Analysis of Vertical Channel Flow and Heat Transfer Using the Finite-Element Method

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(ABSTRACT)

This investigation addresses the problem of numerically predicting the heat transfer rates between parallel surfaces of the type found in electronic equipment. This has been accomplished through a unique application of the finite-element method for transient or steady-state, two-dimensional mixed convection heat transfer with surface radiation. The approach was specifically geared toward implementation on present engineering workstations. Results are presented for mixed-convection in vertical channels using the full-elliptic form of the Navier-Stokes equations with radiation effects included. The results show that the heat transfer and flow solutions can be significantly affected when not using common approximations and simplifications.
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Nomenclature

A  aspect ratio (L/b)
b  channel width
C  a constant (see local text)
Dₜ  hydraulic diameter (2b for parallel plates)
f  friction factor
fᵢ  body force in the i direction
Gr  Grashof number (see text)
g  gravitational acceleration
h  heat transfer coefficient
k  thermal conductivity
L  channel length (height)
m  index (total number of equations)
Nu  Nusselt number (locally defined)
n  index (number of nodes) or order (see local definition)
P  pressure
ΔP*  nondimensional pressure difference, $ΔP^* = \frac{ΔPb^2}{ρv^2}$
\( \text{Pr} \)  Prandtl number
\( q \)  specified heat flux
\( \text{Ra} \)  Rayleigh number (defined in text)
\( \text{Re} \)  Reynolds number based on hydraulic diameter (2b)
\( \text{RHS} \)  right hand side
\( s \)  plate spacing
\( T \)  temperature (K)
\( t \)  time
\( t_i \)  traction term in the i direction (defined in text)
\( U^* \)  nondimensional velocity (local velocity / average velocity)
\( X^* \)  nondimensional axial coordinate \((x/L)\)
\( Y^* \)  nondimensional cross-stream coordinate \((y/L)\)

**Greek Symbols**

\( \alpha \)  thermal diffusivity
\( \beta \)  coefficient of thermal expansion
\( \Gamma \)  represents length or line
\( \gamma \)  penalty parameter
\( \Delta \)  difference
\( \varepsilon \)  emissivity
\( \eta \)  local coordinate (corresponding to \( y \))
\( \theta \)  dimensionless temperature or time approximation scheme (see local text)
\( \mu \)  dynamic viscosity

**Nomenclature**
\( \nu \)  \hspace{1em} \text{kinematic viscosity} \\
\( \Xi \)  \hspace{1em} \sigma T^4 \\
\( \xi \)  \hspace{1em} \text{local coordinate (corresponding to } x) \\
\( \rho \)  \hspace{1em} \text{density} \\
\( \Sigma \)  \hspace{1em} \text{summation} \\
\( \sigma \)  \hspace{1em} \text{Stefan-Boltzmann constant} \\
\begin{align*} 
(5.670 \times 10^{-8} \text{ W/m}^2\text{K}^4, 0.1714 \times 10^{-8} \text{ Btu/ft}^2\text{K}^4) 
\end{align*} \\
\( \psi \)  \hspace{1em} \text{shape or interpolation function (see local text)} \\
\( \Omega \)  \hspace{1em} \text{represents area}
1.0 Introduction

America's race for space in the 1960's necessitated the miniaturization of electronic equipment. From this development the use of electronics has entered every phase of our lives, from work to recreation to transportation. The failure of critical electronic components can trigger a major loss of resources, if not a modern disaster. Paramount to the operation and reliability of these components is the accurate prediction of the heat transfer rates and surface temperatures involved.

With world-wide competition, present-day manufacturers no longer have the luxury of long lead times from product conception to final production. This lack of development time results in less prototype development and testing (if any) and greater dependence on computer simulation or modelling of the design.

This dissertation addresses the problem of numerically predicting the heat transfer rates between parallel surfaces of the type found in electronic equipment. However, the scope of the work presented here reaches into other areas of heat transfer, for it will be shown that to achieve an accurate prediction of the heat transfer rates, the boundary-layer approximations typically employed for this problem can not be used and buoyancy effects along with radiation losses must be included in the analysis.
The governing equations for this problem form a set of coupled partial differential equations, namely the fluid equations (Navier-Stokes and continuity) and the energy equation. An equation of state couples the fluid solution to the energy equation solution and visa-versa. This may be a weak coupling, as in the case when there are small temperature changes compared to the Reynolds number, and the equations may then be solved independently. However, if the temperature changes in the problem are significant compared to the flow rates, a simultaneous solution must be found. Determination of the need for a coupled solution has traditionally required examination of the nondimensional buoyancy term and the Reynolds number in the Navier-Stokes equation. This demands a more detailed knowledge of the flow than a designer may have at the initial design stage. A method for determining when a coupled solution is necessary will be presented.

Heat transfer problems of this type are usually divided into three distinct areas:

(1) pure natural convection - the mass flow is driven by density variation due to temperature differences in the presence of a gravitational field.

(2) forced convection - an external pressure difference or imposed velocity determines the flow field and density variations have little or no influence on the flow solution.

(3) mixed convection - both an imposed pressure difference and density variations determine the flow field and there may be a strong interaction between the two forces.

The work presented here will include pure natural convection and mixed convection up to the limit where forced convection dominates the problem.
Many different techniques have been developed for numerically solving the governing nonlinear partial differential equations. Most of these methods involve changing the partial differential equations into a system of linear algebraic equations and solving the linear system. The linear system is better able to approximate the true solution as the domain is divided into an increasing number of approximating equations. The various methods differ mainly in the coefficients of the matrix for the solution vector. However, this author has concluded that there is a common thread between the different methods. That is, once one has chosen a method, the advantages of that individual technique or method must be vigorously defended so as to instill in others the wisdom of one's choice. With this admitted, a review of the two methods that have received the most attention follows.

1.1 Finite-Difference Method

In the finite-difference method, algebraic expressions for the operators in the partial differential equations are derived from a Taylor series expansion, with the truncation terms giving an estimate of the order of the method (Carnahan et al., 1969). The first finite-difference schemes have been attributed by Roache (1985) to Richardson (1910) in his solution of Laplace's equation. He developed a relaxation technique and defined the difference between problems that required relaxation (elliptic or jury type) and marching (parabolic) problems. Later Courant, et al. (1928) investigated uniqueness and existence questions for PDEs and established stability requirements. This is known as the CFL stability requirement and requires that the Courant number be small for the numerical method to be stable. From this early beginning until the present, finite-difference techniques have dominated numerical methods in fluid dynamics. Finite-difference
methods are easy for the beginning analyst to understand and the implementation for simple problems and geometries is straightforward. However, when the governing partial differential equation becomes nonlinear and the geometries are no longer simple, finite-difference methods lose much of their attraction.

1.2 Variational Methods

With variational methods the governing partial differential equations are put into an equivalent variational form and the solution is assumed to be a combination of given approximation functions (Reddy, 1984). Under the class of variational methods there are finite-element methods (piecewise application of the variational method), control volume methods, and even integral methods (Anderson, et al., 1984).

The finite-element method is a very general solution technique containing many different types of solution methods. The common fundamental concept is that any continuous quantity, such as velocity, temperature, or pressure can be approximated over a domain by a discrete model composed of piecewise continuous functions defined over a finite number of subdomains. Even finite-difference techniques have been shown (Zienkiewicz, 1975) to fall into a subclass of the general finite-element methodology.

The finite-element method which has long dominated the solution of structural problems has only relatively recently been used for nonlinear fluid dynamics problems. The inertia that is natural when investigators have been developing particular solutions methods, which they then pass on to their students, plays no small part in the reluctance to accept the finite-element solution technique for fluid flow problems. The difficulty in
implementing this technique along with the special considerations required for the Navier-Stokes equations have also hindered the number of investigators selecting this method. This difficulty in implementation is actually only extra attention to detail and programming cost, for which the investigator will be more than rewarded in solution quality and robustness. Indeed, the maximum cell Reynolds number restriction of 4 (see Roache, 1982), required for stability by a centered finite-difference scheme can be as high as 200 without artificial viscosity with the finite-element method employed in this dissertation. This last point should not be taken lightly for it will allow solutions to complex problems at greatly reduced CPU cost and resources.

1.3 Outline and Objective

The following chapters will include a review of the literature, finite-element formulation and implementation, validation and results, and a final summary. Chapter 2 explains the different modes of heat transfer being investigated and reviews the literature. The reviews in Chapter 2 contain pertinent papers on theoretical, experimental, and computational investigations. This review shows that most studies in convective cooling are neglecting critical aspects of the problem of developing flow and heat transfer between parallel plates. Chapter 3 provides the finite-element formulation of the problem. Implementation of this finite-element formulation is described in Chapter 4. From this, computer code execution is straightforward and so will not be furnished. Chapter 5 includes validation and results from the computer simulations and Chapter 6 has vertical channel results. Final conclusions and an overall summary are included in Chapter 7.
The objective of this dissertation is to provide an accurate, robust, and full computational scheme that will not require excessive computer cost for the investigation of the cooling of electronic equipment. Accurate is used in the sense that the method is of high enough order so that excessive grid points and therefore excessive cost are not required. Robust means that techniques such as implicit or explicit false diffusion are not inherent or required. Full means that all basic terms and variations that effect the solution are included and not assumed away at the beginning, the only restrictions being that the flows be two dimensional and laminar. The method that will be presented can easily be extended to three-dimensional problems. However, much of the cooling of electronic equipment is in the laminar flow regime and it has been shown from experimental data (Sparrow and Bahrami, 1980 for edge effects; Wirtz and McAuliffe, 1989 for flow over discrete chips) that a two-dimensional analysis is justified for typical component card and chip spacings. This two-dimensional approximation (which greatly reduces computer cost) will allow a more complete investigation into difficult coupled (heat transfer and flow) problems. The objective of this dissertation is to provide a unique application of the finite-element method for transient or steady-state, two-dimensional mixed convection with surface radiation geared toward implementation on present engineering workstations. Results from this implementation will show that many investigators are neglecting terms of primary importance.

Introduction
2.0 Literature Review

2.1 Chapter Introduction

This chapter will review previous investigations of single and combined mode heat transfer that may be applied to the cooling of electronic equipment. Particular attention is given to heat transfer in air or results that may apply to a heat transfer problem in this medium. The media restriction is due to the interest in electronics with its prevalent use of air cooling and the desire to establish design trends which can be property and transport media dependent. Natural convection in a vertical cavity and between parallel plates, along with mixed and forced convection between parallel plates, and these modes combined with surface radiation will be examined. Each mode of heat transfer will be explained and there will be a brief summary after the reviews.

Mother Nature’s attempts to relieve high gradients present engineers with many problems and opportunities. One of the interesting phenomena that will be reviewed in this chapter is the interaction between thermal energy transfer and fluid motion by natural convection. Natural convection occurs when thermal energy transfer causes a temperature gradient to exist in the surrounding fluid. If in the presence of a gravitational field this temperature gradient causes a change of density in the fluid, then buoyant forces will act on the fluid.
causing motion in an otherwise still media. This fluid motion presents us with challenging problems such as heat transfer between building walls, unwanted heating of the upper reaches of a building, energy loss through double glazed windows, and complications in crystal growth; or opportunities such as cooling of electrical and electronic equipment, natural draft furnaces, nuclear reactor and radioactive waste cooling, solar energy collection, and natural convection mixing in many industrial processes. In the application of electronic cooling, natural convection is the preferred method when applicable; and for critical sealed electronic components it may be the only cooling method.

Highlights of previous analytical, experimental, and numerical studies of the differentially heated vertical cavity will be presented. Considerable effort by this author and the many following researchers has been directed at examining this seemingly innocent (to the uninitiated) problem. Do not make the mistake of this author by assuming this to be a trivial numerical problem. The full Navier-Stokes equations need to be solved because there arise primary and secondary recirculating flows for high values of the Rayleigh number. An equation of state or the Boussinesq approximation must be used to couple the fluid equations to the energy equation and a suitable iteration scheme needs to be devised to accomplish convergence.

Pure natural convection between vertical plates will also be reviewed, for this is one of the most common modes of cooling for electronics. It has the advantage of being quiet and reliable without the vibration problems associated with fan cooling. This mode of heat transfer is driven solely by temperature differences and not by an imposed pressure difference other than that caused by buoyancy.
Forced convection is characterized by the fluid flow dominating the solution to the problem. There is only minor or secondary dependence on the temperature field with the fluid motion being caused by an imposed velocity or pressure distribution. This allows uncoupling of the Navier-Stokes equations from the energy equation and separation of the solutions. First the solution of the flow field would be obtained, and then used as input into the energy equation.

Mixed convection combines natural convection and forced convection effects. While there is an imposed pressure or velocity, temperature gradients are such that the velocities and heat transfer are significantly affected by the buoyancy effects. This requires a full coupled fluid and temperature solution.

Radiant heat transfer is one of the three basic modes of transferring energy between surfaces at different temperatures. This energy transfer does not depend on the presence of a medium as do conduction and convection heat transfer. However, there can be strong interdependence of the three modes since all three depend on and affect the surrounding surface temperatures. Techniques for solving radiation heat transfer between surfaces in a vacuum are well established, and are covered by Gebhart (1971) and in greater detail by Siegel and Howell (1981). When there is a medium present that absorbs, emits, and scatters the radiant energy, one has the problem of radiation with a participating medium. The solution techniques and, indeed, the basic governing equations are far more complex. Methods for solving this radiant heat transfer problem are beyond the scope of this investigation. (See Ozisik, 1985 and Chandrasasekhar, 1960). The following reviews on radiation will be on research in mixed mode radiant heat transfer (both natural and/or
forced) in interacting but nonparticipating media between open parallel plates (such as air cooling in electronic applications).

2.2 Review of Publications

The first publication concerning natural convection between vertical surfaces was by W. Nusselt in 1909. Since then this subject has received considerable attention theoretically, experimentally, and by numerical analysis due to the above mentioned significance in practical problems.

Elenbaas (1942) presented a study of the heat transfer between two plane vertical plates with uniform symmetric wall temperatures and open sides. He found the limiting functional relationships of the Nusselt number to the Rayleigh number and aspect ratio for small and large plate spacing.

Batchelor (1954) gave a theoretical analysis of the cavity problem for the case when the Rayleigh number is less than $10^5$. He used Goldstein's (1938) experimental results with air which suggested that the flow is laminar below a Rayleigh number of about $10^9$. The Nusselt number was found to be a function of both Rayleigh number and aspect ratio. However, with his limited experimental data (mostly obtained by Mull and Reiher, 1930). Batchelor postulated that at a large Rayleigh number the assumption of a continuous boundary layer surrounding a core of uniform temperature described the situation. This was later found not to be valid.
Eckert and Carlson (1961) experimentally determined the temperature field in the air layer between two isothermal vertical plates with different temperatures and aspect ratios (height to width). Local heat transfer coefficients were derived using the temperature gradients from a Zehnder-Mach interferometer. The aspect ratio was varied from 2.5 to 20, and the Grashof number (based on width) was varied from 284 to greater than $10^5$. They found the Nusselt number to vary as a function of Grashof number and as a weak function of aspect ratio. Even with the local Grashof number (based on vertical distance) close to $10^7$, they did not find turbulent flow. They did find, however, that the situation of a uniform core temperature (utilized by the analytical papers until that time) did not exist.

One of the first numerical studies of the simultaneously developing fluid flow and heat transfer in the entrance region of a duct was by Hwang and Fan (1963). They employed a finite-difference scheme to solve the boundary-layer equations along with the energy equation for a range of Prandtl numbers and boundary conditions. Results were compared to existing approximation methods and theoretical solutions.

Heaton, et al. (1964) analyzed the simultaneous development of velocity and temperature profiles for laminar flow and heat transfer in annular passages with a specified heat flux. Results of an integral solution method were compared to experimental measurements made for a Prandtl number of 0.7.

