A NONLINEAR VOLTERRA EQUATION OF NONCONVOLUTION TYPE

by

Manfred Charles Smith

Dissertation submitted to the Graduate Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

APPROVED:

K. B. Hannsgen, Chairman

G. W. Crofts

J. A. Cochran

L. W. Johnson

J. K. Shaw

December, 1977

Blacksburg, Virginia

ACKNOWLEDGMENTS

The author is deeply indebted to Professor Kenneth Hannsgen for suggesting the area of research in this dissertation and for his extreme patience and guidance throughout the course of this effort.

It is also my pleasure to thank the committee members for their consideration and interest.

Lastly, I would like to thank my wife,   , for her patient understanding and moral support during the course of this research.

TABLE OF CONTENTS

1. Introduction

The purpose of this paper is to study the asymptotic behavior of bounded solutions x(t) of the integrodifferential equation

$$(1.1)\qquad \begin{cases} x'(t) + \int_0^t a(t,\tau)\, g(x(\tau))\, d\tau = f(t)\ , & 0 \le t < \infty \\ x(0) = x_0 \end{cases}$$

where $a(t,\tau)$, $g(x)$ and $f(t)$ are given real-valued functions. We assume that

$$(1.2)\qquad \begin{cases} x(t) \text{ is a locally absolutely continuous} \\ \text{function which satisfies (1.1) a.e. for} \\ 0 \le t < \infty, \text{ and } \sup\limits_{0 \le t < \infty} |x(t)| < \infty \end{cases}$$

and that

$$(1.3)\qquad \begin{cases} g(x) \in C(-\infty,\infty) \\ f(t) \in L^1(0,\infty) \end{cases}$$

and we find conditions on $a(t,\tau)$ which ensure that $g(x(t)) \to 0$ $(t \to \infty)$. The problem of finding a set of minimal assumptions (even in this classical Volterra equation setting) is open. We improve upon a recent result of Kiffe [7, Theorem 1] by weakening his assumptions on the kernel and yet obtaining the same conclusion, namely, $g(x(t)) \to 0$ as $t \to \infty$. In particular, we remove some of Kiffe's smoothness conditions and place less restrictive growth conditions on $a(t,\tau)$ than he did.

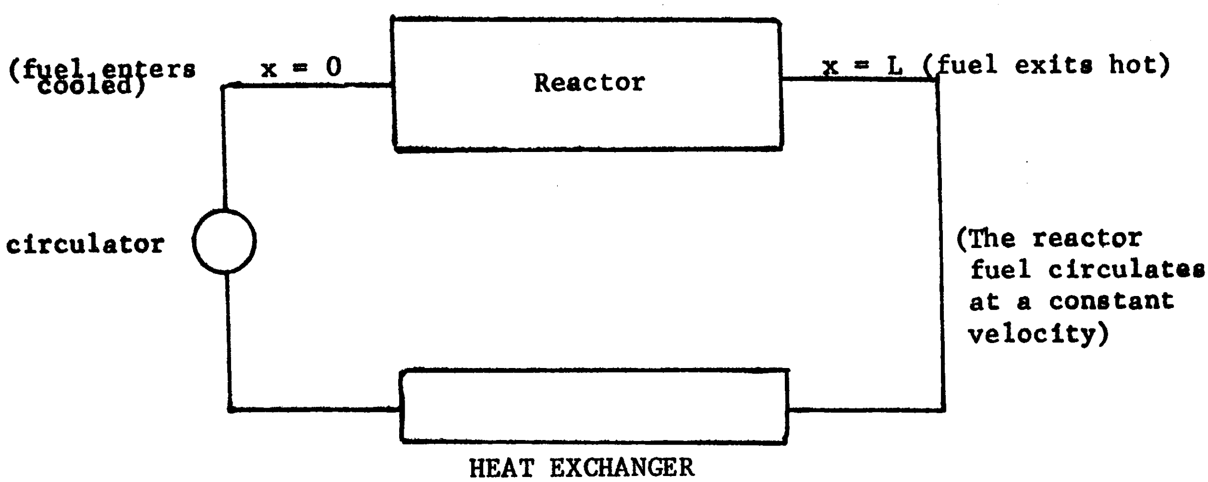
The outline of this paper is as follows. The work done on the equation (1.1) grew from investigations of the convolution case (that is equation (1.1) with $a(t,\tau) = A(t-\tau)$); section 2 is a compendium of the work on both the nonconvolution equation (1.1) and its convolution analogue. We begin the section by tracing the derivation of the nonlinear convolution equation from a problem in reactor dynamics. Secondly, we discuss the previous work, [9], [7], on the nonconvolution equation (1.1) and indicate an application of recent interest.

In section 3 we present the statement of our result and give its proof. Lastly, in section 4 we compare our theorem to the main result in Kiffe [7]. We will show that all of our hypotheses are implied by Kiffe's assumptions except for one, namely (vi). However, we show by means of an example that our assumption (vi) and his corresponding assumption (vii′) are independent. Finally we give an alternate version of our hypothesis (vi), namely (vi*), which, when used in place of (vi) in our theorem, yields a theorem which has Kiffe's main result and our original theorem as corollaries.

2. A History of the Problem

The Convolution Case

Our problem has its roots in a 1959 paper of Ergen and Nohel [2] in which dynamic stability of a continuous-medium nuclear reactor is studied. Specifically, the reactor is considered as a continuous medium in which the power $P(t)$ and the temperature $T(x,t)$ are dependent on time t and on the spatial variable x. The following is a schematic view of the circulating-fuel reactor system:



Remark: We by no means imply that the above system is presently in operation; in fact this system is considered obsolete, however because it motivated studies of equations of type (1.1) we present a discussion of the system.

Since the heating of the reactor is not instantaneous, the negative temperature coefficient does not compensate immediately for an excess in reactivity, nor does the power return to normal immediately after the temperature coefficient begins to act. Because of these delays, oscillations in the reactor power may occur. The original model neglected the following considerations for simplicity:

(i) the effect of delayed neutrons,

(ii) the interaction between mechanical vibrations and power oscillations,

(iii) the interactions between power oscillations and hydrodynamic flow of the fuel,

(iv) the dependence of the reactivity on parameters other than the temperature.

However it was shown earlier (1954) by Ergen that, in most cases, the effect of delayed neutrons is a dampening of the power oscillations.

Under all these considerations the reactor system was described by the system

$$(2.1)\qquad \begin{cases} \dfrac{d}{dt}[\ln P(t)] = -\displaystyle\int_{-\infty}^{\infty} \alpha(x)\ T(x,t)\ dx \\[2ex] \epsilon\ \dfrac{\partial T}{\partial t} = \eta(x)[P(t) - 1] + \dfrac{\partial^2 T}{\partial x^2} \end{cases} \qquad t > 0$$

where (a) P(t), the total reactor power as a function of time is neccessarily positive and at equilibrium is taken to be 1,

(b) T(x,t) is the deviation of the temperature from its equilibrium value (taken to be 0) at position x and time t,

(c) $-\alpha(x)$ is the negative temperature coefficient of reactivity at position x divided by the mean life of the neutrons ($\alpha(x) \geq 0$),

(d) ϵ is the heat capacity per unit volume ($\epsilon > 0$),

(e) $\eta(x)$ is the fraction of the power generated at position x ($\eta(x) \geq 0$).

The initial conditions to be satisfied are:

$$(2.2) \qquad \begin{cases} P(0) = P_0 > 0 \\ T(x,0) = f(x) \qquad -\infty < x < \infty \end{cases}$$

where $f(x)$ is a given initial deviation of temperature from equilibrium. From the physics of the problem we see that for all x and for $t \geq 0$ we may assume:

(2.3) $f(x)$, $T(x,t)$, T_x, T_{xx}, $P(t)$, $P'(t)$ are continuous; $P(t)$ is bounded, $T(x,t)$ and $f(x)$ are uniformly bounded; and T, T_x, T_{xx}, T_t, $\alpha(x)$, $\eta(x)$ are all in $L^1 \cap L^2$ as functions of x.

A brief inspection of (2.1) reveals the obvious solution ($P \equiv 1$, $T \equiv 0$) for the case $P_0 = 1$ and $f(x) = 0$. Thus the basic problem studied in the late 1950's was the question of stability of the power oscillations for large t of the system (2.1). Ergen and Nohel, using a Green's function, rewrote the system (2.1) as

$$(2.4) \qquad \begin{cases} \dfrac{d}{dt}[\ln P(t)] = -\bar{\alpha}\, g(t) \\[2ex] \dfrac{\bar{\alpha}}{\lambda}\, \dfrac{dg}{dt} = P(t) - 1 - \int_0^\infty k(s)[P(t-s) - 1]\, ds \end{cases}$$

where $\bar{\alpha} = \int_{-\infty}^{\infty} \alpha(x)\,dx$, $g(t) = \frac{1}{\bar{\alpha}} \int_{-\infty}^{\infty} \alpha(x)\, T(x,t)\,dx$, $\lambda = \int_{-\infty}^{\infty} \alpha(x)\,\eta(x)\,dx$

and the kernel $k(s)$ was defined by $k(s) = -\frac{1}{\lambda}\int_{-\infty}^{\infty} \alpha(x) \int_{-\infty}^{\infty} \frac{\partial G}{\partial s}(x,\xi,s)\,\eta(\xi)\,d\xi\,dx$

with the Green's function

$$G(x,\xi,s) = \frac{1}{\sqrt{\frac{4\pi s}{\epsilon}}} \exp\{-\frac{\epsilon(x-\xi)^2}{4s}\},$$

and then applied a stability criterion due to T. A. Welton [27]:

> The system (2.4) is stable if the Fourier sine transform of the kernel $k(s)$ satisfies
> $\int_0^{\infty} k(s)\,\sin\omega s\,ds \geq 0$ for all ω.

Substituting the above equations for $k(s)$ into this criterion and using integration by parts they obtained the following sufficient condition for stability of the system (2.1):

$$(2.5)\qquad \int_{-\infty}^{\infty} \alpha(x) \int_{-\infty}^{\infty} \eta(\xi)\, [\int_0^{\infty} \frac{\cos\omega s}{\sqrt{s}} \exp\{-\frac{\epsilon(x-\xi)^2}{4s}\}\,ds]\,d\xi\,dx \geq 0$$

for all real ω.

Welton's criterion was based on physically plausible arguments, but at that time it had not been rigorously justified.

