+T}M_{x_{1+}x_{1+}}\delta x^+ + \delta x^{+T}M_{x_{1+}x_{1-}}\delta x^- + \delta x^{+T}\frac{dM_{x_{1+}}}{dt}dt_1 +
$$

$$
+\dot{x}^{+T}M_{x_{1+}x_{1+}}\delta x^+ dt_1 + \dot{x}^{+T}M_{x_{1+}x_{1-}}\delta x^- dt_1 + \dot{x}^{+T}\frac{dM_{x_{1+}}}{dt}dt_1{}^2 +
$$

$$
+\delta x^{-T}M_{x_{1-}x_{1+}}\delta x^+ + \delta x^{-T}M_{x_{1-}x_{1-}}\delta x^- + \delta x^{-T}\frac{dM_{x_{1-}}}{dt}dt_1 +
$$

$$
+\dot{x}^{-T}M_{x_{1-}x_{1+}}\delta x^+ dt_1 + \dot{x}^{-T}M_{x_{1-}x_{1-}}\delta x^- dt_1 + \dot{x}^{-T}\frac{dM_{x_{1-}}}{dt}dt_1{}^2 +
$$

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$$+ M_{tx_{1+}} \delta x^+ dt_1 + M_{tx_{1-}} \delta x^- dt_1 + \frac{dM_t}{dt} dt_1{}^2 \Bigg]$$

$$= -\frac{1}{2} \left[\left(\frac{dH}{dt} \right)^+ - \dot{\lambda}^{+T} \dot{x}^+ \right] dt_1{}^2 + \dot{\lambda}^{+T} \delta x^+ dt_1 +$$

$$+ \frac{1}{2} \left[\left(\frac{dH}{dt} \right)^- - \dot{\lambda}^{-T} \dot{x}^- \right] dt_1{}^2 - \dot{\lambda}^{-T} \delta x^- dt_1 +$$

$$+ \frac{1}{2} \Bigg[\delta x^{+T} M_{x_{1+} x_{1+}} \delta x^+ + \delta x^{-T} M_{x_{1-} x_{1-}} \delta x^- + 2 \frac{dM_{x_{1+}}}{dt} \delta x^+ dt_1 +$$

$$+ 2 \frac{dM_{x_{1-}}}{dt} \delta x^- dt_1 + 2 \delta x^{+T} M_{x_{1+} x_{1-}} \delta x^- + \left(\dot{x}^{+T} \frac{dM_{x_{1+}}}{dt} + \dot{x}^{-T} \frac{dM_{x_{1-}}}{dt} + + \frac{dM_t}{dt} \right) dt_1{}^2 \Bigg]$$

$$= \frac{1}{2} \delta x^{+T} M_{x_{1+} x_{1+}} \delta x^+ + \frac{1}{2} \delta x^{-T} M_{x_{1-} x_{1-}} \delta x^- +$$

$$+ \frac{1}{2} \left(- \frac{dH}{dt} \Big|_-^+ + \left(\dot{\lambda}^T \dot{x} \right) \Big|_-^+ + \dot{x}^{+T} \frac{dM_{x_{1+}}}{dt} + \dot{x}^{-T} \frac{dM_{x_{1-}}}{dt} + \frac{dM_t}{dt} \right) dt_1{}^2 +$$

$$+ 2 \frac{1}{2} \delta x^{+T} M_{x_{1+} x_{1-}} \delta x^- +$$

$$+ 2 \frac{1}{2} \left(\dot{\lambda}^{+T} + \frac{dM_{x_{1+}}}{dt} \right) \delta x^+ dt_1 +$$

$$+ 2 \frac{1}{2} \left(- \dot{\lambda}^{-T} + \frac{dM_{x_{1-}}}{dt} \right) \delta x^- dt_1$$

$$= + \frac{1}{2} \begin{bmatrix} \delta x^+ \\ \delta x^- \\ dt_1 \end{bmatrix}^T C \begin{bmatrix} \delta x^+ \\ \delta x^- \\ dt_1 \end{bmatrix} ,$$

where

$$
C = \begin{bmatrix}
M_{x_{1+}x_{1+}} & M_{x_{1+}x_{1-}} & \left(+\dot{\lambda}^{+T} + \frac{dM_{x_{1+}}}{dt}\right) \\[2mm]
M_{x_{1-}x_{1+}} & M_{x_{1-}x_{1-}} & \left(-\dot{\lambda}^{-T} + \frac{dM_{x_{1-}}}{dt}\right) \\[2mm]
\left(+\dot{\lambda}^{+T} + \frac{dM_{x_{1+}}}{dt}\right) & \left(-\dot{\lambda}^{-T} + \frac{dM_{x_{1-}}}{dt}\right) & \begin{array}{c} -\frac{dH}{dt}\Big|_{-}^{+} + \left(\dot{\lambda}^T\dot{x}\right)\Big|_{-}^{+} + \\[2mm] +\dot{x}^{+T}\frac{dM_{x_{1+}}}{dt} + \dot{x}^{-T}\frac{dM_{x_{1-}}}{dt} + \frac{dM_t}{dt} \end{array}
\end{bmatrix}.
$$

