

# Chapter 7

## MAXIMUM RESISTANCES OF THE BROWN-MOOD TEST FOR THE TWO-WAY LAYOUT

### § 7.1 Introduction

After studying both the maximum and expected resistance of the Friedman test, a natural question that arises is whether or not the Friedman test is the most resistant test for the two-way layout. Does a test exist that is more resistant than the Friedman test, both to rejection and/or to acceptance? The Friedman test is just one of many nonparametric methods to test the null hypothesis of no treatment effects in the two-way layout. Of the other nonparametric tests, most are based on average correlations among the values within the blocks, with the Friedman test a specific case of a test using an *internal rank correlation* based on Spearman's rho (1904). Other examples of nonparametric statistical tests for the two-way layout are found in Quade (1984). Our objective is to examine an alternative statistical test and study the resistance properties relative to the Friedman test. Specifically, we examine the Brown-Mood test.

## § 7.2 Average Internal Score Correlation and the Brown-Mood Test

As a generalization for some of the nonparametric tests in the two-way layout, we introduce here the *average internal score correlation*. For a given set of  $t$  real values in the  $i^{\text{th}}$  block,  $y_{i1}, \dots, y_{it}$ , not all equal, let the corresponding *scores* of the ranks be defined as  $r_{R_{ij}}$ , where  $R_{ij}$  is the rank of the  $j^{\text{th}}$  observation in the  $i^{\text{th}}$  block. The scores of the ranks are simply a transformation of the ranks. As a simple example, if  $r_{R_{ij}} \equiv j$ , an identity transformation, then the scores of the ranks are the actual ranks. The *score correlation*, denoted  $C_{ii'}$ , is the product moment correlation of the scores between blocks  $i$  and  $i'$ . There are a total of  $\binom{b}{2}$  correlations among the blocks which yields the *average internal score correlation* as

$$\bar{C} = \frac{2}{n(n+1)} \sum_{i < i'} C_{ii'}.$$

For the case where  $r_{R_{ij}} \equiv j$ , the score correlation becomes Spearman's correlation or Spearman's rho, and the internal score correlation becomes an internal *rank* correlation. As previously mentioned the Friedman test is one such nonparametric test that is based on internal rank correlations. Quade shows that the transformation of the average score correlations yielding the variable

$$X^2 = (t-1)[1 + (b-1)\bar{C}] \tag{7.1}$$

is asymptotically  $\chi^2_{(t-1)}$  under  $H_0$ .

As a second example of scores, if we define  $r_{R_{ij}} = 1$  when  $R_{ij} > (t+1)/2$ , and  $r_{R_{ij}} = 0$  otherwise, then (7.1) becomes the Brown-Mood test (1948, 1951) based on the correlation measure of Blomqvist (1950). A more appealing form of the Brown-Mood test statistic (in the sense of sum of squared errors) is

$$X_{BM}^2 = \begin{cases} \frac{4(t-1)}{bt} \sum_{j=1}^t \left( R_{.j} - \frac{b}{2} \right)^2, & \text{if } t \text{ is even,} \\ \frac{4t}{b(t+1)} \sum_{j=1}^t \left( R_{.j} - \frac{b(t-1)}{2b} \right)^2, & \text{if } t \text{ is odd,} \end{cases}$$

where  $R_{.j} = \sum_{i=1}^b r_{R_{ij}}$ , the number of times that the rank of an observation in the  $j^{\text{th}}$  treatment is strictly above the median for the  $i^{\text{th}}$  block.

### § 7.3 The Maximum Resistance to Rejection ( $b = t$ case)

To study the maximum resistance to rejection of the Brown-Mood test, we need an optimal initial configuration of the ranks that would lead to an optimal initial configuration of the scores that is most resistant to perturbations in the data. For the Friedman test, this configuration was a Latin Square arrangement of the ranks. If we consider this arrangement of the ranks, the configuration of the scores for the Brown-Mood test can be seen as cyclic permutations of the zeros and ones that mirror the cyclic permutations of the ranks. As an example, Table 7.1 shows this for eight treatments and eight blocks.

Table 7.1. Conversion of Friedman Ranks to Brown-Mood Scores

Friedman								→	Brown-Mood							
1	2	3	4	5	6	7	8		0	0	0	0	1	1	1	1
8	1	2	3	4	5	6	7	1	0	0	0	0	1	1	1	
7	8	1	2	3	4	5	6	1	1	0	0	0	0	1	1	
6	7	8	1	2	3	4	5	1	1	1	0	0	0	0	1	
5	6	7	8	1	2	3	4	1	1	1	1	0	0	0	0	
4	5	6	7	8	1	2	3	0	1	1	1	1	0	0	0	
3	4	5	6	7	8	1	2	0	0	1	1	1	1	0	0	
2	3	4	5	6	7	8	1	0	0	0	1	1	1	1	0	

