# CASCADE ANALYSIS AND SYNTHESIS OF TRANSFER FUNCTIONS OF INFINITE DIMENSIONAL LINEAR SYSTEMS

by

Lon E. Carpenter

Dissertation submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

APPROVED:

Joseph A. Ball

Dr. Joseph A. Ball, Chairman

John A. Burns

Dr. John A. Burns

David L. Russell

Dr. David L. Russell

Robert L. Wheeler

Dr. Robert L. Wheeler

Terry L. Herdman

Dr. Terry L. Herdman

# CASCADE ANALYSIS AND SYNTHESIS OF TRANSFER FUNCTIONS OF INFINITE DIMENSIONAL LINEAR SYSTEMS

by

Lon E. Carpenter

Dr. Joseph A. Ball
Department of Mathematics

## (Abstract)

Problems of cascade connections (synthesis) and decomposition (analysis) are analyzed for two classes of linear systems with infinite dimensional state spaces, namely, 1) admissible systems in the sense of Bart, Gohberg and Kaashoek and 2) regular systems as recently introduced by Weiss. For the class of BGK-admissible systems, it is shown that the product of two admissible systems is again admissible and that a Wiener-Hopf factorization problem can be solved just as in the finite-dimensional case. For the class of regular systems, it is shown that the cascade connection of a rational stable and antistable system has an additive stable-antistable decomposition; this involves giving a distribution interpretation to the solution of a linear Sylvester equation involving unbounded operator coefficients. As an application, some preliminary work is presented toward obtaining a state space solution of the sensitivity minimization problem for a pure delay plant.

## 0.1 Acknowledgments

I want first to acknowledge my faith in Jesus Christ and what He did for me on the Cross of Calvary. Without the faith that all things are in His hands, I would not have had the perseverance to finish this dissertation.

I would.like to thank my wife for always being by my side during my triumphs as well as my tribulations. I want to thank her for believing in me even when I questioned myself. I would like to thank my parents for instilling in me the strength of character to aim high and to see a task through to its completion no matter how great the obstacles.

Many at Virginia Tech have been influential in my obtaining this degree. The first of course is my thesis advisor, Dr. Joseph Ball, who encouraged my curiosity very early in my graduate career. I wish to thank Dr. Ball for his direction and the opportunities he has provided me. I would like to thank Dr. John Burns and Dr. Terry Herdman for supporting my trip to the Mathematical Theory of Networks and Systems Conference and for bring many interesting and diverse fields of control theory to Virginia Tech. I would also like to thank Dr. John Burns and Dr. Terry Herdman for their editorial work to encourage production of a quality manuscript. I would like to thank Dr. Bob Wheeler for his energy in the classroom and his concern for the development of my research interests. I would like to thank Dr. David Russell for his public words of praise for my Pritchard Salamon lecture series. Those few public words gave this graduate student a much needed boost of confidence. I would like to also issue a special thanks to Dr. George Wiess. I can not overstate the value of his technical assistance in the final phases of this dissertation.

To the dissertation committee as a whole, I would like to thank you for your patience with the slow development of this manuscript. I have not gone out in a blaze of glory, however I do claim the old saying, "slow and steady wins the race".

# Contents

## 0.2 Introduction

For the study of rational matrix functions, a highly successful approach has been the representation of the function as the transfer function of a linear system. By a linear system we mean a system of ordinary differential equations

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
\tag{0.2.1}
$$

where $A$, $B$, $C$, $D$ are finite matrices of appropriate sizes. Here $x(t)$ is the state vector, lying in the state space X, $u(\cdot)$ is the input function with values in the input space $U$, usually taken to be at least locally norm square integrable on $[0, \infty)$, and $y(\cdot)$ is the output function with values in the output space $Y$. The system is said to be finite dimensional if $X$, $U$ and $Y$ are all finite dimensional. One can solve the differential equation for $x(t)$ explicitly to obtain

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

and then substitute this into the second equation to obtain the output $y(t)$ directly in terms of the input $u(t)$ and the initial state $x_0$:

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t).$$

Upon taking the Laplace transform with initial state $x_0$ taken to be 0, we obtain that the transformed output function $\hat{y}(s)$ is related to the transformed input function $\hat{u}(s)$ via multiplication by a matrix function $W(s)$. If $x_0 = 0$, the relationship is given by

$$\hat{y}(s) = W(s)\hat{u}(t)$$

where

$$W(s) = D + C(sI - A)^{-1}B \tag{0.2.2}$$

is called the transfer function of the linear system. Since

$$(sI - A)^{-1} = \frac{adj(sI - A)}{\det(sI - A)}$$

it is clear that $W(s)$ is a proper rational matrix function. Conversely, by the realization theorem of linear systems theory (see [29]), it is known that any proper rational matrix function can be expressed in terms of four matrices $(A, B, C, D)$ as the transfer function of a linear system. This has proved to be a great aid in the mathematical study of rational matrix functions for their own sake. Indeed, cascade connections (or synthesis) of such systems produces the product, sum or linear fractional composition of the corresponding transfer functions while conversely, cascade decomposition (or analysis) of the system lead to factorizations or additive (or more generally linear fractional) decompositions of the transfer function (see Bart, Gohberg and Kaashoek [6], Gohberg, Lancaster and Rodman [23], Wonham [44]). Many practical problems in operator and systems theory (e.g. solving Wiener-Hopf integral equations, matrix Nevanlinna-Pick interpolation, Nehari approximation, problems in $H^{\infty}$-control theory) can be reduced to factorization problems for matrix-valued functions of various types (e.g. Wiener-Hopf or spectral, coprime, or J-inner-outer factorization). More recently it has been shown how factorization per se can be bypassed and one can obtain solutions of many such problems by more direct manipulation of state space objects (see Ball, Gohberg and Rodman [3]). The same technique applies as well to the study of interpolation problems for rational matrix functions (see Ball, Gohberg and Rodman [3]).

However many linear systems in practice are not finite dimensional. Such linear systems are often modeled by a system of partial differential equations or functional ordinary differential equations. The input space U and the output space Y are still physical, natural finite dimensional linear spaces, but the state space necessarily is infinite dimensional and is more delicate to make precise. A representation of the transfer function of the form $W(s) = D + C(sI - A)^{-1}B$ is more problematic. A natural first step is to assume $A, B, C, D$ are all bounded linear operators, but the operator $A$ acts on an infinite dimensional Hilbert (or more generally Banach) space $X$. This generalization is straight forward; all the formalism concerning cascade connections and decompositions goes through as in the finite dimensional case. Of course in practice the issue of finding nontrivial invariant subspaces for infinite dimensional operators remains. However, this class of systems leaves out most physically interesting examples. A next level of generalization is to assume that $B$ and $C$ are bounded operators but that $A$ is a (possible unbounded) generator of a $C_0$-semigroup (see Hille and Philips [26], Pazy [32]). This enlarges the class of systems and transfer functions but still leaves out many physically important examples. A beginning toward a more inclusive theory was made in the influential book by Curtain and Pritchard [16].

Recently Bart, Gohberg and Kaashoek [7]-[10], introduced a class of systems (which we shall call BGK-admissible) with a representation of the form (0.2.1) with only the operator B bounded. On the frequency domain side, Bart, Gohberg and Kaashoek have described the class of functions having realizations as transfer functions of BGK-admissible systems, while Salamon [36], Pritchard and Salamon [33], Callier and Desoer [12] and Logemann [31] have introduced various special classes of functions which arise in stabilization problems for infinite dimensional linear systems. Recently there has been some effort made to unify and clarify the relationships among these various special classes of functions. For example Curtain [14] discussed the connection between transfer functions of Pritchard and Salamon systems and the Callier and Desoer class.

Current works by Weiss [40]-[43] have systematically developed earlier work of Helton [24], Salamon ([36], [37]) and Pritchard and Salamon [33] to describe precisely the sense in which a representation of the form (0.2.1) holds for a general abstract well-posed linear system. The works by Weiss [40]-[43] and Curtain and Weiss [18] identify the class of regular linear systems as a large subclass of abstract linear systems which comprise many abstract linear systems of practical interest. A system is regular if its output function corresponding to a step input function and its zero initial state has the property that its average over $[0,\tau]$ has a limit as $\tau \to 0$. Equivalently, a system is regular if its transfer function has a strong limit at infinity (along the real axis). Weiss has shown that this class of systems has a meaningful representation in the form of (0.2.1), like finite dimensional systems.

The precise meaning of the system (0.2.1) for infinite dimensional systems has been clarified. It remains to investigate to what extent algebraic manipulations of the quadruple of operators $(A, B, C, D)$ can be used to solve factorization, interpolation and optimization problems for infinite dimensional linear systems. The first step is to try techniques analogous to what has been so successful over the past three decades for the finite dimensional case.

There has already been some preliminary work in this direction. Lasiecka and Triggiani [30] and Pritchard and Salamon [33] have results on how to reduce the quadratic optimal control problem to solving a Riccati equation with unbounded coefficients. Also Ran [34] and Curtain and Ran [17] performed a J-spectral factorization of the transfer functions of an infinite dimensional system in state space terms to solve a model reduction problem. Bart, Gohberg and Kaashoek [6] discuss Wiener-Hopf factorization in state space terms for the class of BGK-admissible systems to solve Wiener-Hopf integral equations.

Although the meaning of the system (0.2.1) and its transfer function (0.2.2) have been clarified for the case where the state space is infinite dimensional, manipulation of such systems to solve problems in the same way as is done for finite dimensional systems often leads to technical difficulties. Even a basic question, such as how to compose two systems of the form (0.2.1) to get a system whose transfer function is the product of the transfer functions of the original systems, is nontrivial. The purpose of this thesis is to demonstrate how some of the finite dimensional machinery can be extended to infinite dimensional settings.

Chapters 1 and 2 deal only with BGK-admissible systems while Chapters 3 and 4 deal with the more general class of regular systems. The main goal of Chapter 1 is to obtain rigorous state space formulas for the factors $X_+$ and $X_-$ of a right canonical Wiener-Hopf factorization $W(\lambda) = X_-(\lambda)X_+(\lambda)$ in terms of given BGK-admissible realizations for the factors $Y_+$ and $Y_-$ of a left canonical Wiener-Hopf factorization $W(\lambda) = Y_+(\lambda)Y_-(\lambda)$ of the same function $W(\lambda)$; this is done in Section 1.4. This extends work of Ball and Ran [5] on the case where $Y_+$ and $Y_-$ are rational; the special case where $Y_+ = Y_-(-\bar{\lambda})^*J$ is implicit in the work of Ran [34]. Section 1.2 discusses the solution of operator Sylvester equations where the coefficients may be unbounded. This section also gives a direct constructive state space proof that the product of a stable and an antistable BGK-admissible system is again BGK-admissible. While this is all that is needed for the left versus right Wiener-Hopf factorization issue, the natural question arises as to whether the product of two BGK-admissible systems is again BGK-admissible. This issue is resolved directly in a state space context in Section 1.4. A more implicit form of the product realization can be deduced from the realization theorem in [10]. The proof calls for some subtle results concerning perturbations of generators of semigroups from Hille and Philips [26]. The results of Chapter 1 and 2 have been announced in Ball and Carpenter [2].

The main goal of Chapter 3 and 4 is to develop the theory necessary to obtain a rigorous state space solution of the weighted sensitivity minimization problem for a pure delay plant. While this goal was not achieved in full, we do present some partial results which should be of interest in their own right. The strategy for solving the weighted sensitivity minimization problem is as follows. The first step is to reduce it to a Nehari problem in a standard way (see [19]). The Nehari problem is to find the distance of a given function $\tilde{P}$ to the subspace of all stable matrix functions in the $L^\infty$-norm along the imaginary axis. To solve this problem, it is convenient to decompose $\tilde{P}$ as $\tilde{P} = \tilde{P}_+ + \tilde{P}_-$ where $\tilde{P}_+$ is stable and $\tilde{P}_-$ is antistable. The second step is then to replace $\tilde{P}$ with $\tilde{P}_-$ and consider the Nehari problem for $\tilde{P}_-$. By the method of Ball and Helton [11], a linear fractional parameterization for the set of all solutions can be obtained via a J-inner-outer factorization of the matrix function $L = \begin{bmatrix} I & \tilde{P}_- \\ 0 & I \end{bmatrix}$. The third step is to obtain state space formulas for the J-inner factor $\Theta$ and the J-outer factor $R$ in a J-inner-outer factorization $L = \Theta R$ of $L$. In terms of $R$, one can back solve to obtain a linear fractional formula for the set of stabilizing compensators which achieve the desired level of performance of the weighted sensitivity.

Chapter 3 introduces regular systems, develops their elementary properties and shows that step 2 of the above procedure for solving the weighted sensitivity minimization problem can be done successfully for a pure delay plant. This involved solving a Sylvester equation of a type more general than that treated in Chapter 1. The solution in this case must be interpreted in the sense of distributions. Also addressed in Chapter 3 is an issue not needed for the sensitivity minimization problem, namely, when is the product of a stable and antistable regular system again regular. We give sufficient conditions for this to be true. This is the analogue of a question addressed in Chapter 1 for BGK-admissible systems. Chapter 4 introduces the sensitivity minimization problem, presents the above solution procedure in more detail and implements the procedure to the extent possible at this writing for the case of the plant corresponding to a pure delay. Several basic issues are left open for future work.

We should mention that there has been a series of papers by Foias and Tannenbaum and associates on a "skew-Toeplitz" frequency domain approach to $H^\infty$-control problems for distributed parameter systems (see [20] and [22]). However, given the recent advances in state space theory for distributed parameter systems in general, it is only natural to investigate to what extent state space formulas can also be found; these should lead to complementary insights on the problem.

# Chapter 1

# Left versus Right Wiener-Hopf Factorization for Matrix Functions Analytic in a Strip

## 1.1 Introduction

Section 1.2 contains definitions and important properties previously established for the class of matrix functions analytic in a strip. In Section 1.3 we collect various basic preliminaries needed for the work of Section 1.4; these include a proof of the fact that the product of a stable and an antistable BGK-admissible system is again BGK-admissible, and an analysis of how to solve certain types of Sylvester equations with unbounded coefficients. Finally in Section 1.4, we give explicit state space formulas for construction of a right Wiener-Hopf factorization in terms of BGK-admissible realizations for the factors of a left Wiener-Hopf factorization; this generalizes work of Ball and Ran [5] for the finite dimensional case. The case where the given left Wiener-Hopf factorization $W(\lambda) = Y_+(\lambda)Y_-(\lambda)$ has the form $W(\lambda) = [Y_-(-\bar{\lambda}^*)J]Y_-(\lambda)$ is implicit in Ran [34].

## 1.2 Notations, Definitions and Properties

This section introduces the definitions and notation necessary to define the class of transfer functions that we will refer to as BGK-admissible. Let m be a positive integer and let $\mathbf{C}$ denote the complex numbers. By $D_1^m(\Re)$ we denote the linear subspace of $L_1^m(\Re) = L_1(\Re; \mathbf{C}^m)$ consisting of all $f \in L_1^m(\Re)$ for which there exists $g \in L_1^m(\Re)$ such that

$$f(t) = \begin{cases} \int_{-\infty}^{t} g(s)\, ds & \text{a.e. on } (-\infty, 0], \\ -\int_{t}^{\infty} g(s)\, ds & \text{a.e. on } [0, \infty). \end{cases}$$

Let $X$ be a complex Banach space. The operators $A : D(A) \to X$, $B : \mathbf{C}^m \to X$ and $C : D(A) \to \mathbf{C}^m$ are linear operators and $\omega < 0$ is a negative real number.

**Definition 1.2.1** *The operator $A : X \to X$ is exponentially dichotomous, if $A$ is densely defined (i.e. the domain $D(A)$ of $A$ is dense in $X$) and $X$ admits a topological direct sum decomposition $X = X_- \oplus X_+$ with the following properties:*

1. *the decomposition reduces $A$, (i.e. $D(A) = (D(A) \cap X_-) \oplus (D(A) \cap X_+)$ and $A : D(A) \cap X_- \to X_-$, $A : D(A) \cap X_+ \to X_+$),*
2. *the restriction of $-A$ to $X_-$, denoted $A_-$, is the infinitesimal generator of an exponentially decaying strongly continuous semigroup,*
3. *the restriction of $A$ to $X_+$, denoted $A_+$, is the infinitesimal generator of an exponentially decaying strongly continuous semigroup of the same exponential type as that generated by the restriction of $-A$.*

We define the term bisemigroup to represent the two semigroups generated by the restrictions of the dichotomous operator A in the following manner.

**Definition 1.2.2** *The bisemigroup generated by the dichotomous operator $A$, denoted $E(t; A)$, is defined as follows*

$$E(t; A)x = \begin{cases} -e^{tA_-}\underline{P}x, & t < 0, \\ e^{tA_+}(I - \underline{P})x, & t > 0, \end{cases} \tag{1.2.1}$$

*where the exponential notation is being abused to represent strongly continuous semigroups generated by $A_+$ and $A_-$. Here $\underline{P}$ denotes the projection of $X$ onto $X_-$, parallel to $X_+$.*

**Definition 1.2.3** *(see [7]) Suppose $A : X \to X$ is an exponentially dichotomous operator of exponential type $\omega < 0$, and let $\underline{P}$ be its separating projection. Then*

$$\underline{P}x = \frac{-1}{2\pi i}\int_{h-i\infty}^{h+i\infty} \frac{1}{\lambda^2}(\lambda I - A)^{-1}A^2 x\, d\lambda,$$

*for $x \in D(A^2)$, where $h$ is any real number strictly between 0 and $-\omega$.*

It is restrictive to require both semigroups to have the same exponential decay rate, therefore we define the concept of exponential type.

**Definition 1.2.4** *The bisemigroup $E(t;A)$ is of exponential type $\omega < 0$ if and only if for every $\alpha$, $0 > \alpha > \omega$,*

$$\sup_{t\neq 0} e^{-\alpha|t|} \parallel E(t;A) \parallel \, dt < \infty.$$

**Definition 1.2.5** *(see [10]) The triple $\theta = (A, B, C)$ is <u>BGK-admissible</u> if the following conditions are satisfied:*

1. *$A$ is exponentially dichotomous of exponential type $\omega$,*
2. *$D(C) \supset D(A)$ and $C$ is $A$-bounded,*
3. *there exists a linear operator $\Lambda_\theta : X \to L_1^m(\Re)$ such that*

   *(a)*
   $$\sup_{\|x\|\le 1}\int_{-\infty}^{\infty} e^{-\omega|t|}\|(\Lambda_\theta x)(t)\|\, dt < \infty, \tag{1.2.2}$$

   *(b) $\Lambda_\theta$ maps $D(A)$ into $D_1^m(\Re)$ and*
   $$\Lambda_\theta x = \quad CE(t,A)x \quad \text{for } x \in D(A). \tag{1.2.3}$$

**Remark 1.2.6** *If $\Lambda_\theta$ is bounded from $X$ to $L_1^m(\Re)$ then (1.2.2) follows automatically.*

Since $B$ is a linear operator from the finite dimensional space $\mathbf{C}^m$ into $X$, $B$ is bounded. Also Definition 1.2.5.3a implies that $\Lambda_\theta$ is bounded and maps $X$ into $L^m_{1,\omega}(\Re)$, where $L^m_{1,\omega}(\Re) = \{f \in L^m_1(\Re) \mid e^{-\omega|t|}f(t) \in L^m_1(\Re)\}$. Using 3b from above and the fact that $D(A)$ is dense in $X$, one sees that $\Lambda_\theta$ is determined uniquely. Since $\omega$ is negative, $L^m_{1,\omega}(\Re)$ is a linear subspace of $L^m_1(\Re)$. The spaces $X$ and $\mathbf{C}^m$ are respectively called the state space and the input/output space of the triple $\theta$.

Next we introduce an operator $\Gamma_\theta : L^m_1(\Re) \to X$ which depends only on $A$ and $B$. We define $\Gamma_\theta$ by

$$\Gamma_\theta \phi = \int_{-\infty}^{\infty} E(t; A)B\phi(t)\,dt. \tag{1.2.4}$$

Note in this context the finite dimensionality of $\mathbf{C}^m$ implies the operator function $E(\cdot; A_+)B_+$ is continuous on $i\Re/\{0\}$ with a (possible) jump at the origin, all with respect to the operator norm topology. Thus $\Gamma_\theta$ is well-defined, linear and bounded operator.

**Proposition 1.2.7** *(see [10]) Let $\theta$ be BGK-admissible, and let $\Gamma_\theta$ be as in (1.2.4). Then $\Gamma_\theta$ is compact and maps $D^m_1$ into $D(A)$.*

Suppose $\theta = (A, B, C)$ is BGK-admissible. For fixed $y$ in $\mathbf{C}^m$ (the input/output space of $\theta$), $\Lambda_\theta By \in L^m_{1,\omega}(\Re)$. Thus the expression

$$k_\theta(t)y = \quad \Lambda_\theta By(t) \quad \text{a.e. on } \Re \tag{1.2.5}$$

determines a unique element $k_\theta$ of $L^{m\times m}_{1,\omega}(\Re)$. We call $k_\theta$ the <u>kernel</u> associated with $\theta$.

Since $k_\theta \in L^{m\times m}_{1,\omega}(\Re) \subset L^{m\times m}_1(\Re)$ and $\theta$ satisfies Definition 1.2.5, the bilateral Laplace transform $\hat{k}_\theta$ of $k_\theta$ is an analytic $m \times m$ matrix function on the strip $|Re\lambda| < -\omega$. More explicitly, we have the following.

**Theorem 1.2.8** *(see [10]) Let $\theta$ be BGK-admissible. The transfer function of $\theta$, denoted $W_\theta(\lambda)$ equals $I - \hat{k}(\lambda)$ where $\hat{k}(\lambda)$ is given by*

$$\hat{k}_\theta(\lambda) = \quad -C(\lambda I - A)^{-1}B \quad |Re\lambda| < -\omega. \tag{1.2.6}$$

**Theorem 1.2.9** *(see [10]) Let $\theta$ be BGK-admissible. The fact that $C$ is an $A$-bounded operator implies $C(\lambda I - A)^{-1}$ is a well-defined bounded linear operator depending analytically on $\lambda$ in the strip $|Re\lambda| \leq -\omega$.*

We now establish the relationship between the inverse of the transfer funtion, $W^{-1}(\lambda) = [I - \hat{k}(\lambda)]^{-1}$, and the kernel of the cross system, $\hat{k}^\times(\lambda)$. Let $k \in L_1[\Re, \mathcal{L}(Y)]$, where $Y$ is a finite dimensional complex linear space with a norm. If we assume that $\det(I - \hat{k}(\lambda)) \neq 0$ for all imaginary $\lambda$, then a well-known result of Wiener (see [39], chap. VII) asserts that $[I - \hat{k}(\lambda)]^{-1}$ is of the same form as $I - \hat{k}(\lambda)$. More precisely, we obtain the existence of a unique $k^\times \in L_1(\Re, \mathcal{L}(Y))$ such that

$$[I - \hat{k}(\lambda)]^{-1} = I - \hat{k}^\times(\lambda) = I - \int_{-\infty}^{\infty} e^{\lambda t} k^\times(t)\, dt, \tag{1.2.7}$$

for $\lambda$ on the imaginary axis. Generally $k^\times$ cannot be calculated explicitly, but for the case involving bisemigroups, we are able to calculate $k^\times$ by use of the following theorem.

