HALPHEN'S THEOREM AND RELATED RESULTS

by

George Edward Culbertson

Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in partial fulfillment for the degree of

DOCTOR OF PHILOSOPHY

in

Mathematics

APPROVED:

Dr. J. Layman

Dr. L. Rutland　　Dr. L. McFadden　　Dr. W. Patty

Dr. J. DeCicco　　Dr. C. Kramer　　Dr. C. Williams

Dr. H. Johnson　　Dr. J. Washenberger

March 1970

Blacksburg, Virginia

ACKNOWLEDGEMENTS

JPP 5-18-77

The author is indebted to Dr. John DeCicco of the Illinois Institute of Technology, who suggested and directed the project. His advice and encouragement during the study were invaluable.

Special thanks are also due Dr. John Layman of Virginia Polytechnic Institute who served as Committee Chairman. His constructive suggestions and criticisms and his strong support are highly appreciated.

The project was initiated in 1965 under the support of a National Science Foundation Research Participation Grant. During the summer of 1966, the study was supported by a grant from the Naval Academy Research Council. The author would like to express his gratitude to both institutions for making this study possible.

Finally, the author thanks for her skill in typing the final manuscript.

TABLE OF CONTENTS

INTRODUCTION

This research effort was initiated to explore the possibility of extending a dynamical theorem due to G. Halphen. Halphen's Theorem states that, "Every dynamical trajectory in a positional field of force in E_3 is planar, if and only if the force field is either parallel or central".

Although Halphen's Theorem has been known for some time, a search of the available literature did not reveal a complete proof of the result. It is well known and well documented that every dynamical trajectory in a parallel or central field of force is planar; however, the same cannot be said of the converse implication.

The published works of Dr. Edward Kasner and Dr. John DeCicco served as the primary reference material for this paper. Several of their classical results are contained in Chapter I, which serves as an orientation chapter.

Chapters II and III were designed to support later chapters. A transformation is developed in Chapter II that is used in Chapter VII to show the projective similarity between k-parallel and (k-1)-central fields of force. Chapter III contains several calculational formulas that are useful in Chapters IV and V.

A new analytic proof of the proposition that a field of force in E_3 is either parallel or central if and only if the dynamical trajectories are all plane curves, is presented in Chapter IV. The details of this proof suggested the new concepts of a flat point in a field of force and a flat point on a dynamical trajectory in a field of force, and these led to various new results related to Halphen's Theorem. One of the more important new results in this setting is a less restrictive version of Halphen's Theorem which states that, "At every point P of a positional field of force, in E_3 there exists six distinct directions, not all of which are on a quadric cone with vertex at P, such that each dynamical trajectory which passes through the point P in one of these six directions is flat at the point P, if and only if the field of force is either parallel or central".

It is trivially true that every plane curve is a helix. This suggested that additional results related to Halphen's Theorem could be obtained by considering those fields of force in which every dynamical trajectory is a helix. The results obtained under this condition are contained in Chapter V.

On the basis of these findings the new concept of a Helical point of a positional field of force was defined.

One of the more interesting new results obtained from this effort states that, "If the point P is not a helical point of a positional field of force in E_3 , then in each direction through P there passes at most two dynamical trajectories which have the point P as a helical point".

A close analysis of Halphen's Theorem revealed that the key concepts which required redefinition in order to make a non trivial extension to higher dimensional space possible, were those of a parallel and central field of force. In this connection, definitions were structured for a k-dimensional parallel field of force and a (k-1)-dimensional central field of force in a Euclidean space of n-dimensions with $1 \leq k < n$. These definitions were then used to obtain an extension of Halphen's Theorem to a Euclidean space of 4-dimensions which states that, "Every dynamical trajectory in a positional field of force in E_4 is contained in some k-flat, with $k = 2,3$, if and only if the field of force is either (k-1)-parallel or (k-2)-central". This new result is contained in Chapter VI.

Although it is firmly believed that Halphen's Theorem is extendable to a Euclidean space of n-dimensions with $n > 4$, no complete proof has as yet been developed. A partial extension is presented in Chapter VII. This result states that "Every dynamical trajectory in a k-dimensional parallel or a (k-1)-dimensional central positional field of force in E_n with $k \leq n - 1$, is contained in some (k+1)-dimensional hyperplane".

The applicability of the new definitions of a k-parallel and (k-1)-central fields of force was further established by extending other classical dynamical results in E_3 to spaces of higher dimension. The most noteworthy new result obtained in this setting is an extension of Kepler's second law of motion which states that, "In a (k-1)-dimensional central field of force in E_n with $k \leq n-1$ and a fixed (k-1)-dimensional hyperplane, L^0_{k-1} , as center, the time required in going from point P to point Q along a fixed dynamical trajectory is proportional to the vectorial area OP_0Q_0 swept out by the radius vector of the orthogonal projection of the dynamical trajectory on a plane orthogonal to L^0_{k-1} . The point 0 is the projection of the central flat in the projection plane".

The proof of Halphen's Theorem in both E_3 and E_4 led to the consideration of quadrics. It is strongly anticipated that the generalization of E_n for $n > 4$, will also involve quadrics in E_n .

CHAPTER I

THE MOTION OF A PARTICLE IN A POSITIONAL FIELD OF FORCE IN A EUCLIDEAN SPACE E_n OF DIMENSION $n \geq 2$.

1. Dynamical trajectories in a Euclidean space E_n of dimension $n \geq 2$.

This development begins with the presentation of basic introductory material. This material, for the most part, was suggested by the works of Professors E. Kasner and J. DeCicco.

Consider a particle of mass m moving in a Euclidean space E_n of dimension $n \geq 2$, such that the laws governing this motion are Newtonian. That is, with the proper choice of units of measurement, the mass of the particle multiplied by its acceleration, is equal to the force.

In E_n, let $x = (x_1, x_2, \ldots, x_n)$ denote cartesian coordinates of a point, and let $F = (F_1, F_2, \ldots, F_n)$ be the rectangular components of a force vector acting on the particle situated at the position x.

Definition 1.1. If the force vector F is a single valued and continuous vector function of x such that it possesses continuous partial derivatives of at least the first order in a certain region of E_n and F is not identically zero in this region, then this physical configuration is called a stationary or positional field of force.

For a discussion of this and other fields of force in E_2 and E_3 see Kasner [1], and Kasner and DeCicco [2,7].

The following definition can be found in [2].

Definition 1.2. A Faraday line of force in a given positional field of force is a curve C of the region that has its tangent at each of its points parallel to the corresponding force vector at that point.

This concept of lines of force was first introduced by Faraday, and is of fundamental importance in the mathematical theory of electricity and magnetism.

The lines of force are by definition the solutions to the following set of ordinary defferential equations of first order:

$$\dot{x}_i = \frac{dx_i}{dt} = \alpha F_i \ , \tag{1.1}$$

where $1 \leq i \leq n$, the dot represents the derivative with respect to a continuously varying parameter other than time, and α is a non-zero scalar.

The solution of equation (1.1) gives rise to n constants of integration. However, only (n-1) of these constants are essential.

An existence and uniqueness theorem [4,5] on systems of differential equations states that the solution to (1.1) for a given set of initial conditions is unique. In other words, through any point x of the given region, there passes one and only one line of force.

If C is the unique line of force which passes through the point x_0 , then through any other point x_1 of C there passes a unique line of force, namely C. Because of this fact the number of essential constants is (n-1) and the set of lines of force in E_n is (n-1)-fold infinite. Thus, according to the notation of Kasner [6] there are ∞^{n-1} Faraday lines of force in E_n.

The following definition can be found in [8] and [9].

<u>Definition 1.3</u>. A <u>dynamical trajectory</u> is the path of a freely moving particle in a positional field of force.

Under the previously discussed conditions, the motion of a freely moving particle of mass m is described by the following system of $n \geq 2$, differential equations

$$
\begin{aligned}
m\ddot{x}_1 &= F_1(x_1,x_2,\ldots,x_n),\\
m\ddot{x}_2 &= F_2(x_1,x_2,\ldots,x_n),\\
&\;\;\vdots\\
m\ddot{x}_n &= F_n(x_1,x_2,\ldots,x_n),
\end{aligned}
\tag{1.2}
$$

where the dots represent derivatives with respect to a continuously varying parameter such as time.

Definition 1.4. The set of all solutions to the system (1.2) is defined to be a dynamical system S_0.

A more detailed discussion of dynamical systems S_0 can be found in [14].

Theorem 1.1. Related to a given positional field of force $\vec{F}$ with $|F| > 0$, of Euclidean space E_n of dimension $n \geq 2$, there is a dynamical system S_0 of ∞^{2n-1} dynamical trajectories C.

For, solving such a system of differential equations, 2n constants are introduced. However, not all of these constants are essential. By appealing to an existence and uniqueness theorem [4,5] on systems of ordinary differential equations, it is found that the solution C, corresponding to a particular choice of the 2n constants (that is, the initial position and the initial velocity) is unique. Thus a given dynamical trajectory C may be generated by selecting any of its points as initial position and the proper initial velocity at that point. The set of points that a given dynamical trajectory C passes through is one-fold infinite. This would imply that for a given set of initial conditions there are ∞^1 other initial conditions which will generate the same dynamical trajectory C.

The preceding agrument implies that the number of essential constants is (2n - 1) and the set of all dynamical trajectories satisfying equation (1.2) is (2n - 1) fold infinite.

Consequently, the validity of Theorem 1.1, is established.

2. The system of explicity equations of dynamical trajectories. The problem under consideration involves the geometric properties of dynamical trajectories. In order to study such properties, it is sometimes helpful to eliminate the parameter t, from equations (1.2).

This can be done by first using the implicit function theorem to find that the equations of the dynamical trajectory can be expressed in terms of the independent variable x_i, for sime i. Suppose it is found that x_1 is such an independent variable. Then the chain rule can be used to express the system (1.2) in terms of derivatives with respect to x_1 which will be denoted by primes. This process leads to

$$(2.1) \qquad [F_2 - x_2'F_1]\, x_i'' = [F_i - x_i'F_1]\, x_2'',$$

for $i = 3,4,\ldots,n$, where $F_2 - x_2'\, F_1$ is assumed to be not zero.

By differentiating equations (2.1) with respect to x_1, the following equations free from the parameter t, are obtained:

$$(2.2) \qquad (F_i - x_i'\, F_1)x_i''' = \left[\sum_{j=1}^{n} x_j' \frac{\partial F_1}{\partial x_j} - x_i' \sum_{j=1}^{n} x_j' \frac{\partial F_1}{\partial x_j}\right] x_i'' - 3F_1 x_i''^{2},$$

for $i = 2,3,\ldots,n$.

In describing the dynamical trajectories the equations (2.2) for i= 3,4,...,n are unnecessary as they are dependent on equations (2.1) and (2.2) for i = 2. These results are summarized in the following theorem.

Theorem 2.1. The system S_0 of ∞^{2n-1} dynamical trajectories C of Euclidean space E_n of dimension $n \geq 2$, is composed of the ∞^{2n-1} integral solutions C of the system of (n-1) ordinary differential equations

$$x''_i = K_i x''_2 ,$$

$$x_2''' = Gx''_2 + Hx''^2_2 , \text{ where}$$

$$(2.3) \qquad [F_2 - x_2'F_1]G = \sum_{j=1}^{n} x_j' \frac{\partial F_2}{\partial x_j} - x_2' \sum_{j=1}^{n} x_j' \frac{\partial F_1}{\partial x_j} ,$$

$$[F_2 - x_2'F_1]\, H = -3F_1 ,$$

$$[F_2 - x_2'F_1]\, K_i = F_i - x_i' F_1 ,$$

for $i = 3,4,5,\ldots,n$, and $F_2 - x_2' F_1 \neq 0$.

3. Actual and virtual dynamical trajectories. Suppose that a projectile is fired in a Euclidean space E_3, of three dimensions under the action of gravity which is assumed constant. The ∞^5 system of actual trajectories consists of parabolas with vertical axes and downward concavity. This set does not represent the set of all possible solutions to the system (2.3).

The vertical trajectories with concavity directed upwards also satisfy the same set of equations. This set of solutions is called the aggregate of virtual trajectories of the field of force. The virtual trajectories are the actual trajectories corresponding to the field of force with direction reversed.

In an arbitrary field of force in E_n the same distinction arises. The complete system of trajectories is composed of both the actual and virtual trajectories [1].

From equations (2.3) it is noted that the complete system of trajectories is not changed if the force field acting at every point is multiplied by a non-zero constant. A positive constant multiplier gives rise to the actual trajectories while a negative multiplier leads to the virtual trajectories.

CHAPTER II

GEOMETRICAL PRELIMINARIES AND APPELL'S TRANSFORMATION

4. Hyperplanes in E_n. This paper is concerned with the geometry of paths of particles moving in a field of force in E_n. Hence, it is necessary to extend certain basic geometrical concepts from E_3 to E_n.

For this discussion let $\vec{V}_1, \vec{V}_2, \ldots, \vec{V}_k$ be k linearly independent vectors in E_n.

Definition 4.1. A k-dimensional direction μ_k, is composed of the set of direction numbers of all vectors that can be expressed as a linear combination of a basic set $\{\vec{V}_1, \vec{V}_2, \ldots, \vec{V}_k\}$.

A set of direction numbers for μ_k is given by D_1: D_2: ... D_n, where

$$D_j = \sum_{i=1}^{k} a^i v_i{}^j , \tag{4.1}$$

for $1 \leq j \leq n$, a^i is a scalar for $1 \leq i \leq k$, and $v_i{}^j$ is the j^{th} component (in some basis of E_n) of the i^{th} basis vector $\vec{V}_i$.

Definition 4.2. A k-dimensional hyperplane or k-flat L_k, in E_n is determined by a point $\vec{x}_0$ in E_n and a k-dimensional direction μ_k.

Thus, a point and a 1-dimensional direction determine a straight line; a point and a 2-dimensional direction determine a plane; a point and a 3-dimensional direction determine a 3-flat; etc. An arbitrary point $\vec{x}$ in the k-flat [10] through $\vec{x}_0$ in the direction determined by $\{\vec{V}_1, \vec{V}_2, \ldots, \vec{V}_k\}$ is given by the vector equation

$$\vec{x} = \vec{x}_0 + \sum_{i=1}^{k} a^i \vec{V}_i \ , \tag{4.2}$$

where a^i is a scalar.

A k-flat is observed [11,p.28] to be the totality of points in E_n into which a fixed point $\vec{x}_0$ is transformed by the action of the vectors of a k-dimensional vector space of E_n.

<u>Definition 4.3</u>. Two flats L_k and L_m, where $k \leq m$, are defined to be <u>parallel</u> if every straight line (1-flat) in L_k is parallel to some straight line in L_m.

It is noted that according to Definition 4.3 an arbitrary 0-flat (point) is parallel to every k-flat with $k \geq 0$.

<u>Definition 4.4</u>. A vector $\vec{V}$ in E_n is defined to be <u>parallel to the k-dimensional direction μ_k</u> if every 1-flat generated by $\vec{V}$ is parallel to every k-flat with direction μ_k.

The definitions introduced in this section will serve as the basic framework for later discussions. Other geometrical concepts will be defined in later chapters as needed.

5. Collineations in a projective space S_n of dimension $n \geq 2$. A Euclidean space E_n becomes a projective space S_n by postulating [10.p.4] that any two distinct straight lines in a plane (2-flat) in E_n uniquely determine a point.

Definition 5.1. If the straight lines in a plane in E_n are parallel, then the uniquely determined point is defined to be a ideal point or the point at infinity.

Thus, any straight line in S_n has a point at infinity.

Consider any two non-parallel lines in a plane in E_n. These two lines determine two distinct points at infinity both of which are points at infinity of the given plane. Thus a plane has more than a single point at infinity.

Consider the collection of all straight lines in a plane that pass through a fixed point. An arbitrary straight line in this plane is parallel to one of the lines of the above collection and these two lines have a common point at infinity. That is, there is a 1-1 correspondence between the points at infinity of a plane and the set of all straight lines in a plane passing through a fixed point. Thus the points at infinity of a plane form a one-fold infinite collection.