In the same spirit as with the Friedman test, we would like to contaminate the Brown-Mood scores in a manner that creates at least one treatment difference. However, because the scores are binary, each treatment before contamination has  $b/2$  of each digit, implying that we are allowed to only contaminate at most  $b/2$  observations within each treatment. Because of this constraint, we may need to add contaminants to other treatments to force the test statistic to cross into the rejection region. This is especially true for small  $b$ , and  $t$ . This adds some complexity to the contamination scheme. After studying many different contamination schemes for different numbers of treatments, we found that the optimal way to increase the test statistic as a function of the number of contaminants is difficult to model mathematically. However, we can put a reasonable upper bound on the Brown-Mood test statistic and compare the resistance using this upper bound to the resistance derived using the Friedman test. If the Brown-Mood test using the upper bound test statistic were more resistant than the Friedman test, then obviously the Brown-Mood test using the true statistic would also be more resistant. We start by defining the upper bound.

**Lemma 7.1:** For the case where the number of blocks is equivalent to the number of treatments and the ranks are in a Latin Square type configuration, then an upper bound for the Brown-Mood test statistic,  $X_{BM}^2$ , after  $m$  contaminants is:

$$X_{BM}^2 \leq K(m^2 + m + 2), \quad (7.2)$$

where  $K = 4(t-1)/bt$ , if  $t$  even, and  $K = 4t/b(t+1)$  if  $t$  odd (proof of Lemma 7.1 in Appendix B).

As with deriving the maximum resistance of the Friedman test, we take the test statistic as a function of  $m$ , subtract a critical constant defined by  $b$ ,  $t$ , and  $\chi_{(t-1),\alpha}^2$ , and solve for the roots

of this function. By setting (7.2) to equal the chi-square critical value, we can obtain a lower bound on the number of contaminants necessary to switch an acceptance to a rejection.

Mathematically, we have

$$K(m^2 + m + 2) = \chi_{(t-1),\alpha}^2 \Leftrightarrow m^2 + m + 2 = K^{-1}\chi_{(t-1),\alpha}^2 = k_\alpha.$$

Here,  $K$  is the normalizing constant defined in Lemma 7.1. Since this is a polynomial of order two, a simple expression of the roots of this equation is easily obtained using the quadratic formula and one of these two roots will be a lower bound of the number of contaminants necessary to force a rejection of the null hypothesis. Formally stated, we have the following:

**Theorem 7.1:** Assume the number of blocks,  $b$ , is equivalent to the number of treatments,  $t$ , and the ranks are in a Latin Square configuration. Then for the Brown-Mood test, the minimum number of contaminants necessary to change an acceptance to a rejection is:

$$P_t^{BM} = \frac{1}{2}(\sqrt{4k_\alpha - 7} - 1),$$

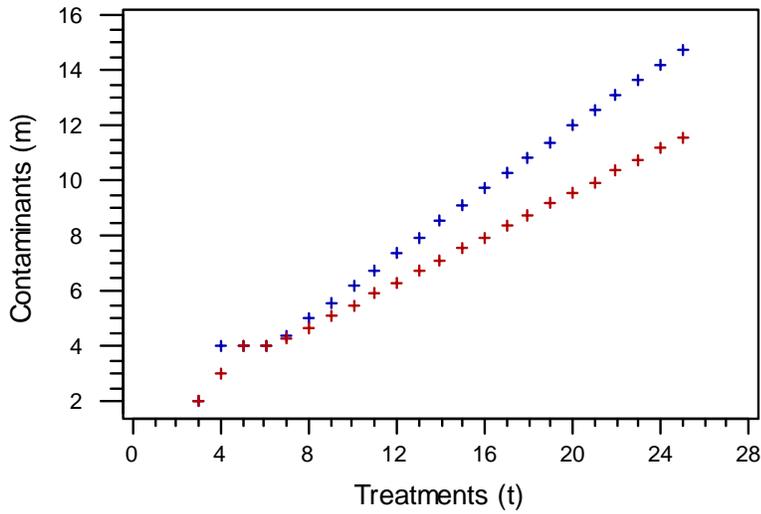
where  $k_\alpha = K^{-1}\chi_{(t-1),\alpha}^2$ ,  $K = 4(t-1)/bt$ , if  $t$  even, and  $K = 4t/b(t+1)$  if  $t$  odd. Therefore, the minimum maximum resistance to rejection is

$$\rho_t^{BM} = \frac{P_t^{BM}}{t^2}.$$

The subscript  $t$  indicates the number of blocks, and the superscript  $BM$  refers to Brown-Mood.

We would like to show that the Brown-Mood test is more resistant to rejection than the Friedman test. To show that one resistance value is greater than another is very difficult mathematically due to the severe complexity of the Friedman derivation. However, we can plot over the treatment values the number of contaminants needed to break down the test as well as the maximum resistance for each test. These values are found using Theorems 5.1 and 7.1.