**Theorem 1.2.10** *(see [9]) Let $Y$ be a finite dimensional complex linear space, and let $k : \Re \backslash \{0\} \to \mathcal{L}(Y)$ have the spectral exponential representation*

$$k(t) = CE(t; A)B, \ t \neq 0.$$

*Set $A^\times = A - BC$. Then $\det(I - \hat{k}(\lambda)) \neq 0$ for all $\lambda$ on the imaginary axis if and only if $A^\times$ is an exponentially dichotomous operator. In this case (1.2.7) is satisfied by*

$$k^\times(t) = -CE(t; A^\times)B, \ t \neq 0,$$

*that is,*

$$[I - \hat{k}(\lambda)]^{-1} = I + \int_{-\infty}^{\infty} e^{\lambda t} CE(t; A^\times)B\, dt,$$

*for $\lambda$ on the imaginary axis.*

We also obtain the following result for the inverse system when $\theta$ is BGK-admissible.

**Theorem 1.2.11** *(see [10]) Let $\theta = (A, B, C)$ be BGK-admissible. The following statements are equivalent:*

1. *$\theta^\times$ is BGK-admissible*
2. *$\det(I + C(\lambda I - A)^{-1}B)$ does not vanish on the imaginary axis*
3. *$A^\times$ has no spectrum on the imaginary axis*

*4. $A^{\times}$ is exponentially dichotomous.*

We now define a canonical Wiener-Hopf factorization. The factorization is with respect to some fixed, general Cauchy contour $\Gamma$ in the complex plane. A Cauchy contour $\Gamma$ is a positively oriented boundary of a bounded Cauchy domain in the complex plane $\mathbf{C}$ such that the contour consists of finitely many nonintersecting closed rectifiable Jordan curves. We denote the interior domain of $\Gamma$ by $F_+$ and the complement of $\bar{F}_+$ in the extended complex plane by $F_-$. An important example is the case where $\Gamma$ is the imaginary line with $F_+$ equal to the open left half plane and $F_-$ equal to the open right half plane.

**Definition 1.2.12** *Let $W(s)$ be a $m \times m$ matrix function. Then a right canonical (spectral) factorization of $W$ with respect to $\Gamma$ is a factorization of the form*

$$W(s) = W_-(s)W_+(s) \tag{1.2.8}$$

*where both $W_-$ and $W_-^{-1}$ are analytic functions on $\bar{F}_-$ and $W_+$ and $W_+^{-1}$ are analytic functions on $\bar{F}_+$.*

Similarly a left canonical factorization just reverses the order of the product (1.2.8).

## 1.3 Preliminaries

We first establish some properties for the convolution of functions from $L_{1,\omega}(\Re)$ for $\omega < 0$. We denote the convolution of f and g by

$$f * g(t) = \int_{-\infty}^{\infty} f(t-s)g(s)\,ds.$$

We note that $L_{1,\omega}(\Re)$ is closed under the convolution operation.

**Lemma 1.3.1** *For $\omega \leq 0$, let $g$ and $f$ be elements of $L_{1,\omega}(\Re)$. Then the convolution of $f$ and $g$ is in $L_{1,\omega}(\Re)$.*

Proof:

A function $v(t)$ is in $L_{1,\omega}(\Re)$ if and only if $e^{-\omega|t|}v(t)$ is in $L_1(\Re)$. We consider $e^{-\omega|t|}f * g(t)$ given by

$$e^{-\omega|t|}f * g(t) = \int_{-\infty}^{\infty} e^{-\omega|t|}f(t-s)g(s)\,ds.$$

Inserting $s - s$ into the absolute value, we obtain

$$e^{-\omega|t|}f * g(t) = \int_{-\infty}^{\infty} e^{-\omega|t-s+s|}f(t-s)g(t).$$

In order to show that $e^{-\omega|t|}f * g(t) \in L_1(\Re)$, we must verify

$$\int_{-\infty}^{\infty} e^{-\omega|t|}f(t-s)g(s)\,ds. < \infty.$$

The inequality $|t - s + s| \leq |t - s| + |s|$ leads to

$$\int_{-\infty}^{\infty} e^{-\omega|t|}|f(t-s)g(s)|\,ds \leq \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\omega|t-s|}f(t-s)e^{-\omega|s|}g(s)\,ds\,dt.$$

The interior integral is the convolution of $e^{-\omega|t|}f(t)$ and $e^{-\omega|t|}g(t)$. These two functions are in $L_1(\Re)$ since $f$ and $g$ are in $L_{1,\omega}(\Re)$. As is well known the convolution of two $L_1(\Re)$ functions is again $L_1(\Re)$. We thus obtain

$$\int_{-\infty}^{\infty} e^{-\omega|t|}|f * g(t)|\,dt < \infty$$

implying $f * g(t) \in L_{1,\omega}(\Re)$.□

Next we look at the convolution of an $L_1^m(\Re)$ function and a $D_1^m(\Re)$ function.

**Lemma 1.3.2** *Let $f \in L_1^m(\Re)$ and $g \in D_1^m(\Re)$ , then $f * g \in D_1^m(\Re)$.*

Proof:

Since $g \in D_1^m(\Re)$ there is a $h \in L_1^m(\Re)$ such that:

$$g(t) = \begin{cases} \int_{-\infty}^{t} h(s)\,ds, & t < 0, \\ -\int_{t}^{\infty} h(s)\,ds, & t > 0. \end{cases} \tag{1.3.1}$$

Let $w(t) = f * h(t) - f(t)\int_{-\infty}^{\infty} h(s)\,ds$. We want to consider $\frac{d}{dt}(f*g)(t)$. Note that $(f*g)(t) = (g*f)(t)$, so $\frac{d}{dt}(f*g)(t) = \frac{d}{dt}(g*f)(t)$ where

$$\frac{d}{dt}(g*f)(t) = \frac{d}{dt}\int_{-\infty}^{\infty} g(t-s)f(s)\,ds.$$

The function $g$ possibly has a jump discontinuity at zero. We handle the discontinuity by writing

$$\tfrac{d}{dt}(f*g)(t) = \tfrac{d}{dt}[\int_t^\infty g(t-s)f(s)\,ds + \int_{-\infty}^t g(t-s)f(s)ds].$$

This gives that $t-s$ is always nonnegative in the first integrand and always nonpositive in the second integrand. We now apply (1.3.1) to obtain

$$\begin{aligned}&\frac{d}{dt}(f*g)(t) = \\ &\quad \tfrac{d}{dt}\int_t^\infty \int_{-\infty}^{t-s} h(\tau)\,d\tau f(s)\,ds \quad + \frac{d}{dt}\int_{-\infty}^{t}[-\int_{t-s}^{\infty} h(\tau)\,d\tau]f(s)\,ds.\end{aligned}$$

Differentiating the integrals with respect to $t$ leads to

$$\begin{aligned}\frac{d}{dt}(f*g)(t) = &\ -\int_{-\infty}^0 h(\tau)\,d\tau f(t) + \int_t^\infty h(t-s)f(s)\,ds \\ &- \int_0^\infty h(\tau)\,d\tau f(t) + \int_{-\infty}^t h(t-s)f(s)\,ds. \end{aligned} \tag{1.3.2}$$

Note that the first and third terms are the same integrand. Similarly, the second and fourth terms have the same integrand. We obtain infinite limits of integration by combining the first with the third term of (1.3.2) and the second with the fourth term of (1.3.2);

$$\frac{d}{dt}(f*g)(t) = \int_{-\infty}^{\infty} h(t-s)f(s)\,ds - \int_{-\infty}^{\infty} h(\tau)\,d\tau f(t). \tag{1.3.3}$$

We observe that the first term of (1.3.3) is the convolution of $h$ and $f$:

$$\frac{d}{dt}(f*g)(t) = h*f(t) - \int_{-\infty}^{\infty} h(\tau)\,d\tau f(t).$$

Thus the derivative of $(f*g)(t)$ is $w(t)$ which is an element of $L_1^m(\Re)$. We recover $(f*g)(t)$ as

$$f*g(t) = \begin{cases} \int_{-\infty}^t w(s)\,ds, & t<0, \\ -\int_t^\infty w(s)\,ds, & t>0. \end{cases} \tag{1.3.4}$$

Hence $f*g(t) \in D_1^m(\Re)$.$\square$

We now address the issue of left versus right Wiener-Hopf factorization, given the transfer functions $Y_+$ and $Y_-$. The transfer funtion $W(\lambda)$ is an $m \times m$ matrix function defined by $W(\lambda) = I - \hat{k}(\lambda)$ where $k = k_\theta \in L_{1,\omega}(\Re)$ is the kernel associated with the realization triple $\theta = (A, B, C)$, i.e. $W(\lambda) = I + C(\lambda I - A)^{-1}B$. Let $W(\lambda)$ have a left canonical Wiener-Hopf factorization $W(\lambda) = Y_+(\lambda)Y_-(\lambda)$ where

$$Y_+(\lambda) = I + C_+(\lambda I - A_+)^{-1}B_+ \tag{1.3.5}$$

and

$$Y_-(\lambda) = I + C_-(\lambda I - A_-)^{-1}B_- \tag{1.3.6}$$

with $\theta_+ = (A_+, B_+, C_+)$ and $\theta_- = (A_-, B_-, C_-)$ being BGK-admissible realizations. This implies $Y_+(\lambda)$ is analytic and invertible for $Re\lambda > -\omega$ and $Y_-(\lambda)$ is analytic and invertible for $Re\lambda < \omega$.

**Theorem 1.3.3** *Let $\theta_+ = (A_+, B_+, C_+)$ and $\theta_- = (A_-, B_-, C_-)$ be BGK-admissible realizations. Then there is a unique solution $Q : D(A_-) \to D(A_+)$ to the Sylvester equation:*

$$A_+Qx - QA_-x = -B_+C_-x \tag{1.3.7}$$

*for every $x \in D(A_-)$. Moreover, $Q$ extends to a bounded operator from $X_-$ to $X_+$ given by $Q = \Gamma_{\theta_+}\Lambda_{\theta_-}$ for $\Gamma$ and $\Lambda$ as in (1.2.4) and Definition 1.2.5.3 respectively.*

Proof:

Set $Q = \Gamma_{\theta_+}\Lambda_{\theta_-}$. Since the $Ran(\Lambda_{\theta_-})$ is the finite dimensional space $\mathbf{C}^m$, the operator $\Lambda_{\theta_-}$ is a compact operator and therefore bounded from $X_-$ to $\mathbf{C}^m$. The operator $\Gamma_{\theta_+}$ is a bounded operator from $\mathbf{C}^m$ to $X_+$ from the discussion surrounding equation (1.2.4). Hence $Q$ defines a bounded operator from $X_-$ to $X_+$.

Let $x \in D(A_-)$. For such $x$ , (1.2.5.3b) gives us $\Lambda_{\theta_-} = C_-E(t, A_-)x$. From (1.2.4) with $\Gamma_{\theta_+}$ applied to $\Lambda_{\theta_-}x$, we obtain

$$Qx = \int_0^\infty E(t, A_+)B_+C_-E(-t, A_-)x\, dt. \tag{1.3.8}$$

Even though formally this is the correct form for the solution of (1.3.7) from the well known finite dimensional case, we must worry about domains. From Definition 1.2.5.3b the operator $\Lambda_{\theta_-}$ takes $D(A_-)$ into $D_1^m(\Re)$ and by Proposition 1.2.7 the operator $\Gamma_{\theta_+}$ takes $D_1^m(\Re)$ into $D(A_+)$, hence $Q$ takes $D(A_-)$ into $D(A_+)$.

Let $S$ be any solution of the Sylvester equation

$$A_+Sx - SA_-x = -B_+C_-x$$

for every $x \in D(A_-)$. Premultiply by $e^{A_+t}$ and postmultiply by $e^{-A_-t}$ to produce

$$e^{A_+t}[A_+S - SA_-]e^{-A_-t}z = -e^{A_+t}B_+C_-e^{-A_-t}z$$

for $z \in D(A_-)$. Integrating both sides from 0 to $\infty$ gives

$$\int_0^\infty e^{A_+t}[A_+S - SA_-]e^{-A_-t}z\,dt = -\int_0^\infty e^{A_+t}B_+C_-e^{-A_-t}z\,dt.$$

Note that since $e^{A_+t}$ and $e^{-A_-t}$ are both exponentially decaying the integral converges. But the integrand on the left can be rewritten as the derivative of a product to give

$$\int_0^\infty \frac{d}{dt}[e^{A_+t}Se^{-A_-t}]z\,dt = -\int_0^\infty e^{A_+t}B_+C_-e^{-A_-t}z\,dt.$$

Evaluating the integral on the left produces

$$-Sx = -\int_0^\infty e^{A_+t}B_+C_-e^{-A_-t}z\,dt.$$

Hence,

$$Sx = \int_0^\infty e^{A_+t}B_+C_-e^{-A_-t}z\,dt$$

for all $z \in D(A_-)$. This $S$ has a unique bounded extension to all $z \in X_-$ given by $Sz = \Gamma_+\Lambda_-z$. Thus any solution of (1.3.7) necessarily must equal Q.

Finally, we show $Q$ is a solution of (1.3.7). Inserting (1.3.8) into the left side of (1.3.7) leads to

$$\begin{aligned} A_+Qx - QA_-x = \quad & A_+ \textstyle\int_0^\infty E(t, A_+)B_+C_-E(-t, A_-)x\,dt \\ & - \textstyle\int_0^\infty E(t, A_+)B_+C_-E(-t, A_-)A_-x\,dt. \end{aligned}$$

Since $A_+$ is the generator of a strongly continuous semigroup, $A_+$ is a closed operator. We take $A_+$ inside the integral to obtain

$$\begin{aligned} A_+Qx - QA_-x = \quad & \textstyle\int_0^\infty A_+E(t, A_+)B_+C_-E(-t, A_-)x\,dt \\ & - \textstyle\int_0^\infty E(t, A_+)B_+C_-E(-t, A_-)A_-x\,dt. \end{aligned} \tag{1.3.9}$$

Since $A_+$ and $A_-$ are generators of the respective semigroups,

$$A_+E(t,A_+)x = \frac{d}{dt}E(t,A_+)x, \text{ when } x \in D(A_+) \tag{1.3.10}$$

and

$$\frac{d}{dt}E(-t,A_-) = E(-t,A_-)(-A_-), \text{ when } x \in D(A_-). \tag{1.3.11}$$

Insert (1.3.10) and (1.3.11) into (1.3.9). Thus

$$A_+Qx - QA_-x =$$

$$\int_0^\infty \tfrac{d}{dt}E(t,A_+)B_+C_-E(-t,A_-)x\,dt - \int_0^\infty E(t,A_+)B_+C_-\tfrac{d}{dt}E(-t,A_-)x\,dt$$

$$= \int_0^\infty \tfrac{d}{dt}[E(t,A_+)B_+C_-E(-t,A_-)x]\,dt.$$

The operators $A_+$ and $-A_-$ are stable so that the fundamental theorem of calculus yields

$$A_+Qx - QA_-x = -B_+C_-x,$$

for every $x \in D(A_-)$. □

By the Definition 1.2.12 of canonical Wiener-Hopf factorization, $Y_+^{-1}(\lambda)$ is analytic on the closed right half plane. Since $Y_+(\lambda)$ is analytic and invertible in the closed right half plane, the equation

$$(\lambda - A^\times)^{-1} = (\lambda - A_+)^{-1} - (\lambda - A_+)^{-1}B_+Y_+^{-1}(\lambda)C_+(\lambda - A_+)^{-1}$$

holds for $\lambda$ in the close right half plane. Therefore $A_+^\times$ has no spectrum in the closed right half plane. By Theorem 1.2.11 the transfer function $Y_+^{-1}(\lambda)$ has a BGK-admissible realization $\theta_+^\times = (A_+^\times, -C_+, B_+)$ where $A_+^\times = A_+ - B_+C_+$. Note that since $C_+$ is $A_+$-bounded and $B_+$ is bounded, $B_+C_+$ is well-defined on $D(A_+)$, so that $D(A_+^\times) = D(A_+)$. Also $Y_+^{-1}(\lambda)$ being analytic on the closed right half plane implies that $A_+^\times$ is exponentially decaying as well. Analogous remarks apply to $Y_-^{-1}(\lambda)$ with respect to the closed left half plane and its BGK-admissible realization $\theta_-^\times = (A_-^\times, -C_-, B_-)$, where $-A_-^\times$ is exponentially decaying.

**Theorem 1.3.4** *There is a unique solution $P : D(A_+) \to D(A_-)$ to the Sylvester equation*

$$A_-^\times Px - PA_+^\times x = \quad B_-C_+x \quad \textit{for every } x \in D(A_+). \tag{1.3.12}$$

*Moreover $P$ extends to a bounded operator from $X_+$ to $X_-$ given by $P = \Gamma_{\theta_-^\times}\Lambda_{\theta_+^\times}$ for $\Gamma$ and $\Lambda$ defined as in (1.2.4) and Definition 1.2.5.3 respectively.*

The proof parallels exactly that of Theorem 1.3.3 and so will be omitted.

The next consideration is the question of whether $W(\lambda)$ has a BGK-admissible realization if $Y_+(\lambda)$ and $Y_-(\lambda)$ have BGK-admissible realizations. From [6] we obtain the form of the product realization for the finite dimensional setting. The product system $\theta = \theta_+\theta_- = (A_p, B_p, C_p)$ is given formally by

$$A_p = \begin{bmatrix} A_+ & B_+C_- \\ 0 & A_- \end{bmatrix}, \quad B_p = \begin{bmatrix} B_+ \\ B_- \end{bmatrix}, \quad C_p = \begin{bmatrix} C_+ & C_- \end{bmatrix}. \tag{1.3.13}$$

Now we need to pick $X_1$ and $X_2$ so that Definition 1.2.1 is satisfied. We take

$$X_1 = \operatorname{Im}\begin{bmatrix} I \\ 0 \end{bmatrix} \quad \text{and} \quad X_2 = \operatorname{Im}\begin{bmatrix} Q \\ I \end{bmatrix}$$

where $Q$ is the solution of the Sylvester equation (1.3.7). $\begin{bmatrix} I & Q \\ 0 & I \end{bmatrix}$ is invertible and hence $X = X_1 \oplus X_2$.

**Theorem 1.3.5** *Let $Y_+(\lambda)$ and $Y_-(\lambda)$ be given and have BGK-admissible realizations $\theta_+ = (A_+, B_+, C_+)$ and $\theta_- = (A_-, B_-, C_-)$ where $A_+$ and $A_+^\times$ are stable and $A_-$ and $A_-^\times$ are antistable. Then the product realization for $W(\lambda)$ is BGK-admissible.*

Proof:

Define $\theta = \theta_+\theta_- = (A_p, B_p, C_p)$ by

$$A_p = \begin{bmatrix} A_+ & B_+C_- \\ 0 & A_- \end{bmatrix}, \quad B_p = \begin{bmatrix} B_+ \\ B_- \end{bmatrix}, \quad C_p = \begin{bmatrix} C_+ & C_- \end{bmatrix} \tag{1.3.14}$$

with $D(A_p) = D(A_+) \oplus D(A_-)$. Set

$$X_1 = \mathrm{Im} \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \text{and} \quad X_2 = \mathrm{Im} \begin{bmatrix} Q \\ I \end{bmatrix}$$

where $Q$ is the solution of the Sylvester equation (1.3.7).

We first check to see if $D(A_p)$ is dense in $X$. Due to the block diagonal structure of $A_p$, the domain of $A_p$ is contained in $D(A_+) \oplus D(A_-)$. The operator $C_-$ is $A_-$-bounded and $B_+$ is a bounded operator, hence $B_+C_-$ does not restrict the domain of $A_p$, more precisely $D(A_p) = D(A_+) \oplus D(A_-)$. Therefore $D(A_p)$ is dense in $X$.

Next we show that $X_1 \oplus X_2$ reduces $A_p$. Since $A_p$ is an unbounded operator, we show $A_p(X_2 \cap D(A_p)) \subset X_2$, specifically

$$A_p(\mathrm{Im} \begin{bmatrix} Q \\ I \end{bmatrix} \cap D(A_p)) \subset \mathrm{Im} \begin{bmatrix} Q \\ I \end{bmatrix}.$$

The vector $x \in X_2 \cap D(A_p)$ implies that there is $y \in D(A_-)$ so that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} Qy \\ y \end{bmatrix}.$$

From Theorem 1.3.3 we know $Qy \in D(A_+)$. We perform the matrix multiplication

$$\begin{bmatrix} A_+ & B_+C_- \\ 0 & A_- \end{bmatrix} \begin{bmatrix} Qy \\ y \end{bmatrix}$$

$$= \begin{bmatrix} A_+Qy + B_+C_-y \\ A_-y \end{bmatrix}.$$

Since $Q$ solves (1.3.7)

$$A_p \begin{bmatrix} Qy \\ y \end{bmatrix} = \begin{bmatrix} QA_-y \\ A_-y \end{bmatrix} \in \mathrm{Im} \begin{bmatrix} Q \\ I \end{bmatrix} = X_2. \tag{1.3.15}$$

Due to the block diagonal structure of the operator $A_p$, $A_p(D(A_p) \cap X_1) \subset X_1$. Hence $X_1 \oplus X_2$ reduces $A_p$.

The operator $A_p$ restricted to $X_1$ is $A_1 = \begin{bmatrix} A_+ \\ 0 \end{bmatrix}$ which generates a forward time strongly continuous semigroup. The operator $A_p$ restricted to $X_2$ by (1.3.15) is $A_2 = \begin{bmatrix} Q \\ I \end{bmatrix} A_-$ which generates a backward time strongly continuous semigroup

$$E(t, A_2) = \begin{bmatrix} QE(t, A_-) \\ E(t, A_-) \end{bmatrix}.$$

Hence $A_p$ is exponentially dichotomous. Since $C_p$ is defined as in (1.3.13) it is clear that $D(C_p) \supset D(A_p)$.

Now comes the task of constructing a $\Lambda_\theta$ to satisfy (3) in Definition 1.2.5. Let

$$\Lambda_\theta = [\Lambda_+, -k_+ * \Lambda_- + \Lambda_-]. \tag{1.3.16}$$

The operator $\Lambda_\theta$ operates on $X_1 \oplus X_2$. For each $x \in X_1 \oplus X_2$, there exists an $x_1 \in X_1$ and $x_2 \in X_2$ such that $x = x_1 + x_2$. The vector $x_1$ being in $X_1$ implies there is an $x_+ \in X_+$ such that $x_1 = \begin{bmatrix} I \\ 0 \end{bmatrix} x_+$. The vector $x_2$ being in $X_2$ implies there is an $x_- \in X_-$ such that $x_2 = \begin{bmatrix} Q \\ I \end{bmatrix} x_-$. We can now write

$$x = x_1 + x_2 = \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} \begin{bmatrix} x_+ \\ x_- \end{bmatrix} = \begin{bmatrix} x_+ + Qx_- \\ x_- \end{bmatrix}.$$

Applying $\Lambda_\theta$ to $x$ we obtain

$$\Lambda_\theta x = \Lambda_+ x_+ + \Lambda_+ Q x_- - k_+ * \Lambda_- x_- + \Lambda_- x_-.$$

Since $L_{1,\omega}(\Re)$ is a subspace of $L_1(\Re)$, $(k_+ * \Lambda_-)x_-(t)$ determines a bounded linear operator from $X_-$ into $L_1^m(\Re)$ such that

$$\sup_{\|x_-\| \leq 1} \int_{-\infty}^{\infty} e^{-\omega|t|} \|(k_+ * \Lambda_-)x_-\| \, dt < \infty. \tag{1.3.17}$$

The operator $\Lambda_+ Q$ also satisfies equation (1.2.5.3a) and maps $X_-$ into $L_1^m(\Re)$ linearly, since $Q$ is a bounded operator from $X_-$ to $X_+$. The operators $\Lambda_+$ and $\Lambda_-$ satisfy equation

(1.2.5.3a) since they are from BGK-admissible realizations. Hence the operator $\Lambda_\theta$ defined in (1.3.16) is a bounded linear operator from $X_1 \oplus X_2$ into $L_1^m(\Re)$ and satisfies equation (1.2.5.3a).

