Comparative Statics Analysis of Some Operations Management Problems

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ABSTRACT

We propose a novel analytic approach for the comparative statics analysis of operations management problems on the capacity investment decision and the influenza (flu) vaccine composition decision. Our approach involves exploiting the properties of the underlying mathematical models, and linking those properties to the concept of stochastic orders relationship. The use of stochastic orders allows us to establish our main results without restriction to a specific distribution. A major strength of our approach is that it is "scalable," i.e., it applies to capacity investment decision problem with any number of non-independent (i.e., demand or resource sharing) products and resources, and to the influenza vaccine composition problem with any number of candidate strains, without a corresponding increase in computational effort. This is unlike the current approaches commonly used in the operations management literature, which typically involve a parametric analysis followed by the use of the implicit function theorem. Providing a rigorous framework for comparative statics analysis, which can be applied to other problems that are not amenable to traditional parametric analysis, is our main contribution.

We demonstrate this approach on two problems: (1) Capacity investment decision, and (2) influenza vaccine composition decision. A comparative statics analysis is integral to the study of these problems, as it allows answers to important questions such as, "does the firm acquire more or less of the different resources available as demand uncertainty increases? does the firm benefit from an increase in demand uncertainty? how does the vaccine composition change as the yield uncertainty increases?" Using our proposed approach, we establish comparative statics results on how the newsvendor's expected profit and optimal capacity decision change with demand risk and demand dependence in multi-product multi-resource newsvendor networks; and how the societal vaccination benefit, the manufacturer's profit, and the vaccine output change with the risk of random yield of strains.

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Dedication

Dedicated to science!

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Chapter 1

Introduction and Motivation

This research focuses on the comparative statics analysis of operations management problems within the context of the capacity investment decision and the influenza (flu) vaccine composition decision. A comparative statics analysis is integral to the study of these problems, as it allows answers to important questions such as, "does the firm acquire more or less of the different resources available as demand uncertainty increases? does the firm benefit (in terms of expected profit) from an increase in demand uncertainty? how does the vaccine composition change as the yield uncertainty increases?, etc." We model the capacity investment decision under uncertainty as a stochastic programming problem with recourse, and we model the influenza vaccine composition problem as a Stackelberg game with two players (i.e., the Committee making decisions on the formulation of the vaccine, and the vaccine manufacturer making decisions on the production quantity), with conflicting interests. For both models, characterization of their optimal solutions and the subsequent comparative statics analysis typically require a "parametric analysis," followed by the use of the implicit function theorem. However, such tools can quickly get "involved" as problem size increases (Van Mieghem and Rudi (2002)). For example, for the capacity investment decision problem modeled as a stochastic programming problem with recourse, the parametric analysis requires determining all possible demand-space partitions ("domains"), which increase *exponentially* with the number of "non-independent" (i.e., demand or resource sharing) products, taking away from analytical tractability. In summary, a major limitation of the existing approaches, commonly used in the operations management literature for comparative statics analysis, is that the computational effort increases exponentially with problem size.

In this research, we propose a *novel* analytic approach for the comparative statics analysis of these problems through exploiting their special structural properties and through the use of stochastic order relationships, e.g., stochastic, convex, and supermodular orders. This enables us to overcome the aforementioned limitations of the current approaches for comparative statics analysis. Specifically, a major strength of our proposed approach is that it is "scalable," i.e., it applies to the capacity investment decision problem with any number of "non-independent" (i.e., demand or resource sharing) products and resources, and to the influenza vaccine composition problem with any number of candidate strains, without a corresponding increase in computational effort. This makes the proposed approach powerful and promising for future work. Another important advantage of our approach is that it is distribution-free; this eliminates the need to restrict the analysis to a specific distribution. We demonstrate our approach on the two aforementioned problems: (1) Capacity investment decision, and (2) influenza vaccine composition decision. We provide rigorous proofs for the comparative statics analysis of the optimal solutions to these two problems.

We provide more details on each problem.

Capacity investment decision. We study the optimal capacity decision in multi-product multi-resource newsvendor networks under responsive and endogenous pricing, and with imperfectly substitutable products. Demands of the substitutable products are linked through the pricing of all products in the product family. We explicitly consider own- and cross-price effects, which contribute linearly to product demands. We formulate this decision problem as a two-stage stochastic programming problem with recourse: Resource capacities are determined *ex-ante*, when demand curves are uncertain but the joint probability distribution of demand intercepts is known (first stage); and pricing and production decisions are postponed and made *ex-post*, when demand curves are observed (second stage). Thus the second stage decision is constrained by the first stage capacity decision and impacted by the demand curve realizations.

Influenza vaccine composition decision. We study the influenza vaccine composition problem, considering the influenza vaccine supply chain, comprised of a single vaccine manufacturer and the Vaccines and Related Biological Products Advisory Committee of the Food and Drug Administration (the "Committee"). Prior to the start of the influenza season, the demand is estimated and the set of candidate strains to be considered in the flu vaccine is determined for the coming influenza season. Then the Committee makes the strain composition decision so as to maximize the expected societal vaccination benefit (i.e., the total cost of prevented flu cases) under uncertainty on future prevalences of the virus strains and under uncertainty on random production yields of the strains. Given the demand and the strain composition, the vaccine manufacturer then determines the input quantity of the strains selected by the Committee, under uncertainty on the yield of the strains, so as to maximize her expected profit. The vaccine output quantity is limited by the least output of the selected strains and the demand. The unit sales price of the vaccine is assumed to be increasing in the number of the strains the vaccine contains. An extension of this model is also studied, where the left-over strains can be used by the manufacturer to produce a secondary "inferior" vaccine, which contains fewer number of strains than the primary vaccine. The secondary vaccine, which is sold at a lower unit price than the primary vaccine, offers the ability to fulfill the demand that cannot be satisfied by the primary vaccine.

As discussed above, a comparative statics analysis of multi-product or multi-strain settings is fraught with analytical challenges. To overcome the limitations of the current approach, which typically involves performing a parametric analysis, utilizing the implicit function theorem, and working with random variable parameters (mean, standard deviation, and correlation), we work explicitly with structural properties of the resulting models, and link these properties directly to the concept of stochastic orders. The use of these stochastic order relationships for comparative statics analysis of operations management problems is greatly facilitated by the works of Müller and Stoyan (2002) and Shaked and Shanthikumar (2007); also see Cooper and Gupta (2006), Van Mieghem and Rudi (2002), and the references therein for some examples. Utilizing this approach on these two operations management problems leads to new structural properties, along with rigorous proofs. Specifically, for the capacity investment decision problem, we generalize the previous results obtained in two-product “flexible-only” networks (Liu (2009)) (Chapter 2), and establish new results for “dedicated” (Chapter 2) and “serial” (Chapter 3) networks. For the influenza vaccine composition decision problem, we establish important comparative statics analysis results in the original game setting, and develop an effective solution technique to solve the game utilizing the structural properties that we derive (Chapter 4). We then extend our analysis to our proposed approach, where the left-over strains are utilized to produce a secondary vaccine. Finally, we conduct an extensive numerical study using published or publicly available data. Our numerical study quantifies the significant societal benefits that can be achieved by utilization of a secondary vaccine (Chapter 4).

In summary, the main contribution of this research comes from providing a rigorous framework for comparative statics analysis, which allows generalization of the existing insights, and more importantly, which can be applied to similar problems that are not amenable to traditional parametric analysis. For the capacity investment decision problem, our work is not the first to derive structural properties of newsvendor networks (e.g., see Van Mieghem and Rudi (2002) and the references therein for exogenous pricing settings), nor it is the first to utilize stochastic order relationships for comparative statics analysis (see the references cited above). However, ours is the first to extensively focus on the structural properties of the capacity dual variable in responsive pricing settings, where the recourse problem is no longer a linear programming problem. Our work represents a significant extension of the work in Liu (2009), which establishes comparative statics analysis results for the flexible network, in that we are able to expand the analysis to other types of networks, such as the dedicated only networks and serial networks. Furthermore, we are able to simplify some of the proofs in Liu (2009) by utilizing results well-known in linear algebra and optimization. For the influenza vaccine composition problem, to our knowledge, our research is the first to analyze the vaccine composition decision by explicitly considering the different decision-makers involved, and under a new approach, of the left-over strain usage possibility. We complete our analysis by a comparative statics analysis that characterizes the impact of yield uncertainty on the societal vaccination benefit, the manufacturer's profit, and the vaccine output. Our numerical analysis indicates significant societal benefits of utilizing a secondary vaccine, comprised of the left-over strains. Further, it suggests that the current practice, of including one strain from each of virus family in the vaccine, does not necessarily maximize the societal benefits. Depending on the relationship between prevalence and yield estimate of the strain, it may be beneficial to deviate from this practice. From that perspective, our study generates important insights and principles for public policy decision-making.

The remainder of this dissertation is organized as follows. In Chapter 2 we present our proposed approach for the comparative statics analysis of multi-product newsvendor networks under responsive pricing for the flexible network and dedicated network. In Chapter 3, we expand our proposed approach to the analysis of a more complicated newsvendor network, the serial network, and to the study of demand dependence through the use of supermodular orders. In Chapter 4, we focus on the influenza vaccine composition problem. We derive structural properties of the optimal solution, and perform a comparative statics analysis and extensive numerical studies to generate insight into the public policy decision-making. For clarity in the presentation, most definitions and proofs are relegated to the Appendix.

Chapter 2

Comparative Statics Analysis of Multi-product Newsvendor Networks under Responsive Pricing – Flexible and Dedicated Networks

2.1 Introduction

This Chapter provides a novel analytic approach for the comparative statics analysis of multi-product multi-resource newsvendor networks under responsive (postponed) pricing. A major strength of this methodology is that it is "scalable," that is, this approach applies to newsvendor networks with any number of non-independent (i.e., demand or resource sharing) products and resources without a corresponding increase in effort. This makes the proposed approach powerful and promising for future work, as a major limitation of the existing approaches, commonly used in the operations management literature for comparative statics analysis, is that the effort increases exponentially with problem size. Another important advantage of our approach is that it is distribution-free, eliminating the need to restrict the analysis to a specific demand distribution. This approach enables us to provide rigorous proofs for the comparative statics analysis of the optimal capacity decision in multi-product multi-resource newsvendor networks.

Specifically, we consider the optimal capacity decision in multi-product multi-resource newsvendor networks under responsive and endogenous pricing. Demands of the substitutable products are linked through the pricing of all products in the product family. We model this dependence through own- and cross-price effects, which contribute linearly to product demands. The firm needs to determine the capacities of the flexible and dedicated resources under uncertainty, but is able to postpone its pricing and production decisions until after demand uncertainty is resolved. This capacity investment decision problem and its variants are well-studied in the operations management literature (see Van Mieghem (2003) for an excellent review of the capacity investment decision problem, as well as Bish and Wang (2004); Chod and Rudi (2005); Van Mieghem and Dada (1999) for responsive pricing). The timing of the capacity investment decision naturally leads to a two-stage stochastic programming problem with recourse.

A comparative statics analysis is integral to the study of the capacity investment decision, as it can provide important insight on questions such as whether or not the firm should acquire more resources as demand risk increases, or if and how the firm can benefit from an increase in demand risk. For stochastic programming problems with recourse, characterization of their optimal solution and the subsequent comparative statics analysis typically require a parametric analysis of the recourse problem, followed by the use of Jacobians and the implicit function theorem. Such tools can quickly get "involved" as problem size increases (Van Mieghem and Rudi (2002)). For example, the parametric analysis requires determining all possible demand-space partitions ("domains"), wherein each domain is uniquely identified by the set of constraints that must be binding in the optimal solution. For the capacity investment problem, which is the focus of Chapters 2 and 3, the number of possible domains for the recourse problem increases exponentially with the number of non-independent (i.e., demand or resource sharing) products and resources, taking away from analytical tractability. Consequently, it is not surprising that, while an analytical study of the capacity investment problem in the single-product setting is relatively straightforward (e.g., Van Mieghem and Dada (1999)), its analysis in multi-product settings is fraught with analytical challenges, and has, therefore, been limited to two-product instances (e.g., Bish and Wang (2004); Bish et al. (2010); Chod and Rudi (2005); Goyal and Netessine (2011); Van Mieghem (1998), and the references in Van Mieghem (2003)) as well as to specific demand distribution functions (mostly to the Normal distribution) (e.g., Chod and Rudi (2005)).

Van Mieghem and Rudi (2002) is a notable exception that generalizes the structure of the optimal capacity decision for newsvendor networks under *exogenous pricing* to an arbitrary number of products and a general resource structure, and performs a comparative statics analysis of the optimal expected profit in demand parameters (standard deviation and correlation). This is done by exploiting the special analytical structure of newsvendor networks and by making use of linear programming theory, as the recourse problem under exogenous pricing is a linear program. However, their comparative statics analysis is still limited to the Normal distribution for the demand vector and to the study of the expected profit function. (As Van Mieghem and Rudi (2002) states, "establishing general comparative statics results of capacity levels is more difficult.") As the main contribution of Chapters 2 and 3, we further their work, by exploiting special analytical properties of multi-product newsvendor networks under *responsive pricing* and establishing comparative statics results on the optimal capacity decision.

Specifically, rather than taking the traditional approach of performing a parametric analysis of the recourse problem, utilizing the implicit function theorem, and working with risk measures such as demand variability, we work explicitly with properties of the primal mathematical programming formulation and of the dual variables, and link these properties directly to the concept of convex orders. The use of stochastic order relationships for comparative statics analysis of operations management problems is greatly facilitated by the works of Müller and Stoyan (2002) and Shaked and Shanthikumar (2007); and these relationships are used for comparative statics studies of risk pooling models with regard to demand risk in various settings, e.g., among others, see Choi and Ruszczynski (2008); Choi et al. (2011);

Zhang (2005) for the newsvendor problem modeled as an *unconstrained* optimization problem (hence, the capacity dual variables, whose properties we extensively study in our setting, are not relevant in those settings); see Bish et al. (2010); Van Mieghem and Rudi (2002), and the references therein, for newsvendor networks in which resource capacity constraints are imposed on the recourse problem (thus, the recourse problem becomes a *constrained* optimization problem). All these works focus on *exogenous* pricing settings (with the exception of Bish et al. (2010), which is restricted to the two-product setting, as we discuss subsequently). However, due to the special structure responsive pricing imposes on the stochastic programming problem, properties that hold under exogenous pricing no longer hold under responsive pricing, and this presents new analytical challenges. For example, the recourse problem under exogenous pricing is a linear programming problem, whereas under responsive pricing it is nonlinear. As a result, we lose properties of the capacity dual variable based on linear programming theory. Nevertheless, our analysis shows that the capacity dual variable possesses other properties under responsive pricing, which we exploit in our comparative statics analysis.

Utilizing this approach on multi-product newsvendor networks under responsive pricing leads to new structural properties and rigorous proofs. In this Chapter we generalize the previous results obtained in two-product **FN** (Liu (2009)), and establish new results for **DN**, for which we are not aware of any comparative statics analysis in the responsive pricing setting. In particular, in the two-product **FN**, it is well-known that both the optimal capacity and the firm's expected profit are increasing in demand risk (Chod and Rudi (2005); Bish et al. (2010)).[1] Our analysis generalizes these results to multi-product newsvendor networks. In addition, we show that in the **DN**, an increase in demand risk forces the firm to acquire more capacity for *at least one* of its resources. However, it may still be optimal for the firm to reduce the capacities of *some* resources as demand risk increases; we demonstrate this through examples.

In summary, continuing along the lines of Van Mieghem and Rudi (2002), our main con-

[1]Both these proofs follow by a traditional comparative statics analysis approach, which requires a parametric analysis of the second stage problem.

tribution comes from providing a rigorous framework for comparative statics analysis, which allows generalization of the existing insights, and more importantly, which can be applied to similar problems that are not amenable to traditional parametric analysis. Our work is not the first to derive structural properties of newsvendor networks (e.g., see Van Mieghem and Rudi (2002) and the references therein for exogenous pricing settings), nor it is the first to utilize stochastic order relationships for comparative statics analysis (see the references cited above). However, ours is the first to extensively focus on the structural properties of the capacity dual variable in responsive pricing settings, where the recourse problem is no longer a linear programming problem. For this purpose, we combine the theory of constrained nonlinear programming problems (in particular, the *Definite Quadratic Programming Problem*) with linear algebra (especially properties of *M-matrices*, which become relevant due to the parameter restrictions imposed by our demand model of substitutable products), and with stochastic order relationships to establish our comparative statics analysis results. This methodological approach, along with several structural properties we derive, is our main contribution. Further, the underlying idea, of extensively working with properties of the stochastic programming formulation and convex order relationships, certainly holds promise in contexts of other operations management problems.

The remainder of this Chapter is organized as follows. In Section 2.2, we introduce the notation and the model. In Section 2.3, we present our proposed approach for the comparative statics analysis of multi-product multi-resource networks and establish our main results. In particular, we derive various properties of the primal mathematical programming formulation and of the dual variables, and show how the optimal expected profit and optimal capacity behave in demand risk. Finally, we conclude in Section 2.4 with a summary of our results and suggestions for future research. For clarity and compactness in the presentation, some definitions, preliminary results, and all proofs are relegated to the Appendix.

2.2 The Notation and the Model

Throughout, we use boldface letters to denote vectors and matrices, and superscripts T and -1 to respectively denote their transpose and inverse. In particular, $\boldsymbol{X}_{(i,j)}$ denotes a matrix of size $i \times j$ (we drop the subscript (i,j) when it is clear from the context). Then, $\boldsymbol{X}_i$ and X_{ij} respectively denote its i-th row and (i,j)-th element. All vectors are column vectors, unless their transpose is taken. We use $|X|$ to denote the cardinality (size) of set X. Finally, we use upper-case letters to denote random variables and lower-case letters to denote their realizations; and we use the terms "increasing" and "decreasing" in the weak sense, to indicate "non-decreasing" and "non-increasing," respectively.

In the remainder of this section, we detail the multi-product multi-resource newsvendor's demand function, resource structure, and capacity investment decision problem.

2.2.1 The Demand Function

The firm offers n (≥ 2) *imperfectly substitutable* products, whose demands are modeled as linear functions of the price vector of the entire set of products, $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)^T$, as is common in the Economics and operations management literature (e.g., Chod and Rudi (2005); Singh and Vives (1984), and the references therein), that is, we model both own- and cross-price effects. Then, D_i, the demand for product $i, i = 1, \cdots, n$, can be expressed as:

$$D_i = \Gamma_i - v_{ii} p_i + \sum_{j=1,\cdots,n, j \neq i} v_{ij} p_j, \ i = 1, \cdots, n, \tag{2.1}$$

where $\Gamma_i, i = 1, \cdots, n$, is the random demand intercept for product i, whose realization we denote by γ_i. Let $\boldsymbol{\Gamma} \equiv \{\Gamma_1, \Gamma_2, \cdots, \Gamma_n\}^T$ and $\boldsymbol{\gamma} \equiv \{\gamma_1, \gamma_2, \cdots, \gamma_n\}^T$. Parameter v_{ii} $(\geq 0), i = 1, \cdots, n$, is the own-price effect of product i, and v_{ij} $(\geq 0), i, j = 1, \cdots, n, \ j \neq i$, is the cross-price effect of product j on product i, with $\sum_{j=1,\cdots,n,j\neq i} v_{ij} < v_{ii}$, $i = 1, \cdots, n$. Thus, while the demand of product i drops with an increase in its own price, its demand rises with an increase in the price of any of the other products (due to the substitution effect);

however, its own-price effect dominates the sum of all cross-price effects – an assumption commonly adopted in the Economics literature for imperfectly substitutable products. For analytical tractability and transparency, we make two assumptions on own- and cross-price effects: (1) cross-price effects are symmetric, i.e., $v_{ij} = v_{ji}$, $i, j = 1, \cdots, n, j \neq i$, and (2) $v_{ii} = 1$, $i = 1, \cdots, n$. Let $\boldsymbol{S}_{(n,n)}$ denote the negative of the Jacobian matrix of demand functions, comprised of the following elements:

$$\text{For } i, j = 1, \cdots, n: \ S_{ij} = \begin{cases} v_{ii}(=1), & \text{if } i = j, \\ \\ -v_{ij}, & \text{otherwise.} \end{cases} \tag{2.2}$$

Notice that since $\boldsymbol{S}$ is symmetric and $S_{ii} = 1 > \sum_{j=1,\cdots,n,\ j\neq i} |S_{ij}|$, $i = 1, \cdots, n$, $\boldsymbol{S}$ is strictly positive definite, hence invertible. Then, its inverse, $\boldsymbol{W} \equiv \boldsymbol{S}^{-1}$, is symmetric, positive, and strictly positive definite. These properties of the $\boldsymbol{S}$ and $\boldsymbol{W}$ matrices follow from the consumer choice model underlying the linear demand curves (see Chod et al. (2010), Endnote 7).

2.2.2 The Resource Structure

The multi-resource network (of m resources) is comprised of flexible and dedicated resources. A flexible resource is capable of producing multiple products, while dedicated resource i can produce product $i, i = 1, \cdots, n$, only. We represent resource capabilities with technology matrix $\boldsymbol{A}_{(m,n)}$, with $A_{ki} = 1$ indicating that resource k has the capability to produce product i, and $A_{ki} = 0$ indicating otherwise, $k = 1, \cdots, m; i = 1, \cdots, n$. For analytical tractability of the subsequent comparative statics analysis, we restrict our study to network structures in which the production quantity vector uniquely determines the resource allocation decision (i.e., how the capacity of each resource is split among the products it produces). These are networks in which the production route of each product is fixed, and there exist no alternative routes. Mathematically speaking, letting η_i denote the number of different resources product i requires, this assumption can be formally stated as follows.

Assumption 1. *For each product $i, i = 1, \cdots, n$, $\sum_{k=1}^{m} A_{ki} = \eta_i$.*

While this assumption imposes certain restrictions on the network structures we can analyze (in Van Mieghem's (2007) terminology, this formulation cannot represent "parallel" networks, for example, in which each product can be produced either by a dedicated resource or by a flexible resource), it allows us to establish our main results through a comparative statics analysis of the challenging n-product (for an arbitrary n) setting on various, commonly studied network structures. Specifically, following the terminology in Van Mieghem (2007), these networks can represent basic networks, including the "dedicated" networks (**DN**), consisting of dedicated resources only, "flexible-only" networks (**FN**),[2] consisting of one flexible resource only, and "serial" networks (**SN**) (See Chapter 3 for a detailed analysis of this kind of networks), where each product requires processing both by its own dedicated resource and by the common flexible resource, see Figure 2.1,[3] as well as more complex networks that can be constructed from combinations of these networks.

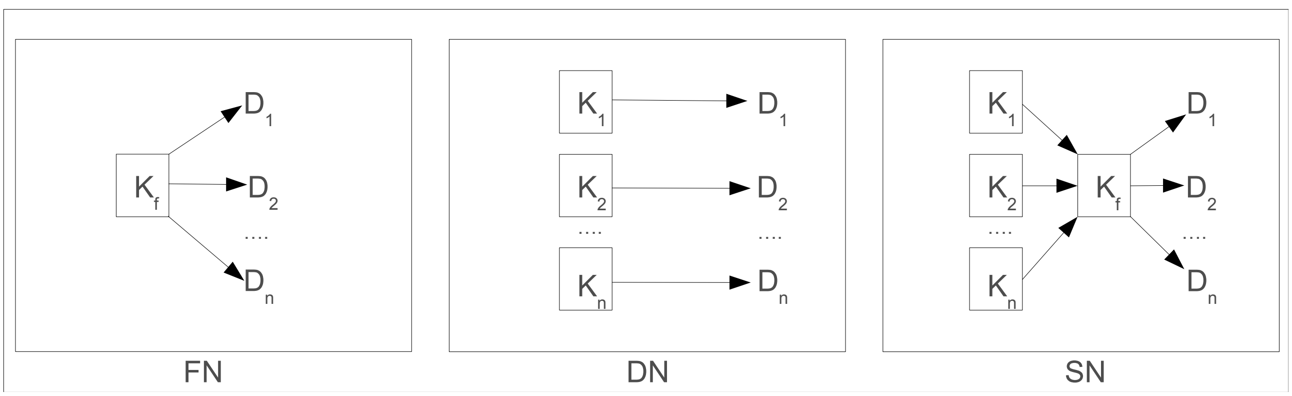

Figure 2.1: Illustration of the three basic networks, the **FN**, **DN**, and **SN** (Chapter 3)

In particular, we establish our first main result, on the relationship between the firm's optimal expected profit and demand risk, for any network structure that satisfies Assumption 1. We then focus on the two basic networks, the **FN** and **DN**, and establish our second main result, on the relationship between the optimal capacity and demand risk. These basic networks, together with **SN** studied in Chapter 3, can be considered as canonical building

[2]This terminology is due to us.

[3]The production sequence of the **SN** in Figure 2.1 can be reversed without loss of generality; our analysis applies to both cases.

blocks for complex multi-resource multi-product newsvendor networks, and as such, have received considerable attention in the literature (albeit all in two-product settings), e.g., Bish et al. (2010, 2009); Callen and Sarath (1995); Chod and Rudi (2005); Goyal and Netessine (2011); Harrison and Van Mieghem (1999); Van Mieghem (2007), as well as the additional references in Van Mieghem (2003).

On the financial side, investment costs are linear and the cost to acquire resource capacity vector $\boldsymbol{K} = (K_1, K_2, \cdots, K_m)^T$ is given by $\boldsymbol{c}^T\boldsymbol{K}$, where $\boldsymbol{c} = (c_1, c_2, \cdots, c_m)^T$. We assume that $\boldsymbol{c} > \boldsymbol{0}$ so that an optimal capacity vector, $\boldsymbol{K^*}$, is finite.

2.2.3 The Decision Problem

The risk-neutral firm determines an optimal capacity vector, $\boldsymbol{K^*}$, under uncertainty on the demand intercept vector, $\boldsymbol{\Gamma}$, whose joint probability distribution function (pdf), $f(\gamma_1, \gamma_2, \cdots, \gamma_n)$, is known at the time of the investment decision. We do not assume any functional forms on $f(\cdot)$, except that Γ is continuous, has positive support, is finite with probability 1, and has finite expectation. Consequently, our analysis does not rely on a particular distribution, and all our results hold for dependent demand intercepts as well. The firm's recourse action (i.e., when the realization, $\boldsymbol{\gamma}$, of the random vector, $\boldsymbol{\Gamma}$, is observed) involves the pricing, $\boldsymbol{p} = (p_1, p_2, \cdots, p_n)^T$, and production quantity, $\boldsymbol{q} = (q_1, q_2, \cdots, q_n)^T$, decisions for all products. This timing of events naturally leads to the following two-stage stochastic programming problem with recourse:

$$\boldsymbol{P_1}: E_{\boldsymbol{\Gamma}}\left[V(\boldsymbol{K}^*, \Gamma)\right] \equiv \text{maximize}_{\boldsymbol{K}\geq\boldsymbol{0}} E_{\boldsymbol{\Gamma}}\left[V\left(\boldsymbol{K}, \boldsymbol{\Gamma}\right)\right], \tag{2.3}$$

$$\text{where } V\left(\boldsymbol{K}, \boldsymbol{\Gamma}\right) \equiv \Pi^*\left(\boldsymbol{K}, \boldsymbol{\Gamma}\right) - \boldsymbol{c}^T\boldsymbol{K}. \tag{2.4}$$

$$\boldsymbol{P_2}: \Pi^*(\boldsymbol{K}, \boldsymbol{\gamma}) \equiv \text{maximize}_{\boldsymbol{p},\boldsymbol{q}} \Pi(\boldsymbol{K}, \boldsymbol{\gamma}) = \boldsymbol{p}^T\boldsymbol{q}$$

$$\text{subject to: } \boldsymbol{q} \leq \boldsymbol{\gamma} - \boldsymbol{S}\boldsymbol{p}, \tag{2.5}$$

$$A\boldsymbol{q} \leq \boldsymbol{K}, \tag{2.6}$$

$$\boldsymbol{q} \geq \boldsymbol{0}, \tag{2.7}$$

$$\boldsymbol{p} \geq \boldsymbol{0}, \tag{2.8}$$

where $\boldsymbol{K}^*$, $\boldsymbol{K}$, and $\boldsymbol{c}$ are $m \times 1$ column vectors; $\boldsymbol{\Gamma}$, $\boldsymbol{\gamma}$, $\boldsymbol{q}$, and $\boldsymbol{p}$ are $n \times 1$ column vectors; and $\boldsymbol{A}$ is an $m \times n$ matrix. We refer to $\boldsymbol{K^*}$ and $E_{\boldsymbol{\Gamma}}[V(\boldsymbol{K^*})]$ respectively as the optimal resource capacity vector and the resulting expected profit. We let $y^*(\boldsymbol{K})$ denote the optimal value of y for a given capacity vector $\boldsymbol{K}$; and let y^* denote the optimal value of y, i.e., $y^* = y^*(\boldsymbol{K^*})$, for $y = \boldsymbol{q}, \boldsymbol{p}, \Pi$. To simplify the notation, we drop the arguments in parentheses when clear from the context.

In this formulation, (2.5) implies that the production of each product does not exceed its demand; (2.6) ensures that the total consumption of each resource does not exceed its capacity acquired in the first stage; (2.7) and (2.8) are the nonnegativity constraints for outputs and prices.

Definition 1. *(Müller and Stoyan (2002) p. 98) Let* $\boldsymbol{X} = (X_1, \cdots, X_n)$ *and* $\boldsymbol{Y} = (Y_1, \cdots, Y_n)$ *be* $n-$*dimensional random vectors with finite expectations. Then* $\boldsymbol{X}$ *is said to be smaller than* $\boldsymbol{Y}$ *in the convex order, written* $\boldsymbol{X} \leq_{cx} \boldsymbol{Y}$*, if* $E[f(\boldsymbol{X})] \leq E[f(\boldsymbol{Y})]$ *for all convex functions* $f : \Re^n \rightarrow \Re$ *such that the expectations exist.*

Following the conventional terminology, we say that a random vector becomes "more risky" if it increases in convex order (e.g., Definition A1 in Mas-Colell et al. (1995); Müller and Stoyan (2002)), which we denote by $\boldsymbol{\Gamma} \leq_{\text{cx}} \overline{\boldsymbol{\Gamma}}$ (i.e., $\boldsymbol{\Gamma}$ is smaller in convex order than $\overline{\boldsymbol{\Gamma}}$); and we use the terms "more risky" and "higher risk exposure" interchangeably (Van Mieghem (2007)). For $\boldsymbol{\Gamma} \leq_{\text{cx}} \overline{\boldsymbol{\Gamma}}$, we let $\boldsymbol{K}^*$ and $\overline{\boldsymbol{K}}^*$ denote their respective optimal capacity vectors. We also utilize properties of *M*- and *Stieltjes* matrices throughout (See Definition B4).

Definition 2. *(Horn and Johnson (1991) p. 114) A square matrix* $\boldsymbol{X}$ *is called an M-matrix if all off-diagonal elements are less than or equal to zero, and it satisfies any one of the following equivalent conditions.*

1. *All principal minors of* $\boldsymbol{X}$ *are positive.*

2. *The leading principal minors of* $\boldsymbol{X}$ *are positive.*

3. $\boldsymbol{X}$ *is non-singular and the inverse of* $\boldsymbol{X}$ *is non-negative.*

A symmetric M-matrix is called a Stieltjes matrix (Varga (2000)).

The remainder of this Chapter focuses on the comparative statics analysis of multi-product ($n \geq 2$) multi-resource ($m \geq 1$) newsvendor networks under responsive pricing. Unlike previous approaches that the extant capacity investment literature utilizes for comparative statics analysis of stochastic programming problems with recourse, our approach does not require a parametric analysis of the recourse problem nor the use of the implicit function theorem.

2.3 A Novel Approach for Comparative Statics Analysis

This section is organized as follows. In Section 2.3.1, we first derive some properties of an optimal solution. We use these properties to perform a comparative statics analysis with regard to demand risk for the optimal expected profit (Section 2.3.2) and the optimal capacity vector (Section 2.3.3).

All the results in Sections 2.3.1 and 2.3.2 hold for general network structures that satisfy Assumption 1, while the analyses in Sections 2.3.3 are restricted to the two basic networks, the **FN** and **DN**.

2.3.1 Properties of an Optimal Solution to Stage 1 and 2 Problems

Lemma 1. *(Proposition 27 in Liu (2009)) There exists an optimal solution to Problem* $\boldsymbol{P_2}$ *in which* $q_i^* = d_i^*, i = 1, \cdots, n$ *(i.e., constraints (2.5) are binding). In addition, price nonnegativity constraints (2.8) are redundant.*

Lemma 1 indicates that in an optimal solution to the stage 2 problem, the firm always sets prices such that all (induced) demand is met. Bish et al. (2009) proves a similar result for the two-product **FN**. Lemma 1 generalizes their result to this more general setting with an arbitrary number of products and a more general resource structure, as discussed in

Section 2.2.2. Mathematically speaking, $q_i^* = \gamma_i - \boldsymbol{S}_i\boldsymbol{p}$, $i = 1, \cdots, n$. While this result is not surprising, it plays a crucial role in our analysis by allowing the recourse problem to be formulated as a *Definite Quadratic Programming Problem (DQP)*. In particular, to establish our main results, we utilize several well-known properties of *DQP*s, and derive further properties of *DQP*s that apply when their parameters satisfy certain conditions that are imposed in our problem setting (see Section 2.2.1).

By Lemma 1 and because $\boldsymbol{W}$ is strictly positive definite, Problem $\boldsymbol{P_2}$ can be reformulated as a *DQP*:

$$\boldsymbol{P_2} \textbf{ :}\ \Pi^*(\boldsymbol{K}, \boldsymbol{\gamma}) = \text{maximize}_{\boldsymbol{q}} \Pi(\boldsymbol{K}, \boldsymbol{\gamma}) = \boldsymbol{\gamma}^T \boldsymbol{W} \boldsymbol{q} - \boldsymbol{q}^T \boldsymbol{W} \boldsymbol{q} \tag{2.9}$$
$$\text{subject to: } A\boldsymbol{q} \leq \boldsymbol{K}, \tag{2.10}$$
$$\boldsymbol{q} \geq \boldsymbol{0}. \tag{2.11}$$

Let $\boldsymbol{\lambda} \geq \boldsymbol{0}$ and $\boldsymbol{\mu} \geq \boldsymbol{0}$ denote the dual variables (KKT multipliers) respectively corresponding to constraints (2.10) and (2.11), and let $\Upsilon \equiv \{\boldsymbol{\lambda}, \boldsymbol{\mu}\}$.

We next prove the following expected result, on the strict concavity of the stage 1 objective function in our n-product setting, establishing the uniqueness of the optimal capacity solution.

Lemma 2. *$E_{\boldsymbol{\Gamma}}[V(\boldsymbol{K})]$ is strictly jointly concave in $\boldsymbol{K}$, and the KKT first-order conditions, given in (2.12)-(2.15), are necessary and sufficient for the optimality of the unique $\boldsymbol{K^*}$ for Problem $\boldsymbol{P_1}$.*

$$\begin{aligned} \partial E_{\boldsymbol{\Gamma}}[\Pi^*(\boldsymbol{K})]/\partial K_i|_{\boldsymbol{K}=\boldsymbol{K^*}} - c_i + \beta_i &= 0, i = 1, \cdots, m, && (2.12)\\ K_i^* &\geq 0, i = 1, \cdots, m, && (2.13)\\ \beta_i &\geq 0, i = 1, \cdots, m, && (2.14)\\ \beta_i K_i^* &= 0, i = 1, \cdots, m, && (2.15) \end{aligned}$$

where β_i, $i = 1, \cdots, m$, is the KKT multiplier corresponding to the primal constraint, $K_i \geq 0$.

So far, our methodology for deriving the concavity of the stage 1 expected profit in

Lemma 2 has not deviated from the extant literature. To study properties of an optimal solution, it is then common, in the extant literature on capacity investment, to proceed with a parametric analysis of the recourse problem, i.e., to partition the demand (or demand intercept, depending on the problem setting) space into mutually exclusive "domains," the union of which spans the entire support set of the random variables, in such a way that each domain is uniquely identified by the primal constraints that are binding in the optimal recourse solution. By this definition, the expression for the optimal recourse solution remains of the same functional form within each domain. Then, the set of domains and their optimal solutions for the recourse problem can be used to derive an expression on the stage 1 objective function, as well as to explicitly write out the necessary optimality conditions. Comparative statics analysis then typically involves the implicit function theorem, which requires derivatives of the stage 1 objective function with respect to certain variables. However, because the number of possible domains for the recourse problem increases exponentially with the number of products and resources, such an approach becomes analytically intractable for a large number of products and resources. Because of this major limitation, papers that study this problem restrict themselves to two-product instances, as discussed in Section 2.1.

As an alternative, the comparative statics analysis we utilize in this Chapter relies solely on properties of the objective functions and of the dual variables of Problems $\boldsymbol{P_1}$ and $\boldsymbol{P_2}$. Not restricting ourselves to a specific demand intercept distribution and linking the concept of risk exposure to the concept of convex orders are crucial for our methodology. See Van Mieghem and Rudi (2002) for such an approach in the context of newsvendor networks with *exogenous* pricing. However, their analysis does not require an extensive analysis of the capacity dual variable, which remains constant within each demand domain under exogenous pricing. In addition, their comparative statics analysis is limited to the optimal expected profit, and the dependence on a specific distribution (Normal) remains. In the next section, we proceed with the application of this methodology.

2.3.2 The Optimal Expected Profit versus Risk Exposure

For our comparative statics study, we will need to derive additional properties of *DQP*s, considering the parameter restrictions imposed in our setting.

Lemma 3. *For any* $\boldsymbol{K}$*,* $\Pi^*(\boldsymbol{K}, \boldsymbol{\gamma})$*, the optimal revenue function of Problem* $\boldsymbol{P_2}$*, is jointly convex in* $\boldsymbol{\gamma}$*.*

We are ready to prove our first main result in the n-product setting.

Theorem 1. *The optimal expected profit,* $E_{\boldsymbol{\Gamma}}[V(\boldsymbol{K}^*, \Gamma)]$*, will increase if* $\boldsymbol{\Gamma}$ *increases in convex order.*

This extends the two-product **FN** results of Chod and Rudi (2005), which assumes that $\boldsymbol{\Gamma}$ follows a Bivariate Normal distribution, and of Bish et al. (2010), which considers any arbitrary continuous distribution for $\boldsymbol{\Gamma}$.

Remark 1. *From Theorem 3.4 in Fiacco and Ishizuka (1990), Lemma 3 holds for more general nonlinear programming problems than Problem* $\boldsymbol{P_2}$*, as long as* $\Pi(\boldsymbol{K}, \boldsymbol{\gamma})$ *is convex in* $\boldsymbol{\gamma}$*. Consequently, we expect Theorem 1 to extend to more general price-demand functions, as long as the convexity of the stage 2 objective function in* $\boldsymbol{\gamma}$ *is preserved.*

Remark 2. *Consider that* $\mathbf{S}$ *is a matrix of zeros, the price vector,* $\boldsymbol{p}$*, is given, and* $\boldsymbol{\Gamma}$ *represents the random demand vector. Then, the formulation of Problems* $\boldsymbol{P_1}$ *and* $\boldsymbol{P_2}$*, in* (2.3)– (2.8)*, corresponds to the exogenous pricing setting, with Problem* $\boldsymbol{P_2}$ *reduced to a linear programming problem having decision variable* $\boldsymbol{q}$*, and with parameter* $\boldsymbol{\gamma}$ *appearing only in constraint* (2.5) *(rather than in the objective function, see* (2.9)*). Then, from linear programming theory, since* $\Pi^*(\boldsymbol{K}, \boldsymbol{\gamma})$ *is piecewise concave in* $\boldsymbol{\gamma}$ *(Bazaraa et al. (2005)), an argument similar to the proof of Theorem 1 will lead to the result that the optimal expected profit,* $E_{\boldsymbol{\Gamma}}[V(\boldsymbol{K}^*, \boldsymbol{\Gamma})]$*, will decrease if* $\boldsymbol{\Gamma}$ *increases in convex order. This recovers the results derived in the earlier literature for the exogenous pricing setting (e.g., see Eppen (1979) as well as the discussion in Chod and Rudi (2005)).*

Remark 2 highlights one of the major differences between the responsive pricing and exogenous (fixed) pricing settings, well-known in the literature. Because the capacity decision in both settings needs to be made under uncertainty on demand curves, the very existence of demand variability leads to under/over-investment costs. In the fixed pricing setting, the higher the demand risk is, the higher the under- and over-investment costs would be, reducing the firm's expected profit. On the other hand, as Chod and Rudi (2005) explains nicely, "Under responsive pricing, the firm will be able to take advantage of high demand levels by selling at a high price, while it can mitigate the impact of low demand levels on sold quantity by charging a low price. In expectation, this effect dominates the effect of under/over-investment, and overall the firm benefits from demand variability." As a result, under responsive pricing, the stage 1 profit function, $V(\boldsymbol{K}, \boldsymbol{\gamma}) = \Pi^* (\boldsymbol{K}, \boldsymbol{\gamma}) - \boldsymbol{c}^T \boldsymbol{K}$, becomes jointly convex in $\boldsymbol{\gamma}$ (Lemma 3), i.e., the impact of high and low realizations of $\boldsymbol{\Gamma}$ on the firm's profit are asymmetric, and Theorem 1 follows. Thus, Theorem 1 shows that this result, established in the literature for single-product single-resource networks and two-product single-resource **FN** networks under responsive pricing (Chod and Rudi (2005); Van Mieghem and Dada (1999)), continues to hold in our setting with an arbitrary number of resources and products, and a similar insight applies in this more general network.

All properties of this section hold for any network structure. To deepen our analysis and derive properties of the optimal capacity solution, we next consider the two types of networks introduced in Section 2.2.

2.3.3 The Optimal Capacity versus Risk Exposure

In this section, we focus on two basic network structures, namely, the **FN** and **DN**, see Figure 2.1:

1. The **FN** with cross-price effects: The flexible-only network consists of n products, with cross-price effects, and one flexible resource ($m = 1$) that is capable of producing all n products.

2. The **DN** with cross-price effects: The dedicated network consists of n products, with cross-price effects, and n dedicated resources ($m = n$), where dedicated resource $i, i = 1, \cdots, n$, is capable of producing product i only.

Thus, in the **FN**, products are dependent through the shared capacity of the common flexible resource and through cross-price effects; in the **DN**, in the absence of a flexible resource, products are dependent through cross-price effects (without cross-price effects, this setting decomposes into n independent single-product single-resource newsvendors). Throughout this section, we use the index f to refer to the flexible resource in the **FN** (e.g., K_f and c_f). The KKT first-order optimality conditions for Problem $\boldsymbol{P_2}$ for each type of network are:

$$\begin{aligned} \textbf{FN}: \quad \boldsymbol{W\gamma} - 2\boldsymbol{Wq}^* - \lambda^f \mathbf{1} + \boldsymbol{\mu} &= \mathbf{0}, && (2.16)\\ \lambda^f (K_f - \sum_{l=1}^{n} q_l^*) &= 0, && (2.17)\\ \mu^l q_l^* &= 0, \quad l = 1 \ldots n. && (2.18) \end{aligned}$$

$$\begin{aligned} \textbf{DN}: \quad \boldsymbol{W\gamma} - 2\boldsymbol{Wq^*} - \boldsymbol{\lambda} + \boldsymbol{\mu} &= \mathbf{0}, && (2.19)\\ \lambda^l (K_l - q_l^*) &= 0, \quad l = 1 \ldots n, && (2.20)\\ \mu^l q_l^* &= 0, \quad l = 1 \ldots n. && (2.21) \end{aligned}$$

Recall that, for a given $\boldsymbol{K}$ and $\boldsymbol{\gamma}$, $\mathbf{q}^*(\boldsymbol{K}, \boldsymbol{\gamma})$ and $\Upsilon \equiv \{\boldsymbol{\lambda}, \boldsymbol{\mu}\}$ $(\geq \mathbf{0})$ respectively denote the optimal primal and dual solutions to Problem $\boldsymbol{P_2}$. In order to simplify the notation, in the following, when convenient, we will simply use the vector form of the dual variables (in boldface, e.g., $\boldsymbol{\lambda}$), with the understanding that the vector will be a scalar in the **FN** (i.e., $\boldsymbol{\lambda}$ will refer to λ^f in the **FN** and to $(\lambda^1, \lambda^2, \cdots, \lambda^n)^T$ in the **DN**.