Motivated by (2.1), as we shall see below, Levin [8] began in 1963 to investigate the nonlinear Volterra integrodifferential equation

$$(2.6)\qquad \begin{cases} \mu'(t) + \int_0^t a(t-\tau)\, g(\mu(\tau))\,d\tau = 0 \\ \mu(0) = \mu_0 \end{cases} \qquad (0 \leq t < \infty).$$

(Here $a(t)$, $g(x)$ are prescribed real-valued functions.) Local existence is easily established by standard techniques. (Either Picard approximations or fixed point theorems.) The primary concern is finding the weakest kernel assumptions sufficient to prove that solutions exist on $[0,\infty)$ and tend to 0 as $t \to \infty$. Using a Liapounov function, Levin proved that under the assumptions

$$(2.7) \qquad \begin{cases} a(t) \in C[0,\infty), \ (-1)^k a^{(k)}(t) \geq 0 \quad (0 < t < \infty;\ k = 0,1,2,3), \\ g(x) \in C(-\infty,\infty), \ x\,g(x) > 0 \ (x \neq 0) \text{ and} \\ G(x) \equiv \int_0^x g(\xi)\, d\xi \to \infty \quad (|x| \to \infty) \end{cases}$$

if $a(t)$ is not constant and $\mu(t)$ is any solution of (2.6) on $[0,\infty)$ then $\lim_{t\to\infty} \mu^{(j)}(t) = 0$ for $j = 0,1,2$. The existence of a solution on $[0,\infty)$ was established from the triply monotonic condition on $a(t)$ in (2.7) and a priori bounds obtained from assumptions in (2.7). The Liapounov function used was

$$(2.8) \qquad E(t) = G(\mu(t)) + \frac{1}{2}\, a(t)\Big[\int_0^t g(\mu(\tau))\, d\tau\Big]^2 - \frac{1}{2}\int_0^t a'(t-\tau)\Big[\int_\tau^t g(\mu(s))\, ds\Big]^2 d\tau .$$

Levin's method of proof consisted of showing $E(t) \geq 0$, $E'(t) \leq 0$ with $E''(t)$ bounded. He was able to show that as $t \to \infty$, $E'(t) \to 0$ and with some further work, $\mu(t) \to 0$. From certain bounds obtained in showing the above and the mean value theorem he deduced that $\mu^{(j)}(t) \to 0$ $(t \to \infty)$ for $j = 1,2$, also.

In the same year, 1963, Levin and Nohel [11] proved by a different method that if $a(t)$ is completely monotonic (i.e. $(-1)^k a^{(k)}(t) \geq 0$ for $k = 0,1,2,\ldots$), then solutions of (2.6) satisfy $\lim_{t\to\infty} \mu^{(j)}(t) \to 0$ $(j = 0,1,2)$. Although this result is weaker than that of [8], it drew together such different notions of positivity as Liapounov functions, completely monotonic functions, and kernels of positive type. The proof provided a new Liapounov function for (2.6), namely

$$V(t) = G(\mu(t)) + \frac{1}{2}\int_0^t \int_0^t a(\tau + s)\; g(\mu(t-\tau))\, g(\mu(t-s))\, d\tau\, ds. \tag{2.9}$$

Levin and Nohel noted that if $V(t)$ is to serve as a Liapounov function for (2.6) ($V(t) \geq 0$, $V'(t) \leq 0$ for $t \in [0,\infty)$) then it must be true that $a(\tau + s)$ and $-a'(\tau + s)$ are of positive type [28, p. 270] on the square $0 < \tau,\ s < t$ for each $0 < t < \infty$. (This can be seen by differentiating (2.9).) However, in [28, pp. 273-275] it is shown that $a(\tau + s)$ and $-a'(\tau + s)$ are of positive type if and only if

$$\left\{\begin{array}{l} a(t) = \int_0^\infty \exp\{-\xi t\}\, d\,\alpha(\xi) \ \text{ where} \\ \alpha(\xi) \text{ is nondecreasing on } (-\infty,\infty) \text{ with} \\ \alpha(0) = 0,\ \alpha(\xi) = \frac{1}{2}\,[\alpha(\xi+) + \alpha(\xi-)] \text{ and} \\ \alpha(-\infty) = \alpha(0-) \end{array}\right. \tag{2.10}$$

and as Levin and Nohel further indicated, a theorem of S. Bernstein [28, p. 160] provides the representation of $a(t)$ found in (2.10) if and only if $a(t)$ is completely monotonic on $[0,\infty)$.

These various notions and their interdependencies influenced the later works of several authors; see for example ([4], [16], [18], [20], [21], [22], [23], [24], [25]). Although these papers produced significant results for the convolution equation (2.6), the techniques they employed involved transforms and therefore have no direct bearing on the method we later use in discussing the nonconvolution equation (1.1). Consequently, we mention their content only briefly. For the earliest discussion of the nonlinear equation (2.6) using transforms we refer the reader to Halanay [4]. However there is an error in this paper which was later partially rectified in a paper by McCamy and Wong [16, theorem 4.3]. McCamy and Wong also extended the main ideas found in [4] for the equation (2.6) to Hilbert space. Not until 1976, in a paper by Nohel and Shea [18] was the error in Halanay's paper fully rectified. This paper by Nohel and Shea contains the most complete discussion of the transform techniques used on equation (2.6) (up to the work of Staffans, which we mention below). In 1960 Levin and Nohel [10] studied the linearized version of (2.1) (obtained by replacing $\ln P(t)$ and $P(t) - 1$ by $\mu(t)$); using transform techniques, they proved that the linear system is stable. Motivated by this paper, Hannsgen [5] considered equation (2.6) with $g(\mu(t)) = \mu(t)$ and, using transform techniques, he found kernel restrictions (weaker kernel conditions than exhibited in (2.7)) that not only yield $\mu(t) \to 0$ $(t \to \infty)$ but also $\int_0^t \mu(s)\,ds \to 0$ $(t \to \infty)$. (Later Shea and Wainger [20] improved Hannsgen's result to $\mu(t) \in L^1(0,\infty)$.) The latest and perhaps most revealing work done on the nonlinear convolution equation (2.6) is found in a collection of papers by Staffans ([21], [22], [23], [24], [25]), which use transform methods entirely.

We now return to a discussion of the works of others on the convolution equation (2.6), whose methods are closely akin to techniques we will use later.

In 1965 Levin and Nohel [12] perturbed the equation (2.6) to

$$(2.11) \qquad \begin{cases} \mu'(t) + \int_0^t a(t-\tau)\, g(\mu(\tau))\, d\tau + b(t) = c(t) \\ \mu(0) = \mu_0 \end{cases}$$

for $0 \leq t < \infty$ where $c(t) \in C[0,\infty) \cap L^1[0,\infty)$ and $b(t) \in C[0,\infty) \cap C^1(0,\infty)$, and again investigated the stability of the solutions. (Here $a(t)$, $g(x)$, $b(t)$, $c(t)$ are prescribed real-valued functions with $g(x)$ satisfying the same conditions as found in (2.7).) In this paper, as was the case in [8] and in many papers to follow, local existence results were not considered in great detail, since usually the hypotheses required for stability are much stronger than what is required for local existence. The importance of this paper was to be illustrated by another result of Levin and Nohel [13], which appeared in 1966, addressing the problem (2.1) and using the main theorem in [12] as a lemma. In this 1966 paper Levin and Nohel rewrote (2.1) as

$$(2.12) \qquad \begin{cases} \mu'(t) = -\int_{-\infty}^{\infty} \alpha(x)\, T(x,t)\, dx \\ \frac{\partial T}{\partial t} = \eta(x)\, g(\mu(t)) + \frac{\partial^2 T}{\partial x^2} \end{cases} \qquad t > 0$$

where $\mu(t)$ is the logarithm of $P(t)$, the power, and $T(x,t)$, $\eta(x)$, $\alpha(x)$

the same as in (2.1), and T, P satisfy the initial conditions (2.2). Here $g(x) = e^x - 1$ is nonlinear. They then related the system (2.12) to the perturbed nonlinear Volterra integrodifferential equation (2.11). To see this relationship one makes the following definitions:

Let the Fourier transform of a function $r(x)$ be given by $\hat{r}(x) = \underset{A \to \infty}{\text{l.i.m.}} \int_{-A}^{A} r(y) \exp\{-i\,x\,y\}dy$, and define $h_1(x) = \Re e\{\hat{\eta}(x)\ \hat{\alpha}(-x)\}$, $h_2(x) = \Re e\{\hat{f}(x)\ \hat{\alpha}(-x)\}$ and

$$
(2.13) \qquad \begin{cases} a(t) = \dfrac{1}{\pi} \displaystyle\int_0^\infty h_1(x) \exp\{-x^2 t\}dx \\ b(t) = \dfrac{1}{\pi} \displaystyle\int_0^\infty h_2(x) \exp\{-x^2 t\}dx \end{cases}
$$

for $0 \leq t < \infty$ and $-\infty < x < \infty$. Since α, η, $f \in L^2(-\infty,\infty)$ and are real-valued, it follows that $h_1(x)$, $h_2(x)$ are even and belong to $L^1(0,\infty)$. Now suppose the equation (2.11) with $c(t) \equiv 0$ and $a(t)$, $b(t)$ given by (2.13) has a solution $\mu(t)$ on $[0,\infty)$; if there is more than one solution pick one and call it $\mu(t)$. Then define

$$
(2.14) \qquad \begin{cases} T(x,t) = \displaystyle\int_{-\infty}^{\infty} G(x-\xi,t)\ f(\xi)\, d\,\xi \\ \qquad\qquad + \displaystyle\int_0^t \int_{-\infty}^{\infty} G(x-\xi,t-\tau)\ \eta(\xi)\ g(\mu(\tau))\, d\,\xi\, d\,\tau \end{cases}
$$

where $G(x,t) = [4\pi\, t]^{-\frac{1}{2}} \exp\{\frac{-x^2}{4t}\}$ is the well-known Green's function for the homogeneous heat equation. The classical results concerning the inhomogeneous heat equation immediately yield: T, T_t, T_{xx} are continuous in (x,t) for $-\infty < x < \infty$, $0 < t < \infty$; the second equation of (2.12) is satisfied there and $T(x,t)$ is continuous in (x,t) for

$-\infty < x < \infty$, $0 \leq t < \infty$, with $T(x,0) = f(x)$. Also from (2.14) one can readily show that for each fixed t, $T(x,t)$ belongs to $L^2(-\infty,\infty)$; moreover, the Fourier transform of $T(x,t)$ with respect to x (denoted by $\hat{T}(x,t)$) is given by the formula

$$\hat{T}(x,t) = \hat{f}(x)\ \exp\{-x^2 t\} + \hat{\eta}(x) \int_0^t g(\mu(\tau))\ \exp\{-x^2(t-\tau)\}d\tau. \tag{2.15}$$

Thus by Parseval's relation, (2.15) and (2.13), we have

$$\int_{-\infty}^{\infty} \alpha(x)\ T(x,t)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\alpha}(-x)\ \hat{T}(x,t)dx =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\alpha}(-x)\ \hat{f}(x)\ \exp\{-x^2 t\}dx +$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\alpha}(-x)\ \hat{\eta}(x) \int_0^t \exp\{-x^2(t-\tau)\}\ g(\mu(\tau))d\tau\ dx$$

$$= \frac{1}{\pi}\int_0^{\infty} h_2(x)\ \exp\{-x^2 t\}dx + \frac{1}{\pi}\int_0^t\int_0^{\infty} h_1(x)\ \exp\{-x^2(t-\tau)\}dx\ g(\mu(\tau))d\tau$$

$$= b(t) + \int_0^t a(t-\tau)\ g(\mu(\tau))\ d\tau = -\mu'(t).$$

(We point out that the previous change of order of integration is easily justified by Fubini's theorem.) Consequently, if $\mu(t)$ is the solution of (2.11) with $c(t) = 0$ and $a(t)$, $b(t)$ given by (2.13) and if $\mu(t) \to 0$ $(t \to \infty)$ then this $\mu(t)$, together wich $T(x,t)$ given by (2.14), is a solution of the system (2.12) in which the power, $\ln P(t)$, is stable. In fact, using Sobolev type inequalities, (2.15), and the physically accurate assumption that α, η, $f \in L^2(-\infty,\infty)$, Levin and Nohel [13] proved that the $T(x,t)$ given by (2.14) satisfies $\lim_{t\to\infty} \sup_{-\infty<x<\infty} |T(x,t)| = 0$.