Dropkin and Somerscales (1965) presented the results of an experimental investigation of the heat transfer in liquids confined by two parallel plates and inclined at various angles with respect to the horizontal. Their work was motivated by a lack of experimental data for liquids. Since the Prandtl number for liquids is a stronger function of temperature than
it is for air, there was question as to the validity of using existing air data. The experiments covered a Rayleigh number range from $5 \times 10^4$ to $7.17 \times 10^3$ and Prandtl numbers between 0.02 and $1.156 \times 10^4$. They reported finding no effect on the Nusselt number as a result of changing the aspect ratio from 4.41 to 16.5. The Nusselt number was found to be a function of Rayleigh number, Prandtl number, and inclination angle. They also reported turbulent flow for the vertical case for Rayleigh numbers above $5 \times 10^5$. Landis (1965) discussed this paper and questioned if turbulent flow did indeed exist, the choice of Rayleigh instead of Grashof number, and the lack of aspect ratio dependence. He concludes that their results demonstrate best the previously observed conclusion that, for natural convection flows in enclosures, the overall Nusselt number correlations are particularly insensitive to geometry, boundary conditions, and flow regimes.

Elder (1965) examined laminar and turbulent free convection in theory and with experiments. He found the flow to be steady only for Rayleigh numbers less than $10^8$. His laminar experiments were restricted to a Prandtl number of $10^3$ and the aspect ratio range of 1 to 60. Flow patterns and temperature details are included.

Emery and Chu (1965) presented heat transfer data for vertical plane layers for fluids having Prandtl numbers from 3 to 30,000. They also presented an integral equation analysis that predicted heat transfer results to within 12 percent in the range in which the equations were applicable. They reported laminar flow up to a Rayleigh number of $10^7$ and only weak aspect ratio dependence. Landis (1965) also reviewed this paper and questioned if laminar flow existed and again the use of the Rayleigh number instead of the Grashof number to determine the Nusselt number.
Cess (1966) was one of the first to investigate the interaction of thermal radiation with natural convection. Boundary-layer equations were used for the free convection flow of a gray gas along a vertical, isothermal, black plate. Only the solution for small interaction between radiation and free convection was presented.

Gill (1966) developed a boundary-layer approximate solution to the enclosed natural convection problem for the case of high Prandtl numbers. He found for his approximate solution that neither the boundary-layer region nor the center core solution depend on the Rayleigh number or the aspect ratio. These parameters determine only the ratio of the thickness of the two regions. He also found that the center core had a vertical temperature gradient contrary to Batchelor’s prediction, but matching existing experimental data.

One of the first numerical studies of natural convection in an enclosure was by Wilkes and Churchill (1966). They used finite-difference computations on the transient vorticity, energy, and stream function equations. Computations were made for Grashof numbers up to $10^5$ and on aspect ratios of 1, 2, and 3. They found “remarkably close” agreement between a 20x20 grid and a 10x10 grid. A comparison of their predictions with the correlation of Jakob (1949) based on the experiments of Mull and Reiher (1930), showed the computed average Nusselt number to be from 0 to 70% higher. The discrepancy was partly attributed to the difference in the aspect ratio range. Computations were unstable for Grashof numbers above $2 \times 10^5$.

Mercer, et al. (1967) presented local and mean Nusselt number measurements for forced convection in the entrance region between isothermal parallel plates. He found that as much as 50% of the heat transfer was taking place in the developing region.
G. De Vahl Davis (1968) numerically solved the enclosed natural convection problem for a maximum value of the Rayleigh number of $2 \times 10^5$ for a square cavity and $1.25 \times 10^6$ for a cavity with an aspect ratio of 5. He found a stabilizing influence from increasing the Prandtl number over a range of four orders of magnitude ($10^{-1}$ to $10^3$) with little effect on the final heat transfer results except at higher values of the Rayleigh number. Calculations were carried out on an 11x11 mesh only. Davis quoted the work of Wilkes (1963) and Thomas (private communication) stating that the use of a finer mesh made relatively little difference in the final results. He reasoned that the use of a finer mesh could reveal secondary flows in the corner regions but that the effects of such flows on the overall heat-transfer rates would be small. Davis also found differences in the average Nusselt number obtained depending on the type of finite-difference formula (3-, 5-, or 7-point central difference approximations).

Schmidt and Zeldin (1969) solved the complete set of Navier-Stokes equations, without heat transfer, using finite-difference techniques. They found significant differences in predicted channel flow rates for low to moderate Reynolds number when using the full elliptic equations versus the parabolic equations.

Aung, et al. (1972) did a numerical and experimental investigation of developing laminar free convection heat transfer in vertical parallel plate channels with asymmetric heating. They presented results for boundary conditions of both uniform wall heat fluxes and uniform wall temperatures. They found that when the Rayleigh number was less than 2, the average Nusselt number can accurately be given by explicit expressions from fully-developed flow solutions. Numerical results were compared to experimental data and it was found that at Rayleigh numbers greater than 400, the numerical solutions over
predict the average Nusselt number. Boundary-layer approximations for the flow equations were used.

Kettleborough (1972) examined the transient laminar free convection between heated vertical plates. He used the vorticity-transport equations to study the effects of entrance boundary conditions. He found that for small time the velocity profile has a local minimum on the channel axis with a pair of symmetrically placed maxima. When the Grashof number was small this entrance profile changed to one with a maximum at the center for larger times. However, for larger Grashof numbers the entrance profile always had a pair of maxima which developed into the more normal distribution as the flow moved up the channel. His computed Nusselt numbers differed greatly from a fully-developed flow value at a Grashof number of $10^4$.

Oden and Wellford (1972) presented general finite-element models for the analysis of both compressible and incompressible viscous fluids. A number of schemes were developed for both steady and transient solutions for incompressible fluids. Results from several test problems showed the great potential of the finite-element method for the analysis of viscous flow problems.

Quon (1972) presented finite-difference computations at high Rayleigh numbers for a variety of boundary conditions, Rayleigh numbers, and Prandtl numbers for the heated cavity problem. He concluded that horizontal boundary conditions have little effect on the vertical boundary-layer flow, that for Prandtl numbers greater than 7 (water) there is little Prandtl number influence on the solution, and that when scaled properly (and within certain restrictions) the vertical velocity and temperature distributions across the
boundary layer were invariant with respect to the Rayleigh number, as predicted by Gill (1966). In comparisons to Gill's theory, Quon believed it to be the best one for this problem, but recommended a modification of the arbitrary constant in the theory. Quon suggested that more numerical experiments be performed to verify whether scaled velocity and temperature are invariant with the aspect ratio as predicted by Gill.

Aihara (1973) did a numerical analysis of the effects of inlet boundary conditions for laminar free convection between vertical parallel plates with uniform wall temperature. He found that the pressure drop due to the acceleration of the fluid in the channel-inlet, which had been neglected in all previous investigations, must be considered to obtain an accurate solution at high Rayleigh numbers. His method neglected axial diffusion and cross-stream pressure gradients. Both flat and parabolic inlet velocity profiles were compared and it was found that these different inlet conditions exert 'predominant influences' on the Nusselt number, especially with decreasing Prandtl number. He also found Bodoia, et al.'s (1962) inlet boundary conditions (i.e., using zero pressure defect and a flat velocity profile at the entrance), which had been used in all previous theoretical studies, to be applicable only for Prandtl numbers greater than 10 for a practical range of Rayleigh numbers. For a Prandtl number of 0.7 center-line velocity inversions were discovered.

Shah and London (1974) attempted to clarify the specifications of thermal boundary conditions and solutions for laminar duct flow forced convection. Nine specific thermal boundary conditions were categorized and the most useful fully-developed flow solutions were summarized.
Carpenter, et al. (1976) did a numerical and theoretical investigation of combined radiation and free convection in air between vertical parallel plates with asymmetric heating. A uniform inlet velocity along with boundary-layer flow approximations were assumed. They found that the inclusion of radiation significantly alters the nonradiation results and that the fully-developed flow solution often could not be obtained with radiation effects included. The Grashof number was varied from $10^{-1}$ to $10^{4}$. They found that radiation could not be neglected for either symmetric or asymmetric heating for Grashof numbers greater than 10. This was true even for low emissivities (0.3) and moderate aspect ratio (10 to 1). With radiation included, the maximum wall temperature was greatly reduced and no longer occurred at the exit. For asymmetric heating, with one wall insulated, radiation was found to have little effect when the Rayleigh number was less than 2. Their numerical results were verified by experiments.

Gray and Giorgini (1976) examined the validity of the Boussinesq approximation for natural convection in a Newtonian liquid or gas. Explicit conditions were given for the using this approximation. Specific examples of the restrictions for water and air at room temperature were provided. However, these restrictions for the proper use of the Boussinesq approximation have been almost universally ignored by numerical analysts!

Quon (1977) attempted to improve on Gill's theory by determining the free constant in the theory by various means. He was successful in only locally improving the predicted solution when compared to experimental data and numerical computations.

Raithby, et al. (1977) presented a simple analysis which predicts the temperature distribution in the differentially heated vertical slot for $0^\circ$ to $20^\circ$ angle of tilt. Their
equations, which are restricted to constant wall temperatures, compared favorably with experimental data for aspect ratios greater than 5, for both laminar and turbulent heat transfer.

One of the first applications of the finite-element method to natural convection in an enclosure was the use of the penalty-function method using primitive variables, by Marshall, Heinrich, and Zienkiewicz (1978). They obtained solutions for a Rayleigh number up to $10^7$ and a Prandtl number of unity on a nonuniform grid of 64 nine-noded rectangular elements. Their results compared well with existing experimental data and confirmed the predictions of Gill's analysis with some adjustments of the free constant in Gill's theory.

Roux, et al. (1978) used a finite-difference method to study the effects of either perfectly conducting or perfectly insulated horizontal walls. Their calculations were for a Prandtl number of 0.7 (air) and over a range of Rayleigh numbers from $10^2$ to $10^5$. They concluded that the finite-difference method has better convergence properties for the Dirichlet-type boundary conditions (perfectly conducting walls) than for the mixed Dirichlet-Neuman type in the insulated case. Emphasis was on the development of empirical correlations for the overall Nusselt number. Horizontal boundary conditions effected the parameters in the correlations, contrary to the finding of Quon (1972).

Schneider, et al. (1978) examined three new finite-element equal-order interpolation schemes for the pressure terms. Natural convection in a cavity along with shear-driven flow were investigated. They reported "superior" performance for a velocity correction procedure.
Hughes, et al. (1979) gave an extensive review of previous work and new developments for the penalty-function, finite-element formulation of viscous incompressible flows. Galerkin and upwind treatments of the convective terms were discussed and numerical results to several engineering flow problems were presented.

Taylor and Ijam (1979) used a finite-element method to study the effects of varying the aspect ratio and Prandtl number for free convection flow within a cavity for Rayleigh numbers up to $10^7$. They claimed to have varied the Prandtl number between $10^{-2}$ to $10^3$ and examined aspect ratios of 1, 10, and 20. Unfortunately, they did not discuss the results or present this data.

Pepper and Cooper (1980) presented numerical results from a time-split finite-element recursion relation for the recirculating flow in a differentially heated enclosure. The resulting system of equations for two or three-dimensional problems are tridiagonal. They presented results for the driven cavity problem up to a Reynolds number of $2 \times 10^4$ and for the glazing problem up to a Rayleigh number of $10^5$ and compared results to numerous previous studies. Their comparison plot for Nusselt number versus Rayleigh number is used in many subsequent papers. One can only assume they used a Prandtl number close to 0.7, for they never give the value used or state that air is the working fluid. *Unfortunately, the physics is very often lost to the numerical analyst.*

Reddy and Satake (1980) did a comparative study of the stream function-vorticity formulation and the penalty-function formulation. They presented results for rectangular and nonrectangular enclosures with Prandtl numbers varying from $10^{-2}$ to 1 and moderate
Rayleigh numbers. Their results showed that the Nusselt number and the rate of convergence depend on the Prandtl number.

Sparrow and Bahrami (1980) presented results from naphthalene sublimation measurements on open rectangular parallel plates. Three type of boundary conditions along the lateral edges were tested: (1) fully open to the environment, (2) blockage of one of the edge gaps, (3) blockage of both of the edge gaps. It was found that when the ratio of the Rayleigh number to the aspect ratio was greater than 10, the lateral edge conditions did not effect the mass transfer (and thus by analogy, heat transfer).

Sparrow, et al. (1980) did a numerical investigation of natural convection in a vertical channel examining the interaction of radiation and convection, with and without shrouding. They found that radiative transport between one isothermal and one adiabatic wall increased the heat transfer by 50-75% in the range of intermediate to large Grashof numbers. The flow was assumed to be boundary-layer type and radiation losses were only approximated.

Acharya and Patankar (1981) numerically investigated the effect of buoyancy on laminar forced convection for both heating and cooling in shrouded fin array. They found significant influences on the Nusselt numbers and friction factors when including buoyancy forces.

Churchill, et al. (1981) used the method of Richardson and Gaunt (1927) for extrapolating a laminar natural convection problem to zero element size. The results of this paper have caused great abuse of this method, for few methods meet his requirement
of consistent order of truncation and nobody seems to have read their last sentence in the addendum: “We conclude that caution should be exercised in applying extrapolation formulas if the partial differential equations are significantly nonlinear (as contrasted to the case we investigated: natural convection for large Pr).”

Gresho and Lee (1981) examined the cause of oscillatory solutions in modelling certain partial differential equations. They gave a stinging condemnation of the increasing practice of upwinding to suppress these “wiggles”, instead of using the information to properly solve the problem at hand.

Heinrich and Marshall (1981) evaluated the penalty-function, finite-element method for the driven cavity flow problem, Jeffery-Hamel flow in a convergent channel, and natural convection in a square cavity for Rayleigh number up to $10^6$. They concluded that this procedure achieved excellent accuracy using rather coarse meshes, effectively imposed the continuity constraint, and was more efficient than the velocity-pressure formulation.

Wirtz and Stutzman (1982) reported on temperature measurements of two-dimensional natural convection between vertical plates in air subjected to symmetric heating. Since the data do not plot as a straight line on logarithmic coordinates, they use a fitting technique suggested by Churchill and Usagi (1972). A correlation is given that allows for the determination of the maximum temperature variation of the plates within ± 5 percent.

Chang, et al. (1983) examined the interaction of radiation and natural convection in two-dimensional enclosures. They studied the effects of both participating (gas radiation) and nonparticipating (surface radiation) media. Calculations were made for Grashof numbers

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of $6.55 \times 10^6$ and $8.20 \times 10^5$ with a Prandtl number of 0.686. For empty enclosures they found the effect of radiation was to increase the bulk temperature of the gas while reducing natural convection flow. The net heat transfer was overwhelmingly increased when including radiation at the temperature levels considered. While the assumption of constant properties was used, the Boussinesq approximation was not utilized due to these large temperature differences. A study by Leonardi and Reizes (1981) (demonstrating that including variable properties in pure natural convection did not significantly alter the overall heat transfer rates) was sighted as validity for the constant-property assumption along with the fact that natural convection was only a small part of the total heat transfer.

Davis and Jones (1983) presided over a numerical comparison exercise for the problem of natural convection in a square cavity. A total of 37 contributions were compared to each other and to what was considered a high-accuracy solution used as a benchmark. The benchmark solution was obtained by Davis using a finite-difference method and extrapolation from successively finer meshes. It is interesting to note the contradiction of Davis' (1968) earlier comment on mesh requirements for this problem. (More will be said later on this extrapolation method.) A Prandtl number of 0.71 was used by all of the participants and the Rayleigh number was varied from $10^3$ to $10^6$. No restrictions were placed on method, mesh size, or effort (CPU time). Contributors were expected to use their best judgement to obtain ‘the answer to the problem’.

The primary comparison was based on the integrated average value of the heat flux, $N_u_{av}$, with secondary considerations on temperatures, velocities, and stream function when given. As the Rayleigh number increased there became a wider range in the contributed $N_u_{av}$ values, with some contributors failing to supply answers for the higher
values of the Rayleigh number, citing failure to reach convergence or excessive computer cost.

The first general conclusion drawn was that even accounting for the ability of mesh refinement the finite-element method out performed the finite-difference method despite the fact that more contributors used the latter. Indeed the best (their rating) contributed results were from Upson, et al. (1983), using the FEM and the Galerkin method of Kessler and Oertel (1983). They stated that scatter in finite-difference methods was too large to draw any conclusions.

Kennedy and Zebib (1983) presented results from a numerical and experimental study into combined free and forced convection between horizontal parallel plates. They found that the effects from axial conduction needed to be included for a Grashof number of the order of \(10^5\).