Definition 5.2. The ∞^1 set of points at infinity of a given plane is defined to be the line at infinity of the given plane.

Two planes π_1 and π_2 in a 3-flat in E_n are seen to be parallel if and only if their line of intersection is the line at infinity of both planes.

Thus, by continuing this process it is noted that every k-flat has a (k-1)-flat at infinity; and in particular, a Euclidean space E_n becomes a projective space S_n by adjoining to E_n an (n-1)-flat at infinity.

Definition 5.3. A one to one transformation [13,p.23] of a projective space S_n onto itself which transforms points into points, lines into lines and in general, k-flats into k-flats if defined to be a collineation in S_n.

In projective n-space, a general collineation is given by

$$(5.1) \qquad y_i = \frac{\sum_{j=1}^{n} a_{ij}\, x_j + b_i}{\sum_{j=1}^{n} c_j\, x_j + b_0}, \quad \text{for } 1 \leq i \leq n,$$

where the (n+1)x(n+1) determinant $\Delta = \begin{vmatrix} c_j ; b_0 \\ a_{ij} ; b_i \end{vmatrix} \neq 0.$

The condition on the determinant Δ is to insure that the transformation is one to one.

The expression in the denominator of (5.1) is used to define an (n-1)-flat as follows:

$$(5.2) \qquad \sum_{j=1}^{n} c_j \, x_j + b_0 = 0 \; .$$

This flat is known as the vanishing (n-1)-flat and corresponds, under (5.1), to the (n-1)-flat at infinity.

By clearing fractions in (5.1) a set of equations in terms of x_i are obtained. This set of equations can be solved to obtain a unique set $(x_1, x_2, \ldots, x_n)$ provided the determinant $\Delta \neq 0$. That solution is given by

$$(5.3) \qquad x_i = \frac{\sum_{j=1}^{n} d_{ij} \, y_j + f_i}{\sum_{j=1}^{n} g_j \, y_j + f_0} \; ,$$

for $1 \leq i \leq n$, where the constants d_{ij}, f_i, g_j, and f_0 are algebraic functions of a_{ij}, b_i, c_j and b_0. Thus, the inverse transformation is itself a general projective collineation.

The set of all transformations of type (5.1) forms a group [15,p.104-105], namely, the projective collineation group of projective n-space depending on n(n+2) essential constants.

6. Appell's transformation. The importance of projective transformations in dynamics was first demonstrated by P. Appell [16]. His work was concerned primarily with positional fields of force in Euclidean spaces of two and three dimensions. E. Kasner and J. DeCicco [17] have

extended Appell's work to generalized fields of force.

Appell was concerned with the class of point transformations which, with an appropriate change in time, could be used to transform a positional field of force into a positional field of force. Such a transformation is called an Appell transformation.

In a Euclidean space E_n with $n \geq 2$, a collineation as described in (5.1) along with the change in time,

$$dt_1 = \frac{dt}{k(\sum_{j=1}^{n} c_j x_j + b_0)^2}, \tag{6.1}$$

where k is a finite non-zero real constant, is an Appell transformation.

To show this, suppose that such a transformation is applied to a positional field of force, i.e.

$$y_i = \frac{P_i}{Q},$$
$$dt_1 = \frac{dt}{KQ^2}, \tag{6.2}$$

where

$$P_i = \sum_{j=1}^{n} a_{ij} x_j + b_i,$$

$$Q = \sum_{j=1}^{n} c_j x_j + b_0 .$$

The velocity components in the transformed field are

$$\text{(6.3)} \qquad \frac{dy_i}{dt_1} = K[Q\dot{P}_i - P_i\dot{Q}]$$

Differentiate (6.3) once more with respect to the time t_1 to obtain the equations representing the force vectors in the transformed force field, i.e.,

$$\text{(6.4)} \qquad \frac{d^2y_i}{dt_1^{\,2}} = K^2\, Q^2\, [Q\ddot{P} - P_i\ddot{Q}] \ .$$

Now P_i and Q depend on $x_1,x_2,\ldots,x_n$ and by transformation (5.3) depend on $y_1,y_2,\ldots,y_n$. $\ddot{P}_i$ and $\ddot{Q}$ depend on $\ddot{x}_1,\ddot{x}_2,\ldots,\ddot{x}_n$ as the force field is positional. Thus, by way of transformation (5.1), $\ddot{P}_i$ and $\ddot{Q}$ also depend on $y_1,y_2,\ldots,y_n$. Therefore, $\frac{d^2y_i}{dt_1^{\,2}}$ is a function of position only and the transformed force field is positional.

It is natural to consider the possibility of the existence of other pointwise transformations which transform positional force fields into positional force fields. Before doing this it will be necessary to present the following definition.

A real valued function f with domain D an open subset of E_n , is of <u>class m or c^m</u> on D if f has continuous partial derivatives of at least order m.

Now consider the following general transformation:

$$y_i = \phi_i(x_1, x_2, \ldots, x_n) \;,$$

(6.5) $$\qquad\qquad\qquad \text{for } 1 \le i \le n \;,$$

$$dt_1 = \lambda(x_1, x_2, \ldots, x_n)\,dt$$

where ϕ_i and λ are of at least class two and one, respectively, the Jacobian matrix $\frac{\partial \phi_i}{\partial x_j}$ is of rank n, and $\lambda(x_1, x_2, \ldots, x_n) \neq 0$.

Differentiate y_i with respect to the time t_1 and it is found that the velocity components are

(6.6) $$\frac{dy_i}{dt_1} = \frac{1}{\lambda} \sum_{j=1}^{n} \frac{\partial \phi_i}{\partial x_j} \dot{x}_j \;.$$

The equations defining the force vectors in the transformed force field are obtained by differentiating (6.6) with respect to t_1, i.e.,

(6.7) $$\frac{d^2 y_i}{dt_1^{\,2}} = \frac{1}{\lambda}\Big[\sum_{j=1}^{n}\Big\{\sum_{e=1}^{n} \frac{\partial}{\partial x_e}\Big\{\frac{1}{\lambda}\frac{\partial \phi_i}{\partial x_j}\Big\}\dot{x}_e\Big\}\dot{x}_j\Big] + \frac{1}{\lambda^2}\sum_{j=1}^{n}\frac{\partial \phi_i}{\partial x_j}\ddot{x}_j \;.$$

In order for the force field to be positional, the first of the above expressions must be identically zero, i.e.,

$$\frac{1}{\lambda}\Big[\sum_{j=1}^{n}\Big\{\sum_{e=1}^{n} \frac{\partial}{\partial x_e}\Big\{\frac{1}{\lambda}\frac{\partial \phi_i}{\partial x_j}\Big\}\dot{x}_e\Big\}\dot{x}_j\Big] = 0 \;.$$

This leads to the following system of partial differential equations:

$$\frac{\partial}{\partial x_j}\{\frac{1}{\lambda}\frac{\partial \phi_i}{\partial x_j}\} = 0$$

(6.8)

$$\frac{\partial}{\partial x_e}\{\frac{1}{\lambda}\frac{\partial \phi_i}{\partial x_j}\} + \frac{\partial}{\partial x_j}\{\frac{1}{\lambda}\frac{\partial \phi_i}{\partial x_e}\} = 0 ,$$

for $1 \leq i \leq n$, $1 \leq e \leq n$, $1 \leq j \leq n$ and $e \neq j$.

The first of (6.8) implies that $\frac{1}{\lambda}\frac{\partial \phi_i}{\partial x_j}$ is not a function of x_j . This then implies that each part of the second of (6.8) is a constant, i.e.,

$$\frac{\partial}{\partial x_e}\{\frac{1}{\lambda}\frac{\partial \phi_i}{\partial x_j}\} = A^i_{je}$$

(6.9)

$$\frac{\partial}{\partial x_j}\{\frac{1}{\lambda}\frac{\partial \phi_i}{\partial x_e}\} = A^i_{ej}$$

$$\text{and } A^i_{ej} = -A^i_{je} .$$

Solving this set of equations gives rise to

(6.10) $$\frac{\partial \phi_i}{\partial x_j} = \lambda[\sum_{e=1}^{n} A^i_{je}\, x_e + B_{ij}] ,$$

where $A^i_{ej} = -A^i_{je}$.

Now ϕ_i is assumed to have continuous partial derivatives of at least the second order and as a result, the following is true:

$$\frac{\partial^2 \phi_i}{\partial x_m \partial x_j} = \frac{\partial^2 \phi_i}{\partial x_j \partial x_m} . \tag{6.11}$$

From (6.10) it is found that

$$\frac{\partial^2 \phi_i}{\partial x_m \partial x_j} = \frac{\partial \lambda}{\partial x_m} \left[\sum_{e=1}^{n} A^i_{je} x_e + B_{ij}\right] + A^i_{jm} ,$$

$$\frac{\partial^2 \phi_i}{\partial x_j \partial x_m} = \frac{\partial \phi}{\partial x_j} \left[\sum_{e=1}^{n} A^i_{me} x_e + B_{im}\right] + \lambda A^i_{mj} .$$

Then by (6.11) and the fact that $A^i_{jm} = -A^i_{mj}$ the following is obtained:

$$2\lambda A^i_{jm} + \frac{\partial \lambda}{\partial x_m} \left[\sum_{e=1}^{n} A^i_{je} x_e + B_{ij}\right] - \frac{\partial \lambda}{\partial x_j} \left[\sum_{e=1}^{n} A^i_{me} x_e + B_{im}\right] = 0 , \tag{6.12}$$

for $j \neq m$.

By selecting two different values for i, say first j and then m, equation (6.12) leads to two linear equations in the two unknowns $\frac{\partial \lambda}{\partial x_j}$ and $\frac{\partial \lambda}{\partial x_m}$. This set of equations can be solved provided that the determinant of the coefficient matrix is not zero.

By first fixing j and letting m take on all possible values, (n-1) expressions for $\frac{\partial \lambda}{\partial x_j}$ are obtained. All of these expressions must be identically equal which leads to

$$(6.13) \qquad \frac{1}{\lambda}\frac{\partial \lambda}{\partial x_j} = \frac{-2c_j}{\sum_{e=1}^{n} c_e x_e + b_0} ,$$

$1 \leq j \leq n$, where c_e and b_0 are constants depending on the constants in equation (6.12).

Next equations (6.13) can be solved to find that

$$(6.14) \qquad \lambda = \frac{1}{K(\sum_{e=1}^{n} c_e x_e + b_0)^2} ,$$

where K is the constant of integration.

From equations (6.10) and (6.14), it is found that $\frac{\partial \phi_i}{\partial x_j}$ is given by

$$(6.15) \qquad \frac{\partial \phi_i}{\partial x_j} = \frac{\sum_{e=1}^{n} A^i_{je} x_e + B_{ij}}{K(\sum_{e=1}^{n} c_e x_e + b_0)^2} ,$$

for $1 \leq i \leq n$, $1 \leq j \leq n$ and $A^i_{je} = 0$ for $e = j$.

The solution of (6.15) gives rise to

$$\phi_i = \frac{\sum_{e=1}^{n} a_{ie} x_e + b_i}{\sum_{e=1}^{n} c_e x_e + b_0} ,$$

for $1 \leq i \leq n$, where a_{ie} and b_i are constants. This is exactly Appell's transformation described in equations (5.1).

The results which were developed in the preceeding paragraphs are summarized in the following theorem.

Theorem 6.1. A transformation $T : R \to E_n$, where R is a subset of E_n and $n \geq 2$, whereby every point of R is mapped into a unique point, and the time is changed in such a manner that the ratio $\frac{dt_1}{dt}$ is a function of position only, sends every dynamical system S_0 of a positional field of force F into a dynamical system S_0' of a positional field of force F' if and only if T is a collineation,

$$y_i = \frac{\sum_{j=1}^{n} a_{ij}\, x_j + b_i}{\sum_{j=1}^{n} c_j\, x_j + b_0},$$

for $1 \leq i \leq n$, for which the corresponding determinant is not zero, and the change in time is given by

$$dt_1 = \frac{dt}{K\left(\sum_{j=1}^{n} c_j\, x_j + b_0\right)^2},$$

where K is a non-zero constant.

In later sections, special cases of this transformation will be studied.

CHAPTER III

SOME GEOMETRICAL RESULTS CONCERNING DYNAMICAL TRAJECTORIES IN A EUDLIDEAN SPACE E_3 OF THREE DIMENSIONS.

7. The arc length derivatives of a positional field of force in E_3. This development is similar to that contained in [14].

Let a positional field of force

$$(7.1)\quad \vec{F} = \vec{i}\Phi(x,y,z) + \vec{j}\Psi(x,y,z) + \vec{k}\chi(x,y,z)\ ,$$

of at least class two be defined on a non-empty simply connected open region of a Euclidean space E_3, of three dimensions. The three vectors, $\vec{i}$, $\vec{j}$ and $\vec{k}$, in (7.1) form an orthonormal basis for the vector space containing the force vectors.

Construct a path C of at least class four, not a straight line, in this non-empty simply-connected open region. Relative to the force F acting at any point P of this path C, the two space derivatives $\frac{d\vec{F}}{ds}$ and $\frac{d^2\vec{F}}{ds^2}$ of the first and second orders are the corresponding arc length derivatives of the force vector $\vec{F}$.

It is evident that

$$\vec{F} = T_1\vec{T} + N_1\vec{N} + B_1\vec{B} ,$$

$$\frac{d\vec{F}}{ds} = T_2\vec{T} + N_2\vec{N} + B_2\vec{B} , \tag{7.2}$$

$$\frac{d^2\vec{F}}{ds^2} = T_3\vec{T} + N_3\vec{N} + B_3\vec{B} ,$$

Where $T\alpha$, $N\alpha$, $B\alpha$, for $\alpha = 1,2,3$, are the corresponding components of the indicated vectors relative to the moving trihedral $(\vec{T}, \vec{N}, \vec{B})$ of the path C.

The three unit vectors $\vec{T}$, $\vec{N}$ and $\vec{B}$ are related by the Serret-Frenet formulas [23,p.14], namely

$$\frac{d\vec{T}}{ds} = \kappa\vec{N} ,$$

$$\frac{d\vec{N}}{ds} = -\kappa\vec{T} + \tau\vec{B} , \tag{7.3}$$

$$\frac{d\vec{B}}{ds} = -\tau\vec{N}$$

where $\kappa > 0$, is the circular curvature and τ is the torsion of the curve C. The radius of circular curvature is given by $\rho = \frac{1}{\kappa} > 0$. If $\tau \neq 0$ the radius of torsion is given by $\sigma = \frac{1}{\tau}$.

<u>Theorem 7.1</u>. <u>If $\vec{F}$ and $\frac{d\vec{F}}{ds}$ are as in (7.2) then</u>

$$(7.4) \qquad T_2 = \frac{dT_1}{ds} - \kappa N_1, \; N_2 = \frac{dN_1}{ds} + \kappa T_1 - \tau B_1, \text{ and}$$

$$B_2 = \frac{dB_1}{ds} + \tau N_1 \; .$$

This can be established by differentiating $\vec{F} = T_1\vec{T} + N_1\vec{N} + B_1\vec{B}$, with respect to s. Thus

$$\frac{d\vec{F}}{ds} = \frac{dT_1}{ds}\vec{T} + T_1\frac{d\vec{T}}{ds} + \frac{dN_1}{ds}\vec{N} + N_1\frac{d\vec{N}}{ds} + \frac{dB_1}{ds}\vec{B} + B_1\frac{d\vec{B}}{ds}$$

Now apply the Serret-Frenet equations and collect terms. The result is

$$(7.5) \quad \frac{d\vec{F}}{ds} = \left[\frac{dT_1}{ds} - \kappa N_1\right]\vec{T} + \left[\frac{dN_1}{ds} + \kappa T_1 - \tau B_1\right]\vec{N} + \left[\frac{dB_1}{ds} + \tau N_1\right]\vec{B}.$$

By these vector equations and (7.2) the three relations in (7.4) are found to be valid.