The main theorem which appears in [12] and which is used as the principal lemma in the above discussion is the following:

> <u>Theorem</u>: Let $g(\mu)$ and $a(t)$ satisfy (2.7); $b(t) \in C[0,\infty)$ be such that $b''(t)$ exists on $0 < t < \infty$, and $a(t) \not\equiv a(0)$. Further let there exist a function $d(t)$ such that $d(t) \in C[0,\infty)$, $d''(t)$ exists on $0 < t < \infty$, and $[b^{(k)}(t)]^2 \le a^{(k)}(t)\ d^{(k)}(t)$ for $0 < t < \infty$ and $k = 0,1,2$. Then for each μ_0 there exists a solution $\mu(t)$ of (2.11) with $c(t) \equiv 0$ on $0 \le t < \infty$ that satisfies $\lim_{t \to \infty} \mu^{(k)}(t) = 0\ (k = 0,1,2)$.

Thus we see that if there exists a measurable function $h_3(x) \in L^1[0,\infty)$ satisfying $h_2^{\,2}(x) \le h_1(x)\ h_3(x)$ and if $h_i(x) \ge 0\ (i = 1,2,3)$ for $0 \le x < \infty$, then if

$$d(t) = \frac{1}{\pi} \int_0^\infty h_3(x) \exp[-x^2 t]dx \quad (0 \le t < \infty),$$

all the hypotheses of this theorem are satisfied and hence a solution $\mu(t)$ of (2.11) (and hence of (2.12)) does exist, and $\mu(t) \to 0$ as $t \to \infty$.

We point out that the above theorem was proved, as in [8], by using an energy function argument (Liapounov's second method); the energy function was a modified form of (2.8).

In 1968, subsequent to the 1966 paper, Levin and Nohel [14] added the effect of delayed neutrons to the equations (2.12). Again using energy function techniques, together with the intimate connections between the system (2.12) and the equation (2.11), they were able to show stability of the solutions under similar kernel assumptions to those of [13]; this corroborated the previous physical arguments of Ergen.

Motivated by these early papers, several people began to investigate stability for solutions of the equation (2.11) under a variety of kernel assumptions.

In 1969 Hannsgen [6] showed that it is not necessary to impose four sign conditions on a(t) and its derivatives in order to achieve asymptotic stability of solutions of (2.11). By changing the Liapounov function used in [12], Hannsgen proved that if

$$\left\{\begin{array}{l} a(t) \in C(0,\infty) \cap L^1(0,\infty) \text{ and } a(t) \text{ is} \\ \text{nonnegative, nonincreasing and} \\ \text{convex on } (0,\infty) \end{array}\right. \tag{2.16}$$

and if either

$$\begin{array}{l} a(0) = a(0+) < \infty, \text{ with } c(t) \text{ locally absolutely} \\ \text{continuous on } [R,\infty) \text{ for some } R < \infty, \text{ and} \\ |c'(t)| + |b'(t)| < \infty \text{ a.e. on } R \le t < \infty, \end{array}$$

or

$$a(t) \in L^1(0,\infty) \text{ with } |c(t)| < \infty \text{ for } 0 \le t < \infty$$

and if the same conditions are placed on $g(x)$, $b(t)$ and $c(t)$ as found in Levin and Nohel [12], then solutions of (2.11) are asymptotically stable provided that $a(t)$ is not piecewise linear with changes of slope at $\{\sigma_k\}_{k=1}^{\infty}$ with $\sigma_{k+1} - \sigma_k \geq \delta > 0$ for every k. Note that not only did Hannsgen improve upon the smoothness restrictions placed on the kernel $a(t)$ but in doing so he was not forced to assume his kernel satisfied a strict convexity assumption for $t \simeq 0$ as is the case if $a(t)$ is triply monotonic. (See [8, Lemma 2].)

Still another paper was to appear improving the results of Levin and Nohel [8]. In 1971, Londen [15] considered the equation

$$(2.17) \qquad \begin{cases} \mu'(t) + \int_0^t a(t-\tau)\; g(\mu(\tau))d\tau = c(t) \\ \mu(0) = \mu_0 \end{cases} \qquad (0 \leq t < \infty),$$

with $g(x) \in C(-\infty,\infty)$ and $c(t) \in L^1[0,\infty)$. He assumed that the kernel satisfies a double monotonicity condition; however, unlike Levin and Nohel [8] and Hannsgen [6], Londen did not exclude the case in which $a(0+) = \infty$ and $a(t) \notin L^1(0,\infty)$. Prior to his paper the proofs for existence of $\lim_{t\to\infty} \mu(t)$ essentially rested upon the Liapounov function (2.8). In the proof of Londen's main theorem the equation (2.17) is written in a form that immediately indicates the importance of kernel monotonicity for the existence of $\lim_{t\to\infty} g(x(t))$. As a result, Londen proves stability without direct recourse to energy function techniques.

The Nonconvolution Case

In 1968 Levin [9] realized that the techniques used in deriving stability results for the convolution equation ([8], [12]) did not depend on the convolution nature of the kernel $a(t)$ but rather on the sign changes in its derivatives. He began to investigate the equation

$$(2.18) \qquad \begin{cases} \mu'(t) + \int_0^t a(t,\tau)\, g(\mu(\tau))\, d\tau + b(t) = 0 \\ \mu(0) = \mu_0 \end{cases}$$

with

$$(2.19) \qquad \begin{cases} g(\mu) \in C^1(-\infty,\infty),\ \mu\, g(\mu) > 0 \ (\mu \neq 0) \\ \text{and } G(\mu) = \int_0^\mu g(y)\, dy \to \infty \quad (|\mu| \to \infty) \end{cases}$$

Again using a Liapounov function argument, Levin showed that the crucial hypothesis required for stability of solutions of (2.18) is that the kernel $a(t,\tau)$ satisfy

$$(2.20) \qquad \begin{cases} a(t,\tau) \geq 0,\ a_t(t,\tau) \leq 0,\ a_{tt}(t,\tau) \geq 0 \\ a_\tau(t,\tau) \geq 0,\ a_{t\tau}(t,\tau) \leq 0,\ a_{tt\tau}(t,\tau) \geq 0 \\ \text{for } (t,\tau) \text{ satisfying } 0 < t < \infty,\ 0 \leq \tau < t \end{cases} ;$$

i.e., successive differentiation of $a(t,\tau)$ with respect to t causes a change in sign while differentiation with respect to τ does not affect the sign. (Which is the case if $a(t,\tau) = A(t-\tau)$ with $(-1)^k A^{(k)}(t) \geq 0$ for $k = 0,1,2,3$.) Of course he had to require certain growth conditions on $a(t,\tau)$ and a similar relation between $a(t,\tau)$ and $b(t)$ as found in [12] for $a(t)$ and $b(t)$. This extends the old stability problem to kernels of the type $a(t,\tau) = \gamma(t)\, \rho(\tau)\, b(t-\tau)$ where $b(t)$ satisfies (2.7) and $\gamma(t) \in C[0,\infty)$ with $(-1)^k\, \gamma^{(k)}(t) \geq 0$

$(0 < t < \infty;\ k = 0,1,2)$, $\gamma(\infty) > 0$ and $\rho(t) \in C^1[0,\infty)$, $\rho^{(k)}(t) \geq 0$ $(0 \leq t < \infty;\ k = 0,1)$, $0 < \rho(\infty) < \infty$; or of the type $a(t,\tau) = b[(t-\tau)\ \gamma(t)]$ with $\gamma(t)$, $b(t)$ satisfying the above and $(t\ \gamma(t))' \geq 0$, $(t\ \gamma(t))'' \leq 0$ for $0 < t < \infty$, $\lim_{t \to \infty} (t\ \gamma(t))' > 0$. These kernels may be viewed as perturbations of the kernel $b(t-\tau)$ in the sense that for large t and τ the differences $\gamma(t)\ \rho(\tau)\ b(t-\tau) - \gamma(\infty)\ \rho(\infty)\ b(t-\tau)$, $b[(t-\tau)\ \gamma(t)] - b[(t-\tau)\ \gamma(\infty)]$ are small.

Prior to our work the latest results on stability for the classical nonconvolution equation are found in Kiffe [7]. His paper generalizes the method of Londen [15] for the convolution equation (2.17) to the equation (1.1), thereby extending the results of Levin [9], in the sense that the kernel smoothness and changes in sign of the derivatives that Levin required, (2.20), are replaced by

$$(2.21) \qquad \left\{ \begin{array}{l} a(t,\tau) \geq 0,\ a_t(t,\tau) \leq 0,\ a_\tau(t,\tau) \geq 0 \\ \text{and } a_{\tau t}(t,\tau) \text{ is continuous in } (t,\tau) \\ \text{with } a_{\tau t}(t,\tau) \leq 0 \text{ for } 0 < t < \infty,\ 0 \leq \tau < t \end{array} \right. .$$

In the following chapter we shall improve the smoothness hypothesis, (2.21), of Kiffe for the equation (1.1), in much the same fashion as Hannsgen [6] did for earlier results concerning the convolution analogue of (1.1).

Because of its applications to problems in linear viscoelasticity, equation (1.1) has recently drawn renewed interest. (See Dafermos [1].) If $a(t,\tau)$, $f(t)$ are given sufficient smoothness to allow differentiation of equation (1.1) with $g(\mu) = \mu$, one obtains the equation

$$\mu''(t) + a(t,t)\,\mu(t) + \int_0^t a_t(t,\tau)\,\mu(\tau)\,d\tau = f'(t) \tag{2.22}$$

which can be envisaged as an abstract form of the equations of linear viscoelasticity. Here $a(t,t)$ for fixed t and $a_t(t,\tau)$ for t and τ fixed are interpreted as linear operators on a Hilbert space and the integral is understood in the sense of Bochner. Again, the interest is in the behavior of solutions of (2.22) as $t \to \infty$, i.e., whether a Boltzmann type dependence of the stress on the history of deformation may induce a damping mechanism.