Bar-Cohen and Rohsenow (1984) developed composite relationships for the variation of the heat transfer coefficient along both isothermal and isoflux vertical natural convection cooled parallel plates. Their relationships were developed for design optimization and were verified with experimental data.

Chung and Kim (1984) investigated conduction, convection, and radiation in an emitting, absorbing, and scattering medium between converging and diverging channels. The velocity profiles were taken from an analytical expression for fully-developed flow and surface temperatures were taken as uniform and constant. They found that standard Galerkin finite elements may be used if convection domination is relatively small
(RePr<1000), but concluded that for large Peclet numbers ill-conditioning of the matrix from the convection terms would cause the solution to deteriorate.

Sparrow, et al. (1984) studied natural convection in an opened-ended vertical channel. One wall was heated while the other was insulated. The experiments which included flow visualization and Nusselt number measurements were performed with water. At Rayleigh numbers above a threshold value a recirculation region was found across from the heated wall at the exit. A finite-difference parabolic numerical scheme, which did not allow for flow reversal, yielded Nusselt numbers in ‘good’ agreement with those of the experiment. Results from the numerical method were then presented for a range of Prandtl numbers from 0.7 to 10.

Wirtz and Dyshoom (1984) reported on experimental measurements of convective heat transfer over sparse arrays of flush-mounted flat packs, similar to surface mounted VLSI chips. They reported a shorter development length to periodically fully-developed flow than Sparrow, et al. (1982). However, much of their flow was turbulent.

Farouk and Fusegi (1985) evaluated a new solution procedure using the vorticity-velocity formulation in solving the enclosed natural convection problem. They presented local Nusselt number plots along the vertical direction and compared results to other investigators for Rayleigh numbers to 10^6.

Braaten and Patankar (1985) analyzed secondary flow laminar mixed convection in shrouded arrays of heated rectangular blocks. The primary flow was assumed to be both
hydrodynamically and thermally fully developed. They demonstrated that secondary flow buoyancy effects have a significant influence on primary flow and heat transfer.

In discussing research needs in electronic equipment, Moffat (1986) pointed out that the simplified scaling laws of dynamic similarity do not necessarily apply to complex geometrical systems containing multiple length scales and encouraged the research community to seek simplified approaches to the what and why components of the research question. He also stressed that natural convection cooling is preferred in many low performance applications because it is quiet, reliable, and does not generate electromagnetic interference or vibration problems.

Moffat and Ortega (1986) used experimental data on free convection cooling of simulated electronic components in multiple vertical parallel plates to show that the elements experienced forced-convection-type heat transfer even though the flow was buoyancy driven.

Wirtz (1986) suggested that research on air cooling should be more responsive to the inherent complexities of the "real world" applications. He expressed concern for the fact that, in the interest of generalizations, much of the technical heat transfer literature is based on idealizations which render the results difficult to apply. Some of the fundamental mechanisms which he stated needed to be addressed included: mixed convection effects, natural convection phenomena due to localized heating, and the effect of recirculating flows in complex geometries.
Guglielmini, et al. (1987) presented results of a numerical and experimental study of natural convection and radiation from staggered vertical fins. An empirical relation was used for the convective heat transfer while the radiation component was calculated. The excellent agreement with experimental data showed that radiation accounted for 25% to 40% of the total heat transfer for highly polished surfaces (low emissivities) when the fin to ambient temperature was in the 10° to 50° C range. At higher temperature rises the contribution from radiation was even greater.

Incropera (1987) addressed the research needs in electronic equipment cooling. He stated that in the computational area more efficient algorithms for computing convection heat transfer need to be developed and that while it will be some time before computer solutions can handle complex flows, it may not be long before they can handle regular arrays of VLSI chips on a substrate.

Nickell, et al. (1987) presented results for using a secant method of calculating natural convection and radiation heat transfer in air from parallel plates with discrete heat sources. They solved the boundary-layer equations for the fluid flow and neglected radiation losses to the inlet and exit. Difficulty in obtaining a solution with the nonlinear net radiation terms in the boundary conditions of the energy equations lead them to approximate the radiation term. Their work confirmed the importance of considering radiation in a typical electronics cooling application.

O’Meara and Poulakakos (1987) presented experimental results examining the natural convection cooling of an array of vertical plates with equal and uniform heat fluxes in air. Plate spacing, heat flux, entrance and exit effects from nearby boundaries, and alignment

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or stagger of a second set of plates were all investigated. They showed that the proximity of a floor or ceiling could greatly affect the flow and heat transfer rates. They also showed that existing proposed analytical expressions did not fit their data for high Rayleigh numbers.

Zengyuan, et al. (1987) did a numerical and experimental study of natural convection between isothermal vertical plates. They used boundary-layer approximations for the flow and also approximated the radiative flux. Radiative heat transfer was found to be of the same order as channel natural convection in some cases. They also found that the ratio of radiation heat transfer to natural convection does not vary monotonically with the Rayleigh number and that the vertical channel flow can be divided into three distinct regions of either developing, transition, or fully-developed flow according to the heat transfer performance.

Buller and McNelis (1988) studied the enhancement of cooling electronic equipment by optimizing the effects of radiation. They stated that for symmetric heating indicative of electronic equipment, "the boundary where radiation can be ignored and fully developed free convection exists is well under the temperature and power limits of existing systems".

An experimental study of the local steady natural convection in an asymmetric heated vertical channel was performed by Hung and Perng (1988). Their investigation spans a range of plate spacings (0.7 to 16.0 cm) and heat fluxes (40 to 270 w/m²) that are typical for electronic packages. They found significant effects of convective heat fluxes compared to the effects of channel spacing on local and average heat transfer results.
Moffat and Anderson (1988) reviewed the various methods in use for defining the heat transfer coefficient. They showed that a misunderstanding of the basis of the reference temperature can lead to underprediction of component temperature rise by 20-30%.

Natural convection from vertical plates, maintained with uniform heat fluxes in a large enclosure was numerically studied by Ramanathan, et al. (1988). Their results indicate that the entrance temperature profile is uniform while the velocity profile is not, but they showed that this does not significantly alter the heat transfer results.

Ramanathan and Kumar (1988) presented correlations for natural convection between heated vertical plates. A parametric study was conducted for various Prandtl numbers and aspect ratios. They found good agreement with the reported results in the literature for large aspect ratios but not for small aspect ratios. The discrepancy was attributed to the neglect of the diffusion of thermal energy in the vertical direction by the published papers. They conclude that vertical conduction should be included for channel aspect ratios less than 10 for Prandtl numbers ≥ 0.7.

Yamada (1988) did both analytical and experimental studies on combined radiation and free convection heat transfer in a vertical channel. Wall emissivities were found to have a major effect on the heat transfer. Even at a low temperature range, surface radiation was found to have dominant importance in the combined mode asymmetric heating problem.

Wirtz and McAuliffe (1989) presented results from the experimental modelling of forced convection in the wake of a closely-spaced electronic package row. Results were presented for Reynolds numbers ranging from $10^3$ to $4\times10^3$. Flow visualization showed
the wake region to be similar to flow past a continuous backwards-facing step. The flow was forced to be fully developed before reaching the test section by a 175 cm long development section.

Peterson and Ortega (1990) gave an extensive review of the thermal control of electronic equipment and devices. Current and future cooling techniques and numerical methods were included along with results from many of the papers covered here.

Ramaswamy (1990) presented a finite-element method for the Navier-Stokes equations and the energy equation based on a segregated velocity-pressure solution method which has obvious storage advantages over full direct solvers. He obtained wiggle-free solutions for free and forced convection problems without resorting to upwinding, asymmetric basis functions, or exponential interpolation. He concluded that the cost of the segregated approach was moderate and competitive with other numerical methods.

2.3 Chapter Summary

The purpose of this review chapter was to give the reader sources for an investigation and an assessment of the current state of numerical modelling of convective heat transfer for the stated problem. The publications presented here are just the highlights of the many investigations into natural and mixed convection heat transfer as applicable to the typical cooling of electronic equipment. Many excellent investigations into similar problems have not been included.
The reader should realize by now that rather than being a well-developed field, numerical modelling of convective heat transfer is still in its infancy. An adequate theory for the differentially heated cavity eluded researchers until Gill's 1966 theory. There is even an ongoing debate over which nondimensional number should be used to classify natural convection. Many of the computational models presented were based on simplifying assumptions which neglected important terms. The numerical results of these simple models are often only compared to specially designed experiments, such as naphthalene sublimation technique experiments, which are related to heat transfer but do not include radiation and often not buoyancy effects (unless heated). Several of the experimental results presented for modelling the cooling of electronic equipment forced the flow to become fully developed with an unrealistically long (for electronics) development section before reaching the test section. The many disagreements are an indication of the complexity of the problem and the vast number of investigators indicates the many practical applications mentioned at the beginning of this chapter.

One should note that the investigators that solved the full Navier-Stokes equations did not include buoyancy and radiation effects, while the analysis that included and showed the importance of radiation only solved the parabolic or boundary-layer form of the Navier-Stokes equations. This investigation will use a full elliptic solution to the Navier-Stokes and energy equations with natural convection and surface radiation effects included. Comparison to simplifying assumptions will be made where appropriate.
3.0 Finite-Element Formulation

3.1 Chapter Introduction

In this chapter the basic governing equations will be introduced. First the continuity and Navier-Stokes equations and next the energy equation will be presented. The variational formulation for each equation will be derived and then the finite-element formulation will be presented. Radiation energy exchange along the boundary will then be discussed. The final section in this chapter will deal with time integration of the basic equations. Solution techniques, details of individual terms, and implementation will be discussed in Chapter 4.

The following assumptions and simplifications will apply for clarity of this development:

i) two-dimensional flow and heat transfer
ii) viscous, incompressible, Newtonian fluid
iii) constant fluid and thermal properties
iv) non-participating medium with gray diffuse surfaces
3.2 Navier-Stokes Equations

Most successful early attempts at numerically solving the full two-dimensional Navier-Stokes equations have utilized the vorticity-stream function method (Roach 1985). Primitive variable methods have had convergence problems partly due to the pressure not appearing directly in the momentum equations. The correct pressure is only obtained indirectly through the continuity constraint. Many investigators have sought solutions to this problem through various techniques, some with mathematical basis, and some, unfortunately, rather ad hoc. The penalty method presented here will circumvent the velocity-pressure coupling problem and still allow the advantages of working with primitive variables and physically meaningful boundary conditions.

Consider a two-dimensional domain, \( \Omega \), with boundary, \( \Gamma \). The domain may be simply connected or multiply connected as shown in Fig. 3.1. Thus, one can consider internal as well as external flows with equal ease. If in \( \Omega \) there is a continuous, viscous, incompressible fluid, then the three basic governing equations for the fluid flow are:

Conservation of mass:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]  \hspace{1cm} (3.1)

Conservation of x-direction momentum:

\[
\rho \frac{\partial u}{\partial t} + \rho (u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}) - 2\mu \frac{\partial^2 u}{\partial x^2} - \mu \frac{\partial}{\partial y} (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) + \frac{\partial P}{\partial x} = f_x
\]  \hspace{1cm} (3.2)
Figure 3.1 Computational Domain and Boundary
Conservation of y-direction momentum:

\[
\rho \frac{\partial v}{\partial t} + \rho(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}) - 2\mu \frac{\partial^2 v}{\partial y^2} - \mu \frac{\partial}{\partial x}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) + \frac{\partial P}{\partial y} = f_y
\]  \hspace{1cm} (3.3)

Where \( f_x \) and \( f_y \) are body forces in the respective directions.

The body forces may be due to a gravitational field, a magnetic source, or an applied external acceleration. The nature of these body forces will be explored later. An advantage of the method presented here is that the direction our system makes with the axes does not need to be included in the governing equations, since the natural geometric flexibility of the finite-element method will treat this without modification of the basic governing equations. Therefore, one can handle inclined flows as easily as a vertical or horizontal situation.

If the continuity requirement (equation 3.1) is viewed as a constraint on the two momentum equations, then application of the penalty finite-element method (Marshall et al., 1978) requires approximating the continuity constraint by:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{1}{\gamma} P,
\]  \hspace{1cm} (3.4)

where \( \gamma \) is the penalty parameter.

Since the pressure, \( P \), is finite and \( \gamma \) can be assigned an arbitrary large value, then the continuity equation is solved approximately. The quality of the continuity constraint, and thus the conservation of mass, will be determined by the value of \( \gamma \). It has been

Finite Element Formulation
recommended (Bertin and Ozoe, 1986) that the value of the penalty parameter be chosen such that:

\[ \gamma = C \max(v, \nu Re) \quad \text{for forced flows} \]

or

\[ \gamma = C \max(1, Pr, PrRa) \quad \text{for natural convection} \]

Where \( C \) is a constant of the order of \( 10^7 \).

In either case \( \gamma \) should not be so large that it dominates the other terms to the extent that round-off errors would give the trivial solution. This upper limit on \( \gamma \) depends on the precision being used in the calculation. For double-precision computations on a 64 bit computer the upper limit for \( \gamma \) is approximately \( 10^{13} \) (Reddy, 1984). Studies on the effects of varying the penalty parameter have found the solution unchanged for a range of \( \gamma \) from \( 10^4 \) to \( 10^{10} \) (Reddy, 1983; Bertin and Ozoe, 1986; and this author). It is possible to piecewise vary the penalty parameter and have its value be a function of the continuity constraint (increasing \( \gamma \) forces the LHS of equation 3.1 closer to zero). Investigations done by this author have found this not to be necessary and, unless specified otherwise, the penalty parameter used in all subsequent calculations will be \( 10^8 \).

A method proposed by Reddy (1984) to ensure that the penalty terms do not overly dominate the nonpenalty terms will be employed in this implementation. The stiffness matrix of the penalty terms will be made singular by reduced integration so that only the total stiffness matrix is invertible, since the nonpenalty matrix is nonsingular. This way a
matrix solver with error checking will flag a situation where the penalty terms overly dominate the other terms.

Equation 3.4 is now introduced into equations 3.2 and 3.3 yielding:

$$\rho \frac{\partial u}{\partial t} + \rho (u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}) - 2\mu \frac{\partial^2 u}{\partial x^2} - \mu \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right) \cdot \gamma \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = f_x$$  \hspace{1cm} (3.5)

$$\rho \frac{\partial v}{\partial t} + \rho (u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}) - 2\mu \frac{\partial^2 v}{\partial y^2} - \mu \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \right) \cdot \gamma \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = f_y$$  \hspace{1cm} (3.6)

The problem of solving one continuity equation and two momentum equations has been reduced into solving the above two equations. More importantly, a variable, pressure, has been eliminated from our system of equations. This will result in considerable saving of CPU time and allow the solution of far more difficult problems with fixed computer resources, since the operations count (and, thus, the computer cost) for Gauss elimination is of the order of $n^3$ (Johnson and Riess, 1982), where $n$ is the total number of nodal values. So one can expect a better than two-fold savings in CPU time along with reduced storage requirements when using the penalty method versus carrying the pressure as a variable to be solved for. The pressure distribution, if needed, can be computed after convergence in a single inexpensive post-processing calculation using equation 3.4.
3.3 Variational Formulation

A general ‘weak’ variational formulation was chosen for equations 3.5 and 3.6 and the subsequent energy equation (3.21). This technique is referred to as ‘weak’ because there is a transfer of differentiation from the dependent variables to the test functions. This transfer results in a weaker continuity (in a functional sense) requirement on the primary variables. The purpose of this transfer is to equalize the continuity requirements on the test functions and the dependent variables. This will minimize the degree requirement of the approximation, which has advantages (Becker, et al., 1981) to be discussed in Chapter 4 on implementation.