<u>Theorem 7.2</u>. <u>If $\vec{F}$, $\frac{d\vec{F}}{ds}$ and $\frac{d^2\vec{F}}{ds^2}$ are as in (7.2) then</u>

$$(7.6) \quad T_3 = \frac{dT_3}{ds} - \kappa N_2, \; N_3 = \frac{dN_2}{ds} + \kappa T_2 - \tau B_2, \text{ and}$$

$$B_3 = \frac{dB_2}{ds} + \tau N_2$$

The technique used in establishing this result parallels that used in Theorem 7.1.

If a particle traces a dynamical trajectory C of the given positional field of Force $\vec{F}$ with $|\vec{F}| > 0$, then the force vector $\vec{F}$ is in the direction of the acceleration vector which is known to be contained in the osculating plane of C at the point of definition. Thus $B_1 = 0$ in the first of (7.2).

The above discussion leads to a consideration of the following theorem.

Theorem 7.3. If a constrained motion is possible along the given path C, such that

$$\vec{F} = T_1\vec{T} + N_1\vec{N} ,$$

then the components (T_2,N_2,B_2) of $\frac{d\vec{F}}{ds}$ are

$$(7.7)\quad T_2 = \frac{dT_1}{ds} - \kappa N_1,\ N_2 = \frac{dN_1}{ds} + \kappa T_1,\ B_2 = \tau N_1$$

and the three components (T_3,N_3,B_3) of $\frac{d^2\vec{F}}{ds^2}$, are given by

$$T_3 = \frac{d^2 T_1}{ds^2} - N_1 \frac{d\kappa}{ds} - 2\kappa \frac{dN_1}{ds} - \kappa^2 T_1,$$

$$(7.8)\quad N_3 = \frac{d^2 N_1}{ds^2} + T_1 \frac{d\kappa}{ds} + 2\kappa \frac{dT_1}{ds} - (\kappa^2 + \tau^2)N_1,$$

$$B_3 = N_1 \frac{d\tau}{ds} + 2\tau \frac{dN_1}{ds} + \kappa\tau T_1 .$$

The results in (7.7) are obtained by letting $B_1 = 0$ in (7.4). Differentiate (7.7) with respect to s, substitute in (7.6) and collect terms to obtain (7.8).

Note that since the force vector $\vec{F}$, in this case, is contained in the osculating plane, then

$$T_1 = \vec{F} \cdot T = \vec{F} \cdot \frac{d\vec{r}}{ds} ,$$

(7.9)

$$N_1^2 = \vec{F}^2 - T_1^2 = (\vec{F} \times \frac{d\vec{r}}{ds})^2 .$$

8. <u>The system S_0 of dynamical trajectories of a positional field of force</u>. If a particle of constant mass $m > 0$, is permitted to move subject <u>only</u> to the influence of the positional field of force $\vec{F}$ of Section 7, then it describes a dynamical trajectory C of the positional field of force $\vec{F}$

The physical system S_0 of ∞^5 dynamical trajectories C of a positional field of force $\vec{F}$ with $|\vec{F}| > 0$, is composed of the ∞^5 integral solutions of the vector differential equation

$$m\vec{A} = m\ddot{\vec{r}} = \vec{F} , \tag{8.1}$$

where $\vec{A}$ is the acceleration vector.

Since the acceleration vector is contained in the osculating plane, the following result is clear.

Theorem 8.1. The physical system S_0 of the ∞^5 dynamical trajectories of a positional field of force $\vec{F}$ with $|\vec{F}| > 0$, is given by the system of equations

$$\vec{F} = T_1\vec{T} + N_1\vec{N} ,$$

$$(8.2) \qquad m\, v\, \frac{dv}{ds} = T_1 = \vec{F} \cdot \frac{d\vec{r}}{ds} ,$$

$$\frac{mv^2}{\rho} = N_1 = \varepsilon\left[\vec{F}^2 - \left(\vec{F} \cdot \frac{d\vec{r}}{ds}\right)^2\right]^{1/2} ,$$

where ε is either +1 or -1.

For this situation, the T_2, N_2, B_2; T_3, N_3, B_3; of Section 7, can be evaluated by obtaining the space derivatives of T_1 and N_1. The procedure is similar to that used by DeCicco in [22]. Thus

$$(8.3) \qquad \begin{aligned} \frac{dT_1}{ds} &= m\left[v\,\frac{d^2v}{ds^2} + \left(\frac{dv}{ds}\right)^2\right] \\ \frac{dN_1}{ds} &= m\left[2v\kappa\,\frac{dv}{ds} + v^2\,\frac{d\kappa}{ds}\right] . \end{aligned}$$

The next step is to substitute these results into the expressions for T_2, N_2, B_2; T_3, N_3, B_3; given in Theorem 7.3 to obtain

$$T_2 = m[v \frac{d^2v}{ds^2} + \left(\frac{dv}{ds}\right)^2 - v^2\kappa^2] ,$$

$$N_2 = m[3v\kappa \frac{dv}{ds} + v^2 \frac{d\kappa}{ds}] ,$$

$$B_2 = m\, v^2\, \kappa\, \tau ,$$

(8.4)

$$T_3 = m[v \frac{d^3v}{ds^3} + 3 \frac{dv}{ds} \frac{d^2v}{ds^2} - 5v\kappa^2 \frac{dv}{ds} - 3v^2\kappa \frac{d\kappa}{ds}] ,$$

$$N_3 = m[4v\kappa \frac{d^2v}{ds^2} + 4\kappa\left(\frac{dv}{ds}\right)^2 + 5v \frac{dv}{ds}\frac{d\kappa}{ds} + v^2 \frac{d^2\kappa}{ds^2} - (\kappa^2+\tau^2)v^2\kappa] ,$$

$$B_3 = m[2v^2\tau \frac{d\kappa}{ds} + 5v\kappa\tau \frac{dv}{ds} + v^2\kappa \frac{d\tau}{ds}] .$$

Now consider two dynamical trajectories C_1 and C_2 described by $\vec{r}_1 = \vec{r}_1(s)$ and $\vec{r}_2 = \vec{r}_2(s^*)$, respectively, such that the two paths intersect at the point P_o . Let P_o correspond to the values $s = s_o$ and $s^* = s^*_o$ on C_1 and C_2, respectively. Then, of course, $\vec{r}_1(s_o) = \vec{r}_2(s^*_o)$.

The following definition can be found in [27,p50].

<u>Definition 8.1</u>. A curve C_1 described by $\vec{r}_1 = \vec{r}_1(s)$, has contact of order n with a curve C_2 described by $\vec{r}_2 = \vec{r}_2(s^*)$, at the point P_o if

$$\vec{r}_1(s_o) = \vec{r}_2(s_o^*) = P_o \text{ , for some } s_o \text{ and } s_o^* \text{ ,}$$

$$(8.5) \qquad \left.\frac{d^k r_1}{ds^k}\right|_{s=s_o} = \left.\frac{d^k \vec{r}_2}{d(s^*)^k}\right|_{s^*=s_o^*} \text{ , for } 1 \leq k \leq n \text{ ,}$$

and, if the derivatives of order (n+1) exist at P_o, then

$$(8.6) \qquad \left.\frac{d^{(n+1)}\vec{r}_1}{ds^{(n+1)}}\right|_{s=s_o} \neq \left.\frac{d^{(n+1)} r_2}{d(s^*)^{n+1}}\right|_{s^*=s^*_o}$$

<u>Definition 8.2</u>. A curve C has <u>contact of order n</u> with a surface S at a point P_o if there exists at least one curve C* on S which has contact of order n with C and there does not exist a curve on S which has contact of order greater than n with C at P_o.

This definition is based on that found in [27,p51].

The osculating plane to a curve C at a point P_o is known to have contact of the second order with C at the point P_o, and second order contact is usually known as <u>osculation</u>. Contact of order higher than two is called <u>super osculation</u> or <u>hyper osculation</u>.

9. A study of the curvature and the torsion of a dynamical trajectory. In a positional field of force, the force vector $\vec{F}$ acting at any point P of a dynamical trajectory C, with $\kappa > 0$, at the point P is contained in the osculating plane of the dynamical trajectory C at this point P. Thus,

$$\vec{F} = T_1\vec{T} + N_1\vec{N} ,$$

where

$$mv\,\frac{dv}{ds} = T_1 \quad \text{and} \quad \frac{mv^2}{\rho} = N_1 .$$

This set of relations implies that formulas (8.4) are applicable. From these it is found that

$$(9.1)\qquad \begin{aligned} &\frac{d\vec{r}}{ds} \times \vec{F} = N_1\vec{B} = mv^2\kappa\vec{B} \\ &\left(\frac{d\vec{r}}{ds} , \vec{F}, \frac{d\vec{F}}{ds} \right) = m^2v^4\kappa^2\tau . \end{aligned}$$

Theorem 9.1. The torsion τ of any dynamical trajectory C of a positional field of force $\vec{F}$, with $|\vec{F}| > 0$, is

$$(9.2)\qquad \tau = \frac{B_2}{N_1} = \frac{\left(\frac{d\vec{r}}{ds} , \vec{F}, \frac{d\vec{F}}{ds} \right)}{\left(\frac{d\vec{r}}{ds} \times \vec{F}\right)^2}$$

where N_1 is assumed to be not equal to zero.

From Theorem 7.3, it is found that $B_2 = \tau N_1$, which gives the first part of the equality. The second part follows immediately from (9.1).

Theorem 9.2. The rate of variation, $\frac{d\kappa}{ds}$, of the circular curvature κ per unit of arc length s along any dynamical trajectory C of a positional field of force $\vec{F}$ with $|\vec{F}| > 0$, is

$$\frac{d\kappa}{ds} = \frac{\kappa}{N_1}[N_2 - 3\kappa T_1] , \tag{9.3}$$

if $N_1 \neq 0$.

In the dynamical case it is known that $mv^2\kappa = N_1$. Differentiate this result with respect to s and solve for $\frac{d\kappa}{ds}$ to obtain

$$\frac{d\kappa}{ds} = \frac{\kappa}{N_1}[N_2 - 3\kappa T_1] .$$

Using the definition of the radius of curvature ($\rho = \frac{1}{\kappa}$), equation (9.3) can be written as

$$N_1 \frac{d\rho}{ds} = 3T_1 - \rho N_2 , \tag{9.4}$$

which is used to obtain the derivation of a theorem on rest trajectories originally stated and proved by Kasner. This will be discussed further in Section 10.

Theorem 9.3. The rate of variation $\frac{d\tau}{ds}$ of the torsion τ per unit of arc length s along any dynamical trajectory C of a positional field of force $\vec{F}$ with $|\vec{F}| > 0$, is

$$\frac{d\tau}{ds} = \frac{1}{N_1}[B_3 - 2\tau N_2 + \tau\kappa T_1], \tag{9.5}$$

if $N_1 \neq 0$.

By differentiating $\tau = \frac{B_2}{N_1}$, with respect to the arc length s of the dynamical trajectory C, it is found that

$$\frac{d\tau}{ds} = \frac{N_1 \frac{dB_2}{ds} - B_2 \frac{dN_1}{ds}}{N_1^2} \tag{9.6}$$

Next, substitute the expressions for $\frac{dB_2}{ds}$ and $\frac{dN_1}{ds}$, found in Theorems 7.2 and 7.3, into (9.6).

Upon simplification, the formula (9.5) is obtained.

10. Rest trajectories.

Definition 10.1. If a particle in a positional field of force starts from rest, the resulting path is termed a rest trajectory.

A discussion of rest trajectories can be found in [28].

The initial velocity vector for a rest trajectory C, is the zero vector. At the initial point of a rest trajectory C, the force vector $\vec{F}$ is tangent to the trajectory. Thus, the initial conditions for a rest trajectory are

$$N_1 = 0 \, ,$$

$$(10.1) \qquad \kappa = \frac{1}{\rho} = \frac{N_2}{3T_1} \, ,$$

where it is assumed that $\frac{d\vec{F}}{ds}$ is <u>not</u> perpendicular to $\vec{N}$ so that $T_1 \neq 0$, and $N_2 \neq 0$.

The Faraday lines of force of a positional field of force $\vec{F}$ with $|\vec{F}| > 0$, (Chapter I) consists of the ∞^2 solutions of the vector differential equation

$$(10.2) \qquad \frac{d\vec{r}}{dt} = \alpha\vec{F} \, ,$$

where α is a non-zero scalar.

Thus, it is noted that a rest trajectory and a Faraday line of force have a common unit tangent vector at the initial point of the rest trajectory. That is, they have contact of at least order one at the initial point of the rest trajectory.

The following result is known as Kasner's Theorem on Rest Trajectories and can be found in [9].

Theorem 10.1. The ratio of the curvature of a rest trajectory to that of the corresponding Faraday line of force at the initial point, is 1/3, provided that the order of contact is exactly one.

From equation (10.2) for the system of ∞^2 Faraday lines of force it is noted that

$$\vec{F} = (\vec{F} \cdot \vec{F})^{1/2} \vec{T} = T_1 \vec{T} \tag{10.3}$$

where $\vec{T}$ is a unit vector in the direction of $\vec{F}$. Now $\vec{T}$ is tangent to the Faraday line of force and the rest trajectory with initial point at the point in question.

Differentiate (10.3) with respect to arc length of the line of force to obtain

$$\frac{d\vec{F}}{ds} = \frac{dT_1}{ds} \vec{T} + T_1 \kappa^* \vec{N}^* , \tag{10.4}$$

where κ^* is the curvature and $\vec{N}^*$ is the principal normal to the Faraday line of force.

The first derivative of the force vector $\vec{F}$ with respect to the arc length s of a dynamical trajectory is given by

$$\frac{d\vec{F}}{ds} = T_2 \vec{T} + N_2 \vec{N} + B_2 \vec{B} , \tag{10.5}$$

where $\vec{T}$, $\vec{N}$, and $\vec{B}$ are the unit tangent vector, unit normal vector and unit principal vector, respectively.

If the dynamical trajectory is a rest trajectory, then at the initial point, (10.5) reduces to

$$\frac{d\vec{F}}{ds} = \frac{dT_1}{ds}\vec{T} + (3T_1\kappa)\vec{N} , \tag{10.6}$$

where κ is the curvature of the rest trajectory at the initial point and $T_1 = (F \cdot F)^{1/2}$.

As the Faraday line of force and the rest trajectory have a common tangent at the initial point of the rest trajectory, the $\frac{d\vec{F}}{ds}$ in (10.4) and (10.6) are identical. Thus, the normal components of the two vectors must be equal which implies that

$$3k = k^* \tag{10.7}$$

This completes the proof of Kasner's Theorem on Rest trajectories.

In addition, Kasner [30] proved that if the order of contact at the initial point between the rest trajectory and the corresponding Faraday line of force is $n \geq 1$, then the ratio ρ of the rate of departures from the common tangent line, is

$$\rho = \frac{1}{(2n+1)} . \tag{10.8}$$

Extensions by Kasner and others [28], [29], [7] have been obtained to physical systems S_k , even when the field of force $\vec{F}$ is <u>not</u> positional.

CHAPTER IV

HALPHEN'S THEOREM AND SOME RELATED RESULTS IN E_3 .

11. The conditions for a planar dynamical trajectory in E_3 . The result of Halphen that is investigated in this chapter concerns the converse of a situation that is discussed in most texts on analytical dynamics. Before discussing Halphen's theorem, necessary and sufficient conditions are determined so that the complete dynamical system S_0 of a positional field of force $\vec{F}$ with $|\vec{F}| > 0$, in a Euclidean space E_3 of three dimensions, be composed entirely of ∞^5 plane dynamical trajectories C.

For the purpose of a unified treatment, a proof is given of the known proposition [26,p.61] that a non-rectilinear curve C, in a Euclidean space E_3 of three dimensions, is planar, if and only if its torsion τ is identically zero.

For, the torsion τ of a non-rectilinear curve C in E_3 defined by the vector equation $\vec{r} = \vec{r}(t)$, where $\vec{r}(t)$ is of at least class three and $\vec{V} = \dot{\vec{r}}(t) \neq 0$, on some interval $a \leq t \leq b$, with $-\infty < a < b < +\infty$, is given by

$$(11.1) \qquad \tau = \frac{(\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}})}{(\dot{\vec{r}} \times \ddot{\vec{r}})^2}$$

provided that $|\dot{\vec{r}} \times \ddot{\vec{r}}| > 0$, on $a \leq t \leq b$.

