

Chapter 4

RESISTANCE DEFINITIONS AND CONCEPTS

§ 4.1 Introduction

As mentioned previously, Ylvisaker first established the concept of test resistance in 1977. Thus, the theory of resistances of statistical tests is still in its infancy. Research in this field remained fairly dormant until the early 1990's when Coakley and Hettmansperger rekindled the study with their definitions of maximum and expected resistance. This chapter details some of the concepts and results from previous works in resistance and illustrates them with examples.

§ 4.2 Test Resistance

Borrowing a line from Ylvisaker's paper: "If the sample is subjected to a test of hypotheses in a strict acceptance or rejection fashion, one might ask whether the decision reached could have been determined through the bad rather than the good observations."

It is from this statement that he develops the initial theory of test resistance. Here, we restate Ylvisaker's definition below along with two of his examples: Defining F_N to be the empirical cumulative distribution function (ecdf), let T be a real-valued function of the ecdf, and c_N be a critical value such that as a rule, one accepts H_0 if $T(F_N) \leq c_N$ and rejects H_0 if $T(F_N) > c_N$.

Definition 4.2.1: The resistance to acceptance (rejection) of the test (T, c_N) is the smallest proportion m_0/N for which no matter what x_{m_0+1}, \dots, x_N are, there are values x_1, \dots, x_{m_0} in \mathfrak{X} with $T(F_N) \leq c_N$ ($T(F_N) > c_N$). Resistance to acceptance (rejection) will be denoted by $\rho_A = \rho_A(N)$ ($\rho_R = \rho_R(N)$).

Example 1: Let the sample space $\mathfrak{X} = \mathfrak{R}^1$ and the rejection rule be to reject the null for $T(F_N) = \bar{x} \geq c_N$. By choosing, say, $x_1 \in \mathfrak{X}$ to be arbitrarily large or small, then it is easy to see that the resistance to acceptance ρ_A , as well as the resistance to rejection ρ_R , is $1/N$, independent of c_N .

Example 2: Let $\mathfrak{X} = \mathfrak{R}^k$ and $T(F_N) = \bar{\underline{x}}_N' S^{-1} \bar{\underline{x}}_N$ be Hotelling's T^2 test such that the null is rejected when $T(F_N) \geq c_N$. Again, it is easy to see that the resistance to acceptance is $1/N$, irrespective of the choice of c_N . For the resistance to rejection, let $\bar{\underline{x}}_m$ and S_m represent the sample mean vector and covariance matrix for the first m

observations, and let \bar{x}_n and S_n represent the sample mean vector and covariance of the final $n = N - m$ observations. Therefore,

$$\bar{x}_N = (m / N)\bar{x}_m + (n / N)\bar{x}_n,$$

$$S_N = (m / N)S_m + (n / N)S_n + (mn / N^2)(\bar{x}_m - \bar{x}_n)(\bar{x}_m - \bar{x}_n)',$$

$$\text{and } \inf_{\bar{x}_n S_n} \sup_{\bar{x}_m S_m} T(F_N) = \bar{x}_N' S^{-1} \bar{x}_N = \frac{m}{n} \text{ (for proof, see Ylvisaker, 1977).}$$

So, if m/n is greater than the critical value, then there exist m observations that produce a rejection of the null hypothesis regardless of the other $n = N - m$ observations. By setting $m_0 / n = m_0 / (N - m_0) = c_N$, a little algebra will show that the resistance to rejection is $\rho_R = m_0 / N = c_N / (1 + c_N)$.

§ 4.3 Maximum and Expected Resistance

Ylvisaker's definition of resistance is based on an addition contamination model. Coakley and Hettmansperger (1994) concentrated on the more realistic replacement contamination model that is frequently used in other definitions of robustness. It is under this model that they define the maximum resistance and expected resistance. In short, the maximum resistance is a measure of robustness of the test statistic when the data are in the least favorable case for a desired conclusion. The expected resistance is then a measure of the robustness of the test statistic in the average unfavorable case for a desired conclusion. The formal definitions are stated below.

Definition 4.3.1: The maximum resistance to acceptance (rejection) of a test (T, c_N) is the smallest proportion m/N such that for any sample S there exists a contaminated sample S_m^* such that $T(S_m^*) \leq c_N$ ($T(S_m^*) > c_N$).

Definition 4.3.2a: For each alternative θ_1 , the expected resistance to acceptance is $ERA_{\theta_1}(T, c_N) = E_{\theta=\theta_1}[M_A | M_A > 0] / N$, where for each sample S , M_A is the number of contaminations needed to force acceptance.

Definition 4.3.2b: The expected resistance to rejection is $ERR(T, c_N) = E_{H_0}[M_R | M_R > 0] / N$, where for each sample S , M_R is the number of contaminations needed to force rejection.

Example 3: Coakley and Hettmansperger consider the maximum resistance to rejection (*MRR*) for the one sample t-test with a right-tailed alternative hypothesis. Thus, the test statistic is $t = \sqrt{N}\bar{x} / s$, where \bar{x} and s are the sample mean and standard deviation, respectively. To derive the maximum resistance to rejection for the t-test, the worst case scenario for rejection is when all N observations equal $-K$, where $K > 0$, thus making the test statistic negative infinity. If m of these N are contaminated to a value B , say, with B arbitrarily large, then the new mean and standard deviation are:

$$\bar{x} = \frac{mB - (N - m)K}{N} \quad \text{and} \quad s^2 = \frac{m(N - m)(B - K)^2}{N(N - 1)}.$$

Setting the test statistic equal to the lowest possible value for rejecting implies that

$$\sqrt{N} \frac{\bar{x}}{s} = \frac{\sqrt{N}(mB - (N-m)K)\sqrt{N(N-1)}}{N\sqrt{m(N-m)(B-K)^2}} \approx \frac{\sqrt{(N-1)m}}{\sqrt{N-m}} = t_{N-1,\alpha},$$

and solving for m shows

$$m = \frac{Nt_{N-1,\alpha}^2}{N-1+t_{N-1,\alpha}^2} \quad \text{and} \quad MRR(t, t_{N-1,\alpha}) = \frac{m}{N} = \frac{t_{N-1,\alpha}^2}{N-1+t_{N-1,\alpha}^2}.$$

Example 4: Coakley and Hettmansperger consider the expected resistance to rejection

for the sign test and right-tailed alternative. The test statistic is $S^+ = \sum_{i=1}^N I(x_i > 0)$, and

assume $S^+ < c_N$, for some α -level critical value c_N . To obtain a rejection, exchange

$m = [c_N]_{gif} - S^+ + 1$ negative observations with positive observations (where $[\cdot]_{gif}$ is

the greatest integer function). So, m is a random variable with the same distribution as

S^+ , which is binomial with parameters N and $1/2$. It is easy to see that

$E_{H_0}(M) = [c_N]_{gif} - (N/2) + 1$, and using a lemma that states if a test (T, c_N) has size

α , the $ERR(T, c_N) = E_{H_0}(M) / N(1-\alpha)$, the expected resistance to rejection can be

written as

$$ERR_{H_0}(S^+, c_N) = \frac{[c_N]_{gif} - (N/2) + 1}{N(1-\alpha)}.$$

For most calculations of the expected resistance, simulations must be performed since it

is not as straightforward to get the distribution of the number of contaminants, M , as it

was in the case of the sign test.