Appendix A

Derivation of the Real and Reactive Power

From a Pi Equivalent Circuit Model
Depicted in Figure A.1 is the model for the Pi Equivalent Circuit.

![Diagram of Pi-Equivalent Circuit](image)

*Figure A.1. Model of a Pi-Equivalent Circuit*

Given are the resistance of the line, $R$, and the reactance of the line, $X$. We desire to derive the formulae for the real power $P$, and the reactive power $Q$. From power systems engineering, the total complex power, $S_{kl}$, leaving bus $k$ through line $k-l$, can be expressed as:

$$S_{kl} = P_{kl} + jQ_{kl},$$

where $P_{kl}$ is the real power from bus $k$ through line $k-l$, $Q_{kl}$ is the reactive power from bus $k$ through line $k-l$, and $j = \sqrt{-1}$. To derive what $P$ and $Q$ are, we use another relationship that the total complex power is equivalent to the voltage potential from the bus to the ground, denoted $\bar{V}_k$, multiplied by the total current leaving the bus, denoted $\bar{I}^*_k$. Formally, we have

$$S_{kl} = P_{kl} + jQ_{kl} = \bar{V}_k \bar{I}^*_k.$$  \hspace{1cm} (A.1)

and for the voltage potential term,

$$\bar{V}_k = V_k e^{j\theta_k}.$$  \hspace{1cm} (A.2)
where $V_k$ is the voltage magnitude, and $\theta_k$ is the voltage phase angle at bus $k$. The total current is also expressed as:

$$I_k = I_k e^{j\psi_k},$$

where $I_k$ is the current magnitude, $\psi_k$ is the current phase angle at bus $k$, and $I_k^* = I_k e^{-j\psi_k}$ is the complex conjugate of $I_k$. Now, the total current can be broken up into current through the branch, and current through the shunt, denoted $I_{kl}$ and $I_{k-sh}$. Substituting in equation (A.1) we get

$$\bar{S}_k = \bar{V}_k (I_{k-sh} + I_{kl})^*, \quad (A.3)$$

where again $\bar{Z}^*$ represents the complex conjugate of $\bar{Z}$. Now, from Ohm’s Law, we can write the current in terms of the voltage and admittance. Letting $Y_{k-sh}$, and $Y_{kl}$ represent the shunt admittance and the branch admittance, we rewrite (A.3) as

$$\bar{S}_k = \bar{V}_k (\bar{Y}_{k-sh} + (\bar{V}_k - \bar{V}_l)\bar{Y}_{kl})^* = \bar{V}_k (\bar{Y}_{k-sh} + (\bar{V}_k^* - \bar{V}_l)^*\bar{Y}_{kl}^*). \quad (A.4)$$

In general, the admittance, $\bar{Y}$, can be written as

$$\bar{Y} = G + jB,$$

where $G$ is the conductance of the line and $B$ is the susceptance of the line. These can be written in terms of the resistance and reactance of the line. Specifically,

$$G = \frac{R}{R^2 + X^2}, \quad \text{and} \quad B = \frac{-X}{R^2 + X^2}.$$
For the admittance of the shunt, $G$ does not exist. Thus, $\bar{Y}_{k-\text{sh}} = jB_{k-\text{sh}}$, and $\bar{Y}_{kl} = G_{kl} + jB_{kl}$.

Expanding (A.4), we get

$$\bar{S}_k = -jV_k^2 B_{k-\text{sh}} + V_k^2 (G_{kl} - jB_{kl}) - (\bar{V}_k \bar{V}_l^*) (G_{kl} - jB_{kl}).$$

From (A.2), $\bar{V}_k \bar{V}_l^* = V_k e^{j\theta_k} V_l e^{-j\theta_l} = V_k V_l e^{j(\theta_k - \theta_l)}$, where $\theta_{kl} = \theta_k - \theta_l$, and using Euler’s formula that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ yields

$$\bar{S}_k = -jV_k^2 B_{k-\text{sh}} + V_k^2 (G_{kl} - jB_{kl}) - (V_k V_l (\cos(\theta_{kl}) + j \sin(\theta_{kl}))) (G_{kl} - jB_{kl}),$$

or

$$\bar{S}_k = -jV_k^2 B_{k-\text{sh}} + (V_k^2 - V_k V_l (\cos(\theta_{kl}) + j \sin(\theta_{kl}))) (G_{kl} - jB_{kl}),$$

or

$$\bar{S}_k = -jV_k^2 B_{k-\text{sh}} + (V_k^2 - V_k V_l \cos(\theta_{kl}) + jV_k V_l \sin(\theta_{kl})) (G_{kl} - jB_{kl}),$$

or

$$\bar{S}_k = -jV_k^2 B_{k-\text{sh}} + (\Delta + j\Gamma) (G_{kl} - jB_{kl}),$$

where $\Delta = V_k^2 - V_k V_l \cos(\theta_{kl})$ and $\Gamma = V_k V_l \sin(\theta_{kl})$. Using foil, we can finally write the total complex power as

$$\bar{S}_k = -jV_k^2 B_{k-\text{sh}} + \Delta G_{kl} + j\Gamma G_{kl} - j\Delta B_{kl} + \Gamma B_{kl}.$$}

Hence,

$$\bar{S}_k = [\Delta G_{kl} + \Gamma B_{kl}] + j[\Gamma G_{kl} - \Delta B_{kl} - V_k^2 B_{k-\text{sh}}] = P + jQ.$$

Thus, the real power, $P$, and reactive power, $Q$, defined in terms of the voltage and phase angle, are expressed as:

$$P_{kl} = [V_k^2 - V_k V_l \cos(\theta_{kl})] G_{kl} + [V_k V_l \sin(\theta_{kl})] B_{kl},$$

$$Q_{kl} = [V_k V_l \sin(\theta_{kl})] G_{kl} - [V_k^2 - V_k V_l \cos(\theta_{kl})] B_{kl} - V_k^2 B_{k-\text{sh}}.$$