A curve C of E_3 , is a plane curve if and only if the vector equation $\vec{r} = \vec{r}(t)$ obeys the scalar identity

$$(11.2) \qquad \vec{B} \cdot \vec{r}(t) = c \ ,$$

where $\vec{B}$ is a constant unit vector, c is a real scalar and $a \leq t \leq b$.

From (11.2) it is clear that $\vec{B} \cdot \dot{\vec{r}}(t) = 0$, $\vec{B} \cdot \ddot{\vec{r}}(t) = 0$, $\vec{B} \cdot \dddot{\vec{r}}\,(t) = 0$. Hence, if the curve C is planar, then $(\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}\,) = 0$, for $a < t < b$. Thus if a curve C in E_3 is planar then it is either a straight line or its torsion τ is identically zero.

Conversely, suppose that $(\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}\,) = 0$, for $a \leq t \leq b$.

If $\dot{\vec{r}} \times \ddot{\vec{r}} = \emptyset$ (null vector), for $a \leq t \leq b$, then since $|\dot{\vec{r}}| > 0$, it follows that a scalar u(t) exists for which $\ddot{\vec{r}} = u(t)\dot{\vec{r}}$. Then

$$(11.3) \qquad r = A_o \, e^{\int_t^t u(\lambda)d\lambda}$$

where $\vec{A}_o$ is a non-null, constant vector and t_o and t are points of $a \leq t \leq b$. Hence

$$(11.4) \quad \vec{r} = \vec{A}_o \int_{t_o}^{t} e^{\int_{t_o}^{\mu} \mu(\lambda)d\lambda} d\mu + \vec{r}_o \ ,$$

where $\vec{r}_o$ is a constant vector.

Therefore, if $\dot{\vec{r}} \times \ddot{\vec{r}} = \emptyset$, for $a \leq t \leq b$, the curve C is a straight line.

Hence forth it is assumed that the curvature $\kappa = |\dot{\vec{r}} \times \ddot{\vec{r}}|^{3/2} / (\dot{r}^2)^{1/2} > 0$, for $a \leq t \leq b$. Thus for $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, it may be supposed that

$$\dot{x}\ddot{y} - \ddot{x}\dot{y} \neq 0 , \tag{11.5}$$

for $a \leq t \leq b$.

Since $(\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}) = 0$, there exist two scalar functions $u(t)$ and $v(t)$ such that

$$\begin{aligned} \dot{z} &= u(\dot{x}) + v(\dot{y}) , \\ \ddot{z} &= u(\ddot{x}) + v(\ddot{y}) , \\ \dddot{z} &= u(\dddot{x}) + v(\dddot{y}) . \end{aligned} \tag{11.6}$$

Then it follows that

$$\begin{aligned} \dot{u}\dot{x} + \dot{v}\dot{y} &= 0 , \\ \dot{u}(\ddot{x}) + \dot{v}(\ddot{y}) &= 0 . \end{aligned} \tag{11.7}$$

The only solution to the system of equations in (11.7) is the zero solution since $\dot{x}\ddot{y} - \dot{y}\ddot{x} \neq 0$. Therefore, for $a \leq t \leq b$, it is seen that $u(t) = u_0$ and $v(t) = v_0$, where u_0 and v_0 are two real constants.

As $\dot{z} = u_0\dot{x} + v_0\dot{y}$, it follows that

$$z(t) = u_0x(t) + v_0y(t) + w_0 , \tag{11.8}$$

where u_0 , v_0 and w_0 , are three real constants.

Consequently, if $|\dot{\vec{r}} \times \ddot{\vec{r}}| > 0$, and $(\dot{\vec{r}}, \ddot{\vec{r}}, \dddot{\vec{r}}) = 0$, for $a \leq t \leq b$. The path is not straight and is a plane curve C in E_3 .

An expression for the torsion τ of a dynamical trajectory C , in terms of the space derivatives of the force vector of a positional field of force was given in (9.2). Thus, the following result is clear.

Theorem 11.1. A dynamical trajectory C, given by the vector equation $\vec{r} = \vec{r}(s)$, with s representing arc length, in a positional field of force $\vec{F}$ with $|\vec{F}| > 0$, in E_3 is planar if and only if

$$(11.9) \qquad \left(\frac{d\vec{r}}{ds}, \vec{F}, \frac{d\vec{F}}{ds}\right) = 0 ,$$

is a scalar identity.

In addition, this scalar equation, if not an identity, defines the tangent directions at an admissible point p, such that if a particle starts from such a point p in anyone of these tangent directions, then it describes a dynamical trajectory C that is initially flat ($\tau=0$) at the given point p .

12. Halphen's Theorem. In most texts on analytical dynamics, it is shown that the ∞^5 dynamical trajectories C, of a parallel or central positional field of force in E_3

are all plane curves. The converse result is not as well known but it is also true.

A new proof of this converse result is presented in the following theorem which is known as Halphen's Theorem [31].

Theorem 12.1. Every dynamical trajectory C of a positional field of force $\vec{F}$ with $|\vec{F}| > 0$, in Euclidean space E_3 of three dimensions, is planar, if and only if the positional field of force is either parallel or central.

The force vector $\vec{F}$ acting at any admissible point is

$$\vec{F} = \phi\vec{i} + \psi\vec{j} + \chi\vec{k} , \tag{12.1}$$

where the three cartesian components (ϕ, ψ and χ) are such that $|\vec{F}| = (\phi^2 + \psi^2 + \chi^2)^{1/2} > 0$.

If a dynamical trajectory C is defined by the pair of explicit equations $y = y(x)$ and $z = z(x)$, then it is a plane curve C if and only if

$$\begin{vmatrix} 1 & y' & z' \\ \phi & \psi & \chi \\ \phi' & \psi' & \chi' \end{vmatrix} = 0 , \tag{12.2}$$

where the primes denote total differentiations with respect to x.

Upon expanding the determinant and simplifying, the preceding is equivalent to the equation

$$(12.3)\qquad \begin{aligned} &a(y')^2 + 2by'z' + c(z')^2 + 2dy' + \\ &\qquad + 2ez' + f = 0 , \end{aligned}$$

where

$$(12.4)\qquad \begin{aligned} a &= \chi \frac{\partial\phi}{\partial y} - \phi \frac{\partial\chi}{\partial y} , \\ b &= \frac{1}{2} \left(\chi \frac{\partial\phi}{\partial z} - \phi \frac{\partial\chi}{\partial z} + \phi \frac{\partial\psi}{\partial y} - \psi \frac{\partial\phi}{\partial y} \right) , \\ c &= \phi \frac{\partial\psi}{\partial z} - \psi \frac{\partial\phi}{\partial z} , \\ d &= \frac{1}{2} \left(\chi \frac{\partial\phi}{\partial x} - \phi \frac{\partial\chi}{\partial x} + \psi \frac{\partial\chi}{\partial y} - \chi \frac{\partial\psi}{\partial y} \right) , \\ e &= \frac{1}{2} \left(\phi \frac{\partial\psi}{\partial x} - \psi \frac{\partial\phi}{\partial x} + \psi \frac{\partial\chi}{\partial z} - \chi \frac{\partial\psi}{\partial z} \right) , \\ f &= \psi \frac{\partial\chi}{\partial x} - \chi \frac{\partial\psi}{\partial x} . \end{aligned}$$

If the ∞^5 dynamical trajectories of a positional field of force $\vec{F}$ with $|\vec{F}| > 0$ are plane curves, then the equation (12.3) is an identity. This is equivalent to saying that each one of the six coefficients a, b, c, d, e, f is identically zero, and conversely.

Since $|\vec{F}| > 0$, there is no loss in generality in assuming that $\phi = \phi(x,y,z) \neq 0$.

Under this supposition, the two slope functions $\alpha = \alpha(x,y,z)$, $\beta = \beta(x,y,z)$, of the positional field of force $F = \phi i + \psi j + \chi k$, are defined by

$$\psi = \alpha\phi ,$$

$$\chi = \beta\phi . \tag{12.5}$$

Then the system of six equations in (12.4) becomes the set

$$
\begin{aligned}
a &= -\phi^2 \frac{\partial\beta}{\partial y} , \\
b &= \frac{\phi^2}{2} \left[-\frac{\partial\beta}{\partial z} + \frac{\partial\alpha}{\partial y} \right] , \\
c &= \phi^2 \frac{\partial\alpha}{\partial z} , \\
d &= \frac{\phi^2}{2} \left[-\frac{\partial\beta}{\partial x} + \alpha \frac{\partial\beta}{\partial y} - \beta \frac{\partial\alpha}{\partial y} \right] , \\
e &= \frac{\phi^2}{2} \left[\frac{\partial\alpha}{\partial x} + \alpha \frac{\partial\beta}{\partial z} - \beta \frac{\partial\alpha}{\partial z} \right] , \\
f &= \phi^2 \left[\alpha \frac{\partial\beta}{\partial x} - \beta \frac{\partial\alpha}{\partial x} \right] .
\end{aligned}
\tag{12.6}
$$

Evidently, $a = 0$ and $c = 0$, if and only if

$$\alpha = \alpha(x,y) ,$$

$$\beta = \beta(x,z) . \tag{12.7}$$

Then $a = 0$, $b = 0$ and $c = 0$ if and only if

$$\alpha = m(x)y + u(x) ,$$

$$\beta = m(x)z + v(x) , \tag{12.8}$$

where $m(x)$, $u(x)$ and $v(x)$ depend only on x .

Hence the three functions, d, e and f , of (12.6) assume the forms

$$d = -\frac{\phi^2}{2}\left[\left(\frac{dm}{dx} + m^2\right)z + \left(\frac{dv}{dx} + mv\right)\right]$$

$$e = -\frac{\phi^2}{2}\left[\left(\frac{dm}{dx} + m^2\right)y + \left(\frac{du}{dx} + mu\right)\right] \tag{12.9}$$

$$f = \phi^2\left[u\frac{dv}{dx} - v\frac{du}{dx}\right] .$$

These three expressions are identically zero if and only if

$$\frac{dm}{dx} + m = 0 ,$$

$$\frac{du}{dx} + mu = 0 , \tag{12.10}$$

$$\frac{dv}{dx} + mv = 0 .$$

If $m(x) = 0$, then u and v are two constants. This means that $\alpha = \alpha_0$, $\beta = \beta_0$, are two real constants.

<u>The corresponding positional field of force $\vec{F}$ with $|\vec{F}| > 0$, is parallel.</u>

If $m(x) \neq 0$, then the solution of (12.10) is

$$m(x) = \frac{1}{x - x_0} ,$$

$$u(x) = -\frac{y_0}{x - x_0} , \tag{12.11}$$

$$v(x) = -\frac{z_0}{x - x_0} ,$$

where x_0, y_0, z_0 are three real constants.

Therefore, (12.8) assumes the form

$$\alpha = \frac{y - y_0}{x - x_0} \ ,$$

(12.12)

$$\beta = \frac{z - z_0}{x - x_0} \ ,$$

where x_0, y_0, z_0, are three real constants.

Thus, the corresponding positional field of force $\vec{F}$ with $|\vec{F}| > 0$, is central with center at $P(x_0,y_0,z_0)$.

Consequently the proof of Halphen's Theorem is completed.

The following sections contain new results related to Halphen's Theorem.

13. The notion of a flat point of a positional field of force E_3. Consider a positional field of force

$$\vec{F} = \phi\vec{i} + \psi\vec{j} + \chi\vec{k} \ ,$$

with $|\vec{F}| = (\phi^2 + \psi^2 + \chi^2)^{1/2} > 0$, of at least class two, defined on a certain open region of a Euclidean space E_3 of three dimensions.

Definition 13.1. A flat point $P(x,y,z)$ is in the given region, and is such that every dynamical trajectory C of the given positional field of force $\vec{F}$, has contact of at least order three with its osculating plane Π constructed at the given point $P(x,y,z)$.

Thus, a point $P(x,y,z)$ of a given positional field of force $\vec{F}$ with $|\vec{F}| > 0$, is flat and only if the equation

$$(13.1) \qquad \left(\frac{d\vec{r}}{ds} , \vec{F} , \frac{d\vec{F}}{ds} \right) = 0 ,$$

is an identity in $\frac{d\vec{r}}{ds} = \vec{i}\,\frac{dx}{ds} + \vec{j}\,\frac{dy}{ds} + \vec{k}\,\frac{dz}{ds}$, at the point $P(x,y,z)$.

Theorem 13.1. A point $P(x,y,z)$ of a positional field of force $\vec{F} = \phi\vec{i} + \psi\vec{j} + \chi\vec{k}$, with $\phi \neq 0$, is flat if and only if the two slope functions $\alpha = \frac{\psi}{\phi}$, and $\beta = \frac{\chi}{\phi}$, obey the set of five equations

$$(13.2) \qquad \begin{array}{l} \frac{\partial\beta}{\partial y} = 0 , \quad \frac{\partial\beta}{\partial z} - \frac{\partial\alpha}{\partial y} = 0 , \quad \frac{\partial\alpha}{\partial z} = 0 , \\ \frac{\partial\alpha}{\partial x} + \alpha\,\frac{\partial\alpha}{\partial y} = 0 , \quad \frac{\partial\beta}{\partial x} + \beta\,\frac{\partial\beta}{\partial z} = 0 , \end{array}$$

at the point $P(x,y,z)$.

For, by the two set of equations (12.3) and (12.4), a point $P(x,y,z)$ of the given positional field of force $\vec{F}$ with $\phi \neq 0$, is flat, if and only if $a = 0$, $b = 0$, $c = 0$, $d = 0$, $e = 0$ and $f = 0$, at the point $P(x,y,z)$.

This set of six conditions is seen to be equivalent to the system of five conditions listed in (13.2).

According to Halphen's Theorem, a positional field of force $\vec{F}$ with $|\vec{F}| > 0$, is either central or parallel, if and only if every one of its points $P(x,y,z)$ is flat.

Theorem 13.2. A point $P(x,y,z)$ of a positional field of force $\vec{F} = \phi\vec{i} + \psi\vec{j} + \chi\vec{k}$, with $\phi \neq 0$, is flat if and only if there exist six distinct directions passing through P, and these directions are not all on a common quadric cone Q with vertex at the given point $P(x,y,z)$, such that the corresponding dynamical trajectories C passing through the point $P(x,y,z)$ in each of these six directions is hyperosculated by its osculating plane Π constructed at the given point $P(x,y,z)$.

If the point $P(x,y,z)$ is a flat point of the positional field of force $\vec{F}$, then every dynamical trajectory C passing through $P(x,y,z)$ is hyperosculated by its osculating plane Π at the point $P(x,y,z)$.

If there exist six distinct directions $(1,y',z')$ passing through the given point P such that they do not all lie on a quadric cone Q with vertex at the given point P, and such that the corresponding dynamical trajectory C passing through the point $P(x,y,z)$ in every one of the six directions is hyperosculated by its osculating plane Π at the point $P(x,y,z)$, then the directions $(1,y',z')$ satisfy equation (12.3). This results in six equations in five essential unknowns. Since these six directions are not on the same quadric cone Q with vertex at $P(x,y,z)$, then the six unknowns a, b, c, d, e and f are identically zero at the given point. Thus, the point $P(x,y,z)$ is a flat point of the positional field of force.

Theorem 13.3. At a flat point $P(x,y,z)$ of a positional field of force F with $|F| > 0$, in E_3, the curvature of the Faraday line of force there is $\kappa^* = 0$.

The Faraday lines of force in E_3 satisfy equations (10.2) and (10.3). Thus,

$$\frac{d\vec{r}}{dt} = \alpha\vec{F} = \alpha T_1\vec{T}^* , \tag{13.3}$$

where α is a non-zero scalar and $\vec{T}^*$ is the unit tangent vector to the Faraday line of force. Differentiate (13.3) with respect to the arc length s_f of the Faraday line of force to obtain

$$\frac{d\vec{F}}{ds_f} = \left(\frac{dT_1}{ds_f}\right)\vec{T}^* + T_1\kappa^*\vec{N}^* , \tag{13.4}$$

where κ^* represents the curvature of the Faraday line of force and $\vec{N}^*$ represents its unit normal.