To motivate our discussion on the effort required for the parametric analysis of the recourse problem, which the traditional comparative statics analysis relies on, consider the **FN**. (This argument readily extends to the **DN**, and **SN** studied in Chapter 3.) For the **FN**, consider the solution to Problem $\boldsymbol{P_2}$ satisfying the primal feasibility constraints and the KKT first-order conditions (Eq. (2.10)–(2.11) and (2.16)–(2.18)), which we denote by $(\boldsymbol{q}^*, \lambda^f, \boldsymbol{\mu})$. Then, for a given $K_f > 0$, the parametric analysis of the recourse problem requires deriving the form of the optimal recourse solution in each mutually exclusive domain in the demand shock space, $\Re^n_+$. As defined above, each domain is uniquely identified by the set of primal constraints in (2.10)–(2.11) that must be binding in the optimal recourse solution, or equivalently, by the subset $\Upsilon^B \subseteq \Upsilon = \{\lambda^f, \mu^1, \cdots, \mu^n\}$, of KKT multipliers having strictly positive values, with all other KKT multipliers, i.e., those in set $\Upsilon \setminus \Upsilon^B$, fixed at the value of zero. Under this restriction, the optimal primal solution, $\boldsymbol{q}^*$, and the corresponding values of the KKT multipliers in set Υ^B can be obtained from Eq. (2.10)–(2.11) and (2.16)–(2.18). Observe that since set Υ has 2^{n+1} subsets, there will be at most 2^{n+1} "feasible domains" (whose optimal recourse solution exists).Since some subsets may not lead to a feasible solution to Eq. (2.10)–(2.11) and (2.16)–(2.18), not every domain will be feasible.

Since Eq. (2.10)–(2.11) and (2.16)–(2.18) are linear in $\boldsymbol{\gamma}$, their solution, $(\boldsymbol{q}^*, \lambda^f, \boldsymbol{\mu})$, will be linear in $\boldsymbol{\gamma}$ within each domain. For each domain, the feasibility requirements for $\boldsymbol{q}^*$ (imposed from the primal formulation), along with the restriction that all KKT multipliers in set Υ^B need to be strictly positive, will lead to a series of linear inequalities, which will serve as the boundaries of that domain.Specifically, each KKT multiplier in set Υ^B will lead to a strict linear inequality of $\boldsymbol{\gamma}$ (i.e., of the form $\sum_{l=1}^n a_l \gamma_l > 0$), while each multiplier in set $\Upsilon \setminus \Upsilon^B$ will lead to a linear inequality of $\boldsymbol{\gamma}$ (i.e., of the form $\sum_{l=1}^n a_l \gamma_l \geq 0$). Then, each domain will be formed by the intersection of up to n half-spaces, with its boundary determined by up to n hyperplanes.Since some (strictly) linear inequalities of $\boldsymbol{\gamma}$ are always satisfied due to the fact that $\boldsymbol{\gamma} \geq \boldsymbol{0}$ and $\boldsymbol{W}$ is a positive matrix, some boundaries may be determined by strictly less than n hyperplanes. See Bish et al. (2010); Chod and Rudi (2005); Van Mieghem (1998) for examples of analysis using this technique. Label all feasible domains such that Ω_j denotes domain $j, j \in \Psi$, where Ψ denotes the set of all feasible domains constructed this way in the demand shock space. In what follows, we perform a comparative statics analysis for the **FN** and **DN**, <u>without</u> performing a parametric analysis, hence without explicitly identifying all possible domains of set Ψ, their boundary hyperplanes, and corresponding optimal solutions for the recourse problem.

We do this by studying properties of the capacity dual variable vector, $\boldsymbol{\lambda}$, in each of the two networks. In particular, in the **FN**, we study properties of only one capacity dual variable (λ^f), whereas in both the **DN**, we simultaneously consider the behavior of multiple capacity dual variables (i.e., λ^l, for $l = 1, \cdots, n$, in the **DN**). For this purpose, it suffices to work with a generic domain, represented by a subset of dual variables having strictly positive values, with the remaining dual variables having values of zero, and derive closed-form expressions for the dual variables. Further, it suffices to establish some properties for its "adjacent" (neighboring) domains only (see Definition 3 below), which allows us to prove our results without resorting to identifying all demand domains. Consequently, the level of effort becomes independent of the problem size (n and m), and, unless stated, our analysis applies to any n and m.

In what follows, if the analysis applies to a particular domain, then we use a subscript to denote the index of the domain. For example, λ^l_j denotes capacity dual variable λ^l (corresponding to resource l) in domain Ω_j. We consider, without loss of generality, that the resource vector $\boldsymbol{K}^* > \mathbf{0}$ in the **DN**. This is without loss of generality, as any $K^*_l = 0$, $l = 1, \cdots, n$, in the **DN** implies that product l cannot be produced at all, effectively reducing the product size to $n - 1$, and a similar analysis applies.

Lemma 4. *The optimal capacity dual variable vector, $\boldsymbol{\lambda}$, in the* **FN** *and* **DN***satisfies the following properties:*

1. *$\boldsymbol{\lambda}$ is continuous in $\boldsymbol{\gamma}$.*

2. *$\boldsymbol{\lambda}$ can be expressed as a linear function of $\boldsymbol{\gamma}$ within each feasible domain Ω_j, $j \in \Psi$.*

Remark 3. *In all our settings, the* **FN** *and* **DN***, non-constant functional forms of $\boldsymbol{\lambda}$ in $\boldsymbol{\gamma}$ are possible within each domain. This is in direct contrast with newsvendor networks with exogenous pricing, where $\boldsymbol{\lambda}$ remains a constant, independent of $\boldsymbol{\gamma}$, within each domain $\Omega_j, j \in \Psi$, and hence, is piecewise constant over the entire domain of $\boldsymbol{\gamma}$ (e.g., Van Mieghem*

(1998); Van Mieghem and Rudi (2002)). Similarly, the property that $\boldsymbol{\lambda}$ is continuous in $\boldsymbol{\gamma}$ in our setting does not hold in newsvendor networks with exogenous pricing. Technically speaking, this is because in exogenous pricing networks, the recourse problem is a linear programming problem, whereas in responsive pricing networks, it reduces to a DQP.

Next, we prove that $\boldsymbol{\lambda}$ is jointly convex in $\boldsymbol{\gamma}$ in both the **FN** and **DN**. We do this in three steps. First, we establish the relationship between $\boldsymbol{\lambda}_i$ and $\boldsymbol{\lambda}_j$ for any pair of *adjacent* domains Ω_i and Ω_j, defined below, and use this result to study the relationship between $\boldsymbol{\lambda}_i$ and $\boldsymbol{\lambda}_j$ for any pair of *non-adjacent* domains Ω_i and Ω_j. These results allow us to express $\boldsymbol{\lambda}$ as a pointwise maximum of a set of convex functions, and the convexity result follows.

Definition 3. *Domains Ω_i and $\Omega_j, i, j \in \Psi, j \neq i$, are adjacent if and only if there exists a unique hyperplane, H_{ij}, that separates Ω_i and Ω_j.*

Hyperplane H_{ij} divides the entire demand intercept space into two half spaces. Let H_{ij}^i denote the half space that contains Ω_i. The following is a direct consequence of the above definition.

Remark 4. *Domains Ω_i and $\Omega_j, i, j \in \Psi, j \neq i$, are adjacent if and only if the sets of dual variables that are strictly positive in each domain differ by exactly one dual variable; that is, there exists exactly one dual variable that switches from the value of zero to a strictly positive value, or vice versa, when moving from one domain to its adjacent domain.*

Lemma 5. *For any pair of adjacent domains Ω_i and Ω_j, $i, j \in \Psi, j \neq i$, we must have:*

1. *In the* **FN**, $\lambda_i^f(\boldsymbol{\gamma}) \geq \lambda_j^f(\boldsymbol{\gamma})$, $\forall \boldsymbol{\gamma} \in H_{ij}^i$.

2. *In the* **DN**, $\lambda_i^l(\boldsymbol{\gamma}) \geq \lambda_j^l(\boldsymbol{\gamma})$, $\forall l = 1, \cdots, n$, *and* $\forall \boldsymbol{\gamma} \in H_{ij}^i$.

Since Lemma 5 will be crucial for the development of our main results for the **FN** and **DN**, we discuss it further. We motivate our discussion with the following example. Example 1 illustrates Lemma 5 for the two-product **FN**.

Example 1. *Consider the two-product* **FN** *with a symmetric cross-price effect* v. *From Chod and Rudi (2005) and Bish et al. (2010), for a given* K^f *and* $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$, *the corresponding optimal dual variable,* λ^f, *in each feasible domain is given below:*

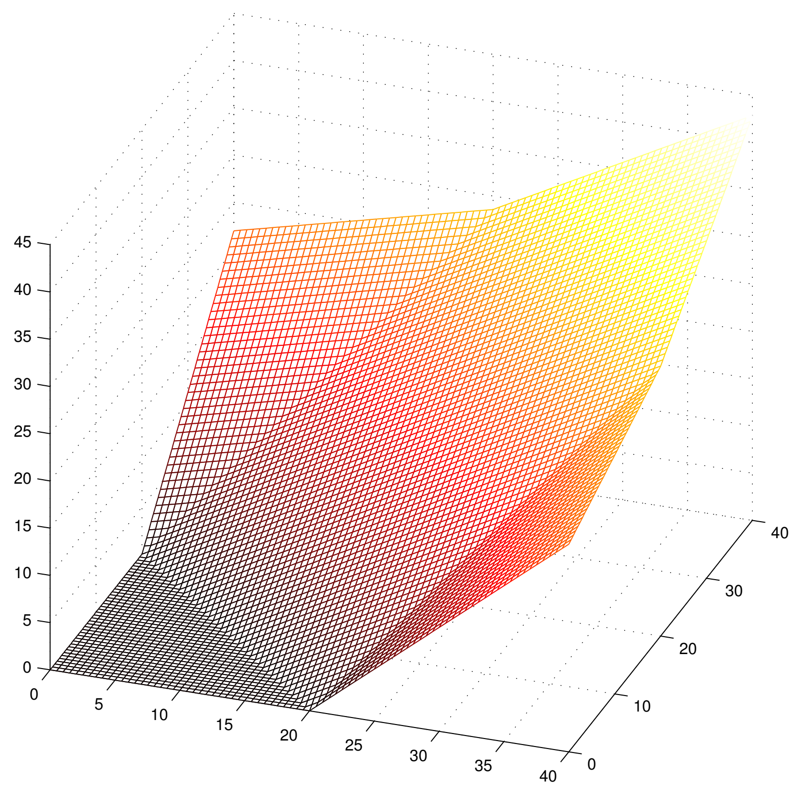

Figure 2.2: The capacity dual variable, λ^f, in the **FN**, plotted in the (γ_1, γ_2)-space (for $K_f = 10$ and $v = 0.3$)

$$\lambda^f = \begin{cases} 0, & \textit{if } (\gamma_1, \gamma_2) \in \Omega_1 \\ \frac{\gamma_1 + \gamma_2 - 2K_f}{2(1-v)}, & \textit{if } (\gamma_1, \gamma_2) \in \Omega_2 \\ \frac{\gamma_1 + v\gamma_2 - 2K_f}{1-v^2}, & \textit{if } (\gamma_1, \gamma_2) \in \Omega_3 \\ \frac{v\gamma_1 + \gamma_2 - 2K_f}{1-v^2}, & \textit{if } (\gamma_1, \gamma_2) \in \Omega_4 \end{cases},$$

where $\Omega_1 = \{\gamma_1 + \gamma_2 \leq 2K_f\}$, $\Omega_2 = \{\gamma_1 + \gamma_2 > 2K_f, -2K_f < \gamma_1 - \gamma_2 < 2K_f\}$, $\Omega_3 = \{\gamma_1 - \gamma_2 \geq 2K_f\}$, *and* $\Omega_4 = \{\gamma_1 - \gamma_2 \leq -2K_f\}$.

It is easy to see that $\lambda^f(\boldsymbol{\gamma})$ *satisfies Lemma 5 for every pair of adjacent domains.*

Continuing with our analysis of the **FN** and **DN**, we extend Lemma 5 to non-adjacent domains. This follows by expressing any pair of non-adjacent domains, Ω_i and Ω_j, as a set of adjacent domain pairs, $(\Omega_i, \Omega_{k_1}), (\Omega_{k_1}, \Omega_{k_2}), \cdots, (\Omega_{k_r}, \Omega_j)$, for some $r \in Z^+$, and utilizing the properties established in Lemmas 4 and 5.

Lemma 6. *For any pair of domains* Ω_i *and* Ω_j*,* $i, j \in \Psi, j \neq i$*, we must have:*

1. *In the* **FN***,* $\lambda_i^f(\boldsymbol{\gamma}) \geq \lambda_j^f(\boldsymbol{\gamma})$*,* $\forall \boldsymbol{\gamma} \in \Omega_i$*.*

2. *In the* **DN***,* $\lambda_i^l(\boldsymbol{\gamma}) \geq \lambda_j^l(\boldsymbol{\gamma})$*,* $\forall l = 1, \cdots, n$*, and* $\forall \boldsymbol{\gamma} \in \Omega_i$*.*

We are ready to prove the convexity result for the capacity dual variables in the **FN** and **DN**. This simply follows because Lemmas 4 and 6 allow us to express the capacity dual variable vector, $\boldsymbol{\lambda}$, as a pointwise maximum of a set of jointly convex functions.

Lemma 7. 1. *In the* **FN***, the dual variable* λ^f *is jointly convex in* $\boldsymbol{\gamma}$*.*

2. *In the* **DN***, the dual variable* $\lambda^l, l = 1, \cdots, n$*, is jointly convex in* $\boldsymbol{\gamma}$*.*

Remark 5. *The joint convexity of* $\boldsymbol{\lambda}$ *in* $\boldsymbol{\gamma}$ *does not hold in newsvendor networks with exogenous pricing. In fact, it does not even hold for all two-stage stochastic programming problems where the recourse problem can be formulated as a DQP, for example, the* **SN***, studied in Chapter 3.*

With the aforementioned properties of the capacity dual variable vector, that $\boldsymbol{\lambda}$ is continuous, linear, and jointly convex in $\boldsymbol{\gamma}$ in the **FN** and **DN**, we can proceed with our second main result, which relates the optimal capacity to demand risk.

Theorem 2. *If* $\boldsymbol{\Gamma}$ *increases in convex order:*

1. *In the* **FN***, the optimal capacity,* K_f^**, will increase.*

2. In the **DN***, at least one of the optimal capacities,* $K_l^*, l = 1, \cdots, n$*, will increase.*

Before we explain the underlying economic principles behind Theorem 2, it is helpful to revisit the exogenous pricing setting in which the optimal capacity in the **FN** is not monotone in demand variability (e.g., Eppen (1979); see also Van Mieghem (2007) and the discussion in Chod and Rudi (2005)). In the exogenous pricing setting, whether the optimal capacity increases or decreases with demand risk depends on the relationship between under- and over-investment costs, which are functions of financial parameters and the demand distribution function. As expected, for high investment costs and low prices, the over-investment cost dominates the under-investment cost, leading to a reduction in capacity as demand risk increases, and vice versa. However, as Chod and Rudi (2005) explains, in the **FN** under responsive pricing, "The under-investment cost will be more significant than the over-investment cost (independent of the cost parameters), inducing the firm to invest more. (This is because) the firm can mitigate the cost of over-investment by holding back some of the resource. In other words, the firm will respond to higher demand variability by acquiring more of the resource because of the option not to use all of it." In summary, our first main result, that the optimal expected profit increases in demand risk (Theorem 1), follows mainly because the stage 1 profit function, $V(\boldsymbol{K}, \boldsymbol{\gamma})$, is jointly convex in $\boldsymbol{\gamma}$ (Lemma 3), i.e., the impact of high and low realizations of $\boldsymbol{\Gamma}$ on the firm's profit are asymmetric. Lemma 7, along with the property that $\partial \Pi^*(\boldsymbol{K}, \boldsymbol{\gamma})/\partial K_l = \lambda^l(\boldsymbol{\gamma})$ (see Property A3 in the Appendix), implies that a similar asymmetry result holds for the marginal optimal revenue, $\boldsymbol{\lambda}$.

In particular, as demand risk increases, the firm adjusts the optimal capacity vector in such a way that the expected marginal revenue equals the marginal cost (see Eq. (B.5) and (B.6 in the proof of Theorem 2). In the **FN**, the only way to achieve this is to acquire more of the flexible resource, the only resource available. In the **DN**, however, with n dedicated resources to adjust the capacities of, it may be optimal for the firm to decrease the capacities of *some* of its resources as demand risk increases. (Example 2 below shows that this is indeed possible.) However, it can never be optimal for the firm to decrease the capacities of *all* of its resources, i.e., it should acquire more capacity for *at least one* resource (as Theorem 2 indicates).

Observe that in the **DN**, the only dependency among the products arises due to their cross-price effects. Then, as $v \to 0$, the n-product **DN** behaves like n single-product single-resource newsvendors; and it is well-known that the single-product newsvendor under responsive pricing acquires more capacity as demand risk increases (Van Mieghem (2007); Van Mieghem and Dada (1999)). In contrast, when v is positive, some resource capacities might decrease with an increase in demand risk.

To construct Example 2, let $\mathbf{\Gamma}$ follow a Bivariate Normal distribution (BVN) with expectation vector $\boldsymbol{\mu}$ and variance-covariance matrix $\mathbf{\Sigma}$, where

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix},$$

and let $\rho \equiv \sigma_{12}/(\sigma_1\sigma_2)$ denote its correlation coefficient. Recall that $\mathbf{\Gamma}$ is assumed to have positive support. While the BVN does not satisfy this assumption, it allows us to utilize the results established in the literature on a convex order increase of this distribution, see Lemma A1. Further, we select distribution parameters such that the probability of negative numbers is extremely small; and also recompute the optimal solution using Truncated Normal distributions and find no difference in results within two decimal point accuracy.

Example 2. *Consider a two-product* **DN***, with capacity cost vector,* $\boldsymbol{c} = (10, 10)$*, and a symmetric cross-price effect,* $v = 0.5$*.* $\mathbf{\Gamma}$ *follows a BVN with* $\boldsymbol{\mu} = (300, 200)$ *and* $\boldsymbol{\sigma} = (100, 30)$*. We consider three settings, with independent (*$\rho = 0$*), positively correlated (*$\rho > 0$*), and negatively correlated (*$\rho < 0$*) demand intercepts. In each setting, a convex order increase in* $\mathbf{\Gamma}$ *is achieved by increasing* σ_2 *only (keeping the covariances,* $\sigma_{12} = \sigma_{21} = \rho\sigma_1\sigma_2$*, the same), see Lemma A1. Thus,* ρ *in the dependent demand settings will also vary as* σ_2 *changes, but we ensure that the sign of* ρ *does not change.*

In particular, in each setting, we increase σ_2 from 30 to 80 and determine the optimal capacity vector, $\boldsymbol{K}^* = (K_1^*, K_2^*)$, for each scenario, see Table 2.1. In all demand dependence settings, as $\boldsymbol{\Gamma}$ increases in convex order, the optimal capacity of at least one resource (Resource 2) increases, as indicated by Theorem 2, but at the same time, the optimal capacity of Resource 1, K_1^*, does decrease. (To ensure that the numerical findings are not distorted by the negative realizations of the Normal distribution, we computed the optimal capacity vectors using the Truncated Normal distributions and obtained the same values as in Table 2.1 within two decimal digit accuracy.)

Table 2.1: Numerical results for the two-product **DN** in Example 2

Independent demand intercepts					Positively correlated demand intercepts					Negatively correlated demand intercepts				
σ_1	σ_2	ρ	K_1^*	K_2^*	σ_1	σ_2	ρ	K_1^*	K_2^*	σ_1	σ_2	ρ	K_1^*	K_2^*
100	30	0	201.95	106.48	100	30	0.30	204.17	107.55	100	30	-0.30	199.78	105.27
100	35	0	201.67	108.91	100	35	0.26	203.73	110.03	100	35	-0.26	199.68	107.69
100	40	0	201.44	111.49	100	40	0.23	203.35	112.64	100	40	-0.23	199.60	110.27
100	45	0	201.25	114.20	100	45	0.20	203.02	115.36	100	45	-0.20	199.53	113.00
100	50	0	201.08	117.04	100	50	0.18	202.75	118.20	100	50	-0.18	199.49	115.86
100	55	0	200.95	119.99	100	55	0.16	202.51	121.15	100	55	-0.16	199.45	118.83
100	60	0	200.83	123.05	100	60	0.15	202.30	124.19	100	60	-0.15	199.41	121.91
100	65	0	200.72	126.20	100	65	0.14	202.12	127.32	100	65	-0.14	199.37	125.09
100	70	0	200.62	129.44	100	70	0.13	201.96	130.55	100	70	-0.13	199.32	128.36
100	75	0	200.52	132.77	100	75	0.12	201.81	133.85	100	75	-0.12	199.26	131.71
100	80	0	200.42	136.18	100	80	0.11	201.67	137.24	100	80	-0.11	199.18	135.14

It is important to understand the main drivers behind this example (that as demand gets riskier, the capacity of a resource might decrease). From the above discussion, we know that the existence of product substitutability is a main driver of this result. Interestingly, one does not need a capacity cost differential among the resources, nor a specific demand dependence structure, for this finding to hold. We can explain this phenomenon as follows. As the demand for product 2 becomes more variable (σ_2 increases), in accordance with Theorem 2, at least one resource capacity will increase (K_2^* in this case). This, in turn, results in a higher output, and hence a lower price for product 2 for any realization of $\boldsymbol{\Gamma}$. Then, since the products are substitutes, a cheaper product 2 leads to a lower price of product 1, and thus a lower marginal revenue for K_1. In other words, the decrease in K_1^* is a result of the increase in K_2^* *and* product substitutability. In general, because at least one capacity will increase in response to an increase in demand risk, some resource capacities might decrease as a result of such substitution effects.

Only when a convex order increase in demand risk does not alter the resource priorities, will all resource capacities increase. The following, quite restrictive situation demonstrates just such a case.

Remark 6. *Consider the n-product* $\mathbf{DN}$ *with $c_i = c, i = 1, \cdots, n$, and suppose that the joint pdf of demand intercepts, $f(\gamma_1, \gamma_2, \cdots, \gamma_n)$, is symmetric, i.e., $f(\gamma_1, \gamma_2, \cdots, \gamma_n) = f(\hat{\gamma_1}, \hat{\gamma_2}, \cdots, \hat{\gamma_n})$ for any permutation $(\hat{\gamma_1}, \hat{\gamma_2}, \cdots, \hat{\gamma_n})$ of $(\gamma_1, \gamma_2, \cdots, \gamma_n)$. If the symmetry property of the pdf is preserved after a convex order increase of* $\mathbf{\Gamma}$*, then the optimal capacity of each resource, $K_l^* = K^*, l = 1, \cdots, n$, will increase.*

Thus, in this setting, resource priorities, which are equal initially, are not affected by a convex order increase of $\mathbf{\Gamma}$, and all capacities increase (and still equal each other).

Finally, observe, in Example 2, that for a given demand standard deviation vector ($\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$), the capacity of both resources increase in ρ (see each row of Table 2.1). This result will be formally proven in Chapter 3, when we study the impact of demand dependence.

2.4 Conclusion

The main contribution of this Chapter is a novel analytic approach for the comparative statics analysis of newsvendor networks modeled as two-stage stochastic programming problems with recourse. Traditional approaches involve a parametric analysis of the recourse problem, followed by the use of Jacobians and the implicit function theorem, and have a level of effort that increases exponentially in problem size. As opposed to this, our approach exploits properties of the primal mathematical programming problem and of the dual variables, and links these properties to the concept of convex orders and to properties of the demand function. This yields a scalable approach, i.e., it applies to newsvendor networks with any number of non-independent (i.e., demand or resource sharing) products and resources, without a corresponding increase in effort, and is a main strength of this approach. Furthermore, our analysis does not rely on a specific demand distribution, and all results are distribution-free.

We use this approach to derive several structural properties of newsvendor networks under responsive pricing, and utilize these properties to establish the behavior of the optimal capacity and expected profit in demand risk for an arbitrary number of products in **FN** and **DN**. These properties can be used as a building block for future research. For example, extending this line of analysis to other newsvendor networks under responsive pricing, such as the parallel networks in which each product can be produced either by a dedicated resource or by a flexible resource, or to more complex resource networks, would be a next step. Furthermore, our approach provides a rigorous framework, which could be applied to other operations management problems that are not amenable to traditional parametric analysis. This is a promising area for future research. Another important research direction would be to explore our findings further. For instance, we show that the capacity of at least one resource increases in the **DN** as demand risk rises. However, we also provide a **DN** example in which a higher demand risk forces some resource capacities to decrease. While we provided some underlying principles that drive this result, it would be interesting to further explore when this situation happens.

Our analysis and insights come with several limitations. First, our approach for comparative statics analysis is successful for the capacity investment decision problem under responsive pricing, mainly because the capacity dual variable vector is well-behaved (in most networks we consider), in the sense that it is continuous, linear, and jointly convex in the demand intercept vector. These properties come from the *Definite Quadratic Programming* formulation of the recourse problem, combined with the parameter restrictions imposed in our setting. This reformulation, however, is possible due to the linear form of the demand function that we utilize here. Considering newsvendor networks under responsive pricing with other functional forms of demand remains a future research direction. Secondly, the model we consider here is too stylized (as most newsvendor models are) for practical decision-making purposes, and more research should be conducted to see if these insights continue to hold in more realistic settings.

In summary, while we acknowledge these limitations, we also believe that this Chapter utilizes a new approach that can be tested, and modified as needed, for comparative statics analysis of other operations management problems that may be modeled as stochastic programming problems with recourse. Eventually, one would like to develop this methodology into a unified framework. If this effort is successful, then it will provide a significant contribution to the operations management literature.

Chapter 3

Comparative Statics Analysis of Multi-product Newsvendor Networks under Responsive Pricing – Serial Network, and Demand Dependence

3.1 Introduction and Motivation

The results in Chapter 2 are encouraging, as they demonstrate that our proposed approach is indeed promising in providing a rigorous framework, which allows the analysis of **FN** and **DN** having an arbitrary number of products and resources. Hence, it is important to understand whether our proposed approach holds promise in the comparative statics study of a more analytically challenging setting. A natural next step would be to move from **FN** and **DN**, to a more complicated newsvendor network, i.e., the "Serial" network (**SN**), and to move from the comparative statics analysis of demand risk to demand dependence in the multi-product newsvendor networks under responsive pricing setting. In summary, in this Chapter, we investigate whether or not the results of Chapter 2 would continue to hold for a more complected newsvendor network, **SN**, and if there exits a similar scalable approach that allows for the comparative statics analysis of demand dependence in multi-product multi-resource newsvendor networks.

3.2 The Optimal Expected Profit and Capacity versus Demand Dependence – Serial Network

Having established our main results for the **FN** and **DN**, we now turn our attention to the **SN** without cross-price effects. The **SN** without cross-price effects consists of n products, *without* cross-price effects, with n dedicated resources and one flexible resource ($m = n+1$). Each product needs to be processed both by its own dedicated resource and by the flexible resource. In the **SN**, products are dependent through the shared capacity of the common flexible resource. We restrict the study of the **SN** to demand functions without cross-price effects to keep the analysis tractable. The KKT first-order optimality conditions for Problem $\boldsymbol{P_2}$ for **SN** are:

$$\begin{aligned}
\textbf{SN}: \quad \boldsymbol{\gamma} - 2\boldsymbol{q}^* - \boldsymbol{A}^T\boldsymbol{\lambda} + \boldsymbol{\mu} &= \mathbf{0}, && (3.1)\\
\lambda^l(K_l - q_l^*) &= 0, \quad l = 1\ldots n, && (3.2)\\
\lambda^f(K_f - \sum_{l=1}^{n} q_l^*) &= 0, && (3.3)\\
\mu^l q_l^* &= 0, \quad l = 1\ldots n. && (3.4)
\end{aligned}$$

where $\boldsymbol{\lambda} = (\lambda^1, \lambda^2, \cdots, \lambda^n, \lambda^f)^T$. Similar to **DN**, we consider, without loss of generality, that the resource vector $\boldsymbol{K^*} > \mathbf{0}$ in the **SN**. This is without loss of generality since if $K_l^* = 0$, $l = 1, \cdots, n$, as explained in Section 2.3.3, the original problem can be effectively reduced to the problem of smaller product size. Further, an optimal capacity solution to the **SN** satisfies $\sum_{l=1,\cdots,n} K_l^* \geq K_f^* \geq \max_{l=1,\cdots,n} K_l^*$ (see Harrison and Van Mieghem (1999) and Van Mieghem (2003) for the exogenous pricing setting; a similar argument applies in our case). Thus, as long as $K_l^* > 0$ for some $l = 1, \cdots, n$, K_f^* in the **SN** will also be positive.

In order to study properties of the optimal capacity dual variable vector, $\boldsymbol{\lambda}$, we need to ensure that it is unique. The following assumption, well-known in the optimization literature, guarantees this.

Assumption 2. Linear Independence Constraint Qualifications (LICQ) *(e.g., Bazaraa et al. (1993)): For any given* $\boldsymbol{\gamma}$*, consider* $\boldsymbol{q}^*(\boldsymbol{K}^*, \gamma)$*, the optimal solution to* $\boldsymbol{P_2(K^*, \gamma)}$*, where* $\boldsymbol{K^*}$ *denotes the optimal stage 1 capacity vector. In this optimal solution, let* $I_R \subseteq \{1, \cdots, m\}$ *denote the indices of resources for which constraint (2.10) is binding, and let* $I_P \subseteq \{1, \cdots, n\}$ *denote the indices of products for which constraint (2.11) is binding, that is, for each* $l \in I_R$*,* $\boldsymbol{A}_l \boldsymbol{q^*} = K_l^*$*, and for each* $i \in I_P$*,* $q_i^* = 0$*. Then, we assume that* $\bigtriangledown(\boldsymbol{A}_l \boldsymbol{q} - K_l^*)|_{\boldsymbol{q}=\boldsymbol{q^*}}$*,* $l \in I_R$*, and* $\bigtriangledown(-q_i)|_{\boldsymbol{q}=\boldsymbol{q^*}}$*,* $i \in I_P$*, are linearly independent, where* $\bigtriangledown f(\boldsymbol{q}) \equiv (\frac{\partial f(\boldsymbol{q})}{\partial q_1}, \cdots, \frac{\partial f(\boldsymbol{q})}{\partial q_n})$*.*

Remark 7.

1. *In the* **FN** *and* **DN***, Assumption 2 is redundant.*
2. *In the* **SN***, Assumption 2 reduces to the assumption that the optimal stage 1 solution,* $\boldsymbol{K^*}$*, is not fully or partially "balanced," that is, there does not exist any binary vector* $\boldsymbol{\tau}_{(n,1)}$ *such that* $(K_1^*, \cdots, K_n^*)^T \cdot \boldsymbol{\tau}_{(n,1)} = K_f^*$*. Thus, Assumption 2 excludes capacity solutions where* K_f^* *can be expressed as the sum of a number of dedicated resource capacities.*

We assume that Assumption (**2**) is satisfied in the **SN** without cross-price effects studied in this Chapter.

3.2.1 Two-product SN without cross-price effects

We start our analysis by studying the two-product **SN** without cross-price effects. Consider the two-product **SN** without cross-price effects ($v = 0$). The following Lemma gives the condition which guarantee the uniqueness of optimal capacity dual variable vector $\boldsymbol{\lambda} = (\lambda^1, \lambda^2, \lambda^f)$.

Lemma 8. *Consider the two-product* **SN** *without cross-price effects ($v = 0$). The optimal capacity dual variable vector* $\boldsymbol{\lambda} = (\lambda^1, \lambda^2, \lambda^f)$ *is unique if and only if* $\max\{K_1^*, K_2^*\} < K_f^* < K_1^* + K_2^*$.

Remark 8. *For 2-product* **SN** *without cross-price effects,* $\boldsymbol{K}^*$ *satisfies the condition given in Lemma 8 as long as Assumption 2 is satisfied for* $\boldsymbol{K}^*$.

On the other hand it is insightful to derive the conditions for the fully balanced **SN**, i.e., $K_f^* = K_1^* + K_2^*$ which implies that the optimal capacity dual variable vector is not unique.

Lemma 9. *In the two-product* **SN** *without cross-price effects, for any feasible solution* $\boldsymbol{K} = (K_1, K_2, K_f)$ *with* $K_1 + K_2 = K_f$ *we must have that*

$$\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_1} + \frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_f} = \int_{2K_1}^{\infty} (\gamma_1 - 2K_1) f_{\Gamma_1}(\gamma_1) d\gamma_1,$$

and

$$\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_2} + \frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_f} = \int_{2K_2}^{\infty} (\gamma_2 - 2K_2) f_{\Gamma_2}(\gamma_2) d\gamma_2,$$

where $f_{\Gamma_i}(.)$ *representing the marginal pdf of* Γ_i.

Lemma 10. *The optimal capacity solution to the two-product* **SN** *is fully balanced, with* $K_f^* = K_1^* + K_2^*$, *if and only if*

$$\begin{aligned} &E[\Gamma_2 - 2K_2^* | \Gamma_1 - 2K_1^* > \Gamma_2 - 2K_2^*, \Gamma_2 > 2K_2^*] \Pr(\Gamma_1 - 2K_1^* > \Gamma_2 - 2K_2^*, \Gamma_2 > 2K_2^*) \\ &+ E[\Gamma_1 - 2K_1^* | \Gamma_2 - 2K_2^* > \Gamma_1 - 2K_1^*, \Gamma_1 > 2K_1^*] \Pr(\Gamma_2 - 2K_2^* > \Gamma_1 - 2K_1^*, \Gamma_1 > 2K_1^*) \geq c_f, \end{aligned} \tag{3.5}$$

where $K_i^*, i = 1, 2$, *are solutions to* $\int_{2K_i^*}^{\infty} (\gamma_i - 2K_i^*) f_{\Gamma_i}(\gamma_i) d\gamma_i = c_i + c_f$, *with* $f_{\Gamma_i}(.)$ *representing the marginal pdf of* Γ_i.

The left-hand side of (3.5) represents the marginal revenue in the capacity-constrained demand domains ($\{(\gamma_1, \gamma_2) : \gamma_1 > 2K_1 \text{ or } \gamma_2 > 2K_2\}$), whereas the right-hand side represents the marginal cost of flexible capacity. Thus, recalling that $K_f^* \leq K_1^* + K_2^*$ in an optimal solution to the **SN**, if the marginal revenue exceeds the marginal cost, then it becomes economically desirable to add that final unit of flexible capacity, thus balancing the capacities. If this condition does not hold, then it is better to keep the flexible capacity lower, which leads to capacity imbalance, allowing the firm to take advantage of risk pooling. From this perspective, the capacity balance condition under responsive pricing and its interpretation is similar to that under exogenous prices, see Bernstein et al. (2007).

Having established the condition for the uniqueness of the capacity dual variable vector, we now derive further properties of it. For a given $\boldsymbol{K} = (K_1, K_2, K_f)$ that satisfies the conditions given in Lemma 8, $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$, the KKT first-order conditions (in (3.1)-(3.4)) imply the following optimal solution for the capacity dual variables λ^f, λ^1, and λ^2 in each feasible domain, see Table 3.1, where,

Table 3.1: Optimal values of the capacity dual variables λ^f, λ^1, and λ^2 in each domain

Domain j	λ_j^f	λ_j^1	λ_j^2	$\lambda_j^f + \lambda_j^1$	$\lambda_j^f + \lambda_j^1 + \lambda_j^2$
Ω_1	0	0	0	0	0
Ω_2	0	$\gamma_1 - 2K_1$	0	$\gamma_1 - 2K_1$	$\gamma_1 - 2K_1$
Ω_3	0	0	$\gamma_2 - 2K_2$	0	$\gamma_2 - 2K_2$
Ω_4	$\frac{\gamma_1+\gamma_2}{2} - K_f$	0	0	$\frac{\gamma_1+\gamma_2}{2} - K_f$	$\frac{\gamma_1+\gamma_2}{2} - K_f$
Ω_5	$\gamma_2 - 2K_f + 2K_1$	$\gamma_1 - \gamma_2 + 2K_f - 4K_1$	0	$\gamma_1 - 2K_1$	$\gamma_1 - 2K_1$
Ω_6	$\gamma_1 - 2K_f + 2K_2$	0	$\gamma_2 - \gamma_1 + 2K_f - 4K_2$	$\gamma_1 - 2K_f + 2K_2$	$\gamma_2 - 2K_2$

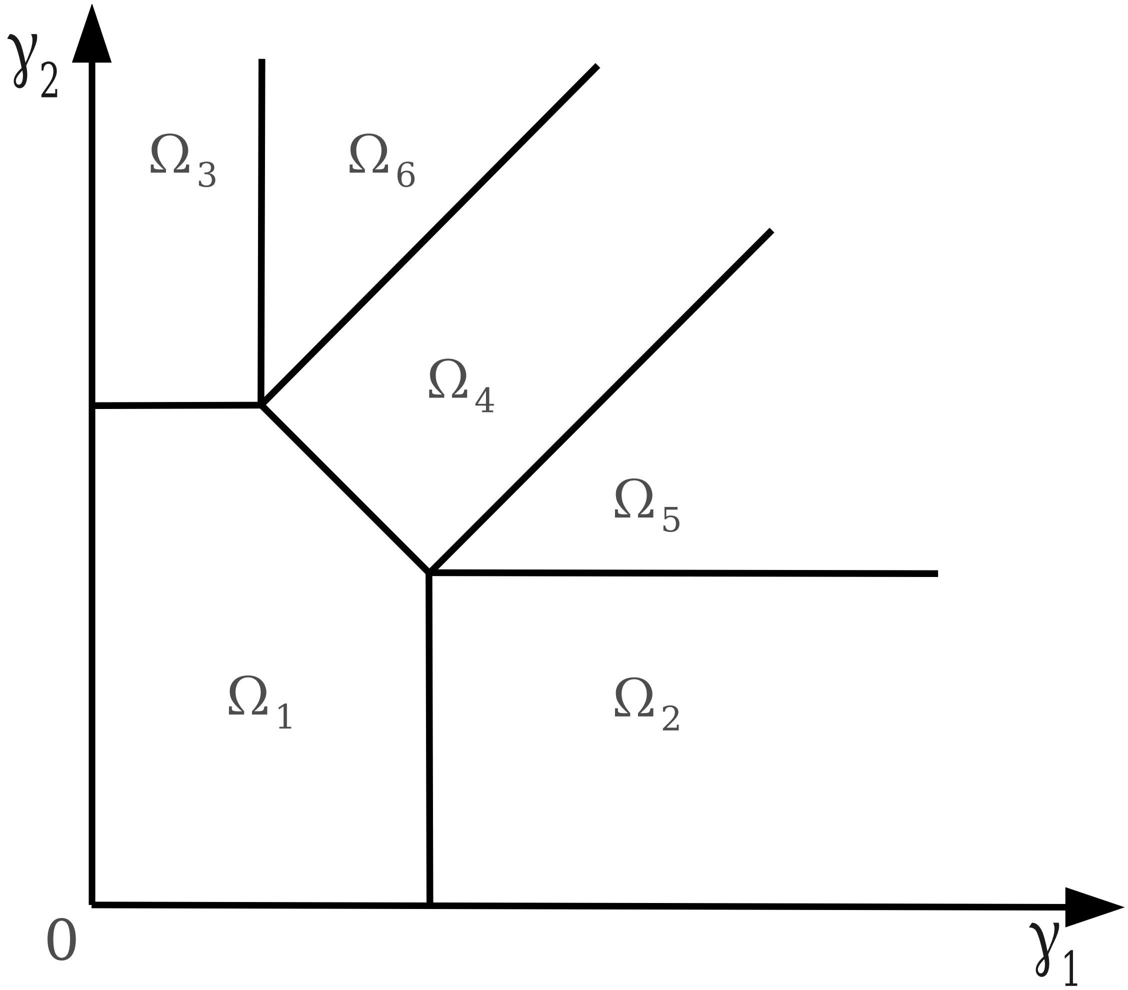

Figure 3.1: The domains for the two-product **SN**

$$\Omega_1 = \{\gamma_1 \leq 2K_1, \gamma_2 \leq 2K_2, \gamma_1 + \gamma_2 \leq 2K_f\}, \tag{3.6}$$

$$\Omega_2 = \{\gamma_1 > 2K_1, \gamma_2 \leq 2K_2, \gamma_2 \leq 2K_f - 2K_1\}, \tag{3.7}$$

$$\Omega_3 = \{\gamma_2 > 2K_2, \gamma_1 \leq 2K_1, \gamma_1 \leq 2K_f - 2K_2\}, \tag{3.8}$$

$$\Omega_4 = \{-2K_f \leq \gamma_1 - \gamma_2 \leq 2K_f, \gamma_1 + \gamma_2 > 2K_f, \gamma_1 - \gamma_2 < 4K_1 - 2K_f, \gamma_2 - \gamma_1 < 4K_2 - 2K_f\}, \tag{3.9}$$

$$\Omega_5 = \{\gamma_1 - \gamma_2 > 4K_1 - 2K_f, \gamma_2 > 2K_f - 2K_1\}, \tag{3.10}$$

$$\Omega_6 = \{\gamma_2 - \gamma_1 > 4K_2 - 2K_f, \gamma_1 > 2K_f - 2K_2\}. \tag{3.11}$$

As we can see from the expressions of $\boldsymbol{\lambda}$ within each domain in Table 3.1, we have a similar result of the continuity and linearity of optimal capacity dual variable vector $\boldsymbol{\lambda}$ in the **SN**.

Lemma 11. *The optimal capacity dual variable vector,* $\boldsymbol{\lambda} = (\lambda^1, \lambda^2, \lambda^f)$ *in the two-product* **SN** *without cross-price effect satisfies the following properties:*

1. $\boldsymbol{\lambda}$ *is continuous in* $\boldsymbol{\gamma}$.

2. $\boldsymbol{\lambda}$ *can be expressed as a linear function of* $\boldsymbol{\gamma}$*, given in Table 3.1, within each feasible domain* Ω_j*,* $j \in \Psi$.

Having established a similar result for **SN**, then the natural next step of utilizing our approach to perform comparative static analysis of **SN**, similar to what have done in Chapter 2, is to prove that $\boldsymbol{\lambda}$ is jointly convex in $\boldsymbol{\gamma}$ in **SN**. However this result does not hold in the **SN**. Consider domains Ω_4 and Ω_5, defined by the following set of dual variables:

$$\Omega_4 : \lambda_4^f > 0, \lambda_4^1 = \lambda_4^2 = \mu_4^1 = \mu_4^2 = 0, \text{ and } \Omega_5 : \lambda_5^f > 0, \lambda_5^1 > 0, \lambda_5^2 = \mu_5^1 = \mu_5^2 = 0.$$

From Remark 4, Ω_4 and Ω_5 are adjacent domains, with $\lambda_4^f - \lambda_5^f = (\gamma_1 - \gamma_2)/2 + K_f - 2K_1$. Notice that by definition of Ω_4, $\boldsymbol{\gamma} \in \Omega_4$ implies that $(\gamma_1 - \gamma_2)/2 + K_f - 2K_1 < 0$, or equivalently, $\lambda_4^f - \lambda_5^f < 0$, contradicting with Lemma 5. Likewise, it is easy to show that both λ^l and $\lambda^l + \lambda^f$, $l = 1, 2$, violate Lemma 5 in this network. Further, based on this observation, we are able to conclude the non-convexity of its $\boldsymbol{\lambda}$ vector in Lemma 12. Interestingly, however, the sum of the capacity dual variables, $\lambda^1 + \lambda^2 + \lambda^f$, does satisfy Lemma 5 for every pair of adjacent domains (see Table 3.1). We will discuss this finding further in the following.