In the variational formulation one multiplies equations 3.5 and 3.6 by test functions \( w_1 \) (variation in \( u \)) and \( w_2 \) (variation in \( v \)) and integrates over a finite two-dimensional domain, \( \Omega^e \), yielding:

\[
0 = \int_{\Omega^e} \left[ w_1 \left( \rho \frac{\partial \bar{u}}{\partial t} + \rho \left( u \frac{\partial \bar{u}}{\partial x} + v \frac{\partial \bar{u}}{\partial y} \right) \right) - 2\mu w_1 \frac{\partial^2 u}{\partial x^2} - \mu w_1 \frac{\partial}{\partial x} \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) \right] dx \, dy
- \gamma w_1 \frac{\partial}{\partial x} \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) - w_1 f_x \right] dx \, dy
\]  

(3.7)

and

\[
0 = \int_{\Omega^e} \left[ w_2 \left( \rho \frac{\partial \bar{v}}{\partial t} + \rho \left( u \frac{\partial \bar{v}}{\partial x} + v \frac{\partial \bar{v}}{\partial y} \right) \right) - 2\mu w_2 \frac{\partial^2 v}{\partial y^2} - \mu w_2 \frac{\partial}{\partial x} \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) \right] dx \, dy
- \gamma w_2 \frac{\partial}{\partial y} \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) - w_2 f_y \right] dx \, dy
\]  

(3.8)

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Next in the weak formulation used here, one integrates the terms of higher order by parts (using Green's theorem in the plane and the divergence theorem of Gauss) which equalizes the continuity requirements on the variables, \( u \) and \( v \), and the weighting functions, \( w_1 \) and \( w_2 \). This results in equations 3.9 and 3.10 below.

\[
0 = \int_{\Omega^e} \left[ w_1 \left[ \rho \frac{\partial u}{\partial t} + \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + 2\mu \frac{\partial w_1}{\partial x} \frac{\partial u}{\partial x} + \mu \frac{\partial w_1}{\partial y} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \gamma \frac{\partial w_1}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + w_1 f_x \right] \, dx \, dy \, - \int_{\Gamma^e} f_x w_1 \, ds \tag{3.9}
\]

\[
0 = \int_{\Omega^e} \left[ w_2 \left[ \rho \frac{\partial v}{\partial t} + \rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + 2\mu \frac{\partial w_2}{\partial y} \frac{\partial v}{\partial y} + \mu \frac{\partial w_2}{\partial x} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] + \gamma \frac{\partial w_2}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) + w_2 f_y \right] \, dx \, dy \, - \int_{\Gamma^e} f_y w_2 \, ds \tag{3.10}
\]

The above equations are the weak variational form of the governing flow equations using the penalty method. These equations will be solved using the finite-element method presented in the next section. The last terms in equations 3.9 and 3.10 (\( f_x \) and \( f_y \)) emerge during the transfer of differentiation to the test functions. These traction boundary terms are classified as natural boundary conditions and will be given below. The natural boundary conditions together with the specification of the dependent variables, \( u \) and \( v \), constitute the total admissible boundary conditions for this formulation. Details of each term will be discussed when appropriate.
3.4 Finite-Element Formulation

The finite-element method makes use of the above weak variational form to discretize the equations over each element. Note that a prior knowledge of a quadratic functional form which is the total potential energy that is minimized in classical finite elements is not necessary for the present formulation. In the Ritz (weak) method being used, the weighting functions \( w_1 \) and \( w_2 \) are of the same form as the approximation functions, \( \psi_i \) (to be introduced) for the variables. Substituting the approximating function, \( \psi_i \), for \( w_1 \) and \( w_2 \) yields:

\[
0 = \int_{\Omega^e} \left[ \psi_i \left[ \rho \frac{\partial \bar{u}}{\partial t} + \rho \left( u \frac{\partial \bar{u}}{\partial x} + v \frac{\partial \bar{u}}{\partial y} \right) \right] + 2\mu \frac{\partial \psi_i}{\partial x} \frac{\partial u}{\partial x} + \mu \frac{\partial \psi_i}{\partial y} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] dxdy - \int_{\Gamma^e} \psi_i f_x ds, \quad i = 1, 2, ..., n \quad (3.11)
\]

and

\[
0 = \int_{\Omega^e} \left[ \psi_i \left[ \rho \frac{\partial \bar{v}}{\partial t} + \rho \left( u \frac{\partial \bar{v}}{\partial x} + v \frac{\partial \bar{v}}{\partial y} \right) \right] + 2\mu \frac{\partial \psi_i}{\partial y} \frac{\partial v}{\partial y} + \mu \frac{\partial \psi_i}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right) \right] dxdy - \int_{\Gamma^e} \psi_i f_y ds, \quad i = 1, 2, ..., n, \quad (3.12)
\]

where the boundary conditions are given by specifying either a primary degree of freedom (velocities) or a secondary degree of freedom (traction forces, see below) and \( n \).
is the number of nodes per element, which will be discussed in Chapter 4. At any boundary node one of the following will be specified (Reddy, 1984):

acceptable boundary conditions

i) \( u \) and \( v \) (\( t_x \) and \( t_y \) unknown)

ii) \( u \) and \( t_y \) (\( t_x \) and \( v \) unknown)

iii) \( t_x \) and \( v \) (\( u \) and \( t_y \) unknown)

iv) \( t_x \) and \( t_y \) (\( u \) and \( v \) unknown)

The traction terms are given by:

\[
t_x = 2\mu \frac{\partial u}{\partial x} n_x + \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) n_y - Pn_x
\]

(3.13)

and

\[
t_y = 2\mu \frac{\partial v}{\partial y} n_y + \mu \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right) n_x - Pn_y.
\]

(3.14)

The traction form of the boundary conditions is far less restrictive and more flexible than their Neumann counterparts used in the finite-difference method. As pointed out by Treventi (1984), the streamlines are not forced to have zero slopes at the boundaries where this value may be unknown.

The following interpolation form is used for the velocities:

\[
u = \sum_{j=1}^{n} u_j \psi_j \quad \text{and} \quad v = \sum_{j=1}^{n} v_j \psi_j,
\]

(3.15a, b)
where \( n \) is the number of nodes per element and is determined by the order of the approximation function, \( \psi \). This will be discussed during the implementation steps.

Substituting the above expressions for the velocities into equations 3.11 and 3.12 and casting into matrix form yields:

\[
\begin{bmatrix}
[K^{11}] & [K^{12}]
\end{bmatrix}
\begin{bmatrix}
\{u\}
\end{bmatrix}
= 
\begin{bmatrix}
\{F^1\}
\end{bmatrix}
- 
\begin{bmatrix}
[M^{11}] & [O]
\end{bmatrix}
\begin{bmatrix}
\{\dot{u}\}
\end{bmatrix}
- 
\begin{bmatrix}
[M^{22}] & [O]
\end{bmatrix}
\begin{bmatrix}
\{\dot{v}\}
\end{bmatrix}.
\tag{3.16}
\]

The above equation has been cast in the traditional form of the finite-element method which has its roots in structural and structural dynamics problems. So one sees first a stiffness matrix \([K]\) multiplied by a displacement vector equated to a force vector minus the system mass times acceleration vector. The stiffness matrix is composed of the coefficients of the velocity components and will be given below along with the mass matrix. The details of the force vector will be given in the next chapter and the method of handling the acceleration terms will be given in the last section on time integration in this chapter.

The overbar terms in the above equation represent non-symmetric submatrices and are given below in equations 3.17 through 3.20.

\[
\overline{K}_{ij}^{11} = K_{ij}^{11} + \int_{\Omega^e} \rho \psi_i \left( \frac{\partial \psi_j}{\partial x} + v \frac{\partial \psi_j}{\partial y} \right) dx \, dy
\tag{3.17}
\]

\( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, n \)
The second term on the RHS of equation 3.16 is from the convection terms in the Navier-Stokes equation and causes the nonlinearity and nonsymmetry of the final system of equations. The techniques used to handle this term are the essence of any Navier-Stokes solver and will be discussed in detail in the next chapter on implementation.

The first term on the RHS of equation 3.17 is given by:

$$K^{11}_{ij} = \int_{\Omega^e} \left[ 2\mu \left( \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + \mu \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) + \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right] dxdy + \gamma \int_{\Omega^e} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} dxdy. \tag{3.17a}$$

The symmetric stiffness submatrices are given by:

$$K^{12}_{ij} = \int_{\Omega^e} \mu \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} dxdy + \gamma \int_{\Omega^e} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} dxdy \tag{3.18}$$

and

$$K^{21}_{ij} = \int_{\Omega^e} \mu \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} dxdy + \gamma \int_{\Omega^e} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} dxdy = K^{12}_{ji}. \tag{3.19}$$

The other nonsymmetric submatrix is given by:

$$\overline{K}^{22}_{ij} = K^{22}_{ij} + \int_{\Omega^e} \rho \psi_i \left( \frac{\partial \psi_j}{\partial x} + \nu \frac{\partial \psi_j}{\partial y} \right) dxdy \tag{3.20}$$

where

$$K^{22}_{ij} = \int_{\Omega^e} \left[ \mu \left( \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + 2\mu \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) + \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right] dxdy + \gamma \int_{\Omega^e} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} dxdy \tag{3.20a}$$

Finite Element Formulation
and finally the mass matrix is computed from:

\[
M_{ij}^{11} = \int_{\Omega^e} \rho \psi_i \psi_j \, dx \, dy \quad \text{and} \quad M_{ij}^{22} = M_{ij}^{11}
\] (3.21)

The mass matrix above is consistent with the variational formulation and more accurate than the method of lumping the mass at the nodes used in some formulations which would result in a diagonal mass matrix.

3.5 Energy Equation

The transient energy equation for the same two-dimensional domain shown in Fig. 3.1, for a medium with constant properties, neglecting viscous dissipation is given by:

\[
\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right). \tag{3.22}
\]

In the variational formulation one multiplies equation 3.22 by the test function \( w_1 \) (variation in \( T \)) and integrates over a finite two-dimensional domain, \( \Omega^e \), yielding:

\[
0 = \int_{\Omega^e} \left[ \frac{\partial T}{\partial t} \left( \frac{\partial T}{\partial t} \right) + \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) + \alpha \left( \frac{\partial w_1}{\partial x} \frac{\partial T}{\partial x} + \frac{\partial w_1}{\partial y} \frac{\partial T}{\partial y} \right) \right] \, dx \, dy \ - \int_{\Gamma^e} q_n w_1 \, ds \quad \tag{3.23}
\]

The following interpolation form is used for the temperature:

\[
T = \sum_{j=1}^{n} T_j \psi_j , \tag{3.24}
\]
where $n$ is the number of nodes per element and is determined by the order of the approximation function, $\psi$.

Substituting the above expression for the temperature into equation 3.23 and casting into matrix form yields:

$$[K'](T) = \{F\} \cdot [M] (\dot{T})$$

(3.25)

where

$$K'_{ij} = K_{ij} + G_{ij}$$

(3.26)

with

$$K_{ij} = \alpha \int_{\Omega^e} \left( \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx dy$$

(3.27)

and

$$G_{ij} = -\int_{\Omega^e} \psi_i \left( \bar{u} \frac{\partial \psi_j}{\partial x} + \bar{v} \frac{\partial \psi_j}{\partial y} \right) dx dy .$$

(3.28)

The overbar velocity terms are the integrated average velocities of each element and are given by:

$$\bar{u} = \sum_{j=1}^{n} u_j \psi_j \quad \text{and} \quad \bar{v} = \sum_{j=1}^{n} v_j \psi_j .$$

(3.29 a, b)

Since the density and specific heat were assumed constant the mass matrix is simply:
\[ M_{ij}^{11} = \int_{\Omega^e} \psi_i \psi_j \, dxdy . \] (3.30)

The force vector for this problem is composed of only boundary heat flux terms since internal generation is not being considered. Thus:

\[ F_{ij} = \int_{\Gamma^e} q_n \psi_i ds , \] (3.31)

where \( q_n = \alpha \frac{\partial T}{\partial n} \).

3.6 Radiation

Any substance above absolute zero temperature continuously emits electromagnetic radiation. Along the boundary, \( \Gamma \), of the domain, there may be many different surfaces continuously emitting and receiving radiant energy to and from all other surfaces along this boundary.

It has been shown (Steinberg, 1980) that radiant energy transfer in electronic equipment is one of the important, if not dominant, modes of energy transfer and thus can not be ignored in any complete analysis. In this section the exchange of energy by radiation between diffuse-gray surfaces will be examined.

A diffuse surface is defined as one which does not have a directional bias for the emitted or reflected radiation from that surface. A gray surface is one in which radiative properties do not vary with wavelength, \( \lambda \). Since only the net exchange of energy between the surfaces is of interest, a method for finding the radiant flux or surface
temperature within an enclosure, first presented by Hottel (1954) and later refined by Gebhart (1957) will be presented.

Consider the \( j \)th surface of the enclosure \( \Gamma \) as shown in Figure 3.2. An energy balance (Siegel and Howell, 1981) on surface \( j \) results in:

\[
Q_j = q_j A_j = (B_j - H_j) A_j,
\]

where

- \( B_j \) is the radiosity or net radiant flux leaving surface \( j \)
- \( H_j \) is the net radiant flux incident on surface \( j \).

The radiosity is composed of directly emitted and reflected energy. When the surfaces are opaque (transmissivity, \( \tau \), is zero) and gray (wavelength, \( \lambda \), independent) so that the reflectivity, \( \rho \), and absorptivity, \( \alpha \), (which is equal to the emissivity, \( \varepsilon \)) sum to unity, one may write:

\[
B_j = \varepsilon_j \sigma T_j^4 + \rho_j H_j = \varepsilon_j \sigma T_j^4 + (1 - \varepsilon_j) H_j
\]

The net incident radiant energy on surface \( j \) can be written as the contribution from all surfaces in the enclosure. So that:

\[
A_j H_j = \sum_{i=1}^{n} A_i B_i F_{i,j},
\]

where \( F_{i,j} \) is the geometric configuration factor and is defined as the total radiant energy
Figure 3.2 Surface Discretization for Radiation
that leaves surface i that is directly incident on surface j. For two surface exchange we have:

\[ F_{i,j} = \frac{1}{A_i} \int_{A_i} \int_{A_j} \frac{\cos \theta_i \cos \theta_j}{\pi S^2} \, dA_j \, dA_i \quad (3.35) \]

where \( \theta \) is the angle from the surface normal and \( S \) is the distance between the two differential elements, \( dA_i \) and \( dA_j \) on the surfaces being considered. More details on calculating these configuration factors will be given in the implementation chapter.

Using reciprocity:

\[ H_j = \sum_{i=1}^{n} B_i F_{j-i} \quad (3.36) \]

yielding:

\[ Q_j = \frac{\varepsilon_j}{1 - \varepsilon_j} A_j (\sigma T_j^4 - B_j) \quad (3.37) \]

and

\[ Q_j = (B_j \cdot \sum_{i=1}^{n} B_i F_{j-i}) A_j \quad (3.38) \]

Equations 3.37 and 3.38 express the balance between the energy supplied by conduction or convection to surface j to the energy loss by radiation from surface j to the other
surfaces along $\Gamma$. A system of equations may be written for $n$ surfaces within an enclosure which relates the net energy loss per unit time, $Q_j$, to the surface temperatures, $T_j$. Using a summation notation:

$$\sum_{i=1}^{n} \left( \delta_{ji} \frac{F_{j,i}}{\varepsilon_i} \right) Q_i = \sum_{i=1}^{n} (\delta_{ji} - F_{j,i}) \sigma T_i^4,$$

(3.39)

where $\delta_{ji}$ is the Kronecker delta.