Consider the bilateral Laplace transform of $\Lambda_\theta x(t)$, denoted by $\hat{\Lambda}_\theta x(\lambda)$, defined by

$$\hat{\Lambda}_\theta x(\lambda) = \hat{\Lambda}_+ x_+(\lambda) + \hat{\Lambda}_+ Q x_-(\lambda) - \hat{k}_+ \hat{\Lambda}_- x_-(\lambda) + \hat{\Lambda}_- x_-(\lambda). \tag{1.3.18}$$

From Theorem 1.2.8 and Definition 1.2.5.3b for $\theta_+$ and $\theta_-$, the Laplace transform of $\Lambda_+$ is

$$\hat{\Lambda}_+ x_+(\lambda) = -C_+(\lambda I - A_+)^{-1} x_+(\lambda).$$

The operator $Q$ takes $X_-$ into $X_+$ implying

$$\hat{\Lambda}_+ Q x_-(\lambda) = -C_+(\lambda I - A_+)^{-1} Q x_-(\lambda).$$

From Theorem 1.2.8 and Theorem 1.2.9, we obtain

$$-\hat{k}_+ \Lambda_- x_-(\lambda) = -C_+(\lambda I - A_+)^{-1} B_+ C_-(\lambda I - A_-)^{-1} x_-(\lambda)$$

and

$$\hat{\Lambda}_- x_-(\lambda) = -C_-(\lambda I - A_-)^{-1} x_-(\lambda).$$

Hence (1.3.18) becomes

$$\begin{aligned} \hat{\Lambda}_\theta x(\lambda) = {} & [-C_+(\lambda I - A_+)^{-1}, -C_+(\lambda I - A_+)^{-1} Q \\ & - C_+(\lambda I - A_+)^{-1} B_+ C_-(\lambda I - A_-)^{-1} - C_-(\lambda I - A_-)^{-1}] \begin{bmatrix} x_+(\lambda) \\ x_-(\lambda) \end{bmatrix}. \end{aligned} \tag{1.3.19}$$

Straightforward algebraic manipulation then leads to

$$\hat{\Lambda}_\theta x(\lambda) = -C_p(\lambda I - A_p)^{-1} \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} \begin{bmatrix} x_+(\lambda) \\ x_-(\lambda) \end{bmatrix}. \tag{1.3.20}$$

The product $\begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} \begin{bmatrix} x_+(\lambda) \\ x_-(\lambda) \end{bmatrix}$ is just an element of $X_1 \hat{\oplus} X_2$. Hence

$$\hat{\Lambda}_\theta x(\lambda) = \quad -C_p(\lambda I - A_p)^{-1} x(\lambda) \quad \text{for } |\mathrm{Re}\lambda| < -\omega \tag{1.3.21}$$

for $x(\lambda) \in X_1 \hat{\oplus} X_2$.

Take $x \in D(A_p)$ and write $x = A^{-1}z$; then

$$C_p(\lambda I - A_p)^{-1}x = C_p A_p^{-1}(\lambda I - A_p)^{-1}z,$$

where $C_p A_p^{-1}$ is a bounded linear operator from $X$ into $\mathbf{C}^m$. From equation (1.5) in [9]

$$\hat{\Lambda}_\theta x(\lambda) =$$
$$-C_p A_p^{-1} \int_{-\infty}^{\infty} e^{\lambda t} E(t; A_p) z\, dt - \int_{-\infty}^{\infty} e^{\lambda t} C_p E(t; A_p) x\, dt.$$

Hence

$$\Lambda_\theta x(t) = C_p E(t; A_p) x(t)) \quad x(t) \in D(A_p).$$

We need to show $\Lambda_\theta$ takes $D(A_p)$ into $D_1^m(\Re)$. We know $\Lambda_+$, and $\Lambda_-$ obey this desired property. By Lemma 1.3.2 the convolution $k_+ * \Lambda_- x_-(t) \in D_1^m(\Re)$. But $Q$ takes $D(A_-)$ into $D(A_+)$ so $\Lambda_+ Q x_-(t) \in D_1^m(\Re)$. Hence $\Lambda_\theta$ satisfies Definition 1.2.5.3b.□

## 1.4 Left and Right Canonical Wiener-Hopf Factorization

In this section we analyze the existence of a right canonical Wiener-Hopf factorization for a given matrix function analytic in a strip in terms of a given left canonical Wiener-Hopf factorization. We assume we know the factors $Y_+$ and $Y_-$ of the left canonical factorization $W(\lambda) = Y_+(\lambda)Y_-(\lambda)$ with $Y_+$ and $Y_-$ formulas given by (1.3.5) and (1.3.6), respectively. We give a necessary and sufficient condition for a right canonical Wiener-Hopf factorization to exist and we compute the factors $W_-$ and $W_+$ of a right canonical factorization $W(\lambda) = W_-(\lambda)W_+(\lambda)$ in terms of the realizations of $Y_+$ and $Y_-$.

**Theorem 1.4.1** *Given $Y_+$ and $Y_-$, $m \times m$ matrix functions, with BGK-admissible realizations*

$$Y_+(\lambda) = I + C_+(\lambda I - A_+)^{-1}B_+ \tag{1.4.1}$$

*and*

$$Y_-(\lambda) = I + C_-(\lambda I - A_-)^{-1}B_-, \tag{1.4.2}$$

*define $W$ by $W = Y_+Y_-$. Let $P$ and $Q$ denote unique solutions of the operator Sylvester equations*

$$A_+Qx - QA_-x = \quad -B_+C_-x \quad \textit{for every } x \in D(A_-) \tag{1.4.3}$$

$$A_-^\times Px - PA_+^\times x = \quad B_-C_+x \quad \textit{for every } x \in D(A_+). \tag{1.4.4}$$

*Then $W$ has a right canonical factorization if and only if either $I - QP$ is invertible or equivalently, if and only if $I - PQ$ is invertible.*

Proof:

Since $W(\lambda) = Y_+(\lambda)Y_-(\lambda)$, by Theorem 1.3.5 the transfer function $W(\lambda)$ has a BGK-admissible realization. The transfer function $W^{-1}(\lambda)$ has a BGK-admissible realization by Theorem 1.2.11. Let $\underline{P}$ be the spectral projection for the left half plane of the $A$ from the BGK-admissible realization of $W(\lambda)$. The operator $\underline{P^\times}$ is the spectral projection for the left half plane of $A^\times$ from the BGK-admissible realization of $W^{-1}(\lambda)$. From the realizations of $Y_+(\lambda)$ and $Y_-(\lambda)$, we obtain the realization given in (1.3.13), for notational convenience we drop the subscript p.

From straightforward calculation the matrix $A^\times = A - BC$ equals

$$A^\times = \begin{bmatrix} A_+^\times & 0 \\ -B_-C_+ & A_-^\times \end{bmatrix} \tag{1.4.5}$$

where $A_+^\times = A_+ - B_+C_+$ and $A_-^\times = A_- - B_-C_-$. The operator $A_+^\times$ is closed with $D(A_+^\times) = D(A_+)$ since $B_+$ is bounded while $C_+$ is $A_+$-bounded. Similarly the operator $A_-^\times$ is closed with $D(A_-^\times) = D(A_-)$. Also $\sigma(A_+)$ is contained in $\{\lambda : Re\lambda < \omega\}$ and $\sigma(A_-)$ is contained in $\{\lambda : Re\lambda > \omega\}$, where $\omega < 0$ is the exponential type of the strongly continuous semigroups generated by $A_+$ and $A_-$.

The block diagonal form of $A$ implies $\sigma(A) = \sigma(A_+) \cup \sigma(A_-)$ and

$$X_+ = \mathrm{Ker}\underline{P} = \mathrm{Im} \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Similarly, the block diagonal form of $A^\times$ implies

$$X_- = \mathrm{Im}\underline{P^\times} = \mathrm{Im}\begin{bmatrix} 0 \\ I \end{bmatrix}.$$

We need the $\mathrm{Im}\underline{P}\oplus\mathrm{Ker}\underline{P^\times}$ decomposition of the state space in order to obtain the right canonical factorization. Since $\underline{P}$ and $\underline{P^\times}$ are projections, the space complementary to $\mathrm{Im}\begin{bmatrix} I \\ 0 \end{bmatrix}$ will have the form $\mathrm{Im}\begin{bmatrix} Q \\ I \end{bmatrix}$. Formally, we require

$$A(\mathrm{Im}\begin{bmatrix} Q \\ I \end{bmatrix}) \subset \mathrm{Im}\begin{bmatrix} Q \\ I \end{bmatrix}.$$

Since $A$ is an unbounded operator, the precise statement is

$$A(\mathrm{Im}\begin{bmatrix} Q \\ I \end{bmatrix}) \cap D(A)) \subset \begin{bmatrix} Q \\ I \end{bmatrix}.$$

Let

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathrm{Im}\begin{bmatrix} Q \\ I \end{bmatrix} \cap D(A);$$

then $x_1 \in D(A_+)$ and $x_2 \in D(A_-)$. The condition

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathrm{Im}\begin{bmatrix} Q \\ I \end{bmatrix}$$

implies that there is a $y \in D(A_-)$ so that

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} Qy \\ y \end{bmatrix}.$$

Therefore $y \in D(A_-)$ and $Qy \in D(A_+)$ and

$$A\begin{bmatrix} Qy \\ y \end{bmatrix} = \begin{bmatrix} A_+ & B_+C_- \\ 0 & A_- \end{bmatrix}\begin{bmatrix} Qy \\ y \end{bmatrix} =$$

$$\begin{bmatrix} A_+Qy + B_+C_-y \\ A_-y \end{bmatrix}.$$

In order for this product to be back in $\operatorname{Im}\begin{bmatrix} Q \\ I \end{bmatrix}$, we require

$$A\begin{bmatrix} Qy \\ y \end{bmatrix} = \begin{bmatrix} QA_-y \\ A_-y \end{bmatrix}.$$

Therefore, we must have

$$A_+Qy + B_+C_-y = \quad QA_-y \quad \text{for all } y \in D(A_-).$$

Thus we need to solve the Sylvester equation (1.3.7) in order to obtain the right canonical factorization. The operators $A_+$ and $-A_-$ are stable implying by Theorem 1.3.3, that Q is a unique solution, bounded from $X_-$ into $X_+$ and takes $D(A_-)$ into $D(A_+)$.

Now by definition of the Wiener-Hopf factorization, $\sigma(A_+^\times)$ is contained in the $\{\lambda : Re\lambda < \omega\}$ and $\sigma(A_-^\times)$ is contained in the $\{\lambda : Re\lambda > -\omega\}$. But Theorem 1.2.8 and Theorem 1.2.11 give us $k^\times(\lambda) \in L_{1,\omega}(\Re)$. We see that the spectral subspace of $A^\times$ for the right half plane is the coordinate space

$$\operatorname{Im}\begin{bmatrix} 0 \\ I \end{bmatrix} = \operatorname{Im}\underline{P^\times}.$$

We compute the spectral subspace of $A^\times$ for the left half plane. We assume that it has the form $\operatorname{Im}\begin{bmatrix} I \\ P \end{bmatrix}$. Invariance under $A^\times$ amounts to

$$A^\times(\operatorname{Im}\begin{bmatrix} I \\ P \end{bmatrix} \cap D(A^\times)) \subset \operatorname{Im}\begin{bmatrix} I \\ P \end{bmatrix}.$$

In more detail, if

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \operatorname{Im}\begin{bmatrix} I \\ P \end{bmatrix} \cap D(A^\times)$$

then by definition

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ Py \end{bmatrix}$$

for some $y \in D(A_+^\times)$. The operator $A^\times$ applied to $\begin{bmatrix} y \\ Py \end{bmatrix}$ yields

$$A^\times \begin{bmatrix} y \\ Py \end{bmatrix} = \begin{bmatrix} A_+^\times y \\ -B_-C_+y + A_-^\times Py \end{bmatrix}. \tag{1.4.6}$$

The requirement that the vector in (1.4.6) be in Im $\begin{bmatrix} I \\ P \end{bmatrix}$ leads to

$$\begin{bmatrix} A_+^\times y \\ -B_-C_+y + A_-^\times Py \end{bmatrix} = \begin{bmatrix} A_+^\times y \\ PA_+^\times y \end{bmatrix}.$$

Hence equating the second component of each vector is the Sylvester equation produces $-B_-C_+y + A_-^\times Py = PA_+^\times y$ for all $y \in D(A_+^\times)$. This matches equation (1.3.12). Theorem 1.3.4 establishes the existence, uniqueness and form of the operator $P$.

Now if we apply Theorem 4.1 from [10], we conclude that the function $W(\lambda)$ has a right canonical factorization $W(\lambda) = W_-(\lambda)W_+(\lambda)$ if and only if $X = \mathrm{Ker}\underline{P} \oplus \mathrm{Im}\underline{P^\times}$ and $\theta^\times$ is BGK-admissible. Combining Theorem (1.3.5) and Theorem (1.2.11), we obtain that our $\theta^\times$ is BGK- admissible. Note that $X = \mathrm{Ker}\underline{P} \oplus \mathrm{Im}\underline{P^\times}$ if and only if

$$\mathrm{Im} \begin{bmatrix} Q \\ I \end{bmatrix} \oplus \mathrm{Im} \begin{bmatrix} I \\ P \end{bmatrix} = X.$$

We check this by testing the invertibility of

$$\begin{bmatrix} I & Q \\ P & I \end{bmatrix}.$$

We decompose this matrix by use of the transformations $\begin{bmatrix} I & Q \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} I & 0 \\ P & I \end{bmatrix}$ as

$$\begin{bmatrix} I & Q \\ P & I \end{bmatrix} = \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} \begin{bmatrix} I-QP & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \tag{1.4.7}$$

and

$$\begin{bmatrix} I & Q \\ P & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & I - PQ \end{bmatrix} \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix}. \tag{1.4.8}$$

Therefore by (1.4.7) and (1.4.8) the invertibility condition is equivalent to the invertibility of either $I - QP$ or equivalently $I - PQ$.□

**Theorem 1.4.2** *When the conditions of Theorem 1.4.1 are satisfied, the factors $W_-(\lambda)$ and $W_+(\lambda)$ for a right canonical factorization $W(\lambda) = W_-(\lambda)W_+(\lambda)$ are given by the formulas*

$$W_-(\lambda) = I + (C_+Q + C_-)(\lambda I - A_-)^{-1}(I - PQ)^{-1}(-PB_+ + B_-) \tag{1.4.9}$$

*and*

$$W_+(\lambda) = I + (C_+ + C_-P)(I - QP)^{-1}(\lambda I - A_+)^{-1}(B_+ - QB_-) \tag{1.4.10}$$

*with inverses given by*

$$W_-(\lambda)^{-1} = \\ I - (C_+Q + C_-)(I - PQ)^{-1}(\lambda I - A_+ + B_-C_-)^{-1}(-PB_+ - QB_-) \tag{1.4.11}$$

*and*

$$W_+(\lambda)^{-1} = I - (C_+ + C_-P)(\lambda I - A_+ + B_+C_+)^{-1}(I - QP)^{-1}(B_+ - QB_-). \tag{1.4.12}$$

To prove Theorem 1.4.2, we use the realization formulas in Theorem II.4.1 of [10]. We therefore define

$$X_1 = \mathrm{Im} \begin{bmatrix} I \\ P \end{bmatrix} \quad \text{and} \quad X_2 = \mathrm{Im} \begin{bmatrix} Q \\ I \end{bmatrix}.$$

Also define the projection $\pi$ of $X$ onto $X_1$ along $X_2$ by

$$\pi = \begin{bmatrix} I \\ P \end{bmatrix} (I - QP)^{-1}[I, -Q] \tag{1.4.13}$$

**Theorem 1.4.3** *The transformation $\pi$ defined by (1.4.13) is the projection of $X$ onto $X_1$ along $X_2$ if and only if*

1. $\pi^2 = \pi$,

2. $\pi \begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ P \end{bmatrix}$,

3. $\pi \begin{bmatrix} Q \\ I \end{bmatrix} = 0.$

Proof:

Let the transformation $\pi$ be the projection of $X$ onto $X_1$ along $X_2$ defined by equation (1.4.13). The operator $\pi^2$ is defined by $\pi^2 x = \pi(\pi x)$. Insert formula (1.4.13) for $\pi$ to produce

$$\pi^2 x = \begin{bmatrix} I \\ P \end{bmatrix} (I - QP)^{-1}[I, -Q] \begin{bmatrix} I \\ P \end{bmatrix} (I - QP)^{-1}[I, -Q]x.$$

Multiplying $[I, -Q] \begin{bmatrix} I \\ P \end{bmatrix}$ out gives $(I - QP)$ which cancels with $(I - QP)^{-1}$ to leave us with

$$\pi^2 x = \begin{bmatrix} I \\ P \end{bmatrix} (I - QP)^{-1}[I, -Q]x = \pi x.$$

Thus 1. holds. Now consider $\pi \begin{bmatrix} I \\ P \end{bmatrix}$ applied to x. Inserting (1.4.13) leads to

$$\pi \begin{bmatrix} I \\ P \end{bmatrix} x = \begin{bmatrix} I \\ P \end{bmatrix} (I - QP)^{-1}[I, -Q] \begin{bmatrix} I \\ P \end{bmatrix} x.$$

Again from the same multiplication as above results in

$$\pi \begin{bmatrix} I \\ P \end{bmatrix} x = \begin{bmatrix} I \\ P \end{bmatrix} (I - QP)^{-1}(I - QP)x = \begin{bmatrix} I \\ P \end{bmatrix} x.$$

Thus 2. holds. Finally we consider $\pi \begin{bmatrix} Q \\ I \end{bmatrix} x$. Inserting the formula (1.4.13), we obtain

$$\pi \begin{bmatrix} Q \\ I \end{bmatrix} x = \begin{bmatrix} I \\ P \end{bmatrix} (I - QP)^{-1}[I, -Q] \begin{bmatrix} Q \\ I \end{bmatrix} x.$$

The multiplication of $[I, -Q]\begin{bmatrix} Q \\ I \end{bmatrix}$ is zero, hence

$$\pi \begin{bmatrix} Q \\ I \end{bmatrix} x = 0.$$

Hence 3. holds.

Now given 1, 2, and 3 from above and the formula (1.4.13), we show $\pi$ is the projection of $X$ onto $X_1$ along $X_2$. Due to the invertibility of $(I - QP)$, we know that $X = X_1 \oplus X_2$. Therefore for $x \in X$ we can write $x = x_1 + x_2$ for some $x_1 \in X_1$ and $x_2 \in X_2$. We now calculate $\pi$ applied to an element of $X$ using the decomposition of $X$ as follows:

$$\pi x = \pi(x_1 + x_2) = \pi x_1 + \pi x_2.$$

Property 3 from above implies $\pi x_2 = 0$, and Property 2 implies $\pi x_1 = y_1$ where $y_1 \in X_1$. Hence $\pi$ is the projection of $X$ onto $X_1$ along $X_2$. □

By similar arguments

$$I - \pi = \begin{bmatrix} Q \\ I \end{bmatrix} (I - PQ)^{-1}[-P, I] \tag{1.4.14}$$

is the projection of $X$ onto $X_2$ along $X_1$.□

Proof of Theorem 1.4.2:

From [10] Theorem II.4.1, the right canonical Wiener-Hopf factorization $W(\lambda) = W_-(\lambda)W_+(\lambda)$ of $W(\lambda)$ is given by $\lambda \in i\Re$ with

$$W_-(\lambda) = I + C(\lambda I - A)^{-1}(I - \pi)B, \tag{1.4.15}$$

$$W_+(\lambda) = I + C\pi(\lambda I - A)^{-1}B, \tag{1.4.16}$$

with inverses given by

$$W_-(\lambda)^{-1} = I - C(I - \pi)(\lambda I - A^\times)^{-1}B, \tag{1.4.17}$$

$$W_+(\lambda)^{-1} = I - C(\lambda I - A^\times)^{-1}\pi B. \tag{1.4.18}$$

We apply (1.4.15)-(1.4.18) to the case where $(A, B, C)$ arise from the product realization formed from the left canonical Wiener-Hopf factorization. Then with $Q$ the solution of the Sylvester equation (1.4.3), we have the identities

$$(\lambda I - A) = \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} \begin{bmatrix} (\lambda I - A_+) & 0 \\ 0 & (\lambda I - A_-) \end{bmatrix} \begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix} \tag{1.4.19}$$

and

$$(\lambda I - A)^{-1} = \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} \begin{bmatrix} (\lambda I - A_+)^{-1} & 0 \\ 0 & (\lambda I - A_-)^{-1} \end{bmatrix} \begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix}. \tag{1.4.20}$$

Substituting (1.4.20), (1.4.14) and

$$S = \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix}$$

into (1.4.15), we obtain

$$W_-(\lambda) = I +$$
$$CS \begin{bmatrix} (\lambda I - A_+)^{-1} & 0 \\ 0 & (\lambda I - A_-)^{-1} \end{bmatrix} S^{-1} \begin{bmatrix} Q \\ I \end{bmatrix} (I - PQ)^{-1}[-P, I]B. \tag{1.4.21}$$

Calculating $S^{-1}$ applied to $\begin{bmatrix} Q \\ I \end{bmatrix}$ gives

$$S^{-1} \begin{bmatrix} Q \\ I \end{bmatrix} = \begin{bmatrix} 0 \\ I, \end{bmatrix}.$$

Multiplying $[-P, I]B$ produces

$$[-P, I]B = (-PB_+ + B_-),$$

and $C$ applied to $S$ is

$$CS = [C_+, C_+Q + C_-].$$

Substituting these calculations into (1.4.21) results in

$$W_-(\lambda) = I + [C_+, C_+Q + C_-] \begin{bmatrix} 0 \\ (\lambda I - A_-)^{-1} \end{bmatrix} (I - PQ)^{-1}(-PB_+ + B_-).$$

Further simplification leads to

$$W_-(\lambda) = I + (C_+Q + C_-)(\lambda I - A_-)^{-1}(I - PQ)^{-1}(-PB_+ + B_-).$$

Hence (1.4.9) is formally verified. However, since we are dealing with unbounded operators we must also verify that the domains of the operators produce a well defined equation. The operator $(-PB_+ + B_-)$ takes $\mathbf{C}^m$ into $X_-$ while $(I - PQ)$ takes $X_-$ into $X_-$ and $(\lambda I - A_-)^{-1}$ takes $X_-$ into $D(A_-)$. Since $Q$ takes $D(A_-)$ into $D(A_+)$, both $C_+$ and $C_-$ are applied to elements in their respective domains. Hence (1.4.9) satisfies domain constraints.