It is important to understand why Lemma 5 holds in the **FN** and **DN**, but not in the **SN**. This is mainly due to the dependencies among the dedicated and flexible resources present in the **SN**. For the **FN**, with only one resource available, there is no such dependency, and a similar argument holds for the **DN**, with only dedicated resources. Mathematically speaking, consider two adjacent domains, Ω_i and Ω_j, where Ω_j can be reached from Ω_i by increasing some γ_k (e.g., Ω_4 and Ω_5 in Table 3.1). Then, a necessary condition for Lemma 5 to hold is that $\frac{\partial \lambda_i^l(\gamma)}{\partial \gamma_k} \leq \frac{\partial \lambda_j^l(\gamma)}{\partial \gamma_k}, l = 1, \cdots, m$. Since $\boldsymbol{\lambda}$ is continuous and linear in $\boldsymbol{\gamma}$, we have that $\lambda_i^l(\boldsymbol{\gamma}) = \lambda_j^l(\boldsymbol{\gamma})$, $\forall l = 1, \cdots, m$ and for any $\boldsymbol{\gamma} \in H_{ij}$. Consider λ^f, the marginal revenue of the flexible resource, which is the sensitivity of the optimal stage 2 revenue, $\Pi^*(\boldsymbol{K})$, to unit change in the capacity of the flexible resource. Observe that λ^f in the two-product **SN** violates this necessary condition, i.e., $\frac{\partial \lambda_4^f(\gamma)}{\partial \gamma_1} = \frac{1}{2} > \frac{\partial \lambda_5^f(\gamma)}{\partial \gamma_1} = 0$. This violation occurs in the **SN**, because while the bottleneck in domain Ω_4 is the flexible resource, the bottleneck becomes dedicated resource 1 (K_1) in domain Ω_5. Consequently, increasing the capacity of the flexible resource on its own in this domain has no value to the firm.

Previous analysis shows that the capacity dual variables in the two-product **SN** violate Lemma 5. However, we have not yet made any claims about the convexity (or the lack of it) of the capacity dual variables in the **SN**. We now do this through Lemma 12, which establishes that the capacity dual variables in the **SN**, λ^f and $\lambda^l, l = 1, \cdots, n$, cannot be jointly convex in $\boldsymbol{\gamma}$. However, this lack of convexity of each capacity dual variable does not readily imply that a result similar to dedicated networks (i.e., as demand risk increases, at least one resource capacity increases, see Theorem 2) would not hold, and we pursue this question here.

Lemma 12. *For some dual variable λ, if there exist any two adjacent domains such that Lemma 5 does not hold, then λ cannot be jointly convex in $\boldsymbol{\gamma}$.*

Interestingly, in the previous discussion we mention that $\lambda^1 + \lambda^2 + \lambda^f$ does satisfy Lemma 5, and as we show below, this indeed becomes sufficient to prove a similar result for the two-product **SN**. This suggests that our comparative statics approach might be somewhat robust to deviations in our key result, Lemma 5.

Our methodology for establishing the capacity result for the two-product **SN** is similar to that utilized for the n-product **FN** and **DN**. Specifically, starting from the result that $\lambda^1 + \lambda^2 + \lambda^f$ satisfies Lemma 5, we first establish its joint convexity in $\boldsymbol{\gamma}$ (similar to the proof of Lemma 7) and show that it is decreasing in $\boldsymbol{K}$, and Theorem 3 follows.

Theorem 3. *Consider the two-product* **SN** *without cross-price effects and suppose that Assumption 2 holds at the optimal capacity solution,* $\boldsymbol{K}^*$*. If* $\boldsymbol{\Gamma}$ *increases in convex order, then at least one of the optimal capacities,* K_1^*, K_2^**, or* K_f^**, will increase, provided that Assumption 2 continues to hold at the new optimal solution.*

Thus, for Theorem 3 to hold, both the current and the new optimal capacity solutions cannot be balanced (see Remark 7 on the definition of a balanced solution), because the proof of Theorem 3 relies on the uniqueness of the optimal dual variable, $\boldsymbol{\lambda}$, and this can no longer be guaranteed when the optimal capacity solution is balanced.

3.2.2 Multi-product SN with no cross-price effects

In Section 3.2.1, we show that in the two-product **SN** without cross-price effects, $\lambda^1 + \lambda^2 + \lambda^f$ does satisfy Lemma 5, hence Theorem 3. To further investigate this result, we now study the conditions (if any) under which this result carries over to n-product ($n > 2$) **SN**s. Specifically, with an additional (albeit restrictive) assumption on the structure of an optimal capacity solution in the n-product **SN** (Assumption **(3)**), we are able to show that there exists a linear combination of all capacity dual variables, i.e., $\sum_{l=1}^{n} \lambda^l + \alpha\lambda^f$, for some scalar α (defined below), that satisfies Lemma 5. Similar to what we have done for the **FN**, **DN**, and two-product **SN** without cross-price effects, we then use this result to show that $\sum_{l=1}^{n} \lambda^l + \alpha\lambda^f$ is jointly convex in $\boldsymbol{\Gamma}$ and decreasing in $\boldsymbol{K}$. This result allows us show that as $\boldsymbol{\Gamma}$ increases in convex order, at least one of the optimal capacities, $K_l^*, l = 1, \cdots, n, f$, in the n-product **SN** will increase (Theorem 4), similar to those proven for the **FN**, **DN**, and two-product **SN** without cross-price effects.

For the **SN** without cross-price effects, matrix $\boldsymbol{W}$ becomes an identity matrix, and matrix $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \begin{vmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & 1 & \cdots & 1 \end{vmatrix}, \tag{3.12}$$

The second stage problem $\boldsymbol{P}_2$ is given by:

$$\boldsymbol{P}_2 : \Pi(\boldsymbol{K}, \boldsymbol{\gamma}) \equiv \text{maximize}_q \Pi\left(\boldsymbol{K}, \boldsymbol{\gamma}\right) = \boldsymbol{\gamma}^T \boldsymbol{q} - \boldsymbol{q}^T \boldsymbol{q} \tag{3.13}$$

$$\text{subject to } \boldsymbol{A}\boldsymbol{q} \leq \boldsymbol{K}, \tag{3.14}$$

$$\boldsymbol{q} \geq \boldsymbol{0}, \tag{3.15}$$

where $\boldsymbol{A}$ is given by (3.12) and $\boldsymbol{K} = (K_1, ..., K_n, K_f)^T$ is the capacity vector, where $K_l, l = 1, ...n$, represents the dedicated resource for product l, and K_f represents the flexible resource. Let $\boldsymbol{\lambda} = (\lambda^1, ..., \lambda^n, \lambda^f)^T \geq \boldsymbol{0}$ and $\boldsymbol{\mu} = (\mu^1, \mu^2, ..., \mu^n)^T \geq \boldsymbol{0}$ denote the dual variables (KKT multipliers) corresponding to Constraints (3.14) and (3.15), respectively. Then, the KKT first-order conditions for problem $\boldsymbol{P}_2$ are given by Eqs. (3.1) - (3.4). For comparison purposes, Table 3.2 provides a summary of the parameters for the **FN** and **DN** (studied in Chapter 2), and **SN**.

Since in the **SN**, $\boldsymbol{\lambda}$ is a vector (in the **FN**, it is a scalar), for ease of discussion, in the **SN**, we partition the set of products into three disjoint sets in any given $\Omega_j, j \in \Psi$, such that:

$$\begin{aligned} \Phi_j^1 &\equiv \{l = 1, \cdots, n : \mu_j^l > 0, \lambda_j^l = 0, \lambda_j^f \geq 0\}, \\ \Phi_j^2 &\equiv \{l = 1, \cdots, n : \mu_j^l = 0, \lambda_j^l = 0, \lambda_j^f \geq 0\}, \\ \Phi_j^3 &\equiv \{l = 1, \cdots, n : \mu_j^l = 0, \lambda_j^l > 0, \lambda_j^f \geq 0\}. \end{aligned} \tag{3.16}$$

Since $\boldsymbol{K}^* > \boldsymbol{0}$ in the **SN**, observe that in any $\Omega_j, j \in \Psi$, in the **SN**, there does not exist any

Table 3.2: Parameters for the three basic networks: The **FN**, **DN**, and **SN**

Network	**FN**	**DN**	**SN**
m	1	n	$n+1$
$\boldsymbol{A}$	$(1,1,\ldots,1)$	$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$
$\boldsymbol{K}$	K_f	$(K_1, K_2, \ldots K_n)^T$	$(K_1, K_2, \ldots K_n, K_f)^T$
$\boldsymbol{\lambda}$	λ^f	$(\lambda^1, \lambda^2, \ldots \lambda^n)^T$	$\left(\lambda^1, \lambda^2, \ldots \lambda^n, \lambda^f\right)^T$
$\boldsymbol{\mu}$	$(\mu^1, \mu^2, \ldots \mu^n)^T$	$(\mu^1, \mu^2, \ldots \mu^n)^T$	$(\mu^1, \mu^2, \ldots \mu^n)^T$
Constraint set (2.10)	$\sum_{l=1}^n q_l \leq K_f$	$q_l \leq K_l,\, l = 1, 2, \ldots n$	$q_l \leq K_l,\, l = 1, 2, \ldots n$ $\sum_{l=1}^n q_l \leq K_f$

product $l \in \{1, \cdots, n\}$ such that both $\mu_j^l > 0$ and $\lambda_j^l > 0$. This is because $\mu_j^l > 0$ implies that $q_l^* = 0$, which, in turn, implies that $\lambda_j^l = 0$ (since $K_l^* > 0$). Thus, $\Phi_j^1 \cup \Phi_j^2 \cup \Phi_j^3 = \{1, \cdots, n\}$ in any feasible domain $\Omega_j, j \in \Psi$. If $j \in \Phi_1^j$ then $q_j = 0$ and if $j \in \Phi_3^j$ then $q_j = K_j$. Then, for $\Omega_j, j \in \Psi$, Eq. (3.1) can be written as:

$$\begin{cases} \vdots \\ \gamma_l - \lambda_l^f + \mu_j^l = 0 & \text{if } l \in \Phi_j^1 \\ \vdots \end{cases}$$

$$\begin{cases} \vdots \\ \gamma_l - 2q_l^* - \lambda_l^f = 0 & \text{if } l \in \Phi_j^2 \\ \vdots \end{cases}$$

$$\begin{cases} \vdots \\ \gamma_l - 2K_l - \lambda_j^l - \lambda_j^f = 0 & \text{if } l \in \Phi_j^3 \\ \vdots \end{cases}$$

Lemma 13. *The optimal capacity dual variable vector, $\boldsymbol{\lambda}$, in the n-product* **SN** *satisfies the following properties:*

1. *$\boldsymbol{\lambda}$ is continuous in $\boldsymbol{\gamma}$.*

2. *$\boldsymbol{\lambda}$ can be expressed as a linear function of $\boldsymbol{\gamma}$ within each feasible domain Ω_j, $j \in \Psi$.*

Lemma 13 establishes the continuity and linearity for the dual capacity variables $\boldsymbol{\lambda}$, hence the continuity and linearity of $\sum_{l=1}^n \lambda^l + \alpha\lambda^f$, $\forall\alpha$, in the n-product **SN** without cross-effects. Our next step is to study the convexity of the $\sum_{l=1}^n \lambda^l + \alpha\lambda^f$. Before we do this, we firstly discuss the aforementioned additional assumption.

For a given capacity vector $\boldsymbol{K} = (K_1, K_2, ...K_n, K_f)$, sort $K_l, l = 1, \cdots, n$, in nondecreasing order and relabel such that $K_{(1)} \leq K_{(2)} \leq ... \leq K_{(n)}$. Define $\bar{i}\,(\boldsymbol{K}) \in \{1, \cdots, n-1\}$ and

$\bar{j}(\boldsymbol{K}) \in \{0, \cdots, n-2\}$ such that

$$\begin{aligned} K_f &\in \left(K_{(1)} + K_{(2)} + \cdots + K_{\left(\bar{i}\right)}, K_{(1)} + K_{(2)} + \cdots + K_{\left(\bar{i}+1\right)}\right), \\ K_f &\in \left(K_{(n)} + K_{(n-1)} + \cdots + K_{\left(n-\bar{j}\right)}, K_{(n)} + K_{(n-1)} + \cdots + K_{\left(n-\bar{j}-1\right)}\right). \end{aligned}$$

Recall that in an optimal solution to the **SN**, $K_l^* \le K_f^* \le \sum_{l=1}^n K_l^*$, for $l = 1, \cdots, n$. In addition, Assumption **(2)** rules out the balanced solutions, further reducing this range to $\max_{l=1,\cdots,n} K_l^* < K_f^* < \sum_{l=1}^n K_l^*$. Then, for any capacity solution that satisfies these conditions, $\bar{i}(\boldsymbol{K})$ and $\bar{j}(\boldsymbol{K})$ always exist within their specified ranges. Now we are ready to present Assumption **(3)**.

Assumption 3. *Let $\boldsymbol{K}^*$ denote the optimal capacity solution in the* **SN** *for the original random vector, $\boldsymbol{\Gamma}$. For any $\boldsymbol{\Gamma} \le_{cx} \overline{\boldsymbol{\Gamma}}$, denote the optimal capacity vector for $\overline{\boldsymbol{\Gamma}}$ as $\overline{\boldsymbol{K}}^*$. Then, corresponding to all random vectors $\overline{\boldsymbol{\Gamma}} \ge_{cx} \boldsymbol{\Gamma}$ and $\forall t : 0 \le t \le 1$, we assume that there exists a constant α such that:*

$$\alpha \in [\bar{i}(\boldsymbol{K}), \bar{j}(\boldsymbol{K}) + 2], \tag{3.17}$$

where $\boldsymbol{K}$ is a convex combination of $\boldsymbol{K}^*$ and $\overline{\boldsymbol{K}}^*$, that is, $\boldsymbol{K} = t\boldsymbol{K}^* + (1-t)\overline{\boldsymbol{K}}^*$.

This assumptionx is crucial in the proof of the convexity of $\sum_{l=1}^n \lambda^l + \alpha\lambda^f$, and in the subsequent analysis of the n-product ($n > 2$) **SN** without cross-price effects.

Example 3. *In this example, without loss of generality, suppose $K_i < K_j$, if $i < j$. Consider a two-product* **SN** *without cross-price effect. We have $K_1 < K_2 < K_f < K_1 + K_2$. Then,*

$$\begin{aligned} K_f &> K_1, \\ K_f &< K_1 + K_2, \\ K_f &> K_2, \\ K_f &< K_2 + K_1. \end{aligned}$$

Consequently, $\bar{i} = 1$ and $\bar{j} = 0$. $\bar{i} = 1 \le \bar{j} + 2 = 2$, which leads to $\alpha \in [\bar{i}, \bar{j} + 2]$, i.e., $\alpha = 1$ or

2. *Recall that in Section 3.2.1, we use the linear combination,* $\lambda^1 + \lambda^2 + \lambda^f$*, which essentially is a special case with* $\alpha = 1$*, to prove the convexity result. Hence, given that* $\alpha = 1$ *or* 2*, we can use the linear combination,* $\lambda^1 + \lambda^2 + 2\lambda^f$*, to establish similar results to Section 3.2.1.*

Now consider a three-product **SN** *without cross-price effect, with* $K_1 < K_2 < K_3 < K_f$*. Then, for the specific range of* K_f*, there are only two cases: (i)* $K_f \in (K_1 + K_2, K_1 + K_2 + K_3)$ *and (ii)* $K_f \in (K_1, K_1 + K_2)$*.*

(i) $K_f \in (K_1 + K_2, K_1 + K_2 + K_3)$*. Since*

$$\begin{aligned} K_f &> K_1 + K_2, \\ K_f &< K_1 + K_2 + K_3, \end{aligned}$$

then $\bar{i} = 2$*. Although* K_f *is not necessarily greater than* $K_3 + K_2$*, we know that* $K_f > K_3$*. Consequently, we have* $\bar{i} \geq 0$*, i.e.,* $\bar{i} + 2 \geq 2 = \bar{i}$*. Therefore,* $\alpha = 2$*.*

(ii) $K_f \in (K_1, K_1 + K_2)$*. Since*

$$\begin{aligned} K_f &> K_1, \\ K_f &< K_1 + K_2, \end{aligned}$$

then $\bar{i} = 1$*. Similarly,* $\bar{j} \geq 0$*, i.e.,* $\bar{j} + 2 \geq 1 = \bar{i}$*. Therefore* $\alpha = 1$ *or* 2*. Consequently, for the three-product* **SN** *without cross-price effects,* $\alpha = 2$*.*

Now consider a four-product **SN** *without cross-price effect, with* $K_1 < ... < K_4 < K_f$*. There are three cases for the specific range of* K_f*: (i)* $K_f \in (K_1 + K_2 + K_3, K_1 + K_2 + K_3 + K_4)$*, (ii)* $K_f \in (K_1 + K_2, K_1 + K_2 + K_3)$*, and (iii)*$K_f \in (K_1, K_1 + K_2)$*. Consider case (i).*

(i) $K_f \in (K_1 + K_2 + K_3, K_1 + K_2 + K_3 + K_4)$*. Since*

$$\begin{aligned} K_f &> K_1 + K_2 + K_3, \\ K_f &< K_1 + K_2 + K_3 + K_4, \end{aligned}$$

then $\bar{i} = 3$*. However,* α *exists if and only if* $\bar{j} \geq 1$*, i.e.,* $K_f > K_4 + K_3$*, which does not* *necessarily hold. Consequently, α does not necessarily exist for a four-product* **SN***.*

From Example 3 we can see that, for the two or three-product **SN**, $\alpha \in [\bar{i}\,(\boldsymbol{K}), \bar{i}\,(\boldsymbol{K})+2]$ exits. However, for the four-product **SN**, α does not necessarily exist. Consequently, for the general n-product **SN**, α does not necessary exist for an arbitrary $\boldsymbol{K}$, i.e., Assumption (**3**) is not necessarily satisfied. The purpose of Assumption (**3**) is to impose a restriction on each optimal capacity solution such that the ordering of the capacities is preserved as $\boldsymbol{\Gamma}$ increases in convex order. Then, with Assumption (**3**), we are able to prove a similar result on the optimal capacity versus risk exposure for the general n-product **SN** without cross-price effects, similar to those proven for the **FN** and **DN**.

Lemma 14. *For a given optimal $\boldsymbol{K}^* > 0$ that satisfies Assumptions* (**2**) *and* (**3**)*, if there exists constant α such that $\alpha \in [\bar{i}\,(\boldsymbol{K}^*), \bar{j}\,(\boldsymbol{K}^*)+2]$, then we must have that $\sum_{l=1}^{n} \lambda_j^l(\boldsymbol{\gamma}) + \alpha\lambda_j^f(\boldsymbol{\gamma}) \geq \sum_{l=1}^{n} \lambda_k^l(\boldsymbol{\gamma}) + \alpha\lambda_k^f(\boldsymbol{\gamma})$, $\forall \boldsymbol{\gamma} \in H_{kj}^j$.*

Then, from Lemmas 13 and 14 we are able to prove the convexity result of $\sum_{l=1}^{n} \lambda^l + \alpha\lambda^f$. Lemma 15 states this result.

Lemma 15. *For a given optimal $\boldsymbol{K}^* > 0$ that satisfies Assumptions* (**2**) *and* (**3**)*, if there exists constant α such that $\alpha \in [\bar{i}\,(\boldsymbol{K}^*), \bar{j}\,(\boldsymbol{K}^*)+2]$, then we must have that $\sum_{l=1}^{n} \lambda^l(\boldsymbol{\gamma}) + \alpha\lambda^f(\boldsymbol{\gamma})$ is jointly convex in $\boldsymbol{\gamma}$.*

In addition, the following lemma shows that $\sum_{l=1}^{n} \lambda^l + \alpha\lambda^f$ is decreasing in $\boldsymbol{K} \in [\boldsymbol{K}_a, \boldsymbol{K}_b]$ if a constant $\alpha \in [\bar{i}\,(\boldsymbol{K}), \bar{j}\,(\boldsymbol{K})+2]$ exits for $\forall \boldsymbol{K} \in [\boldsymbol{K}_a, \boldsymbol{K}_b]$.

Lemma 16. *Suppose that there exists a constant α such that for any $\boldsymbol{K} \in [\boldsymbol{K}_a, \boldsymbol{K}_b]$, $\alpha \in [\bar{i}\,(\boldsymbol{K}), \bar{j}\,(\boldsymbol{K})+2]$. Then, $\sum_{l=1}^{n} \lambda^l + \alpha\lambda^f$ is decreasing in $\boldsymbol{K}$ for any $\boldsymbol{K} \in [\boldsymbol{K}_a, \boldsymbol{K}_b]$.*

Having shown that $\sum_{l=1}^{n} \lambda^l + \alpha\lambda^f$ is continuous, convex, and piecewise linear in $\boldsymbol{\gamma}$, and decreasing in $\boldsymbol{K}$, we are now ready to present our main result for the n-product **SN** on the optimal capacity versus risk exposure.

Theorem 4. *For the $n(\geq 3)$-product* **SN** *without cross-price effects, if* $\mathbf{\Gamma}$ *increases in convex order, then at least one of the optimal capacities, $K_l^*, l = 1, \cdots, n, f$, will increase as long as Assumptions* **(2)** *and* **(3)** *hold.*

Suppose we know that at least one capacity, K_l, $l = 1, \cdots, n, f$, will increase if $\mathbf{\Gamma}$ increases in convex order, but what is the exact behavior of the optimal $\boldsymbol{K}^*$? Without additional information or restriction on an arbitrary n-product **SN**, it is difficult to predict what would happen. As discussed above, we know that neither $\lambda^f(\boldsymbol{\gamma})$ and $\lambda^l(\boldsymbol{\gamma})$, $l = 1, \cdots, n$, is necessarily convex in $\boldsymbol{\gamma}$. As $\mathbf{\Gamma}$ increases in convex order, it is difficult to predict the relationships of $E_{\mathbf{\Gamma}}[\lambda^f(\boldsymbol{\gamma})]$ and c_f, and $E_{\mathbf{\Gamma}}[\lambda^l(\boldsymbol{\gamma})]$ and c_l, $l = 1, \cdots, n$. However, if α does exist, then there will be some restrictions imposed on the relationships of $E[\lambda^f(\boldsymbol{\gamma})]$ and c_f, and $E[\lambda^l(\boldsymbol{\gamma})]$ and c_l, such that $\sum_{l=1}^n E[\lambda^l(\boldsymbol{\gamma})] + \alpha E[\lambda^f(\boldsymbol{\gamma})] \geq \sum_{l=1}^n c_l + \alpha c_f$. In what follows, for ease of presentation, we use λ^f and λ^l to denote $E_{\mathbf{\Gamma}}[\lambda^f(\boldsymbol{\gamma})]$ and $E_{\mathbf{\Gamma}}[\lambda^l(\boldsymbol{\gamma})]$, respectively. Then, both λ^f and λ^l are functions of $\boldsymbol{K}$.

Lemma 17. *Let $\Lambda(\boldsymbol{K}) \equiv E_{\mathbf{\Gamma}}[\Pi(\boldsymbol{K}, \boldsymbol{\gamma})]$, $\frac{\partial \Lambda(\boldsymbol{K})}{\partial K_l} \equiv \Lambda_{K_l}$, and $\frac{\partial^2 \Lambda(\boldsymbol{K})}{\partial K_i \partial K_l} \equiv \Lambda_{K_l K_i}$, $l, i = 1, \cdots, n, f$. Then, the following results hold:*

(i) $\frac{\partial \lambda^l}{\partial K_i} = \Lambda_{K_l K_i}$, $l, i = 1, \cdots, n,\ f$;

(ii) $\Lambda_{K_f K_f} \leq 0$, $\Lambda_{K_l K_l} \leq 0$, $l = 1, \cdots, n$;

(iii) $\Lambda_{K_f K_l} = \Lambda_{K_l K_f} \geq 0$, $\Lambda_{K_l K_i} \leq 0$, $l, i = 1, \cdots, n$, *and* $l \neq i$;

(iv) $\Lambda_{K_f K_l} + \Lambda_{K_l K_l} \leq 0$, $\Lambda_{K_f K_f} + \Lambda_{K_l K_f} \leq 0$, *and* $\Lambda_{K_f K_i} + \Lambda_{K_l K_i} \geq 0$, $l, i = 1, \cdots, n$, *and* $l \neq i$.

Corollary 1. *If $\alpha = n$, then $\sum_{l \in N} \lambda^l + |N|\, \lambda^f$ is decreasing in K_l and K_f, but increasing in $K_{l'}$, where $N \subset \{1, \cdots, n\}$, $l \in N$ and $l' \in \overline{N}$.*

Lemma 17 and Corollary 1 establish some general rules on how $\boldsymbol{K}^*$ changes when $\mathbf{\Gamma}$ increases in convex order. In the following, we provide some examples of how to utilize Lemma 17 and Corollary 1 to study the behavior of $\boldsymbol{K}^*$ when $\mathbf{\Gamma}$ increases in convex order for a special case where $\alpha = n$. When $\alpha = n$, we know that if $\mathbf{\Gamma}$ increases in convex order, then we have that $\sum_{l\in\{1,\cdots,n\}} \lambda^l + n\lambda^f \geq \sum_{l\in\{1,\cdots,n\}} c_l + nc_f$. Therefore there exists at least one l, $l = 1, \cdots, n$, such that $\lambda^l + \lambda^f \geq c_l + c_f$ if $\mathbf{\Gamma}$ increases in convex order. In the following we consider some examples where $\lambda^l + \lambda^f > c_l + c_f$ for all $l = 1, \cdots, n$, and some examples where there exists exactly one l, $l = 1, \cdots, n$, such that $\lambda^l + \lambda^f > c_l + c_f$.

Example 4. *Consider an n-product* **SN** *with $\alpha = n$, and suppose that if $\mathbf{\Gamma}$ increases in convex order, $\lambda^f > c_f$, and $\lambda^l + \lambda^f > c_l + c_f$, $l = 1, \cdots, n$. Then, K_f^* must increase.*

Example 5. *Consider an n-product* **SN** *with $\alpha = n$, and suppose that if $\mathbf{\Gamma}$ increases in convex order, $\lambda^f < c_f$, and $\lambda^l + \lambda^f > c_l + c_f$, $l = 1, \cdots, n$. Then, all K_l^*, where $l = 1, \cdots, n$, must increase if K_f^* decreases or stays the same.*

Example 6. *Consider an n-product* **SN** *with $\alpha = n$, and suppose that if $\mathbf{\Gamma}$ increases in convex order, $\lambda^f < c_f$, $\lambda^i + \lambda^f > c_i + c_f$, $i \in \{1, \cdots, n\}$, and $\lambda^l + \lambda^f < c_l + c_f$, $l = 1, \cdots, n$, $l \neq i$. Then, K_i^* must increase.*

Example 7. *Consider an n-product* **SN** *with $\alpha = n$, and suppose that if $\mathbf{\Gamma}$ increases in convex order, $\lambda^f > c_f$, $\lambda^i + \lambda^f > c_i + c_f$, $i \in \{1, \cdots, n\}$, and $\lambda^l + \lambda^f < c_l + c_f$, $l = 1, \cdots, n$, $l \neq i$. Then, (i) at least one of K_i^* and K_f^* must increase, and (ii) all K_l^*, where $l = 1, \cdots, n$, $l \neq i$, must decrease or stay the same if K_i^* decreases or stays the same.*

3.3 The Optimal Expected Profit and Capacity versus Demand Dependence

The next question is whether there is a similar scalable approach that allows for the comparative statics analysis of demand dependence in multi-product multi-resource newsvendor networks, similar to what we have done above for the demand risk. For demand risk, this was possible by studying properties of the optimal revenue function, $\Pi^*(\boldsymbol{K}, \boldsymbol{\gamma})$, and of the capacity dual variable vector, $\boldsymbol{\lambda}$, and linking them to the concept of convex orders. In a similar way, we start the study of demand dependence through the concept of supermodular (submodular) orders, because an increase in $\mathbf{\Gamma}$ in supermodular order is associated with a higher demand dependence, as we elaborate below.

Definition 4. *(Müller and Stoyan (2002) p.113) Let $\boldsymbol{X} = (X_1, \cdots, X_n)$ and $\boldsymbol{Y} = (Y_1, \cdots, Y_n)$ be $n-$dimensional random vectors with finite expectations. Then $\boldsymbol{X}$ is said to be smaller than $\boldsymbol{Y}$ in the supermodular order, written $\boldsymbol{X} \leq_{sm} \boldsymbol{Y}$, if $E[f(\boldsymbol{X})] \leq E[f(\boldsymbol{Y})]$ for all supermodular functions $f : \Re^n \to \Re$ such that the expectations exits.*

Studying the supermodularity (or submodularity) of the optimal revenue function, $\Pi^*(\boldsymbol{K}, \boldsymbol{\gamma})$, in $\boldsymbol{\gamma}$ requires determining whether or not $\partial\Pi^*(\boldsymbol{K}, \boldsymbol{\gamma})/\partial\gamma_i$ is increasing or decreasing in γ_j, $i, j = 1, \cdots, n, j \neq i$, which, in turn, requires deriving an expression for $\partial\Pi^*(\boldsymbol{K}, \boldsymbol{\gamma})/\partial\gamma_i$ in each and every feasible domain, and as such, this does not yield a scalable approach for the study of demand dependence. Similarly, for $\boldsymbol{\lambda}(\boldsymbol{\gamma})$, we have established that $\boldsymbol{\lambda}(\boldsymbol{\gamma}) = \max\left\{\lambda_1(\boldsymbol{\gamma}), \lambda_2(\boldsymbol{\gamma}), \cdots, \lambda_{|\Psi|}(\boldsymbol{\gamma})\right\}$ in the **FN** and **DN** (Lemma 7), and $\sum_{l=1}^{n} \lambda^l(\boldsymbol{\gamma}) + \lambda^f(\boldsymbol{\gamma}) = \max\left\{\sum_{l=1}^{n} \lambda_1^l(\boldsymbol{\gamma}) + \lambda_1^f(\boldsymbol{\gamma}), \sum_{l=1}^{n} \lambda_2^l(\boldsymbol{\gamma}) + \lambda_2^f(\boldsymbol{\gamma}), \cdots, \sum_{l=1}^{n} \lambda_{|\Psi|}^l(\boldsymbol{\gamma}) + \lambda_{|\Psi|}^f(\boldsymbol{\gamma})\right\}$ in the **SN** (Lemma 15). Since each $\lambda_i(\boldsymbol{\gamma})$ or $\sum_{l=1}^{n} \lambda_i^l(\boldsymbol{\gamma}) + \lambda_i^f(\boldsymbol{\gamma})$, $i \in \Psi$, is a linear function of $\boldsymbol{\gamma}$ (Lemmas 4, 11, and 13), it is both supermodular and submodular in $\boldsymbol{\gamma}$, satisfying condition **(C1)** of Lemma A2. Therefore, it remains to show that conditions **(C2)** and **(C3)** of Lemma A2 are satisfied, which is equivalent to showing that, $\forall i = 1, \cdots, n$, there exists a unique sequence of all the feasible domains, $\Omega_{(1)}, \Omega_{(2)}, ..., \Omega_{(|\Psi|)}$, such that

$$\frac{\partial\lambda_{(1)}(\boldsymbol{\gamma})}{\partial\gamma_i} \geq \frac{\partial\lambda_{(2)}(\boldsymbol{\gamma})}{\partial\gamma_i} \geq \cdots \geq \frac{\partial\lambda_{(|\Psi|)}(\boldsymbol{\gamma})}{\partial\gamma_i}, \text{ or} \tag{3.18}$$

$$\frac{\partial\sum_{l=1}^{n} \lambda_{(1)}^l(\boldsymbol{\gamma}) + \lambda_{(1)}^f(\boldsymbol{\gamma})}{\partial\gamma_i} \geq \frac{\partial\sum_{l=1}^{n} \lambda_{(2)}^l(\boldsymbol{\gamma}) + \lambda_{(2)}^f(\boldsymbol{\gamma})}{\partial\gamma_i} \geq \cdots \geq \frac{\partial\sum_{l=1}^{n} \lambda_{(|\Psi|)}^l(\boldsymbol{\gamma}) + \lambda_{(|\Psi|)}^f(\boldsymbol{\gamma})}{\partial\gamma_i}. \tag{3.19}$$

This, again, requires each feasible domain to be identified and the expression of $\lambda_j(\boldsymbol{\gamma})$ to be derived for each feasible domain $\Omega_j, j \in \Psi$, so that a sequence of all feasible domains (if one exists) that satisfies (3.18) can be identified. As such, this direction does not provide us with a scalable approach for the study of demand dependence either. Therefore, the question, of whether such a scalable approach for the comparative statics analysis of demand dependence exists, remains an important research direction. Consequently, we restrict our study of demand dependence to two-product networks, because, to our knowledge, the following results have not been established in the extant literature.

The following well-known result relates an increase in supermodular order to demand dependence. For any two random vectors $\boldsymbol{X} = (X_1, X_2)$ and $\boldsymbol{Y} = (Y_1, Y_2)$ with the same marginal distributions, if $\boldsymbol{X}$ is smaller than $\boldsymbol{Y}$ in the supermodular order, then $\rho_{X_1,X_2} \leq \rho_{Y_1,Y_2}$, where ρ_{X_1,X_2} and ρ_{Y_1,Y_2} respectively denote the correlation coefficient associated with $\boldsymbol{X}$ and $\boldsymbol{Y}$ (Shaked and Shanthikumar (2007), Chapter 9.A).

Theorem 5. *For two-product networks, we have the following results:*

1. *Consider the optimal expected profit, $E_{\boldsymbol{\Gamma}}[V(\boldsymbol{K}^*)]$. As $\boldsymbol{\Gamma}$ increases in supermodular order:*

1. *In the* **FN** *with no cross-price effects ($v = 0$), $E_{\boldsymbol{\Gamma}}[V(\boldsymbol{K}^*)]$ decreases.*
2. *In the* **DN***, $E_{\boldsymbol{\Gamma}}[V(\boldsymbol{K}^*)]$ increases.*
3. *In the* **SN** *with no cross-price effects ($v = 0$), $E_{\boldsymbol{\Gamma}}[V(\boldsymbol{K}^*)]$ decreases.*

2. *Consider the optimal capacity vector, $\boldsymbol{K}^*$, in the* **DN***. As $\boldsymbol{\Gamma}$ increases in supermodular order, at least one of the optimal capacities, K_1^* or K_2^*, increases.*

For all other two-product settings not included in the theorem, our analysis is inconclusive. Parts 1 and 3 of the first result, on the optimal expected profit, hold for the setting where the products are not substitutable ($v = 0$), and share a common flexible resource (the **FN** and **SN**). In this setting, it is well-known that as the demand correlation increases, the firm benefits less from risk pooling, and its optimal expected profit decreases. The result in part 2, for the **DN** with substitutable products, is more interesting, and follows due to the existence of product substitutability. For any price vector, a higher correlation leads to a higher variability of the overall demand, $D_1 + D_2$ (given by $\Gamma_1 + \Gamma_2 - (1 - \nu)(p_1 + p_2)$), and the optimal expected profit increases under responsive pricing, similar to the well-known result for the single-product newsvendor under responsive pricing.

Finally, for the case of the BVN, noting Theorem 4.1 in Bäuerle (1997) or Theorem 3.13.5 in Müller and Stoyan (2002), Theorem 5 confirms the finding observed in Example 3: As ρ increases, the optimal capacity of at least one resource in the two-product **DN** increases.

3.4 Conclusion

Table 3.3: Summary of the main results in Chapters 2 and 3

	Increase in demand risk				**Increase in demand dependence**			
	Exogenous pricing	Responsive pricing			Exogenous pricing	Responsive pricing		
	FN	**General multi-product networks**			**FN**	**Two-product FN**	**Two-product DN**	**Two-product SN**
Optimal expected profit	$\downarrow$ (Eppen (1979))	General multi-product networks with Assumption 1: $\uparrow$ (Thm 1)			$\downarrow$ (Eppen (1979))	$\downarrow$ (Thm 5)	$\uparrow$ (Thm 5)	$\downarrow$ (Thm 5)
Optimal capacity	Not monotone (Eppen (1979); Van Mieghem (2007))	**FN**: $\uparrow$ (Thm 2)	**DN**: At least one capacity $\uparrow$ (Thm 2)	**SN**: At least one capacity $\uparrow$ (Thm 3)	Not monotone (Eppen (1979); Chod and Rudi (2005))	Inconclusive	At least one capacity $\uparrow$ (Thm 5)	Inconclusive

In this chapter, we show that in the **SN** the convexity of the capacity dual variables disappears due to the dependencies among the resources. Nevertheless, we are able to establish a similar capacity versus demand risk result in the two-product **SN** and n-product **SN**, which relies on the result that a linear combination of the capacity dual variables in this network satisfies the convexity result. This suggests that our comparative statics approach might be somewhat robust to deviations in our key result, Lemma 5. Extending this result to the "parallel network," among other networks, is worthy of future research efforts. In addition, while we are able to extend our comparative statics analysis to demand dependence (through the use of supermodular orders), this approach, in its current form, is not scalable, and is performed only for two-product networks. It is an important research direction to explore the demand dependence study further. Table 3.3 provides a summary of the results of our comparative statics analysis of multi-product newsvendor networks under responsive pricing in Chapters 2 and 3.

In summary, we use our proposed approach to study the impact of demand uncertainty and dependence on the optimal capacity and the resulting expected profit in **FN**, **DN**, and **SN** with an arbitrary number of products and resources. Our proposed approach provides a rigorous framework, which can be applied to other operations management problems that are not amenable to traditional parametric analysis. This is a promising area for future research.

Chapter 4

The Influenza Vaccine Composition Problem

4.1 Introduction and Motivation

In this chapter, we study the influenza vaccine composition decision, which has received recent interest among Operations Researchers, motivated in part by the recent vaccine shortages (see, for example, Chick et al. (2008); Cho (2010); Kornish and Keeney (2008); Özaltın et al. (2011); Wu et al. (2005)). Currently, the seasonal influenza vaccine is trivalent, that is, it contains three strains, one from each of the main influenza families: the Influenza A (H3N2) subtype, the Influenza A (H1N1) subtype, and the Influenza B type. The vaccine is reformulated every year, as new strains emerge and the prevalence of known strains changes. For example, from 1980 to 2010, 16 changes were made for the Influenza A (H3N2) strain, 13 for the Influenza A(H1N1) strain, and 15 for the Influenza B strain (Özaltın et al. (2011)). Consequently, the influenza vaccine is perishable, and the unused quantities of the vaccine are discarded at the end of the flu season. Vaccination not only provides protection against the strains it contains, but also against antigenically similar strains, albeit less effectively ("cross-effective immunity"), CDC (2010). Because many strains from the same family can circulate in a given year, cross-effective immunities should be considered in the vaccine composition decision.

The vaccine manufacturing process, which involves growing the virus in chicken eggs, and harvesting, inactivating, and purifying the vaccine (e.g., Gerdil (2003), Mamani (2008)), takes considerable time. Further, because of the uncertain growth characteristics of the viral strains, there is inherent uncertainty regarding the vaccine yield given the number of eggs used (e.g., Chick et al. (2008), Cho (2010), Gerdil (2003), Özaltın et al. (2011)). As a result, the vaccine composition decision needs to be made prior to the start of the influenza season, under uncertainty on *both* the future strain prevalence and the yield of the production process. Per current practice, the vaccine is trivalent, comprised of one strain from each family. Due to the high yield uncertainty, however, this practice limits the amount of vaccine output to the amount of strain with the least output, resulting in waste of the other non-limiting strains in the composition. At the same time, it is not uncommon to have years in which the demand for the influenza vaccine cannot be fully satisfied due to the limited amount of vaccine available (FDA (2005)).

In the United States, the Vaccines and Related Biological Products Advisory Committee (hereafter, "Committee") is responsible for making a recommendation to the Food and Drug Administration (FDA) on the influenza vaccine composition, usually between February and March for the coming influenza season; see Cho (2010) and Özaltın et al. (2011) for detailed descriptions of this process. Once the strains are selected by the Committee, vaccine manufacturers make their own production quantity decisions so as to maximize their expected profits. First, the individual monovalent strains are manufactured from chicken eggs, and this process typically continues until July. Then, from July to August, the monovalent strains are blended, and the trivalent vaccine, comprised of each of the three strains selected by the Committee, is prepared, followed by the packaging, labeling, and delivery of the vaccine from August to September, in time for the new influenza season.

Given this time-line and the different players involved, important research questions arise:

1. **Is selecting three strains optimal?** Is the current practice of selecting three strains, one from each family, the one that is best for the society? Are there societal bene-

fits of selecting more/less strains from the different families, taking into account the prevalence and production yield estimates of the candidate strains?

2. **Should all vaccines be required to have all selected strains?** Is the current practice of requiring each unit of vaccine to contain all selected strains reasonable, given the high yield uncertainty? For instance, in addition to producing the trivalent vaccine, are there societal benefits of providing the public, if needed, with an inferior, bivalent (two-strain) vaccine produced from the left-over strains that would otherwise have been discarded?

3. **How would the vaccine strain selection and manufacturing decisions change if left-over strains were utilized?** Would the strain composition change if the Committee considered, during the vaccine formulation phase, the possible utilization of the left-over two-strain vaccine? What are the benefits of this novel approach to the society?

To our knowledge, this Chapter is the first to analyze the vaccine composition decision by explicitly considering the different decision-makers involved, and under a novel approach, of the left-over strain usage possibility. We complete our analysis by a comparative statics study that characterizes the impact of yield uncertainty on the societal vaccination benefit, the manufacturer's profit, and the vaccine output.

Research into the above questions is important, as it can start discussion on a novel approach, of utilizing the left-over, otherwise to be discarded, strains by combining them into two-strain vaccines in this perishable-strain setting. A better vaccine composition and a higher utilization of the available resources (i.e., strains produced) may have huge potential benefits to the society, as vaccination is considered to be the most effective way to prevent influenza, a highly contagious and acute respiratory viral disease that can cause mild to severe illness, and even death in some cases (AAP (2012)). Indeed, our numerical study shows that optimizing the strain selection decision considering the secondary bivalent vaccine option can significantly increase the societal vaccination benefit, incurred from the number of influenza cases prevented, by an average of 10.85% for years with high demand (i.e., high attack rate of the virus) under the current practice of formulating the vaccine with one strain from each family. In addition, we show that the current practice of selecting one strain from each family need not be optimal. The optimal vaccine composition is complex, and depends on both the prevalence and yield estimates of all candidate strains. Our numerical study suggests that not limiting the vaccine composition to the current practice may result in significant additional improvement in the societal vaccination benefit (by an average of 33.81%). Further, we also show that considering the secondary vaccine option in the vaccine formulation phase can change the vaccine composition, incurring yet further benefits. From this perspective, our results provide key public policy insight on this important healthcare problem.

The remainder of this Chapter is organized as follows. In Section 4.2, we provide a brief review of the related literature. In Section 4.3, we introduce our models for the "no left-over" setting, where the secondary, left-over vaccine is not used, and for the left-over setting with the secondary vaccine option. Then in Section 4.4, we derive structural properties of the optimal solution and perform a comparative statics analysis. In Section 4.5, we discuss our extensive numerical study, based on realistic data, to answer our research questions on how the optimal vaccine composition generated by our model differs from the current practice, on the benefits of allowing the production of a secondary two-strain vaccine from the left-over strains, and on how the optimal vaccine composition would change if the secondary vaccine option were considered in the vaccine formulation phase. Our results allow us to derive important insights and principles for public policy decision-making. Finally, we conclude, in Section 4.6, with a summary of our main results and suggestions for future research. To improve the presentation, some proofs are relegated to the Appendix.

4.2 Related Literature

This review is not meant to be exhaustive, but rather indicative of the various areas related to this Chapter. This work draws upon and contributes to the following streams of literature. First, we contribute to research that studies the influenza vaccine composition decision and the influenza vaccine manufacturer's production planning problem, where the latter is broadly related to production planning under stochastically proportional yield (see, for example, Yano and Lee (1995)). Our model is also very broadly related to research on horizontally differentiated products, although, due to the characteristics of the influenza vaccine setting, consumer choice does not play a role in the selection of the vaccine (i.e., the primary, trivalent vaccine versus the secondary, bivalent vaccine). All consumers are provided with the more effective, trivalent vaccine; and the bivalent vaccine is used only if the trivalent vaccine turns out to be insufficient to satisfy all demand.