Equation 3.39 actually forms a system of equations where either a surface temperature or flux can be specified. The system is linear in $T^4$ and $q$, therefore one need only pivot the column vector in the matrix to place the appropriate unknown on the LHS and the known quantity on the RHS and solve a linear system of equations. The assumption of uniform radiosity for each surface will become less of an approximation as the boundary is discretized into a larger number of surfaces. An example of a four surface enclosure is given below, where $\Xi_i$ is $\sigma T_i^4$ and $q$ is the surface heat flux.

$$\begin{bmatrix}
\frac{1}{\varepsilon_1} - F_{1,1} & \frac{1-\varepsilon_1}{\varepsilon_2} & \frac{1-\varepsilon_1}{\varepsilon_3} & \frac{1-\varepsilon_1}{\varepsilon_4} \\
\frac{1-\varepsilon_1}{\varepsilon_1} & \frac{1}{\varepsilon_2} - F_{1,2} & \frac{1-\varepsilon_2}{\varepsilon_3} & \frac{1-\varepsilon_2}{\varepsilon_4} \\
\frac{1-\varepsilon_1}{\varepsilon_1} & \frac{1-\varepsilon_2}{\varepsilon_2} & \frac{1}{\varepsilon_3} - F_{1,3} & \frac{1-\varepsilon_3}{\varepsilon_4} \\
\frac{1-\varepsilon_1}{\varepsilon_1} & \frac{1-\varepsilon_2}{\varepsilon_2} & \frac{1-\varepsilon_3}{\varepsilon_3} & \frac{1}{\varepsilon_4} - F_{1,4}
\end{bmatrix}
\begin{bmatrix}
q_1 \\
q_2 \\
q_3 \\
q_4
\end{bmatrix} =
\begin{bmatrix}
\Xi_1 \\
\Xi_2 \\
\Xi_3 \\
\Xi_4
\end{bmatrix}$$

(3.40)

Finite Element Formulation
3.7 Time Approximation

For transient problems the temporal variation of the dependent variables will be treated as an ordinary differential equation in time, i.e., time and spatial variation are separable. To resolve the variation with time will require integrating this ordinary differential equation with an appropriate scheme. Usually a fully implicit or explicit scheme is employed for this integration. Explicit schemes which allow a direct solution for the variable of interest are easier to program but are restricted in the allowable time steps for stability and are not as accurate as the more involved implicit schemes. A hybrid scheme which exploits the benefits of both techniques will be employed. If a weighted average of the time derivative at two consecutive time steps is used, then a time step of $\Delta t$ yields:

$$\theta \left[ \frac{d\Delta}{dt} \right]_{n+1} + (1-\theta) \left[ \frac{d\Delta}{dt} \right]_n = \frac{\Delta}_{n+1} - \frac{\Delta}_n}{\Delta t}, \quad (3.41)$$

where

- $\theta$ is the weighting function (Reddy),
- $\Delta$ is the dependent variable and can represent a vector of variables,
- and $n$ refers to the $n^{th}$ time step.

Letting $^o$ represent the derivative with respect to time and casting into the form of a basic system of equations yields:

$$[M]\{\dot{\Delta} \} + [K]\{\Delta \} = \{F\}. \quad (3.42)$$

Now multiply equation 3.41 by $[M]$ to obtain:

Finite Element Formulation
\[ \theta[M][\Delta]_{n+1} + (1 - \theta)[M][\Delta]_n = [M][\Delta]_{n+1} - [\Delta]_n \Delta t \quad (3.43) \]

Replace \([M][\Delta]\) by \([F] - [K][\Delta]\) yielding:

\[ \theta[F]_{n+1} - [K][\Delta]_{n+1} + (1 - \theta)([F]_n - [K][\Delta]_n) = [M][\Delta]_{n+1} - [\Delta]_n \Delta t, \quad (3.44) \]

which can be rearranged as:

\[ ([M] + \theta\Delta t[K]_{n+1})[\Delta]_{n+1} = \Delta t(\theta(F)_{n+1} + (1 - \theta)F_n) + ([M] - (1 - \theta)\Delta t[K]_n)[\Delta]_n. \quad (3.45) \]

Using simplifying notation:

\[ [\tilde{K}][\Delta]_{n+1} = [\tilde{F}], \quad (3.46) \]

where

\[ [\tilde{K}] = [M] + \theta\Delta t[K]_{n+1} \quad (3.47) \]

and

\[ [\tilde{F}] = \Delta t(\theta(F)_{n+1} + (1 - \theta)F_n) + ([M] - (1 - \theta)\Delta t[K]_n)[\Delta]_n. \quad (3.48) \]

Equation 3.46 is the actual linear algebraic system that is solved at each time step for the variable \(\Delta\), where the symbol \(\Delta\) can represent the variables \(u, v, \) and \(T\).
A variety of time integration schemes can now be chosen by simply varying $\theta$. Table 3.1 below lists some standard schemes along with the appropriate value of theta for each technique.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>fully explicit</td>
</tr>
<tr>
<td>1/2</td>
<td>Crank-Nicolson</td>
</tr>
<tr>
<td>1</td>
<td>fully implicit</td>
</tr>
</tbody>
</table>
4.0 Solution Method and Technique

4.1 Chapter Introduction

The final equations in Chapter 3 still need to be solved by an appropriate method. The flow equations are highly nonlinear due to the convection terms and the energy equation is nonlinear when radiation is included. If buoyancy effects due to temperature differences are included in the body forces in the Navier-Stokes equations, then the equations are coupled. Therefore, a complete investigation will involve the solution of coupled nonlinear partial differential equations, ones which have been notoriously difficult. This chapter will present the method chosen for solving this nonlinear system of equations and the numerical implementation of this method.

First, the important terms that evolve due to the variational formulation and application of the finite-element method will be examined. The type of interpolating functions used, a method handling variable properties, the calculation of the radiant fluxes, and error estimates will all be discussed. Next, the overall solution scheme for the resulting system of equations will be presented. The advantages and disadvantages to the various methods used to solve this system of equations will be discussed. Convergence of an example
problem using various techniques will illustrate the flexibility of the chosen solution scheme.

4.2 Body Force Terms

The body force terms \( f_x \) and \( f_y \) in equations 3.2 and 3.3 that will be considered here are due to hydrostatic pressure and gravity. If the pressure is defined such that the hydrostatic pressure is included, then in the Navier-Stokes equations the body force term acting in the vertical direction is simply:

\[
f_y = (\rho_r - \rho)g,
\]

where \( f_y \) is a buoyancy force in the vertical direction, \( \rho_r \) is the density at a reference state and is constant, \( \rho \) is the actual density and may vary with temperature, and \( g \) is the gravitational acceleration. Equation 4.1 is usually approximated at this stage by employing the Boussinesq approximation. This approximation assumes that properties are constant and expands the density term in a Taylor series about the reference temperature, \( T_r \). Thus:

\[
\rho = \rho_r + \frac{\partial \rho}{\partial T} (T - T_r) + \frac{\partial^2 \rho}{\partial T^2} \frac{(T - T_r)^2}{2!} + \ldots
\]

The coefficient of thermal expansion is defined as:

\[
\beta = -\frac{1}{\rho} \frac{\partial \rho}{\partial T} \bigg|_{p=\text{const.}}
\]
Using this definition and neglecting the higher-order terms, the normalized buoyancy term can be approximated as:

$$\frac{\rho_r - \rho}{\rho_r} = \beta (T - T_r). \quad (4.4)$$

Details of this approximation are included here and will be discussed in an attempt to curtail its current widespread misuse. Without an equation of state, the approximation is restricted to very small temperature changes. Since air is the medium being investigated, the ideal gas equation of state can be used to check the validity of equation 4.4. For an ideal gas at constant pressure:

$$\frac{\rho_r - \rho}{\rho_r} = \frac{T - T_r}{T}, \quad (4.5)$$

where the temperatures are measured in absolute units. Also it can be shown for an ideal gas that:

$$\beta = \frac{1}{T}. \quad (4.6)$$

Substituting this into equation 4.4 yields:

$$\frac{\rho_r - \rho}{\rho_r} \approx \frac{T - T_r}{T_r}. \quad (4.7)$$

The approximation of the normalized buoyancy term (equation 4.7) and the more accurate expression (equation 4.5) are plotted in Fig. 4.1 against the temperature.
difference for a reference temperature of 300 K, which will be the standard reference temperature in this investigation. Figure 4.1 shows the divergence of ideal gas behavior from this common approximation.

The 'standard' allowable engineering error of 10 per cent for the Boussinesq approximation occurs at a temperature difference of 28°C which is below the operation limits of many electronic components. While it has been shown that the assumption of constant properties for air is valid for large temperature differences (Leornardi and Reizes, 1981), the calculation of the buoyancy term is critical in pure natural convection flows (this force is driving the problem). The justification that using the Boussinesq approximation gives a generic solution is not valid, for no matter how the governing equations are normalized, a specific fluid property (i.e., Pr) will have to be chosen and, thus, an equation of state or a polynomial fit can be used for the density variation versus temperature. Equation 4.7 will only be used in validating computations against existing bench mark solutions which use the Boussinesq approximation. The more accurate form for the buoyancy term (equation 4.5) will be used in all other calculations. All other fluid properties will be considered constant and taken at a reference temperature and pressure of 300 K and 100 kPa, respectively. A procedure for implementing variable properties will be discussed in section 4.3. This procedure, while not required for air at moderate temperature differences, would be necessary in the thermal modelling of water, for instance, at very small (less than 2°C) temperature differences due to the strong variation of thermal properties with temperature (Gray and Giorgini, 1982).
Figure 4.1 Buoyancy Term Comparison
4.3 Convection Terms and Interpolation Functions

The convective terms in equations 3.5 and 3.6 require special consideration in any formulation of the Navier-Stokes equations. These terms, shown below in equations 4.8 and 4.9, represent the rate of change of momentum due to convective acceleration and cause the strong nonlinearities associated with the Navier-Stokes equations. When the flow is diffusion dominated such as highly viscous flows or creeping flows, then these terms may be neglected with the resulting equations representing Stokes flow. In this case the system of equations will be linear resulting in a great savings in computer cost. However, in the application of air cooling to electronics these terms can not be neglected. Therefore a suitable method to handle the convective terms will be presented. For the two dimensional problem the convection terms from the momentum equations are:

\[
\begin{align*}
\text{x - convection:} \\
\quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \\
\end{align*}
\]  
(4.8)

and

\[
\begin{align*}
\text{y - convection:} \\
\quad u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \\
\end{align*}
\]  
(4.9)

The nonlinearity is a result of the squared velocity (velocity multiplied by a gradient of velocity) terms in equations 4.8 and 4.9. Since linear algebra, which is highly developed, will be used to solve this system of equations, a method to linearize these terms is necessary. An iteration scheme, which picks up the nonlinearities as it iterates, is also
required. The velocities in these terms will be assumed known either as initial input or as
the solution from the previous iteration and the gradients will be calculated from the
interpolation functions. Therefore equation 3.17 can be written as:

$$\bar{K}_{ij}^{11} = K_{ij}^{11} + \int_{\Omega^e} \rho \psi_i \left( u^* \frac{\partial \psi_j}{\partial x} + v^* \frac{\partial \psi_j}{\partial y} \right) dxdy$$

(4.10)

where:

$$u^* = \sum_{j=1}^{n} u_j^* \psi_j \quad \text{and} \quad v^* = \sum_{j=1}^{n} v_j^* \psi_j.$$

Known velocities are used for the starred velocities and updated as the system of
equations is repeatedly solved until the difference in the velocity and temperature fields
obtained from consecutive iterations is less than an assigned tolerance. The convergence
test used for both the temperature and velocity field is represented by equation 4.11
below:

$$\left[ \frac{\sum_{i=1}^{m} |\Delta_i^\text{new} - \Delta_i^\text{old}|^2}{\sum_{i=1}^{m} |\Delta_i^\text{new}|^2} \right]^{1/2} \leq \text{tolerance},$$

(4.11)

where $m$ is the total number of nodal values and $\Delta_i$ represents the variables $u$, $v$, or $T$ at
the node location $i$ at two consecutive iteration levels (new and old). While this is an
overall convergence test, the value of the tolerance was set such that all local variables
did not vary greater than .01 percent from previous iterations. The type and choice of
interpolating functions will be now discussed.

Solution Method and Technique
In general the degree of the interpolating functions for the coordinate transformation (presented next) is not required to match the degree of the interpolating functions for the dependent variables. A common simplification is to match the degree of these polynomials resulting in isoparametric elements. These are the type elements used in this development. A theory by Malkus and Hughes (1976) suggests that the most effective elements for this problem are the Lagrange isoparametric elements. Hughes (1979) also comments that triangular elements are “serendipity” quadrilateral elements (see Reddy, 1984) and have been shown to exhibit inferior behavior.

The degree of the interpolating function determines the required number of nodes per element. While quadratic elements may capture steep wall gradients with fewer elements, they have the disadvantage of not being consistent with the linear geometry used for the surface radiation and also requiring a larger bandwidth and therefore more storage in a banded solver. A general four, eight, or nine-node bilinear or quadrilateral element was initially used for the Navier-Stokes equations but was removed when the energy equation, along with buoyancy and radiation effects, were added. This was done not only for simplicity and consistency but also to give a better distributed source term for the body forces caused by buoyancy. Since four bilinear elements could represent one quadrilateral element with less storage and a better distribution of body forces, bilinear interpolating functions are used for both the temperatures and velocities and will be presented next.

The four node bilinear element approximates any function, \( f \), by the following general polynomial:

\[
 f(x,y) = c_1 + c_2 x + c_3 y + c_4 xy .
\]  
(4.12)
This function can be represented over an element by the expression:

\[ f = \sum_{j=1}^{n} f_j \psi_j, \]  \hspace{1cm} (4.13)

where \( n \) is number of nodes per element (four for bilinear elements), \( f_j \) are the functional values at the \( j^{th} \) location, and \( \psi_j \) are the linear interpolation functions. Since the function will be differentiated and integrated over a general element shape, a mapping from the physical plane to a fixed computational plane is performed which allows a general numerical scheme to be employed. Gauss-Legendre quadrature integration is used for the integration scheme. The location of the four Gauss points is shown in Fig. 4.2. These points yield exact integration of a second-degree polynomial which results in third-order accuracy for the bilinear elements.

As mentioned earlier the Navier-Stokes equations can be notoriously difficult to solve for some classes of problems. This has lead some investigators to explore methods which help stability at the cost of accuracy. One of the better known of such techniques is upwinding of the convective terms. Instead of using a central difference operator for the derivative of the velocity, this derivative is approximated using information in an 'upstream' sense. Physical justification is usually given as: upstream information is dominating the control volume, or from whence does one smell an odor. A more detailed study reveals that this technique is lowering the order of the approximation and adding an implicit 'false' viscosity. Implicit means this viscosity is built in and can only be lowered by refining the grid. (Some techniques use 'explicit' artificial viscosity which is added by the numerical analyst in an attempt to acquire a converged solution). This added viscosity
Figure 4.2 Element Transformation
can be several orders of magnitude greater than the actual viscosity, resulting in the suppression of any wiggles that may appear when the grid is not sufficiently refined for a difficult problem. This loss of information, that one is trying to get more than he is willing to pay for (i.e., a high Reynolds number solution on a coarse grid), may be a greater tragedy than the hoax of ‘improving’ the solution by upwinding. Leonard (1979) and others have shown that it is possible to properly incorporate upwinding but this requires a third-order differencing scheme, one which requires more effort than the simple solution technique of many investigators. One argument for finite-difference methods has been the ease with which special ‘tricks’ such as upwinding can be implemented.

The finite-element method can just as easily be modified to include any degree of upwinding desired. This only involves an examination of the direction and magnitude of the previous velocity components and moving the location of the Gauss points shown in Fig. 4.2 in the appropriate direction and adjusting the weights (this, of course, will lower the accuracy of the method). Some upwinding methods use a hybrid scheme (Spalding, 1972 and Patankar, 1980) where a function determines the degree of upwinding depending on the cell Peclet or Reynolds number. It is quite easy to show that most of the hybrid schemes are nothing more than a mask because they switch to full upwinding for any difficult problems. Full upwinding was compared to proper integration of the convective terms early in this investigation and judicial grid refinement without upwinding was found to give superior results and was used in all problems.

If it is desirable to vary properties or obtain an integrated average value over any element, application of equation 4.13 is straightforward and easy to implement. For example if the
modelling of the variation in density (other than the buoyancy term) were desired, then for an ideal gas the density term would be calculated based on the nodal temperatures by:

\[ \rho^e = \sum_{j=1}^{n} \frac{C}{l_j} \psi_j, \tag{4.14} \]

where C is a constant depending on the local pressure and molecular weight of the gas being considered. Equation 4.14 is actually an integration over the element and so is performed at the Gauss quadrature points. The extra computer cost (CPU time and storage) is minimal since the density in this case need not be stored and the shape functions, \( \psi_j \), must be calculated in any case. Other property variations can be incorporated in a similar manner.