Now we substitute (1.4.13) and (1.4.20) into (1.4.16) to obtain

$$W_+(\lambda) = I + \\ C \begin{bmatrix} I \\ P \end{bmatrix} (I - QP)^{-1}[I, -Q]S \begin{bmatrix} (\lambda I - A_+)^{-1} & 0 \\ 0 & (\lambda I - A_-)^{-1} \end{bmatrix} S^{-1}B. \qquad (1.4.22)$$

The operator $C$ applied to $\begin{bmatrix} I \\ P \end{bmatrix}$ is

$$C \begin{bmatrix} I \\ P \end{bmatrix} = (C_+ + C_-P),$$

while $[I, -Q]$ applied to $S$ results in

$$[I, -Q]S = [I, 0].$$

Finally, $S^{-1}$ applied to $B$ produces

$$S^{-1}B = \begin{bmatrix} B_+ - QB_- \\ B_- \end{bmatrix}.$$

Substituting these equations into (1.4.22) leads to

$$W_+(\lambda) = I +$$
$$(C_+ + C_-P)(I - QP)^{-1}[I, 0] \begin{bmatrix} (\lambda I - A_+)^{-1} & 0 \\ 0 & (\lambda I - A_-)^{-1} \end{bmatrix} \begin{bmatrix} B_+ - QB_- \\ B_- \end{bmatrix}.$$

Further simplification results in

$$W_+(\lambda) = I + (C_+ + C_-P)(I - QP)^{-1}(\lambda I - A_+)^{-1}(B_+ - QB_-). \tag{1.4.23}$$

We have formally verified the equation. We must still check domain constraints. The operator $Q$ takes $X_-$ into $X_+$ and $(\lambda I - A_+)^{-1}$ takes $X_+$ into $D(A_+)$. However in order for the domains to behave properly we need that $(I - QP)^{-1}$ must map $D(A_+)$ back into the $D(A_+)$. If we have this then $P$ maps $D(A_+)$ into $D(A_-)$ produces an element in $D(C_-)$, then $C_+$ will be operating on its domain. We will prove the above constraints on $(I - QP)^{-1}$ below.

Using the form of $A^\times$ from (1.4.5) with $P$, the solution of the Sylvester equation (1.3.12) leads to

$$(\lambda I - A^\times)^{-1} = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} (\lambda I - A_+^\times)^{-1} & 0 \\ 0 & (\lambda I - A_-^\times)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix}. \tag{1.4.24}$$

Substituting (1.4.24) and (1.4.14) into (1.4.17), we obtain

$$W_-(\lambda)^{-1} = I - C \begin{bmatrix} Q \\ I \end{bmatrix} (I - PQ)^{-1}$$
$$\times [-P, I] \begin{bmatrix} I & 0 \\ P & I \end{bmatrix} \begin{bmatrix} (\lambda I - A_+^\times)^{-1} & 0 \\ 0 & (\lambda I - A_-^\times)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} B. \tag{1.4.25}$$

Define the transformation

$$T = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix}.$$

Applying $C$ to $\begin{bmatrix} Q \\ I \end{bmatrix}$ give us

$$C \begin{bmatrix} Q \\ I \end{bmatrix} = (C_+Q + C_-),$$

while $[-B, I]$ applied to $T$ is

$$[-P, I]T = [0, I].$$

Finally applying the operator $T^{-1}$ to $B$ produces

$$T^{-1}B = \begin{bmatrix} B_+ \\ -PB_+ + B_- \end{bmatrix}.$$

Making these substitutions into (1.4.25) results in

$$W_-(\lambda)^{-1} = I - (C_+Q + C_-)(I - PQ)^{-1}[0, I]$$

$$\times \begin{bmatrix} (\lambda I - A_+^\times)^{-1} & 0 \\ 0 & (\lambda I - A_-^\times)^{-1} \end{bmatrix} \begin{bmatrix} B_+ \\ -PB_+ + B_- \end{bmatrix}. \tag{1.4.26}$$

Multiplying the matrices out, we obtain

$$W_-(\lambda)^{-1} = I - (C_+Q + C_-)(I - PQ)^{-1}(\lambda I - A_-^\times)^{-1}(-PB_+ + B_-). \tag{1.4.27}$$

Hence we have verified formally formula (1.4.11).

Now we need to check domain constraints. The operator $P$ takes $X_+$ into $X_-$, while $(\lambda I - A_-^\times)^{-1}$ takes $X_-$ into $D(A_-)$ and $Q$ takes $D(A_-)$ into $D(A_+)$ produce the proper domain constraints for the $C_+$ if $(I - PQ)^{-1}$ maps $D(A_-)$ back into $D(A_-)$. We will verify the constraint on $(I - PQ)^{-1}$ below.

Finally we substitute (1.4.14) and (1.4.24) into (1.4.18) to obtain

$$W_+(\lambda)^{-1} = I - C \begin{bmatrix} I & 0 \\ P & I \end{bmatrix}$$

$$\times \begin{bmatrix} (\lambda I - A_+^\times)^{-1} & 0 \\ 0 & (\lambda I - A_-^\times)^{-1} \end{bmatrix} \begin{bmatrix} I & 0 \\ -P & I \end{bmatrix} \begin{bmatrix} I \\ P \end{bmatrix} (I - QP)^{-1}[I, -Q]B. \tag{1.4.28}$$

Let

$$T = \begin{bmatrix} I & 0 \\ P & I \end{bmatrix}.$$

Then $C$ applied to $T$ gives $CT = [C_+ + C_-PC_-]$, while $T^{-1}$ applied to $\begin{bmatrix} I \\ P \end{bmatrix}$ gives

$$T^{-1}\begin{bmatrix} I \\ P \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix},$$

and $[I, -Q]$ applied to $B$ is $[I, -Q]B = (B_+ - QB_-)$. Making these substitutions into (1.4.28) leads to

$$W_+(\lambda)^{-1} = I - [C_+ + C_-PC_-]\begin{bmatrix} (\lambda I - A_+^\times)^{-1} & 0 \\ 0 & (\lambda I - A_-^\times)^{-1} \end{bmatrix}$$
$$\times \begin{bmatrix} I \\ 0 \end{bmatrix}(I - QP)^{-1}(B_+ - QB_-). \qquad (1.4.29)$$

Multiplying the matrices out results in

$$W_+(\lambda)^{-1} = I - (C_+ + C_-P)(\lambda I - A_+^\times)^{-1}(I - QP)^{-1}(B_+ - QB_-). \qquad (1.4.30)$$

This verifies (1.4.12) formally .

Now we check domain constraints in (1.4.12). The operator $Q$ takes $X_-$ into $X_+$, while $(I - QP)^{-1}$ takes $X_+$ into $X_+$. The resolvent $(\lambda I - A_+^\times)^{-1}$ takes $X_+$ into $D(A_+)$, while $P$ takes $D(A_+)$ into $D(A_-)$ satisfy the necessary domain constraints so $C_+$ and $C_-$ can be applied. Hence (1.4.12) is verified.

We have left (1.4.10) and (1.4.11) verified under the condition that $(I - QP)^{-1} : D(A_+) \to D(A_+)$ and $(I - PQ)^{-1} : D(A_-) \to D(A_-)$. We now analyze the inverse operator $(I - QP)^{-1}$. The operator $(I - QP)$ maps $D(A_+)$ into $D(A_+)$ since $P : D(A_+) \to D(A_-)$ and $Q : D(A_-) \to D(A_+)$. The inverse exist implying one to one, hence $(I - QP)^{-1} : D(A_+) \to D(A_+)$ and similarly by the same argument $(I - PQ)^{-1} : D(A_-) \to D(A_-)$.□

# Chapter 2

# Cascade Connection of BGK-Admissible Systems

## 2.1 Definitions, Notations and Properties

Different classes of semigroups arise from the precise sense in which the semigroup $[T(\xi)]$ converges to the identity as $\xi \to 0^+$. The classes as defined by Hille and Philips in [26] are

$(C_0)$ $\lim_{\eta \to 0^+} T(\eta)x = x$ for each $x \in X$;

$(A)$ $\lim_{\lambda \to \infty} \lambda R(\lambda)x = x$ for each $x \in X$;

$(1, A)$ $\int_0^1 \|T(t)\| \, dt < \infty$ and $(A)$ is satisfied;

where $R(\lambda)$ is the resolvent operator given by $\int_0^\infty e^{-\lambda t} T(t)x \, dt$. However, when the integral in class $(1, A)$ is not satisfied, Hille and Philips define a generalized resolvent (see [26]) in order to make sense of the limit defining $(A)$ class. The inclusion relationship between these classes is given by Theorem 10.6.1 in [26] as

$$(A) \supset (1, A) \supset (C_0).$$

It is well known that an infinitesimal generator $A$ of a $C_0-$semigroup perturbed by a bounded operator $B$ is an infinitesimal generator of a $C_0-$semigroup. However, there are more general results on perturbation of an infinitesimal generator. We first define a class of perturbation operators.

**Definition 2.1.1** *(see [26]) Let $A$ be the infinitesimal generator of a semigroup of class $(A)$. A linear operator $B$ is said to belong to the class $\mathcal{T}(A)$ if $D(B) = D(A)$ and $BR(\lambda_0; A)$ is a bounded linear operator on $X$ for some $\lambda_0 \in \rho(A)$.*

**Definition 2.1.2** *(see [26]) Let $C$ and $A$ be linear operators and suppose $D(C) \supset D(A)$. We shall use the notation $\| C \|_A = \sup\{\|Cx\| : \|x\| \leq 1, x \in D(A)\}$.*

**Definition 2.1.3** *(see [26]) Let $A$ be the infinitesimal generator of a semigroup $T(\xi; A)$ of class $(1, A)$. A linear operator $B$ is said to be of class $B(A)$ if*

1. *$B \in \mathcal{T}(A)$,*
2. *$BT(\xi; A)$ defined on $D(A)$ is bounded for $\xi > 0$,*
3. *$\int_0^1 \|BT(\xi; A)\|_A d\xi < \infty$.*

We now state a perturbation result found in [26].

**Theorem 2.1.4** *(see [26]) Let $T(\xi; A)$ be a semigroup of class $(C_0)$ and let $B \in B(A)$. Then $A + B$ defined on $D(A)$ generates a semigroup $T(\xi; A + B)$ of class $(C_0)$ and the series*

$$T(\xi; A + B) = \sum_{n=0}^{\infty} S_n(\xi)$$

*where $S_0(\xi) = T(\xi; A)$ and $S_n(\xi)x = \int_0^{\xi} T(\xi - \sigma; A)\tilde{B}S_{n-1}(\sigma)x \, d\sigma$ converges absolutely, uniformly with respect to $\xi$ in each interval of the form $(0, \beta), 0 < \beta < \infty$.*

Here the operator $\tilde{B}$ is the unique extension $B$ such that an element $x$ belongs to $D(\tilde{B})$ if and only if

$$\lim_{\lambda \to \infty} \lambda \tilde{B} R(\lambda; A)x \equiv y$$

exists, in which case $\tilde{B}x = y$. If the operator $B$ is the restriction of a closed operator $B_1$ then $B \subset \tilde{B} \subset B_1$.

## 2.2 BGK-admissibility of Product realization

Let $\theta_1$ and $\theta_2$ be BGK-admissible realizations of the $m \times m$ transfer functions $W_1(\lambda)$ and $W_2(\lambda)$, respectively. The exponential type of the bisemigroup associated with $\theta_1$ is $\omega_1$. Similiarly, $\omega_2$ is the exponential type of the bisemigroup associated with $\theta_2$. In this section we will show that the kernel associated with $\theta$ formed from the product of two BGK-admissible realizations is an element of $L_{1,\omega}(\Re)$ where $\omega = \max\{\omega_1, \omega_2\}$. The section opens by proving a result regarding the perturbation of an infinitesimal generator of a $C_0$-semigroup. The section concludes with the main result of the chapter, the fact that the product of two general BGK-admissible realizations is BGK-admissible.

Let $W_1(\lambda) = I - \hat{k}_1(\lambda)$ and $W_2(\lambda) = I - \hat{k}_2(\lambda)$ be transfer functions where $k_1(t) \in L_{1,\omega_1}(\Re)$ and $k_2(t) \in L_{1,\omega_2}(\Re)$ have BGK-admissible realizations, $\theta_1$ and $\theta_2$. The function $\hat{k}_1(\lambda)$ indicates the bilateral Laplace transform of $k_1(\lambda)$. Let $\omega = \max(\omega_1, \omega_2)$, then $k_1(t) \in L_{1,\omega}(\Re)$ and $k_2(t) \in L_{1,\omega}(\Re)$ and as already been noted by Lemma 1.3.1 $k_1 * k_2 \in L_{1,\omega}(\Re)$.

Since $\theta_1 = (A_1, B_1, C_1)$ and $\theta_2 = (A_2, B_2, C_2)$ are BGK-admissible there exist separating projections so that

$$A_1 = \begin{bmatrix} A_{1+} & 0 \\ 0 & A_{1-} \end{bmatrix}, B_1 = \begin{bmatrix} B_{1+} \\ B_{1-} \end{bmatrix},$$
$$C_1 = [C_{1+}, C_{1-}]$$

and

$$A_2 = \begin{bmatrix} A_{2+} & 0 \\ 0 & A_{2-} \end{bmatrix}, B_2 = \begin{bmatrix} B_{2+} \\ B_{2-} \end{bmatrix},$$
$$C_2 = [C_{2+}, C_{2-}].$$

The plus and minus are notations arising from (1.3.5), (1.3.6) and Definition 1.2.1. Note that the resulting $\theta_{1+}, \theta_{1-}, \theta_{2+}$ and $\theta_{2-}$ are BGK-admissible.

**Theorem 2.2.1** *Given $\theta_{1+}$ and $\theta_{2+}$ from above, then*

$$\begin{bmatrix} A_{1+} & B_{1+}C_{2+} \\ 0 & A_{2+} \end{bmatrix}$$

*generates a $C_0$-semigroup.*

Proof:

Take

$$B = \begin{bmatrix} 0 & B_{1+}C_{2+} \\ 0 & 0 \end{bmatrix}, A = \begin{bmatrix} A_{1+} & 0 \\ 0 & A_{2+} \end{bmatrix}.$$

Note that $D(C_{2+}) \supset D(A_{2+})$ since $\theta_{2+} = (A_{2+}, B_{2+}, C_{2+})$ is a BGK-admissible realization. This implies $D(B_{1+}C_{2+}) \supset D(A_{2+})$. We take $B_{1+}C_{2+}\,|_{D(A_{2+})}$ to produce $D(B) = D(A)$. Applying $B$ to the resolvent of $A$ gives

$$BR(\lambda; A) = \begin{bmatrix} 0 & B_{1+}C_{2+} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (\lambda I - A_{1+})^{-1} & 0 \\ 0 & (\lambda I - A_{2+})^{-1} \end{bmatrix} =$$

$$\begin{bmatrix} 0 & B_{1+}C_{2+}(\lambda - A_{2+})^{-1} \\ 0 & 0 \end{bmatrix} \text{ for } \lambda \text{ such that } \mathrm{Re}\lambda > \omega. \tag{2.2.1}$$

We can show $BR(\lambda; A)$ is a bounded linear operator on $X$ by showing $B_{1+}C_{2+}(\lambda - A_{2+})^{-1}$ is a bounded linear operator on $X_{2+}$. From Theorem 1.2.9 we know $C_{2+}(\lambda I - A_{2+})^{-1}$ is a well-defined bounded linear operator depending analytically on $\lambda$ in the strip $|Re\lambda| < -\omega$. The operator $B_{1+}$ is a bounded linear operator, hence $B_{1+}C_{2+}(\lambda I - A_{2+})^{-1}$ is a bounded linear operator on $X_{2+}$. Therefore we conclude $B \in \mathcal{T}(A)$.

Consider $BT(\xi; A)$ defined on $D(A)$ and $\xi > 0$ by

$$T(\xi; A) = \begin{bmatrix} E(\xi; A_{1+}) & 0 \\ 0 & E(\xi; A_{2+}) \end{bmatrix}.$$

Applying the operator $B$ to the semigroup produces

$$BT(\xi; A) = \begin{bmatrix} 0 & B_{1+}C_{2+}E(\xi; A_{2+}) \\ 0 & 0 \end{bmatrix}. \tag{2.2.2}$$

From Definition 1.2.5.3b

$$\Lambda_{2+}x(t) = C_{2+}E(t; A_{2+})x(t) \text{ for } x \in D(A_{2+}).$$

Definition 1.2.5.3b also implies $\Lambda_{2+}$ is a bounded extension of $E(\xi; A_{2+})$ which maps $X$ into $L^m_{1,\omega}(\Re)$ . Combining (2.2.2) with the boundedness of $\Lambda_{2+}$, we obtain the following inequality

$$\int_0^1 \| BT(\xi; A) \|_A \, d\xi \leq \int_0^1 \| B_{1+}E(\xi; A_{2+}) \| \, d\xi \leq \int_0^1 \| B_{1+}\Lambda_{2+} \| \, dt. \tag{2.2.3}$$

Increasing the limits for a positive integrand leads to

$$\int_0^1 \| B_{1+}\Lambda_{2+} \| \, dt \leq \int_0^\infty \| B_{1+}\Lambda_{2+} \| \, dt \tag{2.2.4}$$

The supremum in the definition of the norm on the right can be written as a limit of an increasing sequence of functions. The B. Levi Theorem implies that we can pull the limit outside the integral to produce

$$\int_0^\infty \| B_{1+}\Lambda_{2+} \| \, dt \leq$$

$$\sup_{\|x\| \leq 1, x \in D(A_{2+})} \int_0^\infty \| B_{1+} \| \, \| \Lambda_{2+}x(t) \| \, dt.$$

Since $1 \leq e^{-\omega t}$ for $t \in (0, \infty)$ and $B_{1+}$ is a bounded operator, we have

$$\int_0^1 \| BT(\xi; A) \|_A \, d\xi \leq \infty.$$

Thus the last inequality coming from 1.2.5.3b is satisfied yielding $B \in B(A)$. Hence by Theorem 2.1.4

$$\begin{bmatrix} A_{1+} & B_{1+}C_{2+} \\ 0 & A_{2+} \end{bmatrix}$$

generates a $C_0$-semigroup.□

By paralleling the proof of Theorem 2.2.1, we obtain

$$\begin{bmatrix} A_{1-} & B_{1-}C_{2-} \\ 0 & A_{2-} \end{bmatrix}$$

generates a $C_0$-semigroup in backward time.

Now we claim the product of two BGK-admissible realizations is BGK-admissible.

**Theorem 2.2.2** *Let $W_1(\lambda)$ and $W_2(\lambda)$ be two $m \times m$ transfer functions with BGK-admissible realizations*

$$\theta_1 = \left( \begin{bmatrix} A_{1+} & 0 \\ 0 & A_{1-} \end{bmatrix}, \begin{bmatrix} B_{1+} \\ B_{1-} \end{bmatrix}, [C_{1+}, C_{1-}] \right) \tag{2.2.5}$$

*and*

$$\theta_2 = \left( \begin{bmatrix} A_{2+} & 0 \\ 0 & A_{2-} \end{bmatrix}, \begin{bmatrix} B_{2+} \\ B_{2-} \end{bmatrix}, [C_{2+}, C_{2-}] \right). \tag{2.2.6}$$

*Then the product realization is BGK-admissible.*

Proof:

From [6] we obtain the form of the product realization. Define $\theta = \theta_1\theta_2 = (A, B, C)$ such that

$$A = \begin{bmatrix} A_{1+} & 0 & B_{1+}C_{2+} & B_{1+}C_{2-} \\ 0 & A_{1-} & B_{1-}C_{2+} & B_{1-}C_{2-} \\ 0 & 0 & A_{2+} & 0 \\ 0 & 0 & 0 & A_{2-} \end{bmatrix} \quad B = \begin{bmatrix} B_{1+} \\ B_{1-} \\ B_{2+} \\ B_{2-} \end{bmatrix} \tag{2.2.7}$$

$$C = [C_{1+} C_{1-} C_{2+} C_{2-}]$$

with $D(A) = D(A_{1+}) \oplus D(A_{1-}) \oplus D(A_{2+}) \oplus D(A_{2-})$. The domain of $A$ is dense in X.

Now we pick $X_\alpha$ and $X_\beta$ so that Definition 1.2.1 is satisfied. Take

$$X_\alpha = \mathrm{Im} \begin{bmatrix} I & 0 \\ 0 & Q_2 \\ 0 & I \\ 0 & 0 \end{bmatrix} \text{ and } X_\beta = \mathrm{Im} \begin{bmatrix} 0 & Q_1 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \tag{2.2.8}$$

where $Q_1$ is the unique solution of the Sylvester equation

$$A_{1+}Q_1x - Q_1A_{2-}x = -B_{1+}C_{2-}x \tag{2.2.9}$$

for $x \in D(A_{2-})$, and the operator $Q_2$ is the unique solution of the Sylvester equation

$$A_{1-}Q_2x - Q_2A_{2+}x = -B_{1-}C_{2+}x \tag{2.2.10}$$

for $x \in D(A_{2+})$. Define

$$S = \begin{bmatrix} I & 0 & 0 & Q_1 \\ 0 & Q_2 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}. \tag{2.2.11}$$

The operator $S$ is invertible and hence $X = X_\alpha \oplus X_\beta$.

Next we show $X_\alpha \oplus X_\beta$ reduces A. Since A is an unbounded operator we need to show $A(D(A) \cap X_\alpha) \subset X_\alpha$. This produces the requirement

$$A(D(A) \cap \mathrm{Im} \begin{bmatrix} I & 0 \\ 0 & Q_2 \\ 0 & I \\ 0 & 0 \end{bmatrix}) \subset Im \begin{bmatrix} I & 0 \\ 0 & Q_2 \\ 0 & I \\ 0 & 0 \end{bmatrix}. \tag{2.2.12}$$

The vector $\bar{x}$ being in $X_\alpha$ implies

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in Im \begin{bmatrix} I & 0 \\ 0 & Q_2 \\ 0 & I \\ 0 & 0 \end{bmatrix} \cap D(A).$$

Therefore there is $y_{1+} \in D(A_{1+}), y_{2+} \in D(A_{2+})$ such that

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & Q_2 \\ 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_{1+} \\ y_{2+} \end{bmatrix} = \begin{bmatrix} y_{1+} \\ Q_2 y_{2+} \\ y_{2+} \\ 0 \end{bmatrix}.$$

Now we apply $A$ to $\bar{x}$ to obtain

$$A\bar{x} = \begin{bmatrix} A_{1+}y_{1+} + B_{1+}C_{2+}y_{2+} \\ A_{1-}Q_2y_{2+} + B_{1-}C_{2+}y_{2+} \\ A_{2+}y_{2+} \\ 0 \end{bmatrix}.$$

Since $Q_2$ solves (2.2.10), we can replace the second row entry in $A\bar{x}$ to obtain

$$A\bar{x} = \begin{bmatrix} A_{1+}y_{1+} + B_{1+}C_{2+} \\ Q_2 A_{2+} y_{2+} \\ A_{2+} y_{2+} \\ 0 \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & Q_2 \\ 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{1+}y_{1+} + B_{1+}C_{2+}y_{2+} \\ A_{2+}y_{2+} \end{bmatrix}, \tag{2.2.13}$$

hence $A\bar{x} \in X_\alpha$ and (2.2.12) is satisfied.