For a study of (1.1) in a real Hilbert space setting with g replaced by a nonlinear multivalued operator we refer the reader to the Ph.D. thesis of C. L. Rennolet [19].

3. Statement and Proof of the Result

THEOREM 3.1: Let $R = \{(t,\tau) \mid 0 < t < \infty,\ 0 \le \tau < t\}$ and suppose the following are satisfied:

(i) $a(t,\tau)$ is a real-valued continuous nonnegative function defined for $(t,\tau) \in R$, nondecreasing as a function of τ and locally absolutely continuous as a function of t for almost every τ, including, in particular, $\tau = 0$; for each $t > 0$, $\int_0^t a(t,\tau)\, d\tau < \infty$; for each $T \ge 0$,

$$\int_0^T \int_0^t a(t,\tau)\, d\tau\, dt < \infty.$$

(ii) The partial derivative $a_t(t,\tau)$ is a measurable, nonpositive function defined almost everywhere on R. For $t > 0$, except possibly on a set E of measure zero, the function $\tau \to a_t(t,\tau)$ $(0 \le \tau < t)$ is defined almost everywhere and is nonincreasing; without loss of generality, we assume that for $t > 0$, $t \notin E$, this function extends to a nonpositive nonincreasing function (still denoted $a_t(t,\tau)$) on all of $[0,t)$.

(iii) $a_t(t,\tau)(t-\tau)^2 \to 0$ as $\tau \to t^-$ for almost all $t > 0$, and for every $T \ge 0$,

$$\int_0^T \int_0^t |a_t(t,\tau)|(t-\tau)\, d\tau\, dt < \infty.$$

(iv) *There exists* $\epsilon_1 > 0$ *such that* $\limsup_{t \to \infty} \int_{[t-\epsilon_1, t)} (t-\tau)\, d_\tau\, a(t,\tau) < \infty$ *where the integral is an improper Riemann-Stieltjes integral.*

(v) *With* ϵ_1 *as in* (iv), *there is a* $\delta > 0$ *such that* $\delta < \epsilon_1$ *and*

$$\limsup_{y \to \infty} a(y, y - \epsilon_1) < \infty,$$

$$\limsup_{y \to \infty} \int_y^{y+\delta} a(t, t - \epsilon_1)\, dt < \infty,$$

$$\limsup_{y \to \infty} (-1)\int_y^{y+\delta} a_t(t, t - \epsilon_1)\, dt < \infty,$$

$$\limsup_{y \to \infty} (-1)\int_y^{y+\delta} \left[\int_{[0, t - \epsilon_1]} (t - \tau)\, d_\tau \{a_t(t,\tau)\}\right] dt < \infty .$$

(vi) *There exists* ϵ_2, η, $M > 0$ *such that for each* $\tau^* > M$ *there is a* $t^* \geq \tau^* + 2\epsilon_2$ *such that the measure* $-\{d_\tau\, a_t(t,\tau) + \eta\, d\tau\}$ *is nonnegative on* $N_{\epsilon_2}(\tau^*)$ *for almost every* $t \in \underline{N}_{\epsilon_2}(t^*)$. *Here* $N_{\epsilon_2}(\tau^*) \equiv \{\tau \mid \tau^* - \epsilon_2 \leq \tau \leq \tau^* + \epsilon_2\}$ *and* $\underline{N}_{\epsilon_2}(t^*) \equiv \{t \mid t^* - \epsilon_2 \leq t \leq t^*\}$.

(vii) (1.2) *and* (1.3) *are satisfied.*

Then $g(x(t)) \to 0$ *as* $t \to \infty$.

The extension of $a_t(t,\tau)$ in (ii) is made merely to permit us to use the Riemann-Stieltjes integrals $\int d_\tau\, a_t(t,\tau)$; in particular, by (i), we may assume that $a(t,0)$ is differentiable on $(0,\infty)\setminus E$. Note also that (v) includes the tacit assumption that $a_t(t, t - \epsilon_1)$ is locally integrable for large t; since $a_t(t,\tau)$ is jointly measurable, this is no restriction.

In our work we will require the use of the improper Riemann-Stieltjes integral. For a discussion of the relevant theory, see L. M. Graves

[3, Chapters 10,12] and H. P. Thielman [26, Chapter 9]. In particular, we shall integrate by parts in expressions such as

(I) $\int_0^t h(\tau)\ \mu(t,\tau)\ d\tau$

where $h(\tau) \in C[0,t]$ with primitive $H(\tau)$ and $\mu(t,\tau)$ is nondecreasing and continuous as a function of τ for t fixed and $\tau \in [0,t)$. We consider (I) as the improper Riemann-Stieltjes integral

$$(I) \equiv \int_{[0,t)} U_t(\tau)\ dH(\tau) = \lim_{\varepsilon \to 0} \int_{[0,t-\varepsilon]} U_t(\tau)\ dH(\tau),$$

where $U_t(\tau) = \mu(t,\tau)$. Now for each $\varepsilon > 0$ with t fixed, $U_t(\tau)$ is of bounded variation on $[0,t-\varepsilon]$ and $H(\tau)$ is continuous. Thus the Stieltjes integrals $\int_{[0,t-\varepsilon]} H(\tau)\ d_\tau U_t(\tau)$ and $\int_{[0,t-\varepsilon]} U_t(\tau)\ dH(\tau)$ exist. Moreover

$$\int_{[0,t-\varepsilon]} U_t(\tau)\ dH(\tau) + \int_{[0,t-\varepsilon]} H(\tau)\ d_\tau U_t(\tau) =$$

$$H(t-\varepsilon)\ U_t(t-\varepsilon) - H(0)\ U_t(0).$$

If we assume for example, that $\mu(t,\tau)(t-\tau)^j \to 0$ as $\tau \to t^-$ and $H(t-\varepsilon) = O(\varepsilon^j)$ as $\varepsilon \to 0^+$ (j is a positive integer), which are precisely the type of conditions we later need, then

$$(3.1) \qquad (I) = -\ H(0)\ U_t(0) + \int_{[0,t)} H(\tau)\ d_\tau U_t(\tau)$$

where the last integral is again an improper Riemann-Stieltjes integral. In our proof we will write equations similar to (3.1) as follows:

$$(3.2) \qquad \int_0^t h(\tau)\ \mu(t,\tau)\ d\tau = -\{H(0)\ \mu(t,0) + \int_0^t H(\tau)\ d_\tau\mu(t,\tau)\}$$

and when we later do integration by parts with Stieltjes integrals, we do so in the sense just described.

A remark concerning existence and uniqueness of solutions of (1.1) satisfying (1.2) is in order. Kiffe [7] extends the concept of positivity found in A. Halanay [4] to functions of two variables. He then proves, using well known arguments [17, Theorem 1.1, p. 87] that if (1.3) holds, together with

(H1) $a(t,\tau)$ <u>is a function of positive type and for every</u> $T \geq 0$

$$\int_0^T \int_0^t |a(t,\tau)|d\tau\ dt < \infty,$$

(H2) <u>There exists a constant</u> $K > 0$ <u>so that</u> $|g(x)| \leq K[1 + G(x)]$ <u>where</u> $G(x) = \int_0^x g(\xi)\ d\xi,\ \inf\limits_{(-\infty,\infty)} G(x) > -\infty,$ <u>and</u>

$$\lim_{|x|\to\infty} \sup G(x) = \infty.$$

then a solution $x(t)$ satisfying (1.2) exists. By hypotheses (i), (ii), (iii) of our theorem and the identity (3.8) found in the proof of our theorem we see that (H1) above is satisfied. Thus one need merely add (H2) to the list of hypotheses in our theorem to guarantee global existence of solutions. If in addition g satisfies the condition $|g(x) - g(y)| \leq K_\Omega\ |x - y|$ for each bounded set Ω in $\mathbb{R}$ and $x,\ y \in \Omega$, then one can prove [17, p. 91] that (1.1) has a unique solution.

In section 4 we compare our result in detail to those of [7] and [9], and we note a weakened form of (vi) under which our theorem remains valid.

Proof of THEOREM 3.1:

We first claim that for almost every $\tau_0 \geq 0$ we have

$$(3.3) \qquad (t - \tau_0)\ a(t,\tau_0) \to 0 \text{ as } t \to \tau_0^+ ,$$

and for each $t > 0$,

$$(3.4) \qquad (t - \tau)\ a(t,\tau) \to 0 \text{ as } \tau \to t^- .$$

To see (3.3) fix $T > 0$, then by (i) we have $\int_0^T \int_0^t a(t,\tau)\, d\tau\, dt < \infty$. Thus by Fubini's Theorem $\int_0^T \int_\tau^T a(t,\tau)\, dt\, d\tau < \infty$. Hence for all most all $\tau_0 \in [0,T]$, $\int_{\tau_0}^T a(t,\tau_0)\, dt < \infty$. But then for $\tau_0 < t < T$, we may use (i) and (ii) to show that,

$$(t - \tau_0)\ a(t,\tau_0) \leq \int_{\tau_0}^t a(\mu,\tau_0)\, d\mu \leq \int_{\tau_0}^T a(t,\tau_0)\, dt < \infty .$$

Consequently, $(t - \tau_0)\ a(t,\tau_0) \to 0$ as $t \to \tau_0^+$ for such τ_0, as claimed. The proof of (3.4) follows by a similar argument using (i).

