Appendix A

Derivation of the Real and Reactive Power

From a Pi Equivalent Circuit Model

Depicted in Figure A.1 is the model for the Pi Equivalent Circuit.

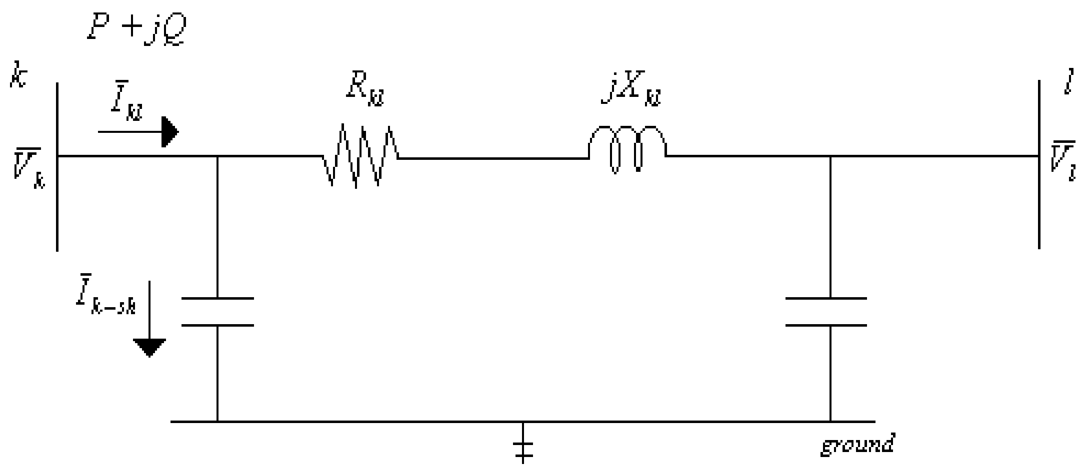

Figure A.1. Model of a Pi-Equivalent Circuit

Given are the resistance of the line, *R*, and the reactance of the line, *X*. We desire to derive the formulae for the real power *P*, and the reactive power *Q*. From power systems engineering, the total complex power, $\bar{S}_{kl}$, leaving bus *k* through line *k-l*, can be expressed as:

$$\bar{S}_{kl} = P_{kl} + jQ_{kl},$$

where P_{kl} is the real power from bus *k* through line *k-l*, Q_{kl} is the reactive power from bus *k* through line *k-l*, and $j = \sqrt{-1}$. To derive what *P* and *Q* are, we use another relationship that the total complex power is equivalent to the voltage potential from the bus to the ground, denoted $\bar{V}_k$, multiplied by the total current leaving the bus, denoted $\bar{I}_k^*$. Formally, we have

$$\bar{S}_{kl} = P_{kl} + jQ_{kl} = \bar{V}_k \bar{I}_k^*. \qquad \text{(A.1)}$$

and for the voltage potential term,

$$\bar{V}_k = V_k e^{j\theta_k}, \qquad \text{(A.2)}$$

where V_k is the voltage magnitude, and θ_k is the voltage phase angle at bus *k*. The total current is also expressed as:

$$\bar{I}_k = I_k e^{j\psi_k},$$

where I_k is the current magnitude, ψ_k is the current phase angle at bus *k*, and $\bar{I}_k^* = I_k e^{-j\psi_k}$ is the complex conjugate of $\bar{I}_k$. Now, the total current can be broken up into current through the branch, and current through the shunt, denoted $\bar{I}_{kl}$ and $\bar{I}_{k-sh}$. Substituting in equation (A.1) we get

$$\bar{S}_k = \bar{V}_k (\bar{I}_{k-sh} + \bar{I}_{kl})^*, \qquad \text{(A.3)}$$

where again $\bar{Z}^*$ represents the complex conjugate of $\bar{Z}$. Now, from Ohm's Law, we can write the current in terms of the voltage and admittance. Letting $\bar{Y}_{k-sh}$, and $\bar{Y}_{kl}$ represent the shunt admittance and the branch admittance, we rewrite (A.3) as

$$\bar{S}_k = \bar{V}_k (\bar{V}_k \bar{Y}_{k-sh} + (\bar{V}_k - \bar{V}_l)\bar{Y}_{kl})^* = \bar{V}_k (\bar{V}_k^* \bar{Y}_{k-sh}^* + (\bar{V}_k^* - \bar{V}_l^*)\bar{Y}_{kl}^*). \qquad \text{(A.4)}$$

In general, the admittance, $\bar{Y}$, can be written as

$$\bar{Y} = G + jB,$$

where *G* is the conductance of the line and *B* is the susceptance of the line. These can be written in terms of the resistance and reactance of the line. Specifically,

$$G = \frac{R}{R^2 + X^2}, \text{ and } B = \frac{-X}{R^2 + X^2}.$$

For the admittance of the shunt, G does not exist. Thus $\overline{Y}_{k-sh} = jB_{k-sh}$, and $\overline{Y}_{kl} = G_{kl} + jB_{kl}$.

Expanding (A.4), we get

$$\overline{S}_k = -jV_k^2 B_{k-sh} + V_k^2 (G_{kl} - jB_{kl}) - (\overline{V}_k \overline{V}_l^*)(G_{kl} - jB_{kl}).$$

From (A.2), $\overline{V}_k \overline{V}_l^* = V_k e^{j\theta_k} V_l e^{-j\theta_l} = V_k V_l e^{j(\theta_k - \theta_l)} = V_k V_l e^{j\theta_{kl}}$, where $\theta_{kl} = \theta_k - \theta_l$, and using Euler's formula that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$ yields

$$\overline{S}_k = -jV_k^2 B_{k-sh} + V_k^2 (G_{kl} - jB_{kl}) - (V_k V_{ll} (\cos(\theta_{kl}) + j\sin(\theta_{kl})))(G_{kl} - jB_{kl}), \text{ or}$$

$$\overline{S}_k = -jV_k^2 B_{k-sh} + (V_k^2 - (V_k V_{ll} (\cos(\theta_{kl}) + j\sin(\theta_{kl}))(G_{kl} - jB_{kl}), \text{ or}$$

$$\overline{S}_k = -jV_k^2 B_{k-sh} + (V_k^2 - V_k V_{ll} \cos(\theta_{kl}) + jV_k V_{ll} \sin(\theta_{kl}))(G_{kl} - jB_{kl}), \text{ or}$$

$$\overline{S}_k = -jV_k^2 B_{k-sh} + (\Delta + j\Gamma)(G_{kl} - jB_{kl}),$$

where $\Delta = V_k^2 - V_k V_l \cos(\theta_{kl})$ and $\Gamma = V_k V_l \sin(\theta_{kl})$. Using foil, we can finally write the total complex power as

$$\overline{S}_k = -jV_k^2 B_{k-sh} + \Delta G_{kl} + j\Gamma G_{kl} - j\Delta B_{kl} + \Gamma B_{kl}.$$

Hence,

$$\overline{S}_k = [\Delta G_{kl} + \Gamma B_{kl}] + j[\Gamma G_{kl} - \Delta B_{kl} - V_k^2 B_{k-sh}] = P + jQ.$$

Thus, the real power, P, and reactive power, Q, defined in terms of the voltage and phase angle, are expressed as:

$$P_{kl} = [V_k^2 - V_k V_l \cos(\theta_{kl})]G_{kl} + [V_k V_l \sin(\theta_{kl})]B_{kl}$$

$$Q_{kl} = [V_k V_l \sin(\theta_{kl})]G_{kl} - [V_k^2 - V_k V_l \cos(\theta_{kl})]B_{kl} - V_k^2 B_{k-sh}.$$