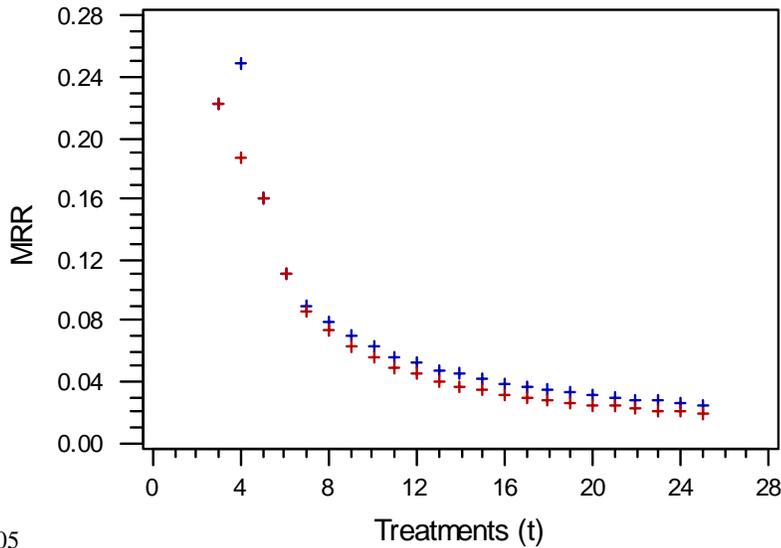
Figures 7.1 and 7.2 show the differences between the two tests. Figure 7.1 compares the number of perturbations in the data that are necessary to force rejection. The symbols for  $t = 3-6$  are the *exact integer* values found by enumeration. Starting at  $t = 7$ , the roots (not integer valued) from Theorems 5.1 and 7.1 are plotted across increasing values of  $t$ , and the Brown-Mood values found from Theorem 7.1 represent the *lower bound* of the number of contaminants. From Figure 7.1, we see that the Friedman values are never above any of the Brown-Mood values and that as the number of treatments increases, the difference in contaminants between the Brown-Mood and Friedman tests increases. Figure 7.2 compares the maximum resistance to rejection for the two tests and is just a plot of the values from 7.1, scaled by the sample size,  $t^2$ . The symbols are the *exact* maximum resistance values for  $t = 3-6$ . Starting at  $t = 7$ , the maximum resistances to rejection from Theorems 5.1 and 7.1 are plotted across values of  $t$ , and again the Brown-Mood values found from Theorem 7.1 represent the *lower bound* of the maximum resistance. Completely analogous to Figure 7.1, we see that the Friedman values are never above any of the lower bound Brown- Mood values. This graphically verifies that the Brown-Mood test has a higher maximum resistance to rejection than the Friedman test, for the case where the number of blocks equals the number of treatments.



$\alpha = 0.05$

Red=Friedman; Blue=Brown-Mood

Figure 7.1. Maximum Number of Contaminants for Rejection ( $b=t$ )



$\alpha = 0.05$

Red=Friedman; Blue=Brown-Mood

Figure 7.2. MRR of the Brown-Mood and Friedman Tests ( $b=t$ )

#### § 7.4 The Maximum Resistance to Rejection ( $b = nt$ case)

For the case where we have the number of blocks equal to an integer multiple of the number of treatments (i.e.  $b = nt$ ,  $n \in \mathbb{Z}^+$ ), we set the initial configuration of the ranks as  $n$  Latin Squares, then convert those ranks into the Brown-Mood scores. Again, we would like to contaminate the Brown-Mood scores in a manner that creates at least one treatment difference. As with the  $b = t$  case described in Section 7.3, we run into the same problem; because the scores are binary, each treatment has  $nt/2$  of each digit, implying that we are allowed to only contaminate at most  $nt/2$  observations within each treatment. Because of this constraint, we may need to add contaminants to other treatments to force the test statistic to cross into the rejection region, thus increasing the complexity of the contamination scheme. The optimal way to increase the test statistic, as a function of the number of contaminants, is again difficult to model mathematically. However, as with the previous case described, we can still put a reasonable upper bound on the Brown-Mood test statistic and compare the maximum resistance of the Brown-Mood test using the upper bound to the Friedman test. If the Brown-Mood test using the upper bound test statistic were more resistant than the Friedman test, then obviously the Brown-Mood test using the true test statistic would be more resistant. We start by defining the upper bound.

**Lemma 7.2:** For the case where the number of blocks,  $b$ , is equivalent to a constant multiple of the number of treatments,  $t$ , such that  $b = nt$ ,  $n \in \mathbb{Z}^+$ , and the ranks are in a Latin Square type configuration, then an upper bound for the Brown-Mood test statistic,  $X_{BM}^2$ , after  $m$  contaminants is:

$$X_{BM}^2 \leq K \cdot 2m^2, \quad (7.3)$$

where  $K = 4(t-1)/bt$ , if  $t$  even, and  $K = 4t/b(t+1)$  if  $t$  odd (proof of Lemma 7.2 in Appendix B)

This upper bound is still valid for  $n = 1$ , but is not nearly as precise as the formula derived in Lemma 7.1. Again, we take the test statistic as a function of  $m$ , subtract a critical constant defined by  $b$ ,  $t$ , and  $\chi_{(t-1),\alpha}^2$ , and solve for the roots of this function. By setting (7.3) to equal the chi-square critical value, we can obtain a lower bound on the number of contaminants necessary to switch an acceptance to a rejection. Mathematically, we have

$$2Km^2 = \chi_{(t-1),\alpha}^2 \Leftrightarrow 2m^2 = K^{-1}\chi_{(t-1),\alpha}^2 = k_\alpha.$$

The correct root of this equation is very easily obtained and formally stated next.