While a detailed error estimate is beyond the scope of this dissertation (see Cuvelier, et al. 1986), an approximate estimate of the difference between the finite-element solution, \( u_h \), and the true solution, \( u \), over an element with characteristic length, \( h \), is given by:

\[ \left\{ \frac{1}{\Omega} \int_{\Omega} (u - u_h)^2 d\Omega \right\}^{1/2} \leq ch^k, \tag{4.15} \]

where the LHS in equation 4.15 is referred to as the \( L_2 \) norm and is one of the many ways to measure distances between two functions (Johnson and Riess, 1984). The constant \( k \) is called the rate of convergence and is dependent on the degree of the equation and the order of the interpolation functions; \( c \) is also a constant which is problem dependent but independent of \( u_h \). If the interpolation functions are complete polynomials of degree \( p \), then for second-order equations (Reddy, 1984) \( k \) is equal to \( p \). While the bilinear
elements are second order they are only complete in first order, so a lower bound on the error estimate would give:

\[ L_2 \leq ch^1. \quad (4.16) \]

In some cases where the data are smooth and boundary conditions are Dirichlet or Neumann on \( \Gamma \), then the estimate for the \( L_2 \) norm can be improved (see Ciarlet, 1978 and Raviart and Thomas, 1983) for this case to:

\[ L_2 \leq ch^2. \quad (4.17) \]

From equations 4.16 and 4.17 it can be seen that the error approaches zero as the element length decreases. Since the interpolation polynomials were estimated as first order, the rate of convergence will be at least first order as given by equation 4.16. Another measure of norms is the energy norm which is a measure of the convergence of the derivatives. This norm converges at a slower rate than the \( L_2 \) norm (Reddy, 1984) and will be explored in a benchmark solution in Chapter 5.

4.4 Surface Radiation

The calculation of the configuration factors needed in the system of equations represented by equation 3.379 would be a formidable task using equation 3.35. Since the domain is two dimensional, a method presented by Hottel (1954) (referred to as Hottel's cross-string method) is used for the surface configuration factors. The method allows the calculation of these configuration factors with straightforward algebraic equations that
are easily implemented. This technique is exact in two dimensions and even allows for blockage of radiant transfer by other surfaces. The cross-string method along with configuration-factor algebra was used to reduce the computations required for the fixed coefficients in the radiant transfer equations. Since all surfaces are approximated as piecewise flat (geometrically matching the bilinear elements), the surface of an element can not see itself, so \( F_{ii} \) is always zero. Therefore, for a general \( n \) surface enclosure only \((n-1)(n-2)/2\) shape factors need to be calculated. This is shown in the example 6x6 shape-factor matrix below, where the diagonal terms represent flat surfaces and the \( x \) terms must be calculated from basic principles, while the \( c \) terms can be derived from shape-factor algebra.

Shape factor matrix for a six surface enclosure:

\[
\begin{bmatrix}
0 & x & x & x & x & c \\
   & c & 0 & x & x & x \\
   & & c & c & 0 & x \\
   & & & c & c & c & 0 \\
   & & & & c & c & c & c & 0 \\
\end{bmatrix}
\]

The general matrix given by equation 3.39 is full and non-symmetric. However, if the emissivities do not vary with temperature over the range of interest, then the two matrices in this radiant transfer equation have constant coefficients. LU decomposition was used to solve this set of equations. Both matrices were calculated once and the LHS matrix was decomposed and stored. The RHS matrix in equation 3.39 was repeatedly multiplied by
the latest temperatures in an iteration scheme to produce a vector that was sent along with the decomposed matrix to the radiant transfer solver routine.

4.5 Solution Technique

Once the coefficients in the general matrix representation for the Navier-Stokes and energy equations presented in Chapter 3 have been found, a variety of solution techniques for a large system of equations can be employed. Since the resulting coefficient matrix is banded and unsymmetric, the method chosen was a banded unsymmetric solver. The arrangement of the variables shown in equation 3.15 would create an excessive bandwidth. In order to minimize storage requirements the equations were rearranged so that appropriate coefficients of the variables were kept as close together as possible in the matrix and node numbering transversed the direction of least stride so that the bandwidth was kept to a minimum. Since the coefficients are in general functions of temperature and velocities (both of which are being continuously updated), only one large work matrix was dimensioned for the banded coefficients of the velocities, and the temperature stiffness matrix destroyed this matrix when storage was needed for the energy equation.

Two of the most common methods for solving the resulting nonlinear system of equations are: Newton’s method and Picard’s method. Newton’s method has a faster rate of convergence while Picard’s method of successive substitutions has a larger radius of convergence and is simpler to implement. Due to the complexity of including the changing nonlinear boundary conditions and the need for a sufficiently close starting point for Newton’s method, the method of successive substitutions will be used to solve the system of equations presented in Chapter 3.
Partial segregation was used in solving the entire system of equations for \( u \), \( v \), and \( T \). A segregated approach solves for one set of variables (\( u \), \( v \), or \( T \)), assuming the other variables are known, and iterates while updating the assumed values. While this approach has obvious storage advantages, it has convergence problems when there is strong coupling of the variables, such as \( u \) and \( v \) in recirculating flows or velocities and temperature in buoyancy driven flows. Only the temperatures were separated in the solution scheme in an attempt to minimize storage and separate nonlinearities. The effects of this segregation will be discussed shortly.

A schematic of the overall solution scheme is shown in Fig. 4.3. The preprocessor unit reads input data such as properties, boundary conditions, integration scheme information, maximum allowable iterations, and the convergence criteria. This unit does initial fixed calculations such as grid generation and LU decomposition of the coefficient matrix for the radiant fluxes. Mapping arrays are created which allow the referencing of variables to the different subroutines. (A fixed nodal location may have as many as four different numbering schemes, two for velocities and two for the temperatures.) If an initial guess to the variables is not supplied, then zero velocities and the reference temperature are used as starting points. With zero velocities as a starting point the first iteration will yield the Stokes flow solution for the velocities or a pure conduction solution from the energy equation. The old values of velocities are immediately updated and used in the next iteration for the convective terms. If the equations are strongly pressure driven as opposed to buoyancy driven then iteration is continued on the Navier-Stokes equations until the convergence criterion is met. The converged velocities are then used in the energy equation to solve for the temperature field. The surface temperatures from this temperature field are then used in the equations of radiant transfer to solve for the radiant
Figure 4.3 Overall Solution Scheme
fluxes. These fluxes are then subtracted from the specified fluxes and this new value is used as a boundary condition on the specified convective fluxes. This process is iterated until the test for convergence is satisfied. The final convective fluxes and radiant fluxes will add to equal the specified boundary fluxes. A return to the Navier-Stokes solver allows for correction of the buoyancy terms and a recalculation of the velocity field. This process is repeated until an overall convergence test is met. The postprocessor unit handles data output and variable manipulation such as gradient or pressure calculations.

The equations for velocities and temperatures are still coupled by virtue of the buoyancy term in the Navier-Stokes equation and the velocities in the energy equation. This decoupled solution technique, while saving storage and giving flexibility, can cause oscillations when the flow is strongly or totally driven by temperature as in the differentially heated cavity. Initial zero velocities give rise to temperature gradients which then drive the fluid to excessively high velocities through buoyancy effects. These high velocities quench the temperatures which then give low velocities and start the process anew.

This effect is shown in Fig. 4.4 for a Rayleigh number of $10^4$ for a square cavity (this problem is more fully described in Chapter 5). Two cures for these oscillations were discovered. Attempts to arrive directly at steady state by iteration using direct substitution could be under-relaxed as shown in this figure. It was found that the best relaxation factor for a given problem was Rayleigh number but not grid dependent. Another technique employed when strong oscillations were encountered was time integration, which is also shown in Fig. 4.4. While the CPU cost for the under-relaxation technique was far less for
Figure 4.4 Convergence Techniques
In this example, a 'best' relaxation factor is not always known and use of relaxation where it is not warranted will increase this cost. It should be stressed that neither of these techniques alter the final solution such as implicitly adding artificial viscosity (upwinding) or explicitly adding viscosity (dispersion control). When solution oscillations were not due to the decoupling mentioned (such as high Reynolds number flows) then grid refinement was found to yield stable solutions in all cases.
5.0 Flow Results And Validation

5.1 Chapter Introduction

The ultimate aim of any computer simulation is to predict, not simply confirm information. However, before solving the complete set of coupled nonlinear equations presented in Chapter 3, verification of the method chosen and the techniques used to solve these equations will be presented. The computations will be compared with existing results and experiments where possible. This validation will be done first to test the Navier-Stokes solution for developing flow through parallel channels without heat transfer. Examination of these results will allow determination of whether an elliptic solution is required or whether a less costly parabolic solution is adequate. Secondly, the glazing problem, which tests the coupling of the energy equation with the flow equations, will be presented and the results analyzed. Finally, the combined problem of forced convection with developing flow and heat transfer through parallel plates will be examined and compared to experimental measurements. The combined mode vertical parallel plate problem will be examined separately in Chapter 6.
5.2 Hydrodynamically Developing Flow Between Parallel Plates

The developing flow between the parallel plate geometry shown in Fig. 5.1 has been examined by many investigators (see Shah and London, 1978). The techniques used to solve this fluids problem have varied from the integral methods of Schiller (1922) to the perturbation method of Schlichting (1968).

These early attempts were later shown to represent a physical paradox by Van Dyke (1970) and to be mathematically inconsistent by Wilson (1971). The linearization method of Sparrow, et al. (1964) gave results which were in closer agreement with the numerical results of Bodoia and Osterle (1961), which were found by linearizing the momentum equation at any cross section. Lui (1974) tabulated the parallel plate results of Shah and Famia (1974) which were calculated using finite differencing.

In the above methods, the velocity distribution at the entrance was considered uniform and the axial diffusion and transverse pressure gradient terms in the Navier-Stokes equations were assumed small enough to be neglected. When the complete Navier-Stokes equations are considered for this problem, an interesting phenomenon occurs. Instead of the uniform or parabolic entrance velocity profile that is often assumed or presented in the literature, one finds a local minimum velocity at the channel center with symmetric maxima near the wall region. This phenomenon is referred to by Shah and London (1978) as velocity overshoots, axial velocity inflections, and bulges, kinks, or concavity in the velocity profile.
Figure 5.1. Geometry for Parallel Plate Flow
The velocity overshoots were first reported by Wang and Longwell (1964) in their numerical solution for parallel plates. They found velocity overshoots at all Reynolds numbers, with the overshoots being more pronounced for the initial condition of a uniform velocity profile. The computations of Schmidt and Zeldin (1969) are sighted by White (1974) as the most accurate for their predictions of the pressure defect (to be presented in this section). They solved the complete set of Navier-Stokes equations for parallel plates with irrotational inlet flow, at Reynolds numbers of 100, 500, and $10^4$, and also observed overshoots in the entrance-region velocity profiles. At first these predictions of concavity in the entrance velocity profiles were thought to be numerical oddities or the results of singularities at the entrance. However, Abarbanel, et al. (1970) theoretically showed that the overshoots exist for Stokes flow and Berman and Santos (1969) found the velocity overshoots in a series of laser-Doppler measurements of the velocity profiles in the entrance region for laminar flow. The possible reasons and situations for such overshoots are discussed fully by Shah and London (1978) and Morihara and Cheng (1973).

The finite-element method presented in the preceding chapters was used to solve for the flow field for a given pressure difference between the supply and exhaust plenums of Fig. 5.1. Since the method solves the full elliptic problem, a guess of the magnitude and shape of the entrance velocity profile is not required as in parabolic solvers (which then must iteratively correct this assumption). The Bernoulli equation is assumed to apply from a stagnation point in the supply plenum to the channel entrance and from the channel exit to a stagnation point in the exhaust plenum. This assumption allows recovery of any entrance and exit pressure losses and, thus, the specified pressure difference between the plenums will be the losses within the channel. To test the validity of this assumption an
expanded grid system was placed in the plenums. It was found that the entrance and exit losses can be assumed recoverable for small to modest Reynolds numbers, but for Reynolds numbers greater than 1800 these losses become irreversible and significant compared to the the channel losses. Indeed it has been shown in the experiments of O'Meara and Poulakis (1987) that even the presence and location of entrance and exit walls can have a profound effect on the channel flow. Early investigations by this author found a large variation in the predicted channel flow rate when the plenum had an imposed velocity across the mouth of the channel. This situation would represent a small supply plenum feeding multiple channels and, thus, would have a cross stream velocity component.

An appropriate question would be: “Where does one stop with the domain of the problem?”. Proper boundaries would be ones in which the boundary conditions are known, such as the plenum walls or far enough into the plenum for the velocities to become approximately zero. The use of these boundaries is usually too expensive and will not be employed in this investigation into the generic problem of channel flow and heat transfer. The results presented here are intended to be used as trends and predictions, not as absolutes. Nevertheless, for all of the test cases and problems that follow, progressive grid refinement was applied starting with a coarse grid of 100 elements and progressively moving to a finer grid until the solution remained unchanged from the previous grid. In some cases this required as many as 2500 elements or 7500 nodal values since there are three degrees of freedom per node when temperature is included.

Typical developing velocity profiles for an aspect ratio of 10 are shown in Fig. 5.2, 5.3, and 5.4 for Reynolds numbers of 144, 838, and 4260, respectively. These are the resulting
Reynolds numbers from the specified nondimensional pressures of $10^4$, $10^5$, and $10^6$. (The velocity profile in Fig. 5.2 was fully developed at a nondimensional axial location of 0.25, therefore other locations were not plotted.) One should compare the shape of the entrance velocity profiles to the typical assumption of fully-developed flow which would result in a parabolic entrance profile that remains constant or a uniform inlet profile often used with parabolic solution methods. Also note that the overshoots are present for the full range of Reynolds number being investigated, which are representative for typical electronic packages. The existence of a second smaller set of ‘humps’ in the entrance velocity profile was detected at higher Reynolds numbers (see Fig. 5.3 and 5.4), whether this is a result of the entrance curvature in the velocity or numerical oscillations, warrants further investigation. Initial convergence was achieved for the Reynolds number case of 4260 on an uniform 10x20 grid, showing the high allowable cell Reynolds number for this formulation.

The effect of the velocity overshoots in the entrance region is twofold. Firstly, there will be less flow for a given pressure drop due to the extra momentum losses in changing the velocity profile. Secondly, the wall shear stress will be greater in the entrance region due to the steeper velocity gradient. This second effect will also result in less flow and (normally) correspondingly less heat transfer. However, this heat transfer loss due to reduced flow may be offset by the greater shear stresses at the wall which results in enhanced heat transfer via Reynolds analogy. This will be investigated further in the mixed convection heat transfer results presented later.

A general equation for calculating the pressure drop between parallel plates is given by

(see, for example, Shah and London, 1978):
Figure 5.2. Normalized Axial Velocity Profile
Figure 5.3. Normalized Axial Velocity Profile
Figure 5.4. Normalized Axial Velocity Profile
The first term, \( f \), on the RHS in equation 5.1 represents the losses due to wall shear stresses and is a function of the shape of the velocity profile (thus the Reynolds number) and also the geometry. The second term, \( K \), on the RHS in equation 5.1 is called the pressure defect and represents pressure losses caused by changes in momentum due to the development of the velocity profile. For fully developed laminar flow:

\[
f = \frac{C}{Re},
\]

(5.2)

where Re is the Reynolds number based on the hydraulic diameter and C is a constant dependent on geometry, i.e., for circular ducts C is 64., and for the parallel plates being considered here C is 96.

While the friction factor, \( f \), obviously will change as the flow develops, the fully developed value which is a constant is used in equation 5.1. Since the fully developed value is less than the integrated average value for any axial distance, the correction for using this value is incorporated into (and just increases) the pressure defect term, \( K \). The pressure defect increases from zero at the entrance to a maximum value which is a function of the Reynolds number.