Next we develop identity (3.8) which is the key to our proof; compare [7, Eq. (2.1)], noting that the expression $a_{t\tau}(t,\tau)$ in [7] means $\frac{\partial}{\partial t}\left(\frac{\partial a(t,\tau)}{\partial \tau}\right)$. By (i) and (vii) we see that for any $T \geq 0$, $\int_0^T \int_0^t |v(t)\ a(t,\tau)\ v(\tau)|\, d\tau\, dt < \infty$ where $v(t) = g(x(t))$; as a consequence Fubini's theorem yields the identity

$$(3.5) \qquad \int_0^T v(t) \int_0^t a(t,\tau)\ v(\tau)\, d\tau\, dt = \int_0^T v(\tau) \int_\tau^T v(t)\ a(t,\tau)\, dt\, d\tau$$

for every $T \geq 0$. Integrating by parts, using the boundedness of $v(t)$ together with (iii) and (3.3), we see that (3.5) may be written

$$(3.6)\qquad \int_0^T v(t) \int_0^t a(t,\tau)\, v(\tau)\, d\tau\, dt = \int_0^T v(\tau)\, a(T,\tau) \left[\int_\tau^T v(s)\, ds\right] d\tau - \int_0^T v(\tau) \int_\tau^T a_t(t,\tau) \left[\int_\tau^t v(s)\, ds\right] dt\, d\tau,$$

for every $T \geq 0$. Integrating the first integral on the right of (3.6) by parts and using (3.4) on the boundary term we find that

$$(3.7)\qquad \int_0^T v(\tau)\, a(T,\tau) \left(\int_\tau^T v(s)\, ds\right) d\tau = \frac{a(T,0)}{2} \left[\int_0^T v(s)\, ds\right]^2 + \tfrac{1}{2} \int_0^T \left[\int_\tau^T v(s)\, ds\right]^2 d_\tau\, a(T,\tau).$$

Next we consider the last integral in (3.6). Using (iii) we may interchange the order of integration so that

$$-\int_0^T v(\tau) \int_t^T a_t(t,\tau) \left(\int_\tau^t v(s)\, ds\right) dt\, d\tau = -\int_0^T \int_0^t a_t(t,\tau)\, v(\tau) \left(\int_\tau^t v(s)\, ds\right) d\tau\, dt\,.$$

Now by (ii) we may integrate by parts again; using (iii) on the boundary term, we see that the above expression is equal to

$$-\int_0^T \frac{a_t(t,0)}{2} \left[\int_0^t v(s)\, ds\right]^2 dt - \tfrac{1}{2} \int_0^T \int_0^t \left[\int_\tau^t v(s)\, ds\right]^2 d_\tau\, \{a_t(t,\tau)\}\, dt.$$

Now substituting back to (3.6) and using (3.7), we obtain the desired identity,

$$(3.8)\qquad \int_0^T v(t) \int_0^t a(t,\tau)\, v(\tau)\, d\tau\, dt = \frac{a(T,0)}{2} \left[\int_0^T v(s)\, ds\right]^2$$

$$+ \tfrac{1}{2} \int_0^T \left[\int_\tau^T v(s)\, ds\right]^2 d_\tau\, a(T,\tau) - \tfrac{1}{2} \int_0^T a_t(t,0)\left[\int_0^t v(s)\, ds\right]^2 dt$$

$$- \tfrac{1}{2} \int_0^T \int_0^t \left[\int_\tau^t v(s)\, ds\right]^2 d_\tau\, \{a_t(t,\tau)\}\, dt,$$

which is true for every $T \geq 0$.

We now return to equation (1.1). Multiply by $g(x(t))$ and integrate, obtaining

$$(3.9)\qquad G(x(T)) - G(x(0)) + \int_0^T g(x(t)) \int_0^t a(t,\tau)\, g(x(t))\, d\tau\, dt$$

$$= \int_0^T g(x(t))\, f(t)\, dt$$

for every $T \geq 0$, where $G(x) = \int_0^x g(\xi)\, d\xi$. From (vii) we see that $\sup\limits_{0 \leq t < \infty} |g(x(t))| \equiv K < \infty$, and $\sup\limits_{T \geq 0} |G(x(T))| < \infty$; thus

$$\sup_{T \geq 0} \left|\int_0^T g(x(t))\, f(t)\, dt\right| < \infty .$$

Therefore by (3.9),

$$(3.10)\qquad \sup_{0 \leq T < \infty} \left|\int_0^T v(t) \int_0^t a(t,\tau)\, v(\tau)\, d\tau\, dt\right| < \infty .$$

Now by (3.10), (3.8), (i) and (ii) we have

$$(3.11)\qquad \sup_{0 \leq T < \infty} a(T,0)\left[\int_0^T g(x(s))\, ds\right]^2 < \infty \text{ and}$$

$$(3.12) \qquad \sup_{0 \le T < \infty} (-1) \int_0^T \int_0^t \left[\int_\tau^t g(x(s))\, ds\right]^2 d_\tau \{a_t(t,\tau)\}\, dt < \infty .$$

We now claim that

$$(3.13) \qquad \limsup_{t \to \infty} \left|\int_0^t a(t,\tau)\, g(x(\tau))\, d\tau\right| < \infty ;$$

we will later use this to show $g(x(t))$ is uniformly continuous on $[0,\infty)$. With $\varepsilon = \varepsilon_1$ from (iv), let

$$B(t) = \int_0^{t-\varepsilon} \left[\int_\tau^t g(x(s))\, ds\right] d_\tau a(t,\tau)$$

$$C(t) = \int_{t-\varepsilon}^t \left[\int_\tau^t g(x(s))\, ds\right] d_\tau a(t,\tau) \text{ and}$$

$$D(t) = a(t,0) \int_0^t g(x(\tau))\, d\tau .$$

Now using (i) and integration by parts we see that

$$\int_0^t a(t,\tau)\, g(x(\tau))\, d\tau = a(t,0) \int_0^t g(x(s))\, ds + \int_0^t \left(\int_\tau^t g(x(s))\, ds\right) d_\tau a(t,\tau).$$

Thus $\int_0^t a(t,\tau)\, g(x(\tau))\, d\tau = B(t) + C(t) + D(t)$. According to assumption (iv),

$$\limsup_{t \to \infty} |C(t)| < \infty .$$

Now suppose it is not true that

$$\limsup_{t \to \infty} |D(t)| < \infty .$$

Then there is a sequence $\{t_n\}$ such that $t_n \to \infty$ as $n \to \infty$ and $|D(t_n)| \to \infty$ as $n \to \infty$. However, by (i) $0 \le a(t_n,0) \le a(\epsilon,0)$ so that we must have $|\int_0^{t_n} g(x(\tau))\, d\tau| \to \infty$ as $n \to \infty$. Therefore $a(t_n,0)[\int_0^{t_n} g(x(\tau))\, d\tau]^2 \to \infty$ as $n \to \infty$, which is a contradiction of (3.11). Next we consider $B(t)$; integrating by parts, and using the nondecreasing property of $a(t,\tau)$ as a function of τ, we see that

$$(3.14) \qquad \begin{aligned} B(t) &= a(t,t-\epsilon) \int_{t-\epsilon}^{t} g(x(s))\, ds - a(t,0) \int_0^t g(x(s))\, ds \\ &\quad + \int_0^{t-\epsilon} a(t,\tau)\, g(x(\tau))\, d\tau \\ &\equiv B_1(t) + B_2(t) + B_3(t). \end{aligned}$$

Suppose our claim (3.13) is not true; by (v) and (vii) we have $\limsup_{t\to\infty} B_1(t) < \infty$; therefore if we define

$$(3.15) \qquad \tilde{B}(t) \equiv B_2(t) + B_3(t)$$

then it must be true that there exist $t_n \to \infty$ as $n \to \infty$ such that $|\tilde{B}(t_n)| \to \infty$ as $n \to \infty$. Let δ be given by (v); by (i) and (vii) we see that $B_2(t)$ is absolutely continuous on $[t_n, t_n + \delta]$ for every n. Thus for $t \in [t_n, t_n + \delta]$ we have

$$(3.16) \qquad \int_t^{t_n} d\, B_2(s) = -\int_t^{t_n} \{a_t(s,0) \int_0^s g(x(r))\, dr + a(s,0)\, g(x(s))\}\, ds.$$

Now by (i), (ii), (vii) and Fubini's Theorem we see that

$$(3.17)\qquad \int_t^{t_n} dB_3(s) = \int_t^{t_n} a(s,s-\varepsilon)\, g(x(s-\varepsilon))\, ds + \int_t^{t_n}\int_0^{s-\varepsilon} a_t(s,\tau)\, g(x(\tau))\, d\tau\, ds.$$

Consequently $B_3(t)$ is absolutely continuous on $[t_n, t_n+\delta]$ for each n, and thus so is $\tilde{B}(t)$. Therefore, we have

$$(3.18)\qquad \int_t^{t_n} d\tilde{B}(s) = \tilde{B}(t_n) - \tilde{B}(t).$$

Since $a_t(t,\tau)$ is nonincreasing as a function of τ for almost every t, we may integrate the last integral in (3.17) by parts to obtain

$$(3.19)\qquad \int_t^{t_n} dB_3(s) = \int_t^{t_n} a(s,s-\epsilon)\, g(x(s-\epsilon))\, ds + \int_t^{t_n} \Big\{ a_t(s,0)\int_0^s g(x(r))\, dr - a_t(s,s-\epsilon)\int_{s-\epsilon}^s g(x(r))\, dr + \int_0^{s-\epsilon}\Big(\int_\tau^s g(x(r))\, dr\Big)\, d_\tau a_t(s,\tau)\Big\}\, ds .$$

By (i) and (v) there exists an integer k so that for $n \geq k$ and $t \in [t_n, t_n+\delta]$ each term in the second integrand on the right in (3.19) is integrable with respect to s on (t_n, t). Thus combining (3.15), (3.16), (3.18) and (3.19) we have

$$(3.20)\qquad \tilde{B}(t_n) - \tilde{B}(t) = \int_t^{t_n} a(s,s-\varepsilon)\, g(x(s-\varepsilon))\, ds$$

$$- \int_t^{t_n} a(s,0)\, g(x(s))\, ds$$

$$- \int_t^{t_n} a_t(s,s-\varepsilon)\left(\int_{s-\varepsilon}^{s} g(x(r))dr\right)ds$$

$$+ \int_t^{t_n} \int_0^{s-\varepsilon}\left(\int_T^{s} g(x(r))dr\right) d_\tau \{a_t(s,\tau)\}\, ds .$$

which is valid for $n \geq k$ and $t \in [t_n, t_n + \delta]$. Therefore, we have for $n \geq k$ and $t \in [t_n, t_n + \delta]$

$$(3.21)\qquad |\tilde{B}(t_n) - \tilde{B}(t)| \leq \int_{t_n}^{t_n+\delta} a(t,t-\varepsilon)|g(x(t-\varepsilon))|dt$$

$$+ \int_{t_n}^{t_n+\delta} a(t,0)|g(x(t))|dt$$

$$- \int_{t_n}^{t_n+\delta} a_t(t,t-\varepsilon)\left|\int_{t-\varepsilon}^{t} g(x(s))ds\right|dt$$

$$- \int_{t_n}^{t_n+\delta} \int_0^{t-\varepsilon}\left|\int_\tau^{t} g(x(s))\, ds\right| d_\tau\{a_t(t,\tau)\}\, dt.$$

However, by (i), (v), (vii), (3.21) and our assumption we see, upon choosing n sufficiently large, that $|\tilde{B}(t_n) - \tilde{B}(t)| < \frac{1}{2}|\tilde{B}(t_n)|$ for $t \in [t_n, t_n + \delta]$. Therefore for $t \in [t_n, t_n + \delta]$ and n chosen large enough we have

$$(3.22)\qquad |\tilde{B}(t)| > \tfrac{1}{2}|\tilde{B}(t_n)|.$$

In particular, for large enough n, $\tilde{B}(t)$ is of one sign on $[t_n, t_n + \delta]$. Now integrating (1.1), we obtain the inequality

$$(3.23)\qquad |x(t_n+\delta) - x(t_n)| > \int_{t_n}^{t_n+\delta} |\tilde{B}(t)|dt$$

$$- \int_{t_n}^{t_n+\delta} |C(t)|dt - \int_{t_n}^{t_n+\delta} |D(t)|dt - \int_{t_n}^{t_n+\delta} |f(t)|dt.$$

Thus by (vii), (3.22), (3.23) and the above we have $|x(t_n+\delta) - x(t_n)| \to \infty$ as $n \to \infty$, which contradicts (vii), and therefore (3.13) must be true.