Next we show $A(X_\beta \cap D(A)) \subset X_\beta$. The vector $\bar{x}$ being in $X_\beta$ implies

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in X_\beta \cap D(A).$$

Therefore there is $y_{1-} \in D(A_{1-})$ and $y_{2-} \in D(A_{2-})$ such that

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & Q_1 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} y_{1-} \\ y_{2-} \end{bmatrix} = \begin{bmatrix} Q_1 y_{2-} \\ y_{1-} \\ 0 \\ y_{2-} \end{bmatrix}.$$

Applying the operator $A$ to $\bar{x}$ produces

$$A\bar{x} = \begin{bmatrix} A_{1+}Q_1 y_{2-} + B_{1+}C_{2-}y_{2-} \\ A_{1-}y_{1-} + B_{1-}C_{2-}y_{2-} \\ 0 \\ A_{2-}y_2 \end{bmatrix}.$$

Since $Q_1$ solves (2.2.9), we replace the first row entry in $A\bar{x}$ to obtain

$$A\bar{x} = \begin{bmatrix} Q_1 A_{2-} y_{2-} \\ A_{1-}y_{1-} + B_{1-}C_{2-}y_{2-} \\ 0 \\ A_{2-}y_{2-} \end{bmatrix} = \begin{bmatrix} 0 & Q_1 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{1-}y_{1-} + B_{1-}C_{2-}y_{2-} \\ A_{2-}y_{2-} \end{bmatrix}. \tag{2.2.14}$$

Hence $A\bar{x} \in X_\beta$ satisfies $A(X_\beta \cap D(A)) \subset X_\beta$. Thus $X_\beta \oplus X_\beta$ reduces A.

Factoring the column vector $\begin{bmatrix} y_{1+} \\ y_{2+} \end{bmatrix}$ to the right in (2.2.13) leads to

$$A\bar{x} = \begin{bmatrix} I & 0 \\ 0 & Q_2 \\ 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{1+} & B_{1+}C_{2+} \\ 0 & A_{2+} \end{bmatrix} \begin{bmatrix} y_{1+} \\ y_{2+} \end{bmatrix} \tag{2.2.15}$$

for $\bar{x} \in X_\alpha \cap D(A)$. Factoring the column vector $\begin{bmatrix} y_{1-} \\ y_{2-} \end{bmatrix}$ to the right in (2.2.14) results in

$$A\bar{x} = \begin{bmatrix} 0 & Q_1 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{1-} & B_{1-}C_{2-} \\ 0 & A_{2-} \end{bmatrix} \begin{bmatrix} y_{1-} \\ y_{2-} \end{bmatrix} \tag{2.2.16}$$

for $\bar{x} \in X_\beta \cap D(A)$.

Now when we consider $A$ restricted to $X_\alpha$ ,we obtain from (2.2.15)

$$A|_{X_\alpha} = \begin{bmatrix} I & 0 \\ 0 & Q_2 \\ 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{1+} & B_{1+}C_{2+} \\ 0 & A_{2+} \end{bmatrix}$$

where

$$A_+ = \begin{bmatrix} A_{1+} & B_{1+}C_{2+} \\ 0 & A_{2+} \end{bmatrix}, \tag{2.2.17}$$

but Theorem 2.2.1 gives that this generates a $C_0$-semigroup for $t > 0$. The operator $A$ restricted to $X_\beta$ from (2.2.16) is

$$A|_{X_\beta} = \begin{bmatrix} 0 & Q_1 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{1-} & B_{1-}C_{2-} \\ 0 & A_{2-} \end{bmatrix}$$

where

$$A_- = \begin{bmatrix} A_{1-} & B_{1-}C_{2-} \\ 0 & A_{2-} \end{bmatrix} \tag{2.2.18}$$

which generates a backward time $C_0$-semigroup. Hence by Definition 1.2.1, A is dichotomous. Thus we have satisfied 1 of Definition 1.2.5.

It is clear $D(C) \supset D(A)$ since $D(C_{1+}) \supset D(A_{1+})$, $D(C_{1-}) \supset D(A_{1-})$, $D(C_{2+}) \supset D(A_{2+})$ and $D(C_{2-}) \supset D(A_{2+})$. Thus we have satisfied 2 of Definition 1.2.5.

Next we would like to calculate $(\lambda I - A)^{-1}$. The similarity transformation $S^{-1}AS$ produces a block diagonal structure that is easy to invert. The operator $S^{-1}$ is given by

$$S^{-1} = \begin{bmatrix} I & 0 & 0 & -Q_1 \\ 0 & 0 & I & 0 \\ 0 & I & -Q_2 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \tag{2.2.19}$$

then

$$S^{-1}AS = \begin{bmatrix} A_+ & 0 \\ 0 & A_- \end{bmatrix}. \tag{2.2.20}$$

Calculating the similarity transform of the resolvent of $A$ produces

$$S^{-1}(\lambda I - A)^{-1}S = \tag{2.2.21}$$

$$\begin{bmatrix} (\lambda I - A_+)^{-1} & 0 \\ 0 & (\lambda I - A_-)^{-1} \end{bmatrix} \tag{2.2.22}$$

where

$$(\lambda I - A_+)^{-1} = \begin{bmatrix} (\lambda I - A_{1+})^{-1} & (\lambda I - A_{1+})^{-1}B_{1+}C_{2+}(\lambda I - A_{2+})^{-1} \\ 0 & (\lambda I - A_{2+})^{-1} \end{bmatrix}$$

and

$$(\lambda I - A_-)^{-1} = \begin{bmatrix} (\lambda I - A_{1-})^{-1} & (\lambda I - A_{1-})^{-1}B_{1-}C_{2-}(\lambda I - A_{2-})^{-1} \\ 0 & (\lambda I - A_{2-})^{-1} \end{bmatrix}.$$

The resolvent $(\lambda I - A)^{-1}$ can then be obtained by premultiplying (2.2.21) by S and post-multiplying by $S^{-1}$ to obtain

$$\begin{bmatrix} (\lambda I - A_1)^{-1} & E \\ 0 & (\lambda I - A_2)^{-1} \end{bmatrix}$$

where E is the following matrix

$$\begin{bmatrix} (\lambda I - A_{1+})^{-1}B_{1+}C_{2+}(\lambda I - A_{2+})^{-1} & Q_1(\lambda I - A_{2-})^{-1} - (\lambda I - A_{1+})^{-1}Q_1 \\ Q_2(\lambda I - A_{2+})^{-1}Q_2 & (\lambda I - A_{1-})^{-1}B_{1-}C_{2-}(\lambda I - A_{2-})^{-1} \end{bmatrix}. \tag{2.2.23}$$

If we take the Sylvester equation (2.2.9) and identify $x \in D(A_{2-})$ as $x = (\lambda I - A_{2-})^{-1}z$ for $z \in X_{2-}$, this leads to

$$(\lambda I - A_{1+})Q_1(\lambda I - A_{2-})^{-1}z - Q_1 z = B_{1+}C_{2-}(\lambda I - A_{2-})^{-1}z. \tag{2.2.24}$$

Now for the Sylvester equation (2.2.9), identifying $x = (\lambda I - A_{2+})^{-1}y$ for $y \in X_{2+}$ yields

$$(\lambda I - A_{1-})Q_2(\lambda I - A_{2+})^{-1}y - Q_2 y = B_{1-}C_{2+}(\lambda I - A_{2+})^{-1}y. \tag{2.2.25}$$

Premultiplying (2.2.24) by $(\lambda I - A_{1+})^{-1}$ and (2.2.25) by $(\lambda I - A_{1-})^{-1}$, we obtain

$$Q_1(\lambda I - A_{2-})^{-1}z - (\lambda I - A_{1+})^{-1}Q_1 z = (\lambda I - A_{1+})^{-1}B_{1+}C_{2-}(\lambda I - A_{2-})^{-1}z$$

for $z \in X_{2-}$ and

$$Q_2(\lambda I - A_{2+})^{-1}y - (\lambda I - A_{1-})^{-1}Q_2 y = (\lambda I - A_{1-})^{-1}B_{1-}C_{2+}(\lambda I - A_{2+})^{-1}y$$

for $y \in X_{2+}$. Substituting these into (2.2.23) produces

$$E = \begin{bmatrix} (\lambda I - A_{1+})^{-1}B_{1+}C_{2+}(\lambda I - A_{2+})^{-1} & (\lambda I - A_{1+})^{-1}B_{1+}C_{2-}(\lambda I - A_{2-})^{-1} \\ (\lambda I - A_{1-})^{-1}B_{1-}C_{2+}(\lambda I - A_{2+})^{-1} & (\lambda I - A_{1-})^{-1}B_{1-}C_{2-}(\lambda I - A_{2-})^{-1} \end{bmatrix}.$$

We can now use this form to construct the transfer function for the product system and formally the form of $\Lambda$.

Now we construct $\Lambda$ to satisfy Definition 1.2.5.3;

$$\Lambda = [\Lambda_{1+},\ \Lambda_{1-},\ -k_{1+} * \Lambda_{2+} - k_{1-} * \Lambda_{2+} + \Lambda_{2+},\ -k_{1+} * \Lambda_{2-} - k_{1-} * \Lambda_{2-} + \Lambda_{2-}].$$

The operator $\Lambda$ is defined on $X_\alpha \oplus X_\beta$. The vector $x_1$ is in $X_\alpha$ if there exist $x_{1+} \in X_{1+}$ such that

$$x_1 = \begin{bmatrix} I & 0 \\ 0 & Q \\ 0 & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{1+} \\ x_{2+} \end{bmatrix} = \begin{bmatrix} x_{1+} \\ Q_2 x_{2+} \\ x_{2+} \\ 0 \end{bmatrix}. \tag{2.2.26}$$

The vector $x_2$ is in $X_\beta$ if there exist $x_{1-} \in X_{1-}$ and $x_{2-} \in X_{2-}$ such that

$$x_2 = \begin{bmatrix} 0 & Q_1 \\ I & 0 \\ 0 & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} x_{1-} \\ x_{2-} \end{bmatrix} = \begin{bmatrix} Q_1 x_{2-} \\ x_{1-} \\ 0 \\ x_{2-} \end{bmatrix}. \tag{2.2.27}$$

Forming the vector $\bar{x} = x_1 + x_2$ yields

$$\bar{x} = \begin{bmatrix} x_{1+} + Q_1 x_{2-} \\ Q_2 x_{2+} + x_{1-} \\ x_{2+} \\ x_{2-} \end{bmatrix}.$$

Applying $\Lambda$ to $\bar{x}$ we obtain

$$\begin{aligned} \Lambda\bar{x} = & \\ & \Lambda_{1+}x_{1+} + \Lambda_{1+}Q_1 x_{2-} + \Lambda_{1-}Q_2 x_{2+} + \Lambda_{1-}x_{1-} \\ & -k_{1+} * \Lambda_{2+}x_{2+} - k_{1-} * \Lambda_{2+}x_{2+} + \Lambda_{2+}x_{2+} \\ & -k_{1+} * \Lambda_{2-}x_{2-} - k_{1-} * \Lambda_{2-}x_{2-} + \Lambda_{2-}x_{2-}. \end{aligned} \tag{2.2.28}$$

The compositions in (2.2.28) are bounded by Lemma (1.3.1), hence they satisfy Definition 1.2.5.3a. Since $Q_1$ and $Q_2$ are bounded operators and the $\Lambda$ operators come from BGK-admissible realization, the remaining terms satisfy Definition 1.2.5.3a.

We now calculate the bilateral Laplace transform of $\Lambda\bar{x}(t)$ from (2.2.28) producing

$$\begin{aligned}
\hat{\Lambda}x = & \\
& \hat{\Lambda}_{1+}x_{1+}(\lambda) + \hat{\Lambda}_{1-}Q_2x_{2+}(\lambda) \\
& -\hat{k}_{1+}\hat{\Lambda}_{2+}x_{2+}(\lambda) - \hat{k}_{1-}\hat{\Lambda}_{2+}x_{2+}(\lambda) \\
& +\hat{\Lambda}_{1-}x_{1-}(\lambda) + \hat{\Lambda}_{1+}Q_1x_{2-}(\lambda) + \hat{k}_{1+}\hat{\Lambda}_{2-}x_{2-}(\lambda) \\
& -\hat{k}_{1-}\hat{\Lambda}_{2-}x_{2-}(\lambda) + \hat{\Lambda}_{2-}x_{2-}(\lambda).
\end{aligned} \tag{2.2.29}$$

From Theorem 1.2.8 and Definition 1.2.5.3b for $\theta_{1+}$, $\theta_{2+}$, $\theta_{1-}$ and $\theta_{2-}$,

$$\hat{\Lambda}_{1+}x_{1+}(\lambda) = -C_{1+}(\lambda I - A_{1+})^{-1}x_{1+}(\lambda),$$

$$\hat{\Lambda}_{2+}x_{2+}(\lambda) = -C_{2+}(\lambda I - A_{2+})^{-1}x_{2+}(\lambda),$$

$$\hat{\Lambda}_{1-}x_{1-}(\lambda) = -C_{1-}(\lambda I - A_{1-})^{-1}x_{1-}(\lambda),$$

$$\hat{\Lambda}_{2-}x_{2-}(\lambda) = -C_{2-}(\lambda I - A_{2-})^{-1}x_{2-}(\lambda,$$

$$\hat{k}_{1+}\hat{\Lambda}_{2+}x_{2+}(\lambda) = C_{1+}(\lambda I - A_{1+})^{-1}B_{1+}C_{2+}(\lambda I - A_{2+})^{-1}x_{2+}(\lambda),$$

$$\hat{k}_{1-}\hat{\Lambda}_{2+}x_{2+}(\lambda) = C_{1-}(\lambda I - A_{1-})^{-1}B_{1-}C_{2+}(\lambda I - A_{2+})^{-1}x_{2+}(\lambda),$$

$$\hat{k}_{1+}\hat{\Lambda}_{2-}x_{2-}(\lambda) = C_{1+}(\lambda I - A_{1+})^{-1}B_{1+}C_{2-}(\lambda I - A_{2-})^{-1}x_{2-}(\lambda),$$

$$\hat{k}_{1-}\hat{\Lambda}_{2-}x_{2-}(\lambda) = C_{1-}(\lambda I - A_{1-})^{-1}B_{1-}C_{2-}(\lambda I - A_{2-})^{-1}x_{2-}(\lambda).$$

Making the above substitution into (2.2.29) together with the fact $Q_1 : X_{2-} \to X_{1+}$ and $Q_2 : X_{2+} \to X_{1-}$ results in

$$\begin{aligned}
\hat{\Lambda}Sx(\lambda) = \quad & -C_{1+}(\lambda I - A_{+1})^{-1}x_{1+}(\lambda) - C_{1-}(\lambda I - A_{1-})^{-1}Q_2x_{2+}(\lambda) \\
& -C_{1+}(\lambda I - A_{1+})^{-1}B_{1+}C_{2+}(\lambda I - A_{2+})^{-1}x_{2+}(\lambda) \\
& -C_{1-}(\lambda I - A_{1-})^{-1}x_{1-}(\lambda) - C_{1+}(\lambda I - A_{1+})^{-1}Q_1x_{2-}(\lambda) \\
& -C_{1+}(\lambda I - A_{1+})^{-1}B_{1+}C_{2-}(\lambda I - A_{2+})^{-1}x_{2-}(\lambda) \\
& -C_{1-}(\lambda I - A_{1-})^{-1}B_{1-}C_{2-}(\lambda I - A_{2-})^{-1}x_{2-}(\lambda)
\end{aligned}$$

After algebraic manipulation, we obtain

$$\hat{\Lambda}Sx(\lambda) = -C(\lambda I - A)^{-1}Sx(\lambda).$$

Hence

$$\hat{\Lambda}x(\lambda) = -C(\lambda I - A)^{-1}x(\lambda)$$

for $| Re\lambda |< -\omega$ and $x(\lambda) \in \hat{X}_+ \oplus \hat{X}_-$.

Take $x \in D(A)$ and write $x = A^{-1}z$ then $C(\lambda I - A)^{-1}x = CA^{-1}(\lambda I - A)^{-1}z$, where $CA^{-1}$ is a bounded linear operator from $X$ into $\mathbf{C}^m$. From (1.5) in [9]

$$\hat{\Lambda}x(\lambda) = -CA_p^{-1}\int_{-\infty}^{\infty} e^{\lambda t}E(t;A)zdt = -\int_{-\infty}^{\infty} e^{\lambda t}CE(t;A)xdt$$

Hence $(\Lambda x)(t) = CE(t;A)x$ for all $x \in D(A)$. We need to show $\Lambda_\theta$ takes $D(A)$ into $D_1^m(\Re)$. We know the $\Lambda$ operators take $D(A)$ into $D_1^m(\Re)$ since they are from BGK-admissible realizations. By Lemma 1.3.2 the convolutions are in $D_1^m(\Re)$ for x out of D(A). The operator $Q_1$ maps $D(A_{2+})$ into $D(A_{1+})$ and the operator $Q_2$ maps $D(A_{2+})$ into $D(A_{1-})$, therefore $\Lambda$ satisfies Definition 1.2.5.3b. Thus we have proved that the product realization of two BGK-admissible realizations is BGK-admissible.□

# Chapter 3

# Regular Systems

## 3.1 Introduction

In this section we introduce a class of linear systems formalized by Weiss [40]-[43] called regular systems; this class is roughly the largest class of linear systems which have transfer functions and a realization theory analogous to the finite dimensional case. Preliminary versions of the ideas involved have appeared in work by Helton [24], Salamon [37], Pritchard and Salamon [33] and Ho and Russell [27].

## 3.2 Elementary Properties

We wish to consider a large subclass of abstract linear systems namely the class of regular systems introduced by Weiss. First we introduce concatenation on $L^2_{loc}([0,\infty),W)$, where W is a Hilbert space. Let $u,v \in L^2_{loc}([0,\infty),W)$ and let $\tau \geq 0$, then the $\tau$-concatenation of u and v, denoted $u\diamond_\tau v$, is an element of $L^2_{loc}([0,\infty),W)$ defined by

$$(u\diamond_\tau v)(t) = \begin{cases} u(t), & \text{for } t \in [0,\tau), \\ v(t-\tau), & \text{for } t \geq \tau. \end{cases}$$

Now we give the formal definition of an abstract linear system in the Hilbert space context.

**Definition 3.2.1** *Let $U$, $X$, $Y$ be Hilbert spaces, $\Omega = L^2([0,\infty), U)$ and $\Gamma = L^2([0,\infty), Y)$. An abstract linear system on $\Omega$, $X$ and $\Gamma$ is a quadruple $\Sigma = (\mathbf{T}, \mathbf{\Phi}, \mathbf{L}, \mathbf{F})$, where*

1. *$\mathbf{T} = (\mathbf{T}_t)_{t\geq 0}$ is a strongly continuous semigroup of bounded linear operators on $X$,*

2. *$\mathbf{\Phi} = (\mathbf{\Phi}_t)_{t\geq 0}$ is a family of bounded linear operators from $\Omega$ to $X$ such that*

$$\mathbf{\Phi}_{\tau+t}(u \diamond_\tau v) = \mathbf{T}_t \mathbf{\Phi}_\tau u + \mathbf{\Phi}_t v, \tag{3.2.1}$$

   *for any $u, v \in \Omega$ and any $\tau, t \geq 0$,*

3. *$\mathbf{L} = (\mathbf{L}_t)_{t\geq 0}$ is a family of bounded linear operators from $X$ to $\Gamma$ such that*

$$\mathbf{L}_{\tau+t} x = \mathbf{L}_\tau x \diamond_\tau \mathbf{L}_t \mathbf{T}_\tau x, \tag{3.2.2}$$

   *for any $x \in X$ and any $\tau, t \geq 0$, and $\mathbf{L}_0 = 0$,*

4. *$\mathbf{F} = (\mathbf{F}_t)_{t\geq 0}$ is a family of bounded linear operators from $\Omega$ to $\Gamma$ such that*

$$\mathbf{F}_{\tau+t}(u \diamond_\tau v) = \mathbf{F}_\tau u \diamond_\tau (\mathbf{L}_t \mathbf{\Phi}_\tau u + \mathbf{F}_t v), \tag{3.2.3}$$

   *for any $u, v \in \Omega$ and any $\tau, t \geq 0$, and $\mathbf{F}_0 = 0$.*

We will refer to $\mathbf{T}$ as the semigroup, $\mathbf{\Phi}$ as the input map, $\mathbf{L}$ as the output map and $\mathbf{F}$ as the input-output map.

The motivating example for this definition of an abstract linear system is the finite dimensional case which is characterized by the system of linear equations

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0 \\ y &= Cx + Du \end{aligned}$$

where $A$, $B$, $C$, $D$ are finite matrices of appropriate sizes. The associated abstract linear system is given by $\Sigma = (\mathbf{T}, \mathbf{\Phi}, \mathbf{L}, \mathbf{F})$ where

the semigroup is

$$\mathbf{T}_t = e^{At},$$

the input map is

$$\Phi_\tau u = \int_0^\tau e^{A(\tau-\sigma)} Bu(\sigma)\, d\sigma,$$

the output map is

$$(\mathbf{L}_\tau x)(t) = Ce^{At}x, \text{ for } t \in [0,\tau),$$

and the input-output map is

$$(\mathbf{F}_\tau u)(t) = C\int_0^t e^{A(t-\sigma)} Bu(\sigma)\, d\sigma + Du(t), \text{ for } t \in [0,\tau).$$

We will make use of the spaces $X_1$ and $X_{-1}$, as introduced in Salamon [36] or Weiss [41], to extend the above definition to infinite dimensional systems. They are defined as follows. Let $\mathbf{T}$ be a semigroup on $X$ with generator $A$.

**Definition 3.2.2** *The space $X_1$ is $D(A)$ with norm $\| x \|_1 = \| (\beta I - A)x \|$, where $\beta \in \rho(A)$ is fixed.*

The $\| \cdot \|_1$ norm is equivalent to the graph norm.

**Definition 3.2.3** *The space $X_{-1}$ is defined as the completion of $X$ with respect to the norm $\| x \|_{-1} = \| (\beta I - A)^{-1}x \|$, $\beta \in \rho(A)$ fixed.*

The restriction $\mathbf{T}$ to $X_1$, is a semigroup isomorphic to the original semigroup, and $\mathbf{T}$ has an extension to $X_{-1}$ which is also isomorphic to the original semigroup. We will use the same symbol to denote all three semigroups. The semigroup on $X_{-1}$ has a generator that is an extension of $A \in L(X_1, X)$.

Next we define what will be an admissible control operator and an admissible observation operator for the semigroup $\mathbf{T}$.