Now consider an arbitrary dynamical trajectory C in this field of force $\vec{F}$ and let s represent the arc length of C. Differentiate the result in (13.3) with respect to the arc length s to obtain

$$\frac{d\vec{F}}{ds} = \frac{dT_1}{ds}\vec{T}^* + T_1\kappa^*\frac{ds_f}{ds}\vec{N}^* \tag{13.5}$$

where κ^* and $\vec{N}^*$ are as before.

The cross product of $\vec{F}$ and $\frac{d\vec{F}}{ds}$ is then seen to be

$$(13.6) \qquad \vec{F} \times \frac{d\vec{F}}{ds} = T_1^2 \kappa^* \left(\frac{ds_f}{ds} \right) \vec{B}^* ,$$

where $\vec{B}^*$ is the unit binormal to the Faraday line of force.

By definition a point $P(x,y,z)$ is a flat point of the positional field of force $\vec{F}$ if and only if

$$(13.7) \qquad \frac{d\vec{r}}{ds} \cdot \left(\vec{F} \times \frac{d\vec{F}}{ds}\right) = 0$$

is an identity in $\frac{d\vec{r}}{ds} = \frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j} + \frac{dz}{ds}\vec{k}$, at the given point $P(x,y,z)$. From (13.6) this is seen to be true if and only if the curvature of the Faraday line of force is equal to zero at the given point.

This completes the proof of Theorem 13.3.

14. The theory of a flat point of a dynamical trajectory of a positional field of force in E_3 .

Consider a positional field of force

$$\vec{F} = \phi\vec{i} + \psi\vec{j} + \chi\vec{k} ,$$

with $|\vec{F}| = (\phi^2 + \psi^2 + \chi^2)^{1/2} > 0$.

Definition 14.1. A flat point P(x,y,z) of a dynamical trajectory C of a positional field of force is a point on C at which the dynamical trajectory is hyperosculated by its osculating plane Π .

At a flat point P(x,y,z) of a positional field of force $\vec{F}$ with $|\vec{F}| > 0$, every dynamical trajectory C passing through P , possesses this point as a flat point.

If (X,Y,Z) with $(X^2 + Y^2 + Z^2)^{1/2} > 0$, denotes a set of direction numbers of a direction $\frac{d\vec{r}}{ds} = \vec{i}\,\frac{d\vec{x}}{ds} + \vec{j}\,\frac{d\vec{y}}{ds} + \vec{k}\,\frac{d\vec{z}}{ds}$, in E_3 , then

$$\frac{d\vec{r}}{ds} = \frac{\vec{i}X + \vec{j}Y + \vec{k}Z}{(X^2 + Y^2 + Z^2)^{1/2}} \tag{14.1}$$

Theorem 14.1. If (X,Y,Z) represents a tangent direction of a dynamical trajectory C passing through a given point P(x,y,z) of a positional field of force

$$\vec{F} = \phi\vec{i} + \psi\vec{j} + \chi\vec{k} ,$$

with $|\vec{F}| > 0$, then P(x,y,z) is a flat point of this dynamical trajectory C if and only if the equation

$$AX^2 + BY^2 + CZ^2 + 2DYZ + 2EZX + 2GXY = 0 , \tag{14.2}$$

where

$$
\begin{aligned}
&\rho A = \chi \frac{\partial\psi}{\partial x} - \psi \frac{\partial\chi}{\partial x} \ , \\
&\rho B = \phi \frac{\partial\chi}{\partial y} - \chi \frac{\partial\phi}{\partial y} \ , \\
&\rho C = \psi \frac{\partial\phi}{\partial z} - \phi \frac{\partial\psi}{\partial z} \ , \\
(14.3) \qquad &\rho D = \frac{1}{2} [\phi \frac{\partial\chi}{\partial z} - \chi \frac{\partial\phi}{\partial z} + \psi \frac{\partial\phi}{\partial y} - \phi \frac{\partial\psi}{\partial y}] \ , \\
&\rho E = \frac{1}{2} [\psi \frac{\partial\phi}{\partial x} - \phi \frac{\partial\psi}{\partial x} + \chi \frac{\partial\psi}{\partial z} - \psi \frac{\partial\chi}{\partial z}] \ , \\
&\rho G = \frac{1}{2} [\phi \frac{\partial\chi}{\partial x} - \chi \frac{\partial\phi}{\partial x} + \chi \frac{\partial\psi}{\partial y} - \psi \frac{\partial\chi}{\partial y}] \ ,
\end{aligned}
$$

<u>for which $\rho \neq 0$, is a factor of proportionality, is satisfied at the lineal element $E(x,y,z; X,Y,Z)$</u> .

This is an immediate consequence of (14.1) and (12.3).

The set of six equations (14.3), may be written in terms of the slope functions α and β , as the following system of six conditions:

$$
\begin{aligned}
&\rho A = \phi^2 [\beta \frac{\partial\alpha}{\partial x} - \alpha \frac{\partial\beta}{\partial x}] \ , \\
&\rho B = \phi^2 \frac{\partial\beta}{\partial y} \ , \\
&\rho C = -\phi^2 \frac{\partial\alpha}{\partial z} \ , \\
(14.4) \qquad &\rho D = \frac{\phi^2}{2} [\frac{\partial\beta}{\partial x} + \beta \frac{\partial\alpha}{\partial y} - \alpha \frac{\partial\beta}{\partial y}] \ , \\
&\rho E = \frac{\phi^2}{2} [-\frac{\partial\alpha}{\partial x} - \alpha \frac{\partial\beta}{\partial z} + \beta \frac{\partial\alpha}{\partial z}] \ , \\
&\rho G = \frac{\phi^2}{2} [\frac{\partial\beta}{\partial z} - \frac{\partial\alpha}{\partial y}] \ .
\end{aligned}
$$

Theorem 14.2. At a non-flat point $P(x,y,z)$ of a positional field of force $\vec{F} = \phi\vec{i} + \psi\vec{j} + \chi\vec{k}$, with $|\vec{F}| > 0$, the tangent directions (X,Y,Z) of all possible dynamical trajectories C passing through the point $P(x,y,z)$ and having a flat point at $P(x,y,z)$, describe a quadric cone Q with vertex at $P(x,y,z)$.

The quadric cone Q, is described by equation (14.2).

The next theorem is important in that it represents a less restrictive form of Halphen's Theorem.

Theorem 14.3. A positional field of force $\vec{F}$ with $|\vec{F}| > 0$, is either central or parallel if and only if at every point P of the region of definition there exists at least six distinct directions, not all of which are on the same quadric cone with vertex at P, such that the dynamical trajectories C passing through P in each of these six directions have P as a flat point.

If the field of force is either central or parallel, then this result is trivially true since every point on a dynamical trajectory C in such a field is a flat point of the dynamical trajectory C.

The converse follows by appealing to Theorem 13.2 to find that every point P in the region of definition is a flat point of the positional field of force. Thus every dynamical trajectory in such a field is planar and the field of force is either central or parallel.

15. <u>A study of the quadric cone Q of Theorem 14.2</u>. The characteristic equation of the quadric cone Q, represented by (14.2), is

$$\begin{vmatrix} A-\rho & G & E \\ G & B-\rho & D \\ E & D & C-\rho \end{vmatrix} = 0 \tag{15.1}$$

This characteristic equation is a cubic in ρ, namely,

$$\rho^3 - I\rho^2 + J\rho - \Delta = 0 , \tag{15.2}$$

where the three invariants I, J and Δ are

$$\begin{aligned} I &= A + B + C \\ J &= BC + CA + AB - D^2 - E^2 - G^2 \\ \Delta &= ABC - AD^2 - CG^2 + 2EDG - BE^2 . \end{aligned} \tag{15.3}$$

The quadric Q represented by (14.2) has centers of symmetry and is homogeneous in X, Y and Z. Thus, its reduced canonical form is given by $\sum_{i=1}^{r} \rho_i z_i = 0$, with $1 \leq r \leq 3$. The following is taken from [15].

<u>Theorem 15.1</u>. <u>If $I \neq 0$, $J = 0$, and $\Delta = 0$, then the quadric Q described by (14.2) is composed of two identical planes π passing through the point P(x,y,z)</u>.

If $I \neq 0$, $J = 0$ and $\Delta = 0$ in (14.2) , it is apparent that two of the characteristic roots are zero. Thus, the rank of the coefficient matrix of (14.2) is one.

The result then follows by appealing to the reduced canonical form.

Theorem 15.2. If $J \neq 0$ and $\Delta = 0$, then the quadric Q described by (14.2) is composed of two distinct intersecting planes or a single line passing through $P(x,y,z)$.

Under the conditions of the hypothesis, it is noted that the rank of the coefficient matrix is two. The conclusion is then a consequence of the reduced canonical form knowing that $r = 2$.

Theorem 15.3. If $\Delta \neq 0$, the quadric cone Q of (14.2) is a non degenerate quadric cone Q . The quadric cone Q is real or imaginary depending on the signs of the ρ_α for $\alpha = 1,2,3$.

If $\Delta \neq 0$, the rank of the coefficient matrix of the quadric Q represented by (14.2) is three. Thus $r = 3$ in the reduced canonical form and the result is obvious.

If $\Delta \neq 0$ and the roots of (14.2) are not of the same sign then the quadric Q is a non-degenerate, real quadric cone Q . This quadric cone Q is right circular if and only if the characteristic equation (15.1) has a double root. The quadric cone Q is elliptic if and only if the characteristic equation (15.1) does not have repeated roots.

Theorem 15.4. If $P(x,y,z)$ is not a flat point of the positional field of force $\vec{F}$ with $|\vec{F}| > 0$, and $\Delta \neq 0$ then the related quadric cone Q of (14.2), is rectangular if and only if

$$\vec{F} \cdot (\nabla \times \vec{F}) = 0 \ , \tag{15.4}$$

at the point $P(x,y,z)$.

If the force vector $\vec{F}$ is represented by $\vec{F} = \phi\vec{i} + \psi\vec{j} + \chi\vec{k}$, with $(\phi^2 + \psi^2 + \chi^2)^{1/2} > 0$, then,

$$\vec{F} \cdot \nabla \times F = \phi\left(\frac{\partial\chi}{\partial y} - \frac{\partial\psi}{\partial z} \right) + \psi\left(\frac{\partial\phi}{\partial z} - \frac{\partial\chi}{\partial x} \right) + \chi\left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y} \right) . \tag{15.5}$$

By rearranging terms and using equation (14.3) it is found that

$$\vec{F} \cdot \nabla \times \vec{F} = \rho(A + B + C) \ , \tag{15.6}$$

where $\rho \neq 0$, is the constant of proportionality in (14.3).

Thus, the inner product of the force vector $\vec{F}$ and the curl $\nabla \times \vec{F}$, is equal to a constant times the trace of the coefficient matrix of the related quadric Q described by (14.2).

The conclusion then follows directly from [21,p.127].

It is also found that if the quadric Q in (14.2) is rectangular, then at least one of the generators of the asymptotic quadric cone Γ has $\pm\frac{1}{\sqrt{3}}$, $\pm\frac{1}{\sqrt{3}}$, $\pm\frac{1}{\sqrt{3}}$, as direction cosines, where the angles are measured from the principal axes of the quadric Q .

CHAPTER V

SOME RESULTS CONCERNING HELICAL TRAJECTORIES IN E_3

16. The theory of helices in a Euclidean space E_3 .

The next definition is found in [25,p.106].

Definition 16.1. A helix H is a curve of at least class three, whose tangent makes a constant angle with a fixed direction.

Let $\vec{A}$ represent a unit vector in the fixed direction and $\vec{T}$ represent the unit tangent vector to the helix H. Then by definition

$$(16.1) \qquad \vec{T} \cdot \vec{A} = \cos \alpha ,$$

where α is a constant.

If the curvature κ is not zero, then clearly $\vec{N} \cdot \vec{A} = 0$ and $\vec{A}$ is in the plane (rectifying) defined by $\vec{T}$ abd $\vec{B}$. This, along with (16.1), implies that

$$(16.2) \qquad \vec{A} = (\cos \alpha)\vec{T} + (\sin \alpha)\vec{B} .$$

Differentiate this result with respect to the arc length s. The result is

$$(16.3) \qquad 0 = (\kappa \cos \alpha - \tau \sin \alpha)\vec{N} .$$

This implies that

(16.4) $$\tau/\kappa = \cot\alpha = \text{constant}.$$

Therefore it has been shown that (16.4) is a necessary condition for a curve to be a helix. This leads to the following well known proposition.

Theorem 16.1. A curve H of at least class three in a Euclidean space E_3 of three dimensions, is a helix if and only if it is either a straight line or else, when $\kappa \neq 0$, then

(16.5) $$\tau/\kappa = \cot\alpha ,$$

where α is a given constant angle.

In the case where H is a straight line ($\kappa = 0$) the result is clear. Henceforth suppose that the curve H is not a straight line ($\kappa \neq 0$) .

Then by equations (16.1) through (16.4) the result listed in (16.5) is found to be a necessary condition for a curve H to be a helix.

Next consider the sufficiency part of the theorem.

If H is a straight line then it is a helix.

Henceforth assume that $\kappa \neq 0$ and that $\tau/\kappa = \cot\alpha$, where α is a constant.

Since $\tau \sin \alpha = \kappa \cos \alpha$, it is seen that

$$(-\tau\vec{N})\sin \alpha + (\kappa\vec{N})\cos \alpha = 0 \; . \tag{16.6}$$

From the Serret-Frenet formulas in (7.3) of Section 7, it is noted that the above equation can be written as

$$\frac{d\vec{B}}{ds}(\sin \alpha) + \frac{d\vec{T}}{ds}(\cos \alpha) = 0 \; . \tag{16.7}$$

Hence,

$$\vec{B}(\sin \alpha) + \vec{T} \cos \alpha = \vec{A} \; ,$$

where $\vec{A}$ is a constant unit vector.

Since $\vec{T} \cdot \vec{A} = \cos \alpha$, it follows that the curve H is a helix.

The next theorem is an exercise in [23].

<u>Theorem 16.2.</u> <u>A curve $\vec{r} = \vec{r}(t)$ of at least class four in a Euclidean space E_3 of three dimensions is a helix if and only if</u>

$$\left(\frac{d^2 r}{ds^2} , \frac{d^3 r}{ds^3} , \frac{d^4 r}{ds^4} \right) = 0 \; . \tag{16.8}$$

For, by the Serret-Frenet formulas, it is seen that

$$\frac{d\vec{r}}{ds} = \vec{T}$$

$$\frac{d^2\vec{r}}{ds^2} = \kappa\vec{N}$$

$$\frac{d^3\vec{r}}{ds^3} = -\kappa^2\vec{T} + \frac{d\kappa}{ds}\vec{N} + \kappa\tau\vec{B} \tag{16.9}$$

$$\frac{d^4\vec{r}}{ds^4} = -3\kappa\frac{d\kappa}{ds}\vec{T} + \left[\frac{d^2\kappa}{ds^2} - \kappa^3 - \kappa\tau^2\right]\vec{N} + \left[\kappa\frac{d\tau}{ds} + 2\tau\frac{d\kappa}{ds}\right]\vec{B} .$$

Therefore

$$\left(\frac{d^2 r}{ds^2}, \frac{d^3 r}{ds^3}, \frac{d^4 r}{ds^4}\right) = \kappa^3\left[\kappa\frac{d\tau}{ds} - \tau\frac{d\kappa}{ds}\right] . \tag{16.10}$$

If (16.8) is satisfied then either $\kappa = 0$, or else, $\kappa \neq 0$, and $\frac{d}{ds}(\tau/\kappa) = 0$.

In the first case ($\kappa = 0$) the curve is a straight line which is a helix by definition.