**Theorem 7.2:** Assume the number of blocks,  $b$ , is equivalent to a constant multiple of the number of treatments,  $t$ , such that  $b = nt$ ,  $n \in Z^+$ , and the ranks are in a Latin Square configuration. Then for the Brown-Mood test, the minimum number of contaminants necessary to change an acceptance to a rejection is:

$$P_R^{BM_t} = \sqrt{\frac{k_\alpha}{2}}$$

where  $k_\alpha = K^{-1}\chi_{(t-1),\alpha}^2$ ,  $K = 4(t-1)/bt$ , if  $t$  even, and  $K = 4t/b(t+1)$  if  $t$  odd. Therefore, the minimum maximum resistance to rejection is

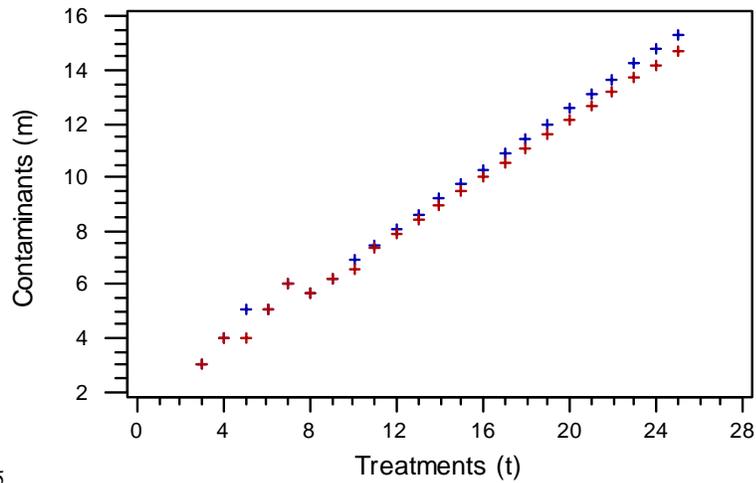
$$\rho_R^{BM_t} = \frac{P_R^{BM_t}}{nt^2}.$$

As with the  $b = t$  case, we would like to show that the Brown-Mood test is more resistant to rejection than the Friedman test. Again, this is mathematically very difficult to show one

resistance value is greater than another due to the severe complexity of the Friedman derivation. Graphically we plot over the treatment values the number of contaminants to break down the test as well as the maximum resistance for each test, and examine the trend. These values are found using the Friedman approximation for the  $n$ -group case from Section 5.4.2, and Theorem 7.2. Figures 7.3 and 7.4 show the differences between the two tests, when the number of blocks is *twice* the number of treatments. Figure 7.3 compares the number of perturbations in the data that are necessary to force rejection. The symbols for  $t = 3-7$  are the *exact integer* values found by enumeration. Starting at  $t = 8$ , the roots (not integer valued) from the Friedman approximation and Theorem 7.2 are plotted across increasing values of  $t$ , and the Brown-Mood values found from Theorem 7.2 represent the *lower bound* of the number of contaminants. From Figure 7.3, we see that the Friedman values are never above any of the Brown-Mood values and that as the number of treatments increases, the difference in contaminants between the Brown-Mood and Friedman tests increases. Figure 7.4 compares the maximum resistance to rejection for the two tests and is just a plot of the values from 7.3, scaled by the sample size,  $2t^2$ . The symbols are the *exact* maximum resistance values for  $t = 3-7$ . Starting at  $t = 8$ , the maximum resistances to rejection from the Friedman approximation and Theorem 7.2 are plotted across increasing values of  $t$ , and again the Brown-Mood values found represent the *lower bound* of the maximum resistance to rejection. Analogous to Figure 7.3, we see that the Friedman values are never above any of the lower bound Brown-Mood values. This verifies that the Brown-Mood test has a higher maximum resistance to rejection than the Friedman test, for the case where the number of blocks equals twice the number of treatments.