**Definition 3.2.4** *Suppose that $X$ and $Y$ are Hilbert spaces and $\mathbf{T}$ is a $C_0$-semigroup of operators on $X$ and the space $X_1$ is as defined in Definition 3.2.3. An operator $C \in L(X_1, Y)$ is an admissible observation operator for $\mathbf{T}$ if for some (and hence any) $\tau > 0$, the operator $\mathbf{L}_\tau : X_1 \to L^2([0,\infty), Y)$ defined by*

$$(\mathbf{L}_\tau x)(t) = \begin{cases} C\mathbf{T}_t x, & \text{for } 0 \le t < \tau, \\ 0, & \text{for } t \ge \tau, \end{cases} \tag{3.2.4}$$

*for $x \in X_1$, has a continuous extension to $X$.*

**Definition 3.2.5** *(see [41]) Suppose that $U$ and $W$ are Hilbert spaces and $\mathbf{T}$ is a strongly continuous semigroup of operators on $X$ and space $X_{-1}$ as defined in Definition 3.2.3. An operator $B \in L(U, X_{-1})$ is an admissible control operator for $\mathbf{T}$ if for some (and hence for any) $\tau > 0$, the operator $\mathbf{\Phi}_\tau : L^2([0,\infty), U) \to X_{-1}$ defined by*

$$\mathbf{\Phi}_\tau u = \int_0^\tau \mathbf{T}_{\tau-\sigma} B u(\sigma)\, d\sigma \tag{3.2.5}$$

*has its range in $X$.*

Briefly we mention the infinite time extensions of $\mathbf{L}_\tau$ and $\mathbf{F}_\tau$. We consider $\tilde{\Omega} = L^2_{loc}([0,\infty), U)$ as a Frechét space with the family of seminorms $p_n(u) = \| P_n u \|_{L^2}$, $n \in \mathbf{N}$ where $P_n u$ is the projection defined by

$$(P_\tau u)(t) = \begin{cases} u(t), & \text{for } t \in [0,\tau), \\ 0, & \text{for } t \geq \tau. \end{cases} \tag{3.2.6}$$

From Definition 3.2.1 we can derive the following formulas from the concept of causality:

$$\mathbf{\Phi}_\tau P_\tau = \mathbf{\Phi}_\tau \mathbf{F}_\tau P_\tau \tag{3.2.7}$$

for any $\tau \geq 0$. The operators $\mathbf{\Phi}_\tau$ and $\mathbf{F}_\tau$ which are defined on $\Omega$, can be extended to $\tilde{\Omega}$ by continuity using (3.2.7) to define the extensions. We will use the same symbols to denote the extensions. Again from [42], let the space $\tilde{\Gamma} = L^2_{loc}([0,\infty), Y)$ be a Frechét space defined similar to $\tilde{\Omega}$, then the following limits exist in $\tilde{\Gamma}$, for any $x \in X$ and any $u \in \tilde{\Omega}$:

$$\mathbf{L}_\infty x = \lim_{\tau\to\infty} \mathbf{L}_\tau x,$$

$$\mathbf{F}_\infty u = \lim_{\tau\to\infty} \mathbf{F}_\tau u.$$

Now we give a form for $\mathbf{L}_\infty$ in terms of an observation operator. The formula (3.2.4) for $\mathbf{L}$ makes sense only for $x \in X_1$; our goal is to give an expression for $\mathbf{L}_\infty$ applied to a general element $x \in X$. We extend our representation (3.2.4) of $L$ by using the

Lebesgue extension of C as introduced in Weiss [40]. The Lebesgue extension of an operator $C \in L(X_1, Y)$ is defined by

$$C_L x = \lim_{\tau \to 0} C \frac{1}{\tau} \int_0^\tau \mathbf{T}_\sigma x \, d\sigma, \tag{3.2.8}$$

with domain $D(C_L) = \{x \in X : \text{ limit in (3.2.8 ) exists}\}$. The space $D(C_L)$ with norm

$$\| x \|_{D(C_L)} = \| x \| + \sup_{\tau \in (0,1)} \| C \frac{1}{\tau} \int_0^\tau \mathbf{T}_\sigma x \, d\sigma \|$$

is a Banach space. We have $X_1 \subset D(C_L) \subset X$ with continuous embeddings, and $C_L \in L(D(C_L), Y)$. For any $x \in X$ and any $\tau \geq 0$, we have that $\mathbf{T}_\tau x \in D(C_L)$ if and only if $\mathbf{L}_\infty x$ has a Lebesgue point at $\tau$, hence $\mathbf{L}_\infty$ has the following representation:

$$(\mathbf{L}_\infty x)(t) = C_L \mathbf{T}_t x, \text{ for a.e. } t \geq 0. \tag{3.2.9}$$

From [42] if we have a regularity assumption, then the operator $C_L(\beta I - A)^{-1} B$ makes sense and is an element of $L(X, Y)$ for $\beta \in \rho(A)$. We obtain a representation of $\mathbf{F}_\infty$ as

$$(\mathbf{F}_\infty u)(t) = C_L \int_0^\tau \mathbf{T}_{t-\sigma} B u(\sigma) \, d\sigma + D u(t), \tag{3.2.10}$$

for any $u \in \tilde{\Omega}$ and a.e. $t \geq 0$, just as for finite dimensional systems. Let us now give the required regularity assumption. For any $v \in U$, let $\chi_v$ denote the constant function on $[0, \infty)$ equal to $v$ everywhere, so $\chi_v \in \tilde{\Omega}$. The function

$$y_v = \mathbf{F}_\infty \chi_v \tag{3.2.11}$$

will be called the step response of $\Sigma$ corresponding to $v$.

**Definition 3.2.6** *(see [42]) Let $\Sigma$ be an abstract linear system, with input space $U$ and output space $Y$. We say $\Sigma$ is regular if for any $v \in U$, the corresponding step response $y_v$ (given by (3.2.11)) has a Lebesgue point at 0 (i.e. the following limit exists in $Y$).*

$$D v = \lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau y_v(\sigma) \, d\sigma. \tag{3.2.12}$$

*In that case, the operator $D$ defined by (3.2.12) is called the feedthrough operator of the system $\Sigma$.*

Most systems arising in practice (for example from PDE's) actually satisfy a much stronger condition than (3.2.12) namely $\lim_{\tau \to 0} \chi_v(\tau)$ exists for any $v \in U$. The next proposition gives further insight into the properties of regular systems.

**Proposition 3.2.7** *(see [42]) Let $\Sigma = (\mathbf{T}, \mathbf{\Phi}, \mathbf{L}, \mathbf{F})$ be an abstract linear system, with input space $U$, state space $X$, and output space $Y$. Let $A$ be the generator of $\mathbf{T}$, let $B$ be the control operator of $\Sigma$ and let $C$ be the observation operator of $\Sigma$. We denote by $C_L$ the Lebesgue extension of $C$. The following conditions are equivalent:*

1. *$\Sigma$ is regular,*
2. *for some $s \in \rho(A)$ and any $v \in U$, $(sI - A)^{-1}Bv \in D(C_L)$,*
3. *$C_L(sI - A)^{-1}B$ is an analytic $L(U, Y)$-valued function of $s$ on $\rho(A)$.*

Finally from Weiss [42], we have that the feedthrough operator $D$ is an element of $L(U, Y)$.

## 3.3 Regular Admissible Triples

This section considers under what conditions does the triple (A,B,C) form a regular system. In ([40]-[43]) Weiss assumes that an abstract linear system is given and that one obtains C and B from formulas contained in [40] and [41] respectively. Curtain and Weiss in [18] give sufficient conditions for a triple (A,B,C) to form an abstract linear system; we will call such a triple an admissible triple. The following is an adaptation of the result of Curtain and Weiss.

**Theorem 3.3.1** *If $U$, $X$ and $Y$ are Hilbert spaces, $\omega$ is a real number less than zero and (A,B,C) is a triple of operators such that:*

1. *$A$ is the generator of a strongly continuous semigroup $\mathbf{T}$ on $X$ such that $\mathbf{T}$ determines two new Hilbert spaces, $X_1$ and $X_{-1}$ given by Definitions 3.2.2 and 3.2.3,*
2. *$B \in L(U, X_{-1})$ is an admissible control operator for $\mathbf{T}$ on $X$ (see Definition 3.2.5),*
3. *$C \in L(X_1, Y)$ is an admissible observation operator for $\mathbf{T}$ (see Definition 3.2.4),*

*4. The Lebesgue extension $C_L$ of $C$ exists, and the expression $C_L(\lambda I - A)^{-1}B \in L(U,Y)$ for $\lambda \in \rho(A)$ and is bounded as a function of $\lambda$ in some right half plane $\{\lambda : Re\lambda > -\omega\}$.*

*Then (A,B,C) is a regular admissible triple.*

Before we prove this, we need a definition from Curtain's paper on representations of infinite dimensional linear systems [15].

**Definition 3.3.2** *Let U, X and Y be Hilbert spaces and let $\mathbf{T} = (\mathbf{T}_\tau)_{\tau \geq 0}$ be a $C_0$-semigroup on X with appropriate spaces $X_1$ and $X_{-1}$. Let $B \in L(U, X_{-1})$ and $C \in L(X_1, Y)$. Suppose B is an admissible control operator for $\mathbf{T}$ and that C is an admissible observation operator for $\mathbf{T}$. Then we define the transfer function for the triple (A,B,C) to be a solution $H : \rho(A) \to L(U,Y)$ of*

$$\frac{H(s) - H(\beta)}{s - \beta} = -C(sI - A)^{-1}(\beta I - A)^{-1}B \tag{3.3.13}$$

*for $s, \beta \in \rho(A)$, and $s \neq \beta$.*

Note that $H$ is determined only up to an additive constant.

Proof of Theorem 3.3.1:

Conditions (1)-(3) coincide with the first three hypothesis of Theorem 5.1 from [18]. We verify that $(A, B, C)$ generates an abstract linear system by showing that our condition (4) implies condition (iv) in Theorem 5.1 of [18], namely that there exists an $\alpha \in \Re$ such that some (and hence any) solution $H : \rho(A) \to \mathcal{L}(U,Y)$ of equation (3.3.13) is bounded on $\{s : Re\, s > \alpha\}$. Finally, condition (4) combined with Proposition 3.2.7 implies that $(A, B, C)$ is regular.

Let H be the transfer function of the triple (A,B,C). From Definition 3.3.2 we have that H solves

$$\frac{H(s) - H(\beta)}{s - \beta} = -C(sI - A)^{-1}(\beta I - A)^{-1}B \tag{3.3.14}$$

for $s, \beta \in \rho(A)$ and $s \neq \beta$. Choose s with $Re\, s > -\omega$ and fix $\beta$. Now solve (3.3.14) for $H(s)$ to obtain

$$H(s) = (\beta - s)C(sI - A)^{-1}(\beta I - A)^{-1}B + H(\beta). \tag{3.3.15}$$

Now consider $H(s)$ applied to $u \in U$:

$$H(s)u = (\beta - s)C(sI - A)^{-1}(\beta I - A)^{-1}Bu + H(\beta)u.$$

The double resolvent expression $(sI - A)^{-1}(\beta I - A)^{-1}Bu$ is in $X_1$, hence it is in the domain of C. If the operator $C_L$ is the Lebesgue extension of C, then we replace C by $C_L$ in (3.3.15) to give

$$H(s)u = (\beta - s)C_L(sI - A)^{-1}(\beta I - A)^{-1}Bu + H(\beta)u.$$

We insert the double resolvent identity to obtain

$$H(s)u = C_L[(sI - A)^{-1} - (\beta I - A)^{-1}]Bu + H(\beta)u.$$

Multiply through by $Bu$ to produce

$$H(s)u = C_L[(sI - A)^{-1}Bu - (\beta I - A)^{-1}Bu] + H(\beta)u.$$

But property (4) guaranties that $C_L(\lambda I - A)^{-1}Bu$ makes sense for all $\lambda \in \rho(A)$, hence we can apply $C_L$ over the difference:

$$H(s)u = C_L(sI - A)^{-1}Bu - C_L(\beta I - A)^{-1}Bu + H(\beta)u.$$

Since $\beta$ is a fixed value we have that $D = -C_L(\beta I - A)^{-1}B + H(\beta)$ is a bounded (constant) operator from U to Y. The operator $C_L(sI - A)^{-1}Bu$ is a bounded function of s on the right half plane $\{s : Re\, s > -\omega\}$, hence $H(s)$ is a bounded function on the right half plane $\{s : Re\, s > -\omega\}$. This verifies condition (iv) of Theorem 5.1 from [18] as required. Thus the triple $(A, B, C)$ generates an abstract linear system. Finally, condition (4) also asserts that we satisfy (2) of Proposition 3.2.7; this implies that the triple generates a regular system.

## 3.4 Additive Decomposition of the Product realization of a Rational Stable and Antistable Regular System

In this section we present some partial results on the realization of the cascade product of a rational stable and antistable regular system as a regular system. Let

$(A_+, B_+, C_+)$ be a rational stable admissible triple (i.e. $\sigma(A_+) \subset \{\lambda : Re\lambda < \omega_+\}$). Let $(A_-, B_-, C_-)$ be an antistable regular admissible triple (i.e. $\sigma(A_-) \subset \{\lambda : Re\lambda > -\omega\}$) in the sense that

$$(\lambda I + A_-)^{-1} B_- u \in D(C_{-L})$$

for $u \in U$ with $-A_-$ the generator of a $C_0$-semigroup of exponential type $\omega_-$. Take $\omega = \max(\omega_-, \omega_+)$. Let both systems have feedthrough operators equal to $I$. We now calculate the product realization of the antistable and stable system. From [6] we obtain the formal realization of the product

$$A_p = \begin{bmatrix} A_- & B_-C_+ \\ 0 & A_+ \end{bmatrix}, B_p = \begin{bmatrix} B_- \\ B_+ \end{bmatrix} C_p = [C_-, C_+]$$

We make rigorous sense of this formal realization by obtaining an additive stable-antistable decomposition. To do this we must split $A_p$ into a stable and an antistable part.

We first proceed formally as if all operators were bounded. From the upper triangular structure of $A_p$, it is easy to identify the spectral subspace associated with the right half plane as $X_\alpha = \mathrm{Im} \begin{bmatrix} I \\ 0 \end{bmatrix}$. The spectral subspace of the left half plane must be complementary to $X_\alpha$, therefore it must be of the form $X_\beta = \mathrm{Im} \begin{bmatrix} Q \\ I \end{bmatrix}$. Form $\bar{A}$ as $\begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix} A_p \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix}$. Substituting $A_p$ into this form produces

$$\bar{A} = \begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix} \begin{bmatrix} A_- & B_-C_+ \\ 0 & A_+ \end{bmatrix} \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix}.$$

Postmultiply $A_p$ by $\begin{bmatrix} I & Q \\ 0 & I \end{bmatrix}$:

$$\bar{A} = \begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix} \begin{bmatrix} A_- & A_-Q + B_-C_+ \\ 0 & A_+ \end{bmatrix}.$$

Further manipulation yields

$$\bar{A} = \begin{bmatrix} A_- & A_-Q + B_-C_+ - QA_+ \\ 0 & A_+ \end{bmatrix}.$$

Hence we can diagonalize $A_p$ if Q satisfies the Sylvester equation

$$QA_+z - A_-Qz = B_-C_+z \tag{3.4.16}$$

for $z \in X_+$. From the fact that the spectra of $A_+$ and $A_-$ are disjoint, we have that Q exists in the form $Qz = -\int_0^\infty \mathbf{T}_t^- B_-C_+\mathbf{T}_t^+ z\, dt$ for $z \in X_+$. We rewrite (3.4.16) in an alternate form. Let $| Re\lambda |< -\omega$ then (3.4.16) becomes

$$Q(\lambda I - A_+)z - (\lambda I + A_-)Qz = -B_-C_+z \tag{3.4.17}$$

for $z \in X_+$. By premultiplying both sides of (3.4.17) by $(\lambda I + A_-)^{-1}$ and identifying $z = (\lambda I - A_+)^{-1}x$ for $x \in X_+$, we obtain

$$(\lambda I + A_-)^{-1}Qx - Q(\lambda I - A_+)^{-1}x = -(\lambda I - A_-)^{-1}B_-C_+(\lambda I - A_+)^{-1}x \tag{3.4.18}$$

for $x$ in $X_+$.

We want to be able to interpret the transfer function $(C_p)_L(\lambda I - A_p)^{-1}B_pu$, for $u \in U$. Calculate

$$\bar{B} = \begin{bmatrix} I & -Q \\ 0 & I \end{bmatrix} \begin{bmatrix} B_- \\ B_+ \end{bmatrix} = \begin{bmatrix} B_- - QB_+ \\ B_+ \end{bmatrix}$$

and

$$\begin{aligned} \bar{C} &= [C_-, C_+] \begin{bmatrix} I & Q \\ 0 & I \end{bmatrix} \\ &= [C_-, C_-Q + C_+]. \end{aligned}$$

The transfer function $(C_p)_L(\lambda I - A_p)^{-1}B_pu$ equals $\bar{C}_L(\lambda I - \bar{A})^{-1}\bar{B}u$ if Q satisfies (3.4.18). We multiply the resolvent of $\bar{A}$ against $\bar{B}$:

$$(C_p)_L(\lambda I - A_p)^{-1}B_pu = \bar{C}_L(\lambda I - \bar{A})^{-1}\bar{B}u$$

$$= \bar{C}_L \begin{bmatrix} (\lambda I - A_-)^{-1}(B_- u - QB_+ u) & 0 \\ 0 & (\lambda I - A_+)^{-1} B_+ u \end{bmatrix}$$

where $u \in U$. Now we premultiply by $\bar{C}_L$ to obtain

$$\begin{aligned} &\bar{C}_L(\lambda I - \bar{A})^{-1}\bar{B}u = \\ &\quad (C_-)_L(\lambda I - A_-)^{-1}(B_- u - QB_+ u) \\ &\quad +((C_-)_L Q + C_+)(\lambda I - A_+)^{-1} B_+ u. \end{aligned} \tag{3.4.19}$$

The main technical difficulty is the interpretation of the transfer functions involving $QB_+$ and $(C_-)_L Q$.

The feedthrough operator $D = I$ for each transfer function presents no technical difficulty and so can be ignored in the following discussions. In [37] Salamon introduces a very useful vector space $Z$, that is defined for a controllable pair of operators $A$ and $B$ by $Z = D(A) + (\lambda I - A)^{-1}BU$ for some $\lambda \in \rho(A)$. (Note that $Z$ does not depend on the choice of $\lambda$). It is clear for a regular system that $Z$ must be contained in $D(C_L)$ since $C_L(\lambda I - A)^{-1}B$ is a bounded operator for $\lambda \in \rho(A)$. The space $Z_-$ indicates the $Z$ space associated with the triple $\Theta_-$.

Now that we have formally derived the additive decomposition of the product of a rational stable and antistable regular system and identified some technical difficulties to be addressed, we state a sufficient condition to verify the existence of the stable-antistable additive decomposition of the product of a rational stable admissible system with an antistable regular admissible system. The approach for the proof is to appeal to Theorem 3.3.1 to verify the terms in the additive decomposition are regular admissible systems. Some of the verifications of hypotheses are rather involved and are separated into propositions. Proposition 3.4.2 verifies that $(C_-)_L Q$ is an admissible observation operator for $\mathbf{T}^+$ which satisfies hypothesis (3) of Theorem 3.3.1. The point of Proposition 3.4.3 is to verify that $QB_+$ is an admissible control operator satisfying hypothesis (2) in Theorem 3.3.1 for $\mathbf{T}^-$. In Proposition 3.4.4 we verify that the transfer function of $(A_+, B_+, (C_-)_L Q)$ is bounded on a right half plane, a key part of hypothesis (4) of Theorem 3.3.1, and Proposition 3.4.5

verifies that the transfer function of $(A_-, QB_+, C_-)$ is bounded on a left half plane, a part of hypothesis (4) of Theorem 3.3.1 for an antistable regular triple.

**Theorem 3.4.1** *Let $(A_+, B_+, C_+)$ be a finite dimensional stable triple and $(A_-, B_-, C_-)$ an antistable regular triple, where both are assumed to have feedthrough operators equal to $I$. If the solution $Q$ of (3.4.16) takes $X_+$ and into $Z_-$ then the cascade connection of the antistable system $(A_-, B_-, C_-)$ with the finite dimensional stable system $(A_+, B_+, C_+)$ has an additive decomposition into stable and antistable regular systems.*

Before proving Theorem 3.4.1 we establish the needed preliminary propositions.

**Proposition 3.4.2** *The operator $\tilde{C} = (C_-)_L Q$ (restricted to $X_+$) is an admissible observation operator for $\mathbf{T}^+$.*

Proof:

If $x \in D(A_+)$ then $y(t) = \tilde{C}\mathbf{T}_t^+ x$ is continuous. We need to show that for some $\tau > 0$

$$\| y \|_{L^2[0,\tau]} \leq M \| x \|_{X_+}$$

in order to guarantee a continuous extension to $X_+$. Substituting the form of $Q$ into $y(t) = (C_-)_L Q\mathbf{T}_t^+ x$, we obtain

$$y(t) = (C_-)_L \int_0^\infty \mathbf{T}_\mu^- B_- C_+ \mathbf{T}_\mu^+ \mathbf{T}_t^+ x \, d\mu.$$

We make the change of variable $-(\mu + t) = \sigma$ we obtain

$$y(t) = (C_-)_L \int_{-\infty}^{-t} \mathbf{T}_{-t-\sigma}^- B_- C_+ \mathbf{T}_{-\sigma}^+ x \, d\sigma.$$

Let $v(\sigma) = C_+ \mathbf{T}_{-\sigma}^+ x$. Since $C_+$ is an admissible observation operator for $\mathbf{T}^+$, we have

$$\| v(\cdot) \|_{L^2[-\infty,0]} \leq K \| x \|_{X_+} \tag{3.4.20}$$

and

$$y(t) = (C_-)_L \int_{-\infty}^{-t} \mathbf{T}_{-t-\sigma}^- B_- v(\sigma) \, d\sigma$$

Take $\tau = -t$ and set $\tilde{y}(t) = y(-t)$ to produce

$$\tilde{y}(\tau) = (C_-)_L \int_{-\infty}^{\tau} \mathbf{T}^-_{\tau-\sigma} B_- v(\sigma) d\,\sigma.$$

The operator $-A_-$ generates an exponentially stable semigroup and $(A_-, B_-, C_-)$ is a regular admissible triple leading to

$$\| \tilde{y}(\cdot) \|_{L^2[-\infty,0]} \leq \acute{K} \| v \|_{L^2[-\infty,0]} . \tag{3.4.21}$$

Insert (3.4.20) into (3.4.21) and replacing $\tau$ with -t to yield

$$\| y(t) \|_{L^2[0,\infty]} \leq \acute{K} K \| x \|_{X_+} . \square \tag{3.4.22}$$

**Proposition 3.4.3** *The operator $QB_+$ is an admissible control operator for the exponentially decaying semigroup $\mathbf{T}^-$.*

Proof:

Since $\mathbf{T}^-$ is exponentially stable, we can define the operator $\Phi : L^2([0,\infty), U) \to (X_-)_{-1}$ by

$$\Phi u = \int_0^\infty \mathbf{T}^-_\sigma Q B_+ u(\sigma)\, d\sigma. \tag{3.4.23}$$

It is easy to verify that $QB_+$ is admissible for $\mathbf{T}^-$ if and only if the range of $\Phi$ is in $X_-$. This is equivalent to the statement that for any step function $u \in L^2([0,\infty), U)$

$$\| \Phi u \|_{X_-} \leq M \| u \|_{L^2[0,\infty)} . \tag{3.4.24}$$

(Note that for step functions $u$ we always have $\Phi u \in X_-$, so the left-hand side of (3.4.24) makes sense.) Since the set of step functions is dense in $L^2([0,\infty), U)$, (3.4.24) implies that $\Phi$ extends continuously to an operator from $L^2([0,\infty), U)$ to $X_-$).

To prove (3.4.24) we must establish some properties. First from Theorem 2.2.2 in [32], we have that there exists a negative real number $\omega$ (the exponential type) and a real number $N \geq 1$ such that

$$\|\mathbf{T}^-_\sigma\| \leq N e^{\omega\sigma}. \tag{3.4.25}$$

Let $x(\sigma) = QB_+u(\sigma)$; we claim $\|x(\sigma)\|_{X_-}$ is in $L^2[0,\infty)$ as a function of $\sigma$. Since $B_+$ is a bounded operator from $U$ to $X_-$, and the operator $Q$ is a bounded operator from $X_+$ into $Z_-$, we obtain

$$\|x(\sigma)\|_{X_-} \leq \|QB_+\| \, \|u(\sigma)\| \tag{3.4.26}$$

for each $\sigma$. We square and integrate both sides to produce

$$\int_0^\infty \|x(\sigma)\|_{X_-}^2 \, d\sigma \leq \|QB_+\|^2 \int_0^\infty \|u(\sigma)\|^2 \, d\sigma. \tag{3.4.27}$$

The right hand side of equation (3.4.27) is finite since $QB_+$ is a bounded operator and since $u \in L^2([0,\infty), U)$; hence $\|x(\cdot)\|_{X_-} \in L^2[0,\infty)$.