Both wall shear stress losses and the pressure defect can be incorporated into a single term called the apparent friction factor, \( f_{\text{app}} \). Then the nondimensional form of equation 5.1 becomes:
\[ \Delta P^* = (f_{app} \text{Re}) \frac{\text{Re} \cdot h}{16b}, \]  

or for parallel plates when the flow becomes fully developed:

\[ f_{app} = \frac{96}{\text{Re}}. \]

So once the flow becomes fully developed:

\[ \frac{h}{\text{Re}^2} = \frac{\Delta P^*}{6}. \]  

Equation 5.4 is a simple expression for the nondimensional flow rate or Reynolds number for fully developed flow. This formula is plotted along with the calculated values in Fig. 5.5 and 5.6 for the aspect ratios of 5, 10, and 20. Notice that all four curves coalesce for the low pressure (and Re) range in Fig. 5.5, but start diverging at a nondimensional pressure of $10^3$. A dimensional analysis of the Navier-Stokes equations will show that when viscous diffusion is dominate, the aspect ratio can be removed as a parameter of the problem. This is not the case when the convection terms also become important as can be seen in Fig. 5.6. Since a log-log plot tends to mask the magnitude of the variations, a list of the values used in Fig. 5.5 and 5.6 is included in Table 5.1. It can be seen from these values that equation 5.4 or the fully-developed flow assumption is accurate to within 2% up to a nondimensional pressure of $10^3$ (which yields a Reynolds number range of 32.8, 16.7, and 8.35 for the aspect ratios of 5, 10, and 20, respectively). However, at a nondimensional pressure greater than $10^3$ the fully-developed flow assumption is no longer valid as can be seen from Table 5.1 or Fig. 5.6.
Table 5.1 Reynolds Number versus Nondimensional Pressure

<table>
<thead>
<tr>
<th>$\Delta P^*$</th>
<th>$h/b=5$</th>
<th>$h/b=10$</th>
<th>$h/b=20$</th>
<th>Eq. 5.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^0$</td>
<td>0.16312</td>
<td>0.16610</td>
<td>0.16647</td>
<td>0.1667</td>
</tr>
<tr>
<td>$10^1$</td>
<td>1.7099</td>
<td>1.6804</td>
<td>1.6695</td>
<td>1.667</td>
</tr>
<tr>
<td>$10^2$</td>
<td>17.150</td>
<td>16.824</td>
<td>16.700</td>
<td>16.67</td>
</tr>
<tr>
<td>$10^3$</td>
<td>164.03</td>
<td>167.18</td>
<td>166.90</td>
<td>166.7</td>
</tr>
<tr>
<td>$10^4$</td>
<td>1098.2</td>
<td>1439.2</td>
<td>1612.4</td>
<td>1667.</td>
</tr>
<tr>
<td>$10^5$</td>
<td>5555.5</td>
<td>8386.8</td>
<td>12160</td>
<td>1.667x10^4</td>
</tr>
<tr>
<td>$10^6$</td>
<td>33794</td>
<td>42610</td>
<td>67831</td>
<td>1.667x10^5</td>
</tr>
</tbody>
</table>
An equation for the development length, $L_{hy}$, when the maximum velocity becomes 99% of fully-developed maximum velocity, is given by Chen (1973) in the following relationship:

$$\frac{L_{hy}}{D_h} = \frac{0.315}{0.0175Re + 1} + 0.011Re.$$  \hspace{1cm} (5.5)

This equation is tabulated in Table 5.2 below for the Reynolds number that would give fully-developed conditions at the channel exit (for the range of aspect ratios of interest).

<table>
<thead>
<tr>
<th>$h/b$</th>
<th>Re$_{FD}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>52.2</td>
</tr>
<tr>
<td>5.0</td>
<td>212</td>
</tr>
<tr>
<td>6.0</td>
<td>259</td>
</tr>
<tr>
<td>10.0</td>
<td>446</td>
</tr>
<tr>
<td>15.0</td>
<td>676</td>
</tr>
<tr>
<td>20.0</td>
<td>905</td>
</tr>
</tbody>
</table>

The fully-developed values from Table 5.2 are marked in Fig. 5.6. It should be noted that while the flow may become fully developed at the channel exit, the fully-developed flow assumption is still not accurate until fully developed flow prevails over most of the channel. This occurs at a nondimensional pressure or Reynolds number a full order of magnitude less than that given by equation 5.4.
Figure 5.5 Low Reynolds Number versus Nondimensional Pressure Difference
Figure 5.6. High Reynolds Number versus Nondimensional Pressure Difference
5.3 Natural Convection In A Square Cavity

A trial case that thoroughly tests the coupling of the Navier-Stokes equations and the energy equation, and is used as a validation of numerical methods for solving two-dimensional pure natural convection problems will now be presented. The differentially heated cavity or glazing problem, so called for its application to double glazed windows, tests this coupling. Without the body forces caused by the temperature differences there would be no fluid movement and therefore only conduction heat transfer. A discussion of previous analytical, numerical, and experimental studies involving this problem is provided in the literature review; therefore only a description and results will be presented in this section. A 'square' cavity was chosen since this has been used as a standard benchmark (Davis and Jones, 1983) for convective heat transfer problems. While this eliminates the aspect ratio as a parameter in the problem, the corner regions where the fluid must change direction effect the solution more than the high aspect ratio case, therefore this is only a parameter simplification and requires the retention of all terms in the governing equations. The geometry and boundary conditions are shown in Fig. 5.7. The top and bottom of the cavity are insulated while the opposite sides are held at different temperatures, \( T_H \) and \( T_L \), and both velocity components are zero on the boundaries. This problem can be characterized by the following three parameters:

1) a characteristic length, \( h \), (usually chosen as the vertical length for the non-square cavity),
Figure 5.7 Cavity Geometry and Boundary Conditions
2) the Grashof number, $Gr$, which is a measure of the ratio of buoyancy forces to viscous forces. Using the Boussinesq approximation the Grashof number is defined as:

$$Gr = \frac{g \beta (T_H - T_C) h^3}{\nu^2}.$$  \hspace{1cm} (5.6)

3) the Prandtl number, $Pr$, which is the ratio of the molecular momentum and thermal diffusivity, i.e., $\frac{\nu}{\alpha}$.

Besides the Grashof number, which is the driving potential in this problem, results often are cast in terms of the Rayleigh number, $Ra$, which is the product of the Grashof number and the Prandtl number. Creeping flow solutions do not depend on the Prandtl number and therefore for those cases the use of the Rayleigh number minimizes the number of parameters. For the case that will be presented, use of either number as a characterizing parameter is justified and so the Rayleigh number with a constant Prandtl number of 0.71 (air) will be used in this numerical test case.

As the Rayleigh number is increased, buoyancy forces are amplified and cause greater fluid movement, until at a Rayleigh number of $10^6$ the primary recirculating flow pattern is joined by secondary flows. The solution for this problem is considered numerically costly if not difficult to obtain (Davis and Jones, 1983) and so will be presented as an appropriate test. The overall flow patterns are shown in Fig. 5.8 for a Rayleigh number of $10^6$. Upward vertical flow is created by the higher temperature surface. This fluid gains momentum until it reaches the higher pressure stagnant region in the upper corner and
Figure 5.8 Velocity Vectors for a Ra of $10^6$
must turn towards the opposing wall where cooling causes the downward motion near this wall. Secondary recirculation can be seen and a section giving greater detail is presented in Fig. 5.9. For low values of the Rayleigh number ($< 10^3$) fluid motion is small and the predominant mode of heat transfer is conduction, and isotherms remain vertical across the cavity. However, at a Rayleigh number of approximately $10^4$ convective heat transfer becomes significant and the vertical isotherms in the center region no longer exist. Energy is transported across the top of the channel by the fluid motion and the horizontal temperature gradient is replaced by a vertical gradient in a stagnant center core. Some energy is actually conducted from the hot fluid leaving the left face through the center section to the cooler fluid returning to this face. While this conduction phenomenon reduces the heat transfer somewhat, the overall effect due to convection is an increase in the energy transport. The effects just described can be seen from the isotherm plot in Fig. 5.10. This figure shows the isotherms for a Rayleigh number of $10^5$. The close spacing of the isotherms in the lower region of the hotter wall and the upper region of the cooler wall indicate the dominant regions of energy transport. Instead of the uniform energy transport from the high temperature face to the cold temperature face (that would result from pure conduction), energy at this high Rayleigh number is being transported selectively through different wall regions. A study of the local heat transfer shows that the energy transport is actually reduced in the two ‘dead’ corner regions when compared to pure conduction. As pointed out before, all early theories failed to predict the physics of this problem, while the first computer simulations gave accurate heat transfer and variable information. This information (from numerical studies such as this) can give an electronics board designer insight into critical component placement.
Figure 5.9 Secondary Flow Region Velocity Vectors for a Ra of $10^6$

Dotted region enlarged and shown above.
Figure 5.10 Isotherms for a $Ra$ of $10^6$
Comparisons for this problem are usually based on the rate of heat transfer across the cavity. A nondimensional Nusselt number, Nu, which is the integrated average of the temperature gradients at the wall will give the ratio of the actual heat transfer to the pure conduction heat transfer. This has been calculated for a Rayleigh number range $10^3$ to $10^6$ for grid sizes from 10x10 elements to 50x50 elements. Results of these computer runs are given in Table 5.3 and are shown in Fig. 5.11 for the test case of $10^6$. For this case the flow is enhancing the heat transfer by a factor greater than eight.

In Fig. 5.11 three different methods for calculating the gradients of the temperature at the wall are plotted. This figure shows the differences in the predicted overall Nusselt number with the same output data using a first, second, and third-order expression for the wall derivative with the different grid sizes. (Various integration methods for the integrated averages were used and showed little if any influence on the overall Nusselt number). The dangers of the popular technique of extrapolation to zero grid size (Ozoe et. al., 1981), should be quite obvious (note the change in the sign of the slope) for this problem. (Data from practitioners of this method plot in a similar fashion). This technique assumes that the truncation error is proportional to the $n^{th}$ power of the grid size and that $n$ is a constant (truncated derivatives must be constant). With this assumption the value of a variable or quantity, $X_0$, at zero grid size is extrapolated by:

$$X_0 = \frac{\left[ \frac{h_2^n}{h_1} \right] X_1 - X_2}{\left[ \frac{h_2^n}{h_1} \right] - 1}, \quad (5.7)$$

where

$n$ is found from a fit of three different grid sizes,
$X_0$ is the extrapolated value to an infinitely fine grid,

$X_1$ is a value at a fine grid,

$X_2$ is from the course grid,

$h_i$ is the grid spacing at $X_i$.

<table>
<thead>
<tr>
<th>Ra #</th>
<th>grid</th>
<th>first order derivative</th>
<th>second order derivative</th>
<th>third order derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^3$</td>
<td>10x10</td>
<td>1.1046</td>
<td>1.1196</td>
<td>1.1072</td>
</tr>
<tr>
<td></td>
<td>16x16</td>
<td>1.1137</td>
<td>1.1186</td>
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</tr>
<tr>
<td></td>
<td>25x25</td>
<td>1.1163</td>
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*nonuniform grid

Table 5.3 Average Nusselt Numbers versus Rayleigh Number

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Figure 5.11 Nusselt Number versus Grid and Technique
The value of \( n \) can be obtained from three different grid spacings and used to reduce the computer cost by extrapolation instead of making a very fine grid run (which then must be checked with an even finer run to prove convergence). While it is obvious that the grids used should be fine enough to capture the flow and heat transfer characteristics (points should be in the boundary layers), even this is not sufficient. It can be seen from Fig. 5.11 that for the secondary variable, \( \text{Nu} \), the method of calculating the variable used in equation 5.7 greatly affects the value of \( n \). This extrapolation technique should be performed on primary variables with as many as possible grid points in common. Davis (1983) reported that his value of \( n \) varied throughout the cavity ("from around 1 or even a little less to about 3 or even a little more"). It was stated by Davis that the extrapolation method failed for a Rayleigh number of \( 10^6 \). This is where one would want to use extrapolation (on the difficult problems). A quadratic rate of convergence was assumed by Davis and he proceeded to extrapolate to the 'true' solution to this problem from his two finest grids, after some further data manipulation. This is being pointed out because Davis' solution is held as a benchmark test and this author feels that extrapolation is akin to upwinding (one is attempting to acquire a difficult solution without paying the cost). The finite-element method easily allows a less precarious and less costly technique; selective grid refinement. A 40x40 grid was refined in the wall regions and gave the final values in Table 5.3 and the .01 grid results in Fig. 5.11. While this grid refinement was manually done (since it is obvious where the high gradients will be on this problem), runtime adaptive grid refinement based on local gradients can easily be implemented in the present method and is an area of future interest.
5.4 Simultaneously Developing Flow Between Parallel Plates

The flow field and heat transfer in the entrance region between parallel plates will now be discussed and comparisons of predicted values with experimental results will be made. Developing flow between parallel plates is usually divided into three distinct types depending on the flow and property characteristics. If there is a sufficiently long unheated length before reaching the heat transfer section or if the Prandtl number is high enough, the velocity profile will be developed before heat transfer begins. This is referred to as thermally developing (hydrodynamically developed) flow and there have been numerous studies of this (Kays, 1966). The other extreme limiting situation is when the Prandtl number is very low such as for liquid metals. In this case the momentum boundary layer is slow to develop compared to the thermal boundary layer so that the changes in the velocity profile can be ignored with this profile often being approximated as uniform (slug) flow. Classical transient heat conduction methods give solutions for this case. When the Prandtl number is close to one as is the case with air and most gases, then the simultaneous development and possible interaction of both the momentum and thermal boundary layers needs to be considered. Any theoretical solution for this simultaneously developing situation will have to make assumptions about boundary conditions and neglect or approximate terms in the governing equations. Therefore, comparisons will be made to experimental measurements that closely approximate the flow situation between component cards such as may be found in computers or other electronic devices.

An experimental study by Mercer, et al. (1967), was found for this desired test. This experiment was chosen because air was the heat transfer medium, the channel was a finite length (entrance and exit effects may not be neglected), and the geometry and
temperatures used fall in the range of interest stated. The test geometry is shown in Fig. 5.12. A supply and exhaust plenum connected two parallel horizontal copper (low emissivity) plates (6 5/8" long and spaced 1/2" apart by 12" wide). The entrance to the channel was fluted and preceded by screens. The top and bottom plates could be separately insulated or held at a constant temperature. Local Nusselt number (defined below) comparisons will be made for the case when both plates were held at a constant temperature (135°F). This local Nusselt number is defined as:

$$N_{uL_0} = \frac{\theta_0}{\frac{\partial \theta}{\partial Y}} Y = 0, 1,$$  \hspace{1cm} (5.8)

where:

$$\theta = (T_w - T)/(T_w - T_0),$$

$T_w$ is the constant wall temperature,

$T$ is the local fluid temperature,

$T_0$ is the reference entrance temperature (70°F),

$Y$ is the nondimensional normal coordinate (y/s),

$s$ is the plate spacing.

Figure 5.13 shows the excellent agreement between the numerically predicted values of the local heat transfer versus the dimensionless coordinate $2(x/s)/Re_d Pr$. The Reynolds number is based on the hydraulic diameter $(2s)$ and $x$ is the axial coordinate. The experimental results were derived from Mach-Zehnder interferometer measurements. For Fig. 5.13, experimental values of 1474 and 682 for the Reynolds number are plotted together in the nondimensional independent variable. Computed values past the range of
Figure 5.12 Developing Flow and Heat Transfer Experimental Geometry
Figure 5.13 Isothermal Plates - Comparison with Experimental Results
measured values (closer to the entrance and exit than actually measured) are also plotted in this figure. The computer code did not have buoyancy effects turned on, since it was reported that there were only obvious free convection effects when only the bottom plate was heated and the Reynolds number was less than 300. Radiation was not accounted for in the computer simulation or the experiments for this case, but due to the excellent agreement and lack of reported emissivities for the copper plates this will not be pursued further here, but these combined effects will be included in Chapter 6.