The composition of the influenza vaccine and the timing of this decision have been studied considering the immunity protection offered by the vaccine (i.e., cross-effectiveness) (Cho (2010); Kornish and Keeney (2008); Özaltın et al. (2011)). In particular, Wu et al. (2005) formulates individualized vaccine compositions, considering each individual's vaccination history. Cho (2010) considers the selection of a *single strain* within a commit-or-defer model where, at each stage, the Committee either selects the current vaccine strain, or selects the most prevalent new strain, or defers the selection to the subsequent period. Özaltın et al. (2011) is a notable exception that studies the vaccine composition decision under a *multi-strain* model, but not within a game setting and not considering the secondary vaccine possibility, as we do here. Özaltın et al. (2011) proposes a multi-stage stochastic mixed-integer programming problem that integrates the vaccine composition and timing decisions so as to maximize the societal benefit, based on prevented flu cases. The Committee can select strains in each period. However the earlier a strain is selected, the higher its production quantity will be, which is modeled as a random variable. In the final period (the flu season), the vaccine output is limited by the least amount of strain produced. Our model builds upon Özaltın et al. (2011) by considering the following important characteristics. First, we consider a game setting in which the manufacturer and the Committee are independent decision-makers, acting on their self-interest. As a result, in our model the strain production quantity, hence the vaccine output, becomes an outcome of the manufacturer's optimal action, which is based on the Committee's strain selection. More importantly, we propose a new approach, of utilizing the left-over strains to produce a secondary vaccine. Explicitly modeling the interplay between the Committee and the manufacturer under the secondary vaccine option offers important insight. To focus our attention on the benefits of the secondary vaccine strategy under optimal vaccine composition, we do not model the timing aspect of the vaccine composition decision modeled in Özaltın et al. (2011), and collapse the multi-stage decision-making process of the Committee into a single-stage process. Our model provides important insight on the benefits of utilizing the left-over strains, and on the benefits of an integrated approach that considers the secondary vaccine option in the vaccine composition decision. In our numerical studies, we utilize the data used in Özaltın et al. (2011).

The influenza vaccine manufacturer's problem and its variants are well-studied in the literature (e.g., Chick et al. (2008)), albeit mostly in single-strain settings. Hence, the existence of a limiting strain and the waste of the other strains are not modeled in these former papers. The main focus of these papers is not on vaccine composition, but rather on determining coordinating contracts between the manufacturer and a centralized authority, which is outside the scope of this Chapter. Finally, Deo and Corbett (2009) utilizes a competitive entry model to study the number of vaccine manufacturers entering the market under yield uncertainty. Their analysis shows how yield uncertainty affects the number of firms in the market. We do not model macro-level decisions, and to simplify our model, we consider a single manufacturer.

4.3 The Models

We first provide our model for the "no left-over" setting in which the secondary vaccine is not used. Then, we extend our model to the "left-over" setting, which considers the option of a secondary vaccine, composed of the strains having quantities left over after the production of the primary vaccine.

We consider the influenza vaccine supply chain, comprised of a single vaccine manufacturer and the Committee. At the beginning of the vaccine production cycle, the Committee determines the vaccine composition (i.e., the strains to be included in the vaccine), I, from a set of candidate strains, Θ, given the demand for the vaccine, d, so as to maximize the expected societal vaccination benefit (i.e., the total cost of prevented flu cases) under uncertainty on the prevalence vector of the virus strains, $\boldsymbol{E} = (E_k)_{k\in\Theta}$, and for a given menu of manufacturer's unit sales prices, $w(|I|)$, $I \in \Theta$. We model $w(\cdot)$ as an increasing function of the number of strains to be included in the vaccine, that is, the greater the number of strains the Committee requires to be included in the vaccine, the higher sales price per vaccine the manufacturer charges. We take the $w(.)$ function as given and do not study contracting issues between the Committee (or another central authority) and the manufacturer, as this is not in the scope of this Chapter. We leave this as a future research direction.

Given the demand, d, and the Committee's vaccine composition, I, the manufacturer determines the input quantity vector, $\boldsymbol{Q} = (Q_k)_{k\in I}$, so as to maximize her expected profit under uncertainty on the yield. The manufacturer only produces the vaccine that contains the strains selected by the Committee. The total number of eggs used in production cannot exceed the number of available eggs, N, which is determined prior to production. The yield of the production process is random, with Q_kY_k, $k \in I$, denoting the amount of strain k produced when Q_k eggs are input, where Y_k follows a continuous distribution with probability density function $f_k(\cdot)$, cumulative density function $F_k(\cdot)$, mean μ_k, and variance σ_k^2. In the no left-over model, the Committee requires *all* selected strains ($k \in I$) to be included in the vaccine. Consequently, the number of vaccines produced by the manufacturer will be limited by the least amount of strain produced, $\min_{k\in I}\{Y_kQ_k\}$, and the demand satisfied will be given by $DS \equiv \min\{d, \min_{k\in I}\{Y_kQ_k\}\}$, at unit price $w(|I|)$.

Vaccination with strain $i \in \Theta$ may also provide some immunization against certain other strains $k \in \Theta$, $k \neq i$. Following the cross-effectiveness modeling in Özaltın et al. (2011), we let b_{ik} denote the cross-effectiveness between strains i and k, which is the the ratio of the immunity protection against strain k of a person who is vaccinated with strain i, to that she would have gained if she were vaccinated with strain k. Note that b_{ik} is not necessarily symmetric. Thus, for a person receiving a vaccine composed of the strains in I, the effective immunity protection against strain $k \in \Theta$ is given by $\max_{i \in I}\{\, b_{ik}\}$. As in Özaltın et al. (2011), we model the societal vaccination benefit as a linear function of the number of cases prevented, where t denotes the average cost of an influenza case. We let c_k denote the processing cost per egg inputted for strain k, $k \in \Theta$. Table 4.1 summarizes the notation used.

Table 4.1: Summary of the notation

Parameter	Definition
Θ	Set of candidate strains
$m = \lvert\Theta\rvert$	Size (cardinality) of set Θ
d	Seasonal demand for the influenza vaccine
b_{ik}	Cross-effectiveness between strains i and k, $i, k \in \Theta$ (for $i = k$, b_{ii} represents own-effectiveness, which is assumed to equal 1)
E_k	Random prevalence of virus strain k, $k \in \Theta$
$w(\lvert I\rvert)$	Unit sales price of vaccine comprised of strains in set I
t	Average cost of a flu case
c_k	Unit cost per egg processed for strain k, $k \in \Theta$
N	Number of available eggs
Y_k	Random yield of strain k, $k \in \Theta$
Decision variable	**Definition**
$I \subseteq \Theta$	The set of strains selected by the Committee
$\boldsymbol{Q} = (Q_k)_{k \in I}$	Input quantity vector of strains in set I determined by the manufacturer

The Game in the No Left-over Setting

The timing of the events leads to a sequential Stackelberg game, with the Committee as the leader. We can formulate the game in the no left-over setting, where the secondary vaccine is not an option, as follows:

The Committee's Problem: Problem Strain Composition (**SC**):

$$\text{maximize}_{I\subseteq\Theta}\Pi_g(I) \equiv t\,\underbrace{\mathbb{E}_{\boldsymbol{Y}}[\min\{d,\min_{k\in I}\{Y_kQ_k\}\}]}_{\mathbb{E}_{\boldsymbol{Y}}[(DS(I))]}\,\mathbb{E}_{\boldsymbol{E}}[\sum_{k\in\Theta} E_k \underbrace{\max_{i\in I}\{b_{ik}\}}_{\substack{\text{effective immunity}\\ \text{protection against}\\ \text{strain } k}}] - w(|I|)\,\underbrace{\mathbb{E}_{\boldsymbol{Y}}[\min\{d,\min_{k\in I}\{Y_kQ_k\}\}]}_{\mathbb{E}_{\boldsymbol{Y}}[(DS(I))]}$$

$$= \mathbb{E}_{\boldsymbol{Y}}[(DS(I))]\left(t\sum_{k\in\Theta}\{\mathbb{E}[E_k]\max_{i\in I}\{b_{ik}\}\} - w(|I|)\right), \tag{4.1}$$

where the objective is to select strain set I so as to maximize the expected societal vaccination benefit, which corresponds to the expected cost benefit incurred from prevented flu cases less the total cost of purchasing the vaccine. Let I^* denote the optimal strain composition selected by the Committee.

The Manufacturer's Problem: Problem Quantity Selection (**QS**):

$$\text{maximize}_{Q_k,k\in I}\Pi_m(\boldsymbol{Q}) \equiv -\sum_{k\in I} c_kQ_k + w(|I|)\,\mathbb{E}_{\boldsymbol{Y}}[\min\{d,\min_{k\in I}\{Y_kQ_k\}] \tag{4.2}$$

$$\text{subject to} \sum_{k\in I} Q_k \le N \quad \longleftarrow \lambda \tag{4.3}$$

$$Q_k \ge 0, k\in I, \tag{4.4}$$

where the objective is to select the input quantity vector $\boldsymbol{Q}$ so as to maximize the manufacturer's expected profit. Let λ denote the KKT multiplier corresponding to Constraint (4.3). We let $\boldsymbol{Q}^* = (Q_k^*)_{k\in I}$ denote the optimal input vector determined by the manufacturer.

The Game under a Secondary Vaccine Option (Left-over Setting)

In the no left-over setting, each unit of the vaccine is required to contain *all* the strains selected by the Committee. Consequently, the vaccine production is limited by the least amount of strain output, with the left-over strains discarded. This represents the current practice. However, the left-over quantities of the strains can still be utilized by the manufacturer to combine into a secondary inferior vaccine, containing fewer number of strains than the primary vaccine recommended by the Committee. While the secondary vaccine will have less effectiveness (in terms of immunity benefit) than the primary vaccine, it may still benefit the society if the primary vaccine output turns out to be insufficient to satisfy all demand. In this case, the manufacturer's increased revenue from the secondary vaccine may also provide an incentive to the manufacturer to input more strains for production. This may further increase the societal vaccination benefit. Therefore, in this section we expand our model to consider the secondary vaccine option.

In the left-over setting, the manufacturer produces not only the primary vaccine, but also a secondary vaccine using the left-over strains, but sells the secondary vaccine at a lower unit price than the primary vaccine. During the influenza season, demand for the vaccine, d, will be met first by the primary vaccine to the extent possible, and then by the secondary vaccine if necessary. To make the analysis tractable, we consider only one level of left-over production, that is, only one type of secondary vaccine is produced, and it is comprised of one less strain than the primary vaccine.

We use the superscript LO to denote the left-over setting. Define the random variable L as the strain that limits the primary vaccine output quantity, that is, $L \equiv \arg\min_{k\in I}\{Y_k Q_k\}$. Then, the output quantities for the primary and secondary vaccine are respectively given by $Z_P \equiv \min_{k\in I}\{Y_k Q_k\}$ and $Z_S \equiv \min_{k\in I\setminus\{L\}}\{Y_k Q_k - Z_P\}$; and the demand satisfied by the primary and secondary vaccine are respectively given by $DS_P^{LO} \equiv \min\{d, Z_P\}$ (at unit price $w(|I|)$) and $DS_S^{LO} \equiv \min\{(d-Z_P)^+, Z_S\}$ (at unit price $w(|I|-1)$).

The Committee's problem and the manufacturer's problem in the setting with one-level left-over usage can be formulated as follows:

<u>The Committee's Problem: Problem Strain Composition with Left-overs (**SC-LO**):</u>

$$\begin{aligned} &\text{maximize}_{I\subseteq\Theta}\Pi_g^{LO}(I) \\ &\equiv \underbrace{\mathbb{E}_{\boldsymbol{Y}}\left[\min\{d, Z_P\}\right]\left(t\sum_{k\in\Theta}\mathbb{E}[E_k]\max_{i\in I}\{b_{ik}\} - w(|I|)\right)}_{\Pi_g} \qquad (4.5) \\ &+ \mathbb{E}_{\boldsymbol{Y}}\left[\sum_{k'\in I}\left(\min\{(d-Z_P)^+, Z_S\}|L=k'\right)\Pr(L=k')\left(t\sum_{k\in\Theta}\mathbb{E}[E_k]\max_{i\in I\setminus\{k'\}}\{b_{ik}\} - w(|I|-1)\right)\right]. \end{aligned}$$

The Manufacturer's Problem: Problem Quantity Selection with Left-overs (**QS-LO**):

$$\begin{aligned}
&\text{maximize}_{Q_k, k\in I}\Pi_m^{LO}(\boldsymbol{Q}) \\
&\equiv \underbrace{-\sum_{k\in I} c_k Q_k + w(|I|)\,\mathbb{E}_{\boldsymbol{Y}}[\min\{d, Z_P\}]}_{\Pi_m} + w(|I|-1)\,\mathbb{E}_{\boldsymbol{Y}}[\min\{(d-Z_P)^+, Z_S\}] \qquad (4.6)\\
&= -\sum_{k\in I} c_k Q_k + \mathbb{E}_{\boldsymbol{Y}}\left[\sum_{k'\in I}\left(w(|I|)\min\{d, Z_P\} + w(|I|-1)\min\{(d-Z_P)^+, Z_S\}|L=k'\right)\Pr\left(L=k'\right)\right]
\end{aligned}$$

$$\text{subject to } \sum_{k\in I} Q_k \le N \longleftarrow \lambda^{LO} \qquad (4.7)$$

$$Q_k \ge 0, k\in I, \qquad (4.8)$$

where λ^{LO} denotes the KKT multiplier corresponding to Constraint (4.7). We let I^{LO*} and $\boldsymbol{Q}^{LO*}$ respectively denote the Committee's optimal strain selection set and the manufacturer's optimal strain input vector when the secondary vaccine possibility is considered in the game.

4.4 Structural Properties and Comparative Statics Analysis

We start by analyzing the manufacturer's problem.

Lemma 18. *For a given strain set I, the manufacturer's profit functions in both the no left-over and left-over settings, $\Pi_m(\boldsymbol{Q})$ and $\Pi_m^{LO}(\boldsymbol{Q})$, are jointly concave in $\boldsymbol{Q}$. Then, a feasible $\boldsymbol{Q}^*$ ($\boldsymbol{Q}^{LO*}$) vector is optimal for Problem* **QS** *(***QS-LO***) if and only if there exists scalar $\lambda \ge 0$ ($\lambda^{LO} \ge 0$) such that the KKT first-order conditions in (B.28)–(B.29) ((B.30)–(B.31)) are satisfied (see the Appendix for the KKT first-order conditions).*

The concavity result in Lemma 18 greatly facilitates determining the solution to the manufacturer's problem in both the no left-over and left-over settings. We next reformulate the Committee's problem in both settings as integer programming problems, which we can solve efficiently, especially in realistic settings described in Lemmas 19 and 20 below.

The No Left-over Setting: The Committee's Problem: Problem **SC**:

$$\text{maximize}_{\boldsymbol{x},\boldsymbol{z}}\Pi_g(\boldsymbol{x},\boldsymbol{z}) = \mathbb{E}_{\boldsymbol{Y}}\left[DS(\boldsymbol{x})\right]\left(t\sum_{k\in\Theta}\mathbb{E}[E_k](\sum_{i\in\Theta}b_{ik}z_{ik}) - w\left(\sum_{k\in\Theta}x_k\right)\right) \tag{4.9}$$

$$\text{subject to } z_{ik} \le x_i,\ \forall i,k\in\Theta \tag{4.10}$$

$$\sum_{i\in\Theta} z_{ik} = 1,\ \forall k\in\Theta \tag{4.11}$$

$$x_i = 0 \text{ or } 1,\ \forall i\in\Theta \tag{4.12}$$

$$z_{ik} = 0 \text{ or } 1,\ \forall i,k\in\Theta. \tag{4.13}$$

The Left-over Setting: The Committee's Problem: Problem **SC-LO**:

$$\begin{aligned}
&\text{maximize}_{\boldsymbol{x},\boldsymbol{z},\boldsymbol{v}}\Pi_g(\boldsymbol{x},\boldsymbol{z},\boldsymbol{v}) \\
&= \mathbb{E}_{\boldsymbol{Y}}\left[DS_P^{LO}(\boldsymbol{x})\right]\left(t\sum_{k\in\Theta}\mathbb{E}[E_j]\sum_{i\in\Theta}b_{ik}z_{ik} - w\left(\sum_{k\in\Theta}x_k\right)\right) \\
&+ \sum_{k'\in\Theta}\left(\mathbb{E}_{\boldsymbol{Y}}\left[DS_S^{LO}(\boldsymbol{x})|L(\boldsymbol{x})=k']Pr(L(\boldsymbol{x})=k')\right]\left(t\sum_{k\in\Theta}\mathbb{E}[E_k]\sum_{i\in\Theta\setminus\{k'\}}b_{ik}v_{ik'k} - w\left(\sum_{k\in\Theta}x_k - 1\right)\right)\right.
\end{aligned} \tag{4.14}$$

$$\text{subject to } z_{ik} \le x_i,\ \forall i,k\in\Theta \tag{4.15}$$

$$v_{ik'k} \le x_i,\ \forall i,k',k\in\Theta, k'\neq i \tag{4.16}$$

$$\sum_{i\in\Theta} z_{ik} = 1,\ \forall k\in\Theta \tag{4.17}$$

$$\sum_{i\in\Theta\setminus\{k'\}} v_{ik'k} = 1,\ \forall k,k'\in\Theta \tag{4.18}$$

$$x_i = 0 \text{ or } 1,\ \forall i\in\Theta \tag{4.19}$$

$$z_{ik} = 0 \text{ or } 1,\ \forall i,k\in\Theta \tag{4.20}$$

$$v_{ik'k} = 0 \text{ or } 1,\ \forall i,k',k\in\Theta, k'\neq i. \tag{4.21}$$

Observe that to solve these integer programming problems, one needs to calculate *a priori* the manufacturer's best response coefficients, which include, in the no left-over setting $\mathbb{E}_{\boldsymbol{Y}}[DS(\boldsymbol{x})]$ for each possible binary $\boldsymbol{x}$ vector; and in the left-over setting $\mathbb{E}_{\boldsymbol{Y}}\left[DS_P^{LO}(\boldsymbol{x})\right]$ for each possible binary $\boldsymbol{x}$ vector, and $\mathbb{E}_{\boldsymbol{Y}}\left[DS_S^{LO}(\boldsymbol{x})|L(\boldsymbol{x}) = k'\right]$ and $Pr(L(\boldsymbol{x}) = k')$ for each possible binary $\boldsymbol{x}$ vector and for each $k' : x_{k'} = 1$. With $|\Theta| = m$ candidate strains, the number of best response coefficients to be calculated is proportional to 2^m in the no left-over setting, and to $2^m + \sum_{n=1}^{m} 2n\binom{m}{n} = 2^m + m2^m = (m+1)2^m$ in the no left-over setting.

In practice, data on distributions of random yield for candidate strains are often not available (Özaltın et al. (2011)). Further, for new candidate strains that have not been manufactured in previous years, such data simply do not exist. Consequently, it is reasonable to categorize the candidate strains into "high," "moderate," or "low" yield groups such that all strains in each group follow an identical distribution. For this practical situation, the number of best response coefficients that need to be calculated *a priori* is bounded from above by polynomial functions of the number of candidate strains, m, in both left-over and no left-over settings.

Lemma 19. *Suppose that the set of candidate strains, Θ, can be decomposed into three mutually exclusive sets, of high yield, moderate yield, and low yield strains, respectively denoted by Θ_H, Θ_M, and Θ_L, where $\Theta_H \bigcup \Theta_M \bigcup \Theta_L = \Theta$. Assume that the random yield of each strain in set Θ_k, denoted by Y_k, follows an identical distribution, for $k \in \{H, M, L\}$. Suppose also that the correlation coefficients between any pair of candidate strains in Θ are equal. Then, the number of best response coefficients to be calculated a priori is bounded from above by*

1. $\sum_{n=1}^{|\Theta|} \frac{(n+2)(n+1)}{2} = \frac{m(m+1)(m+5)}{6} + m$ *for Problem* **SC***;*
2. $\sum_{n=1}^{|\Theta|} \frac{(n+2)(n+1)(2n+1)}{2} = \frac{m^4+6m^3+11m^2+6m}{4}$ *for Problem* **SC-LO***.*

Proof: In the no left-over setting, the number of best response coefficients, $\mathbb{E}_{\boldsymbol{Y}}[(DS(I)]$, $I \subseteq \Theta$, to be calculated *a priori* is proportional to the number of distinct combinations of the ordered triplet of nonnegative integers (i, j, k), where i, j, and k respectively refer to the number of strains selected from sets Θ_L, Θ_M, and Θ_H such that $i + j + k = n$, for all $n \in \mathbb{Z}^+, 0 \leq n \leq |\Theta| = m$. Let Γ_n denote the number of such distinct ordered triplets, (i, j, k). Consider first the case where $n < \min\{|\Theta_L|, |\Theta_M|, |\Theta_H|\}$. We derive a recursive relationship for Γ_n, $n = 1, \cdots, m$, by expressing Γ_{n+1} in terms of Γ_n. For a given n, for each ordered triplet (i, j, k) whose sum is n, we can obtain an ordered triplet $(i+1, j, k)$ with sum $n + 1$. All these ordered triplets are distinct, and have their first element non-zero. On the other hand, for the ordered triplets with sum $n + 1$, whose first element i is zero, j can be any number between 0 and $n+1$, in which case k has to equal $n+1-j$. Hence, the number of ordered triplets with sum $n+1$, whose first element i is 0, is $n+2$. Consequently, we have the recurrence relationship $\Gamma_{n+1} = \Gamma_n + (n+2)$, $n = 1, \cdots, m$, with $\Gamma_0 = 1$. The solution to the recurrence is $\Gamma_n = \binom{n+2}{2} = \frac{(n+2)(n+1)}{2}$, $n = 1, \cdots, m$. When $n \geq \min\{|\Theta_L|, |\Theta_M|, |\Theta_H|\}$, the number of distinct ordered triplets will be at most $\frac{(n+2)(n+1)}{2}$, since some ordered triples (i, j, k) will not be feasible. Thus, the number of best response coefficients that need to be calculated *a priori* is bounded from above by $\sum_{n=1}^{m} \frac{(n+2)(n+1)}{2} = \frac{m(m+1)(m+5)}{6} + m$. Similarly, in the left-over setting, for each distinct combination of the ordered triplet of nonnegative integers (i, j, k) with $i + j + k = n$, the number of best response coefficients $\mathbb{E}_{\boldsymbol{Y}}\left[DS_P^{LO}(I)\right]$, $\mathbb{E}_{\boldsymbol{Y}}\left[DS_S^{LO}(I)|L(I) = k'\right]$, and $Pr(L(I) = k')$, $k' \in I \subseteq \Theta$, to be calculated is given by $2n+1$, leading to $\sum_{n=1}^{m} \frac{(n+2)(n+1)(2n+1)}{2} = \frac{m^4+6m^3+11m^2+6m}{4}$. This completes the proof. $\square$

We next consider another important special case, with exactly one strain selected from each family, which corresponds to the current practice.

Lemma 20. *Consider the restriction where exactly one strain from each family needs to be included in the vaccine composition. Let n_i denote the number of distinct random yield distributions for candidate strains in family $i, i = 1, 2, 3$, with $n_1 + n_2 + n_3 \leq m$. Then, the number of best response coefficients to be calculated a priori is bounded from above by $n_1 \times n_2 \times n_3$ for Problem* **SC**; *and $7 \times n_1 \times n_2 \times n_3$ for Problem* **SC-LO**.

Proof: Similar to Lemma 19, the number of best response coefficients in the no left-over and left-over setting are proportional to the number of distinct combinations of the random yield distributions given respectively by $n_1 \times n_2 \times n_3$ and $7 \times n_1 \times n_2 \times n_3$. This completes the proof. $\square$

The special, yet realistic, cases studied in Lemmas 19 and 20 lead to significant reductions in the computational effort required to solve Problems **SC** and **SC-LO**. In particular, for realistic sized, $m = 9$ candidate strain scenarios, we were able to solve Problems **SC** and **SC-LO** respectively within 7.8 hours and 8.7 hours under the condition imposed by Lemma 19; and within 1 hour and 1.5 hour under the condition imposed by Lemmas 20. If we further restrict ourselves to conditions imposed by both lemmas, solution times drop even more drastically . These computational times are acceptable, given that vaccine composition is a strategic decision that needs to be made once a year.

Next, we turn our attention to the general case, not restricted by the conditions in Lemmas 19 and 20, and perform a comparative statics analysis to understand the impact of the random yield. For this purpose, we use the concept of stochastic orders, whose definitions are provided below for completeness.

Definition 5. *(Müller and Stoyan (2002)) Let* $\boldsymbol{X} = (X_1, \cdots, X_n)$ *and* $\boldsymbol{Y} = (Y_1, \cdots, Y_n)$ *be* $n-$*dimensional random vectors with finite expectations. Then,*

1. $\boldsymbol{X}$ *is said to be smaller than* $\boldsymbol{Y}$ *in the stochastic order, written* $\boldsymbol{X} \leq_{st} \boldsymbol{Y}$*, if* $\mathbb{E}[f(\boldsymbol{X})] \leq \mathbb{E}[f(\boldsymbol{Y})]$ *for all bounded increasing functions* $f : \Re^n \to \Re$ *such that the expectations exist.*

2. $\boldsymbol{X}$ *is said to be smaller than* $\boldsymbol{Y}$ *in the convex order, written* $\boldsymbol{X} \leq_{cx} \boldsymbol{Y}$*, if* $\mathbb{E}[f(\boldsymbol{X})] \leq \mathbb{E}[f(\boldsymbol{Y})]$ *for all convex functions* $f : \Re^n \to \Re$ *such that the expectations exist.*

Thus, an increase in stochastic order implies an increase in mean, whereas an increase in convex order implies higher variability.

Lemma 21. *As the random yield vector* $\boldsymbol{Y}$ *increases in stochastic order or in convex order, the expected societal vaccination benefit, the manufacturer's profit, and the demand satisfied in the optimal solution change as depicted in Table 4.2.*

Table 4.2: Comparative statics analysis results with respect to the random yield vector, $\boldsymbol{Y}$

			$\boldsymbol{Y}$ increases in stochastic order	$\boldsymbol{Y}$ increases in convex order
No left-over setting	Manufacturer's expected profit	Π_m	$\nearrow$	$\searrow$
	Expected demand satisfied	$\mathbb{E}_{\boldsymbol{Y}}[(DS]^{\#}$	$\nearrow$	$\searrow$
	Expected societal vaccination benefit	$\Pi_g^{\#}$	$\nearrow$	$\searrow$
Left-over setting	Manufacturer's expected profit	Π_m^{LO}	$\nearrow$	

#: Results hold under the assumption that $c_k = c$, $k \in \Theta$, and Constraint (4.3) is binding in the optimal solution to Problem **QS** (Assumption 1).

##: if $w(|I|) \geq 2w(|I| - 1)$.

Lemma 21 is not surprising in that a strain with a higher mean yield or lower yield uncertainty benefits both players, and leads to a higher amount of demand satisfied.[1] This property, for the manufacturer's expected profit function, carries over to the left-over setting under some restrictions on the $w(.)$ function.[2]

The next question is how the number of strains selected in set I affects the manufacturer in both the no left-over and left-over settings. In the no left-over setting, although a larger composition $I' \supseteq I$ will likely reduce the manufacturer's expected vaccine output, which is limited by the least strain output, the increase in per unit price $(w(|I'|) - w(|I|))$ may compensate this loss, and the manufacturer may benefit from an increase in the number of strains selected by the Committee. Indeed, our numerical results shows that due to this trade-off, Π_m is not monotone in the size of set I; please see Table 4.4, e.g., Scenario 1 where

[1] In the absence of publicly available data on the production cost of the different strains, the equal cost assumption (Assumption 1) is reasonable, as all strains go through a similar process. The second part of the assumption states that the manufacturer utilizes all available chicken eggs, which is common, especially for years with a high virus attack rate (FDA (2005)).

[2] We conjecture that the result on how Π_m^{LO} changes with a convex order increase of $\boldsymbol{Y}$ holds for more general $w(.)$ functions; in fact, we were not been able to find a counter-example that violated this property for general increasing functions of $w(.)$.

Π_m is always increasing as more strains are added to set I; and Scenario 10 where Π_m is first increasing, then decreasing as more strains are added. Interestingly, the expected demand satisfied, $\mathbb{E}_{\boldsymbol{Y}}[DS]$, from the sales of the primary vaccine in the no left-over model is not monotone decreasing as more strains are added to set I; see again Table 4.4, e.g., Scenario 1, where $\mathbb{E}_{\boldsymbol{Y}}[DS]$ is first increasing, then decreasing as more strains are added. To see this, consider the optimal $\boldsymbol{Q^*}$ vector for Scenario 1, where $I = \{5\}$ results in $Q_5^* = 135.22$; and $I = \{5, 9\}$ results in $Q_5^* = 143.03, Q_9^* = 143.37$. Thus, the optimal input quantities may sometimes increase as set I is expanded (if the egg stock is sufficient) because of the existence of a limiting strain (note that for a given $\boldsymbol{Q}(I)$, $\mathbb{E}_{\boldsymbol{Y}}[\{min_{k\in I} Y_k Q_k\}]$ is decreasing in I) and also because of the higher incentive for the manufacturer, as the unit sales price $w(.)$ is increasing in I. However, increasing the input quantities as I expands may not be always possible or desirable due to the limited egg supply, and also because the increase in sales price may not be sufficient to cover the increasing manufacturing cost.

4.5 Numerical Studies on the Optimal Vaccine Composition and Benefits of a Secondary Vaccine

Our objectives in this section are three-fold: (i) to understand how the optimal strain composition generated by our model differs from the current practice, and to quantify the benefits of using such an optimization-based approach on vaccine composition; (ii) to quantify the benefits of utilizing a secondary vaccine from the left-over strains; and (iii) to study the benefits of an integrated approach in which both the Committee and the manufacturer make their decisions (vaccine composition and the input quantities) considering the secondary vaccine option.

Most data used in our numerical study come from the published literature or from publicly available data to the extent possible (Özaltın et al. (2011), Chick et al. (2008), and Cho (2010)). However, as in other similar papers (e.g., Özaltın et al. (2011)), the data that are not publicly available needed to be estimated as described below.

4.5.1 Data and Sources

As in Özaltın et al. (2011), the data used in our study are based on the 2008–2009 flu season in the United States. Among the nine candidate strains considered in the Committee's meeting in February 2008, four belonged to Influenza A/H1N1 family, two to Influenza A/H3N2 family, and three to Influenza B family; see Table 4.3 (from Özaltın et al. (2011)), which depicts the candidate strains, their family, cross-effectiveness parameters (see Özaltın et al. (2011) for how the cross-effectiveness parameters were calculated), and yield rates, classified into high, moderate, and low yield groups. Table B.2 in the Appendix provides the remaining parameter values and the data sources used in our study.

Table 4.3: Candidate strain data (Özaltın et al. (2011))

Virus family	Strain	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	Yield Rates
		cross–effectiveness parameters between strains ($b_{ik}, i, k \in \Theta$)									
A/H1N1	(1) A/New Caledonia/20/99	1	0.333	0.278	0.266						High
	(2) A/Solomon Islands/3/06	0.341	1	0.498	0.261						Moderate
	(3) A/Brisbane/59/07	0.232	0.329	1	0.614						Moderate/High
	(4) A/South Dakota/6/07	0.227	0.325	0.661	1						Low/Moderate
A/H3N2	(5) A/Wisconsin/67/2005					1	0.221				Moderate/High
	(6) A/Brisbane/10/2007					0.265	1				Moderate/High
B	(7) B/Malaysia/2506/2004							1	0.097	0.054	Moderate
	(8) B/Florida/04/2006							0.339	1	0.603	Low/Moderate
	(9) B/Brisbane/03/2007							0.334	0.462	1	Moderate

The random prevalence. As in Özaltın et al. (2011), we model the random prevalence using the *relative* prevalence of a strain in a given season, which represents the ratio of the number of people infected by that particular strain to the total number of people infected. In our model, only the mean prevalence rate of a strain has an impact on the vaccine composition. In accordance with Özaltın et al. (2011), we consider three relative prevalence levels: low prevalence with a mean of 10%, moderate prevalence with a mean of 35%, and high prevalence with a mean of 65%. For each problem instance (scenario), we normalize the means of the prevalence rates such that their sum is 100%.

The random yield. As stated in Özaltın et al. (2011), no academic studies or publicly available data exist on the yield rates for the influenza vaccine production. Consequently, we use the data suggested in Özaltın et al. (2011); see the last column of Table 4.3. In particular, each candidate strain is classified into groups of low, moderate, and high yield strains based on the availability and efficiency of high-growth seed strains. The yield in each group follows a truncated Normal distribution in $[0, 1]$, with a mean of 0.7 for the low yield group, a mean of 0.8 for the moderate yield group, and a mean of 0.9 for the high yield group.

Specifically, in our numerical study, we consider three demand levels (low, moderate, and high), and for each demand level, we randomly generate various scenarios. For each scenario, the prevalence group of each strain is randomly selected among the high, moderate, and low prevalence groups; and for those strains whose yield is given by two possible groups in Table 4.3, the yield group is randomly selected among the two groups, and for all other strains the yield group given in Table 4.3 is used. In all scenarios, we use a yield standard deviation of 0.1 and a yield correlation coefficient of 0 between any pair of strains (we consider correlated yields in Section 4.5.4). The parameter values for each scenario are depicted in Tables B.3 and B.4 (in the Appendix), where H, M, and L respectively refer to high, moderate, and low group. The unit sales price of the vaccine is assumed to be \$4 $\times$ the number of strains included in the vaccine, i.e., $w(|I|) = 4|I|$ (Chick et al. (2008)).

4.5.2 Optimal Vaccine Composition in the No Left-over Setting

For each scenario, we solve Problem **SC** and obtain the optimal vaccine composition (set I^*) and the corresponding expected societal vaccination benefit (Π_g), the manufacturer's expected profit (Π_m), and the expected demand satisfied ($\mathbb{E}_{\boldsymbol{Y}}[DS]$). In particular, for each scenario, for each possible strain set $I \subseteq \Theta$, we first solve the manufacturer's problem using the KKT first-order conditions (see Lemma 18) and obtain the optimal input vector, $\boldsymbol{Q}$, which we then use to numerically calculate the manufacturer's best response function, $\mathbb{E}_{\boldsymbol{Y}}[DS(I)]$. We then input all best response functions to the integer programming problem given in Section 4.4, and solve the Committee's problem to obtain the optimal vaccine composition. Note that Scenarios 1-16 all satisfy the restriction under which Lemma 19 holds (see Table B.4). Consequently, the number of best response functions to be calculated *a priori* is bounded from above by a polynomial function of the number of strains ($m = 9$ in all scenarios). As a result, we are able to obtain the optimal solution for each scenario within 7.8 hours on a 3.06-GHz PC using the optimization modules of Scipy 0.10.0 Library.

Table 4.4 reports the optimal solution to Problem **SC** (bolded). To study how the objective functions and the expected demand satisfied change with the selected strain set I, for each scenario we also report the best solution to Problem **SC** among all subsets of the candidate strain set, Θ, comprised exactly of n strains, for $n = 1, \cdots, 5$ (strain sets comprised of six and higher number of strains did not produce better results). We depict the optimal strain set as $I^* = \{\{\}, \{\}, \{\}\}$, where the first curly bracket contains the strains selected for the A/H1N1 Family, the second curly bracket for the A/H3N2 Family, and the third for the B Family. To represent the composition under the current practice of selecting three strains, one from each family, we then solve Problem **SC** with this constraint and obtain the optimal vaccine composition ($I^{*\#}$) and the corresponding expected societal benefit ($\Pi_g^{\#}$); see Table 4.4. The last column of Table 4.4 provides the % increase in the expected societal benefit obtained by the **SC** Model over current practice.

Table 4.4: Comparison of the optimal vaccine composition generated by the **SC** model with the current practice

		Problem **SC**				Current practice		
Demand	Scenario	I^*	Π_g	Π_m	$\mathbb{E}_{\boldsymbol{Y}}[DS]$	$I^{*\#}$	$\Pi_g^{\#}$	% increase in Π_g
Low ($d = 103$)	1	{{}, {}, {9}}	981.34	263.51	99.68	{{2}, {5}, {9}}	1,867.09	0.00%
		{{}, {5}, {9}}	1,581.71	511.19	99.70			
		{{2}, {5}, {9}}	**1,867.09**	**753.30**	**99.72**			
		{{2}, {5}, {7, 9}}	1,815.26	990.75	99.75			
		{{2}, {5}, {7, 8, 9}}	1,646.32	1,186.18	96.81			
	2	{{4}, {}, {}}	983.32	241.28	98.44			
		{{4}, {6}, {}}	1,378.44	486.82	99.18			
		{{4}, {6}, {8}}	1,606.33	701.41	98.94			

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Table 4.4 – continued from previous page

Demand	Scenario	I^*	Π_g	Π_m	$\mathbb{E}_{\boldsymbol{Y}}[DS]$	$I^{*\#}$	$\Pi_g^{\#}$	% increase in Π_g
		{{2, 4}, {6}, {8}}	**1,688.57**	**937.33**	**99.16**	{{4}, {6}, {8}}	1,606.33	5.12%
		{{2, 3}, {5, 6}, {8}}	1,517.95	1,215.93	98.30			
		{{}, {6}, {}}	841.06	264.44	99.77			
		{{2}, {6}, {}}	1,462.64	512.76	99.86			
	3	{{2, 4}, {6}, {}}	1,522.93	754.62	99.88			
		{{2, 4}, {5, 6}, {}}	**1,556.18**	**1,012.55**	**100.06**	{{2}, {6}, {8}}	1,387.57	12.15%
		{{1, 2, 4}, {5, 6}, {}}	1,543.24	1,268.03	100.13			
		{{2}, {}, {}}	752.56	264.11	99.65			
		{{2}, {5}, {}}	1,345.04	511.61	99.69			
	4	**{{2}, {5}, {9}}**	**1,646.21**	**753.33**	**99.74**	{{2}, {5}, {9}}	1,646.21	0.00%
		{{2}, {5, 6}, {9}}	1,618.92	991.98	99.74			
		{{1, 2}, {5, 6}, {9}}	1,536.54	1,246.82	99.84			
		{{2}, {}, {}}	1,617.06	528.32	199.35			
		{{2}, {}, {9}}	2,614.67	1,022.37	199.42			
	5	**{{2}, {6}, {9}}**	**3,008.84**	**1,462.98**	**184.42**	{{2}, {6}, {9}}	3,008.84	0.00%
		{{1, 2}, {6}, {9}}	2,336.46	1,462.57	138.29			
		{{1, 2, 4}, {6}, {9}}	1,741.64	1,394.98	107.25			
		{{1}, {}, {}}	2,151.98	558.35	200.95			
		{{1}, {}, {9}}	**4,005.42**	**1,056.19**	**200.23**	{{1}, {5}, {9}}	3,467.30	15.52%
	6	{{1, 3}, {}, {9}}	3,906.48	1,456.76	183.90			
		{{1, 3}, {5}, {9}}	2,688.85	1,459.89	138.12			
Moderate ($d = 206$)		{{1, 3}, {5, 6}, {9}}	1,813.23	1,397.45	107.37			
		{{4}, {}, {}}	1,901.51	526.54	199.40			
		{{4}, {5}, {}}	2,664.56	1,057.46	200.36			
	7	**{{4}, {5}, {9}}**	**2,869.63**	**1,458.24**	**184.02**	{{4}, {5}, {9}}	2,869.63	0.00%
		{{2, 4}, {5}, {9}}	2,298.22	1,397.59	134.22			
		{{1, 2, 4}, {5}, {8}}	1,699.32	1,327.18	103.86			
		{{}, {5}, {}}	2,020.38	527.70	199.42			
		{{2}, {5}, {}}	3,151.02	1,023.45	199.39			
	8	**{{2}, {5}, {9}}**	**3,712.06**	**1,387.90**	**178.16**	{{2}, {5}, {9}}	3,712.06	0.00%
		{{2, 3}, {5}, {9}}	2,504.89	1,400.88	134.43			
		{{1, 2, 3}, {5}, {9}}	1,721.08	1,398.79	107.44			
		{{}, {}, {8}}	2,305.60	719.89	272.18			
		{{3}, {}, {8}}	**3,593.06**	**1,388.88**	**267.36**	{{3}, {5}, {8}}	2,978.31	20.64%
	9	{{3}, {5}, {8}}	2,978.31	1,472.74	185.23			

Continued on next page

Table 4.4 – continued from previous page

Demand	Scenario	I^*	Π_g	Π_m	$\mathbb{E}_{\boldsymbol{Y}}[DS]$	$I^{*\#}$	$\Pi_g^{\#}$	% increase in Π_g
		{{3}, {5}, {7, 8}}	2,187.98	1,394.57	134.04			
		{{1, 3}, {5}, {7, 8}}	1,606.33	1,392.73	107.14			
	10	{{4}, {}, {}}	3,273.85	718.36	272.15	{{1}, {6}, {8}}	2,450.48	82.05%
		{{1, 4}, {}, {}}	**4,461.07**	**1,441.61**	**273.43**			
		{{1, 4}, {}, {8}}	3,624.38	1,468.97	184.91			
		{{1, 2, 4}, {}, {8}}	2,605.80	1,395.91	134.12			
		{{1, 2, 4}, {6}, {8}}	1,834.27	1,345.89	104.79			
	11	{{}, {}, {8}}	1,545.85	720.49	272.21	{{2}, {5}, {8}}	2,681.74	5.94%
		{{2}, {}, {8}}	**2,841.10**	**1,393.09**	**267.89**			
		{{2}, {5}, {8}}	2,681.74	1,477.59	185.63			
		{{2}, {5, 6}, {8}}	2,078.65	1,402.79	134.55			
		{{2, 4}, {5, 6}, {8}}	1,659.62	1,279.41	101.47			
	12	{{1}, {}, {}}	1,879.54	762.06	274.18	{{1}, {5}, {8}}	3,297.68	0.32%
		{{1}, {}, {8}}	**3,308.13**	**1,443.83**	**273.37**			
		{{1}, {5}, {8}}	3,297.68	1,559.04	192.42			
		{{1, 3}, {5}, {8}}	2,537.24	1,531.84	142.61			
		{{1, 3}, {5, 6}, {8}}	1,940.73	1,509.52	112.98			
	13	{{4}, {}, {}}	1,634.02	658.67	268.69	{{3}, {6}, {8}}	2123.87	28.92%
		{{3}, {}, {8}}	**2,738.08**	**1,286.65**	**254.58**			
		{{3}, {6}, {8}}	2,123.87	1,285.98	169.67			
		{{1 3}, {6}, {8}}	1,551.68	1,312.90	128.93			
		{{1 3}, {5 6}, {8}}	1,154.16	1,277.83	101.39			
	14	{{4}, {}, {}}	4,036.98	658.67	268.69	{{3}, {6}, {9}}	1917.79	147.07%
		{{1 3}, {}, {}}	**4,738.36**	**1,443.25**	**273.35**			
		{{1 2 3}, {}, {}}	3,529.99	1,471.43	185.12			
		{{1 2 3 4}, {}, {}}	2,365.05	1,308.01	128.63			
High ($d = 281.1$)		{{1 2 3 4}, {}, {9}}	1,695.99	1,273.55	101.18			
	15	{{}, {6}, {}}	3,499.28	720.73	272.19	{{3}, {6}, {9}}	2319.37	115.17%
		{{}, {5 6}, {}}	**4,990.69**	**1,390.78**	**267.60**			
		{{1}, {5 6}, {}}	3,424.70	1,471.69	185.14			
		{{3}, {5 6}, {9}}	2,451.19	1,337.17	130.45			
		{{1 3}, {5 6}, {9}}	1,715.46	1,345.46	104.77			
	16	{{}, {}, {8}}	4,380.91	660.60	268.84	{{3}, {6}, {8}}	2,345.93	108.12%
		{{}, {}, {7 8}}	**4,882.38**	**1,283.91**	**254.24**			
		{{}, {}, {7 8 9}}	3,274.71	1,280.52	169.21			

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Table 4.4 – continued from previous page

Demand	Scenario	I^*	Π_g	Π_m	$\mathbb{E}_{\boldsymbol{Y}}[DS]$	$I^{*\#}$	$\Pi_g^{\#}$	% increase in Π_g
		{{1}, {}, {7 8 9}}	2,362.66	1,307.75	128.61			
		{{3}, {6}, {7 8 9}}	1,672.27	1,230.90	99.05			
#: $\frac{\Pi_g - \Pi_g^{\#}}{\Pi_g^{\#}} \times 100\%$								

As Table 4.4 indicates, it is not necessarily optimal to follow the current practice of selecting one strain from each family. The optimal set may sometimes omit any strain from a particular family (e.g., Scenarios 9 and 11, where strains 3 and 8, and strains 2 and 8 are respectively selected), or may include more than one strain from a particular family (e.g., Scenario 3, with strains 2 and 4 from Family 1, and strains 5 and 6 from Family 2). Interestingly, even the best three-strain set of the **SC** model does not necessarily correspond to the current practice solution. For example, consider Scenario 3, where the best three-strain solution is $\{\{2,4\},\{6\},\{\}\}$, with strains 2 and 4 from Family 1, strain 6 from Family 2, and no strain from Family 3, compared to the best current practice solution of $\{\{2\},\{6\},\{8\}\}$, which includes exactly one strain from each family. The optimal set depends on *both* the prevalence and yield levels. Roughly speaking, the optimal strain set often contains the strains whose yield and prevalence levels are high, omitting the strains otherwise. Consequently, when strains of one family all have lower yield and/or prevalence, the current practice, which dictates inclusion of a strain from that particular family also, may result in a sub-optimal set (e.g., Scenario 10, where the yields and the prevalences of the strains of Family 2 are all relatively lower than the other two families; as a result, the optimal set to the **SC** model omits all strains from Family 2, resulting in a 82.05% increase in the expected societal vaccination benefit over current practice.) More importantly, composing the strain set under no restrictions provides a significant increase in the expected societal vaccination benefit, with a maximum of 147.07% increase and with a 33.81% increase on average. These results suggest the significant societal benefit of moving from the family-restricted current practice of strain selection to an unrestricted optimal selection.