To conclude the proof of the proposition, we consider

$$\|\mathbf{\Phi} u\|_{X_-} = \| \int_0^\infty \mathbf{T}_\sigma^- QB_+u(\sigma) \, d\sigma \|_{X_-}. \tag{3.4.28}$$

We make the substitution $x(\sigma) = QB_+u(\sigma)$ and bring the norm inside the integral to obtain

$$\|\mathbf{\Phi} u\|_{X_-} = \int_0^\infty \|\mathbf{T}_\sigma^- x(\sigma)\| \, d\sigma. \tag{3.4.29}$$

Since $\mathbf{T}_\sigma^-$ is a semigroup of bounded operators, the definition of the operator norm leads to

$$\|\mathbf{T}_\sigma^- x(\sigma)\|_{X_-} \leq \|\mathbf{T}_\sigma^-\| \, \|x(\sigma)\|_{X_-}. \tag{3.4.30}$$

We apply (3.4.25) to the inequality (3.4.30):

$$\|\mathbf{T}_\sigma^- x(\sigma)\|_{X_-} \leq N e^{\omega\sigma} \|x(\sigma)\|_{X_-}. \tag{3.4.31}$$

Integration of this estimate produces

$$\int_0^\infty \|\mathbf{T}_\sigma^- x(\sigma)\|_{X_-} \, d\sigma \leq \int_0^\infty N e^{\omega\sigma} \|x(\sigma)\|_{X_-} \, d\sigma. \tag{3.4.32}$$

The function $Ne^{\omega\sigma}$ is in $L^2[0,\infty)$ and we already observed that $\|x(\sigma)\|_{X_-} \in L^2[0,\infty)$; therefore by the Hölder's inequality,

$$\int_0^\infty \|\mathbf{T}_\sigma^- x(\sigma)\|_{X_-} \, d\sigma \leq N \|e^{\omega\sigma}\|_2 \| \|x(\cdot)\|_{X_-} \|_2 < \infty. \tag{3.4.33}$$

Recalling (3.4.29) we obtain

$$\|\mathbf{\Phi} u\|_{X_-} \leq N(-\frac{1}{2\omega}) \|QB_+\| \|u(\sigma)\|_2. \tag{3.4.34}$$

Thus $\mathbf{\Phi} u$ can be extended as needed and $QB_+$ is an admissible control operator for $\mathbf{T}^-$.□

**Proposition 3.4.4** *The transfer function of the triple $(A_+, B_+, (C_-)_L Q)$ is bounded and analytic on $\{\lambda : Re\lambda > \omega\}$.*

Proof:

The transfer function of $(A_+, B_+, (C_-)_L Q)$ is $(C_-)_L Q(\lambda I - A_+)^{-1} B_+$. Since $\sigma(A_+)$ is contained in $\{\lambda : Re\lambda < \omega\}$,

$$(\lambda I - A_+)^{-1} = \int_0^\infty e^{\lambda t} e^{A_+ t}\, dt \tag{3.4.35}$$

is bounded and analytic on $\{\lambda : Re\lambda > \omega\}$. Since $B_+$ is a bounded admissible control operator,

$$x(\lambda) = \int_0^\infty e^{\lambda t} e^{A_+ t} B_+\, dt \tag{3.4.36}$$

is bounded and analytic on $\{\lambda : Re\lambda > \omega\}$. The operator $(C_-)_L$ is bounded from $D((C_-)_L)$ to $Y$ and is an admissible observation operator for the antistable regular admissible triple $(A_-, B_-, C_-)$ (i. e. $Z_- \subset D((C_-)_L)$). The operator $Q$ is a bounded operator from $X_+$ into $Z_-$, therefore it follows that the operator $(C_-)_L Q$ is bounded from $X_+$ into $Y$. Hence

$$(C_-)_L Q(\lambda I - A_+)^{-1} B_+ = (C_-)_L Q \int_0^\infty e^{\lambda t} e^{A_+ t} B_+\, dt \tag{3.4.37}$$

is a bounded analytic function on $\{\lambda : Re\lambda > \omega\}$.□

**Proposition 3.4.5** *The transfer function of the triple $(A_-, QB_+, (C_-)_L)$ is bounded and analytic on $\{\lambda : Re\lambda < -\omega\}$.*

Proof:

The transfer function for the triple $(A_-, QB_+, (C_-)_L)$ is

$$(C_-)_L(\lambda I - A_-)^{-1} Q B_+ u \tag{3.4.38}$$

for $u \in U$. Since $B_+$ is a bounded operator, $B_+ u$ is in $X_+$. The operator $Q$ is a bounded operator from $X_+$ into $Z_-$ producing $z = QB_+ u$ is in $Z_-$. The space $Z_-$ is defined to be $(\mu I - A_-)^{-1} BU + D(A_-)$ where $\mu$ is any fixed number in $\rho(A_-)$. We can therefore find $u$ in $U$ and $x$ in $D(A_-)$ such that

$$z = (\mu I - A_-)^{-1} B_- u + x. \tag{3.4.39}$$

Substitution of (3.4.39) into (3.4.38) yields

$$(C_-)_L(\lambda I - A_-)^{-1}(\mu I - A_-)^{-1}B_-u + (C_-)_L(\lambda I - A_-)^{-1}x. \tag{3.4.40}$$

The resolvents are bounded and analytic on $\{\lambda : Re\lambda < -\omega\}$. In the first term $B_-u$ is an element of $X_{-1}$ since $B_-$ is a bounded map of $U$ into $X_{-1}$. The double resolvent expression $(\lambda I - A_-)^{-1}(\mu I - A_-)^{-1}$ is a bounded map from $X_{-1}$ into $D(A_-)$. The operator $(C_-)_L$ is a bounded operator from $D((C_-)_L) \supset D(A_-)$ into $Y$. Hence the first term in (3.4.40) is a bounded analytic, U-valued function on $\{\lambda : Re\lambda < -\omega\}$.

Next note that $x$ is in $D(A_-)$, the resolvent $(\lambda I - A_-)^{-1}$ is a bounded operator from $D(A_-)$ into $D(A_-)$ and $(C_-)_L$ is bounded on $D(A_-)$. Hence the second term in (3.4.40) is bounded and analytic on $\{\lambda : Re\lambda < -\omega\}$. Thus the transfer function for the triple $(A_-, QB_+, (C_-)_L)$ is bounded and analytic on $\{\lambda : Re\lambda < -\omega\}$. □

We are now ready to complete the proof of 3.4.1.

Proof of Theorem 3.4.1:

From (3.4.19) we have a candidate form for the additive decomposition given by

$$\begin{aligned}
&\bar{C}_L(\lambda I - \bar{A})^{-1}\bar{B}u = \\
&\qquad (C_-)_L(\lambda I - A_-)^{-1}(B_-u - QB_+u) \\
&\qquad +((C_-)_LQ + C_+)(\lambda I - A_+)^{-1}B_+u.
\end{aligned} \tag{3.4.41}$$

It is given that $(A_-, B_-, C_-)$ is a regular antistable admissible triple and $(A_+, B_+, C_+)$ is a finite dimensional stable triple, hence we need only show that $(A_-, QB_+, C_-)$ and $(A_+, B_+, (C_-)_LQ)$ are regular admissible triples. We show this by satisfying the conditions of Theorem 3.3.1.

We have already noted that $A_+$ and $A_-$ generate $C_0$-semigroups. The operator $A_-$ generating a backward time semigroup and since $A_+$ is finite dimensional it generates the semigroup $e^{A_+t}$. Thus we have satisfied condition (1) of Theorem 3.3.1.

Since $(A_+, B_+, C_+)$ is a finite dimensional stable triple, by definition $B_+$ is a bounded admissible control operator for the semigroup $\mathbf{T}^+$ (see [42]). From Proposition 3.4.3 we have that $QB_+$ is an admissible control operator for $\mathbf{T}^-$. Thus $(A_-, QB_+, C_-)$ and $(A_+, B_+, (C_-)_LQ)$ satisfy condition (2) of Theorem 3.3.1.

Since $(A_-, B_-, C_-)$ is a regular admissible triple, by definition $C_-$ is an admissible observation operator for $\mathbf{T}^-$ (see [40]). By Proposition 3.4.2 it follows that $(C_-)_L Q$ is an admissible observation operator for $\mathbf{T}^+$. Thus $(A_-, QB_+, C_-)$ and $(A_+, B_+, (C_-)_L Q)$ both satisfy condition (3) of Theorem 3.3.1.

The transfer function of the triple $(A_-, QB_+, C_-)$ given by $(C_-)_L(\lambda I - A_-)^{-1} QB_+$ is defined and bounded on $\{\lambda : Re\lambda < -\omega\}$ by Proposition 3.4.5. Proposition 3.4.4 demonstrates that the transfer function of the triple $(A_+, B_+, (C_-)_L Q)$ is a bounded operator from $U$ into $Y$ and is a bounded function of $\lambda$ for $\{\lambda : Re\lambda > \omega\}$. Thus we have satisfied condition (4) of Theorem 3.3.1

For both $(A_-, QB_+, C_-)$ and $(A_+, B_+, (C_-)_L Q)$, we have now verified all the hypotheses for Theorem 3.3.1 implying that the triples $(A_-, QB_+, C_-)$ and $(A_+, B_+, (C_-)_L Q)$ are regular admissible triples. We conclude that there exists an additive stable-antistable decomposition of the product of a finite dimensional stable triple with an antistable regular triple as asserted. □

An interesting open question is whether Theorem 3.4.1 holds if the finite dimensional stable triple is replace by a general regular stable triple.

# Chapter 4

# Sensitivity Minimization of the Delay Transmission line.

## 4.1 Introduction

In this chapter we consider the physical example of the delay transmission line. We use this example to show how to apply some of the previously developed theory. In Section 2 we present and develop the sensitivity minimization problem. Section 3 will setup the mathematical notation for the delay transmission line problem. This section contains calculations of the state spaces $X_1$ and $X_{-1}$ as well as the calculation of the adjoint system and its appropriate state spaces. In Section 4 we show how the weighted sensitivity problem with an outer rational weight (i.e. a rational weight function $W$ such that $W$ and $W^{-1}$ are in $H^\infty$) for the delay transmission line reduces to the Nehari problem. The reduction process requires use of the additive decomposition developed in Chapter 3. Section 4 contains many calculations in an attempt to adapt the formulas of Ball and Ran [5] as generalized by Curtain and Ran [17]. In conclusion we are left short of a method of interpreting the formulas even though the computation of the building blocks goes smoothly.

## 4.2 Introduction to the sensitivity minimization problem

The sensitivity minimization problem is to construct for a given plant P (see Figure 4.1) stabilizing compensators C such that the transfer function from $u_1$ to $e_1$ is minimized in the infinity norm. This is the optimal version of the problem. The finite dimensional case results in a unique compensator C which unfortunately may not be proper. An improper compensator is not a physically implementable solution to the sensitivity minimization problem. Often a suboptimal version of the problem namely, to construct compensators C so that the norm of the transfer function from $u_1$ to $e_1$ is less than or equal to some tolerance $\sigma$ slightly larger than the optimal value $\sigma_0$ of the norm, is solved instead. This results in a linear fractional parameterization of a family of compensators which satisfy the suboptimal problem, many of which are proper. In practice, this parameterization gives added flexibility to pick the appropriate compensator based on other design constraints.

The weighted sensitivity minimization problem involves controlling the infinity norm of the transfer function from $u_1$ to $e_1$ multiplied by a weighting function. Weighting functions are used in emphasizing control of certain frequencies. We develop the problem more precisely. Let the known plant be given by an $m \times m$ matrix function $P(s)$. We obtain the following equations from Figure 4.1

$$u_1 - Pe_2 = e_1$$

$$u_2 + Ce_1 = e_2.$$

Solve the system of equations for $u_1$ and $u_2$:

$$u_1 = e_1 + Pe_2$$

$$u_2 = -Ce_1 + e_2.$$

We write this system of equation as the following matrix equation:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} I & P \\ -C & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

When $I + PC$ is invertible, we can solve for $e_1$, $e_2$ in terms of $u_1$, $u_2$; the result is

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} (I+PC)^{-1} & -(I+PC)^{-1}P \\ (I+CP)^{-1}C & (I+CP)^{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.2.1}$$

The compensator $C$ is said to stabilize the closed loop system if $e_1$ and $e_2$ are in vector-valued $H^2$ with respect to the right half plane whenever $u_1$ and $u_2$ are in $H^2$, (i.e., if the four functions $(I + PC)^{-1}$, $(I + PC)^{-1}$, $(I + CP)^{-1}C$ and $(I + CP)^{-1}$ are all in matrix valued $H^\infty$ with respect to the right half plane). The Hardy space $H^\infty$ of the right half plane possesses the norm

$$\| F \|_\infty = \sup\{|F(j\omega)| : \omega \in \Re\}.$$

The $H^2$ norm of $\| f \|_2$ is

$$\| f(s) \|_2 = (\int_{-i\infty}^{+i\infty} \| f(s) \|^2 \, ds)^{\frac{1}{2}}.$$

For details on Hardy spaces $H^2$ and $H^\infty$ see Hoffman [28] and see Rosenblum-Rovnyak [35] for the vector and matrix valued case. From (4.2.1) we see that the transfer function from $u_1$ to $e_1$ is given by the equation

$$e_1 = (I + PC)^{-1}u_1.$$

Let

$$\sigma_0 = \inf\{\| W(I + PC)^{-1} \|_\infty : C \text{ stabilizes } P\},$$

where W is an outer weighting function (e.g. $W, W^{-1} \in H^\infty$). The suboptimal weighted sensitivity minimization problem is then to choose C stabilizing so that

$$\sup\{\| We_1 \|_2 : \| u_1 \| < 1\} < \sigma$$

where $\sigma > \sigma_0$. By rescaling the weight, the problem can be posed in the form

$$\sup\{\| \sigma^{-1}We_1 \|_2 : \| u_1 \| < 1\} < 1.$$

In the case where P itself is stable, it turns out (see Zames [45])) that C is internally stabilizing for P if and only if $Q = C(I + PC)^{-1}$ is stable (i.e. $Q \in H^\infty$). Solving for C in terms of Q, we first obtain

$$C = Q(I + PC). \tag{4.2.2}$$

We collect the terms involving C on the left:

$$C - QPC = Q.$$

Factoring C to the right produces

$$(I - QP)C = Q.$$

We solve for $C$ by use of the inverse $(I - QP)^{-1}$ :

$$C = (I - QP)^{-1}Q = Q(I - PQ)^{-1}. \tag{4.2.3}$$

Equating expressions (4.2.2) and (4.2.3) for C yields

$$Q(I + PC) = Q(I - PQ)^{-1}.$$

Thus

$$(I + PC) = (I - PQ)^{-1}.$$

Making this substitution into the weighted transfer function from $u_1$ to $e_1$ produces

$$\sigma^{-1}W(I + PC)^{-1}u_1 = (\sigma^{-1}W - \sigma^{-1}WPQ)u_1.$$

Our main interest in this chapter will be the case of a scalar stable plant. Then we observe that $\tilde{Q} = \sigma^{-1}WQ$ is in $H^\infty$ if and only if $Q$ is in $H^\infty$. Let $\sigma^{-1}WPQ = P\tilde{Q}$ and $\bar{W} = \sigma^{-1}W$, we then obtain

$$\| \sigma^{-1}W(I + PC)^{-1}u_1 \|_2 = \| \bar{W}u_1 - P\tilde{Q}u_1 \|_2 .$$

This implies that the sensitivity minimization problem has the alternative formulation

$$\min\{\| \bar{W} - P\tilde{Q} \|_\infty : \tilde{Q} \in H^\infty\},$$

and that the suboptimal problem is to characterize all $\tilde{Q} \in H^\infty$ such that

$$\| \bar{W} - P\tilde{Q} \|_\infty \leq 1.$$

In Section 4.4 we will reduce the problem for the case where $P(s) = e^{-hs}$ further to a Nehari problem and seek in Section 4.5 a state space implementation of the solution via J-inner-outer factorization due to Ball and Helton [11]. The state space formulas for the solution have been obtained via this route by Ball and Ran [5] for the rational case, by Ran [34] for the class of BGK-admissible plants and by Curtain and Ran [17] for the class of Pritchard-Salamon systems.

## 4.3 The mathematical setting for the delay transmission line

The delay transmission line was presented by Weiss in [42] as an example of a regular system. We present the abstract linear system for the delay transmission line. Let $X$, the state space, be $L^2[-h,0]$, where $h > 0$, and let $\mathbf{T}$ be the left shift semigroup on X with zero entering from the right, i.e., for any $\tau \geq 0$ and $\zeta \in [-h,0]$,

$$(\mathbf{T}_\tau x)(\zeta) = \begin{cases} x(\zeta+\tau), & \text{for } \zeta+\tau \leq 0, \\ 0, & \text{for } \zeta+\tau > 0. \end{cases}$$

Set the input space $U$ be equal to the complex numbers $\mathbf{C}$, and for any $\tau \geq 0$ and $\zeta \in [-h,0]$, define the input map by

$$(\mathbf{\Phi}_\tau u)(\zeta) = \begin{cases} u(\zeta+\tau), & \text{for } \zeta+\tau \geq 0, \\ 0, & \text{for } \zeta+\tau < 0. \end{cases}$$

Let the output space $Y$ also equal to $\mathbf{C}$ and for any $\tau \geq 0$ and $t \in [0,\tau]$, define the output map by

$$(\mathbf{L}_\tau x)(t) = \begin{cases} x(t-h), & \text{for } t-h \leq 0, \\ 0 & \text{for } t-h > 0. \end{cases}$$

For $t \geq \tau$ we require $(\mathbf{L}_\tau x)(t) = 0$. Finally, for any $\tau \geq 0$ and $t \in [0,\tau]$, the input-output map is given by

$$(\mathbf{F}_\tau u)(t) = \begin{cases} x(t-h), & \text{for } t-h \geq 0, \\ 0, & \text{for } t-h < 0. \end{cases}$$

For $t \geq \tau$ we require $(\mathbf{F}_\tau u)(t) = 0$. The system $\Sigma = (\mathbf{T}, \mathbf{\Phi}, \mathbf{L}, \mathbf{F})$ forms an abstract linear system.

Now we present a realization triple to describe this regular system as in Weiss [42]. We define the concept of absolutely continuous functions in order to define the domain of the semigroup generator.

**Definition 4.3.1** *Let $f$ be a complex-valued function defined on a subinterval $J$ of the $\Re$. Suppose that for every $\varepsilon > 0$, there is $\delta > 0$ such that*

$$\sum_{k=1}^{n} |f(d_k) - f(c_k)| < \varepsilon$$

*for every finite, pairwise disjoint family $\{(c_k, d_k)\}_{k=1}^{n}$ of open subintervals of $J$ for which*

$$\Sigma_{k=1}^{\infty}(d_k - c_k) < \delta.$$

*Then $f$ is said to be absolutely continuous on $J$.*

We denote by $AC[-h,0]$ the space of all complex-valued functions $f$ absolutely continuous on $[-h,0]$. By a standard result from real analysis (see [25]), a function $f(x)$ in $AC[-h,0]$ can be written in the form

$$f(t) = f(-h) + \int_{-h}^{0} g(s)\, ds$$

where $g \in L^1[-h,0]$. Define the operator $A$ to be $\frac{d}{dt}$ with domain equal to

$$X_1 = \{x \in AC[-h,0] : \dot{x}(t) \in L^2[-h,0], x(0) = 0\}.$$

The space $X_{-1}$ is the completion of $L^2[-h,0]$ with respect to the norm

$$\| x \|_{-1} = \| \int_0^{\zeta} x(\sigma) d\sigma \|_{L^2[-h,0]} .$$

This space can be identified concretely as the space of continuous linear functionals on $X_1^* = \{x \in AC[-h,0] : \dot{x} \in L^2[-h,0]$ and $x(-h) = 0\}$, where an element $f \in L^2[-h,0]$ is associated with the linear functional

$$L_f(g) = \int_{-h}^{0} f(s)g(s)\, ds,$$

for $g \in X_1^*$. Since U is one dimensional, the control operator can be identified with an element $B \in X_{-1}$, namely the "delta function" at 0 considered as a linear functional on $X_1^*$

which we will denote $\delta_0(\tau)$. The observation operator C is just point evaluation at -h acting on $X_1$ which we will denote $\chi_{-h}(\tau)$. We now define the Lebesgue extension of C, denoted $C_L$, by

$$C_L x = \lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau x(-h+\zeta)d\zeta.$$

The $D(C_L)$ is the set of $x \in L^2[-h,0]$ which have a Lebesgue point at -h. From Definition 3.2.6 of a regular system, we need to check that

$$Dv = \lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau y_v(\sigma)\, d\sigma$$

exists, where $y_v = \mathbf{F}_\infty \chi_v$. The function $\chi_v$ is the constant function on $[0, \infty]$. If $\tau < h$ then $\mathbf{F}_\infty \chi_v = 0$ therefore

$$Dv = \lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau 0\, d\sigma$$

exists and is zero.

Now we want to calculate the set up for the adjoint system. Let $u \in X_1$ then

$$< Au, z >= \int_{-h}^0 u\acute{(}t)z(t)\, dt.$$

Integrating by parts we obtain

$$< Au, z >= u(t)z(t) \mid_{-h}^0 - \int_{-h}^0 u(t)z\acute{(}t)dt = u(0)z(0) - u(-h)z(-h) + \int_{-h}^0 u(t)(-z\acute{(}t))\, dt.$$

The evaluation at zero produces $u(0)z(0) = 0$ since $u \in X_1$. Therefore in order to insure $< Au, z >=< u, A^* z >$ for all $z$ in the domain of $A^*$, we define $A^* z = -\dot{z}$ with domain equal to $X_1^*$. Hence $z \in X_1^*$ if it is an absolutely continuous function on $[-h, 0]$ with $z\acute{(}t) \in L^2[-h, 0]$ and $z(-h) = 0$. This produces $A^* = -\frac{d}{dt}$ and the $\mathbf{T}^*$ semigroup generated by $A^*$ is the right shift semigroup with the zero entering on the left. For $\zeta \in [-h, 0]$, $\tau > 0$

$$(\mathbf{T}_\tau^* x)(\zeta) = \begin{cases} x(\zeta - \tau), & \text{for } \zeta - \tau \geq -h, \\ 0, & \text{for } \zeta - \tau < -h. \end{cases}$$

The space $X_{-1}^*$ is the set

$$\{f \in L^2[-h, 0] : L_f(g) = \int_{-h}^0 f(s)g(s)\, ds \text{ is a continuous linear functional for } g \in X_1\},$$

with the $\|\cdot\|_{-1}$ norm. Note that the dual of $X_{-1}$, denoted $(X_{-1})^*$, under the pairing

$$< f, g >= \int_{-h}^{0} f(\sigma)g(\sigma)\, d\sigma$$

is $X_1^*$, and the dual of $X_1$, denoted $(X_1)^*$, is $X_{-1}^*$. We have that $B^*$ is therefore just point evaluation at 0 acting on $X_1^*$. The operator $C^*$ is the "delta function" at -h thought of as a linear functional on $X_1$. The operator $C^*$ maps $\mathbf{C}$ into $X_{-1}^*$. The domain $D(B_L^*)$ of the Lebesgue extension $B_L^*$ of $B^*$ is the set of $L^2[-h,0]$ functions with a Lebesgue point at 0 in the classical sense. By Remark 5.4 in Curtain and Weiss [18] (A,B,C) is a regular admissible triple if and only if $(A^*, C^*, B^*)$ is a regular admissible triple.