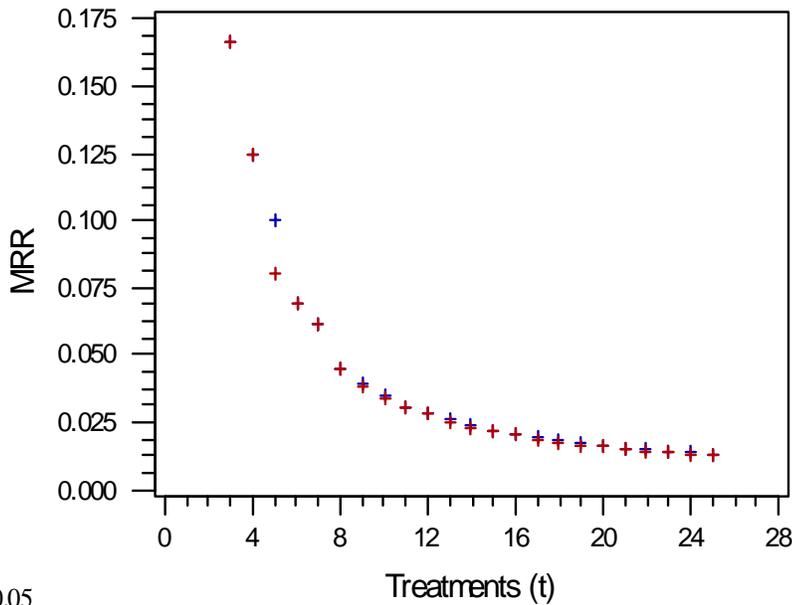
Figures 7.5 and 7.6 mimic Figures 7.3 and 7.4 except with the number of blocks equal to five times the number of treatments. The analysis of these graphs is the same as with  $n = 2$ ,

except that the difference between the lower bound for the Brown-Mood test and the Friedman test is more prominent. (This is primarily due to the fact that for the much larger designs, the upper bound test statistic of  $K \cdot 2m^2$  becomes much more accurate to the true test statistic). Enumerating the maximum resistance to rejection for either test was not necessary since the lower bound number for the number of contaminants to force rejection, and equivalently the maximum resistance of the Brown-mood test was uniformly higher for all values of  $t$ ,  $t \geq 3$ . With higher values of  $n$ , the pattern would still mirror Figures 7.5 and 7.6. Thus we infer that the Brown-Mood test has a higher maximum resistance to rejection than the Friedman test.



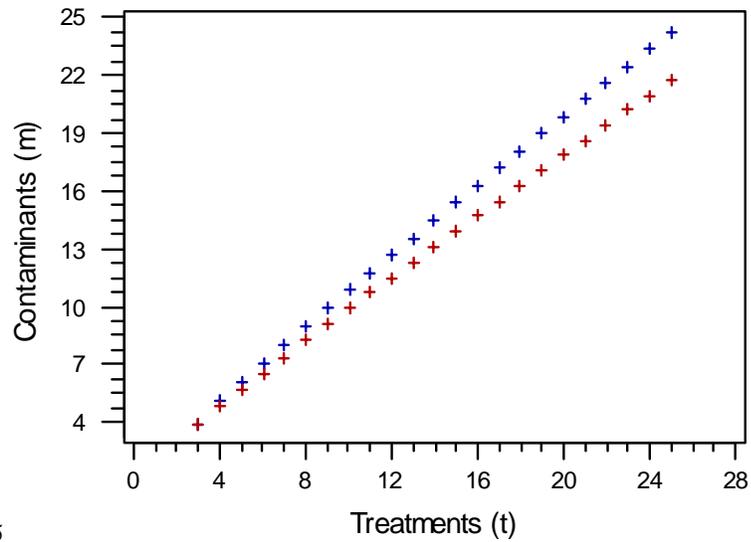
$\alpha = 0.05$   
 Red=Friedman; Blue=Brown-Mood

Figure 7.3. Maximum Number of Contaminants for Rejection ( $b=2t$ )



$\alpha=0.05$   
 Red=Friedman; Blue=Brown-Mood

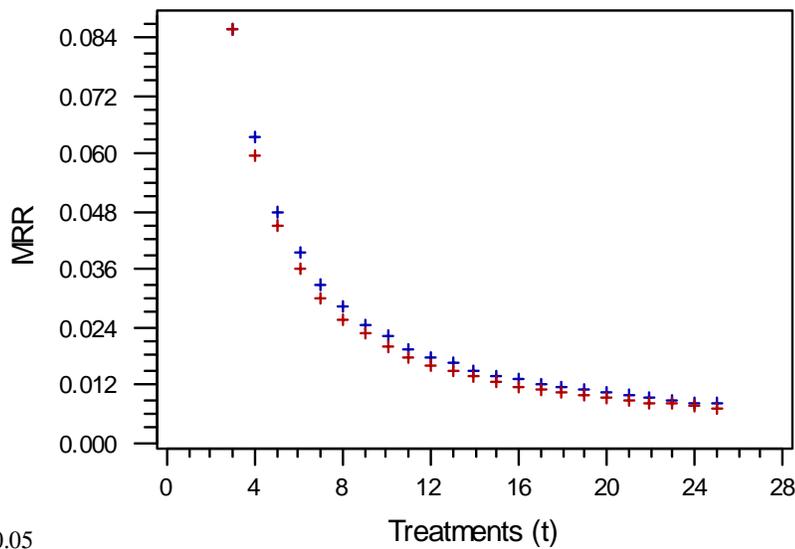
Figure 7.4. MRR of the Brown-Mood and Friedman Tests ( $b=2t$ )