By (1.1) and (3.13) it follows that $x(t)$ is uniformly continuous on $[0,\infty)$. But $g(x) \in C(-\infty,\infty)$ and $x(t)$ is bounded, so $g(x(t))$ is uniformly continuous on $[0,\infty)$.

Now suppose it is not true that $g(x(t)) \to 0$ as $t \to \infty$. Then there is a sequence $\{T_k\}$, $T_k \nearrow \infty$ as $k \nearrow \infty$ and η_1, $\delta > 0$ such that $|g(x(t))| \geq \eta_1$ for $T_k - 2\delta < t < T_k + 2\delta$ and for every k. Without loss of generality, we may assume that $T_{k+1} - T_k > 4\delta$. Now for each k let $\tau_k = T_k - \delta$. By (vi) there exist ϵ_2, η_2, $M > 0$ such that for each $\tau_k > M$ there exist $t_k > \tau_k + 2\epsilon_2$ so that the measure $-[d_\tau\{a_t(t,\tau)\} + \eta_2\, d\tau]$ is nonnegative on $N_{\epsilon_2}(\tau_k)$ for almost every $t \in \underline{N}_{\epsilon_2}(t_k)$. Since $T_k \uparrow \infty$, we may assume that $t_k > t_{k-1} + \epsilon_2$. Let $2\epsilon = \min\{\delta,\epsilon_2\}$ and for each k consider $G_t^k(\tau) = \int_\tau^t g(x(s))ds$ defined for $\tau \in N_{2\epsilon}(\tau_k)$, $t \in \underline{N}_{2\epsilon}(t_k)$. We then have $\left|\frac{d}{d\tau} G_t^k(\tau)\right| = |g(x(\tau))| \geq \eta_1$ for each k. Thus for each k and all t in $\underline{N}_{2\epsilon}(t_k)$ we can find an interval $(\alpha_k(t), \alpha_k(t) + \epsilon)$ where $|G_t^k(\tau)| \geq \epsilon\,\eta_1$ for all pairs (t,τ) with $t \in \underline{N}_{2\epsilon}(t_k)$, $\tau \in (\alpha_k(t), \alpha_k(t)+\epsilon)$. (Either $\alpha_k(t) = \tau_k - 2\epsilon$ or $\alpha_k(t) = \tau_k + \epsilon$, depending on t.) Therefore for every $\tau_k > M$ and $t \in \underline{N}_{2\epsilon}(t_k)$ we have

$$(-1)\int_0^t \left[\int_\tau^t g(x(s))\,ds\right]^2 d_\tau \{a_t(t,\tau)\} \geq$$

$$(-1)\int_{\alpha_k(t)}^{\alpha_k(t)+\epsilon} \left[\int_\tau^t g(x(s))\,ds\right]^2 d_\tau \{a_t(t,\tau)\} \geq \epsilon^3\,\eta_1^2\,\eta_2 .$$

Thus

$$(-1)\int_{t_k-2\epsilon}^{t_k} \int_0^t \left[\int_\tau^t g(x(s))\,ds\right]^2 d_\tau \{a_t(t,\tau)\}\,dt \geq$$

$$\int_{t_k-2\epsilon}^{t_k} \epsilon^3\,\eta_1^2\,\eta_2\,dt = 2\epsilon^4\,\eta_1^2\,\eta_2 .$$

Now choose N' large enough so that $\tau_{N'} > M$. But then since $t_k > t_{k-1} + \epsilon_2 \geq t_{k-1} + 2\epsilon$ and $t_k - 2\epsilon \geq \tau_k + 2\epsilon$ for every k, we have upon choosing $N > N'$,

(3.24)
$$(-1)\int_0^{T_N} \int_0^t \left[\int_\tau^t g(x(s))\,ds\right]^2 d_\tau \{a_t(t,\tau)\}\,dt \geq$$

$$(-1)\sum_{k=N'}^{N} \int_{t_k-2\epsilon}^{t_k} \int_0^t \left[\int_\tau^t g(x(s))\,ds\right]^2 d_\tau \{a_t(t,\tau)\}\,dt \geq$$

$$\sum_{k=N'}^{N} 2\epsilon^4\,\eta_1^2\,\eta_2 \to \infty \text{ as } N \to \infty;$$

so we have contradicted (3.12). Therefore we must have $g(x(t)) \to 0$ as $t \to \infty$ and hence our theorem is established.

4. Dicussion of the Result and a Generalization

Levin [9, Theorem 1] proves a theorem similar to ours under rather strong hypotheses on the kernel $a(t,\tau)$. One of Kiffe's results, stated below as Theorem A, uses the methods of Londen [15], which require fewer smoothness restrictions on the kernel than Levin imposed. As Kiffe points out, Levin's hypotheses imply Kiffe's. Except for our hypothesis (vi), we shall show that our hypotheses are in turn implied by Kiffe's. Moreover, by weakening (vi) to (vi*) (see below) we easily obtain Corollary 4.1, which contains both Theorem 3.1 and Theorem A.

THEOREM A [7, Theorem 1]: *Let* R *be as defined earlier and assume that the following hypotheses are satisfied*:

(i′) $a(t,\tau)$ *is a real-valued function defined for* $(t,\tau) \in R$, $a(t,0) \geq 0$ *for* $t > 0$, $t^2\, a(t,0) \to 0$ *as* $t \to 0^+$, *for each fixed* $t > 0$ $(t-\tau)\, a(t,\tau) \to 0$ *as* $\tau \to t^-$, *for each* $t > 0$ $\int_0^t |a(t,\tau)|\, d\tau < \infty$, *and for every* $T > 0$ $\int_0^T \int_0^t |a(t,\tau)|\, d\tau dt < \infty$,

(ii′) $a_t(t,0) \in C(0,\infty)$, $a_t(t,0) \leq 0$ *for* $t > 0$, *and* $(-1) \int_0^1 t^2\, a_t(t,0) dt < \infty$,

(iii′) $a_\tau(t,\tau) \in C(R)$, $a_\tau(t,\tau) \geq 0$ for $(t,\tau) \in R$, *for almost every* τ, $0 < \tau < \infty$, $(t-\tau)^2\, a_\tau(t,\tau) \to 0$ as $t \to \tau^+$, *and for every* $T > 0$, $\int_0^T \int_0^t (t-\tau)\, a_\tau(t,\tau)\, d\tau\, dt < \infty$,

(iv′) $a_{\tau t}(t,\tau) \in C(R)$, $a_{\tau t}(t,\tau) \leq 0$ *for* $(t,\tau) \in R$, *and for every* $T > 0$ $(-1) \int_0^T \int_0^t (t-\tau)^2\, a_{\tau t}(t,\tau)\, d\tau\, dt < \infty$, (v′) <u>There is an</u> $\epsilon > 0$ <u>such that</u>

$$\limsup_{t \to \infty} \int_{t-\epsilon}^{t} (t - \tau)\, a_\tau(t,\tau)\, d\tau < \infty,$$

(vi′) <u>Given the set</u> ϵ <u>of</u> (v′), <u>there is a</u> $\delta > 0$ <u>such that</u> $\delta < \epsilon$,

$$\limsup_{y \to \infty} \int_{y}^{y+\delta} a(t,t-\epsilon)\, dt < \infty,$$

$$\limsup_{y \to \infty} \int_{y}^{y+\delta} a_\tau(t,t-\epsilon)\, dt < \infty, \text{ \underline{and}}$$

$$\limsup_{y \to \infty} (-1) \int_{y}^{y+\delta} \int_{0}^{t-\epsilon} (t - \tau)\, a_{\tau t}(t,\tau)\, d\tau\, dt < \infty,$$

(vii′) <u>For every</u> $\eta > 0$,

$$\liminf_{t \to \infty} (-1) \int_{t-\eta}^{t} (t - \tau)^2\, a_{\tau t}(t,\tau)\, d\tau > 0,$$

(viii′) (1.3) <u>is satisfied</u>.

(ix′) (1.2) <u>is satisfied</u>.

<u>Then</u> $g(x(t)) \to 0$ <u>as</u> $t \to \infty$.

(Again, we point out that Kiffe [7] uses the notation $a_{t\tau}(t,\tau) \equiv (\frac{\partial}{\partial t})(\frac{\partial a(t,\tau)}{\partial \tau})$; throughout our discussion, including our statement of Theorem A, we use the conventional notation $a_{\tau t} \equiv (\frac{\partial}{\partial t})(\frac{\partial a(t,\tau)}{\partial \tau})$; in fact (ii′), (iii′), (iv′) imply $a_{\tau t} = a_{t\tau}$).