4.5.3 Benefits of a Secondary Vaccine Option and its Impact on the Optimal Vaccine Composition

To study the benefits of a secondary vaccine, we first restrict ourselves to the current practice of selecting one strain from each family. Thus, the primary vaccine is a trivalent (three-strain) vaccine, and the secondary vaccine is a bivalent (two-strain) vaccine. For this purpose, we now obtain the optimal solution to Problem **SC**, which does not utilize the secondary vaccine, under the restriction that exactly one strain is selected from each family, and compare it with the optimal solution to the left-over problem, Problem **SC-LO**, which determines the optimal strain set considering the potential utilization of the secondary vaccine for all 16 scenarios detailed in Tables B.3 and B.4. We use the integer programming formulations given in Section 4.4.

Tables 4.5 and 4.6 provide the optimal strain set, I^*, the expected societal vaccination benefit, Π_g, and the expected manufacturer profit, Π_m, and the corresponding input quantity vector, Q^*, and the expected demand satisfied ($\mathbb{E}_{\boldsymbol{Y}}[DS]$ in the no left-over model, and $\mathbb{E}_{\boldsymbol{Y}}[DS_P^{LO} + DS_S^{LO}]$ in the left-over model) for both the no left-over and left-over models.

Table 4.5: Comparison of the no left-over model with the left-over model

		Without left-over			With left-over				
Demand	Scenario	I^*	$\boldsymbol{Q}^*$	$\mathbb{E}_{\boldsymbol{Y}}[DS]$	I^{LO*}	$\boldsymbol{Q}^{LO*}$	$\mathbb{E}_{\boldsymbol{Y}}[DS_P^{LO}]$	$\mathbb{E}_{\boldsymbol{Y}}[DS_S^{LO}]$	% increase in exp. demand satisfied#
Low	1	$\{2,5,9\}$	[147.72, 147.70, 147.93]	99.72	$\{2,5,9\}$	[137.53, 137.71, 138.14]	96.28	5.56	2.13%
	2	$\{4,6,8\}$	[169.93, 146.99, 169.01]	98.94	$\{4,6,8\}$	[156.64, 138.25, 155.77]	95.00	6.45	2.54%
	3	$\{2,6,8\}$	[147.81, 147.26, 147.88]	99.85	$\{2,6,8\}$	[137.46, 137.94, 137.77]	96.43	5.45	2.03%
	4	$\{2,5,9\}$	[147.86, 147.82, 147.91]	99.74	$\{2,5,9\}$	[137.25, 137.78, 138.10]	96.25	5.59	2.11%
Moderate	5	$\{2,6,9\}$	[257.62, 233.36, 259.02]	184.42	$\{2,6,9\}$	[256.93, 236.28, 256.80]	184.35	15.49	8.36%
	6	$\{1,5,9\}$	[241.71, 240.61, 267.68]	190.21	$\{1,5,9\}$	[242.55, 242.11, 265.33]	190.17	12.40	6.50%
	7	$\{4,5,9\}$	[258.41, 233.84, 257.75]	184.02	$\{4,5,8\}$	[249.60, 229.75, 270.65]	174.89	19.43	5.60%
	8	$\{2,5,9\}$	[249.41, 250.15, 250.44]	178.16	$\{2,5,9\}$	[249.63, 249.93, 250.44]	178.16	17.61	9.88%
High	9	$\{3,5,8\}$	[257.70, 234.35, 257.95]	185.23	$\{3,5,8\}$	[254.83, 240.48, 254.69]	184.87	20.90	11.09%
	10	$\{1,6,8\}$	[235.02, 257.38, 257.59]	185.33	$\{1,6,8\}$	[240.65, 254.92, 254.43]	185.04	20.59	10.95%
	11	$\{2,5,8\}$	[257.16, 234.80, 258.04]	185.63	$\{2,5,8\}$	[254.96, 240.56, 254.48]	185.31	20.70	10.98%
	12	$\{1,5,8\}$	[242.52, 241.65, 265.82]	192.42	$\{1,5,8\}$	[242.85, 242.00, 265.14]	190.37	12.20	5.27%
	13	$\{3,6,8\}$	[240.00, 239.42, 270.58]	169.67	$\{3,6,8\}$	[244.00, 243.88, 262.11]	169.20	21.72	13.19%
	14	$\{3,6,9\}$	[250.21, 249.60, 250.19]	178.46	$\{1,6,8\}$	[304.61, 332.42, 112.98]	79.02	169.20	39.09%
	15	$\{3,6,9\}$	[250.21, 249.60, 250.19]	178.46	$\{1,6,8\}$	[304.61, 332.42, 112.98]	79.02	169.20	39.09%
	16	$\{3,6,8\}$	[240.00, 239.42, 270.58]	169.67	$\{3,6,8\}$	[244.00, 243.88, 262.11]	169.20	21.72	13.19%

#: $\frac{\mathbb{E}_{\boldsymbol{Y}}[DS_P^{LO}+DS_S^{LO}]-\mathbb{E}_{\boldsymbol{Y}}[DS]}{\mathbb{E}_{\boldsymbol{Y}}[DS]} \times 100\%$

Table 4.6: Comparison of the no left-over model with the left-over model

		Without left-over			With left-over				
Demand	Scenario	I^*	Π_g	Π_m	I^*	Π_g^{LO}	Π_m^{LO}	% increase in $\Pi_g^{\#}$	% increase in $\Pi_m^{\#\#}$
Low	1	$\{2,5,9\}$	1,867.09	753.30	$\{2,5,9\}$	1,872.12	786.50	0.27%	4.41%
	2	$\{4,6,8\}$	1,606.33	701.41	$\{4,6,8\}$	1,610.70	740.85	0.27%	5.62%
	3	$\{2,6,8\}$	1,387.57	755.29	$\{2,6,8\}$	1,390.79	787.59	0.23%	4.28%
	4	$\{2,5,9\}$	1,646.21	753.33	$\{2,5,9\}$	1,649.90	786.57	0.22%	4.41%
Moderate	5	$\{2,6,9\}$	3,008.84	1,462.98	$\{2,6,9\}$	3,170.99	1,586.06	5.39%	8.41%
	6	$\{1,5,9\}$	3,467.30	1,532.52	$\{1,5,9\}$	3,609.87	1,631.26	4.11%	6.44%
	7	$\{4,5,9\}$	2,869.63	1,458.24	$\{4,5,8\}$	3,048.11	1,504.07	6.22%	3.14%
	8	$\{2,5,9\}$	3,712.06	1,387.90	$\{2,5,9\}$	3,956.39	1,528.77	6.58%	10.15%
High	9	$\{3,5,8\}$	2,978.31	1,472.74	$\{3,5,8\}$	3,182.85	1,635.68	6.87%	11.06%
	10	$\{1,6,8\}$	2,450.48	1,473.94	$\{1,6,8\}$	2,659.00	1,635.14	8.51%	10.94%
	11	$\{2,5,8\}$	2,681.74	1,477.59	$\{2,5,8\}$	2,871.91	1,639.35	7.09%	10.95%
	12	$\{1,5,8\}$	3,297.68	1,559.04	$\{1,5,8\}$	3,402.99	1,632.05	3.19%	4.68%
	13	$\{3,6,8\}$	2,123.87	1,285.98	$\{3,6,8\}$	2,294.31	1,454.11	8.02%	13.07%
	14	$\{3,6,9\}$	1,917.79	1,391.51	$\{1,6,8\}$	2,321.70	1,551.81	21.06%	11.52%
	15	$\{3,6,9\}$	2,319.37	1,391.51	$\{1,6,8\}$	3,151.85	1,551.81	35.89%	11.52%
	16	$\{3,6,8\}$	2,345.93	1,285.98	$\{3,6,8\}$	2,460.51	1,454.11	4.88%	13.07%

#: $\frac{\Pi_g^{LO}-\Pi_g}{\Pi_g}$
##: $\frac{\Pi_m^{LO}-\Pi_m}{\Pi_m}$

First, utilizing the two left-over strains to produce the secondary vaccine leads to a significant increase in the demand satisfied (from 2.03% to 39.09%, Table 4.5). While the left-over model achieves this through a slight reduction in the demand satisfied by the primary vaccine, overall this leads to a significant increase in the expected societal vaccination benefit (ranging from 0.22% to 35.89%, see Table 4.6). Thus, as expected, the secondary vaccine production benefits both the society and the manufacturer by providing them with more flexibility for utilizing the strain outputs, but what is interesting is that this flexibility leads to huge benefits for the society.

It is insightful to understand how the secondary vaccine option changes the manufacturer's input vector. In the no left-over setting, the requirement that all selected strains must be included in each dose of the vaccine plays an important role on the manufacturer's input quantity decision, and almost always drives the input quantity of each strain up. Consequently, when this requirement is relaxed in the left-over setting, the manufacturer's input quantity vector goes down, which, in turn, reduces the primary vaccine output, while increasing the total demand satisfied. This impact is more significant when the strain set contains low-yield strains or when the demand level is high. However, we also see scenarios where in the left-over model the input quantity of a pair of high-yield strains go up, while that of the low-yield strain goes down (e.g., see Scenario 6, where the input levels of both high-yield strains, $\{1, 5\}$, increase in the left-over model, at the expense of a reduction in the input level of the moderate-yield strain $\{9\}$). Interestingly, however, for moderate and high demand scenarios, the reduction in the primary vaccine output is negligible, and there is a significant output of the secondary vaccine, which greatly expands the vaccine coverage. There are two exceptions to this, and in both cases the vaccine composition differs in the no left-over and left-over models: In Scenarios 14 and 15, the strain selection changes significantly from the no left-over to the left-over model (i.e., from $\{3, 6, 9\}$ in the no left-over model to $\{1, 6, 8\}$) in the left-over model) and this has a high impact on the primary vaccine output: In the left-over model, the primary vaccine output is halved, and is dominated by the secondary vaccine output. To understand why this happens, consider Scenario 14, where the prevalences of strains from Family 1 are much higher than the prevalences of strains from the other two families. As a result, the major immunity protection of both the primary and the secondary vaccine should come from the protection they offer for Family 1. In other words, it is highly desirable that the strain with the least output, which will not be part of the secondary vaccine, not belong to Family 1 under the secondary vaccine option. Consequently, under the secondary vaccine option, the strain selected for Family 1 switches from a moderate-yield strain ($\{3\}$) to a high-yield strain ($\{1\}$), while the strain selected for Family 3 switches from a moderate-yield strain ($\{9\}$) to a low-yield strain ($\{8\}$). As a result, in the left-over model, the output of the primary vaccine is now more likely to be limited by a Family 3 strain ($\{8\}$) than a Family 1 strain, and the secondary vaccine is now more likely to include a strain from Family 1, and have an effectiveness comparable to the primary vaccine in this scenario (as the expected prevalence of Family 1 strains are high).

In general, the vaccine composition can change significantly in the left-over model when the demand level is moderate or high, and the prevalence level of strains from a particular family are much higher than the strains from the other families (see Scenarios 7, 14, and 15 in Tables 4.5 and 4.6). As detailed above, this happens because, with the left-over option, the Committee may deliberately include both high-prevalence strains with a higher yield and low-prevalence strains with a lower yield in the composition. Such a strategy would not work well in the no left-over setting, where the primary vaccine output, the only vaccine produced, may be significantly reduced.

4.5.4 Impact of Correlated Yields

We next study the case of correlated yields and summarize the findings of our studies here. We restrict ourselves to the current practice of selecting one strain from each family, and obtain the optimal solutions to Problems **SC** and **SC-LO** with the parameter values given in Scenario 1 (see Tables B.3 and B.4) for various correlation coefficients -0.5, -0.8, 0 ,$+0.5$, and $+0.8$. The results are reported in Tables 4.7 and 4.8. Our numerical studies indicate that, roughly speaking, the societal vaccination benefit increases as the yields become more correlated. Since realizations of highly correlated yields will differ less, there will be less chance that the vaccine output is limited by a low yield realization for only one of the strains. However the results in Tables 4.7 and 4.8 should be interpreted with caution, since, for the truncated normal distribution, which is used to model the random production yield, increasing correlation coefficients is not necessarily corresponding to increasing dependences among the random yields. We include the results here for the purpose of completeness.

Table 4.7: The impact of correlated yields on $\boldsymbol{Q}^*$, $\mathbb{E}_{\boldsymbol{Y}}[DS]$, $\mathbb{E}_{\boldsymbol{Y}}[DS_P^{LO}]$, and $\mathbb{E}_{\boldsymbol{Y}}[DS_S^{LO}]$

		Without left-over			With left-over				
Scenario	Correlation	I^*	$\boldsymbol{Q}^*$	$\mathbb{E}_{\boldsymbol{Y}}[DS]$	I^{LO*}	$\boldsymbol{Q}^{LO*}$	$\mathbb{E}_{\boldsymbol{Y}}[DS_P^{LO}]$	$\mathbb{E}_{\boldsymbol{Y}}[DS_S^{LO}]$	% increase in $\mathbb{E}_{\boldsymbol{Y}}[DS]^{\#}$
1	-0.8	$\{2,5,9\}$	[161.69, 162.05, 161.63]	96.65	$\{2,5,9\}$	[146.88, 147.30, 147.06]	91.93	8.49	3.90%
	-0.5	$\{2,5,9\}$	[154.42, 155.27, 155.71]	98.00	$\{2,5,9\}$	[142.16, 142.56, 142.91]	93.85	7.16	3.07%
	0	$\{2,5,9\}$	[147.72, 147.70, 147.93]	99.72	$\{2,5,9\}$	[137.53, 137.71, 138.14]	96.28	5.56	2.13%
	0.5	$\{2,5,9\}$	[147.00, 146.92, 147.29]	99.65	$\{2,5,9\}$	[140.45, 139.97, 140.27]	97.53	3.63	1.52%
	0.8	$\{2,5,9\}$	[143.50, 143.47, 143.40]	99.62	$\{2,5,9\}$	[139.30, 139.18, 138.83]	98.38	2.12	0.88%

#: $\frac{\mathbb{E}_{\boldsymbol{Y}}[DS_P^{LO}+DS_S^{LO}]-\mathbb{E}_{\boldsymbol{Y}}[DS]}{\mathbb{E}_{\boldsymbol{Y}}[DS]} \times 100\%$

Table 4.8: The impact of correlated yields on Π_g, Π_m, Π_g^{LO}, and Π_m^{LO}

		Without left-over			With left-over				
Scenario	Correlation	I^*	Π_g	Π_m	I^*	Π_g^{LO}	Π_m^{LO}	% increase in $\Pi_g^{\#}$	% increase in $\Pi_m^{\#\#}$
1	-0.8	$\{2,5,9\}$	1,800.53	674.40	$\{2,5,9\}$	1,822.25	729.87	0.98%	8.23%
	-0.5	$\{2,5,9\}$	1,833.91	710.63	$\{2,5,9\}$	1,844.62	755.89	0.64%	6.37%
	0	$\{2,5,9\}$	1,866.09	753.30	$\{2,5,9\}$	1,877.12	786.50	0.27%	4.41%
	0.5	$\{2,5,9\}$	1,866.74	754.57	$\{2,5,9\}$	1,877.01	778.74	0.28%	3.20%
	0.8	$\{2,5,9\}$	1,866.15	765.05	$\{2,5,9\}$	1,866.30	780.18	0.17%	1.98%

#: $\frac{\Pi_g^{LO}-\Pi_g}{\Pi_g}$

##: $\frac{\Pi_m^{LO}-\Pi_m}{\Pi_m}$

4.6 Conclusion

In this chapter, we study the influenza vaccine composition decision, which involves different independent decision-makers, and under a novel approach, of the left-over strain usage possibility. We derive structural properties of the optimal solution, and perform a comparative statics analysis to study the impact of random yield of strain on the expected societal vaccination benefit, the expected manufacturer's profit, and the expected vaccine output. Using the published and publicly available data, we perform extensive numerical studies to quantify the cost and societal vaccination benefit of relaxing the restriction (i.e., trivalent vaccine requirement) imposed by the current practice, and the novel approach, of utilizing the left-over strains to produce a secondary vaccine. Our analysis and numerical studies show that it may not be optimal to follow the current practice of selecting one strain from each family, especially when strains of one family all have lower yield and/or prevalence. Composing the strain set under no restrictions provides a significant increase in the expected societal vaccination benefit (with a 33.81% increase on average). In addition, the novel approach of using the left-over strains can benefit both the society and the manufacturer, and provides an increase in the expected societal vaccination benefit (with a 11.38% increase on average).

Our analysis and insights come with several limitations. First, we use a simplified cost structure to model the societal vaccination benefit by assuming that the influenza cost is linear in the number of flu cases prevented. However, in practice, this may not be the case. A more sophisticated way of modeling the cost structure remains a promising area for future research; this will provide a more accurate picture of the cost and societal vaccination benefits. Second, many of the insights in this chapter are derived from numerical studies, i.e., they are dependent on the availability and accuracy of the data. As stated above, we have to make some estimations and approximations due to the lack of data. In summary, while we acknowledge these limitations, we also believe that the results of the comparative statics analysis and the numerical studies in this chapter provide valuable insight into public policy decision-making.

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Appendix A

Preliminaries

In addition to the notation defined in the dissertation, we use the following notation in the Appendix. $\boldsymbol{X}_{(i,j)(p,q)}$ denotes the submatrix of $\boldsymbol{X}$ formed by rows i to j and columns p to q, $det(\boldsymbol{X})$ denotes its determinant, $C_{ij}(\boldsymbol{X})$ denotes the cofactor of element X_{ij} of $\boldsymbol{X}$, $\mathbf{1}_n$ denotes an $n \times 1$ unit column vector, and $\boldsymbol{I}_{(n,n)}$ denotes the $n \times n$ identity matrix.

The following definitions and results are commonly used in our analysis.

A.1 Definitions and Preliminaries on Stochastic Order Relationships

Definition A1. *(Müller and Stoyan (2002) p. 98 and 113) Let* $\boldsymbol{X} = (X_1, \cdots, X_n)$ *and* $\boldsymbol{Y} = (Y_1, \cdots, Y_n)$ *be* $n-$*dimensional random vectors with finite expectations. Then,*

1. $\boldsymbol{X}$ *is said to be smaller than* $\boldsymbol{Y}$ *in the convex order, written* $\boldsymbol{X} \leq_{cx} \boldsymbol{Y}$*, if* $E[f(\boldsymbol{X})] \leq E[f(\boldsymbol{Y})]$ *for all convex functions* $f : \Re^n \to \Re$ *such that the expectations exist.*

2. $\boldsymbol{X}$ *is said to be smaller than* $\boldsymbol{Y}$ *in the supermodular order, written* $\boldsymbol{X} \leq_{sm} \boldsymbol{Y}$*, if* $E[f(\boldsymbol{X})] \leq E[f(\boldsymbol{Y})]$ *for all supermodular functions* $f : \Re^n \to \Re$ *such that the expectations exits.*

Lemma A1. *Let $\boldsymbol{X}$ and $\boldsymbol{Y}$ follow the Bivariate Normal distribution (BVN), with expectation vectors $\boldsymbol{\mu_X} = \boldsymbol{\mu_Y}$ and variance-covariance matrices $\boldsymbol{\Sigma_X}$ and $\boldsymbol{\Sigma_Y}$, with $\sigma_{X_1} \leq \sigma_{Y_1}$, $\sigma_{X_2} \leq \sigma_{Y_2}$, $\sigma_{X_{i,3-i}} = \sigma_{Y_{i,3-i}}$, $i = 1, 2$. Then, $\boldsymbol{X} \leq_{cx} \boldsymbol{Y}$.*

Proof: Because both $\boldsymbol{X}$ and $\boldsymbol{Y}$ follow the BVN, the following two statements are equivalent (Müller and Stoyan (2002) p. 100): (1) $\boldsymbol{X} \leq_{\text{cx}} \boldsymbol{Y}$; (2) $\boldsymbol{\mu_X} = \boldsymbol{\mu_Y}$ and $\boldsymbol{\Sigma_X} - \boldsymbol{\Sigma_Y}$ is positive semi-definite. Then the result follows because $\boldsymbol{\mu_X} = \boldsymbol{\mu_Y}$ and $\boldsymbol{\Sigma_Y} - \boldsymbol{\Sigma_X}$ is positive semi-definite when the $\boldsymbol{\sigma_X}$ and $\boldsymbol{\sigma_Y}$ parameters satisfy the restrictions given in the lemma. $\square$

Definition A2. *(Topkis (1998) p. 43) Suppose that $f(x)$ is a real-valued function on lattice X. If $f(x_1) + f(x_2) \leq f(x_1 \vee x_2) + f(x_1 \wedge x_2)$ for all x_1 and x_2, where $x_1 \vee x_2$ and $x_1 \wedge x_2$ respectively denote the componentwise maximum and componentwise minimum of x_1 and x_2, then $f(x)$ is supermodular on X.*

Lemma A2. *Let $f_i, i = 1, \cdots, n : \Re^2 \to \Re$, and suppose that all of the following properties hold for any $\boldsymbol{x} = (x_1, x_2)^T \in \Re^2$ and $\triangle > 0$:*

1. *$f_i, i = 1, \cdots, n$, is supermodular.* **(C1)**
2. *$f_i(x_1 + \triangle, x_2) - f_i(x_1, x_2) \geq f_{i-1}(x_1 + \triangle, x_2) - f_{i-1}(x_1, x_2)$, $i = 2, \cdots, n$.* **(C2)**
3. *$f_i(x_1, x_2 + \triangle) - f_i(x_1, x_2) \geq f_{i-1}(x_1, x_2 + \triangle) - f_{i-1}(x_1, x_2)$, $i = 2, \cdots, n$.* **(C3)**

Then, $F(i, x_1, x_2) \equiv f_i(x_1, x_2)$ is supermodular in (i, x_1, x_2), for $i = 1, \cdots, n$.

Proof: Pick any i and i' in $\{1, \cdots, n\}$, and any (x_1, x_2) and (x_1', x_2') in $\Re^2$. Then, by Definition A2, it suffices to show that $F(i, x_1, x_2) + F(i', x_1', x_2') \leq F(i \vee i', x_1 \vee x_1', x_2 \vee x_2') + F(i \wedge i', x_1 \wedge x_1', x_2 \wedge x_2')$, where $x \vee y = \max(x, y)$ and $x \wedge y = \min(x, y)$.
<u>Case 1.</u> If $i = i'$, then $F(i, x_1, x_2) + F(i', x_1', x_2') = f_i(x_1, x_2) + f_i(x_1', x_2') \leq f_i(x_1 \vee x_1', x_2 \vee x_2') + f_i(x_1 \wedge x_1', x_2 \wedge x_2')$ by **(C1)**.
<u>Case 2.</u> If $i \neq i'$, then assume, without loss of generality, that $i < i'$. Then, by definition, we have $F(i, x_1, x_2) + F(i', x_1', x_2') = f_i(x_1, x_2) + f_{i'}(x_1', x_2')$ and $F(i \vee i', x_1 \vee x_1', x_2 \vee x_2') + F(i \wedge$

$i', x_1 \wedge x_1', x_2 \wedge x_2') = f_{i'}(x_1 \vee x_1', x_2 \vee x_2') + f_i(x_1 \wedge x_1', x_2 \wedge x_2')$. We need to show that

$$[f_i(x_1, x_2) - f_i(x_1 \wedge x_1', x_2 \wedge x_2')] - [f_{i'}(x_1 \vee x_1', x_2 \vee x_2') - f_{i'}(x_1', x_2')] \leq 0.$$

Since $(x_j \wedge x_j') \leq x_j \leq (x_j \vee x_j')$ for $j = 1, 2$ and $i < i'$, we can write

$$\begin{aligned} & f_i(x_1, x_2) - f_i(x_1 \wedge x_1', x_2 \wedge x_2') \\ = & [f_i(x_1, x_2) - f_i(x_1 \wedge x_1', x_2)] + [f_i(x_1 \wedge x_1', x_2) - f_i(x_1 \wedge x_1', x_2 \wedge x_2')] \\ \leq & [f_{i'}(x_1, x_2) - f_{i'}(x_1 \wedge x_1', x_2)] + [f_{i'}(x_1 \wedge x_1', x_2) - f_{i'}(x_1 \wedge x_1', x_2 \wedge x_2')], \qquad \text{by } \textbf{(C2)} \text{ and } \textbf{(C3)} \\ = & f_{i'}(x_1, x_2) - f_{i'}(x_1 \wedge x_1', x_2 \wedge x_2') \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} & [f_{i'}(x_1, x_2) - f_{i'}(x_1 \wedge x_1', x_2 \wedge x_2')] - [f_{i'}(x_1 \vee x_1', x_2 \vee x_2') - f_{i'}(x_1', x_2')] \\ = & [f_{i'}(x_1, x_2) + f_{i'}(x_1', x_2')] - [f_{i'}(x_1 \wedge x_1', x_2 \wedge x_2') + f_{i'}(x_1 \vee x_1', x_2 \vee x_2')] \\ \leq & 0, \qquad \text{by } \textbf{(C1)} \text{ and Definition A2.} \end{aligned} \tag{A.2}$$

It follows, from (A.1) and (A.2), that

$$f_i(x_1, x_2) + f_{i'}(x_1', x_2') - f_{i'}(x_1 \vee x_1', x_2 \vee x_2') - f_i(x_1 \wedge x_1', x_2 \wedge x_2') \leq 0,$$

that is, $F(i, x_1, x_2)$ is supermodular. □

Corollary A1. *Let* $f_i, i = 1, \cdots, n : \Re^2 \to \Re$ *and* $g = \max\{f_1, f_2, ..., f_n\}$. *Then, g is supermodular if each $f_i, i = 1, \cdots, n$, satisfies conditions* **(C1), (C2)**, *and* **(C3)** *given in Lemma A2.*

Proof: Letting $F(i, x_1, x_2) = f_i(x_1, x_2), i = 1, \cdots, n$, we have $g(x_1, x_2) = \max_{i=1,2,\cdots,n} F(i, x_2, x_2)$, and the result directly follows from Lemma A2 and Theorem 2.7.6 in Topkis (1998). □

Our analysis also utilizes properties of the *Definite Quadratic Programming Problem*, whose definition is provided below for completeness.

Definition A3. *(Bazaraa et al. (1993)) A Quadratic Programming Problem,* $\{\min \boldsymbol{c}^T\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^T\boldsymbol{H}\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$, *where* $\boldsymbol{c}$ *and* $\boldsymbol{x}$ *are* $n \times 1$ *vectors,* $\boldsymbol{b}$ *is an* $m \times 1$ *vector,* $\boldsymbol{A}$ *is an* $m \times n$ *matrix, and* $\boldsymbol{H}$ *is an* $n \times n$ *symmetric matrix, is called a Definite Quadratic Programming Problem (DQP) when* $\boldsymbol{H}$ *is positive definite.*

A.2 Definitions and Preliminaries from Linear Algebra

Definition B4. *(Horn and Johnson (1991) p. 114) A square matrix* $\boldsymbol{X}$ *is called an M-matrix if all off-diagonal elements are less than or equal to zero, and it satisfies any one of the following equivalent conditions.*

1. *All principal minors of* $\boldsymbol{X}$ *are positive.*
2. *The leading principal minors of* $\boldsymbol{X}$ *are positive.*
3. $\boldsymbol{X}$ *is non-singular and the inverse of* $\boldsymbol{X}$ *is non-negative.*

A symmetric M-matrix is called a Stieltjes matrix (Varga (2000)).

(1) If square matrices $\boldsymbol{A}$ and $\boldsymbol{D}$ are both invertible, then

$$\begin{aligned}\begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{C} & \boldsymbol{D}\end{bmatrix}^{-1} &= \begin{bmatrix}(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C})^{-1} & -(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C})^{-1}\boldsymbol{B}\boldsymbol{D}^{-1}\\ -\boldsymbol{D}^{-1}\boldsymbol{C}(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C})^{-1} & \boldsymbol{D}^{-1}+\boldsymbol{D}^{-1}\boldsymbol{C}(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C})^{-1}\boldsymbol{B}\boldsymbol{D}^{-1}\end{bmatrix}\\ &= \begin{bmatrix}\boldsymbol{A}^{-1}+\boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D}-\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{C}\boldsymbol{A}^{-1} & -\boldsymbol{A}^{-1}\boldsymbol{B}(\boldsymbol{D}-\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\\ -(\boldsymbol{D}-\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\boldsymbol{C}\boldsymbol{A}^{-1} & (\boldsymbol{D}-\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B})^{-1}\end{bmatrix}\end{aligned} \tag{A.3}$$

(2) Cramer's rule: Given a system of linear equations, $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A}$ is an invertible square matrix, $\boldsymbol{x} = (x_1, x_2, ..., x_n)^T$ is the column vector of the variables, and $\boldsymbol{b}$ is a column vector,

$$x_i = \frac{\det\left(\hat{\boldsymbol{A}}(i)\right)}{\det(\boldsymbol{A})} \qquad i = 1, \cdots, n, \tag{A.4}$$

where $\hat{\boldsymbol{A}}(i)$ is the matrix formed by replacing the i-th column of $\boldsymbol{A}$ by column vector $\boldsymbol{b}$.
(3) Laplace expansion (cofactor expansion): The determinant of matrix $\boldsymbol{A}_{(n,n)}$ can be expanded along any row i: $\det(\boldsymbol{A}) = \sum_{k=1}^{n} A_{ik}C_{ik}$, or along any column j: $\det(\boldsymbol{A}) = \sum_{k=1}^{n} A_{kj}C_{kj}$, where C_{kl} is the cofactor of element A_{kl} of $\boldsymbol{A}$.
(4) If $\boldsymbol{A}$ is an $n \times n$ invertible matrix, then

$$\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}. \tag{A.5}$$

Appendix B

Some Results, Tables, and Proofs

B.1 Appendix for Chapter 2

B.1.1 Properties A1-A2 and Proofs

Property A1. *Matrices $\boldsymbol{S}$ and $\boldsymbol{W}$ exhibit the following properties:*

1. *$\boldsymbol{S}$ is a Stieltjes matrix.*

2. *For the $\boldsymbol{W}$ matrix and any distinct triplet (i, j, k), we must have $W_{kk}W_{ij} - W_{kj}W_{ik} \geq 0$ and $W_{kk}W_{ii} - W_{ki}W_{ik} \geq 0$.*

3. *Let $\boldsymbol{G}(l) \equiv \boldsymbol{W}^{-1}_{(1,l)(1,l)}, l = 1, \cdots, n$. Then, we must have $G_{ii}(l) \geq 0$, $G_{ij}(l) \leq 0$, and $\det(\boldsymbol{G}(l)) \geq 0, i, j = 1, \cdots, l,\ i \neq j$.*

Proof: Part 1 follows from the definition of a Stieltjes matrix since matrix $\boldsymbol{S}$ satisfies the second condition of Definition B4 and is symmetric. Part 2 directly follows from Theorem 3.9 of McDonald and Tsatsomeros (1995) and Proposition 1 of Johnson and Smith (2007). For part 3, $\boldsymbol{W}_{(1,l)(1,l)}$, $l = 1, \cdots, n-1$, is the principal submatrix of $\boldsymbol{W}$. Thus, $\boldsymbol{G}(l)$, $l = 1, \cdots, n-1$, is an M-matrix from Corollary 3 of Johnson (1982). Therefore, we have $G_{ii}(l) \geq 0$, $G_{ij}(l) \leq 0$, and $\det(\boldsymbol{G}(l)) \geq 0, i, j = 1, \cdots, l,\ i \neq j$. $\square$

Property A2. $\Pi(\boldsymbol{K},\boldsymbol{\gamma})$ *is strictly jointly concave in* $\boldsymbol{q}$*. Therefore, the KKT first-order conditions are necessary and sufficient for the optimality of* $\boldsymbol{q^*}$ *for Problem* $\boldsymbol{P_2}$*, and* $\boldsymbol{q^*}$ *is unique.*

Proof: The strict joint concavity of $\Pi(\boldsymbol{K},\boldsymbol{\gamma})$ in $\boldsymbol{q}$ trivially follows because $\boldsymbol{W}$ is strictly positive definite (see Section 2.2.1). Then, since constraints (2.10) and (2.11) are linear, the KKT first-order conditions are necessary and sufficient for optimality, and the optimal solution to Problem $\boldsymbol{P_2}$ is unique (Bazaraa et al. (1993)). $\square$

B.1.2 Proof of Lemma 2:

It directly follows from Property A2 and Proposition 29 in Liu (2009). $\square$

B.1.3 Proof of Lemma 3:

The result follows from Theorem 3.4 in Fiacco and Ishizuka (1990) because $\Pi(\boldsymbol{K},\boldsymbol{\gamma})$ is linear in $\boldsymbol{\gamma}$. $\square$

B.1.4 Lemma A3 and its Proof

Lemma A3. *In the* **DN***, in every feasible domain* $\Omega_j, j \in \Psi$*, we must have that, there does not exist any feasible domain such that* $\mu_j^l > 0$*, for any* $l \in \{1, \cdots, n\}$*.*

Proof: In the **DN**, consider any feasible domain $\Omega_j, j \in \Psi$, for Problem $\boldsymbol{P_2}$, characterized by the dual variables $\mu_j^1 > 0, \mu_j^2 > 0, ..., \mu_j^l > 0$ and $\mu_j^{l+1} = \mu_j^{l+2} =, ..., \mu_j^n = 0$, for some $l : 0 \leq l \leq n$. This implies an optimal solution of $(\boldsymbol{p}^*, \boldsymbol{q}^*)$ in this domain, with $q_1^* = q_2^* = ... = q_l^* = 0$, and $q_{l+1}^* \geq 0, q_{l+2}^* \geq 0, ..., q_n^* \geq 0$ (by Eq. (2.19)). Further, from Lemma 1, $q_i^* = d_i^* = \gamma_i - p_i^* + \sum_{k=1,\cdots,n,k\neq i} v_{ik} p_k^*$, $i = 1, \cdots, n$. Now consider an alternate solution, $(\boldsymbol{p'}, \boldsymbol{q'})$, still satisfying Lemma 1, with $p_i' = p_i^* - \triangle$, $i = 1, 2, ..., l$, and $p_i' = p_i^*$, $i = l+1, ..., n$, for some constant $\triangle > 0$. Then, for any $i \in \{1, 2, ..., l\}$,

$$\begin{aligned}
d_i' = q_i' &= \gamma_i - p_i' + \sum_{k=1,\cdots,n,k\neq i} v_{ik} p_k' \\
&= \gamma_i - (p_i^* - \triangle) + \sum_{k=1,2,...,l,k\neq i} v_{ik}\left(p_k^* - \triangle\right) + \sum_{k=l+1,l+2,...,n} v_{ik} p_k^* \\
&= \triangle - \sum_{k=1,2,...,l,k\neq i} v_{ik} \triangle \;\; (>0), \\
&\Rightarrow \mu_i'^j = 0.
\end{aligned}$$

On the other hand, for any $i \in \{l+1, ..., n\}$,

$$\begin{aligned}
d_i' = q_i' &= \gamma_i - p_i' + \sum_{k=1,\cdots,n,k\neq i} v_{ik} p_k' \\
&= \gamma_i - p_i^* + \sum_{k=1,2,...,l} v_{ik}\left(p_k^* - \triangle\right) + \sum_{k=l+1,l+2,...,n,k\neq i} v_{ik} p_k^* \\
&= d_i^* - \sum_{k=1,2,...,l} v_{ik} \triangle\,.
\end{aligned}$$

Observe that $p_i' = p_i^* \geq 0$, $i = l+1, \cdots, n$. In addition, for sufficiently small $\triangle > 0$, $p_i' = p_i^* - \triangle \geq 0$, $i = 1, \cdots, l$, $0 < q_i' = \triangle(1 - \sum_{k=1,2,...,l,k\neq i} v_{ik}) < K_i$, $i = 1, \cdots, l$, and $0 < q_i' < q_i^* - \sum_{k=1,2,...,l} v_{ik}\triangle < K_i$ ($\Rightarrow \mu_i'^j = 0$), $i = l+1, \cdots, n$. Hence $(\boldsymbol{p'}, \boldsymbol{q'})$ is a feasible solution to Problem $\boldsymbol{P_2}$, with a corresponding $\boldsymbol{\mu'} = \mathbf{0}$. In addition

$$
\begin{aligned}
&\Pi(\boldsymbol{p'}, \boldsymbol{q'}) - \Pi(\boldsymbol{p^*}, \boldsymbol{q^*}) \\
=&\sum_{i=1}^{n} \left(d_i' p_i' - d_i^* p_i^*\right) \\
=&\sum_{i=1}^{l} \left(\left(\triangle - \sum_{k=1,2,\ldots,l,k\neq i} v_{ik} \triangle \right) (p_i^* - \triangle) \right) - \sum_{i=l+1}^{n} \sum_{k=1,2,\ldots,l} v_{ik} \triangle p_i^* \\
=&\triangle \sum_{i=1}^{l} \left(p_i^* - \sum_{k=1,2,\ldots,l,k\neq i} v_{ik} p_i^* - \sum_{k=l+1}^{n} v_{ki} p_k^* - \triangle \left(1 - \sum_{k=1,2,\ldots,l,k\neq i} v_{ik} \right) \right) \\
=&\triangle \sum_{i=1}^{l} \left(p_i^* - \sum_{k=1,\cdots,n,k\neq i} v_{ik} p_k^* - \triangle \left(1 - \sum_{k=1,2,\ldots,l,k\neq i} v_{ik} \right) \right) \quad \text{by symmetricity of cross-price effects} \\
=&\triangle \sum_{i=1}^{l} \left(\gamma_i - \triangle \left(1 - \sum_{k=1,2,\ldots,l,k\neq i} v_{ik} \right) \right).
\end{aligned}
$$

Clearly, for sufficiently small $\triangle > 0$, we have that $\sum_{i=1}^{n} \left(d_i' p_i' - d_i^* p_i^*\right) > 0$, which contradicts with the optimality of $(\boldsymbol{p^*}, \boldsymbol{q^*})$. This completes the proof. $\square$

B.1.5 Lipschitz Continuity of $\Pi^*(\boldsymbol{K}, \boldsymbol{\gamma})$

We first establish that the capacity dual variables are bounded.

Remark A1. *An immediate consequence of Assumption 2 is that the capacity dual variable vector, $\boldsymbol{\lambda}$, is bounded. This follows because one can show that as long as Assumption 2 holds in the* **FN**, **DN**, *and* **SN**, *an optimal solution to Problem $\boldsymbol{P_2}$ satisfies the Mangasarian-Fromovitz Constraint Qualification (MFCQ), and in any nonlinear programming problem, the MFCQ holds if and only if the set of KKT multiplier vectors is compact (Gauvin (1977)).*

Note that function $\Pi^*(\boldsymbol{K}, \boldsymbol{\gamma})$ has a finite expected value (this follows because $\boldsymbol{\Gamma}$ has finite expectation and $\boldsymbol{c} > \mathbf{0}$ by assumption, and hence, the optimal stage 1 capacity solution is finite); that is, it is an integrable function of $\boldsymbol{\gamma}$ for every finite $\boldsymbol{K}$. Furthermore, from Property A3, $\Pi^*(\boldsymbol{K}, \boldsymbol{\gamma})$ is differentiable with respect to $K_l, l = 1, \cdots, m$, with bounded derivatives, $\partial\Pi^*(\boldsymbol{K},\boldsymbol{\gamma})/\partial K_l = \lambda^l(\boldsymbol{\gamma})$, and in addition, $\Pi^*(\boldsymbol{K})$ is continuous (Lemma 4). Therefore, $\Pi^*(\boldsymbol{K},\boldsymbol{\gamma})$ satisfies the Lipschitz continuity condition of order one, and it follows that,

$$\partial E_{\boldsymbol{\Gamma}}[\Pi^*(\boldsymbol{K},\boldsymbol{\Gamma})]/\partial K_l = E_{\boldsymbol{\Gamma}}[\partial\Pi^*(\boldsymbol{K},\boldsymbol{\Gamma})/\partial K_l],\ l = 1,\cdots,m, \tag{B.1}$$

that is, differentiation and expectation can be interchanged (e.g., Glasserman (1994), Lemma 6.3.1, p. 245, see also Harrison and Van Mieghem (1999), Appendix A, and Van Mieghem (2007) for detailed arguments on Lipschitz continuity).

B.1.6 Proof of Theorem 1:

From Lemma 3, $V(\boldsymbol{K}^*,\boldsymbol{\gamma}) = \Pi^*(\boldsymbol{K}^*,\boldsymbol{\gamma}) - \boldsymbol{c}^T\boldsymbol{K^*}$ is jointly convex in $\boldsymbol{\gamma}$ (see (2.4)). Consider $\boldsymbol{\Gamma} \leq_{\text{cx}} \overline{\boldsymbol{\Gamma}}$, with respective optimal capacities $\boldsymbol{K}^*$ and $\overline{\boldsymbol{K}}^*$. Then, since $\boldsymbol{\Gamma} \leq_{\text{cx}} \overline{\boldsymbol{\Gamma}}$, we have $E_{\boldsymbol{\Gamma}}[V(\boldsymbol{K}^*,\boldsymbol{\Gamma})] \leq E_{\overline{\boldsymbol{\Gamma}}}[V(\boldsymbol{K}^*,\overline{\boldsymbol{\Gamma}})] \leq E_{\overline{\boldsymbol{\Gamma}}}[V(\overline{\boldsymbol{K}}^*,\overline{\boldsymbol{\Gamma}})]$, where the first inequality follows from the definition of convex order (see Definition A1), and the second inequality follows by the optimality of $\overline{\boldsymbol{K}}^*$ for $\overline{\boldsymbol{\Gamma}}$. $\square$

B.1.7 Proof of Remark 7:

For the **FN** and **DN**, in any optimal solution $\boldsymbol{q^*}$, the gradients of constraint sets (2.10) and (2.11) are given by $[\mathbf{1}_n, -\boldsymbol{I}_{(n,n)}]^T$ and $[\boldsymbol{I}_{(n,n)}, -\boldsymbol{I}_{(n,n)}]^T$, respectively. It is easy to show that both have a rank of n. In addition, observe that in any optimal solution to the **FN** or **DN**, *at most* n of constraints in (2.10) and (2.11) can be binding, and hence, Assumption 2 automatically holds. In the **DN**, where (2.10) and (2.11) reduce to $q_i \leq K_i^*, i = 1,\cdots,n$, and $-q_i \leq 0, i = 1,\cdots,n$, this follows because if $q_i = K_i^*(> 0)$ (we assume, without loss of generality, that $K_i^* > 0, i = 1,\cdots,n$), for any $i, i = 1,\cdots,n$, then the $-q_i \leq 0$ constraint cannot be binding. Similarly, for the **FN**, where (2.10) and (2.11) reduce to $\sum_{i=1}^n q_i \leq K_f^*$ and $-q_i \leq 0, i = 1,\cdots,n$, this follows because if $\sum_{i=1}^n q_i = K_f^*(> 0)$, then at least one q_i must be positive, and hence, not all $-q_i \leq 0,\ i, i = 1,\cdots,n$, constraints can be binding.