$\alpha = 0.05$

Red=Friedman; Blue=Brown-Mood

Figure 7.5. Maximum Number of Contaminants for Rejection ( $b=5t$ )



$\alpha = 0.05$

Red=Friedman; Blue=Brown-Mood

Figure 7.6. MRR of the Brown-Mood and Friedman Tests ( $b=5t$ )

## § 7.5 The Maximum Resistance to Acceptance

Here, we start by examining the generic  $b = nt$  case due to the extremely straightforward extension from the  $b = t$  case. For the maximum resistance to acceptance, we need to start the Brown-Mood test statistic at the worst possible case. Since, this test statistic is based on the scores of the ranks, which is finite, the test statistic has a finite maximum. For the Brown-Mood test, the worst case scenario for acceptance is where all the treatments either have a set of all zeros, or all ones. This will give the highest possible observed  $X_{BM}^2$  value. One such configuration of the ranks that would yield the highest  $X_{BM}^2$  is where all the highest ranks are associated with one treatment, the next highest with another, and so on. In studying the maximum resistance to acceptance, we will assume the ranks of the data will be in this format. We are not concluding that this is the most resistant configuration of the ranks for the Brown-Mood scores. And if this configuration is not the most resistant, it is certainly very close to the most resistant. As with the maximum resistance to rejection, if we show that the Brown-Mood test is more resistant with this configuration of the ranks, then we can conclude the Brown-Mood test has a higher maximum resistance to acceptance.

The Brown-Mood test statistic is different for even and odd numbers of treatments. We start by examining the case with an even number of treatments. For this case it is very straightforward to contaminate the ranks such that the Brown-Mood test statistic is minimized. As a small example, consider a six-block, six-treatment design. Table 7.2 shows the initial configuration of the ranks and scores such that the Brown-Mood test statistic is at the maximum. To bring the test statistic down to its minimum of zero, we would need to invert the zeros and ones for half of the total blocks, as shown in Table 7.3.

Table 7.2. Brown Mood Scores Before Contamination (even  $t$  case)

Friedman						→	Brown-Mood					
1	2	3	4	5	6		0	0	0	1	1	1
1	2	3	4	5	6	0	0	0	1	1	1	
1	2	3	4	5	6	0	0	0	1	1	1	
1	2	3	4	5	6	0	0	0	1	1	1	
1	2	3	4	5	6	0	0	0	1	1	1	

Table 7.3. Brown Mood Scores After Contamination

Friedman						→	Brown-Mood					
4	5	6	1	2	3		1	1	1	0	0	0
4	5	6	1	2	3	1	1	1	0	0	0	
4	5	6	1	2	3	1	1	1	0	0	0	
1	2	3	4	5	6	0	0	0	1	1	1	
1	2	3	4	5	6	0	0	0	1	1	1	
1	2	3	4	5	6	0	0	0	1	1	1	

One pattern of contamination to accomplish this, such that the test statistic is minimized, would focus on the ranks within a block that are lower, say, than the median rank. In successive steps, change each of those  $t/2$  lower ranks to the highest rank of  $t$ . This is then repeated within another block, and so on until  $X_{BM}^2$  drops below the critical value. The change in the test statistic can be modeled mathematically using the contamination scheme just described. Formally we state the following.

**Lemma 7.3:** Assume the number of blocks is equivalent to an integer multiple of the number of treatments (i.e.  $b = nt$ ,  $t$  even), the number of treatments is even, the ranks are such that all the highest ranks are within one treatment, then next highest ranks with another treatment, etc., and the optimal contamination scheme is used. Then the Brown-Mood test statistic after  $m$  contaminants can be expressed as:

$$X_{BM}^2 = K \left[ \frac{t}{4} (nt - 2g + 2)^2 + m^* (4g - 2nt - 2) \right], \quad (7.4)$$

where  $K = 4(t-1)/bt$ , if  $t$  even,  $K = 4t/b(t+1)$  if  $t$  odd,  $g = [(2m-1)/t]_{gif} + 1$ ,  $gif$  is greatest integer function,  $m^* = m - (g-1)\frac{t}{2}$ . (Proof of Lemma 7.3 in Appendix B).