## 4.4 Reduction to the Nehari problem

The plant for the delay transmission line is $P(s) = e^{-hs}$, which is an inner function (i.e. $P^*(s)P(s) = I$ on the imaginary axis), hence $P^{-1}(s) = P^*(s)$ on the imaginary axis. The suboptimal sensitivity minimization problem has been reduced in section 4.2 to finding $\tilde{Q} \in H^\infty$ such that

$$\| \bar{W} - P\tilde{Q} \|_\infty \leq 1.$$

Multiplication by $P$ is an isometry on $L^2$ since $P^*(s) = P^{-1}(s)$ on the imaginary axis. This gives

$$\| \bar{W} - P\tilde{Q} \|_\infty = \| P(P^*\bar{W} - \tilde{Q}) \|_\infty = \| P^*\bar{W} - \tilde{Q} \|_\infty .$$

Hence for any P such that $P^*P = I$ on the imaginary axis, we may reduce the sensitivity minimization problem to finding $\tilde{Q} \in H^\infty$ such that

$$\| P^*\bar{W} - \tilde{Q} \| \leq 1. \tag{4.4.4}$$

The operator $P^*\bar{W}$ is in $L^\infty$ (the space of functions bounded on the imaginary axis). This is the suboptimal version of the Nehari problem.

For the delay transmission line problem we are using a rational outer plant W. Therefore $P^*\bar{W}$ is the product of an antistable regular system having the transfer function

$P^*$ with a stable rational system having the transfer function $\bar{W}$. We will calculate the additive decomposition of

$$[P^*\bar{W}] = [P^*\bar{W}]_- + [P^*\bar{W}]_+$$

from Theorem 3.4.1. Since $[P^*\bar{W}]_+ \in H^\infty$ and $[P^*\bar{W}]_+ - \tilde{Q} = \bar{Q} \in H^\infty$, the Nehari type suboptimal problem is reduced further to

$$\| P^*\bar{W} - \tilde{Q} \| = \| [P^*\bar{W}]_- - \bar{Q} \| \leq 1.$$

We apply Theorem 3.4.1 to the calculation of $[P^*\bar{W}]_-$ as follows. Since W is a rational matrix function there exists a finite dimensional realization $(A_w, B_w, C_w, I)$. It is an assumption to take $D_w = I$. The realization for $P^*$ is $(-A^*, C^*, -B^*, 0)$. When we form the product realization, we obtain

$$A = \begin{bmatrix} -A^* & C^*C_w \\ 0 & A_w \end{bmatrix}, B = \begin{bmatrix} C^* \\ B_w \end{bmatrix}$$

$$C = [-B^*, 0]$$

Substitution of these expressions into (3.4.16) produces the Sylvester equation

$$A^*Qx + QA_wx = C^*C_wx \tag{4.4.5}$$

for $x \in X_w$. Note $X_w$ is the finite dimension state space for W. From the finite dimensional theory, we expect the form of Q should be

$$Qz = -\int_0^\infty \mathbf{T}_t^* C^* C_w e^{A_w t} z dt$$

for $z \in \mathbf{C}^{\mathbf{m}}$. Let $X_1^* \subset X \subset X_{-1}^*$ be the spaces associated with the triple $(-A^*, C^*, -B^*)$. The operator Qz must be interpreted as an element of $X_{-1}^*$. The space $X_{-1}^*$ is a set of continuous linear functionals on $X_1$. The operator $C^*$ is the delta function at -h and $C_w e^{A_w t} z \in \mathbf{C}$. Therefore we must apply $\mathbf{T}_t^* \delta_{-h}(\tau)$ against a test function in $X_1$ in order to interpret the integral

$$[Qz][u] = [-\int_0^\infty C_w e^{A_w t} z \mathbf{T}_t^* \delta_{-h}(\tau)\, dt][u]$$

as a function. We note that $[\mathbf{T}_t^*\delta_{-h}(\tau)][u] = [\delta_{-h}(\tau)][\mathbf{T}_t u]$ produces

$$[Qz][u] = -\int_{-h}^{0}\int_{0}^{\infty} C_w e^{A_w t} z\delta_{-h}(\tau)\mathbf{T}_t u(\tau)\,dt\,d\tau.$$

We apply the semigroup to u giving

$$[Qz][u] = -\int_{-h}^{0}\int_{0}^{\infty} C_w e^{A_w t} z\delta_{-h}(\tau)\begin{cases} u(\tau+t), & \text{for } \tau+t \le 0, \\ 0, & \text{for } \tau+t > 0. \end{cases}$$

The fact that $\tau + t \le 0$ implies $t \le -\tau$. Therefore the nonzero portion of the integral is

$$[Qz][u] = -\int_{-h}^{0}\int_{0}^{-\tau} C_w e^{A_w t} z\delta_{-h}(\tau)u(\tau+t)\,dt\,d\tau.$$

We introduce a characteristic function in t in order to obtain square limits producing

$$[Qz][u] = -\int_{-h}^{0}\int_{0}^{h} \chi_{[0,-\tau]}(t) C_w e^{A_w t} z\delta_{-h}(\tau)u(\tau+t)\,dt\,d\tau.$$

By interchanging order of integration, we obtain

$$[Qz][u] = -\int_{0}^{h} \chi_{[0,h]}(t) C_w e^{A_w t} u(t-h)\,dt.$$

The change of variable $s = t - h$ leads to

$$[Qz][u] = -\int_{-h}^{0} \chi_{[0,h]}(s+h) C_w e^{A_w(s+h)} z u(s)\,ds. \tag{4.4.6}$$

The characteristic function is identically 1 since $s + h \in [0, h]$ for $s \in [-h, 0]$. In general the distribution $Qz$ may be identified with the function $f$ if the identity

$$[Qz][u] = \int_{-h}^{0} f(s)u(s)\,ds$$

holds. From (4.4.6) we identify $Qz$ as the function

$$Qz = -C_w e^{A_w(s+h)} z.$$

In order to verify Qz solves the Sylvester equation (4.4.5), the equation must be interpreted as an element of $X_{-1}^*$. For $u \in D(A)$ this produces

$$[A^*Qz + QA_w z][u]$$

$$= [Qz][Au] + [QA_w z][u] = \int_{-h}^{0} -C_w e^{A_w(t+h)} z\acute{u}(t) dt$$

$$+ \int_{-h}^{0} -C_w e^{A_w(t+h)} A_w z u(t)\, dt. \qquad (4.4.7)$$

Integrating the first term by parts, (4.4.7) becomes

$$[A^* Qz + QA_w z][u] =$$

$$-C_w e^{A_w(t+h)} z u(t) \mid_{-h}^{0} \qquad + \int_{-h}^{0} C_w e^{A_w(t+h)} A_w z u(t)\, dt$$

$$- \textstyle\int_{-h}^{0} C_w e^{A_w(t+h)} A_w z u(t)\, dt$$

$$[A^* Qz + QA_w z][u] = -C_w e^{A_w(h)} z u(0) + C_w z u(-h). \qquad (4.4.8)$$

By definition $u(0) = 0$ whenever u is in $D(A)$. Inserting $u(0) = 0$ into (4.4.8) produces

$$C_w z u(-h) = \int_{-h}^{0} \delta_{-h}(t) C_w z u(t)\, dt = [C^* C_w z][u]. \qquad (4.4.9)$$

From (4.4.9) we conclude

$$[A^* Qz + QA_w z][u] = [C^* C_w z][u].$$

Since Q satisfies the necessary condition in Theorem 3.4.1 obtain the following theorem.

**Theorem 4.4.1** *The transfer function $P^*(s)$ of the adjoint system for the delay transmission line has the regular antistable realization triple $(-A^*, C^*, -B^*)$, where*

$$A^* = -\frac{d}{dt}, \qquad C^* = \delta_{-h},$$

$$B^* = \chi_0(t) \qquad (4.4.10)$$

*with spaces*

$$X_1^* = \{x \in AC[-h, 0] : \dot{x} \in L^2[-h, 0] \text{ and } x(-h) = 0\}$$

*and*

$X_{-1}^* =$

$$\{f \in L^2[-h, 0] : L_f(g) = \textstyle\int_{-h}^{0} f(s) g(s)\, ds \text{ is a continuous linear functional for } g \in X_1\}.$$

*The domain $D(B_L^*)$ of the Lebesgue extension $B_L^*$ of $B^*$ is the set of $L^2[-h,0]$ functions with a Lebesgue point at 0 in the classical sense. Define $W$ to be the transfer function of a rational outer plant with realization $(A_w, B_w, C_w, I)$. Let $Q$ solve (4.4.5). The operator $Q$ takes $X_+$ into $D(B_L^*)$ since $Qz = -C_w e^{A_w(t+h)} z$. We therefore obtain from Theorem 3.4.1 that the product $P^*W$ has an additive decomposition as stable and antistable systems of the transfer function of*

$$P^*W = [P^*W]_- + [P^*W]_+$$

*where*

$$[P^*W]_+ = -B_L^* Q(\lambda - A_w)^{-1} B_w$$

*and*

$$[P^*W]_- = -B_L^*(\lambda I + A^*)^{-1}(C^* - QB_w).$$

## 4.5 State space formulas for the J-spectral factorization.

Rather than present precise results, in this section we merely formulates the questions to be resolved for a successful completion of the state space approach to the sensitivity minimization problem for a delay system. This section addresses the Nehari problem for the function $[P^*\bar{W}]_-$ namely; find $\tilde{Q} \in H^\infty$ for which $\| [P^*\bar{W}]_- - \tilde{Q} \|_\infty \leq 1$. The starting point for our analysis is the following prescription for the linear fractional description of the set of all solutions in the suboptimal case from Ball and Helton [11] (see also Ball and Ran [5]).

**Theorem 4.5.1** *Let $K$ be a given function in $L^\infty$. Then there exists functions $\tilde{Q} \in H^\infty$ for which*

$$\| K - \tilde{Q} \|_\infty \leq 1$$

*if and only if the Hankel operator $H_K : H^2 \to H^{2\perp}$ given by*

$$H_K : h \to P_{H^{2\perp}}(Kh)$$

*has* $\| H_K \| < 1$. *Moreover in this case there exist* $2 \times 2$ *matrix functions*

$$\Theta(s) = \begin{bmatrix} \Theta_{11}(s) & \Theta_{12}(s) \\ \Theta_{21}(s) & \Theta_{22}(s) \end{bmatrix}$$

*and*

$$H(s) = \begin{bmatrix} H_{11}(s) & H_{12}(s) \\ H_{21}(s) & H_{22}(s) \end{bmatrix}$$

*such that the set of all errors*

$$\{F = K - \tilde{Q} : \tilde{Q} \in H^\infty \text{ and } \| F \|_\infty \leq 1\}$$

*is given by*

$$F = (\Theta_{11}G + \Theta_{12})(\Theta_{21}G + \Theta_{22})^{-1}$$

*and the set of all approximates*

$$\{\tilde{Q} : \tilde{Q} \in H^\infty \text{ and } \| K - \tilde{Q} \|_\infty \leq 1\}$$

*is given by*

$$\tilde{Q} = (H_{11}G + H_{12})(H_{21}G + H_{22})^{-1}$$

*where* $G$ *is a free parameter* $H^\infty$*-function with* $\| G \|_\infty \leq 1$. *Here the matrix functions* $\Theta$ *and* $H$ *are determined by the requirements that*

1. $\Theta^* J \Theta = J$ *on the imaginary axis, where* $J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$.

2. $H$ *and* $H^{-1}$ *have matrix entries in* $H^\infty$ *and,*

3. $L = \Theta H^{-1}$, *where* $L = \begin{bmatrix} I & K \\ 0 & I \end{bmatrix}$.

If $K$ is a rational and antistable with realization

$$K(z) = C(zI - A)^{-1}B \tag{4.5.11}$$

then $H$ is given by

$$H(\lambda) = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} -C\hat{P} \\ B^* \end{bmatrix} (\lambda I + A^*)^{-1}(I - \hat{Q}\hat{P})^{-1}[C^*, \hat{Q}B] \tag{4.5.12}$$

where $\hat{P}$ and $\hat{Q}$ are the controllability and observability gramians defined as unique solutions of the Lyapunov equations

$$A\hat{P} + \hat{P}A^* = -BB^*$$

$$A^*\hat{Q} + \hat{Q}A = -C^*C.$$

We would like to use the theory of regular systems to make sense of the state space representation (4.5.12) for $H(\lambda)$ for the case where $(A,\ B,\ C)$ is the state space representation for $[P^*\bar{W}_-]$ given in Theorem 4.4.1. However due to the twisting induced by the factor $(I - \hat{Q}\hat{P})^{-1}$ in the formula, this goal up to now has proved to be elusive, so we leave this as an open question.

# Bibliography

[1] Admjan,V. M., Arov, D. Z., and Krein, M. G., Analytic Properties of Schmidt Pairs for a Hankel Operator and the Generalized Schur-Takagi Problem, Math. USSR-Sb15:31-73, 1971.

[2] Ball, J. A., and Carpenter, L. E., Realizations of Products and Wiener-Hopf Factors for a Class of Matrix Functions Analytic in the Strip, Signal Processing, Scattering and Operator Theory, and Numerical Methods, Proceedings of the International Symposium MTNS-89, Vol. III, Eds: Kaashoek, M. A., van Schuppen, J. H., and Ran, A. C. M., Birkhäuser Verlag, Basel, pp. 291-300, 1990.

[3] Ball, J. A., Gohberg, I. and Rodman, L., Interpolation of Rational Matrix Functions, Operator Theory: Advances and Applications, Vol. 45, Birkhäuser Verlag, Basel, 1990.

[4] Ball, J. A. and Helton, J. W., Beurling-Lax representations using classical Lie groups with many applications II: $GL(n, \mathbf{C})$ and Wiener-Hopf factorization, Integral Equations Operator Theory, Vol. 7, pp. 291-309, 1984.

[5] Ball, J. A. and Ran, A. C. M., Left versus Right Canonical Wiener-Hopf Factorization, Operator Theory: Advances and Applications, Vol. 21, Birkhäuser Verlag, Basel, pp. 9-38, 1986.

[6] Bart, H., Gohberg, I. and Kaashoek, M. A., Minimal Factorization of Matrix and Operator Functions, Operator Theory: Advances and Applications, Vol. 1, Birkhäuser Verlag, Basel, 1979.

[7] Bart, H., Gohberg, I. and Kaashoek, M. A., Exponentially Dichotomous Operators and Inverse Fourier Transforms, Report 8511/m, Econometric Institute, Erasmus University Rotterdam, The Netherlands, 1985.

[8] Bart, H., Gohberg, I., and Kaashoek, M. A., Fredholm Theory of Wiener-Hopf Equations in Terms of Realization Symbols , Integral Equations and Operator Theory, Vol. 8, pp. 590-613, 1985.

[9] Bart, H., Gohberg, I. and Kaashoek, M. A., Wiener-Hopf Factorization, Inverse Fourier Transforms and Exponentially Dichotomous Operators, J. Functional Analysis, Vol. 68, No. 1, pp. 1-41, 1986.

[10] Bart, H., Gohberg, I. and Kaashoek, M. A., Wiener-Hopf Equations with Symbols Analytic in a Strip, Operator Theory: Advances and Applications, Vol. 21, Birkhäuser Verlag, Basel, pp. 39-75, 1986.

[11] Ball, J. A. and Helton, J. W., A Beurling-Lax Theorem for the Lie Group $U(m,n)$ which contains most Classical Interpolation Theory, J. Operator Theory, Vol. 9, pp. 107-142, 1983.

[12] Callier, F. M. and Desoer, C. A., Stabilization, Tracking and Disturbance Rejection in Multivariable Convolution Systems, Annales de la Soc. Scient. de Bruxelles, T. 94, I, pp. 7-51, 1980.

[13] Conway, J. B., A Course in Functional Analysis, Graduate Texts in Mathematics 96, Springer-Verlag, New York, 1985.

[14] Curtain, R. F., A Synthesis of Time and Frequency Domain Methods for the Control of Infinite-Dimensional System: A System Theoretic Approach, (to appear in the series SIAM Frontiers in Applied Mathematics).

[15] Curtain, R. F., Representations of Infinite Dimensional Linear Systems, Three Decades of Mathematical Systems Theory, Eds: H. Nijmijer and J. M. Schumacher, Springer-Verlag, Berlin, pp. 101-128, 1989.

[16] Curtain, R. F. and Pritchard, A. J., Infinite Dimensional Linear Systems Theory, Lecture Notes in Information Sciences 8, Springer-Verlag, Berlin, 1978

[17] Curtain, R. F. and Ran, A. C. M., Explicit formulas for Hankel Norm Approximation of Infinite-Dimensional Systems (to appear in the series SIAM Frontiers in Applied Mathematics).

[18] Curtain, R. F. and Weiss, G., Well Posedness of Triples of Operators (in the Sense of Linear Systems Theory), Distributed Parameters Systems, proc. of the conference in Vorau, Austria, July 1988, Eds: F. Kappel, K. Kunisch, W. Schappacher, Birkhäuser, Basel, pp. 41-59, 1989.

[19] Francis, B. A., A Course in $H_\infty$ Control, Lecture Notes in Control and Informations Sciences 88, Springer-Verlag, Berlin, 1987.

[20] Foias, C. and Tannenbaum, A., On the Parameterization of the Suboptimal Solution in General Interpolation, Linear Algebra and its Applications, 122/123/124: pp. 145-164, 1989.

[21] Foias, C. and Tannenbaum, A., On the Nehari problem for a certain class of $L^\infty$ functions appearing in control theory, Journal of Functional Analysis, Vol. 74, pp. 146-159, 1987.

[22] Foias, C., Tannenbaum, A. and Zames, G., On the $H^\infty$-optimal sensitivity problem for systems with delays, SIAM Journal Control and Optimization, Vol. 25, pp. 686-706, 1987.

[23] Gohberg, I., Lancaster, P., and Rodman, L., Invariant Subspaces of Matrices with Applications, Wiley, 1986.

[24] Helton, J.W., System with Infinite-Dimensional State Space: The Hilbert Space Approach, Proc. of the IEEE 64, pp. 145-160, 1976.

[25] Hewit, E. and Stromberg, K., Real and Abstract Analysis, Graduate Texts in Mathematics, Vol. 25, Springer-Verlag, New York, NY, 1965.

[26] Hille, E. and Phillips, R., Functional Analysis and Semi-groups, AMS, Providence, RI 1957.

[27] Ho, L. F. and Russel, D. L., Admissible Input Elements for Systems in Hilbert Spaces and Carleson Measure Criterion, SIAM Journal Control and Optimization, Vol. 21, July, pp. 614-640, 1983.

[28] Hoffman, K., Banach Spaces of Analytic Functions, Prentice-Hall, Englewood Cliffs, NJ, 1962.

[29] Kailath, T., Linear Systems, Prentice-Hall, Englewood Cliffs, NJ,1980.

[30] Lasiecka, I. and Triggiani, R., Algebraic Riccati Equations with Applications to Boundary-Point Control Problems: Continuous Theory and Approximation Theory. (to appear in the series SIAM Frontiers of Applied Mathematics).

[31] Logemann, H., On the Transfer Matrix of a Neutral System: Characterizations of Exponential Stability in Input-Output Terms, System and Control Letters, Vol. 9, pp. 393-400, 1987.

[32] Pazy, A., Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, Berlin, 1983.

[33] Pritchard, A. J. and Salamon, D., The Linear Quadratic Optimal Control Problem for infinite dimensional systems with unbounded input and output operators, SIAM J. Control and Optimiz., Vol. 25, pp. 121-144, 1987.

[34] Ran, A. C. M., Hankel norm approximation for infinite-dimensional systems and Wiener-Hopf factorization, Modeling Robustness and Sensitivity Reduction in Control Systems, Ed. R. F. Curtain, NATO ASI Series, Springer-Verlag, Berlin, pp. 57-70, 1986.

[35] Rosenblum, J. and Rovnyak, M., Hardy Classes and Operator Theory, Oxford University Press, 1985.

[36] Salamon, D., Control and Observation of Nuclear Systems, RNM91, Pitman, London, 1984.

[37] Salamon, D., Infinite-dimensional Linear Systems with Unbounded Control and Observation: A Functional Analytic Approach, Trans. AMS, Vol. 300, pp. 383-431, 1987.

[38] Salamon, D., Realization theory in Hilbert space, Mathematical Systems Theory, Vol. 21, pp. 147-164, 1989.

[39] Taylor, A. E. and Lay, D. C., Introduction to Functional Analysis, Wiley, New York, 1980.

[40] Weiss, G., Admissibility Observation Operators for Linear Semigroups, Isreal J. Math, Vol. 65, pp. 17-43, 1989.

[41] Weiss, G., Admissibility of Unbounded Control Operators, SIAM Journal Control & Optimization, Vol. 27, pp. 527-545, 1989

[42] Weiss, G., The Representation of Regular Linear Systems on Hilbert Spaces, Distributed Parameter Systems, proc. of the conference in Vorau, Austria, July 1988, Eds: F. Kappel, K. Kunish, W. Schappacher, Birkhäuser, Basel, pp. 401-416, 1989.

[43] Weiss, G., Transfer functions of regular linear systems: Part I: Characterization of Regularity. Submitted.

[44] Wonham, W.M., Linear Multivariable Control: A Geometric Approach, second edition, Springer-Verlag, Berlin, 1979

[45] Zames, G., Feedback and Optimal Sensitivity: Model, Reference Transformations, Multiplicative seminorms, and approximate inverses, IEEE Control Transactions on AC, Vol. AC26, pp. 301-320, 1981.

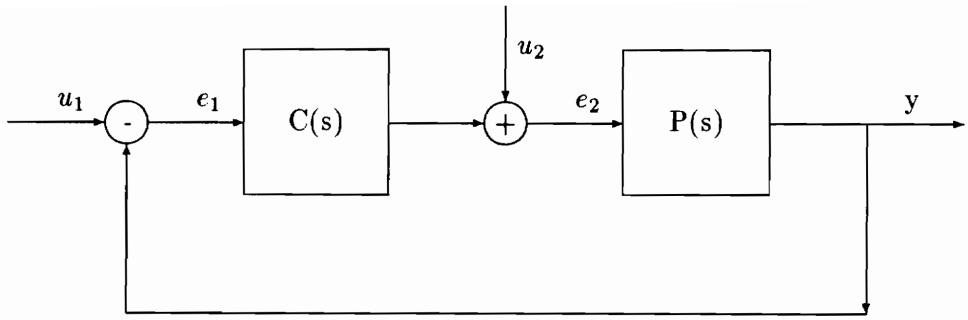


**Figure 4.1 Block Diagram**

## Vita

Lonnie Eugene Carpenter was born 22 September 1962 in Macon, MO. He received his Bachelor of Science degree in Mathematics from Bob Jones University in Greenville, SC in May 1984. He received the Master of Science degree in Mathematics in June 1986 from Virginia Polytechnic Institute and State University, Blacksburg, VA. He will be receiving his Doctor of Philosophy in Mathematics from Virginia Polytechnic Institute and State University, Blacksburg, VA in May of 1992. Mr. Carpenter is currently resides in Fredericksburg, VA and is employed by Logicon, Strategic & Information Systems, P.O. Box 1420, Dahlgren, Va 22448.

Lon E Carpenter