After reading both sets of hypotheses, one can readily verify that the following implications are true:

1) (i′,ii′,iii′) → (i)

2) (i′,ii′,iii′,iv′) → (ii)

3) (v′) → (iv)

4) (viii′,ix′) → (vii)

We now show Kiffe's hypotheses imply our hypotheses (iii) and (v). It is apparent that several of Kiffe's hypotheses are not obtainable from what we have assumed. First, we show that (ii′,iii′,iv′) → (iii). We show that for every $T \geq 0$,

$$(4.1) \qquad \int_0^T \int_0^t |a_t(t,\tau)|(t-\tau)\, d\tau\, dt < \infty .$$

Let $T \geq 0$ and consider $F_n(t) = \int_0^{t-1/n} -(t-\tau)^2\, a_{t\tau}(t,\tau)\, d\tau$ and $F(t) = \int_0^t -(t-\tau)^2\, a_{t\tau}(t,\tau)\, d\tau$. Clearly $F_n(t) \uparrow F(t)$ as $n \uparrow \infty$, moreover, by (iv′) $\int_0^T |F(t)|\, dt < \infty$. Therefore by the monotone convergence theorem we have $0 \leq \int_0^T F_n(t)\, dt \uparrow \int_0^T F(t)\, dt \equiv Q < \infty$, as $n \uparrow \infty$. Now for each n we may use integration by parts to write

$$F_n(t) = -\frac{1}{n^2}\, a_t(t, t-\frac{1}{n}) + t^2\, a_t(t,0)$$

$$- 2\int_0^{t-1/n} (t-\tau)\, a_t(t,\tau)\, d\tau .$$

Thus for <u>each</u> <u>n</u> we have

$$Q \geq \int_0^T \{-\frac{1}{n^2}\, a_t(t,t-\frac{1}{n}) + t^2\, a_t(t,0) - 2\int_0^{t-1/n} (t-\tau)\, a_t(t,\tau)\, d\tau\}\, dt.$$

By (ii′), $\int_0^T t^2\, a_t(t,0)\, dt > -\infty$ and since $-(t-\tau)\, a_t(t,\tau) \geq 0$ we see upon taking the limit as $n \uparrow \infty$ termwise that

$$-\int_0^T \int_0^t (t-\tau)\, a_t(t,\tau)\, d\tau\, dt < \infty;$$

thus (4.1) is satisfied. We now show that $a_t(t,\tau)(t-\tau)^2 \to 0$ as $\tau \to t^-$ for almost all $t > 0$. Fix $T > 0$. By (4.1) we have

$$\int_0^T \int_0^t |a_t(t,\tau)|(t-\tau)\,d\tau dt < \infty$$

Thus for almost every $0 < t_0 < T$ we have

$$\int_0^{t_0} |a_t(t_0,\mu)|(t_0-\mu)\,d\mu < \infty \,. \tag{4.2}$$

Let $I(s) = \int_s^{t_0} |a_t(t_0,\mu)|(t_0-\mu)\,d\mu$; then $I(s)$ is absolutely continuous on $[0,t_0]$ and $\lim_{s\to t_0^-} I(s) = 0$. Also $I'(s) = a_t(t_0,s)(t_0-s)$, so we see that $I(s) \in C[\tau,t_0]$ and $I(s)$ is differentiable on (τ,t_0) for any $0 < \tau < t_0$. Thus by the mean value theorem there exists $\theta \in (\tau,\xi(\tau))$ such that $I(\xi(\tau)) - I(\tau) = I'(\theta)(\xi(\tau)-\tau)$, where $\xi(\tau)$ is the midpoint of the interval (τ,t_0). But then by (iv′) $I(\xi(\tau)) - I(\tau) \leq a_t(t_0,\tau)$ $(t_0-\theta)(\frac{t_0-\tau}{2}) \leq a_t(t_0,\tau)(\frac{t_0-\tau}{4})^2 \leq 0$. Letting $\tau \to t_0^-$ we see that $\lim_{\tau\to t_0^-} a_t(t_0,\tau)(t_0-\tau)^2 = 0$ for almost every $0 < t_0 < T$, as desired.

We now show that Kiffe's hypotheses imply our assumption (v). In view of (vi′) we need only show that

(a) $\limsup_{y\to\infty} a(y,y-\epsilon_1) < \infty$ and

(b) $\limsup_{y\to\infty} (-1)\int_y^{y+\delta} a_t(t,t-\epsilon_1)\,dt < \infty$

where ϵ_1 is as chosen in (iv). Let $F(t) = a(t,t-\epsilon_1)$. By (ii′), (iii′), and (iv′) we have $F'(t) = a_t(t,t-\epsilon_1) + a_\tau(t,t-\epsilon_1) \leq a_\tau(t,t-\epsilon_1)$. Thus with δ as in (vi′),

$$\limsup_{y\to\infty} \left\{ \sup_{0<\lambda<\delta} \int_{y+\lambda}^{y+\delta} F'(t)\, dt \right\} \leq \limsup_{y\to\infty} \int_{y}^{y+\delta} a_{\tau}(t,t-\epsilon_1)\, dt < \infty.$$

Consequently, if $F(y_n) \to \infty$ as $n \to \infty$ then $\inf_{y_n \leq t \leq y_n + \delta} F(t) \to \infty$, which contradicts (vi′), hence (a) must be valid. To see (b) consider the identity,

$$\int_{y}^{y+\delta} \{a_t(t,t-\epsilon_1) + a_{\tau}(t,t-\epsilon_1)\}\, dt = \int_{y}^{y+\delta} F'(t)\, dt$$
$$= a(y+\delta,\ y+\delta-\epsilon_1) - a(y,\ y-\epsilon_1).$$

Thus we have

$$(-1) \int_{y}^{y+\delta} a_t(t,t-\epsilon_1)\, dt = a(y,y-\epsilon_1) - a(y+\delta,\ y+\delta-\epsilon_1)$$
$$+ \int_{y}^{y+\delta} a_{\tau}(t,t-\epsilon_1)\, dt.$$

Therefore,

$$(-1) \int_{y}^{y+\delta} a_t(t,t-\epsilon_1)\, dt \leq a(y,y-\epsilon_1) + \int_{y}^{y+\delta} a_{\tau}(t,t-\epsilon_1)\, dt,$$

and so (a) and (vi′) yield $\limsup_{y\to\infty} (-1) \int_{y}^{y+\delta} a_t(t,t-\epsilon_1)\, dt < \infty$ as desired.

We now discuss our hypothesis (vi) and compare it to (vii′) of Kiffe (which is the same as part 3 of Levin's hypothesis (H4) in [9]).

As Levin points out, his hypothesis (H4) may not simply be dropped. He cites the example $a(t,\tau) = \exp\{-(t + 2)^2(t - \tau)\}$, and by elementary calculations he shows that (H4) is violated and that, with this kernel, solutions of

$$\begin{cases} x'(t) + \int_0^t x(\tau)\, a(t,\tau)\, d\tau = 0 \\ x(0) = 1 \end{cases}$$

do not tend to 0 as $t \to \infty$, while this kernel does satisfy the remaining hypotheses of his Theorem 1. By our earlier remarks, this kernel also satisfies all of our hypotheses except (vi). To illustrate the merit of our hypothesis (vi), we exhibit a family of kernels which satisfy (vi) but not (vii$'$) or Levin's (H4).

Again let $R = \{(t,\tau) \mid t > 0,\ 0 \leq \tau < t\}$ and let Γ be the region bounded by the curve $\tau = \varphi(t-c) < t$ for $t \geq c > 0$ and the t-axis, where $\varphi(t)$ is a C^1 function with $\varphi'(t) > 0$, $\varphi(0) = 0$ and $0 \leq t - \varphi(t) = O(1)$, $(t \to \infty)$.

Define:

$$a(t,\tau) = \begin{cases} F'(0)\,[\tau - \varphi(t-c)] + f(0) & \text{in } R\backslash\Gamma \\ F(\tau - \varphi(t-c)) & \text{in } \Gamma \end{cases}$$

where $F(\mu)$ is any twice differentiable function on $(-\infty,0]$ with $F^{(k)}(\mu) > 0$ in $(-\infty,0)$ for $k = 0,1,2,$; $F(0) = k_0 > 0$, $F'(0) > 0$ and $F''(0) = 0$.

Thus for (t,τ) in Γ we have

$$a_t(t,\tau) = -F'(\tau - \varphi(t-c))\, \varphi'(t-c) < 0$$

$$a_\tau(t,\tau) = F'(\tau - \varphi(t-c)) > 0$$

$$a_{t\tau}(t,\tau) = -\varphi'(t-c)\, F''(\tau - \varphi(t-c)) < 0$$

and along $\tau = \varphi(t-c)$ we have

$$a(t,\tau) = F(0) > 0$$

$$a_t(t,\tau) = -F'(0)\,\varphi'(t-c) < 0$$

$$a_\tau(t,\tau) = F'(0) \qquad\qquad > 0$$

$$a_{t\tau}(t,\tau) = 0$$

and so we see that a, a_t, a_τ and $a_{t\tau}$ are all in $C(R)$. Moreover they satisfy Kiffe's and Levin's sign conditions. However, $a(t,\tau)$ does not satisfy (vii$'$) of Kiffe's Theorem 1 or the third condition of (H4) in Levin's Theorem 1. Our hypothesis (vi) is satisfied by $a(t,\tau)$ as one easily verifies by considering the tube

$$D_{\epsilon_1\epsilon_2} \equiv R \cap \{c \le t < \infty, \varphi(t-c) - \epsilon_2 < \tau < \varphi(t-c) - \epsilon_1\}$$

where $\epsilon_2 > \epsilon_1 > 0$. Clearly there is an $\eta > 0$ so that

$$-d_\tau\,\{a_t(t,\tau)\} = -a_{t\tau}(t,\tau)\,d\tau \ge \eta\,d\tau > 0$$

for $(t,\tau) \in D_{\epsilon_1\epsilon_2}$, and thus our (vi) is satisfied by $a(t,\tau)$.

If $F(\mu)$ and $\varphi(t)$ are judiciously chosen, it is a trivial matter to verify all other hypotheses of Kiffe's Theorem 1 and hence all our hypotheses. For example let $F(\mu) = c\,e^{-(\mu - 1/\sqrt{2})^2}$ on $(-\infty,0]$ and $\varphi(t) = t - \frac{1}{1+t}$ on $[0,\infty)$. We then have

$$a(t,\tau) = \begin{cases} c\sqrt{\frac{2}{e}}\,[\tau - t+c - \frac{1}{1+t-c}] + \frac{c}{\sqrt{e}} & \text{in } R\setminus\Gamma \\ c\,\exp\{-[\tau - t+c - \frac{1}{1+t-c} - \frac{1}{\sqrt{2}}]^2\} & \text{in } \Gamma \end{cases}$$

Therefore it is easy to check that the kernel satisfies all our hypotheses. We point out that $\varphi(t-c)$ was chosen so that $\limsup_{t \to \infty} \{t - \varphi(t-c)\} \leq C < \infty$, in order that (v) part 4 is not violated. In fact, the t^* mentioned in hypothesis (vi) and (vi^*) below must all satisfy $\limsup_{\tau^* \to \infty} \{\tau^* - t^*\} \leq d < \infty$. Note also that this kernel has more smoothness than our theorem requires; by choosing $F(\mu)$ to be less smooth we may construct an example that satisfies our hypotheses minimally.

Finally, we give the weakened form of (vi) mentioned earlier (suggested to the author by K. B. Hannsgen).

<u>COROLLARY 4.1</u>: <u>Theorem</u> 4.1 <u>remains valid if we weaken</u> (vi) <u>to</u>

(vi^*) <u>For each sufficiently small</u> $\epsilon_2 > 0$ <u>there exist</u> $M > 0$, $\eta > 0$, $0 < \epsilon_3 < \epsilon_2$ <u>such that if</u> $\tau^* > M$, <u>there is a</u> $t^* \geq \tau^*$ <u>with either</u>

(I) $t^* - \epsilon_2 < \tau^* \leq t^*$ <u>and</u>

$$(-1) \int_{t^*}^{t^*+\epsilon_3} \int_{t-\epsilon_2}^{t} (t-\tau)^2 \, d_\tau \, \{a_t(t,\tau)\} \, dt \geq \eta$$

<u>or</u>

(II) $\tau^* \leq t^* - \epsilon_2$ <u>and</u>

$$\int_{t^*}^{t^*+\epsilon_3} \left| \int_{\tau^*}^{\tau^*+\epsilon_2 q(t)} (\tau^* - \tau)^2 \, d_\tau \, \{a_t(t,\tau)\} \right| dt \geq \eta$$

<u>whenever</u> $q(t)$ <u>is a function on</u> $[t^*, t^* + \epsilon_3]$ <u>taking the values</u> ± 1 <u>only, with</u> $q^{-1}(1)$ <u>closed</u>.