On the other hand, for the **SN**, where (2.10) and (2.11) reduce to $q_i \leq K_i^*(> 0), i =$

$1, \cdots, n$, $\sum_{i=1}^n q_i = K_f^*(>0)$, and $-q_i \leq 0, i = 1, \cdots, n$, their gradients are given by $[\boldsymbol{I}_{(n,n)}, \mathbf{1}_n, -\boldsymbol{I}_{(n,n)}]^T$. Then, the gradients of the binding constraints are linearly dependent if and only if *all* the following conditions hold:

(1) Constraint $\sum_{i=1}^n q_i \leq K_f^*$ is binding;

(2) For each of $i = 1, 2, \cdots, n$, either constraint $q_i \leq K_i^*$ is binding or constraint $-q_i \leq 0$ is binding.

These, in turn, imply that $\sum_{i \in I_R} K_i^* = K_f^*$, where I_R is the set of dedicated resources $i = 1, \cdots, n$, for which constraint $q_i \leq K_i^*$ is binding; that is, the corresponding $\boldsymbol{K}^*$ in the **SN** is either partially or fully "balanced" (as defined in the remark). Thus, for all capacity vectors that are not partially or fully balanced, the gradients of the binding constraints in a feasible solution to Problem $\boldsymbol{P_2}$ cannot be linearly dependent; hence Assumption 2 will be satisfied. □

B.1.8 Property A3

The following result directly follows from Theorem 3.4.1 and Corollary 3.4.4 in Fiacco (1983).

Property A3. $\Pi^*(\boldsymbol{K}, \boldsymbol{\gamma})$ *is differentiable with respect to* $K_l, l = 1, \cdots, m$, *with* $\partial \Pi^*(\boldsymbol{K}, \boldsymbol{\gamma})/\partial K_l = \lambda^l(\boldsymbol{\gamma})$.[1]

B.1.9 Proof of Lemma 4:

1. The first part directly follows from Lemma 33 in Liu (2009)

2. <u>**FN**</u>: It follows from Lemma 33 in Liu (2009)

   <u>**DN**</u>: Consider domain Ω_i corresponding to $\lambda_i^1 = 0, \lambda_i^2 = 0 = ... = \lambda_i^j = 0, \lambda_i^{j+1} > 0, \lambda_i^{j+2} > 0, ..., \lambda_i^n > 0$ for some $j : 0 \leq j \leq n$. If $j = n$, then $\lambda_i^1 = \lambda_i^2 = \cdots = \lambda_i^n = 0$.

[1] Technically, $\Pi^*(\boldsymbol{K}, \boldsymbol{\gamma})$ is differentiable with respect to $K_l, l = 1, \cdots, m$, everywhere except on a possible set L of Lebesgue measure zero. However, since $\boldsymbol{\Gamma}$ is a continuous random vector, L has $P-$measure zero (Van Mieghem (2007)).

If $j = 0$, then $\lambda_i^1 > 0, \lambda_i^2 > 0, ..., \lambda_i^n > 0$, which imply, by KKT conditions (2.20) and (2.21), that $q_l^* = K_l$, $l = 1, \cdots, n$. Then, (2.19) reduces to $\boldsymbol{W\gamma - 2WK - \lambda = 0}$, and hence, $\lambda_i^l = \boldsymbol{W}_l\boldsymbol{\gamma} - 2\boldsymbol{W}_l\boldsymbol{K}$, for $l = 1, \cdots, n$. Next consider the case where $0 < j < n$. Then, for any $1 \leq l \leq j$, $\lambda_i^l = 0$, by construction. For any $j+1 \leq l \leq n$, consider the following system of linear equations:

$$
\begin{aligned}
2W_{11}q_1^* + 2W_{12}q_2^* + ... + 2W_{1j}q_j^* + 2W_{1l}K_l &= W_1\boldsymbol{\gamma} - 2\sum_{k=j+1}^{n} W_{1k}K_k + 2W_{1l}K_l \\
2W_{21}q_1^* + 2W_{22}q_2^* + ... + 2W_{2j}q_j^* + 2W_{2l}K_l &= W_2\boldsymbol{\gamma} - 2\sum_{k=j+1}^{n} W_{2k}K_k + 2W_{2l}K_l \\
\vdots \quad & \quad \vdots \\
2W_{j1}q_1^* + 2W_{j2}q_2^* + ... + 2W_{jj}q_j^* + 2W_{jl}K_l &= W_j\boldsymbol{\gamma} - 2\sum_{k=j+1}^{n} W_{jk}K_k + 2W_{jl}K_l \\
2W_{l1}q_1^* + 2W_{l2}q_2^* + ... + 2W_{lj}q_j^* + 2W_{ll}K_l &= W_l\boldsymbol{\gamma} - 2\sum_{k=j+1}^{n} W_{lk}K_k + 2W_{ll}K_l - \lambda_i^l,
\end{aligned}
$$

which consists of $j+1$ linear equations, with the variable vector, $(q_1^*, q_2^*, ..., q_n^*, K_l)^T$. Let $\tilde{b}$ denote the right-hand-side vector. By Cramer's rule (see Eq. (A.4)), we have

$$
2K_l = \frac{\det\left(\widehat{\boldsymbol{W}}_l\right)}{\det\left(\widetilde{\boldsymbol{W}}_l\right)}, \text{ where}
$$

$$
\widetilde{\boldsymbol{W}}_l = \begin{bmatrix} \boldsymbol{W}_{(1,j)(1,j)} & \begin{bmatrix} W_{1l} \\ W_{2l} \\ \vdots \\ W_{jl} \end{bmatrix} \\ \begin{bmatrix} W_{l1} & W_{l2} & \cdots & W_{lj} \end{bmatrix} & W_{ll} \end{bmatrix},
$$

and $\widehat{\boldsymbol{W}}_l$ is formed by replacing the last column of $\widetilde{\boldsymbol{W}}_l$ with $\tilde{b}$.

$$\det\left(\widehat{\boldsymbol{W}}_l\right) = \det\begin{bmatrix} W_{11} & \cdots & W_{jj} & W_1\boldsymbol{\gamma} - 2\sum_{k=j+1}^n W_{1k}K_k + 2W_{1l}K_l \\ W_{21} & & W_{2j} & W_2\boldsymbol{\gamma} - 2\sum_{k=j+1}^n W_{2k}K_k + 2W_{2l}K_l \\ \vdots & & \vdots & \vdots \\ W_{j1} & & W_{jj} & W_j\boldsymbol{\gamma} - 2\sum_{k=j+1}^n W_{jk}K_k + 2W_{jl}K_l \\ W_{l1} & \cdots & W_{lj} & W_l\boldsymbol{\gamma} - 2\sum_{k=j+1}^n W_{lk}K_k + 2W_{ll}K_l - \lambda_i^l \end{bmatrix}$$

$$= \sum_{g=1}^{j}\left(\left(\boldsymbol{W}_g\boldsymbol{\gamma} - 2\sum_{k=j+1}^{n} W_{gk}K_k + 2W_{gl}K_l\right)C_{g,j+1}\left(\widetilde{\boldsymbol{W}}_l\right)\right)$$
$$+C_{j+1,j+1}\left(\widetilde{\boldsymbol{W}}_l\right)\left(\boldsymbol{W}_l\boldsymbol{\gamma} - 2\sum_{k=j+1}^{n} W_{lk}K_k + 2W_{ll}K_l - \lambda_i^l\right).$$

Thus,

$$2K_l = \frac{\sum_{g=1}^{j}\left(\left(\boldsymbol{W}_g\boldsymbol{\gamma} - 2\sum_{k=j+1}^{n} W_{gk}K_k + 2W_{gl}K_l\right)C_{g,j+1}\left(\widetilde{\boldsymbol{W}}_l\right)\right)}{\det\left(\widetilde{\boldsymbol{W}}_l\right)} + \frac{C_{j+1,j+1}\left(\widetilde{\boldsymbol{W}}_l\right)\left(\boldsymbol{W}_l\boldsymbol{\gamma} - 2\sum_{k=j+1}^{n} W_{lk}K_k + 2W_{ll}K_l - \lambda_i^l\right)}{\det\left(\widetilde{\boldsymbol{W}}_l\right)},$$

which implies that

$$\lambda_i^l = \frac{\sum_{g=1}^{j}\left(\left(\boldsymbol{W}_g\boldsymbol{\gamma} - 2\sum_{k=j+1}^{n} W_{gk}K_k + 2W_{gl}K_l\right)C_{g,j+1}\left(\widetilde{\boldsymbol{W}}_l\right)\right)}{C_{j+1,j+1}\left(\widetilde{\boldsymbol{W}}_l\right)} + \frac{C_{j+1,j+1}\left(\widetilde{\boldsymbol{W}}_l\right)\left(\boldsymbol{W}_l\boldsymbol{\gamma} - 2\sum_{k=j+1}^{n} W_{lk}K_k + 2W_{ll}K_l\right)}{C_{j+1,j+1}\left(\widetilde{\boldsymbol{W}}_l\right)} - \frac{2\det\left(\widetilde{\boldsymbol{W}}_l\right)K_l}{C_{j+1,j+1}\left(\widetilde{\boldsymbol{W}}_l\right)},$$

and the result follows. $\square$

B.1.10 Proof of Lemma 5:

Proof of Part 1: It directly follows from Proposition 34 in Liu (2009).

Proof of Part 2:

Consider two adjacent domains Ω_i and Ω_j given by the following:

$$\begin{aligned}
\Omega_i &: \lambda_i^1 = 0, ..., \lambda_i^g = 0, \lambda_i^{g+1} > 0, ..., \lambda_i^n > 0 \text{ and } \mu^g = 0, g = 1, ..., n. \\
\Omega_j &: \lambda_j^1 = 0, ..., \lambda_j^g = 0, \lambda_j^{g+1} = 0, \lambda_j^{g+2} > 0..., \lambda_j^n > 0 \text{ and } \mu^g = 0, g = 1, ..., n.
\end{aligned}$$

Note that from Ω_i to Ω_j, only one dual variable λ^{g+1}, corresponding to product $g+1$, changes from a value of strictly positive to zero, for $1 \leq g+1 \leq n$. For any given $\boldsymbol{\gamma}$ in Ω_i and $l = 1, ..., g+1$, we always have that $\lambda_i^l - \lambda_j^l \geq 0$. As for the case where $l = g+2, ..., n$, it is sufficient to show that $\lambda_i^{g+2} - \lambda_j^{g+2} \geq 0$ for any given $\boldsymbol{\gamma}$ in Ω_i.

$$\lambda_j^{g+2} = \frac{-2K_{g+2}\det(\boldsymbol{W}_{(1,g+2)(1,g+2)}) + \sum_{k=1}^{g+2}\Big(C_{k,g+2}(\boldsymbol{W}_{(1,g+2)(1,g+2)})(\boldsymbol{W}_k\gamma - 2\sum_{m=g+3}^n W_{km}K_m)\Big)}{C_{g+2,g+2}(\boldsymbol{W}_{(1,g+2)(1,g+2)})}.$$

On the other hand, in Ω_i, we know that

$$\begin{bmatrix} W_{11} & \cdots & W_{1,g+1} & W_{1,g+2} \\ \vdots & & \vdots & W_{2,g+2} \\ W_{g+1,1} & \cdots & W_{g+1} & \vdots \\ W_{g+2,1} & W_{g+2,2} & \cdots & W_{g+2,g+2} \end{bmatrix} \begin{bmatrix} 2q_1 \\ \vdots \\ 2q_g \\ 2K_{g+1} \\ 2K_{g+2} \end{bmatrix} = \begin{bmatrix} \boldsymbol{W}_1\gamma - 2\sum_{m=g+3}^n W_{1m}K_m \\ \vdots \\ \boldsymbol{W}_g\gamma - 2\sum_{m=g+3}^n W_{gm}K_m \\ \boldsymbol{W}_{g+1}\gamma - 2\sum_{m=g+3}^n W_{g+1,m}K_m - \lambda_i^{g+1} \\ \boldsymbol{W}_{g+2}\gamma - 2\sum_{m=g+3}^n W_{g+2,m}K_m - \lambda_i^{g+2} \end{bmatrix}$$

Thus,

$$\lambda_i^{g+2} = \frac{-2K_{g+2}\det(\boldsymbol{W}_{(1,g+2)(1,g+2)}) + \sum_{k=1}^{g+2}\Big(C_{k,g+2}(\boldsymbol{W}_{(1,g+2)(1,g+2)})(\boldsymbol{W}_k\gamma - 2\sum_{m=g+3}^{n} W_{km}K_m)\Big)}{C_{g+2,g+2}(\boldsymbol{W}_{(1,g+2)(1,g+2)})} - \frac{C_{g+1,g+2}(\boldsymbol{W}_{(1,g+2)(1,g+2)})}{C_{g+2,g+2}(\boldsymbol{W}_{(1,g+2)(1,g+2)})}\lambda_i^{g+1}.$$

For any given $\boldsymbol{\gamma}$ in Ω_i,

$$\lambda_i^{g+2} - \lambda_i^{g+2} = -\frac{C_{g+1,g+2}(\boldsymbol{W}_{(1,g+2)(1,g+2)})}{C_{g+2,g+2}(\boldsymbol{W}_{(1,g+2)(1,g+2)})}\lambda_i^{g+1}.$$

Let $\boldsymbol{G}(g+2) = \boldsymbol{W}^{-1}_{(1,g+2)(1,g+2)}$. Then $C_{g+1,g+2}(\boldsymbol{W}_{(1,g+2)(1,g+2)}) = G_{g+2,g+1}(g+2)\det(\boldsymbol{W}_{(1,g+2)(1,g+2)})$ and $C_{g+2,g+2}(\boldsymbol{W}_{(1,g+2)(1,g+2)}) = G_{g+2,g+2}(g+2)\det(\boldsymbol{W}_{(1,g+2)(1,g+2)})$. By Property A1, $G(g+2)_{g+2,g+1} \le 0$, $G(g+2)_{g+2,g+2} \ge 0$, and $\det(\boldsymbol{W}_{(1,g+2)(1,g+2)}) \ge 0$. Hence $-\frac{C_{g+1,g+2}(\boldsymbol{W}_{(1,g+2)(1,g+2)})}{C_{g+2,g+2}(\boldsymbol{W}_{(1,g+2)(1,g+2)})} \ge 0$. In addition, $\lambda_i^{g+1} \ge 0$. Thus, for any given γ in Ω_i, we have $\lambda_i^{g+2} - \lambda_i^{g+2} \ge 0$. This completes the proof. □

B.1.11 Proof of Lemma 6:

This proof is a revised version of the proof of Proposition 35 in Liu (2009). We correct some typos and provide the new version here for completeness.

To prove the first part, consider the **FN**. If Ω_i and Ω_j are adjacent, then the result holds by Lemma 5. Next, consider the case where Ω_i and Ω_j are not adjacent. $\forall \boldsymbol{\gamma}_0 \in \Omega_i$, arbitrarily pick an $\overline{\boldsymbol{\gamma_0}} \in \Omega_j$, and consider the line segment $L = \{\boldsymbol{\gamma} : \boldsymbol{\gamma} = \overline{\boldsymbol{\gamma_0}} + t(\boldsymbol{\gamma_0} - \overline{\boldsymbol{\gamma_0}}), 0 \le t \le 1\}$. Suppose line segment L, between points $\boldsymbol{\gamma_0}$ and $\overline{\boldsymbol{\gamma_0}}$, only crosses through one other domain (in addition to domains Ω_i and Ω_j), which we denote by Ω_k. Then Ω_k must be adjacent to both Ω_i and Ω_j. Hence, by Definition 3, there must be a hyperplane H_{ki}, which forms the boundary between domains Ω_i and Ω_k, and a hyperplane H_{jk}, which forms the boundary between domains Ω_j and Ω_k. Let $\boldsymbol{\gamma_1} = H_{jk} \cap L$ and $\boldsymbol{\gamma_2} = H_{ki} \cap L$. Then, for any $\boldsymbol{\gamma_0} \in \Omega_i$

and some linear functions $\boldsymbol{g_i}, \boldsymbol{g_j}, \boldsymbol{g_k}$,

$$\begin{aligned}
\lambda_j^f(\boldsymbol{\gamma_0}) &= \lambda_j^f(\overline{\boldsymbol{\gamma_0}}) + \boldsymbol{g_j}(\boldsymbol{\gamma_0} - \overline{\boldsymbol{\gamma_0}}), && \text{by Lemma 4} \\
&= \lambda_j^f(\overline{\boldsymbol{\gamma_0}}) + \boldsymbol{g_j}(\boldsymbol{\gamma_1} - \overline{\boldsymbol{\gamma_0}}) + \boldsymbol{g_j}(\boldsymbol{\gamma_0} - \boldsymbol{\gamma_1}) \\
&= \lambda_j^f(\boldsymbol{\gamma_1}) + \boldsymbol{g_j}(\boldsymbol{\gamma_0} - \boldsymbol{\gamma_1}) \\
&\leq \lambda_k^f(\boldsymbol{\gamma_1}) + \boldsymbol{g_k}(\boldsymbol{\gamma_0} - \boldsymbol{\gamma_1}), && \text{by Lemma 5} \\
&= \lambda_k^f(\boldsymbol{\gamma_2}) + \boldsymbol{g_k}(\boldsymbol{\gamma_0} - \boldsymbol{\gamma_2}) \\
&\leq \lambda_i^f(\boldsymbol{\gamma_2}) + \boldsymbol{g_i}(\boldsymbol{\gamma_0} - \boldsymbol{\gamma_2}), && \text{by Lemma 5} \\
&= \lambda_i^f(\boldsymbol{\gamma_0}).
\end{aligned}$$

The case where line segment L crosses through more than one domain can be proven in a similar way. The second part of the lemma, for the **DN**, can be proven in a similar way given that $\boldsymbol{\lambda}$ in the **DN** exhibits the same properties as λ^f in the **FN** used in the proof (i.e., those established in Lemmas 4 and 5). This completes the proof. $\square$

B.1.12 Proof of Lemma 7:

It follows similarly to the proof of Proposition 36 in Liu (2009). $\square$

B.1.13 Proof of Lemma 8:

It follows directly from part 2 of Remark 7. $\square$

B.1.14 Lemma A4 and its Proof

Lemma A4. *In the* **DN**, $E_{\boldsymbol{\Gamma}}\left[\lambda^l(\boldsymbol{K}, \boldsymbol{\gamma})\right]$, $l = 1, \cdots, n$, *is decreasing in any* $K_j, j = 1, \cdots, n$.

Proof: Clearly, $E_{\boldsymbol{\Gamma}}\left[\lambda^l(\boldsymbol{K}, \boldsymbol{\Gamma})\right]$, $l = 1, \cdots, n$, is decreasing in K_l by the strict concavity of $E_{\boldsymbol{\Gamma}}[\Pi^*(\boldsymbol{K})]$ (Lemma 2). To show that $E_{\boldsymbol{\Gamma}}\left[\lambda^l(\boldsymbol{K}, \boldsymbol{\Gamma})\right]$, $l = 1, \cdots, n$, is decreasing in K_j, for any $j \neq l, j = 1, \cdots, n$, we first show that $\lambda^l(\boldsymbol{K}, \boldsymbol{\gamma})$ is Lipschitz continuous of order one. Clearly, the derivatives, $\partial\lambda^l(\boldsymbol{K},\boldsymbol{\gamma})/\partial K_j$, $j = 1,\cdots,n$, are bounded since $\lambda_i^l(\boldsymbol{K},\boldsymbol{\gamma})$ is linear in $\boldsymbol{K}$ in any feasible domain $\Omega_i, i \in \Psi$. In addition, in the **DN**, the dual solution, $\lambda^l(\boldsymbol{K},\boldsymbol{\gamma})$, corresponding to every primal solution, is unique. $\lambda^l(\boldsymbol{K},\boldsymbol{\gamma}) = \boldsymbol{W}\boldsymbol{\gamma} - 2\boldsymbol{W}\boldsymbol{q}^*(\boldsymbol{K},\boldsymbol{\gamma})$ by Eq. (2.16) and Lemma A3. Since $\boldsymbol{W}$ is positive definite, the primal solution, $\boldsymbol{q}^*(\boldsymbol{K},\boldsymbol{\gamma})$, is continuous in $\boldsymbol{K}$ by Corollary 3.1 in Lee et al. (2006).[2] Hence, $\lambda^l(\boldsymbol{K},\boldsymbol{\gamma})$ is also continuous is $\boldsymbol{K}$. Then we have that $\lambda^l(\boldsymbol{K},\boldsymbol{\gamma})$ is Lipschitz continuous of order one. Hence, we can write $\partial E_{\boldsymbol{\Gamma}}\left[\lambda^l(\boldsymbol{K},\boldsymbol{\Gamma})\right]/\partial K_j = E_{\boldsymbol{\Gamma}}\left[\partial\lambda^l(\boldsymbol{K},\boldsymbol{\Gamma})/\partial K_j\right]$, and it is sufficient to show that $\partial\lambda^l(\boldsymbol{K},\boldsymbol{\gamma})/\partial K_j \leq 0$ in every feasible domain $\Omega_i, i \in \Psi$. Obviously, for any domain Ω_i with $\lambda_i^l = 0$, the result trivially follows. Next, consider a domain $\Omega_i, i \in \Psi$, with $\lambda_i^l > 0$. Suppose, in this domain, that $\lambda_i^1 = 0, ..., \lambda_i^{l-1} = 0$.[3] We derive

$$\lambda_i^l = \frac{-2K_l \det(\boldsymbol{W}_{(1,l)(1,l)}) + \sum_{g=1}^{l}\left(C_{gl}(\boldsymbol{W}_{(1,l)(1,l)})(\boldsymbol{W}_g\boldsymbol{\gamma} - 2\sum_{h=l+1}^{n} W_{gh}K_h)\right)}{C_{ll}(\boldsymbol{W}_{(1,l)(1,l)})}. \tag{B.2}$$

Note that $\frac{\partial\lambda_i^l}{\partial K_j} = 0$, for $j = 1,\cdots,l$. Next consider $\frac{\partial\lambda_i^l}{\partial K_j}$, for $j = l+1,\cdots,n$, and observe, from (B.2), that it is sufficient to show that $\frac{\partial\lambda_i^l}{\partial K_{l+1}} \leq 0$. (The result for any $j = l+2,\cdots,n$, will follow similarly.) We have

$$\frac{\partial\lambda_i^l}{\partial K_{l+1}} = -\frac{2\sum_{g=1}^{l} C_{gl}(\boldsymbol{W}_{(1,l)(1,l)})W_{g,l+1}}{C_{ll}(\boldsymbol{W}_{(1,l)(1,l)})}. \tag{B.3}$$

It follows, from Property A1, that $C_{ll}(\boldsymbol{W}_{(1,l)(1,l)}) = \det(\boldsymbol{W}_{(1,l-1)(1,l-1)})W_{ll} > 0$. Therefore, it

[2] Although we require that the perturbation of $\boldsymbol{K}$ will always result in a nonempty constraint set for Problem $\boldsymbol{P_2}$, this is always satisfied for any $\boldsymbol{K} > \boldsymbol{0}$.

[3] Observe that this is without loss of generality, as for any domain with $\lambda_i^1 = 0, ..., \lambda_i^h = 0$ for some $h : 0 \leq h \leq l-1$, we can relabel the superscripts such that $\lambda_i^l > 0, ..., \lambda_i^n > 0$ holds.

remains to show that $\sum_{g=1}^{l} C_{gl}(\boldsymbol{W}_{(1,l)(1,l)})W_{g,l+1} \geq 0$. By some algebra, we can write

$$
\begin{aligned}
\sum_{g=1}^{l} C_{gl}(\boldsymbol{W}_{(1,l)(1,l)})W_{g,l+1} &\equiv \det(\hat{\boldsymbol{W}}) \\
&= \det \begin{bmatrix} W_{11} & \cdots & W_{1,l-1} & W_{1,l+1} \\ \vdots & & \vdots & \vdots \\ W_{l-1,1} & & W_{l-1,l-1} & W_{l-1,l+1} \\ W_{l1} & & W_{l,l-1} & W_{l,l+1} \end{bmatrix}
\end{aligned}
$$

Notice that $\hat{\boldsymbol{W}}$ is essentially the matrix formed by replacing the last column of $\boldsymbol{W}_{(1,l)(1,l)}$ with $(W_{1,l+1}, W_{2,l+1}, ..., W_{l,l+1})^T$. Now consider the matrix $\boldsymbol{G}(l+1) = \boldsymbol{W}^{-1}_{(1,l+1)(1,l+1)}$. We have that

$$
\begin{aligned}
G(l+1)_{l,l+1} &= \frac{C_{l+1,l}\left(\boldsymbol{W}_{(1,l+1)(1,l+1)}\right)}{\det(\boldsymbol{W}_{(1,l+1)(1,l+1)})} \\
&= \frac{(-1)^{l+1+l}\det(\hat{\boldsymbol{W}})}{\det(\boldsymbol{W}_{(1,l+1)(1,l+1)})}, \text{ from the definition of co-factor} \\
&\leq 0, \quad \text{by Property A1.}
\end{aligned}
$$

Further, from Property A1, $\det(\boldsymbol{W}_{(1,l+1)(1,l+1)}) > 0$. Then, $(-1)^{l+1+l}\det(\hat{\boldsymbol{W}}) \leq 0$, or equivalently, $\det(\hat{\boldsymbol{W}}) \geq 0$, and hence, $\sum_{g=1}^{l} C_{gl}(\boldsymbol{W}_{(1,l)(1,l)})W_{g,l+1} \geq 0$. This then implies that $\frac{\partial \lambda_i^l}{\partial K_{l+1}} \leq 0$, reaching the desired result. Thus, $E_{\boldsymbol{\Gamma}}\left[\lambda^l(\boldsymbol{K}, \boldsymbol{\Gamma})\right], l = 1, \cdots, n$, is decreasing in $\boldsymbol{K}$. $\square$

B.1.15 Proof of Theorem 2:

Consider the **FN**. For $K_f^* = 0$, the result trivially follows. Next consider $K_f^* > 0$, which, by Lemma 2, is the unique solution to Eq. (2.12) (with the corresponding dual variable $\beta_f = 0$). We can show that $\Pi^*(\boldsymbol{K}, \boldsymbol{\gamma})$ is Lipschitz continuous of order one (see Appendix B.1.5), which allows us to interchange integration and differentiation (e.g., Glasserman (1994)). Then, from Property A3, the optimality condition in (2.12) can be written as:

$$E_{\mathbf{\Gamma}}[\partial\Pi^*(K_f,\mathbf{\Gamma})/\partial K_f]|_{K_f=K_f^*} - c_f \tag{B.4}$$

$$= E_{\mathbf{\Gamma}}[\lambda^f(K_f,\mathbf{\Gamma})]|_{K_f=K_f^*} - c_f = 0. \tag{B.5}$$

Consider that $\mathbf{\Gamma} \leq_{\text{cx}} \overline{\mathbf{\Gamma}}$, with respective optimal capacities K_f^* and $\overline{K}_f^*$. Because $\lambda^f(\boldsymbol{\gamma})$ is convex in $\boldsymbol{\gamma}$ (Lemma 7), it follows, by definition of convex order, that

$$E_{\overline{\mathbf{\Gamma}}}[\ \lambda^f(K_f,\overline{\mathbf{\Gamma}})\ ]|_{K_f=K_f^*} \geq E_{\mathbf{\Gamma}}[\ \lambda^f(K_f,\mathbf{\Gamma})\ ]|_{K_f=K_f^*} = c_f.$$

From Lemma 2, $E[\Pi^*(K_f,\mathbf{\Gamma})])$ is strictly concave in K_f, and therefore, $E_{\overline{\mathbf{\Gamma}}}[\ \lambda^f(K_f,\overline{\mathbf{\Gamma}})\ ]$ is decreasing in K_f. In addition, from (2.12), at optimality we must have $E_{\overline{\mathbf{\Gamma}}}[\ \lambda^f(K_f,\overline{\mathbf{\Gamma}})\ ]|_{K_f=\overline{K}_f^*} - c_f = 0$. Consequently, it follows that $\overline{K}_f^* \geq K_f^*$.

Next consider the **DN**. For $\mathbf{\Gamma} \leq_{\text{cx}} \overline{\mathbf{\Gamma}}$, let $\boldsymbol{K^*}$ denote the optimal capacity vector corresponding to $\mathbf{\Gamma}$. Assume, without loss of generality, that $K_l^* > 0, l = 1,\cdots,n$ ($\Rightarrow \beta_l = 0$). Then, because $\lambda^l(\boldsymbol{K},\boldsymbol{\gamma})$ is convex in $\boldsymbol{\gamma}$ (Lemma 7), similar to the first part, we can write, from (2.12),

$$\begin{aligned}
\partial E_{\overline{\mathbf{\Gamma}}}[\Pi^*\left(\boldsymbol{K},\overline{\mathbf{\Gamma}}\right)]/\partial K_l\big|_{\boldsymbol{K}=\boldsymbol{K}^*} &= E_{\overline{\mathbf{\Gamma}}}\left[\partial\Pi^*\left(\boldsymbol{K},\overline{\mathbf{\Gamma}}\right)/\partial K_l\right]\big|_{\boldsymbol{K}=\boldsymbol{K}^*} \\
&= E_{\overline{\mathbf{\Gamma}}}\left[\lambda^l\left(\boldsymbol{K},\overline{\mathbf{\Gamma}}\right)\right]\big|_{\boldsymbol{K}=\boldsymbol{K}^*} \\
&\geq E_{\mathbf{\Gamma}}\left[\lambda^l\left(\boldsymbol{K},\mathbf{\Gamma}\right)\right]\big|_{\boldsymbol{K}=\boldsymbol{K}^*} \\
&= E_{\mathbf{\Gamma}}\left[\partial\Pi^*\left(\boldsymbol{K},\mathbf{\Gamma}\right)/\partial K_l\right]\big|_{\boldsymbol{K}=\boldsymbol{K}^*} \\
&= c_l, \quad l = 1,\cdots,n.
\end{aligned} \tag{B.6}$$

From Lemma A4, $E_{\mathbf{\Gamma}}\left[\lambda^l(\boldsymbol{K},\mathbf{\Gamma})\right], l = 1,\cdots,n$, is decreasing in $\boldsymbol{K}$. Then, at least one of $K_l^*, l = 1,\cdots,n$, must increase to satisfy the optimality condition in (2.12), and the result follows. $\square$

B.2 Appendix for Chapter 3

B.2.1 Proof of Lemma 9:

Since $E[\Pi(\boldsymbol{K})]$ is given by

$$
\begin{aligned}
E[\Pi(\boldsymbol{K})] = & \iint_{\Omega_1} (\frac{\gamma_1^2}{4} + \frac{\gamma_2^2}{4}) f_{\boldsymbol{\Gamma}}(\gamma_1, \gamma_2) d\gamma_1 d\gamma_2 \\
& + \iint_{\Omega_4} ((\frac{\gamma_1 - \gamma_2 + 2K_f}{4})(\frac{3\gamma_1 + \gamma_2 - 2K_f}{4}) \\
& + (\frac{\gamma_2 - \gamma_1 + 2K_f}{4})(\frac{3\gamma_2 + \gamma_1 - 2K_f}{4})) f_{\boldsymbol{\Gamma}}(\gamma_1, \gamma_2) d\gamma_1 d\gamma_2 \\
& + \iint_{\Omega_2} (K_1(\gamma_1 - K_1) + \frac{\gamma_2^2}{4}) f_{\boldsymbol{\Gamma}}(\gamma_1, \gamma_2) d\gamma_1 d\gamma_2 \\
& + \iint_{\Omega_3} (K_2(\gamma_2 - K_2) + \frac{\gamma_1^2}{4}) f_{\boldsymbol{\Gamma}}(\gamma_1, \gamma_2) d\gamma_1 d\gamma_2 \\
& + \iint_{\Omega_5} (K_1(\gamma_1 - K_1) + (K_f - K_1)(\gamma_2 - K_f + K_1)) f_{\boldsymbol{\Gamma}}(\gamma_1, \gamma_2) d\gamma_1 d\gamma_2 \\
& + \iint_{\Omega_6} (K_2(\gamma_2 - K_2) + (K_f - K_2)(\gamma_1 - K_f + K_2)) f_{\boldsymbol{\Gamma}}(\gamma_1, \gamma_2) d\gamma_1 d\gamma_2,
\end{aligned}
$$

then,

$$
\begin{aligned}
\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_1} &= E[\frac{\partial \Pi(\boldsymbol{K})}{\partial K_1}] \\
&= E[\Gamma_1 - 2K_1 |\, \Omega_2] \Pr(\Omega_2) + E[\Gamma_1 + 2K_f - \Gamma_2 - 4K_1 |\, \Omega_5] \Pr(\Omega_5), \quad \text{(B.7)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_2} &= E[\frac{\partial \Pi(\boldsymbol{K})}{\partial K_2}] \\
&= E[\Gamma_2 - 2K_2 |\, \Omega_3] \Pr(\Omega_3) + E[\Gamma_2 + 2K_f - \Gamma_1 - 4K_2 |\, \Omega_6] \Pr(\Omega_6), \quad \text{(B.8)}
\end{aligned}
$$

and

$$\begin{aligned}
\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_f} &= E[\frac{\partial \Pi(\boldsymbol{K})}{\partial K_f}] \\
&= E[-K_f + \frac{\Gamma_1 + \Gamma_2}{2}\Big|\,\Omega_4]\,\Pr(\Omega_4) \\
&\quad + E[\Gamma_2 - 2K_f + 2K_1|\,\Omega_5]\,\Pr(\Omega_5) + E[\Gamma_1 - 2K_f + 2K_2|\,\Omega_6]\,\Pr(\Omega_6) \qquad \text{(B.9)}
\end{aligned}$$

And we know that

$$\Omega_1 = \{\gamma_1 \leq 2K_1, \gamma_2 \leq 2K_2, \gamma_1 + \gamma_2 \leq 2K_f\} \qquad \text{(B.10)}$$

$$\Omega_2 = \{\gamma_1 > 2K_1, \gamma_2 \leq 2K_2, K_1 + \frac{\gamma_2}{2} \leq K_f\} \qquad \text{(B.11)}$$

$$\Omega_3 = \{\gamma_2 > 2K_2, \gamma_1 \leq 2K_1, K_2 + \frac{\gamma_1}{2} \leq K_f\} \qquad \text{(B.12)}$$

$$\Omega_4 = \{-2K_f \leq \gamma_1 - \gamma_2 \leq 2K_f, \gamma_1 + \gamma_2 > K_f, K_1 - (\frac{\gamma_1 - \gamma_2}{4} + \frac{K_f}{2}) \geq 0, K_2 - (\frac{\gamma_2 - \gamma_1}{4} + \frac{K_f}{2}) \geq 0\} \qquad \text{(B.13)}$$

$$\Omega_5 = \{\gamma_1 - \gamma_2 + 2K_f - 4K_1 > 0, \gamma_2 - 2K_f + 2K_1 > 0\} \qquad \text{(B.14)}$$

$$\Omega_6 = \{\gamma_2 - \gamma_1 + 2K_f - 4K_2 > 0, \gamma_1 - 2K_f + 2K_2 > 0\} \qquad \text{(B.15)}$$

From Eqs. (B.7)-(B.15) and $K_f = K_1 + K_2$, we have that

$$\begin{aligned}
\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_1} &= E[\Gamma_1 - 2K_1|\,\Gamma_1 > 2K_1, \Gamma_2 \leq 2K_2]\,\Pr(\Gamma_1 > 2K_1, \Gamma_2 \leq 2K_2) \\
&\quad + E[\Gamma_1 - 2K_1 - \Gamma_2 + 2K_2|\,\Omega_{11}]\,\Pr(\Omega_{11}) \qquad \text{(B.16)}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_2} &= E[\Gamma_2 - 2K_2|\,\Gamma_2 > 2K_2, \Gamma_1 \leq 2K_1]\,\Pr(\Gamma_2 > 2K_2, \Gamma_1 \leq 2K_1) \\
&\quad + E[\Gamma_2 - 2K_2 - \Gamma_1 + 2K_1|\,\Omega_{11'}]\,\Pr(\Omega_{11'}) \qquad \text{(B.17)}
\end{aligned}$$

$$\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_f} = E[\Gamma_2 - 2K_2|\,\Omega_{11}]\,\Pr(\Omega_{11}) + E[\Gamma_1 - 2K_1|\,\Omega_{11'}]\,\Pr(\Omega_{11'}) \qquad \text{(B.18)}$$

Adding Eq. (B.16) to Eq. (B.18) leads to

$$\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_1} + \frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_f} = \int_{2K_1}^{\infty} (\gamma_1 - 2K_1) f_{\Gamma_1}(\gamma_1) d\gamma_1.$$

Similarly,

$$\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_2} + \frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_f} = \int_{2K_2}^{\infty} (\gamma_2 - 2K_2) f_{\Gamma_2}(\gamma_2) d\gamma_2.$$

This completes the proof. □

B.2.2 Proof of Lemma 10:

Define K_1^0 and K_2^0 as the following:

$$\begin{aligned}
\int_{2K_1^0}^{\infty} (\gamma_1 - 2K_1^0) f_{\Gamma_1}(\gamma_1) d\gamma_1 &= c_1 + c_f \\
\int_{2K_2^0}^{\infty} (\gamma_2 - 2K_2^0) f_{\Gamma_2}(\gamma_2) d\gamma_2 &= c_2 + c_f,
\end{aligned}$$

and in addition let $\boldsymbol{K}^0 = (K_1^0, K_2^0, K_1^0 + K_2^0)$. Since the condition given in Eq. (3.5) is equivalent to the following:

$$\left. \frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_f} \right|_{\boldsymbol{K}=\boldsymbol{K}^0} \geq c_f, \tag{B.19}$$

then we need to show that the optimal $\boldsymbol{K}^* = (K_1^*, K_2^*, K_f^*)$ satisfies the boundary condition $K_f^* = K_1^* + K_2^*$ and $\boldsymbol{K}^* = \boldsymbol{K}^0$ if and only if the condition given in Eq. (B.19) is satisfied. Suppose that the optimal $\boldsymbol{K}^* = (K_1^*, K_2^*, K_f^*)$ satisfies the boundary condition $K_f^* = K_1^* + K_2^*$. By the KKT condition, we have,

$$\begin{aligned}
\left. \frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_1} \right|_{\boldsymbol{K}=\boldsymbol{K}^*} + \theta_f - c_1 &= 0, && \text{(B.20)} \\
\left. \frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_2} \right|_{\boldsymbol{K}=\boldsymbol{K}^*} + \theta_f - c_2 &= 0, && \text{(B.21)} \\
\left. \frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_f} \right|_{\boldsymbol{K}=\boldsymbol{K}^*} - \theta_f - c_f &= 0, && \text{(B.22)}
\end{aligned}$$

where θ_f the KKT multiplier corresponding to the constraint $K_f \leq K_1 + K_2$. Then, adding Eq. (B.20) to (B.22), we have,

$$\begin{aligned} & \left.\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_1}\right|_{\boldsymbol{K}=\boldsymbol{K}^*} + \left.\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_f}\right|_{\boldsymbol{K}=\boldsymbol{K}^*} \\ = & \; c_1 + c_f. \end{aligned}$$

On the other hand, by Lemma 9, we have,

$$\left.\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_1}\right|_{\boldsymbol{K}=\boldsymbol{K}^*} + \left.\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_f}\right|_{\boldsymbol{K}=\boldsymbol{K}^*} = \int_{2K_1^*}^{\infty} (\gamma_1 - 2K_1^*) f_{\Gamma_1}(\gamma_1) d\gamma_1.$$

Then,

$$K_1^* = K_1^0$$

Similarly,

$$K_2^* = K_2^0$$

Consequently,

$$\boldsymbol{K}^* = \boldsymbol{K}^0$$

From Eq. (B.22), we have,

$$\begin{aligned} \left.\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_f}\right|_{\boldsymbol{K}=\boldsymbol{K}^*} &= \left.\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_f}\right|_{\boldsymbol{K}^*=\boldsymbol{K}^0} \\ &= \theta_f + c_f \\ &\geq c_f. \end{aligned}$$

Conversely, suppose that $\left.\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_f}\right|_{\boldsymbol{K}=\boldsymbol{K}^0} \geq c_f$ but the optimal solution $\boldsymbol{K}^* \neq \boldsymbol{K}^0$. Since $\boldsymbol{K}^0 = (K_1^0, K_2^0, K_1^0 + K_2^0)$ is a feasible solution but not the optimal solution, then there exits a feasible direction $\boldsymbol{d} = (d_1, d_2, d_3) \neq \boldsymbol{0}$ such that

$$\nabla V(\boldsymbol{K}^0)^t \boldsymbol{d} > \boldsymbol{0},$$

or equivalently,

$$\left(\frac{\partial E[\Pi(\boldsymbol{K}^0)]}{\partial K_1} - c_1\right)d_1 + \left(\frac{\partial E[\Pi(\boldsymbol{K}^0)]}{\partial K_2} - c_2\right)d_2 + \left(\frac{\partial E[\Pi(\boldsymbol{K}^0)]}{\partial K_f} - c_f\right)d_3 > 0. \tag{B.23}$$

Since, by Lemma 9

$$\begin{aligned} & \frac{\partial E[\Pi(\boldsymbol{K}^0)]}{\partial K_1} + \frac{\partial E[\Pi(\boldsymbol{K}^0)]}{\partial K_f} \\ = & \int_{2K_1^0}^{\infty} (\gamma_1 - 2K_1^0) f_{\Gamma_1}(\gamma_1) d\gamma_1 \\ = & \ c_1 + c_f \end{aligned}$$

then

$$\frac{\partial E[\Pi(\boldsymbol{K}^0)]}{\partial K_1} - c_1 = c_f - \frac{\partial E[\Pi(\boldsymbol{K}^0)]}{\partial K_f}.$$

Similarly,

$$\frac{\partial E[\Pi(\boldsymbol{K}^0)]}{\partial K_2} - c_2 = c_f - \frac{\partial E[\Pi(\boldsymbol{K}^0)]}{\partial K_f}$$

Consequently,

$$\begin{aligned} & \left(\frac{\partial E[\Pi(\boldsymbol{K}^0)]}{\partial K_1} - c_1\right)d_1 + \left(\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_2} - c_2\right)d_2 + \left(\frac{\partial E[\Pi(\boldsymbol{K})]}{\partial K_f} - c_f\right)d_3 \\ = & \left(c_f - \frac{\partial E[\Pi(\boldsymbol{K}^0)]}{\partial K_f}\right)d_1 + \left(c_f - \frac{\partial E[\Pi(\boldsymbol{K}^0)]}{\partial K_f}\right)d_2 + \left(\frac{\partial E[\Pi(\boldsymbol{K}^0)]}{\partial K_f} - c_f\right)d_3 \\ = & (d_3 - d_1 - d_2)\left(\frac{\partial E[\Pi(\boldsymbol{K}^0)]}{\partial K_f} - c_f\right). \end{aligned}$$

Since $\boldsymbol{d}$ is a a feasible direction, then,

$$d_3 < d_1 + d_2,$$

and $\frac{\partial E[\Pi(\boldsymbol{K}^0)]}{\partial K_f} \geq c_f$, then

$$(d_3 - d_1 - d_2)\left(\frac{\partial E[\Pi(\boldsymbol{K}^0)]}{\partial K_f} - c_f\right) \leq 0 \tag{B.24}$$

Since Eq. (B.23) contradicts with Eq. (B.24), we must have $\boldsymbol{K}^* = \boldsymbol{K}^0$. This completes the proof. □

B.2.3 Proof of Lemma 11:

For part 1, it follows similarly to the proof of part 1 of Lemma 4. For part 2, it follows directly from Table 3.1. □

B.2.4 Proof of Theorem 3:

First, we show that for any two adjacent domains, Ω_i and Ω_j, we have that $\sum_{l=1,2,f} \lambda_i^l - \sum_{l=1,2,f} \lambda_j^l \geq 0$, for $\forall(\gamma_1, \gamma_2) \in \Omega_i$. Since each $\lambda^l, l = 1, 2, f$, is continuous and linear in $\boldsymbol{\gamma}$ in every feasible domain (Lemma 4), this also implies that $\sum_{l=1,2,f} \lambda_j^l - \sum_{l=1,2,f} \lambda_i^l \geq 0$, for $\forall(\gamma_1, \gamma_2) \in \Omega_j$. Consequently, we only analyze the cases where we move from domain Ω_i to its adjacent domain Ω_j by changing the value of exactly one dual variable from zero to a strictly positive value. Thus, it suffices to consider the following cases:

Case 1. Domain Ω_i is characterized by dual variables $\lambda_i^l = 0, l = 1, 2$, and $\lambda_i^f = 0$. By Remark 4, we can move from Ω_i to its adjacent domain Ω_j by changing the value of exactly one dual variable λ_i^l, $l = 1, 2, f$, from zero to a strictly positive value in the following two ways:

Case 1.1. $\lambda_i^l = 0 \to \lambda_j^l > 0$, for $l = 1$ or 2. Then $\sum_{r=1,2,f} \lambda_i^r - \sum_{r=1,2,f} \lambda_j^r = 2K_l - \gamma_l \geq 0$ for $\forall(\gamma_1, \gamma_2) \in \Omega_i$.