The formula is still only a function of  $m$ , since  $m^*$  and  $g$  are a function of  $m$ . As done with the Friedman derivations, we drop the greatest integer function in  $g$  and substitute  $m^*$  and  $g$  in (7.4) with their respective functions. Once this is done, the test statistic in (7.4) becomes approximate, but now is strictly a function of  $m$  and after simplification is expressed as

$$X_{BM}^2 \approx \frac{n^2 t^3}{4} + \frac{4m^2 - 1}{t} - 2mnt - 1. \quad (7.5)$$

We can approximate the number of contaminants necessary to force an acceptance from a rejection by setting (7.5) equal to  $k_\alpha$ , defined in Theorems 7.1 and 7.2, and solving for  $m$ . By doing so, we can express the approximate number of contaminants as:

$$\frac{1}{4} \left( nt^2 - 2\sqrt{t(k_\alpha - 1) + 1} \right),$$

and thus the approximate maximum resistance to acceptance as

$$\frac{1}{4} - \frac{2\sqrt{t(k_\alpha - 1) + 1}}{nt^2}.$$

We can easily see that as the number of blocks approaches infinity (equivalent to the number of groups,  $n$ , approaching infinity), the limit of the maximum resistance to acceptance is  $1/4$ , identical to that of the Friedman test.

For the case where the number of treatments is odd, the configuration of the ranks and the scores parallels that of the even numbered case, except for one slight difference; the number of treatments with the zero scores is one higher than the number of treatments with scores of one. Table 7.4 shows this difference for  $b = t = 5$ . This complicates the optimal contamination scheme slightly. However, for every contaminant added, we can change one of the zeros into a one, and vice versa, such that the sums of the lower scores are always increasing and the sums of the higher scores are always decreasing. Thus, we contaminate in such a way that the sum of scores that is farthest away in magnitude is always brought toward the mean sum of scores. With this in mind, we can succinctly formalize the Brown-Mood test statistic for an odd number of treatments, and an integer multiple of the number of treatments as blocks.

Table 7.4. Brown Mood Scores Before Contamination (odd  $t$  case)

Friedman					→	Brown-Mood				
1	2	3	4	5		0	0	0	1	1
1	2	3	4	5	0	0	0	1	1	
1	2	3	4	5	0	0	0	1	1	
1	2	3	4	5	0	0	0	1	1	
$\Sigma$					0	0	0	5	5	

**Lemma 7.4:** Assume the number of blocks is equivalent to an integer multiple of the number of treatments (i.e.  $b = nt$ ,  $t$  odd), the ranks are such that all the highest ranks are within one treatment, then next highest ranks with another treatment, etc., and the optimal contamination scheme is used. Then the Brown-Mood test statistic after  $m$  contaminants can be expressed as:

$$X_{BM}^2 = K \left[ \begin{array}{l} m^* (g_1 - n(t-1)/2)^2 + ((t+1)/2 - m^*) (g_1 - 1 - n(t-1)/2)^2 \\ + m^{**} (nt - g_2 - n(t-1)/2)^2 + ((t-1)/2 - m^{**}) (nt - g_2 + 1 - n(t-1)/2)^2 \end{array} \right], \quad (7.6)$$

where  $K = 4(t-1)/bt$ , if  $t$  even,  $K = 4t/b(t+1)$  if  $t$  odd  $g_1 = [(2m-1)/(t+1)]_{gif} + 1$ ,

$g_2 = [(2m-1)/(t-1)]_{gif} + 1$ ,  $gif$  is greatest integer function,  $m^* = m - (g-1)\frac{t+1}{2}$ , and

$m^{**} = m - (g-1)\frac{t-1}{2}$ . (Proof of Lemma 7.4 in Appendix B).

As done with previous Friedman derivations, and with Lemma 7.3, we drop the greatest integer function in  $g_1$  and  $g_2$  and substitute for  $m^*$ ,  $m^{**}$ ,  $g_1$  and  $g_2$  in (7.6) their respective functions.

Once this is done, the test statistic in (7.6) becomes approximate, but now is strictly a function of  $m$  and is expressed as (after some algebra)

$$X_{BM}^2 \approx \frac{t(4(t-1) + (4m + n - nt^2)^2) - 4}{4(t^2 - 1)}. \quad (7.7)$$

We can approximate the number of contaminants necessary to force an acceptance from a rejection by setting (7.7) equal to  $k_\alpha$ , defined in Theorems 7.1 and 7.2, and solving for  $m$ . By doing so, we can express the approximate number of contaminants as

$$\frac{nt^3 - nt - 2\sqrt{t}\sqrt{t^2(k_\alpha - 1) + t - k_\alpha + 1}}{4t}$$

and thus the approximate maximum resistance to acceptance as

$$\frac{1}{4} - \frac{nt - 2\sqrt{t}\sqrt{t^2(k_\alpha - 1) + t - k_\alpha + 1}}{4nt^3}.$$

It is interesting to see that as the number of blocks approaches infinity (equivalent to the number of groups,  $n$ , approaching infinity), the limit of the maximum resistance to acceptance is  $\frac{1}{4} - t^{-2}$ , slightly less than 0.25 for the even  $t$  case.

Figure 7.7 and Figure 7.8 display the differences in the exact *MRA* between the Friedman and Brown-Mood tests, for the cases of  $b = t$  and  $b = 5t$ , respectively. The lack of ‘smoothness’ with the Brown-Mood function is due to the different formulae for even and odd numbers of treatments. As with the *MRR*, the Friedman values are never above any of the Brown-Mood values, graphically supporting the fact that the Brown-Mood test has a higher maximum resistance to acceptance. Thus we can infer that the Brown-Mood test is more resistant than the Friedman test, with regards to both the maximum resistance to rejection *and* the maximum resistance to acceptance.