In the second case, if $\kappa \neq 0$, then $\frac{d}{ds}(\tau/\kappa) = 0$. Hence τ/κ = constant. This implies that the curve H is a helix.

The necessary part of this theorem is an obvious consequence of (16.10) and Theorem 16.1.

17. <u>A study of the conditions for a helical trajectory in a positional field of force in E_3</u>. The results in this and the following two sections are new. The development was suggested by [32], [33] and [34].

It is well known that every plane curve is a helix; and therefore, every dynamical trajectory in a parallel or central field of force is a helix. This suggests the following definition.

<u>Definition 17.1</u>. In a Euclidean space E_3 of three dimensions a curve H is termed a <u>helical trajectory</u> if and only if it is a helix and a dynamical trajectory of some positional field of force $\vec{F}$ with $|\vec{F}| > 0$.

<u>Theorem 17.1</u>. <u>A necessary condition that a dynamical trajectory of a positional field of force $\vec{F}$ with $|\vec{F}| > 0$, be a helix is</u>

$$(17.1) \qquad B_3 - 3\tau N_2 + 4\tau\kappa T_1 = 0 .$$

If such a dynamical trajectory H , is a helix, then τ/κ is a constant and $\frac{d}{ds}(\tau/\kappa) = 0$.

Thus, a necessary condition that a dynamical trajectory H , be a helix is

$$(17.2) \qquad \kappa \frac{d\tau}{ds} - \tau \frac{d\kappa}{ds} = 0 .$$

By substituting the formulas for $\frac{d\tau}{ds}$ and $\frac{d\kappa}{ds}$ found in

(9.5) and (9.3) of Section 9, into the left side of (17.2) and then collecting terms, it is found that

$$(17.3)\qquad \kappa \frac{d\tau}{ds} - \tau \frac{d\kappa}{ds} = \frac{\kappa}{N_1} [B_3 - 3\tau N_2 + 4\tau\kappa T_1] \ .$$

If a dynamical trajectory H is a helix, it follows that the equation (17.1) is satisfied.

The next condition obtained is a consequence of Theorem 16.2 and requires the calculation of $\frac{d^2\vec{r}}{ds^2}$, $\frac{d^3\vec{r}}{ds^3}$ and $\frac{d^4\vec{r}}{ds^4}$. It is known that

$$(17.4)\qquad \begin{aligned} \dot{\vec{r}} &= v \frac{d\vec{r}}{ds} , \\ \ddot{\vec{r}} &= \dot{v} \frac{d\vec{r}}{ds} + v^2 \frac{d^2\vec{r}}{ds^2} . \end{aligned}$$

Since the path is a dynamical trajectory C of the positional field of force $\vec{F}$ with $|\vec{F}| > 0$, then

$$(17.5)\qquad v^2 m \frac{d^2\vec{r}}{ds^2} = \vec{F} - mv \frac{d\vec{r}}{ds} ,$$

where $m > 0$, represents the constant mass of the particle under consideration. As a consequence of equation (8.2) of Section 8, the previous equation can be written as

$$(17.6)\qquad mv^2 \frac{d^2\vec{r}}{ds^2} = \vec{F} - T_1 \frac{d\vec{r}}{ds} .$$

Differentiate (17.6) with respect to arc length s , collect terms and simplify to obtain

$$(17.7)\quad mv^2 \frac{d^3\vec{r}}{ds^3} = \frac{d\vec{F}}{ds} - 3T_1 \frac{d^2\vec{r}}{ds^2} - [T_2 + (\vec{F} \cdot \frac{d^2\vec{r}}{ds^2})] \frac{d\vec{r}}{ds} .$$

Next, multiply through by mv^2 and use (17.6) to reduce (17.7) to

$$(17.8)\quad m^2v^4 \frac{d^3\vec{r}}{ds^3} = mv^2 \frac{d\vec{F}}{ds} - 3T_1\vec{F} + [4T_1^2 - \vec{F}^2 - mv^2T_2] \frac{d\vec{r}}{ds} .$$

The same process can be repeated starting with (17.8) to obtain an expression involving $\frac{d^4\vec{r}}{ds^4}$. That expression is

$$(17.9)\quad \begin{aligned} m^3v^6 \frac{d^4\vec{r}}{ds^4} = {} & m^2v^4 \frac{d^2\vec{F}}{ds^2} - 5mv^2T_1 \frac{d\vec{F}}{ds} \\ & + [19T_1^2 - 4\vec{F}^2 - 4\, mv^2T_2]\vec{F} \\ & + [-28T_1^3 + 13\vec{F}^2T_1 + 12mv^2T_2T_1 \\ & \qquad -3mv^2\vec{F} \cdot \frac{d\vec{F}}{ds} - m^2v^4T_3] \frac{d\vec{r}}{ds} . \end{aligned}$$

By direct calculation, the next result is immediately obvious.

Theorem 17.2. The triple scalar product of $\frac{d^2\vec{r}}{ds^2}$, $\frac{d^3\vec{r}}{ds^3}$, and $\frac{d^4\vec{r}}{ds^4}$ satisfies

$$m^5v^{10}\left(\frac{d^2\vec{r}}{ds^2}, \frac{d^3\vec{r}}{ds^3}, \frac{d^4\vec{r}}{ds^4}\right) = A\left(\vec{F}, \frac{d\vec{F}}{ds}, \frac{d^2\vec{F}}{ds^2}\right)$$

$$+ B\left(\vec{F}, \frac{d\vec{r}}{ds}, \frac{d^2\vec{F}}{ds^2}\right) + C\left(\frac{d\vec{F}}{ds}, \frac{d\vec{r}}{ds}, \frac{d^2\vec{F}}{ds^2}\right) \tag{17.10}$$

$$+ D\left(\frac{d\vec{r}}{ds}, \vec{F}, \frac{d\vec{F}}{ds}\right),$$

where the four expressions A, B, C, D, are

$$A = m^2v^4,$$

$$B = mv^2[T_1^2 - \vec{F}^2 - mv^2T_2]$$

$$C = m^2v^4T_1 \tag{17.11}$$

$$D = -4T_1^3 + 4\vec{F}^2T_1 - 3mv^2\left(\vec{F}\cdot\frac{d\vec{F}}{ds}\right)$$

$$- m^2v^4T_3 + 3mv^2T_2T_1 .$$

It is true that every plane curve is a helix, and consequently every dynamical trajectory in a central or parallel field of force is a helical trajectory. This can be shown through an application of Theorem 17.2.

<u>Theorem 17.3.</u> <u>If a dynamical trajectory C of a positional field of force $\vec{F}$ with $|\vec{F}| > 0$, is planar, then it is a helix.</u>

For $\left(\frac{d\vec{r}}{ds}, \vec{F}, \frac{d\vec{F}}{ds}\right) = 0$, for such a planar trajectory C. Therefore

$$(17.12) \qquad \begin{aligned} \left(\vec{F}, \frac{d\vec{F}}{ds}, \frac{d^2\vec{F}}{ds^2}\right) &= \left(\vec{F}, \frac{d\vec{r}}{ds}, \frac{d^2\vec{F}}{ds^2}\right) \\ &= \left(\frac{d\vec{F}}{ds}, \frac{d\vec{r}}{ds}, \frac{d^2\vec{F}}{ds^2}\right) = 0 \end{aligned}$$

Consequently every planar dynamical trajectory C is a helical dynamical trajectory.

It is conjectured that if every dynamical trajectory is helical then the field of force is either 1-parallel, 0-central or 2-parallel (see Chapter VII).

18. <u>A study of the dynamical trajectories in E_3 that are locally helical</u>. In Theorem 16.2 it was shown that a necessary and sufficient condition for a dynamical trajectory, described by $\vec{r} = \vec{r}(t)$, to be a helix is that

$$(18.1) \qquad \left(\frac{d^2\vec{r}}{ds^2}, \frac{d^3\vec{r}}{ds^3}, \frac{d^4\vec{r}}{ds^4}\right) = 0 ,$$

identically.

It is possible for (18.1) to be satisfied at a particular point on a dynamical trajectory and not be satisfied at all possible points.

This suggests the study of those points on a dynamical trajectory for which (18.1) is satisfied.

Definition 18.1. A point P on a dynamical trajectory C for which (18.1) is satisfied is defined to be a helical point of the dynamical trajectory C .

Definition 18.2. If every dynamical trajectory in a positional field of force has a point P as a helical point, then the point P is a helical point of the positional field of force.

By Theorem 17.3, every point in a central or parallel positional field of force in E_3 is a helical point of the field of force.

The study of helical points on dynamical trajectories in positional fields of force that are not of the central or parallel type is based on a modification of formula (17.10). This modification is accomplished with the tools developed in the next theorem.

Theorem 18.1. In a positional field of force $\vec{F}$ in E_3 with

$$\vec{F} = \phi\vec{i} + \psi\vec{j} + \chi\vec{k} ,$$

if

$$\vec{G} = \left(\frac{dx}{ds}\frac{\partial}{\partial x} + \frac{dy}{ds}\frac{\partial}{\partial y} + \frac{dz}{ds}\frac{\partial}{\partial z} \right)^2\vec{F} ,$$

$$\vec{H} = \phi\frac{\partial\vec{F}}{\partial x} + \psi\frac{\partial\vec{F}}{\partial y} + \chi\frac{\partial\vec{F}}{\partial z} , \tag{18.2}$$

<u>then</u>

$$(18.3) \qquad v^2 \frac{d^2\vec{F}}{ds^2} = v^2\vec{G} + \vec{H} - T_1 \frac{d\vec{F}}{ds} .$$

This result is easily established by first computing $\frac{d^2\vec{F}}{ds^2}$ in the straight forward manner and then substituting for the indicated expressions for $\vec{G}$ and $\vec{H}$.

One additional formula which will prove useful is

$$(18.4) \quad v^2 T_3 = v^2 \frac{d\vec{F}}{ds} \cdot \frac{d\vec{r}}{ds} = v^2\vec{G} \cdot \frac{d\vec{r}}{ds} + \vec{H} \cdot \frac{d\vec{r}}{ds} - T_1 T_2$$

Substitute (18.4) and (18.3) in (17.10) and collect terms to obtain the result stated in the next theorem.

Theorem 18.2. The triple scalar product of $\frac{d^2\vec{r}}{ds^2}$, $\frac{d^3\vec{r}}{ds^3}$, and $\frac{d^4\vec{r}}{ds^4}$ satisfies

$$(18.5)\quad m^5v^{10}\left(\frac{d^2\vec{r}}{ds^2}, \frac{d^3\vec{r}}{ds^3}, \frac{d^4\vec{r}}{ds^4}\right) = Am^2v^4 + Bmv^2 + C,$$

where

$$A = \left(\vec{F}, \frac{d\vec{F}}{ds}, \vec{G}\right) - T_1\left(\frac{d\vec{r}}{ds}, \frac{d\vec{F}}{ds}, \vec{G}\right) + T_2\left(\frac{d\vec{r}}{ds}, \vec{F}, \vec{G}\right)$$
$$- \left(\vec{G}\cdot\frac{d\vec{r}}{ds}\right)\left(\frac{d\vec{r}}{ds}, \vec{F}, \frac{d\vec{F}}{ds}\right),$$

$$B = m\left(\vec{F}, \frac{d\vec{F}}{ds}, \vec{H}\right) - mT_1\left(\frac{d\vec{r}}{ds}, \frac{d\vec{F}}{ds}, \vec{H}\right)$$
$$+ (\vec{F}^2 - T_1^2)\left(\frac{d\vec{r}}{ds}, \vec{F}, \vec{G}\right)$$
$$+ mT_2\left(\frac{d\vec{r}}{ds}, \vec{F}, \vec{H}\right)$$
$$+ \left(3T_2T_1 - 3\vec{F}\cdot\frac{d\vec{F}}{ds}\right.$$
$$\left. - m\vec{H}\cdot\frac{d\vec{r}}{ds}\right)\left(\frac{d\vec{r}}{ds}, \vec{F}, \frac{d\vec{F}}{ds}\right),$$

$$C = m(\vec{F}^2 - T_1^2)\left(\frac{d\vec{r}}{ds}, \vec{F}, \vec{H}\right)$$
$$+ (m-4)(T_1^3 - \vec{F}^2T_1)\left(\frac{d\vec{r}}{ds}, \vec{F}, \frac{d\vec{F}}{ds}\right).$$

From equation (8.2) of Section 8, it is noted that $mv^2 = \rho N_1$, where ρ is the radius of circular curvature of the dynamical trajectory and N_1 is the normal component of the force vector $\vec{F}$ at the point in question.

A point r in the region of definition of a positional field of force and a direction $\frac{d\vec{r}}{ds}$ is called a lineal element [22] and is symbolized by $E(r, \frac{d\vec{r}}{ds})$.

Choose a lineal element $E(r, \frac{d\vec{r}}{ds})$ in the region of definition of the field of force such that $N_1 \neq 0$ at $E(r, \frac{d\vec{r}}{ds})$. The point r of the lineal element $E(r, \frac{d\vec{r}}{ds})$ is a helical point if and only if

$$AN_1^2\rho^2 + BN_1\rho + C = 0 , \tag{18.7}$$

where A, B, and C are as in (17.6).

Equation (18.7) is a quadratic in ρ and has at most two real solutions provided that (18.7) is not an identity in ρ. Thus the following theorem has been proven.

Theorem 18.3. If (18.7) is not an identity in ρ at $E(r, \frac{d\vec{r}}{ds})$ then there are at most two dynamical trajectories C_1 and C_2 (actual or virtual) which pass through the lineal element $E(r, \frac{d\vec{r}}{ds})$ and have the point r as a helical point.

Such a figure may be called a helical lineal element $E(r, \frac{d\vec{r}}{ds})$, of the given positional field of force $\vec{F}$.

19. A further study of the dynamical trajectories in E_3 that satisfy equation (18.7). A closer examination of the possible solutions to equation (18.7) is accomplished by considering the following special cases.

Case 1. If $A = 0$, $B = 0$ and $C \neq 0$ at $E(r, \frac{d\vec{r}}{ds})$, then the quadratic equation (18.7) has $\kappa = \frac{1}{\rho} = 0$ as a double root. Thus there is only one dynamical trajectory C passing through $E(r, \frac{d\vec{r}}{ds})$ with r as a helical point. Since $\kappa = 0$, the point r is also an inflection point.

Case 2. If $A = 0$, $B \neq 0$ and $C = 0$ at an admissible point $E(r, \frac{d\vec{r}}{ds})$, then (18.7) has $\kappa = 0$ and $\rho = 0$ as solutions. This implies that there are two dynamical trajectories C_1 and C_2, passing through $E(r, \frac{d\vec{r}}{ds})$ with r as a helical point. The dynamical trajectory C_1 corresponding to $\kappa = 0$ has r as an <u>inflection point</u>. The point r is a <u>cusp</u> of the dynamical trajectory C_2.

<u>Case 3</u>. If $A \neq 0$, $B = 0$ and $C = 0$ at $E(r, \frac{d\vec{r}}{ds})$ then equation (18.7) has $\rho = 0$ as a double root. Hence there is <u>only one</u> dynamical trajectory C passing through $E(r, \frac{d\vec{r}}{ds})$ with r as a helical point. Since $\rho = 0$, the point r is also a <u>cusp</u> of the dynamical trajectory.

<u>Case 4</u>. If $A = 0$, $B \neq 0$ and $C \neq 0$ at $E(r, \frac{d\vec{r}}{ds})$, then equation (18.7) has $\kappa = 0$ and $\rho = - C/BN_1$ as solutions. Thus there are <u>two</u> dynamical trajectories C_1 and C_2 which pass through $E(r, \frac{d\vec{r}}{ds})$ and have r as a helical point. The dynamical trajectory C_1 corresponding to $\kappa = 0$ also has r as an <u>inflection point</u>.

<u>Case 5</u>. If $A \neq 0$, $B \neq 0$ and $C = 0$ at $E(r, \frac{d\vec{r}}{ds})$, then the two solutions of (18.7) are $\rho = 0$

and $\rho = -B/AN_1$. This implies that there are two dynamical trajectories C_1 and C_2 which pass through $E(r, \frac{d\vec{r}}{ds})$ and have r as a helical point. The dynamical trajectory corresponding to $\rho = 0$ also has r as a cusp.