Case 1.2. $\lambda_i^f = 0 \to \lambda_j^f > 0$. Then $\sum_{l=1,2,f} \lambda_i^l - \sum_{l=1,2,f} \lambda_j^l = K_f - (\gamma_1 + \gamma_2)/2 \geq 0$ for $\forall(\gamma_1, \gamma_2) \in \Omega_i$.

Case 2. Domain Ω_i is characterized by dual variables $\lambda_i^l > 0$ and $\lambda_i^{3-l} = 0$, for $l = 1$ or 2, and $\lambda_i^f = 0$. We can move from Ω_i to its adjacent domain Ω_j only by changing the dual variable $\lambda_i^f = 0$ to $\lambda_j^f > 0$. This is because there does not exist a feasible domain with $\lambda_j^l > 0$, $l = 1$ and 2, and $\lambda_j^f = 0$. Then $\sum_{r=1,2,f} \lambda_i^r - \sum_{r=1,2,f} \lambda_j^r = 0 \geq 0$ for $\forall(\gamma_1, \gamma_2) \in \Omega_i$.

Case 3. Domain Ω_i is characterized by dual variables $\lambda_i^l = 0$, $l = 1, 2$, and $\lambda_i^f > 0$. Then we can move from Ω_i to its adjacent domain Ω_j by changing only one of the dual variables $\lambda_i^l = 0$ to $\lambda_j^l > 0$, for $l = 1$ or 2. Then $\sum_{r=1,2,f} \lambda_i^r - \sum_{r=1,2,f} \lambda_j^r = (\gamma_{3-l} - \gamma_l)/2 - K_f + 2K_l \geq 0$ for

$\forall(\gamma_1,\gamma_2)\in\Omega_i$.

Then, since $\sum_{l=1,2,f}\lambda_i^l$ is linear and continuous in $\boldsymbol{\gamma}$, it can be expressed as a pointwise maximum of a set of convex functions and its joint convexity in $\boldsymbol{\gamma}$ follows (similar to the proofs of Lemmas 6 and 7).

We next show that $\sum_{l=1,2,f}\lambda^l$ is a decreasing function in each of K_1, K_2, and K_f. Since $\frac{\partial E_{\boldsymbol{\Gamma}}[\sum_{l=1,2,f}\lambda^l]}{\partial K_k}=E_{\boldsymbol{\Gamma}}[\frac{\partial\sum_{l=1,2,f}\lambda^l}{\partial K_k}]$ (by the Lipschitz continuity of $\lambda^l(\boldsymbol{K},\boldsymbol{\gamma})$, $l=1,2,f$, see the detailed argument in the proof of Lemma A4), it is sufficient to show that $\sum_{l=1,2,f}\lambda^l$ is decreasing in $K_k, k=1,2,f$, in every feasible domain.

$$\frac{\partial\sum_{l=1,2,f}\lambda^l}{\partial K_f}=\begin{cases}0 & \text{if }(\gamma_1,\gamma_2)\in\Omega_1,\Omega_2,\Omega_3,\Omega_5\text{ or }\Omega_6\\ -1 & \text{if }(\gamma_1,\gamma_2)\in\Omega_4\end{cases}$$

Clearly $\frac{\partial\sum_{l=1,2,f}\lambda^l}{\partial K_f}\le 0$ in every domain. Similarly, for K_1, we can write:

$$\frac{\partial\sum_{l=1,2,f}\lambda^l}{\partial K_1}=\begin{cases}0 & \text{if }(\gamma_1,\gamma_2)\in\Omega_1,\Omega_3,\Omega_4\text{ or }\Omega_6\\ -2 & \text{if }(\gamma_1,\gamma_2)\in\Omega_2,\Omega_5\end{cases},$$

where $\frac{\partial\sum_{l=1,2,f}\lambda^l}{\partial K_1}\le 0$ in every domain, and by symmetry, we have $\frac{\partial\sum_{l=1,2,f}\lambda^l}{\partial K_2}\le 0$. To sum up, $\sum_{l=1,2,f}\lambda^l$ is a decreasing function of K_1, K_2, and K_f.

Then, since $\sum_{l=1,2,f}\lambda^l$ is convex in $\boldsymbol{\gamma}$, we have, for $\boldsymbol{\Gamma}\le_{cx}\overline{\boldsymbol{\Gamma}}$,

$$\begin{aligned}&E_{\overline{\boldsymbol{\Gamma}}}[\textstyle\sum_{l=1,2,f}\partial\Pi^*(\boldsymbol{K},\overline{\boldsymbol{\Gamma}})/\partial K_l]|_{\boldsymbol{K}=\boldsymbol{K}^*}\\ \ge\;&E_{\boldsymbol{\Gamma}}[\textstyle\sum_{l=1,2,f}\partial\Pi^*(\boldsymbol{K},\boldsymbol{\Gamma})/\partial K_l]|_{\boldsymbol{K}=\boldsymbol{K}^*}\\ =\;&\textstyle\sum_{l=1,2,f}c_l\\ =\;&E_{\overline{\boldsymbol{\Gamma}}}[\textstyle\sum_{l=1,2,f}\partial\Pi^*(\boldsymbol{K},\overline{\boldsymbol{\Gamma}})/\partial K_l]|_{\boldsymbol{K}=\overline{\boldsymbol{K}}^*}.\end{aligned}\tag{B.25}$$

Suppose none of K_1, K_2, or K_f increases. That is, $K_1^*\ge\overline{K}_1^*$, $K_2^*\ge\overline{K}_2^*$, and $K_f^*\ge\overline{K}_f^*$. Let

$G(\boldsymbol{K}) \equiv E_{\overline{\boldsymbol{\Gamma}}}[\sum_{l=1,2,f} \partial\Pi^*(\boldsymbol{K}, \overline{\boldsymbol{\Gamma}})/\partial K_l]$. By the mean value theorem, there must exist a $\boldsymbol{K}'$, which is a convex combination of $\boldsymbol{K}^*$ and $\overline{\boldsymbol{K}}^*$, such that

$$\begin{aligned} G(\boldsymbol{K}^*) - G(\overline{\boldsymbol{K}}^*) &= G'(\boldsymbol{K}') \cdot (\boldsymbol{K}^* - \overline{\boldsymbol{K}}^*) \\ &= \frac{\partial G(\boldsymbol{K}')}{\partial K_1}(K_1^* - \overline{K}_1^*) + \frac{\partial G(\boldsymbol{K}')}{\partial K_2}(K_2^* - \overline{K}_2^*) + \frac{\partial G(\boldsymbol{K}')}{\partial K_f}(K_f^* - \overline{K}_f^*). \end{aligned} \tag{B.26}$$

Since, by assumption, both $\boldsymbol{K}^*$ and $\overline{\boldsymbol{K}}^*$ are not balanced solutions, then a convex combination of them, $\boldsymbol{K}'$, cannot be a balanced solution either. That is, $\boldsymbol{K}'$ satisfies Assumption 2. Then, $G(\boldsymbol{K})$ is decreasing at $\boldsymbol{K}'$, as $\sum_{l=1,2,f} \lambda^l$ is decreasing in K_1, K_2, and K_f, as shown above. That is, $\frac{\partial G(\boldsymbol{K}')}{\partial K_i} \leq 0, i = 1, 2, f$. Since none of K_1, K_2, or K_f increases, we must have $G'(\boldsymbol{K}') \cdot (\boldsymbol{K}^* - \overline{\boldsymbol{K}}^*) \leq 0$ which implies, by Eq. (B.26), that $G(\boldsymbol{K}^*) - G(\overline{\boldsymbol{K}}^*) \leq 0$. This, however, contradicts with the fact that $G(\boldsymbol{K}^*) - G(\overline{\boldsymbol{K}}^*) \geq 0$, see Eq.(B.25). Then, at least one of K_1, K_2, or K_f must increase to satisfy the optimality conditions in Eq.(B.25) when $\boldsymbol{\Gamma}$ increases in convex order to $\overline{\boldsymbol{\Gamma}}$. This completes the proof. $\square$

B.2.5 Proof of Lemma 12:

Suppose, to the contrary, that the dual variable $\lambda(\boldsymbol{\gamma})$ is jointly convex in $\boldsymbol{\gamma}$, and that there exist two adjacent domains, Ω_i and Ω_j, $i, j \in \Psi, j \neq i$, such that Lemma 5 does not hold for $\lambda(\boldsymbol{\gamma})$. Consider any two vectors, $\boldsymbol{\gamma}_j \in \Omega_j$ and $\boldsymbol{\gamma}_i \in \Omega_i$. Then, since Lemma 5 does not hold for $\lambda(\boldsymbol{\gamma})$ in Ω_j and Ω_i, and because both $\lambda_j(\boldsymbol{\gamma}_j)$ and $\lambda_i(\boldsymbol{\gamma}_i)$ are linear in $\boldsymbol{\gamma}$ (by Lemma 4), it follows that

$$\lambda_i(\boldsymbol{\gamma}_i) < \lambda_j(\boldsymbol{\gamma}_i).$$

Then, for any $t : 0 < t < 1$,

$$\begin{aligned} t\lambda_j(\boldsymbol{\gamma}_j) + (1-t)\lambda_i(\boldsymbol{\gamma}_i) &< t\lambda_j(\boldsymbol{\gamma}_j) + (1-t)\lambda_j(\boldsymbol{\gamma}_i) \\ &= \lambda_j(t\boldsymbol{\gamma}_j + (1-t)\boldsymbol{\gamma}_i), \quad \text{since } \lambda_j(\boldsymbol{\gamma}) \text{ is linear in } \boldsymbol{\gamma}. \end{aligned} \tag{B.27}$$

On the other hand, since $\lambda(\boldsymbol{\gamma})$ is convex in $\boldsymbol{\gamma}$, we have

$$\begin{aligned}
\lambda(t\boldsymbol{\gamma}_j + (1-t)\boldsymbol{\gamma}_i) &\leq t\lambda(\boldsymbol{\gamma}_j) + (1-t)\lambda(\boldsymbol{\gamma}_i), \text{ by convexity of } \lambda(\boldsymbol{\gamma}) \\
&= t\lambda_j(\boldsymbol{\gamma}_j) + (1-t)\lambda_i(\boldsymbol{\gamma}_i),
\end{aligned}$$

which is a contradiction with (B.27) when $t\boldsymbol{\gamma}_j + (1-t)\boldsymbol{\gamma}_i \in \Omega_j$. Therefore, $\lambda(\boldsymbol{\gamma})$ cannot be convex in $\boldsymbol{\gamma}$. This completes the proof. $\square$

B.2.6 Proof of Lemma 13:

For part 1, it follows similarly to the proof of part 1 of Lemma 4. For part 2, consider, in domain i with $\lambda_i^f = 0$, the result trivially holds. Now consider domain Ω_i, $i \in \Psi$, with $\lambda_i^f > 0$. Adding all the equations, $\gamma_l - 2q_l^* - \lambda_i^l - \lambda_i^f + \mu_i^l = 0$, $l \in \Phi_i^2$, gives $\sum_{l\in\Phi_i^2}(\gamma_l - 2q_l^*) = |\Phi_i^2|\,\lambda_i^f$. Since $\lambda_i^f > 0$, we have $\sum_{l\in\Phi_i^1\cup\Phi_i^2\cup\Phi_i^3} q_l^* = K_f$. Observe that by contradiction, if $l \in \Phi_i^1$ then $q_l^* = 0$, and if $l \in \Phi_i^3$, then $q_l^* = K_l$. We have $\sum_{l\in\Phi_i^2} q_l^* = K_f - \sum_{l\in\Phi_i^3} K_l$, which leads to $|\Phi_i^2|\,\lambda_i^f = \sum_{l\in\Phi_i^2}\gamma_l - 2(K_f - \sum_{l\in\Phi_i^3} K_l)$. Note that with Assumption 2 on $\boldsymbol{K}$, we must have $|\Phi_i^2| \geq 1$ in any feasible domain. Therefore, we have $\lambda_i^f = \frac{1}{|\Phi_i^2|}[\sum_{l\in\Phi_i^2}\gamma_l - 2(K_f - \sum_{l\in\Phi_i^3} K_l)]$. Now for λ_i^l, if $l \in \Phi_i^1\cup\,\Phi_i^2$, $\lambda_i^l = 0$, and the result trivially follows. If $l \in \Phi_i^3$, then $\lambda_i^l = \gamma_l - 2K_l - \lambda_i^f$. This completes the proof. $\square$

B.2.7 Proof of Lemma 14:

For $\Omega_j, j \in \Psi$, let $\overline{V}$ and $\overline{I}$ be the maximum possible size of Φ_j^1 and Φ_j^3 respectively. Since $\alpha \in [\bar{i}\,(\boldsymbol{K}^*)\,, \bar{j}\,(\boldsymbol{K}^*) + 2]$, we have that $\overline{V} \leq \bar{i} \leq \alpha \leq \bar{j} + 2 \leq n - \overline{I}$. Then for any feasible domain $\Omega_j, j \in \Psi$, we must have that $V_j \leq \alpha \leq n - I_j$ always holds, where $V_j = |\Phi_j^3|$ and $I_j = |\Phi_j^1|$. Clearly, in any Ω_j, $\sum_{l=1}^n \lambda_j^l + \alpha\lambda_j^f$ is continuous in $\boldsymbol{\gamma}$ and is a linear function of $\boldsymbol{\gamma}$, then,

$$
\begin{aligned}
&\sum_{l=1}^{n} \lambda_j^l + \alpha\lambda_j^f \\
= &\begin{cases}
\sum_{l\in\Phi_j^3}(\gamma_l - 2K_l) & \Omega_j \text{ with } \lambda^f = 0 \\
\\
\sum_{l\in\Phi_j^3}(\gamma_l - 2K_l) + \frac{\alpha - V_j}{J_j}(\sum_{l\in\Phi_j^2}\gamma_l - 2(K_f - \sum_{l\in\Phi_j^3} K_l)) & \Omega_j \text{ with } \lambda_j^f > 0
\end{cases}
\end{aligned}
$$

Consider any two adjacent domains Ω_j and Ω_k, $j,k \in \Psi$, we only need to study five cases:

$$
\begin{aligned}
\lambda_j^f &= 0 \rightarrow \lambda_k^f > 0, \\
\lambda_j^l &= 0 \rightarrow \lambda_k^l > 0\text{, when } \lambda_j^f = 0, \\
\lambda_j^l &= 0 \rightarrow \lambda_k^l > 0\text{, when } \lambda_j^f > 0, \\
u_j^l &= 0 \rightarrow u_k^l > 0\text{, when } \lambda_j^f = 0, \\
u_j^l &= 0 \rightarrow u_k^l > 0\text{, when } \lambda_j^f > 0
\end{aligned}
$$

Let $\{\Phi_j^1,\Phi_j^2,\Phi_j^3\}$ and $\{\Phi_k^1,\Phi_k^2,\Phi_k^3\}$ be their corresponding partitions and $\{I,J,V\}$ and $\{I',J',V'\}$ be the size of their corresponding sets.

Case 1. $\lambda_j^f = 0 \rightarrow \lambda_k^f > 0$.

Clearly $\{\Phi_j^1,\Phi_j^2,\Phi_j^3\} = \{\Phi_k^1,\Phi_k^2,\Phi_k^3\}$. Then,

$$
\begin{aligned}
\sum_{l\in\Phi_j^1\cup\Phi_j^2\cup\Phi_j^3} \lambda_j^l + \alpha\lambda_j^f &= \sum_{l\in\Phi_j^3}(\gamma_l - 2K_l), \\
\sum_{l\in\Phi_k^1\cup\Phi_k^2\cup\Phi_k^3} \lambda_k^l + \alpha\lambda_k^f &= \sum_{l\in\Phi_k^3}(\gamma_l - 2K_l) + \frac{\alpha - V}{J}(\sum_{l\in\Phi_k^2}\gamma_l - 2(K_f - \sum_{l\in\Phi_k^3} K_l)).
\end{aligned}
$$

Thus,

$$\begin{aligned}&\left(\sum_{l\in\Phi_j^1\cup\Phi_j^2\cup\Phi_j^3}\lambda_j^l+\alpha\lambda_j^f\right)-\left(\sum_{l\in\Phi_k^1\cup\Phi_k^2\cup\Phi_k^3}\lambda_k^l+\alpha\lambda_k^f\right)\\ = \quad & -\frac{\alpha-V}{J}(\sum_{l\in\Phi_j^2}\gamma_l-2(K_f-\sum_{l\in\Phi_j^3}K_l)).\end{aligned}$$

$l\in\Phi_j^2$ and $\lambda_j^f=0$ imply that $\gamma_l-2q_l=0$. Then $\gamma_l=2q_l$. $\lambda_j^f=0$ implies that $\Phi_j^1=\emptyset$, and $\sum_{l\in\Phi_j^3}K_l=\sum_{l\in\Phi_j^3}q_l$. Thus, $\sum_{l\in\Phi_j^2}\gamma_l-2(K_f-\sum_{l\in\Phi_j^3}K_l)=2(\sum_{l\in\Phi_j^1\cup\Phi_j^2\cup\Phi_3}q_l-K_f)\leq 0$. Since $V\leq\alpha$, we always have that $-\frac{\alpha-V}{J}\leq 0$. Thus we have that $\left(\sum_{l\in\Phi_j^1\cup\Phi_j^2\cup\Phi_j^3}\lambda_j^l+\alpha\lambda_j^f\right)\geq\left(\sum_{l\in\Phi_k^1\cup\Phi_k^2\cup\Phi_k^3}\lambda_k^l+\alpha\lambda_k^f\right)$ for any $\boldsymbol{\gamma}\in\Omega_j$.

Case 2. $\lambda_j^l=0\rightarrow\lambda_k^l>0$, when $\lambda_j^f=0$.

Clearly $l\in\Phi_j^2$, and $I=I'$, $J-1=J'$, $V+1=V'$. In the domain with $J=1$, there is no adjacent feasible domain by changing from $\lambda_j^l=0$ to $\lambda_k^l>0$ alone. Otherwise, we have a feasible domain with $J=0$, i.e., there exists a vector $\tau_{(n,1)}$ such that $(K_1,..,K_n)\cdot\tau=K_f$ where the component of τ could only be either 0 or 1 violating Assumption **(2)**. Now consider the domain with $J\geq 2$.

$$\sum_{l\in\Phi_j^1\cup\Phi_j^2\cup\Phi_j^3}\lambda_j^l+\alpha\lambda_j^f=\sum_{l\in\Phi_j^3}(\gamma_l-2K_l)$$

$$\sum_{l\in\Phi_k^1\cup\Phi_k^2\cup\Phi_k^3}\lambda_k^l+\alpha\lambda_k^f=\sum_{l\in\Phi_j^3}(\gamma_l-2K_l).$$

Thus,

$$\begin{aligned}&\left(\sum_{l\in\Phi_j^1\cup\Phi_j^2\cup\Phi_j^3}\lambda_j^l+\alpha\lambda_j^f\right)-\left(\sum_{l\in\Phi_k^1\cup\Phi_k^2\cup\Phi_k^3}\lambda_k^l+\alpha\lambda_k^f\right)\\ = \quad & \sum_{l\in\Phi_j^3}(\gamma_l-2K_l)-(\sum_{l\in\Phi_j^3}(\gamma_l-2K_l)+\gamma_m-2K_m),\end{aligned}$$

where $m \in \Phi_j^2$. $m \in \Phi_j^2 \Rightarrow \gamma_m - 2q_m = 0 \Rightarrow \gamma_m - 2K_m = 2q_m - K_m \leq 0 \Rightarrow$

$$\sum_{l \in \Phi_j^3}(\gamma_l - 2K_l) - (\sum_{l \in \Phi_j^3}(\gamma_l - 2K_l) + \gamma_m - 2K_m) \geq 0.$$

Therefore, we have that $\left(\sum_{l \in \Phi_j^1 \cup \Phi_j^2 \cup \Phi_j^3} \lambda_j^l + \alpha\lambda_j^f\right) \geq \left(\sum_{l \in \Phi_k^1 \cup \Phi_k^2 \cup \Phi_k^3} \lambda_k^l + \alpha\lambda_k^f\right)$ for any $\boldsymbol{\gamma} \in \Omega_j$.
Case 3. $\lambda_j^l = 0 \to \lambda_k^l > 0$, when $\lambda_j^f > 0$.

Likewise $l \in \Phi_j^2$, and $I = I'$, $J - 1 = J'$, $V + 1 = V'$. Then we only need to consider the domain with $J \geq 2$.

$$\begin{aligned} & \sum_{l \in \Phi_j^1 \cup \Phi_j^2 \cup \Phi_j^3} \lambda_j^l + \alpha\lambda_j^f \\ = & \sum_{l \in \Phi_j^3}(\gamma_l - 2K_l) + \frac{\alpha - V}{J}(\sum_{l \in \Phi_j^2} \gamma_l - 2(K_f - \sum_{l \in \Phi_j^3} K_l)) \\ & \sum_{l \in \Phi_k^1 \cup \Phi_k^2 \cup \Phi_k^3} \lambda_k^l + \alpha\lambda^f \\ = & \sum_{l \in \Phi_k^3}(\gamma_l - 2K_l) + \frac{\alpha - V'}{J'}(\sum_{l \in \Phi_k^2} \gamma_l - 2(K_f - \sum_{l \in \Phi_k^3} K_l)) \end{aligned}$$

Thus,

$$\begin{aligned} & \left(\sum_{l \in \Phi_j^1 \cup \Phi_j^2 \cup \Phi_j^3} \lambda_j^l + \alpha\lambda_j^f\right) - \left(\sum_{l \in \Phi_k^1 \cup \Phi_k^2 \cup \Phi_k^3} \lambda_k^l + \alpha\lambda_k^f\right) \\ = & (1 - \frac{\alpha - V - 1}{J - 1})(\lambda_j^f - \gamma_m + 2K_m). \end{aligned}$$

$m \in \Phi_j^2 \Rightarrow \gamma_m - 2q_m - \lambda_j^f = 0 \Rightarrow \lambda_j^f - \gamma_m = -2q_m \Rightarrow$

$$\begin{aligned} & \lambda_j^f - \gamma_m + 2K_m \\ = \ & -2q_m + 2K_m \\ \geq \ & 0 \end{aligned}$$

On the other hand, $\alpha \leq n - I$, i.e., $\alpha \leq V + J$, in every feasible domain, we always have

$$\begin{aligned} & 1 - \frac{\alpha - V - 1}{J - 1} \\ = \ & \frac{J - 1 - \alpha + V + 1}{J - 1} \\ = \ & \frac{J + V - \alpha}{J - 1} \geq 0. \end{aligned}$$

Therefore, we have that $\left(\sum_{l \in \Phi_j^1 \cup \Phi_j^2 \cup \Phi_j^3} \lambda_j^l + \alpha \lambda_j^f\right) \geq \left(\sum_{l \in \Phi_k^1 \cup \Phi_k^2 \cup \Phi_k^3} \lambda_k^l + \alpha \lambda_k^f\right)$ for any $\boldsymbol{\gamma} \in \Omega_j$.

Case 4. $\mu_j^l = 0 \to \mu_k^l > 0$, when $\lambda_j^f = 0$.

$u_j^l > 0$ and $\lambda_j^f = 0$ cannot both be true at the same time, since if we have $\gamma_l + \mu_j^l = 0$, which contradicts with $u_j^l > 0$.

Case 5. $u_j^l = 0 \to u_k^l > 0$, when $\lambda_j^f > 0$.

Clearly $l \in \Phi_j^2$ and $I' = I + 1, J' = J - 1, V' = V$. Likewise, we only need to consider the domain with $J \geq 2$.

$$\begin{aligned} \sum_{l \in \Phi_j^1 \cup \Phi_j^2 \cup \Phi_j^3} \lambda_j^l + \alpha \lambda_j^f &= \sum_{l \in \Phi_j^3} (\gamma_l - 2K_l) + \frac{\alpha - V}{J} (\sum_{l \in \Phi_j^2} \gamma_l - 2(K_f - \sum_{l \in \Phi_j^3} K_l)) \\ \sum_{l \in \Phi_k^1 \cup \Phi_k^2 \cup \Phi_k^3} \lambda_k^l + \alpha \lambda^f &= \sum_{l \in \Phi_k^3} (\gamma_l - 2K_l) + \frac{\alpha - V'}{J'} (\sum_{l \in \Phi_k^2} \gamma_l - 2(K_f - \sum_{l \in \Phi_k^3} K_l)) \\ &= \sum_{l \in \Phi_j^3} (\gamma_l - 2K_l) + (\alpha - V)\lambda_j^f + \frac{\alpha - V}{J - 1} (\lambda_j^f - \gamma_m), \end{aligned}$$

where $m \in \Phi_j^2$. Thus,

$$\left(\sum_{l \in \Phi_j^1 \cup \Phi_j^2 \cup \Phi_j^3} \lambda_j^l + \alpha \lambda_j^f \right) - \left(\sum_{l \in \Phi_k^1 \cup \Phi_k^2 \cup \Phi_k^3} \lambda_k^l + \alpha \lambda_k^f \right)$$
$$= \quad -\frac{\alpha - V}{J-1}(\lambda_j^f - \gamma_m).$$

$m \in \Phi_j^2 \Rightarrow \gamma_m - 2q_m - \lambda_j^f = 0 \Rightarrow \lambda_j^f - \gamma_m = -2q_m \Rightarrow$

$$\lambda_j^f - \gamma_m = -2q_m \leq 0$$

Since $V \leq \alpha$, we always have that $\left(\sum_{l \in \Phi_j^1 \cup \Phi_j^2 \cup \Phi_j^3} \lambda_j^l + \alpha \lambda_j^f \right) \geq \left(\sum_{l \in \Phi_k^1 \cup \Phi_k^2 \cup \Phi_k^3} \lambda_k^l + \alpha \lambda_k^f \right)$ for any $\boldsymbol{\gamma} \in \Omega_j$. This completes the proof. $\square$

B.2.8 Proof of Lemma 15:

It follows directly from Lemma 13 and Lemma 14. $\square$

B.2.9 Summary of Change of the KKT Multiplier

Table B.1 summaries, in $\Omega_j, j \in \Psi$, what the optimal solution to Problem $\mathbf{P_2}$, $\boldsymbol{q}^*$, looks like, the sizes of its corresponding Φ_j^1, Φ_j^2, and Φ_j^3, which KKT multiplier will change when exactly one γ_l, $l = 1, \cdots, n$, increases, and what's its resulting immediate adjacent domain. In order to avoid the possible double counting it is sufficient to only consider exactly one γ increases in $\Omega_j, j \in \Psi$.

Table B.1: Summary of change of KKT multiplier (capacity dual variable, $\lambda_l, l = 1, \cdots, n, f$) in $\Omega_j, j \in \Psi$ when γ_l, $l = 1, \cdots, n$ increases

Optimal solution $\boldsymbol{q}^*$ and $\{\lvert\Phi_j^1\rvert, \lvert\Phi_j^2\rvert, \lvert\Phi_j^3\rvert\}$	Change of KKT Multiplier
$\boldsymbol{q}^* = (\frac{\gamma_1}{2}, ..., \frac{\gamma_n}{2})$ $\{\lvert\Phi_j^1\rvert, \lvert\Phi_j^2\rvert, \lvert\Phi_j^3\rvert\} = (0, n, 0)$	$l \in \Phi_j^2, \gamma_l \uparrow \Rightarrow$ $\lambda_j^f = 0 \rightarrow \lambda_j^f > 0$ $(0, n, 0)$
	$l \in \Phi_j^2, \gamma_l \uparrow \Rightarrow$ $\lambda_j^l = 0 \rightarrow \lambda_j^l > 0$ $(0, n-1, 1)$
$\boldsymbol{q}^* = (\frac{\gamma_1}{2}, ..., \frac{\gamma_J}{2}, K_{J+1}, ..., K_{J+V})$ $1 \leq J < n, 1 \leq V < n$ $\{\lvert\Phi_j^1\rvert, \lvert\Phi_j^2\rvert, \lvert\Phi_j^3\rvert\} = ((0, J, V)$	$l \in \Phi_j^2, \gamma_l \uparrow \Rightarrow$ $\lambda_j^f = 0 \rightarrow \lambda_j^f > 0$ $(0, J, V)$
	$l \in \Phi_j^2, \gamma_l \uparrow \Rightarrow$ $\lambda_j^l = 0 \rightarrow \lambda_j^l > 0$ $(0, J-1, V+1)$
	$l \in \Phi_j^3, \gamma_l \uparrow \Rightarrow$ no adjacent domain
$\boldsymbol{q}^* = (q_1, ..., q_n)$ $0 \leq q_l < \frac{\gamma_l}{2}$ $\{\lvert\Phi_j^1\rvert, \lvert\Phi_j^2\rvert, \lvert\Phi_j^3\rvert\} = ((0, n, 0)$	$l \in \Phi_j^2, \gamma_l \uparrow \Rightarrow$ $\lambda_j^l = 0 \rightarrow \lambda_j^l > 0$ $(0, n-1, 1)$
	$l \in \Phi_j^2, \gamma_l \uparrow \Rightarrow$ $\mu_j^i = 0 \rightarrow \mu_j^i > 0, i \neq l$ $(1, n-1, 0)$

Continued on next page

Table B.1 – continued from previous page

Optimal solution $\boldsymbol{q}^*$ and $\{\lvert\Phi_j^1\rvert, \lvert\Phi_j^2\rvert, \lvert\Phi_j^3\rvert\}$	Change of KKT Multiplier
$\boldsymbol{q}^* = (q_1, ..., q_J, K_{J+1}, ..., K_{J+V})$ $0 \le q_l < \frac{\gamma_l}{2}$ $1 \le J < n, 1 \le V < n$ $\{\lvert\Phi_j^1\rvert, \lvert\Phi_j^2\rvert, \lvert\Phi_j^3\rvert\} = ((0, J, V)$	$l \in \Phi_j^2, \gamma_l \uparrow \Rightarrow$ $\lambda_j^l = 0 \rightarrow \lambda_j^l > 0$ $(0, J-1, V+1)$
	$l \in \Phi_j^2, \gamma_l \uparrow \Rightarrow$ $\mu_j^i = 0 \rightarrow \mu_j^i > 0, i \neq j$ $(1, J-1, V)$
	$l \in \Phi_j^3, \gamma_l \uparrow \Rightarrow$ no adjacent domain
$\boldsymbol{q}^* = (0, ..., 0, q_{I+1}, ..., q_{I+J})$ $0 \le q_l < \frac{\gamma_l}{2}$ $1 \le I < n, 1 \le J < n$ $\{\lvert\Phi_j^1\rvert, \lvert\Phi_j^2\rvert, \lvert\Phi_j^3\rvert\} = ((I, J, 0)$	$l \in \Phi_j^2, \gamma_l \uparrow \Rightarrow$ $\lambda_j^l = 0 \rightarrow \lambda_j^l > 0$ $(I, J-1, 1)$
	$l \in \Phi_j^2, \gamma_l \uparrow \Rightarrow$ $\mu_j^i = 0 \rightarrow \mu_j^i > 0, i \neq l$ $(I+1, J-1, 0)$
	$l \in \Phi_1, \gamma_l \uparrow \Rightarrow$ $\mu_l > 0 \rightarrow \mu_l = 0$ $(I-1, J+1, 0)$

Continued on next page

Table B.1 – continued from previous page

Optimal solution $\boldsymbol{q}^*$ and $\{\lvert\Phi_j^1\rvert, \lvert\Phi_j^2\rvert, \lvert\Phi_j^3\rvert\}$	Change of KKT Multiplier
$\boldsymbol{q}^* = (0, ..., 0, q_{I+1}, ..., q_{I+J}, K_{I+J+1}, ..., K_{I+J+V})$ $0 \le q_l < \frac{\gamma_l}{2}$ $1 \le I, J$, or $V < n$ $\{\lvert\Phi_j^1\rvert, \lvert\Phi_j^2\rvert, \lvert\Phi_j^3\rvert\} = ((I, J, V)$	$l \in \Phi_j^2, \gamma_l \uparrow \Rightarrow$ $\lambda_j^l = 0 \to \lambda_j^l > 0$ $(I, J-1, V+1)$
	$l \in \Phi_j^2, \gamma_l \uparrow \Rightarrow$ $\mu_j^i = 0 \to \mu_j^i > 0, i \neq l$ $(I+1, J-1, V)$
	$l \in \Phi_1, \gamma_l \uparrow \Rightarrow$ $\mu_l > 0 \to \mu_l = 0$ $(I-1, J+1, V)$
	$l \in \Phi_j^3, \gamma_l \uparrow \Rightarrow$ no adjacent domain

B.2.10 Proof of Lemma 16:

Similar to **DN**, since $E_{\mathbf{\Gamma}}[\sum_{l=1}^n \lambda^l + \alpha\lambda^f]$ is Lipschitz continuous of order one, $\frac{\partial E_{\mathbf{\Gamma}}[\sum_{l=1}^n \lambda^l + \alpha\lambda^f]}{\partial K_f} = E_{\mathbf{\Gamma}}[\frac{\partial \sum_{l=1}^n \lambda^l + \alpha\lambda^f}{\partial K_f}]$. It is sufficient to show that $\frac{\partial \sum_{l=1}^n \lambda^l + \alpha\lambda^f}{\partial K_f} \le 0$ in every feasible domain.

$$\frac{\partial \sum_{l=1}^n \lambda^l + \alpha\lambda^f}{\partial K_f}$$
$$= \begin{cases} 0 & \Omega_j \text{ with } \lambda^f = 0, \\ -\frac{2(\alpha - V_j)}{J_j} & \Omega_j \text{ with } \lambda^f > 0, \end{cases}$$

where $V_j = |\Phi_j^3|$ and $J_j = |\Phi_j^2|$. $\alpha \geq V_j$ thus $\frac{\partial \sum_{l=1}^n \lambda^l + \alpha\lambda^f}{\partial K_f} \leq 0$ in every domain which leads to that $\frac{\partial E_{\mathbf{\Gamma}}[\sum_{l=1}^n \lambda^l + \alpha\lambda^f]}{\partial K_f} \leq 0$. Likewise $\frac{\partial E_{\mathbf{\Gamma}}[\sum_{l=1}^n \lambda^l + \alpha\lambda^f]}{\partial K_l} = E_{\mathbf{\Gamma}}[\frac{\partial \sum_{l=1}^n \lambda^l + \alpha\lambda^f}{\partial K_l}]$. We need to show that $\frac{\partial \sum_{l=1}^n \lambda^l + \alpha\lambda^f}{\partial K_l} \leq 0$ in every feasible domain.

$$\frac{\partial \sum_{l=1}^n \lambda^l + \alpha\lambda^f}{\partial K_l} = \begin{cases} 0 & \Omega_j \text{ with } \lambda_j^f = 0 \text{ and } l \notin \Phi_j^3, \\ -2 & \Omega_j \text{ with } \lambda_j^f = 0 \text{ and } l \in \Phi_j^3, \\ 0 & \Omega_j \text{ with } \lambda_j^f > 0 \text{ and } l \notin \Phi_j^3, \\ \frac{2(\alpha - V_j - J_j)}{J_j} & \Omega_j \text{ with } \lambda_j^f > 0 \text{ and } l \in \Phi_j^3, \end{cases}$$

where $V_j = |\Phi_j^3|$ and $J_j = |\Phi_j^2|$. $\alpha \leq V_j + J_j$ in every domain thus $\frac{\partial \sum_{l=1}^n \lambda^l + \alpha\lambda^f}{\partial K_l} \leq 0$ in every domain which leads to that $\frac{\partial E_{\mathbf{\Gamma}}[\sum_{l=1}^n \lambda^l + \alpha\lambda^f]}{\partial K_l} \leq 0$. This completes the proof. $\square$

B.2.11 Proof of Theorem 4:

Suppose Assumptions **(2)** and **(3)** hold for $n(\geq 3)$-product **SN** without cross-price effects. Then, by Lemma 15 and Lemma 16, there exists a constant α such that the linear combination, $\sum_{l=1}^n \lambda^l + \alpha\lambda^f$, is convex in $\boldsymbol{\gamma}$ and is also a decreasing function of $\boldsymbol{K}$ in $[\boldsymbol{K}^*, \bar{\boldsymbol{K}}^*]$. Then, for $\mathbf{\Gamma} \leq_{cx} \bar{\mathbf{\Gamma}}$,

$$\begin{aligned} & (E_{\bar{\mathbf{\Gamma}}}[\textstyle\sum_{l=1}^n(\partial\Pi^*(\boldsymbol{K},\boldsymbol{\gamma})/\partial K_l + \alpha\partial\Pi^*(\boldsymbol{K},\boldsymbol{\gamma})/\partial K_f])|_{\boldsymbol{K}=\boldsymbol{K}^*} \\ \geq\ & (E_{\mathbf{\Gamma}}[\textstyle\sum_{l=1}^n(\partial\Pi^*(\boldsymbol{K},\boldsymbol{\gamma})/\partial K_l + \alpha\partial\Pi^*(\boldsymbol{K},\boldsymbol{\gamma})/\partial K_f])|_{\boldsymbol{K}=\boldsymbol{K}^*} \\ =\ & \textstyle\sum_{l=1}^n c_l + \alpha c_f \\ =\ & (E_{\bar{\mathbf{\Gamma}}}[\textstyle\sum_{l=1}^n \partial\Pi^*(\boldsymbol{K},\boldsymbol{\gamma})/\partial K_l + \alpha\partial\Pi^*(\boldsymbol{K},\boldsymbol{\gamma})/\partial K_f])|^*_{\boldsymbol{K}=\bar{\boldsymbol{K}}}. \end{aligned}$$

Let $Z(\boldsymbol{K}) \equiv E_{\bar{\mathbf{\Gamma}}}[\sum_{l=1}^n \partial\Pi^*(\boldsymbol{K},\boldsymbol{\gamma})/\partial K_l + \alpha\partial\Pi^*(\boldsymbol{K},\boldsymbol{\gamma})/\partial K_f]$, which we show to be a decreasing function of $\boldsymbol{K}$ in $[\boldsymbol{K}^*, \bar{\boldsymbol{K}}^*]$. Then, at least one of $K_1^*, \cdots, K_n^*, K_f^*$ must increase to satisfy the optimality condition when $\mathbf{\Gamma}$ increases in convex order to $\bar{\mathbf{\Gamma}}$. $\square$

B.2.12 Proof of Lemma 17:

(i) By Lipschitz continuity condition of order one, $\frac{\partial \lambda^l}{\partial K_i} = \frac{\partial E_{\mathbf{\Gamma}}[\lambda^l(\boldsymbol{K},\boldsymbol{\gamma})]}{\partial K_i} = E_{\mathbf{\Gamma}}(\frac{\partial \lambda^l(\boldsymbol{K},\boldsymbol{\gamma})}{\partial K_i})$, $l, i = 1, \cdots, n, f$, and $\Lambda_{K_l K_i} = \frac{\partial^2 E_{\boldsymbol{\gamma}}[\Pi(\boldsymbol{K},\boldsymbol{\gamma})]}{\partial K_l \partial K_i} = \partial E_{\mathbf{\Gamma}}[\frac{\partial \Pi(\boldsymbol{K},\boldsymbol{\gamma})}{\partial K_l}]/\partial K_i = \frac{\partial E_{\mathbf{\Gamma}}[\lambda^l(\boldsymbol{\gamma})]}{\partial K_i} = E_{\mathbf{\Gamma}}(\frac{\partial \lambda^l(\boldsymbol{K},\boldsymbol{\gamma})}{\partial K_i})$, $l, i = 1, \cdots, n, f$.

(ii) By the concavity of $\Lambda(\boldsymbol{K})$ or $E_{\mathbf{\Gamma}}[\Pi(\boldsymbol{K},\boldsymbol{\gamma})]$.

(iii) $\Lambda_{K_f K_l} = \Lambda_{K_l K_f} = \frac{\partial^2 E_{\mathbf{\Gamma}}[\Pi(\boldsymbol{K},\boldsymbol{\gamma})]}{\partial K_f \partial K_l} = E_{\mathbf{\Gamma}}(\frac{\partial \lambda^f(\boldsymbol{K},\boldsymbol{\gamma})}{\partial K_l})$, then, for any domain $\Omega_j, j \in \Psi$,

$$\frac{\partial \lambda^f(\boldsymbol{K},\boldsymbol{\gamma})}{\partial K_l} = \begin{cases} \frac{2}{J_j} & l \in \Phi_j^3 \text{ and } \left|\Phi_j^2\right| = J_j, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\frac{\partial \lambda^f(\boldsymbol{K},\boldsymbol{\gamma})}{\partial K_l} \geq 0$ in any domain $\Omega_j, j \in \Psi$, then $E_{\boldsymbol{\gamma}}[\frac{\partial \lambda^f(\boldsymbol{K},\boldsymbol{\gamma})}{\partial K_l}]$ or $\Lambda_{K_f K_l} = \Lambda_{K_l K_f} \geq 0$, $l = 1, \cdots, n$.

$\Lambda_{K_l K_i} = \frac{\partial^2 E_{\mathbf{\Gamma}}[\Pi(\boldsymbol{K},\boldsymbol{\gamma})]}{\partial K_l \partial K_i} = E_{\mathbf{\Gamma}}(\frac{\partial \lambda^l(\boldsymbol{K},\boldsymbol{\gamma})}{\partial K_i})$, then, for any domain $\Omega_j, j \in \Psi$,

$$\frac{\partial \lambda^l(\boldsymbol{K},\boldsymbol{\gamma})}{\partial K_i} = \begin{cases} 0 & \text{if } l \in \Phi_j^1, \\ 0 & \text{if } l \in \Phi_j^2, \\ -\frac{\partial \lambda^f(\boldsymbol{K},\boldsymbol{\gamma})}{\partial K_i} & \text{if } l \in \Phi_j^3. \end{cases}$$

Since $\frac{\partial \lambda^l(\boldsymbol{K},\boldsymbol{\gamma})}{\partial K_i} \leq 0$ in any domain $\Omega_j, j \in \Psi$, then $E_{\mathbf{\Gamma}}(\frac{\partial \lambda^l(\boldsymbol{K},\boldsymbol{\gamma})}{\partial K_i})$ or $\Lambda_{K_l K_i} \leq 0$, $l, i = 1, \cdots, n, l \neq i$.

(iv) $\Lambda_{K_f K_l} + \Lambda_{K_l K_l} = \frac{\partial^2 E_{\mathbf{\Gamma}}[\Pi(\boldsymbol{K},\boldsymbol{\gamma})]}{\partial K_f \partial K_l} + \frac{\partial^2 E_{\mathbf{\Gamma}}[\Pi(\boldsymbol{K},\boldsymbol{\gamma})]}{\partial K_l \partial K_l} = E_{\mathbf{\Gamma}}(\frac{\partial(\lambda^f(\boldsymbol{K},\boldsymbol{\gamma})+\lambda^l(\boldsymbol{K},\boldsymbol{\gamma}))}{\partial K_l})$, then, for any domain $\Omega_j, j \in \Psi$,

$$\frac{\partial(\lambda^f(\boldsymbol{K},\boldsymbol{\gamma})+\lambda^l(\boldsymbol{K},\boldsymbol{\gamma}))}{\partial K_l} = \begin{cases} 0 & \text{if } \lambda^f(\boldsymbol{K},\boldsymbol{\gamma}) = \lambda^l(\boldsymbol{K},\boldsymbol{\gamma}) = 0, \\ -2 & \text{if } \lambda^f(\boldsymbol{K},\boldsymbol{\gamma}) = 0, \lambda^l(\boldsymbol{K},\boldsymbol{\gamma}) > 0, \\ 0 & \text{if } \lambda^f(\boldsymbol{K},\boldsymbol{\gamma}) > 0, \lambda^l(\boldsymbol{K},\boldsymbol{\gamma}) = 0, \\ -2 & \text{if } \lambda^f(\boldsymbol{K},\boldsymbol{\gamma}) > 0, \lambda^l(\boldsymbol{K},\boldsymbol{\gamma}) > 0. \end{cases}$$

Since $\frac{\partial(\lambda^f(\boldsymbol{K},\boldsymbol{\gamma})+\lambda_l(\boldsymbol{K},\boldsymbol{\gamma}))}{\partial K_l} \leq 0$ for any domain $\Omega_j, j \in \Psi$, then $\Lambda_{K_f K_l} + \Lambda_{K_l K_l} \leq 0$, $l = 1, \cdots, n$.