The proof requires only minor changes in the argument given for (3.24), and so we give a sketch: again, the other hypotheses in Theorem 3.1

imply that $g(x(t))$ is uniformly continuous on $[0,\infty)$. Suppose it is not true that $g(x(t)) \to 0$ as $t \to \infty$. Then there exist $\{T_k\} \uparrow \infty$, η_1, $\delta > 0$ such that $|g(x(t))| \geq \eta_1$ for $T_k - \delta < t < T_k + \delta$. Without loss of generality we assume $T_{k+1} - T_k \geq 2\delta$. We then have $\left|\frac{d}{d\tau} \int_\tau^t g(x(s))\, ds\right| = |g(x(\tau))| \geq \eta_1$ for $\tau \in [T_k - \delta, T_k + \delta]$. Now by (vi^*) we choose $\epsilon_2 < \delta/2$ and corresponding M, η, ϵ_3. Without loss of generality assume that $T_1 > M$; then for each T_k there exists $t_k \geq T_k$ satisfying case I or II of (vi^*). We may also assume that $t_k \geq t_{k-1} + \delta (k \geq 2)$. Define $C_1^N \equiv \{k \leq N | t_k$ satisfies case I$\}$ and $C_2^N \equiv \{k \leq N | t_k$ satisfies case II$\}$ for any positive integer N. We then have

$$(-1) \int_0^{t_N+\delta} \int_0^t [\int_\tau^t g(x(s))\, ds]^2\, d_\tau \{a_t(t,\tau)\}\, dt \geq$$

$$(-1) \sum_{k=1}^N \int_{t_k}^{t_k+\delta} \int_0^t [\int_\tau^t g(x(s))\, ds]^2\, d_\tau \{a_t(t,\tau)\}\, dt \geq$$

$$(-1) \sum_{k \in C_1^N} \eta_1^2 \int_{t_k}^{t_k+\epsilon_3} \int_{t-\epsilon_2}^t (t-\tau)^2\, d_\tau \{a_t(t,\tau)\}\, dt +$$

$$\sum_{k \in C_2^N} \eta_1^2 \int_{t_k}^{t_k+\epsilon_3} \left|\int_{T_k}^{T_k+\epsilon_3 q_k(t)} (\tau - T_k)^2\, d_\tau \{a_t(t,\tau)\}\right| dt$$

$\geq N \eta_1^2 \eta \to \infty$ as $N \to \infty$, which contradicts (3.12). Here

$$q_k(t) \equiv \begin{cases} 1 \text{ if } g(x(T_k)) [\int_{T_k}^t g(x(s))\, ds] \leq 0. \\ -1 \text{ otherwise} \end{cases}$$

Thus $g(x(t)) \to 0$ as $t \to \infty$, and our corollary is established.

In addition to including Theorem A and 3.1, this corollary admits some additional kernels, such as the convolution kernels considered in [6]:

$a(t,\tau) = b(t-\tau)$ with $b(t)$ nonnegative, nonincreasing, convex and piecewise linear with changes of slope at $t = t_k \uparrow T \leq \infty$, with $\inf_k (t_{k+1} - t_k) = 0$.

References Cited

1. C. M. Dafermos, "An Abstract Volterra Equation with Applications to Linear Visoelasticity," J. Diff. Equations 7 (1970), 554-568.

2. W. K. Ergen and J. A. Nohel, "Stability of a Continuous Medium Reactor," J. Nucl. Energy, Part A: Reactor Science 10 (1959), 14-18.

3. L. M. Graves, The Theory of Functions of Real Variables, McGraw-Hill Co., New York, 1956.

4. A. Halanay, "On the Asymptotic Behavior of the Solutions of an Integrodifferential Equation," J. Math. Anal. Appl. 10 (1965), 319-324.

5. K. B. Hannsgen, "Indirect Abelian Theorems and a Linear Volterra Equation," Trans. Amer. Math. Soc. 142 (1969), 539-555.

6. __________. "On a Nonlinear Volterra Equation," Mich. Math. J. 16 (1969), 365-376.

7. T. R. Kiffe, "On Nonlinear Volterra Equations of Nonconvolution Type," J. Diff. Equations 22 (1976), 349-367.

8. J. J. Levin, "The Asymptotic Behavior of the Solution of a Volterra Equation," Proc. Amer. Math. Soc. 14 (1963), 534-541.

9. __________. "A Nonlinear Volterra Equation Not of Convolution Type," J. Diff. Equations 4 (1968), 176-186.

10. __________ and J. A. Nohel, "On a System of Integrodifferential Equations Occurring in Reactor Dynamics," J. Math. and Mech. 9 (3) (1960), 347-368.

11. __________ and __________. "Note on a Nonlinear Volterra Equation," Proc. Amer. Math. Soc. 14 (1963), 924-929.

12. __________ and __________. "Perturbations of a Nonlinear Volterra Equation," Mich. Math. J. 12 (1965), 431-447.

13. __________ and __________. "A System of Nonlinear Integrodifferential Equations," Mich. Math. J. 13 (1966), 257-270.

14. ____________ and ____________. "The Integrodifferential Equations of a Class of Nuclear Reactors with Delayed Neutrons," Arch. Rat. Mech. Anal. 31 (1968), 151-172.

15. S. O. Londen, "The Qualitative Behavior of the Solutions of a Nonlinear Volterra Equation," Mich. Math. J. 18 (1971), 321-330.

16. R. C. MacCamy and J. S. W. Wong, "Stability Theorems for Some Functional Equations," Trans. Amer. Math. Soc. 164 (1972), 1-37.

17. R. K. Miller, Nonlinear Volterra Integral Equations, W. A. Benjamin, Inc., Menlo Park, California, 1971.

18. J. A. Nohel and D. F. Shea, "Frequency Domain Methods for Volterra Equations," Advances in Math. 22 (1976), 278-305.

19. C. L. Rennolet, "Abstract Nonlinear Volterra Integrodifferential Equations of Nonconvolution Type," Thesis, University of Wisconsin, Madison, Wisconsin, 1977.

20. D. F. Shea and S. Wainger, "Variants of the Wiener-Levy Theorem, with Applications to Stability Problems for Some Volterra Integral Equations," Amer. J. Math. 97 (1975), 312-343.

21. O. J. Staffans, "Nonlinear Volterra Integral Equations with Positive Definite Kernels," Proc. Amer. Math. Soc. 51 (1975), 103-108.

22. ______________. "Positive Definite Measures with Applications to a Volterra Equation," Trans. Amer. Math. Soc. (to appear).

23. ______________. "Tauberian Theorems for a Positive Definite Form," Trans. Amer. Math. Soc. (to appear).

24. ______________. "On the Asymptotic Spectra of the Bounded Solutions of a Nonlinear Volterra Equation," Trans. Amer. Math. Soc. (to appear).

25. ______________. "On the Holomorphic Properties of the Nonlinearity in a Volterra Equation," Trans. Amer. Math. Soc. (to appear).

26. H. P. Thielman, Theory of Functions of Real Variables, Prentice-Hall, Inc., New York, 1953.

27. T. A. Welton, "A Stability Criterion for Reactor Systems," Report ORNL 1894; Proceedings of the International Conference on the Peaceful Uses of Atomic Energy 5 (1956), 377.

28. D. V. Widder, The Laplace Transform, Princeton Univ. Press, Princeton, N. J., 1946.

The vita has been removed from the scanned document

A NONLINEAR VOLTERRA
EQUATION OF NONCONVOLUTION
TYPE

by

Manfred Charles Smith

(ABSTRACT)

The purpose of this paper is to study the asymptotic behavior of bounded solutions $x(t)$ of the integrodifferential equation

$$\begin{aligned} &x'(t) + \int_0^t a(t,\tau)\, g(x(\tau))\, d\tau = f(t) \\ &x(0) = x_0 \end{aligned} \qquad (0 \le t < \infty)$$

where $a(t,\tau)$, $g(x)$ and $f(t)$ are given real-valued functions. We assume that

> $x(t)$ is a locally absolutely continuous function which satisfies the above equation a.e. for $0 \le t < \infty$, and $\sup_{0 \le t < \infty} |x(t)| < \infty$

and that $g(x) \in C(-\infty,\infty)$, $f(t) \in L^1(0,\infty)$ and we find conditions on $a(t,\tau)$ which ensure that $g(x(t)) \to 0$ $(t \to \infty)$. The problem of finding a set of minimal assumptions (even in this classical Volterra equation setting) is still open.

Our improvements on the recent results for this equation are twofold:

(i) We achieve the existence of $\lim_{t \to \infty} g(x(t))$ with fewer smoothness assumptions placed on the kernel $a(t,\tau)$

than previous works have required.

(ii) We move the restriction that $-a_{\tau t}(t,\tau)$ be weighted along $\tau = t$ to the same assumption along a curve $\tau = \varphi(t-c) \leq t-c$ such that $t-\varphi(t) = O(1)$ $(t \to \infty)$.

Our improvement (i) centers around proving the following identity for any $T > 0$ and $v(t) \in \mathcal{L}^{\infty}[0,T]$:

$$\int_0^T v(t) \int_0^t a(t,\tau)\, v(\tau)\, d\tau\, dt = \frac{a(T,0)}{2} \left[\int_0^T v(s)\, ds\right]^2$$

$$+ \tfrac{1}{2} \int_0^T \left[\int_\tau^T v(s)\, ds\right]^2 d_\tau\, a(T,\tau) - \tfrac{1}{2} \int_0^T a_t(t,0) \left[\int_0^t v(s)\, ds\right]^2 dt$$

$$- \tfrac{1}{2} \int_0^T \int_0^t \left[\int_\tau^t v(s)\, ds\right]^2 d_\tau\, \{a_t(t,\tau)\}\, dt,$$

where $d_\tau\, a(T,\tau)$ and $d_\tau\, \{a_t(t,\tau)\}$ are Stieltjes measures.

The benefit of improvement (ii) is illustrated by exhibiting an example of a kernel not satisfying this weighted assumption along $\tau = t$ but rather along a curve $\tau = \varphi(t-c) \leq t-c$. We also compare our result to those of others and we show that our assumptions are weaker than theirs to date.

We have exhibited only kernel conditions of the sufficient nature to achieve $\lim_{t \to \infty} g(x(t)) = 0$. The problem of showing what assumptions are necessary for the above limit to exist and be 0 is a difficult and still open question.