Case 6. If $A \neq 0$, $B = 0$ and $C \neq 0$ at $E(r, \frac{d\vec{r}}{ds})$, then (18.7) reduces to

$$AN_1^2\rho^2 + C = 0 . \tag{19.1}$$

Equation (19.1) either has no solutions or the solutions are $\rho = \pm(-C/AN_1^2)^{1/2}$. Thus, there are no dynamical trajectories passing through $E(r, \frac{d\vec{r}}{ds})$ with r as a helical point or there are two such dynamical trajectories. If there are two such dynamical trajectories, then one is an actual dynamical trajectory and the other is a virtual dynamical trajectory.

Case 7. If $A \neq 0$, $B \neq 0$ and $C \neq 0$ at $E(r, \frac{d\vec{r}}{ds})$, then the quadratic equation (18.7) has either no real roots, one double root or two distinct roots. Thus the number of dynamical trajectories (actual or virtual) that pass through $E(r, \frac{d\vec{r}}{ds})$ and have r as a helical point is either 0, 1 or 2 . In any case the point r is neither an inflection point nor a cusp of the admissible dynamical trajectories.

CHAPTER VI

HALPHEN'S THEOREM IN A EUCLIDEAN SPACE E_4

20. Conditions for a curve in E_4 to be contained in a k-flat, with k = 2 or 3. Halphen's Theorem in E_4 involves the determination of those force fields that generate k-flat dynamical trajectories, with k = 2 or 3. As a result, necessary and sufficient conditions that a curve C in E_4 be contained in some k-flat, with k = 2 or 3, must be developed.

Consider a curve C in E_4 of at least class 3 that is described by $\vec{r} = \vec{r}(t)$. Suppose this curve is not a straight line and is contained in some 2-flat. Then there exists fixed unit vectors $\vec{A}$ and $\vec{B}$, and scalar constants C_1 and C_2, such that

$$\begin{aligned} \vec{r}(t) \cdot \vec{A} &= C_1 \\ \vec{r}(t) \cdot \vec{B} &= C_2 \end{aligned} \qquad (20.1)$$

where $\vec{A}$ and $\vec{B}$ are not parallel.

Differentiate those equations in (20.1) with respect to time, three times to obtain

$$\begin{aligned} \frac{d^i\vec{r}}{dt^i} \cdot \vec{A} &= 0 \\ \frac{d^i\vec{r}}{dt^i} \cdot \vec{B} &= 0 \,, \end{aligned} \qquad (20.2)$$

for $i = 1,2,3$. This implies that the three vectors $\frac{d\vec{r}}{dt}$, $\frac{d^2\vec{r}}{dt^2}$ and $\frac{d^3\vec{r}}{dt^3}$, are each perpendicular to the plane of $\vec{A}$ and $\vec{B}$. The three vectors must then be linearly dependent and the rank of the matrix

$$(20.3) \qquad M = \begin{bmatrix} \dot{x}_1 & \dot{x}_2 & \dot{x}_3 & \dot{x}_4 \\ \ddot{x}_1 & \ddot{x}_2 & \ddot{x}_3 & \ddot{x}_4 \\ \dddot{x}_1 & \dddot{x}_2 & \dddot{x}_3 & \dddot{x}_4 \end{bmatrix},$$

is two. It can easily be shown that the rank of the matrix in (20.3) is one if and only if the curve C is a straight line.

Suppose the matrix M of (20.3) has rank equal to two. That is, the curve C is not a straight line and row two of M is not a multiple of row one. Thus at least one 2 x 2 submatrix selected from the first two rows of M has determinant different from zero. Suppose without loss in generality, that

$$(20.4) \qquad \dot{x}_1 \ddot{x}_2 - \ddot{x}_1 \dot{x}_2 \neq 0 .$$

Since M has rank two, there exists scalar functions $U_i(t)$ and $V_i(t)$, for $i = 1,2$, such that

$$\dot{x}_3 = U_1(t)\ \dot{x}_1 + U_2(t)\ \dot{x}_2$$

(20.5) $$\ddot{x}_3 = U_1(t)\ \ddot{x}_1 + U_2(t)\ \ddot{x}_2$$

$$\dddot{x}_3 = U_1(t)\ \dddot{x}_1 + U_2(t)\ \dddot{x}_2 \quad ,$$

and

$$\dot{x}_4 = V_1(t)\ \dot{x}_1 + V_2(t)\ \dot{x}_2$$

(20.6) $$\ddot{x}_4 = V_1(t)\ \ddot{x}_1 + V_2(t)\ \ddot{x}_2$$

$$\dddot{x}_4 = V_1(t)\ \dddot{x}_1 + V_2(t)\ \dddot{x}_2$$

From (20.5) and (20.6) it follows that

(20.7) $$\dot{U}_1(t)\ \dot{x}_1 + \dot{U}_2(t)\ \dot{x}_2 = 0$$
$$\dot{U}_1(t)\ \ddot{x}_1 + \dot{U}_2(t)\ \ddot{x}_2 = 0$$

and

(20.8) $$\dot{V}_1(t)\ \dot{x}_1 + \dot{V}_2(t)\ \dot{x}_2 = 0$$
$$\dot{V}_1(t)\ \ddot{x}_1 + \dot{V}_2(t)\ \ddot{x}_2 = 0$$

The only solution to these two systems of equations is the zero solution since $\dot{x}_1\ \ddot{x}_2 - \ddot{x}_1\ \dot{x}_2 \neq 0$. Therefore, it is found that $U_1(t) = C_1$, $U_2(t) = C_2$, $V_1(t) = C_3$ and $V_2(t) = C_4$ where C_1, C_2, C_3 and C_4 are scalar constants. It then follows from (20.5) and (20.6) that

(20.9) $$x_3(t) = C_1\ x_1(t) + C_2\ x_2(t) + C_5$$
$$x_4(t) = C_3\ x_1(t) + C_4\ x_2(t) + C_6 \quad ,$$

where C_5 and C_6 are constants.

Consequently, if the matrix M of (20.3) is of rank two, then the path C described by $\vec{r} = \vec{r}(t)$ is contained in the 2-flat defined by (20.9). Therefore, the following result has been proven.

Theorem 20.1. A necessary and sufficient condition that a curve C described by $\vec{r} = \vec{r}(t)$, of a least class three in E_4 be contained in some 2-flat but not in any 1-flat is that the matrix M of (20.3) have rank equal to two.

Next, consider a curve C in E_4 of at least class four that is described by $\vec{r} = \vec{r}(t)$. Suppose this curve is contained in some 3-flat but not in any 2-flat. Thus, the rank of the matrix M in (20.3) is three. Since C is contained in some 3-flat, there exist a fixed unit vector $\vec{A}$ and a scalar constant C such that

$$\vec{r}(t) \cdot \vec{A} = C \, . \tag{20.10}$$

By differentiating (20.10) four times, with respect to time it is found that

$$\frac{d^i\vec{r}}{dt^i} \cdot \vec{A} = 0 \, , \tag{20.11}$$

for i = 1,2,3,4.

This implies that the four vectors $\frac{d\vec{r}}{dt}$, $\frac{d^2\vec{r}}{dt^2}$, $\frac{d^3\vec{r}}{dt^3}$ and $\frac{d^4\vec{r}}{dt^4}$ are linearly dependent as they are all contained in the same 3-flat. Thus a necessary condition for a curve C defined by $\vec{r} = \vec{r}(t)$, to be contained in a 3-flat and not a 2-flat is that the matrix

$$(20.12)\quad P = \left[\frac{d\vec{r}}{dt} , \frac{d^2\vec{r}}{dt^2} , \frac{d^3\vec{r}}{dt^3} , \frac{d^4\vec{r}}{dt^4} \right] ,$$

have rank equal to three. From Theorem 20.1, it is noted that the curve C is a plane curve if and only if the rank of P is two

Now suppose that the matrix P of (20.12) has rank equal to three. Then the rank of the matrix M of (20.3) is three and at least one 3 x 3 submatrix of M has determinant different from zero. Suppose without loss in generality, that

$$(20.13)\quad \Delta = \begin{vmatrix} \frac{dx_1}{dt} & \frac{dx_2}{dt} & \frac{dx_3}{dt} \\ \frac{d^2x_1}{dt^2} & \frac{d^2x_2}{dt^2} & \frac{d^2x_3}{dt^2} \\ \frac{d^3x_1}{dt^3} & \frac{d^3x_2}{dt^3} & \frac{d^3x_3}{dt^3} \end{vmatrix} \neq 0$$

If P has rank equal to three, then there exists scalar functions $U_i(t)$, for $i = 1,2,3$, such that

$$(20.14)\quad \frac{d^jx_4}{dt^j} = \sum_{i=1}^{3} U_i(t) \frac{d^jx_i}{dt^j} ,$$

for $j = 1,2,3,4$.

From (20.14) the following system of equations in $\dot{U}_1$, $\dot{U}_2$ and $\dot{U}_3$ is easily obtained:

$$(20.15) \qquad \sum_{i=1}^{3} \dot{U}_i(t) \frac{d^j x_i}{dt^j} = 0 ,$$

for j = 1,2,3.

The only solution to the system of equations in (20.15) is the zero solution since the determinant of the coefficient matrix (Δ of 20.13) is not zero. Therefore, it is found that $U_1(t) = C_1$, $U_2(t) = C_2$ and $U_3(t) = C_3$, where C_1, C_2 and C_3 are scalar constants. From the first (j=1) of (20.14), it follows that

$$(20.16) \quad x_4(t) = C_1 x_1 + C_2 x_2 + C_3 x_3 + C_4 ,$$

where C_4 is a constant. This equation is known to describe a 3-flat in E_4. Therefore, the following important result has been proven.

Theorem 20.2. A necessary and sufficient condition that a curve C, described by $\vec{r} = \vec{r}(t)$, of at least class four in E_4 be contained in some 3-flat but not in any 2-flat is that the rank of the matrix P of (20.12) be equal to three.

21. Positional fields of force in E_4 that generate planar dynamical trajectories. Consider a curve C in E_4 that is described by $\vec{r} = \vec{r}(t)$. The curve C is contained

in a plane and not on any straight line if and only if the matrix M of (20.3) has rank equal to two. The following theorem follows immediately from that result.

Theorem 21.1. A dynamical trajectory C given by $\vec{r} = \vec{r}(s)$, with s representing arc length, in a positional field of force $\vec{F}$ with $|\vec{F}| > 0$, in E_4 is planar if and only if the rank of the matrix $\left[\frac{d\vec{r}}{ds}, \vec{F}, \frac{d\vec{F}}{ds}\right]$, is equal to two.

From Theorem 21.1 it is noted that every dynamical trajectory of a positional field of force $\vec{F}$ with $|\vec{F}| > 0$, is planar if and only if the rank of the matrix $\left[\frac{d\vec{r}}{ds}, \vec{F}, \frac{d\vec{F}}{ds}\right]$ is less than or equal to two at every point on each dynamical trajectory.

Theorem 21.2. Every dynamical trajectory C of a positional field of force $\vec{F}$ with $|\vec{F}| > 0$, in E_4 is planar if and only if the field of force is either parallel or central.

The force vector acting at any admissible point is

$$(21.1) \qquad \vec{F} = (F_1, F_2, F_3, F_4) ,$$

where the cartesian components F_i , are functions of position only and are such that $\sum_{i=1}^{4} F_i^2 \neq 0$.

Suppose a dynamical trajectory C in such a force field can be defined by the explicit equations

(21.2) $$x_i = x_i\,(x_1)\ ,$$

for $i = 1,2,3$. Then it is a plane curve if and only if

(21.3) $$\Delta_1 = \begin{vmatrix} 1 & x_2' & x_3' \\ 0 & x_2'' & x_3'' \\ 0 & x_2''' & x_3''' \end{vmatrix} = 0\ ,$$

and

(21.4) $$\Delta_2 = \begin{vmatrix} 1 & x_2' & x_4' \\ 0 & x_2'' & x_4'' \\ 0 & x_2''' & x_4''' \end{vmatrix} = 0\ ,$$

where the primes in (21.3) and (21.4) indicate derivatives with respect to x_1 . This condition follows immediately from Theorem 20.1.

The determinants in (21.3) and (21.4) after expanding and simplifying are found to be equivalent to

(21.5) $$\begin{aligned} x_2''\, x_3''' - x_2'''x_3'' &= 0\ , \\ x_2''\, x_4''' - x_2'''x_4'' &= 0\ . \end{aligned}$$

The first of equations (2.3) indicates that

(21.6) $$\begin{aligned} x_3'' &= K_3\, x_2''\ , \\ x_4'' &= K_4\, x_2''\ . \end{aligned}$$

Substitute the equations in (21.6) and their derivatives in (21.5) to find that

$$x_2'' K_3' = 0 ,$$

(21.7)

$$x_2'' K_4' = 0 .$$

If $x_2'' = 0$, then $x_3'' = 0$ and $x_4'' = 0$, which implies that the curve under consideration is contained on some straight line. Thus every dynamical trajectory in a positional field of force $\vec{F}$ with $|\vec{F}| > 0$ is planar if and only if

(21.8) $$K_i' = 0 ,$$

for $i = 3,4$, where the K_i are described in equations (2.3).

Since $|\vec{F}| > 0$, there is no loss in generality in assuming that $F_1 \neq 0$. Under this supposition the slope functions α_i, $i = 1,2,3$, of this positional field of force are defined by

$$F_2 = \alpha_1 F_1 ,$$

(21.9) $$F_3 = \alpha_2 F_1 ,$$

$$F_4 = \alpha_3 F_1 .$$

The K_i of equations (2.3) can then be described in terms of the slope functions as

$$K_i = \frac{\alpha_{i-1} - x_i'}{\alpha_1 - x_2'} , \tag{21.10}$$

where $\alpha_1 - x_2' \neq 0$ and $i = 3,4$.

The equations in (21.10) can be differentiated and substituted in (21.8) to obtain the following two simultaneous equations.

$$\sum_{j=1}^{4} \sum_{i=1}^{4} A_{ij} \, x_i' \, x_j' = 0 , \tag{21.11}$$

$$\sum_{j=1}^{4} \sum_{i=1}^{4} b_{ij} \, x_i' \, x_j' = 0 , \tag{21.12}$$

where

$$A_{11} = \alpha_1 \frac{\partial \alpha_2}{\partial x_1} - \alpha_2 \frac{\partial \alpha_1}{\partial x_1} ,$$

$$A_{22} = - \frac{\partial \alpha_2}{\partial x_2} ,$$

$$A_{33} = \frac{\partial \alpha_1}{\partial x_3} ,$$

$$A_{44} = 0 ,$$

$$A_{12} = A_{21} = \frac{1}{2}\left[\alpha_1 \frac{\partial\alpha_2}{\partial x_2} - \frac{\partial\alpha_2}{\partial x_1} - \alpha_2 \frac{\partial\alpha_1}{\partial x_2}\right] ,$$

$$A_{13} = A_{31} = \frac{1}{2}\left[\alpha_1 \frac{\partial\alpha_2}{\partial x_3} - \alpha_2 \frac{\partial\alpha_1}{\partial x_3} + \frac{\partial\alpha_1}{\partial x_1}\right] ,$$

$$A_{14} = A_{41} = \frac{1}{2}\left[\alpha_1 \frac{\partial\alpha_2}{\partial x_4} - \alpha_2 \frac{\partial\alpha_1}{\partial x_4}\right] , \tag{21.13}$$

$$A_{23} = A_{32} = \frac{1}{2}\left[- \frac{\partial\alpha_2}{\partial x_3} + \frac{\partial\alpha_1}{\partial x_2}\right] ,$$

$$A_{24} = A_{42} = \frac{1}{2}\left[- \frac{\partial\alpha_2}{\partial x_4}\right] ,$$

$$A_{34} = A_{43} = \frac{1}{2}\left[\frac{\partial\alpha_1}{\partial x_4}\right]$$

and

$$b_{11} = \alpha_1 \frac{\partial\alpha_3}{\partial x_1} - \alpha_3 \frac{\partial\alpha_1}{\partial x_1}$$

$$b_{22} = - \frac{\partial\alpha_3}{\partial x_2}$$

$$b_{33} = 0$$

$$b_{44} = \frac{\partial\alpha_1}{\partial x_4} \tag{21.14}$$

$$b_{12} = b_{21} = \frac{1}{2}\left[\alpha_1 \frac{\partial\alpha_3}{\partial x_2} - \frac{\partial\alpha_3}{\partial x_1} - \alpha_3 \frac{\partial\alpha_1}{\partial x_2}\right]$$

$$b_{13} = b_{31} = \frac{1}{2}\left[\alpha_1 \frac{\partial\alpha_3}{\partial x_3} - \alpha_3 \frac{\partial\alpha_1}{\partial x_3}\right]$$

$$b_{14} = b_{41} = \frac{1}{2}\left[\alpha_1 \frac{\partial\alpha_3}{\partial x_4} - \alpha_3 \frac{\partial\alpha_1}{\partial x_4} + \frac{\partial\alpha_1}{\partial x_4}\right]$$

$$b_{23} = b_{32} = \frac{1}{2}\left[- \frac{\partial\alpha_3}{\partial x_3}\right]$$

$$b_{24} = b_{42} = \frac{1}{2} \left[\frac{\partial\alpha_1}{\partial x_2} - \frac{\partial\alpha_3}{\partial x_4} \right]$$

$$b_{34} = b_{43} = \frac{1}{2} \left[\frac{\partial\alpha_1}{\partial x_3} \right] .$$

The ∞^7 dynamical trajectories of a positional field of force $\vec{F}$ in E_4 are plane curves if and only if the two equations (21.11) and (21.12) are both identities. Thus, the dynamical trajectories are all plane curves if and only if each of the coefficients A_{ij} and b_{ij} for $i = 1,2,3,4$ and $j = 1,2,3,4$, is identically zero.