$\Lambda_{K_f K_f} + \Lambda_{K_l K_f} = \frac{\partial^2 E_{\boldsymbol{\Gamma}}[\Pi(\boldsymbol{K},\boldsymbol{\gamma})]}{\partial K_f \partial K_f} + \frac{\partial^2 E_{\boldsymbol{\Gamma}}[\Pi(\boldsymbol{K},\boldsymbol{\gamma})]}{\partial K_l \partial K_f} = E_{\boldsymbol{\Gamma}}(\frac{\partial(\lambda^f(\boldsymbol{K},\boldsymbol{\gamma})+\lambda^l(\boldsymbol{K},\boldsymbol{\gamma}))}{\partial K_f})$, then, for any domain $\Omega_j, j \in \Psi$,

$$\frac{\partial(\lambda^f(\boldsymbol{K},\boldsymbol{\gamma})+\lambda^l(\boldsymbol{K},\boldsymbol{\gamma}))}{\partial K_f} = \begin{cases} 0 & \text{if } \lambda^f(\boldsymbol{K},\boldsymbol{\gamma}) = \lambda^l(\boldsymbol{K},\boldsymbol{\gamma}) = 0 \\ 0 & \text{if } \lambda^f(\boldsymbol{K},\boldsymbol{\gamma}) = 0, \lambda^l(\boldsymbol{K},\boldsymbol{\gamma}) > 0 \\ -\frac{2}{J_j} & \text{if } \lambda^f(\boldsymbol{K},\boldsymbol{\gamma}) > 0, \lambda^l(\boldsymbol{K},\boldsymbol{\gamma}) = 0 \\ 0 & \text{if } \lambda^f(\boldsymbol{K},\boldsymbol{\gamma}) > 0, \lambda^l(\boldsymbol{K},\boldsymbol{\gamma}) > 0 \end{cases}$$

Since $\frac{\partial(\lambda^f(\boldsymbol{K},\boldsymbol{\gamma})+\lambda_l(\boldsymbol{K},\boldsymbol{\gamma}))}{\partial K_f} \leq 0$ in any domain $\Omega_j, j \in \Psi$, then $\Lambda_{K_f K_f} + \Lambda_{K_l K_f} \leq 0$, $l = 1, \cdots, n$.

$\Lambda_{K_f K_i} + \Lambda_{K_l K_i} = \frac{\partial^2 E_{\boldsymbol{\Gamma}}[\Pi(\boldsymbol{K},\boldsymbol{\gamma})]}{\partial K_f \partial K_i} + \frac{\partial^2 E_{\boldsymbol{\Gamma}}[\Pi(\boldsymbol{K},\boldsymbol{\gamma})]}{\partial K_l \partial K_i} = E_{\boldsymbol{\Gamma}}(\frac{\partial(\lambda^f(\boldsymbol{K},\boldsymbol{\gamma})+\lambda^l(\boldsymbol{K},\boldsymbol{\Gamma}))}{\partial K_i})$, then, for any domain $\Omega_j, j \in \Psi$,

$$\frac{\partial(\lambda^f(\boldsymbol{K},\boldsymbol{\Gamma}) + \lambda^l(\boldsymbol{K},\boldsymbol{\Gamma}))}{\partial K_i}$$

$$= \begin{cases} 0 & \text{if } \lambda^f(\boldsymbol{K},\boldsymbol{\Gamma}) = \lambda^l(\boldsymbol{K},\boldsymbol{\Gamma}) = 0 \\ 0 & \text{if } \lambda^f(\boldsymbol{K},\boldsymbol{\Gamma}) = 0, \lambda^l(\boldsymbol{K},\boldsymbol{\Gamma}) > 0 \\ \frac{2}{J_j} & \text{if } \lambda^f(\boldsymbol{K},\boldsymbol{\Gamma}) > 0, \lambda^l(\boldsymbol{K},\boldsymbol{\Gamma}) = 0, \text{ and } i \in \Phi_j^3 \\ 0 & \text{if } \lambda^f(\boldsymbol{K},\boldsymbol{\Gamma}) > 0, \lambda^l(\boldsymbol{K},\boldsymbol{\Gamma}) = 0, \text{ and } i \notin \Phi_j^3 \\ 0 & \text{if } \lambda^f(\boldsymbol{K},\boldsymbol{\Gamma}) > 0, \lambda^l(\boldsymbol{K},\boldsymbol{\Gamma}) > 0 \end{cases}$$

Since $\frac{\partial(\lambda^f(\boldsymbol{K},\boldsymbol{\Gamma})+\lambda_l(\boldsymbol{K},\boldsymbol{\gamma}))}{\partial K_i} \geq 0$ in any domain in any domain $\Omega_j, j \in \Psi$, then $\Lambda_{K_f K_i} + \Lambda_{K_l K_i} \geq 0$, $l, i = 1, \cdots, n, l \neq i$.

This completes the proof. $\square$

B.2.13 Proof of Corollary 1:

$\sum_{l \in N} \lambda^l + |N| \lambda^f = \sum_{i \in \{1,\cdots,n\}} \lambda^i + n\lambda^f - (\sum_{l' \in \overline{N}} \lambda^{l'} + (n - |N|)\lambda^f)$. Since $\alpha = n$, $\sum_{i \in \{1,\cdots,n\}} \lambda^i + n\lambda^f = \sum_{i \in \{1,\cdots,n\}} \lambda^i + a\lambda^f$ and it is decreasing in K_l, $l \in N$. While $\sum_{l' \in \overline{N}}(\lambda^{l'} + \lambda^f)$ is increasing in K_l since $l \notin \overline{N}$. Thus we have $\sum_{l \in N} \lambda^l + |N| \lambda^f$ is decreasing in K_l, where $l \in N$. In addition we have $\frac{\partial \sum_{l \in N}(\lambda^l + \lambda^f)}{\partial K_f} = \sum_{l \in N}(\Lambda_{K_l K_f} + \Lambda_{K_f K_f}) \leq 0$ and $\frac{\partial \sum_{l \in N}(\lambda^l + \lambda^f)}{\partial K_{l'}} = \sum_{l \in N}(\Lambda_{K_l K_{l'}} + \Lambda_{K_f K_{l'}}) \geq 0$. This completes the proof. $\square$

B.2.14 Proof of Examples 4-7:

Example 4: Suppose $\lambda^f > c_f$, and $\lambda^l + \lambda^f > c_l + c_f$, where $l = 1, \cdots, n$ but K_f decreases or stays the same. λ^f is decreasing in K_f and increasing in K_l. K_f decreases or stays the same thus λ^f must be reduced by decreasing at least one K_l. Now consider $\lambda^l + \lambda^f > c_l + c_f$. $\lambda^l + \lambda^f$

is decreasing in K_l and K_f thus must be reduced by decreasing another K_i $(i \neq l)$. Now consider $\lambda^i + \lambda^l + 2\lambda^f > c_i + c_l + 2c_f$, similarly we must decrease another $K_k(k \neq l, k \neq i)$. Repeating these we have that all K_l, where $l = 1, \cdots, n$, must decrease. While we must have at least one of resources increases. We have a contradiction. Consequently, K_f must increase.

Example 5: If K_f decreases or stays the same then at least one K_i, $i \in \{1, \cdots, n\}$, must increase. Consider $\sum_{l\in\{1,\cdots,n\}/\{i\}} \lambda^l + (n-1)\lambda^f > \sum_{i\in\{1,\cdots,n\}/\{i\}} c_l + (n-1)c_f$. Since K_f decreases or stays the same and K_i increases, then at least one $K_{i'} \in \{1, \cdots, n\}/\{i\}$ must increase. Consider $\sum_{i\in\{1,\cdots,n\}/\{i,i'\}} \lambda^l + (n-2)\lambda^f > \sum_{i\in\{1,\cdots,n\}/\{i,i'\}} c_l + (n-2)c_f$. Similarly at least one $K_{i''} \in \{1, \cdots, n\}/\{i, i'\}$ must increase. Repeating these we have that all K_l, where $i = 1, \cdots, n$, must increase.

Example 6: Only two cases to consider: (i) K_f increases or (ii) K_f decreases or stays the same.

(i) Suppose K_i decreases or stay the same and K_f increases. First of all we know that at least one K_l, $l = 1, \cdots, n, l \neq i$, must increase to increase λ^f. Now consider $\sum_{l\in\{1,\cdots,n\}/\{i\}} \lambda^l + (n-1)\lambda^f < \sum_{l\in\{1,\cdots,n\}/\{i\}} c_l + (n-1)c_f$. At least one $K_{i'}$, where $i' \in \{1, \cdots, n\}/\{i\}$ must decrease to increase $\sum_{l\in\{1,\cdots,n\}/\{i\}} \lambda^l + \lambda^f$. while consider $\sum_{l\in\{1,\cdots,n\}/\{i,i'\}} \lambda^l + (n-2)\lambda^f$, Similarly at least one $K_{i''}$ must decrease, where $i'' \in \{1, \cdots, n\}/\{i, i'\}$. Repeating these we have that all K_l,where $l = 1, \cdots, n$ and $l \neq i$, must decrease or stay the same which contradicts with that at least one K_l, where $l = 1, \cdots, n$ and $l \neq i$, must increase.

(ii) Suppose K_i decreases or stay the same and K_f decreases or stays the same. First of all we know that at least one K_l, where $l = 1, \cdots, n, l \neq i$, must increase. Consider $\lambda^i + \lambda^f > c_i + c_f$, at least one $K_{i'}$, $i' \in \{1, \cdots, n\}/\{i\}$, must decrease to decrease $\lambda^i + \lambda^f$. Now consider $\lambda^i + \lambda^{i'} + 2\lambda^f$ which must be greater than $c_i + c_{i'} + 2c_f$, otherwise we have $\sum_{l\in\{1,\cdots,n\}} \lambda^l + n\lambda^f < \sum_{i\in\{1,\cdots,n\}} c_l + nc_f$. Since $\lambda^i + \lambda^{i'} + 2\lambda^f > c_i + c_{i'} + 2c_f$, and K_i, $K_{i'}$, and K_f decrease or stay the same, then at least one $K_{i''}$, $i'' \in \{1, \cdots, n\}/\{i, i'\}$, must decrease. Now consider $\lambda^i + \lambda^{i'} + \lambda^{i''} + 3\lambda^f > c_i + c_i + c_{i''} + 3c_f$, similarly, we have another $K_{i'''}$, $i''' \in \{1, \cdots, n\}/\{i, i', i''\}$ must decrease. Repeating these we have that all K_l, where $l = 1, \cdots, n, l \neq i$, must decrease which again contradicts with that at least one K_l, where $l = 1, \cdots, n, l \neq i$, must increase.

Example 7: Only two cases to consider:

(i) Suppose both K_i and K_f decrease or stay the same. By the same arguments as the second case in Example 6. We have that K_f and all K_l, where $l = 1, \cdots, n$, must decrease which contradicts with that at least one resources must increase.

(ii) From the first part of the proof if K_i decrease or stays the same, then K_f must increase. By the same arguments as the first case in Example 6 we have that all K_l, $l = 1, \cdots, n$ and $l \neq i$, must decrease or stay the same.

This completes the proof. $\square$

B.2.15 Proof of Theorem 5:

Part 1. Consider $\mathbf{\Gamma} \leq_{\text{sm}} \overline{\mathbf{\Gamma}}$, with respective optimal capacities $\boldsymbol{K}^*$ and $\overline{\boldsymbol{K}}^*$. From Eq. (2.4), $V(\boldsymbol{K}^*) = \Pi^*(\boldsymbol{K}^*) - \boldsymbol{c}^T\boldsymbol{K}^*$. Then, if $\Pi^*(\boldsymbol{K}^*)$ is supermodular in $\boldsymbol{\gamma}$, then we have $E_{\mathbf{\Gamma}}[V(\boldsymbol{K}^*)] \leq E_{\overline{\mathbf{\Gamma}}}[V(\boldsymbol{K}^*)] \leq E_{\overline{\mathbf{\Gamma}}}[V(\overline{\boldsymbol{K}}^*)]$, where the first inequality follows by definition of supermodular order (see Definition A1) and the second inequality follows by the optimality of $\overline{\boldsymbol{K}}^*$ for $\overline{\mathbf{\Gamma}}$. Similarly, if $\Pi^*(\boldsymbol{K}^*)$ is submodular in $\boldsymbol{\gamma}$, then we have $E_{\overline{\mathbf{\Gamma}}}[V(\overline{\boldsymbol{K}}^*)] \leq E_{\mathbf{\Gamma}}[V(\overline{\boldsymbol{K}}^*)] \leq E_{\mathbf{\Gamma}}[V(\boldsymbol{K}^*)]$, where the first inequality follows by definition of supermodular order and the second inequality follows by the optimality of $\boldsymbol{K}^*$ for $\mathbf{\Gamma}$. Hence, for the proof of part 1, we only need to show that $\Pi^*(\boldsymbol{K}^*)$ is submodular in the **FN**$(v = 0)$, supermodular in the **DN**, and submodular in the **SN**$(v = 0)$.

For the two-product **FN**, from Bish et al. (2010), we have

$$\frac{\partial \Pi^*(K_f, \boldsymbol{\gamma})}{\partial \gamma_2} = \begin{cases} \frac{v\gamma_1+\gamma_2}{2(1-v^2)}, & \text{if } \Omega_1 \\ \frac{-\gamma_1+\gamma_2}{4(1+v)} + \frac{K_f}{2(1-v)}, & \text{if } \Omega_2 \\ \frac{K_f v}{1-v^2}, & \text{if } \Omega_3 \\ \frac{K_f}{1-v^2}, & \text{if } \Omega_4 \end{cases},$$

where Ω_i, $i = 1, 2, 3, 4$, are as defined in Example 1. Then, for $v > 0$, $\frac{\partial \Pi^*(q,S^h)}{\partial \gamma_2}$ is increasing in γ_1 in Ω_1, but decreasing in γ_2 in Ω_2. Therefore, we cannot conclude whether $V(K_f, \boldsymbol{\gamma})$ is supermodular, submodular, or neither in $\boldsymbol{\gamma}$. On the other hand, for $v = 0$, it is easy to show that $\frac{\partial \Pi^*(q,S^h)}{\partial \gamma_2}$ is non-increasing in every domain, and thus, $V(K_f, \boldsymbol{\gamma})$ is submodular for any $K_f \geq 0$.

For the two-product **DN**, we have

$$\Pi^*(\boldsymbol{K}, \boldsymbol{\gamma}) = \begin{cases} \frac{\gamma_1^2+2v\gamma_1\gamma_2+\gamma_2^2}{4(1-v^2)}, & \text{if } \Omega_1 \\ -(K_2 - \frac{\gamma_2}{2})^2 + \frac{\gamma_1^2+2v\gamma_1\gamma_2+\gamma_2^2}{4(1-v^2)}, & \text{if } \Omega_2 \\ -(K_1 - \frac{\gamma_1}{2})^2 + \frac{\gamma_1^2+2v\gamma_1\gamma_2+\gamma_2^2}{4(1-v^2)}, & \text{if } \Omega_3 \\ \frac{K_1(\gamma_1+v\gamma_2)+K_2(\gamma_2+v\gamma_1)-K_1^2-K_2^2-2vK_1K_2}{(1-v^2)}, & \text{if } \Omega_4 \end{cases},$$

$$\begin{aligned} \text{where } \Omega_1 &= \{\gamma_1 \leq 2K_1, \gamma_2 \leq 2K_2\}, \\ \Omega_2 &= \{\gamma_1 + v\gamma_2 \leq 2K_1 + 2vK_2, \gamma_2 > 2K_2\}, \\ \Omega_3 &= \{v\gamma_1 + \gamma_2 \leq 2vK_1 + 2K_2, \gamma_1 > 2K_1\}, \\ \Omega_4 &= \{\gamma_1 + v\gamma_2 > 2K_1 + 2vK_2, v\gamma_1 + \gamma_2 > 2vK_1 + 2K_2\}. \end{aligned}$$

Then,

$$\frac{\partial \Pi^*(\boldsymbol{K},\boldsymbol{\gamma})}{\partial \gamma_2} = \begin{cases} \frac{2v\gamma_1+2\gamma_2}{4(1-v^2)}, & \text{if } \Omega_1 \\ 2(K_2-\frac{\gamma_2}{2})+\frac{2v\gamma_1+2\gamma_2}{4(1-v^2)}, & \text{if } \Omega_2 \\ \frac{2v\gamma_1+2\gamma_2}{4(1-v^2)}, & \text{if } \Omega_3 \\ \frac{K_1 v+K_2}{(1-v^2)}, & \text{if } \Omega_4 \end{cases}.$$

Clearly, $\frac{\partial \Pi^*(\boldsymbol{K},\boldsymbol{\gamma})}{\partial \gamma_2}$ is an increasing function of γ_1 for $v \geq 0$. Thus, $V(\boldsymbol{K},\boldsymbol{\gamma})$ is supermodular for any $\boldsymbol{K} \geq \mathbf{0}$.

For the two-product **SN** with $v = 0$, we have

$$\Pi^*(\boldsymbol{K},\boldsymbol{\gamma}) = \begin{cases} (\frac{\gamma_1^2}{4}+\frac{\gamma_2^2}{4}) & \text{if } \boldsymbol{\gamma} \in \Omega_1 \\ K_1(\gamma_1-K_1)+\frac{\gamma_2^2}{4} & \text{if } \boldsymbol{\gamma} \in \Omega_2 \\ K_2(\gamma_2-K_2)+\frac{\gamma_1^2}{4} & \text{if } \boldsymbol{\gamma} \in \Omega_3 \\ (\frac{\gamma_1-\gamma_2+2K_f}{4})(\frac{3\gamma_1+\gamma_2-2K_f}{4})+(\frac{\gamma_2-\gamma_1+2K_f}{4})(\frac{3\gamma_2+\gamma_1-2K_f}{4}) & \text{if } \boldsymbol{\gamma} \in \Omega_4 \\ K_1(\gamma_1-K_1)+(K_f-K_1)(\gamma_2-K_f+K_1) & \text{if } \boldsymbol{\gamma} \in \Omega_5 \\ K_2(\gamma_2-K_2)+(K_f-K_2)(\gamma_1-K_f+K_2) & \text{if } \boldsymbol{\gamma} \in \Omega_6 \end{cases},$$

where Ω_i, $i = 1, \cdots, 6$, are as defined in Eqs. (3.6)–(3.11). Then,

$$\frac{\partial \Pi^*(\boldsymbol{K},\boldsymbol{\gamma})}{\partial \gamma_2} = \begin{cases} \frac{\gamma_2}{2} & \text{if } \boldsymbol{\gamma} \in \Omega_1 \\ \frac{\gamma_2}{2} & \text{if } \boldsymbol{\gamma} \in \Omega_2 \\ K_2 & \text{if } \boldsymbol{\gamma} \in \Omega_3, \\ \frac{-\gamma_1+\gamma_2+2K_f}{4} & \text{if } \boldsymbol{\gamma} \in \Omega_4 \\ K_f - K_1 & \text{if } \boldsymbol{\gamma} \in \Omega_5 \\ K_2 & \text{if } \boldsymbol{\gamma} \in \Omega_6 \end{cases}.$$

Since $\frac{\partial \Pi^*(\boldsymbol{K},\boldsymbol{\gamma})}{\partial \gamma_2}$ is a decreasing function of γ_1, $V(\boldsymbol{K},\boldsymbol{\gamma})$ is submodular for any $\boldsymbol{K} \geq \mathbf{0}$.

Part 2. Consider the two-product **DN**. In the following, we consider, without loss of generality, $\lambda^1(\boldsymbol{\gamma})$. We derive, for $l = 1, 2$:

$$\frac{\partial \lambda^1(\boldsymbol{\gamma})}{\partial \gamma_l} = \begin{cases} 0, & \text{if } \boldsymbol{\gamma} \in \Omega_1 \\ 0, & \text{if } \boldsymbol{\gamma} \in \Omega_2 \\ W_{1l} - \frac{W_{12}}{W_{22}} W_{2l}, & \text{if } \boldsymbol{\gamma} \in \Omega_3 \\ W_{1l}, & \text{if } \boldsymbol{\gamma} \in \Omega_4 \end{cases}.$$

Thus, we have the desired sequence: $\Omega_4, \Omega_3, \Omega_2, \Omega_1$, which satisfies (3.18), and hence, by Lemma A2 and Corollary A1, $\lambda_1(\boldsymbol{\gamma})$ is supermodular. Then, by Definition A1, we have that if $\mathbf{\Gamma}$ increases in supermodular order, then $E_{\mathbf{\Gamma}}[\lambda^l(\boldsymbol{K},\mathbf{\Gamma})]$ will increase. Then, by Lemma A4, it follows, similar to the proof of part 2 of Theorem 2, that at least one of the optimal capacities K_1^* or K_2^* will increase. $\square$

B.3 Appendix for Chapter 4

B.3.1 KKT First-order Conditions in Lemma 18

The No Left-over Setting: KKT First-order Conditions for Problem **QS**:

$$- c_k + w(|I|)\,\mathbb{E}[Y_k|\boldsymbol{Y} \in \Omega_k]\,\Pr(\Omega_k) - \lambda = 0, \quad k \in I \tag{B.28}$$

$$\lambda\,(\sum_{k\in I} Q_k^* - N) = 0, \tag{B.29}$$

where $\Omega_k = \{\boldsymbol{Y}|Y_k Q_k^* = \min_{j\in I}\{Y_j Q_j^*\}, Y_k Q_k^* < d\}$, $k \in I$.

The Left-over Setting: KKT First-order Conditions for Problem **QS-LO**:

$$\begin{aligned} &-c_k + (w(|I|) - w(|I|-1))\,\mathbb{E}[Y_k|\boldsymbol{Y} \in \Omega_k]\,\Pr(\Omega_k) \\ &+w(|I|-1)\,\mathbb{E}[Y_k|\boldsymbol{Y} \in \bigcup_{i\in I\setminus\{k\}} \Omega_{ik}]\,\Pr(\bigcup_{i\in I\setminus\{k\}} \Omega_{ik}) - \lambda = 0,\ k \in I \end{aligned} \tag{B.30}$$

$$\lambda^{LO}\,(\sum_{k\in I} Q_k^{LO*} - N) = 0, \tag{B.31}$$

where $\Omega_{ki} = \{\boldsymbol{Y}|Y_k Q_k^{LO*} = \min_{j\in I}\{Y_j Q_j^{LO*}\}, Y_i Q_i^{LO*} = \min_{j\in I\setminus\{k\}}\{Y_j Q_j^{LO*}\}, Y_k Q_k^{LO*} < d, Y_i Q_i^{LO*} < d\}$, $i,k \in I, i \neq k$, and $\Omega_k = \{\boldsymbol{Y}|Y_k Q_k^{LO*} = \min_{j\in I}\{Y_j Q_j^{LO*}\}, Y_k Q_k^{LO*} < d\}$, $k \in I$.

B.3.2 Data and Sources

B.3.3 Proof of Lemma 18:

The Manufacturer's Problem: **QS**: The expression, $\min\{d, \min_{k\in I}\{Y_k Q_k\}\}$, is jointly concave in $\boldsymbol{Q}$, and expectation preserves concavity. Then, since Constraint (4.3) is linear in $\boldsymbol{Q}$, the result follows.

Table B.2: Summary of parameter values and data sources

Parameter	Value and data source
d (demand)	(i)Low attack rate scenario: 103 million (Cho (2010)) (ii) Moderate attack rate scenario: 206 million (Cho (2010)) (iii) High attack rate scenario: 281.1 million (CDC recommendation for the US in 2006)
N (number of available eggs)	750 million (Özaltın et al. (2011))
t (expected cost of a flu infection, which includes the expected health care cost, work loss cost, and lost earnings)	\$41 (Weycker et al. (2005))
$w(\|I\|)$ (manufacturer's unit sales price per vaccine)	\$4 $\times$ $\|I\|$ (Derived from the estimate of \$12 per dose of trivalent influenza vaccine (Chick et al. (2008))).
c_k (manufacturer's unit production cost per input quantity) $= c$, $k \in \Theta$	\$1 (Derived from the estimate of \$3 per dose of trivalent influenza vaccine (Deo and Corbett (2009)))

Table B.3: Prevalance group for each candidate strain in Scenarios 1-16

		Scenario															
Virus family	Strain	$d = 103$				$d = 206$				$d = 281.1$							
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A/H1N1	(1)	L	L	M	M	H	H	M	L	M	H	L	H	H	H	L	L
	(2)	M	H	M	H	H	L	H	M	M	M	H	L	H	H	L	L
	(3)	L	H	L	L	M	M	M	L	H	L	L	M	H	H	L	L
	(4)	L	H	M	L	M	L	H	L	L	H	H	M	H	H	L	L
A/H2N2	(5)	H	M	M	H	L	L	H	H	H	L	H	H	H	L	H	L
	(6)	L	H	H	M	H	L	L	L	L	L	H	M	H	L	H	L
B	(7)	M	L	L	M	L	L	L	L	H	L	M	L	H	L	L	H
	(8)	M	M	L	L	M	L	M	L	H	M	H	H	H	L	L	H
	(9)	H	M	L	M	H	H	M	M	H	L	M	L	H	L	L	H

Table B.4: Yield group for each candidate strain in Scenarios 1-16

		Scenario															
Virus family	Strain	$d = 103$				$d = 206$				$d = 281.1$							
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A/H1N1	(1)	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H	H
	(2)	M	M	M	M	M	M	M	M	M	M	M	M	M	M	M	M
	(3)	M	H	M	H	M	M	H	H	M	M	H	H	M	M	M	M
	(4)	L	L	M	M	M	L	M	L	M	M	L	M	L	L	L	L
A/H2N2	(5)	M	M	H	M	H	H	H	M	H	M	H	H	M	M	M	M
	(6)	M	M	M	M	H	M	M	M	H	M	M	H	M	M	M	M
B	(7)	M	M	M	M	M	M	M	M	M	M	M	M	M	M	M	M
	(8)	L	L	M	L	L	L	L	L	M	M	M	M	L	L	L	L
	(9)	M	M	M	M	M	M	M	M	M	M	M	M	M	M	M	M

The Manufacturer's Problem: **QS-LO**: Relabel the strains in set I as $1, \cdots, |I|$, and denote function $V(\boldsymbol{Q}, I, \boldsymbol{Y})$ such that $\Pi_m^{LO}(\boldsymbol{Q}(I, \boldsymbol{Y})) = \mathbb{E}_{\boldsymbol{Y}}[V(\boldsymbol{Q}, I, \boldsymbol{Y})]$. We show that $V(\boldsymbol{Q}, I, \boldsymbol{Y})$ is concave in $\boldsymbol{Q}$ only if $w(|I|) \geq 2w(|I|-1)$. We can write,

$$
\begin{aligned}
&V(\boldsymbol{Q}, I, \boldsymbol{Y}) \qquad\qquad (B.32)\\
&= -\sum_{k \in I} c_k Q_k + (w(|I|) - |I| w(|I|-1)) \min\{d, \min_{k \in I}\{Y_k Q_k\}\} + \\
&+ w(|I|-1)\left(\min\{d, \min_{k \in I, k \neq 1}\{Y_k Q_k\}\} + \cdots + \min\{d, \min_{k \in I, k \neq |I|}\{Y_k Q_k\}\}\right)
\end{aligned}
$$

$$
= \begin{cases}
w(|I|)d, & \text{if } \boldsymbol{Y} \in \Psi_0 \\
w(|I|)Y_1Q_1 - w(|I|-1)Y_1Q_1 + w(|I|-1)\min\{d, \min_{k \in I, k \neq 1}\{Y_k Q_k\}\}, & \text{if } \boldsymbol{Y} \in \Psi_1 \\
w(|I|)Y_2Q_2 - w(|I|-1)Y_2Q_2 + w(|I|-1)\min\{d, \min_{k \in I, k \neq 2}\{Y_k Q_k\}\}, & \text{if } \boldsymbol{Y} \in \Psi_2 \\
\vdots & \\
w(|I|)Y_{|I|}Q_{|I|} - w(|I|-1)Y_{|I|}Q_{|I|} + w(|I|-1)\min\{d, \min_{k \in I, k \neq |I|}\{Y_k Q_k\}\}, & \text{if } \boldsymbol{Y} \in \Psi_{|I|},
\end{cases}
\qquad (B.33)
$$

where,

$$\Psi_0 = \{\boldsymbol{Y}|d \le \min\{d, \min_{k\in I}\{Y_kQ_k\}\}$$
$$\Psi_1 = \{\boldsymbol{Y}|Y_1Q_1 \le \min\{d, \min_{k\in I}\{Y_kQ_k\}\}$$
$$\Psi_2 = \{\boldsymbol{Y}|Y_2Q_2 \le \min\{d, \min_{k\in I}\{Y_kQ_k\}\}$$
$$\vdots$$
$$\Psi_{|I|} = \{\boldsymbol{Y}|Y_{|I|}Q_{|I|} \le \min\{d, \min_{k\in I}\{Y_kQ_k\}\}$$

We show that $V(\boldsymbol{Q}, I, \boldsymbol{Y})$ can be rewritten as a pointwise minimum of a set of concave functions of $\boldsymbol{Q}$:

$$V(\boldsymbol{Q}, I, \boldsymbol{Y}) = \min\Bigg\{ w(|I|)d,$$
$$w(|I|)Y_1Q_1 - w(|I|-1)Y_1Q_1 + w(|I|-1)\min\{d, \min_{k\in I, k\neq 1}\{Y_kQ_k\}\},$$
$$\vdots$$
$$w(|I|)Y_{|I|}Q_{|I|} - w(|I|-1)Y_{|I|}Q_{|I|} + w(|I|-1)\min\{d, \min_{k\in I, k\neq |I|}\{Y_{|I|}Q_{|I|}\}\Bigg\}. \tag{B.34}$$

It suffices to show that for any $\boldsymbol{Q}$, Eqs. $(B.33)$ and $(B.34)$ are equivalent when $w(|I|) \ge 2w(|I|-1)$.

Case 1: $d \le \min\{d, \min_{k\in I}\{Y_kQ_k\}\}$. Then $V(\boldsymbol{Q}, I, \boldsymbol{Y}) = w(|I|)d$ by Eq. (B.33). On the other hand, since,

$$\begin{aligned}
& w(|I|)d - \Big(w(|I|)Y_1Q_1 - w(|I|-1)Y_1Q_1 + w(|I|-1)\min\{d, \min_{k\in I, k\neq 1}\{Y_kQ_k\}\}\Big) \\
=& w(|I|)d - w(|I|)Y_1Q_1 + w(|I|-1)Y_1Q_1 - w(|I|-1)d \\
=& (w(|I|) - w(|I|-1))(d - Y_1Q_1) \le 0,
\end{aligned}$$

we have $w(|I|)d \leq w(|I|)Y_1Q_1 - w(|I|-1)Y_1Q_1 + w(|I|-1)\min\{d, \min_{k\in I, k\neq 1}\{Y_kQ_k\}\}$. Similarly, $w(|I|)d \leq w(|I|)Y_iQ_i - w(|I|-1)Y_iQ_i + w(|I|-1)\min\{d, \min_{k\in I, k\neq i}\{Y_kQ_k\}\}$, $i = 2, \cdots, |I|$. Then, we have that $V(\boldsymbol{Q}, I, \boldsymbol{Y}) = w(|I|-1)d$ by Eq. (B.34).

Case 2: $Y_1Q_1 \leq \min\{d, \min_{k\in I}\{Y_kQ_k\}\}$. Then, $V(\boldsymbol{Q}, I, \boldsymbol{Y}) = w(|I|)Y_1Q_1 - w(|I|-1)Y_1Q_1 + w(|I|-1)\min\{d, \min_{k\in I, k\neq 1}\{Y_kQ_k\}\}$ by Eq. (B.33). On the other hand, since,

$$\begin{aligned}
&\Big(w(|I|)Y_1Q_1 - w(|I|-1)Y_1Q_1 + w(|I|-1)\min\{d, \min_{k\in I, k\neq 1}\{Y_kQ_k\}\}\Big) - w(|I|)d \\
\leq& w(|I|)Y_1Q_1 - w(|I|-1)Y_1Q_1 + w(|I|-1)d - w(|I|)d \\
=& (w(|I|) - w(|I|-1))(Y_1Q_1 - d) \leq 0,
\end{aligned}$$

and

$$\begin{aligned}
&\Big(w(|I|)Y_1Q_1 - w(|I|-1)Y_1Q_1 + w(|I|-1)\min\{d, \min_{k\in I, k\neq 1}\{Y_kQ_k\}\}\Big) \\
-&\Big(w(|I|)Y_2Q_2 - w(|I|-1)Y_2Q_2 + w(|I|-1)\min\{d, \min_{k\in I, k\neq 2}\{Y_kQ_k\}\}\Big) \\
\leq& w(|I|)Y_1Q_1 - w(|I|-1)Y_1Q_1 + w(|I|-1)Y_2Q_2 - w(|I|)Y_2Q_2 + w(|I|-1)Y_2Q_2 - w(|I|-1)Y_1Q_1 \\
=& (w(|I|)(Y_1Q_1 - Y_2Q_2) + 2w(|I|-1)(Y_2Q_2 - Y_1Q_1) \\
=& (w(|I|) - 2w(|I|-1))(Y_1Q_1 - Y_2Q_2) \leq 0,
\end{aligned}$$

We have $w(|I|)Y_1Q_1 - w(|I|-1)Y_1Q_1 + w(|I|-1)\min\{d, \min_{k\in I, k\neq 1}\{Y_kQ_k\}\} \leq \Big(w(|I|)Y_2Q_2 - w(|I|-1)Y_2Q_2 + w(|I|-1)\min\{d, \min_{k\in I, k\neq 2}\{Y_kQ_k\}\}$. Similarly, $w(|I|)Y_1Q_1 - w(|I|-1)Y_1Q_1 + w(|I|-1)\min\{d, \min_{k\in I, k\neq 1}\{Y_kQ_k\}\} \leq w(|I|)Y_iQ_i - w(|I|-1)Y_iQ_i + w(|I|-1)\min\{d, \min_{k\in I, k\neq i}\{Y_kQ_k\}\}$, $i = 3, \cdots, |I|$. Then, we have that $V(\boldsymbol{Q}, I, \boldsymbol{Y}) = w(|I|)Y_1Q_1 - w(|I|-1)Y_1Q_1 + w(|I|-1)\min\{d, \min_{k\in I, k\neq 1}\{Y_kQ_k\}\}$ by Eq. (B.34).

For cases where $Y_iQ_i \leq \min\{d, \min_{k\in I}\{Y_kQ_k\}\}$, $i = 2, \cdots, |I|$, we can show the equivalence of Eq.s (B.33) and (B.34) similarly. Consequently, $V(\boldsymbol{Q}, I, \boldsymbol{Y})$ can be expressed by Eq. (B.34) when $w(|I|) \geq 2w(|I|-1)$. Since $w(|I|)d$, and $w(|I|)Y_iQ_i - w(|I|-1)Y_iQ_i + w(|I|-1)\min\{d, \min_{k\in I, k\neq i}\{Y_kQ_k\}\}$, $i = 1, \cdots, |I|$, are concave in $\boldsymbol{Q}$. Then, the result follows since the expectation preserves the concavity. This completes the proof. □

B.3.4 Proof of Lemma 21:

The manufacturer's expected profit, Π_m. Suppose $\boldsymbol{Y} \geq_{st} \boldsymbol{Y'}$, and denote their corresponding optimal solutions by $\boldsymbol{Q}^*$ and $\boldsymbol{Q'}^*$, respectively. Then,

$$\begin{aligned}
\Pi_m(\boldsymbol{Q'}^*(I, \boldsymbol{Y'})) &= -\sum_{k\in I} c_k Q_k'^* + w(|I|)\mathbb{E}_{\boldsymbol{Y'}}[\min\{d, \min_{k\in I}\{Y_k' Q_k'^*\}] \\
&\leq -\sum_{k\in I} c_k Q_k'^* + w(|I|)\,\mathbb{E}_{\boldsymbol{Y}}[\min\{d, \min_{k\in I}\{Y_k Q_k'^*\}] \\
&\leq -\sum_{i\in I} c_k Q_k^* + w(|I|)\,\mathbb{E}_{\boldsymbol{Y}}[\min\{d, \min_{k\in I}\{Y_k Q_k^*\}] \\
&= \Pi_m(\boldsymbol{Q}^*(I, \boldsymbol{Y})),
\end{aligned}$$

where the first inequality follows because $\boldsymbol{Y} \geq_{st} \boldsymbol{Y'}$ and $\min\{d, \min_{k\in I}\{Y_k Q_k^*\}$ is an increasing function of $\boldsymbol{Y}$, and the second inequality follows by the optimality of $\boldsymbol{Q}^*$ to $\Pi_m(\boldsymbol{Q}(I, \boldsymbol{Y}))$. The proof of the convex order part follows similarly, because $\min\{d, \min_{k\in I}\{Y_k Q_k^*\}$ is jointly concave in $\boldsymbol{Y}$.

Expected demand satisfied, $\mathbb{E}_{\boldsymbol{Y}}[(DS]$. When Assumption 1 holds, we have $\mathbb{E}_{\boldsymbol{Y}}[(DS(I, (\boldsymbol{Q}^*(I, \boldsymbol{Y}))] = \Pi_m(\boldsymbol{Q}^*(I, \boldsymbol{Y})) + cN$. Then the results follows directly from the above result for Π_m, since the production cost, cN, is a constant.

As defined in the lemma, Assumption 1 states that $c_k = c$, $k \in \Theta$, and Constraint (4.3) is binding in the optimal solution to Problem **QS**.

Expected societal vaccination benefit, Π_g. When Assumption 1 holds, suppose $\boldsymbol{Y} \geq_{st} \boldsymbol{Y'}$ and denote their corresponding optimal strain selection sets by I^* and I'^*, respectively. Then,

$$\begin{aligned}
\Pi_g(I'^*) &= \mathbb{E}_{\boldsymbol{Y'}}[(DS(I'^*))]\left(t\sum_{k\in\Theta}\{\mathbb{E}[E_k]\max_{i\in I'^*}\{b_{ik}\}\} - w(|I'^*|)\right)\\
&\leq \mathbb{E}_{\boldsymbol{Y}}[(DS(I'^*))]\left(t\sum_{k\in\Theta}\{\mathbb{E}[E_k]\max_{i\in I'^*}\{b_{ik}\}\} - w(|I'^*|)\right)\\
&\leq \mathbb{E}_{\boldsymbol{Y}}[(DS(I^*))]\left(t\sum_{k\in\Theta}\{\mathbb{E}[E_k]\max_{i\in I^*}\{b_{ik}\}\} - w(|I^*|)\right)\\
&= \Pi_g(I^*),
\end{aligned}$$

where the first inequality follows due to Assumption 1, and the second inequality follows by the optimality of I^* to $\Pi_g(I^*)$. The proof of the convex order part follows similarly.

The manufacturer's expected profit, Π_g^{LO}. To prove stochastic order part, suppose, without lost of generality, that $I = \{1, \cdots, |I|\}$, and denote function $V(\boldsymbol{Q}, I, \boldsymbol{Y})$ such that $\Pi_m^{LO}(\boldsymbol{Q}(I, \boldsymbol{Y})) = \mathbb{E}_{\boldsymbol{Y}}[V(\boldsymbol{Q}, I, \boldsymbol{Y})]$. By Eq. (B.33), we can rewrite $V(\boldsymbol{Q}, I, \boldsymbol{Y})$ as following:

$$\begin{aligned}
&V(\boldsymbol{Q}, I, \boldsymbol{Y}) \qquad\qquad \text{(B.35)}\\
&= -\sum_{k\in I} c_k Q_k + (w(|I|) - |I|w(|I|-1))\min\{d, \min_{k\in I}\{Y_k Q_k\}\}+\\
&+w(|I|-1)\left(\min\{d, \min_{k\in I, k\neq 1}\{Y_k Q_k\}\} + \cdots + \min\{d, \min_{k\in I, k\neq |I|}\{Y_k Q_k\}\}\right)
\end{aligned}$$

$$= \begin{cases}
w(|I|)d, & \text{if } \boldsymbol{Y}\in\Psi_0\\
w(|I|)Y_1Q_1 - w(|I|-1)Y_1Q_1 + w(|I|-1)\min\{d, \min_{k\in I, k\neq 1}\{Y_kQ_k\}\}, & \text{if } \boldsymbol{Y}\in\Psi_1\\
w(|I|)Y_2Q_2 - w(|I|-1)Y_2Q_2 + w(|I|-1)\min\{d, \min_{k\in I, k\neq 2}\{Y_kQ_k\}\}, & \text{if } \boldsymbol{Y}\in\Psi_2\\
\vdots & \\
w(|I|)Y_{|I|}Q_{|I|} - w(|I|-1)Y_{|I|}Q_{|I|} + w(|I|-1)\min\{d, \min_{k\in I, k\neq |I|}\{Y_kQ_k\}\}, & \text{if } \boldsymbol{Y}\in\Psi_{|I|},
\end{cases}$$

(B.36)

where,

$$\Psi_0 = \{\boldsymbol{Y} | d \leq \min\{d, \min_{k \in I}\{Y_k Q_k\}\}$$

$$\Psi_1 = \{\boldsymbol{Y} | Y_1 Q_1 \leq \min\{d, \min_{k \in I}\{Y_k Q_k\}\}$$

$$\Psi_2 = \{\boldsymbol{Y} | Y_2 Q_2 \leq \min\{d, \min_{k \in I}\{Y_k Q_k\}\}$$

$$\vdots$$

$$\Psi_{|I|} = \{\boldsymbol{Y} | Y_{|I|} Q_{|I|} \leq \min\{d, \min_{k \in I}\{Y_k Q_k\}\}$$

$V(\boldsymbol{Q}, I, \boldsymbol{Y})$ is continuous in $\boldsymbol{Y}$ and is increasing in $\boldsymbol{Y}$ when $w(|I|) \geq w(|I|-1)$ in each domain. Hence, $V(\boldsymbol{Q}, I, \boldsymbol{Y})$ is increasing in $\boldsymbol{Y}$. Suppose $\boldsymbol{Y} \geq_{st} \boldsymbol{Y'}$ and denote their corresponding optimal solutions by $\boldsymbol{Q}^{LO*}$ and $\boldsymbol{Q'}^{LO*}$, respectively. Then,

$$\begin{aligned}
\Pi_m^{LO}(\boldsymbol{Q'}^{LO*}(I, \boldsymbol{Y'})) &= \mathbb{E}_{\boldsymbol{Y'}}\left[V(\boldsymbol{Q'}^{LO*}, I, \boldsymbol{Y'})\right] \leq \mathbb{E}_{\boldsymbol{Y}}\left[V(\boldsymbol{Q'}^{LO*}, I, \boldsymbol{Y})\right] \\
&\leq \mathbb{E}_{\boldsymbol{Y}}\left[V(\boldsymbol{Q}^{LO*}, I, \boldsymbol{Y})\right] = \Pi_m^{LO}(\boldsymbol{Q}^{LO*}(I, \boldsymbol{Y})),
\end{aligned}$$

where the first inequality follows because $V(\boldsymbol{Q}^{LO*}, I, \boldsymbol{Y})$ is increasing in $\boldsymbol{Y}$ if $w(|I|) \geq w(|I|-1)$. The second inequality follows by the optimality of $\boldsymbol{Q}^{LO*}$ to $\Pi_m^{LO}(\boldsymbol{Q}^{LO*}(I, \boldsymbol{Y}))$.

For the convex order part, it follows similarly to the proof of Lemma 18 that $V(\boldsymbol{Q}, I, \boldsymbol{Y})$ is jointly concave in $\boldsymbol{Y}$, then if $w(|I|) \geq 2w(|I|-1)$. Then, the result follows similarly to the proof of convex order part of Π_m. This completes the proof. □