$$\text{(B.5)}$$

Hence

$$
\begin{aligned}
\overline{J} - J^* =\ & \tfrac{1}{2}[\delta x_f, dt_f]
\begin{bmatrix}
\Psi_{xx} & \left(\frac{d\Psi_x}{dt} - \dot{\lambda}\right)_f \\[2mm]
\left(\frac{d\Psi_x}{dt} - \dot{\lambda}\right)_f & \left(\frac{dH}{dt} - \dot{\lambda}^T\dot{x} + \dot{x}^T\frac{d\Psi_x}{dt} + \frac{d\Psi}{dt}\right)_f
\end{bmatrix}
\begin{bmatrix}
\delta x_f \\[2mm]
dt_f
\end{bmatrix} + \\[3mm]
& + \tfrac{1}{2}\int_{t_0}^{t_1}[\delta x, \delta u]
\begin{bmatrix}
H_{xx} & H_{xu} \\[2mm]
H_{ux} & H_{uu}
\end{bmatrix}
\begin{bmatrix}
\delta x \\[2mm]
\delta u
\end{bmatrix} dt + \\[3mm]
& + \tfrac{1}{2}\int_{t_1}^{t_f}[\delta x, \delta u]
\begin{bmatrix}
H_{xx} & H_{xu} \\[2mm]
H_{ux} & H_{uu}
\end{bmatrix}
\begin{bmatrix}
\delta x \\[2mm]
\delta u
\end{bmatrix} dt + \\[3mm]
& + \tfrac{1}{2}
\begin{bmatrix}
\delta x^+ \\[2mm]
\delta x^- \\[2mm]
dt_1
\end{bmatrix}^T
C
\begin{bmatrix}
\delta x^+ \\[2mm]
\delta x^- \\[2mm]
dt_1
\end{bmatrix} \\[3mm]
& + \mathcal{O}^3.
\end{aligned}
$$

$$\text{(B.6)}$$

In the expressions above all quantities δx, δu, δx_f, δx_{1+}, δx_{1-}, dt_1, dt_f denote the difference between quantities associated with the reference solution and quantities associated

with the perturbed solution, i.e.

$$\delta x(t) = \overline{x}(t) - x^*(t)$$

$$\delta u(t) = \overline{u}(t) - u^*(t)$$

$$\delta x_f = \overline{x}(t_f^*) - x^*(t_f^*)$$

$$\delta x^- = \overline{x}(t_1^{-*}) - x^*(t_1^{-*})$$

$$\delta x^+ = \overline{x}(t_1^{+*}) - x^*(t_1^{+*})$$ (B.7)

$$dt_f = \overline{t_f} - t_f^*$$

$$dt_1 = \overline{t_1} - t_1^*.$$

All other terms appearing in (B.6) are evaluated along the reference solution * and hence are either fixed numbers or fixed functions of time. As the right-hand side of (B.6) is a quadratic form in the quantities (B.7) it is clear that (B.6) remains correct in the leading (second order) term if we replace the quantities (B.7) by any first order approximation. Hence it suffices to determine the quantities δx, δu, δx_f, δx_{1+}, δx_{1-}, dt_1, dt_f from the linear conditions

$$\delta \dot{x} = f_x \delta x + f_u \delta u \text{ on } [t_0, t_1^{-*}] \cup [t_1^{+*}, t_f]$$

$$\Psi_x (\delta x_f + \dot{x} dt_f) + \Psi_t dt_f = 0$$ (B.8)

$$N_{x_{1-}} (\delta x^- + \dot{x}^- dt_1) + N_{x_{1+}} (\delta x^+ + \dot{x}^+ dt_1) + N_t dt_1 = 0.$$

Note that quantities evaluated along the reference solution may change discontinuously across the switching point t_1.

Now the problem

$$\min \overline{J} - J^*$$

with $\overline{J} - J^*$ given by (B.4), (B.6), subject to the linear constraints (B.8) constitutes a linear quadratic optimal control problem, the so-called Accessory Minimum Problem (AMP) associated with the non-linear optimal control problem (B.1). By construction it is clear that

(i) If there is a solution to the AMP that furnishes negative cost then the reference solution * cannot furnish a local minimum to (B.1). Then in any neighbourhood of the reference solution * a competitive solution to (B.1) can be found that furnishes a cost better than the cost associated with reference solution *.

(ii) If all non-trivial solutions to the AMP furnish cost greater than zero then reference solution * furnishes at least a weak local minimum to problem (B.1).

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