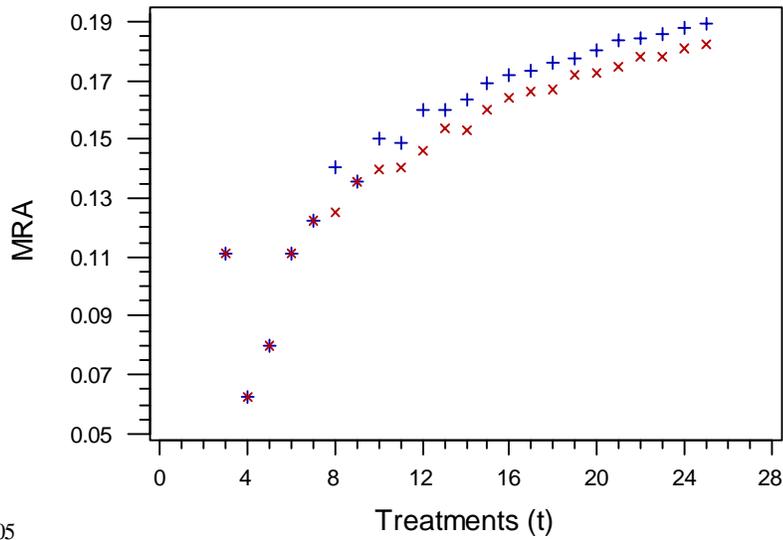


Figure 7.7. MRA of the Brown-Mood and Friedman Tests ( $b=t$ )

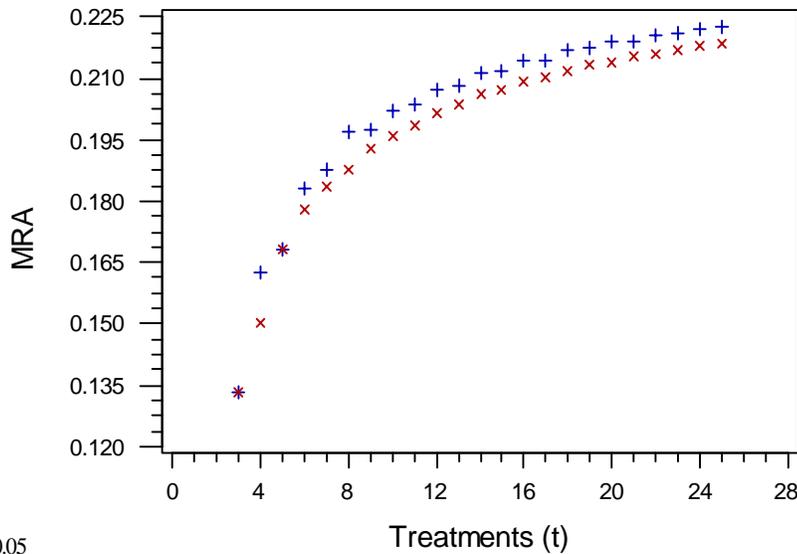


Figure 7.8. MRA of the Brown-Mood and Friedman Tests ( $b=5t$ )

## § 7.6 Concluding Remarks

With regards to the maximum resistance to rejection, although the Brown-Mood test is more resistant, practically speaking, the difference is nominal. As shown with Figures 7.1, 7.3, and 7.5, the additional number of discordant measurements that the Brown-Mood test allows is not substantial until the number of treatments becomes very large (i.e. above 15-20). And even with a substantial increase in the number of bad data allowed, relative to the sample size this is quite nominal, as displayed in Figures 7.2, 7.4, and 7.6. As a concrete example, consider a very large design of 10 treatments and 50 blocks. The Friedman test breaks down with 10 bad data, while the Brown-Mood test breaks down with 12. However, relative to the sample sizes, the percentage of bad data that breaks down each test is 2.0% for the Friedman and 2.4% for the Brown-Mood. This is not a very significant gain.

With regards to the maximum resistance to acceptance, this was surprising that the Brown-Mood test was greater than the Friedman test. At first, it would appear that a gain in resistance to rejection would be a loss in resistance to acceptance. This was not the case. However, the same question arises of the practical gain. Highlighting the aforementioned 10-treatment, 50-block example, the Friedman test breaks down with 98 bad data, while it takes 101 to break down the Brown-Mood test. But while this is an increase of three bad observations, relative to the sample size this is only an increase of 0.6%, from 19.6% to 20.2%. Similar to the *MRR*, using the Brown-Mood test yields only a nominal gain in robustness. Of course, this is only one measure of each test's robustness. Comparisons with the expected resistances should shed some more light in the differences between these two tests. Finally, relative to other consistent tests, the Friedman and Brown-Mood appear to be very robust. However, a more thorough study of other tests would need to be done to make any global affirmations.