5.5 Radiation Validation

Validation for pure radiation within a two-dimensional diffuse-gray enclosure was accomplished by comparison to examples in radiant transfer texts. Shape factors, radiant fluxes, and unknown surface temperatures were calculated and compared with results presented in Siegel and Howell (1981) and Gebhart (1971).
6.0 Heat Transfer Between Vertical Parallel Plates

6.1 Chapter Introduction

Numerical results for mixed convection heat transfer between vertical parallel plates will be presented in this chapter. The geometry and boundary conditions that will be considered are shown in Fig. 6.1 and are as follows. A specified flux, $q_s$, is applied at the vertical left wall while the opposite wall is considered insulated. This is referred to as asymmetric heating and models the situation of low profile heat producing devices mounted on a relatively low-thermal-conductivity board, and is a closer approximation of typical component cards than the assumption of symmetric heating. Asymmetric heating has been studied for pure natural and pure forced convection but has received little attention for the mixed convection case that will be presented. The aspect ratio, $A$, is defined as the ratio of the channel height to the spacing. An inlet velocity profile or a specified pressure difference or traction can be applied between the entrance and exit to the channels. Since typically a pressure difference is known instead of a velocity profile, the pressure difference between the plenums will be specified (most previous investigations have required specification of the inlet velocity profile and, thus, the mass flow rate or Reynolds number). The inlet temperature is considered fixed at $T_{ref}$ (300K), while the outflow thermal boundary condition is one of zero specified flux (convected energy loss but no conducted losses).
Figure 6.1 Vertical Plate Geometry
First, a series of numerical results for mixed convection heat transfer without radiation will be presented for a range of aspect ratios, heat fluxes, and pressure differences. The results will show the point at which forced convection dominates natural convection. This information indicates the point where the Navier-Stokes equations are decoupled from the energy equation, and allows a separated solution technique to be employed. Finally, in this chapter the effects of including radiant heat transfer with the other modes will be presented.

6.2 Mixed Convection

Mixed convection heat transfer through vertical channels, representative of the cooling of electronic equipment, will now be examined. This problem falls into one of three distinct regimes:

1) all natural convection where buoyancy is driving the flow,

2) pure forced convection were the pressure difference or a specified velocity governs the flow field, or

3) mixed convection where buoyancy and a specified velocity or pressure both govern the flow field.

In an actual vertical channel problem natural convection effects would always be present. However, when the pressure forces are large compared to the buoyancy forces, the effects of natural convection can be neglected. This section will provide some quantitative
information for deciding whether a problem is in pure natural, mixed, or forced convection.

The parameters that govern this problem (other than fluid properties) are:

1) the aspect ratio \( A = b/L \),

2) the specified heat flux \( q_s = -k \partial T / \partial x \), and

3) the pressure difference, \( \Delta P \), between the supply and exhaust plenums.

Three values of the aspect ratio (5, 10, and 20) typical of the card height and spacing used in communication equipment will be studied. The specified heat flux and pressure difference will be varied to included the full range from all natural to forced convection. This problem was examined in Chapter 5 for the case when the specified heat flux was zero. It was found that for low values of \( \Delta P \) the aspect ratio was not a parameter of the problem. This was not found to be the case when the effects of heat transfer (due to the buoyancy coupling) were included.

In order to generalize the results the parameters were cast into nondimensional variables. The Nusselt number, \( Nu \), is a dimensionless heat transfer coefficient and is defined as:

\[
Nu = \frac{hL_c}{k}, \tag{6.1}
\]
where \( h \) is the local convective heat transfer coefficient, \( L_c \) is a characteristic length, and \( k \) is the thermal conductivity of the fluid.

The dimensionless temperature is defined as:

\[
\theta = k \frac{\Delta T}{q_w b},
\]

(6.2)

where \( q_w \) is the wall heat flux, \( b \) is the channel width, and \( \Delta T \) is the difference between the local temperature and a selected reference temperature. The knowledge of the selection of this reference temperature is critical, for it effects the value of the heat transfer coefficient and the Nusselt number. The reference temperature used will be the supply (inlet) reference temperature, \( T_{ref} \), which is constant. Other commonly used reference temperatures are the integrated weighted (with velocity) average or 'mixing cup' temperature of the fluid (which is not constant in the flow direction) and the adiabatic wall temperature (which is typically the value used in experimental measurements). With the heat transfer coefficient based on \( T_{ref} \), the wall flux can be written as:

\[
q_w = h(T - T_{ref}).
\]

(6.3)

Selection of the plate height, \( L \), as the characteristic length (so the nondimensional coordinates will be normalized) yields:

\[
Nu_{xo} = \frac{A}{\theta},
\]

(6.4)
where the subscript $x$ refers to the axial location and the $0$ subscript refers to the inlet plenum reference temperature.

The form of the channel Rayleigh number for a specified heat flux that is used here is:

\[
Ra = \frac{g\beta q_w L^2 b^2 \text{Pr}}{\nu^2 k}.
\]  

(6.5)

The nondimensional pressure difference (see Chapter 5) is defined as:

\[
\Delta P^* = \frac{\Delta P b^2}{\rho \nu^2}.
\]  

(6.6)

While the results will be presented in nondimensional variables, the following test cases were run in 'real world' variables and parameters. This was done for three reasons:

1) to monitor the basic assumption of constant properties,

2) to include a better approximation for the buoyancy effects than the Boussinesq approximation (as pointed out earlier in Chapter 4), and

3) to insure that the test cases were realistic tests.

Grids were refined from 10x20 elements to 25x80 elements until changes in the results were insignificant (less than 1%). Figure 6.2 shows the variation of the channel Reynolds number as the nondimensional pressure difference, $\Delta P^*$, is varied from the natural-convection limit to the point where the flow is considered forced convection. For a given
Figure 6.2 $\text{Re}_{2b}$ Versus $\Delta \text{P}^*$ ($A=5$)
aspect ratio and plate height the heat input to the plate is characterized by the Rayleigh number. A value of the Rayleigh number of zero is given for reference and varied from $10^3$ to $10^6$ in the following three figures. One should notice in Fig. 6.2 that the Reynolds number is no longer a function of the pressure as the ratio of Rayleigh number to $\Delta P^*$ becomes greater than approximately $10^3$ for a Rayleigh number of $10^6$. At this point the channel flow and heat transfer are dominated by natural convection. For other Rayleigh numbers, a similar effect is shown to occur at lower values of this ratio. However, when the ratio of Rayleigh number to $\Delta P^*$ becomes less than 1.0, the flow is experiencing pure forced convection and the buoyancy effects can be ignored. In this case a less costly separated solution to the Navier-Stokes and energy equations can be used. Figure 6.3 and 6.4 show the relationship between $\Delta P^*$, $Ra$, and the Reynolds number for the aspect ratios of 10 and 20, respectively. The point where the channel is experiencing pure forced convection is still at a Rayleigh-number-to-$\Delta P^*$ ratio of about 1. This point, however, is a function of the Rayleigh number also, not just the ratio of $Ra$ to $\Delta P^*$; as can be seen from Fig. 6.3, the limiting ratio increases with increasing Rayleigh number from about 1 at a Rayleigh number of $10^3$ to about 10 at a Rayleigh number of 10.

In Fig. 6.5 through 6.7 the relationship between the Nusselt number, $Nu_{Lo}$ (as defined by equation 6.4 at the channel exit, $x=L$), the Rayleigh number, and the nondimensional pressure difference is presented. This Nusselt number (which is inversely proportional to the maximum temperature difference between the heated wall and the plenum reference temperature) is seen to level out at a lower ratio of the Rayleigh number to nondimensional pressure difference (about $10^2$) than the value for the Reynolds number. (Heat transfer is not such a strong function of flow rates, allowing an 'accurate' solution even when the details of the flow are missed). The different Rayleigh number lines are
Figure 6.3 $Re_{2b}$ Versus $\Delta P^*$ ($A=10$)
Figure 6.4 $Re_{2b}$ Versus $\Delta P^*$ ($\Lambda=20$)
Figure 6.5 $\text{Nu}_{Lo}$ Versus $\Delta P^*$ ($A=5$)
Figure 6.6 $Nu_{Lo}$ Versus $\Delta P^*$ ($A=10$)
Figure 6.7 \( \text{Nu}_{Lo} \) Versus \( \Delta P^* \) (A=20)
seen to come together as $\Delta P^*$ increases, finally reaching a pure forced convection limit depending on $Ra$ and $\Delta P^*$ (at a $Ra$ to $\Delta P^*$ ratio of approximately 10 for each aspect ratio).

It has always been tempting for engineers to represent the physical world as a linear model, and when this fails an attempt is made to find a power-law relationship, such as equation 6.7 below.

$$Nu(x) = C \left( \frac{L}{x} \right)^m Ra^n$$  \hspace{1cm} (6.7)

This is an expression for the local Nusselt number as a function a dimensionless geometric ratio (plate total length to local axial location) and the Rayleigh number.

This relationship for Nusselt number versus Rayleigh number was taken from an attempted correlation of the experimental results of Wirtz and Stutzman (1982). These experiments were for free convection only between vertical plates, but a review of the results is justified here. The exponents, $m$ and $n$, and the constant, $C$, in equation 6.7 were determined from a series of experimental measurements with air as the medium. The value of the recommended ‘constant’, $C$, was 0.520 for large plate spacings and 0.144 for small plate spacings. The postulated range of the exponents in equation 6.7 was $0.2 < m < 1.0$ and $0.2 < n < 0.5$. This variation in $m$ and $n$ was verified by natural convection experiments. Analysis of the results presented here shows that $n$ does indeed fall within this range when the plates are experiencing all natural convection.
When the plates are experiencing forced convection the Nusselt number is usually cast in terms of the Peclet number (the Reynolds number times the Prandtl number). A relationship from Gnielinski (1982) for thermally developing flow between infinite parallel plates is:

\[ Nu_x = C \left( \frac{Pe_x^{b \cdot n}}{x} \right), \tag{6.8} \]

where \( n \) is given as 1/3.

Comparison of the computed forced convection limit compares more favorably to this relationship giving a constant value of \( C \) consistently close to 1.05 and an exponent ranging between 0.36 to 0.37. It should be pointed out that the value of \( C \) will depend on the Nusselt number definition.

The computed results presented here are for quantitative information and have been compared with experimental results where possible in order to show the predictive power of a properly implemented numerical method and technique. No attempt will be made to present a simple formula for the general case of mixed convection heat transfer. This would be essentially inconsistent with the use of a numerical method (If the problem can be described by a simple formula, a theoretical solution is lurking in the wings!) It is also obvious from reviewing the experimental results that a general, accurate formula is not possible, even for the case of all natural convection.
6.1 Radiation Effects

In the previous section radiation effects were not included. While a simple calculation will show that for the typical electronic cooling application the effects of radiation can not be ignored, they have rarely been included in numerical analyses. This section will present the results, including radiation, within a typical range of the governing parameters. Due to the large number of parameters and the high solution cost for this highly nonlinear problem, the heat flux will be held constant at a value corresponding to a Rayleigh number of $10^5$ and the aspect ratio will be fixed at the median value of 10 used in the previous section. (This is a typical spacing for standard electronic equipment.) The nondimensional pressure will be varied from the point of natural convection (found in the previous section) to the point of forced convection. The geometry, parameters, and boundary conditions are shown in Fig. 6.8 which is similar to Fig. 6.1 except that radiation effects will now be included. The emissivities, $\varepsilon_s$, for the heated surface and the insulated surface can be specified but will be considered the same for each surface. The radiation losses at the entrance and exit are included, with the entrance and exit being considered as black surfaces at 300K.

The Reynolds number or total induced flow rate was found to be an extremely weak function of the emissivity (increasing only slightly with increasing emissivity), and so is not presented. Figure 6.9 presents the local Nusselt number at the channel exit, $Nu_{Le}$ (based on $T_{ref}$ as defined by equation 6.1), as a function of $\Delta P^*$ for an emissivity range of 0. to 0.8. As expected, the Nusselt number increases with pressure for a given Rayleigh number. Recall this form of the Nusselt number is a measure of the maximum temperature that the heated plate is experiencing. (The lower $Nu_{Le}$, the higher $\Delta T_{max}$ for
Figure 6.8 Geometry and Boundary Conditions Including Radiation
Figure 6.9 Nu vs ΔP*
a given geometry and heat flux.) The nondimensional pressure difference in Fig. 6.9 spans the range from natural convection to forced convection for this geometry and Rayleigh number (typical of the stated interest). The important feature of this figure is that radiation has a significant impact on the local Nusselt number in both the natural convection and forced convection situation presented. As before, when the nondimensional pressure difference is high enough, a decoupled solution to the Navier-Stokes and energy equations is possible.

In Fig. 6.10 the ratio of the integrated average heat transfer from convection and radiation from the heated plate to the total specified input is presented for a given emissivity of 0.6, with the nondimensional pressure being varied from $10^3$ to $10^5$. At a low pressure difference, approximately 30% of the heat transfer from the plate is by radiation, with the remaining 70% being by convection. It should be pointed out that this radiant energy never goes directly into fluid as in the participating medium case; rather, it is absorbed by the insulated plate and then conducted/convected into the medium. A portion of this energy goes into end losses to the plenum regions. These losses include any conducted losses to the entrance (very small) and radiant energy leaving the ends. The end losses must be considered when calculating a Nusselt number based on the bulk fluid temperature (not used here), since all of the heat transfer from the plate is not absorbed by the fluid.

Finally, Fig. 6.11 presents the effects of varying the plate emissivities for a fixed Rayleigh number of $10^5$ and $\Delta P^*$ of $10^4$. The ratio of the heat transfer from either mode to the total heat transfer is plotted on the ordinate. It should be noted that the fraction of heat transfer by radiation varies from being insignificant at an emissivity of .01 to
Figure 6.10 Energy Transfer Ratio Versus $\Delta P^*$
Figure 6.11 Energy Transfer Ratio Versus Emissivity
approximately 30% at an emissivity of 0.8. This latter value is obviously a significant fraction of the total heat load. The inclusion of radiant transfer into an analysis may allow the designer to reduce the supply fan size, if not eliminate the fan altogether, resulting in less power use and a quieter operating system.

While interesting effects of radiation exchange, such as the asymmetry of the flow (due to buoyancy effects) changing to a more symmetrical profile, or the location of the maximum plate temperature moving down the heated plate as the emissivity is increased, were discovered during this investigation, these ‘novelty’ effects did not show up for the typical range presented for cooling of electronics. The velocity profile shift with increasing radiant transfer was for a very low aspect ratio and the movement of the location of the maximum temperature was for a temperature range outside that of electronic components or the assumption of constant properties.

The purpose of this investigation was to determine the local and overall effects of including radiant transfer. The local effects of including radiant transfer are presented in Fig. 6.9 as the $Nu_{LO}$. The effects on the overall heat transfer are presented as a percent of the total heat transfer in Fig. 6.10 and 6.11. In this section radiant transfer is seen to affect both the local temperature and the overall heat transfer. Similar results were discovered for asymmetric heating, by Carpenter, et al. (1976), by linearizing the radiant transport terms, and for the isothermal case by Sparrow, et al. (1980), see Chapter 2.

Discrete heat sources would be simple to model with the present method. (The heat flux on the heated wall is actually put in discretely, so it is but a simple matter to turn some locations on and some off.) Complex geometry due to discrete devices can also be
modelled (with adequate grid resolution). A general tool has been developed which can be applied to a multitude of specific configurations. This author also wants to emphasize the basic message that this problem is an elliptic problem, contrary to what many numerical analyses assume, and that radiation effects, which are so often neglected, must be included in a complete and accurate analysis of electronic cooling design.
7.0 Summary

The purpose of this investigation was to address the problem of numerically predicting the heat transfer rates between parallel surfaces of the type found in electronic equipment. This has been accomplished through a unique application of the finite-element method for transient or steady-state, two-dimensional laminar mixed convection heat transfer with surface radiation. The approach was specifically geared toward implementation on present engineering workstations.

Simple calculations early in the course of this research suggested that the fully-developed-flow assumption, so commonly used, and the neglect of either the buoyancy effects or the radiant effects cannot be justified. A rigorous investigation culminating in the development of a numerical method (that included these often neglected effects) found that to achieve an accurate prediction of the heat transfer rates, a full elliptic solution to the Navier-Stokes equations with buoyancy effects along with radiation losses must be included in the analysis. It was found that as much as 30% of the heat transfer was by radiant transfer for the cases examined.
The work presented here includes pure natural convection and mixed convection up to the limit where forced convection dominates the vertical parallel plate problem. This investigation examined the influence on both flow rate (Re) and heat transfer (Nu) of varying the aspect ratio and including radiation (for a given aspect ratio) from the buoyancy-dominated to the pressure-dominated regime.

While problems with simple boundary conditions and geometries were examined, the solution of more complex geometries and boundary conditions is extremely straightforward with the finite-element method. The method that was presented can also easily be extended to three-dimensional problems.
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References
Vita

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