If $A_{22} = A_{33} = A_{24} = A_{34} = b_{22} = b_{44} = b_{23} = b_{34} = 0$, it follows that

$$\alpha_1 = \alpha_1\ (x_1,x_2) ,$$

(21.15) $$\alpha_2 = \alpha_2\ (x_1,x_3) ,$$

$$\alpha_3 = \alpha_3\ (x_1,x_4) .$$

The next step in the solution process is to consider $A_{23} = b_{24} = 0$. From this condition it is found that α_1 , α_2 and α_3 are linear in x_2 , x_3 and x_4 , respectively, and there exists function $M(x_1)$ and $V_i(x_1)$ for $i = 1,2,3,$ such that

(21.16) $$\alpha_i = M(x_1)\ x_{i+1} + V_i(x_1) ,$$

for $i = 1,2,3.$

Finally, substitute the equations in (21.16) in the identities $A_{12} = 0$, $A_{13} = 0$ and $b_{12} = 0$, to find that

$$
\begin{aligned}
&\frac{dM}{dx_1} + M^2 = 0 \,, \\
&\frac{dV_1}{dx_1} + MV_1 = 0 \,, \\
(21.17)\qquad &\frac{dV_2}{dx_1} + MV_2 = 0 \,, \\
&\frac{dV_3}{dx_1} + MV_3 = 0 \,.
\end{aligned}
$$

If $M(x_1) = 0$, then from (21.17) it is noted that the functions $V_i(x_1)$ for $i = 1,2,3$, are real constants. This implies that

$$(21.18)\qquad \alpha_i = C_i \,,$$

where C_i is a real constant for $i = 1,2,3$. The positional field of force in this situation is then parallel.

Now suppose that $M(x_1) \neq 0$, then the ordinary differential equations in (21.17) can be solved to obtain

$$
\begin{aligned}
&M = \frac{1}{x_1 - d_1} \,, \\
(21.19)\qquad &V_i = \frac{-d_{i+1}}{x_1 - d_1} \,, \text{ for } i = 1,2,3,
\end{aligned}
$$

where the d_i's are real constants.

Therefore, the α_i of (21.16) can be written as

$$\alpha_i = \frac{x_{i+1} - d_{i+1}}{x_1 - d_1},$$

for i = 1,2,3.

Thus, the field of force is <u>central</u> with the point (d_1,d_2,d_3,d_4) as center.

Each of the steps in this development is reversible and hence the converse follows immediately.

In summary it has been proven that every dynamical trajectory in a positional field of force $\vec{F}$ in E_4 with $|\vec{F}| > 0$, is planar if and only if the field of force is either parallel or central.

22. <u>Positional fields of force that generate 3-flat dynamical trajectories</u>. Before proceeding directly to the main result of this section, it will be necessary to extend the concepts of a central and a parallel field of force.

<u>Definition 22.1</u>. A field of force in E_4 is <u>2-dimensional parallel</u> if and only if the straight line determined by each force vector at its point of definition is parallel to a given 2-flat and not parallel to a given 1-flat.

<u>Definition 22.2</u>. If the straight lines determined by the force vectors operating at their points of definition all intersect a given 1-flat, L° , and do not all intersect a fixed point, then the field of force is <u>1-central</u> with L° as center.

<u>Theorem 22.1</u>. <u>Every dynamical trajectory C of a positional field of force $\vec{F}$ with $|\vec{F}| > 0$, in E_4 is contained in some 3-flat but not in any 2-flat if and only if the field of force is either 2-parallel or 1-central.</u>

Suppose the force vector acting at any admissible point is given by (21.1). Suppose further that a dynamical trajectory in such a force field can be defined by the explicit equations

$$x_i = x_i(x_1) , \tag{22.2}$$

for i = 1,2,3.

The dynamical trajectories are contained in some 3-flat if and only if

$$\begin{vmatrix} 1 & x_2' & x_3' & x_4' \\ 0 & x_2'' & x_3'' & x_4'' \\ 0 & x_2''' & x_3''' & x_4''' \\ 0 & x_2^{(iv)} & x_3^{(iv)} & x_4^{(iv)} \end{vmatrix} = 0 , \tag{22.3}$$

where the primes indicate derivatives with respect to x_1 .

This condition follows immediately from Theorem 20.2.

The identity in (22.3) is, with the equations in (21.6), equivalent to the condition that

$$(22.4) \qquad (x_2'')^2 \, [\, K_3'K_4'' - K_3''K_4' \,] = 0 \; .$$

If $x_2'' = 0$ then $x_3'' = x_4'' = 0$. This is true if and only if the dynamical trajectories are all on some straight line. This is impossible by hypothesis. Thus every dynamical trajectory in a positional field of force $\vec{F}$ with $|\vec{F}| > 0$ is contained in some 3-flat if and only if

$$(22.5) \qquad K_3'K_4'' - K_3''K_4' = 0 \; .$$

The condition in (22.5) is satisfied if and only if either

(1) $K_3' = K_4' = 0$,

(2) $K_3' = 0$ and $K_4' \neq 0$,

(3) $K_3' \neq 0$ and $K_4' = 0$, or

(4) $K_3' \neq 0$, $K_4' \neq 0$ and $K_4' = C_0 K_3'$, where C_0 is a real constant,

is true.

Condition (1) is impossible since this, by the proof of Theorem 21.1, implies that the dynamical trajectories must all be plane curves which contradicts the hypothesis.

Conditions (2) and (3) both reduce to an identity similar to (12.3). This identity as shown in Section 12 is equivalent to the requirement that either exactly two of the slope functions α_i , described in (21.9) are constants or two of the three equation

$$(22.6) \qquad \alpha_i = \frac{x_{i+1} - d_{i+1}}{x_1 - d_i}, \text{ for } i = 1,2,3,$$

are satisfied, where the d_i's are real constants. Thus the field force in this situation is either 2-parallel or 1-central.

Suppose that condition (4) is satisfied. Then using the slope functions of (21.9) and the equations in (21.10) this condition is equivalent to the requirement that

$$(22.7) \qquad \sum_{j=1}^{4} \sum_{i=1}^{4} A_{ij}\, x_i'\, x_j' = 0 ,$$

where

$$A_{11} = \alpha_1 \left(\frac{\partial \alpha_3}{\partial x_1} - C_0 \frac{\partial \alpha_2}{\partial x_1} \right) + \frac{\partial \alpha_1}{\partial x_1} (C_0\, \alpha_2 - \alpha_3) ,$$

$$A_{22} = C_0 \frac{\partial \alpha_2}{\partial x_2} - \frac{\partial \alpha_3}{\partial x_2} ,$$

$$A_{33} = -C_0 \frac{\partial \alpha_1}{\partial x_3} ,$$

$$A_{44} = \frac{\partial \alpha_1}{\partial x_4} ,$$

$$A_{12} = A_{21} = \frac{1}{2}\,[\alpha_1\left(\frac{\partial\alpha_3}{\partial x_2} - C_0\frac{\partial\alpha_2}{\partial x_2}\right) + C_0\frac{\partial\alpha_2}{\partial x_1} - \frac{\partial\alpha_3}{\partial x_1} + \frac{\partial\alpha_1}{\partial x_2}(C_0\alpha_2 - \alpha_3)] \,,$$

$$A_{13} = A_{31} = \frac{1}{2}\,[\alpha_1\left(\frac{\partial\alpha_3}{\partial x_3} - C_0\frac{\partial\alpha_2}{\partial x_3}\right) - C_0\frac{\partial\alpha_1}{\partial x_1} + \frac{\partial\alpha_1}{\partial x_3}(C_0\alpha_2 - \alpha_3)] \,,$$

$$(22.8)\qquad A_{14} = A_{41} = \frac{1}{2}\,[\alpha_1\left(\frac{\partial\alpha_3}{\partial x_4} - C_0\frac{\partial\alpha_2}{\partial x_4}\right) + \frac{\partial\alpha_3}{\partial x_1} + \frac{\partial\alpha_1}{\partial x_4}(C_0\alpha_2 - \alpha_3)] \,,$$

$$A_{23} = A_{32} = \frac{1}{2}\,[C_0\frac{\partial\alpha_2}{\partial x_3} - \frac{\partial\alpha_3}{\partial x_3} - C_0\frac{\partial\alpha_1}{\partial x_2}] \,,$$

$$A_{24} = A_{42} = \frac{1}{2}\,[C_0\frac{\partial\alpha_2}{\partial x_4} - \frac{\partial\alpha_3}{\partial x_4} + \frac{\partial\alpha_1}{\partial x_2}] \,,$$

$$A_{34} = A_{43} = \frac{1}{2}\,[\frac{\partial\alpha_1}{\partial x_3} - C_0\frac{\partial\alpha_1}{\partial x_4}] \,.$$

The ∞^7 dynamical trajectories in such a field of force are contained in some 3-flat if and only if equation (22.7) is an identity in direction. Equation (22.7) is an identity in direction if and only if each of the coefficient functions in (22.8) is identically zero.

If $A_{22} = A_{33} = A_{44} = 0$, then it follows that

$$\alpha_1 = \alpha_1(x_1,x_2) \tag{22.9}$$
$$C_0\alpha_2 - \alpha_3 = f(x_1,x_3,x_4) \ .$$

Next, suppose that $A_{32} = A_{24} = 0$. This condition implies that α_1 is linear in x_2 , $C_0\alpha_2 - \alpha_3$ is linear in x_3 and x_4 , and there exists functions $M(x_1)$, $U(x_1)$ and $V(x_1)$ such that

$$\alpha_1 = M(x_1)x_2 + U(x_1) \tag{22.10}$$
$$C_0\alpha_2 - \alpha_3 = C_0\, M(x_1)x_3 - M(x_1)x_4 + V(x_1) \ .$$

Consider the condition resulting from $M(x_1)$ being identically zero. For this case, $A_{12} = A_{13} = 0$ implies that $U(x_1) = C_1$ and $V(x_1) = C_2$, where C_1 and C_2 are real constants. Thus equations (22.10) reduce to

$$\alpha_1 = C_1 \ , \tag{22.11}$$
$$C_0\alpha_2 - \alpha_3 = C_2 \ ;$$

and the force field is recognized to be 2-parallel.

Now suppose that $M(x_1) \neq 0$ and consider $A_{11} = 0$. This implies that $\dfrac{\alpha_1}{C_0\alpha_2 - \alpha_3}$ is independent of x_1 .

Therefore it follows from (22.10) that

$$U(x_1) = -\, b_0\, M(x_1) \ , \tag{22.12}$$
$$V(x_1) = -\, d_0\, M(x_1) \ ,$$

where b_o and d_o are real constants. With this result (22.10) reduces to

$$(22.13)\qquad \begin{aligned} \alpha_1 &= M(x_1)(x_2 - b_o) \,, \\ C_o\alpha_2 - \alpha_3 &= M(x_1)[(C_o x_3 - x_4) - d_o] \,. \end{aligned}$$

If (22.13) is substituted into the equations $A_{12} = 0$, it is found that

$$(22.14)\qquad \frac{dM}{dx_1} + M^2 = 0 \,.$$

The ordinary differential equation in (22.14) is readily solved to obtain

$$(22.15)\qquad M = \frac{1}{x_1 - A_o} \,,$$

where A_o is a real constant.

The equations in (22.13) then reduce to

$$(22.16)\qquad \begin{aligned} \alpha_1 &= \frac{x_2 - b_o}{x_1 - A_o} \,, \\ C_o\alpha_2 - \alpha_3 &= \frac{(C_o x_3 - x_4) - d_o}{x_1 - A_o} \,. \end{aligned}$$

This implies that the field of force is 1-central with center given by $x_1 = A_o$, $x_2 = b_o$ and $C_o x_3 - x_4 = d_o$.

If the field of force is 2-parallel or 1-central, the latter steps can be reversed to find that (22.7) is satisfied. This implies that the trajectories are contained in some 3-flat.

This completes the proof of Halphen's theorem in E_4.

CHAPTER VII

POSITIONAL FIELDS OF FORCE IN E_n THAT RESTRICT PARTICLE MOTION TO (k+1)-FLATS

23. Positional fields of force that are k-dimensional parallel. The concept introduced in this section is an extension of the classical parallel field of force.

Definition 23.1. A field of force in E_n is k-dimensional parallel if and only if the straight line determined by each force vector at its point of definition is parallel to a given k-flat and k is the smallest such integer.

Definition 23.2. The direction μ_k of such a fixed k-flat is defined to be the parallel direction of the k-parallel field of force.

It can be assumed without loss in generality that the parallel direction of a given k-dimensional parallel field of force is defined by

$$x_i = 0 ,$$

for $i = k+1, k+2, \ldots, n$.

Under this assumption, the force vector has the form

$$F = (F_1, F_2, \ldots, F_k \; ; \; 0, 0, \ldots, 0) .$$

The ∞^{2n-1} dynamical trajectories in this situation are represented by the following system of ordinary differential equations:

$$(23.1) \qquad m\ddot{x}_i = \begin{cases} F_i & \text{for } i = 1,2,\dots,k \\ 0 & \text{for } i = k+1,k+2,\dots,n \end{cases},$$

where m is the constant mass of the particle.

<u>Theorem 23.1.</u> <u>In E_n, each dynamical trajectory of a k-dimensional parallel positional field of force with direction μ_k, where $1 \leq k \leq n-1$, is contained in some (k+1) dimensional flat parallel to the direction μ_k.</u>

Suppose that at t = 0 the particle is at $x_0 = [(x_1)_0,(x_2)_0,\dots,(x_n)_0]$, with velocity

$$\dot{x}_0 = [(\dot{x}_1)_0,(\dot{x}_2)_0,\dots,(\dot{x}_n)_0] .$$

The trajectory described by the motion of this particle must be contained in that part of E_n defined by

$$(23.2) \qquad x_i = (x_i)_0 + t(\dot{x}_i)_0 ,$$

for $i = k+1, k+2, \dots, n$.

Now consider the two different situations arising from the velocity vector $\dot{x}_0$.

<u>Case 1.</u> <u>Suppose x_0 is parallel to the direction μ_k.</u> This is equivalent to saying that $(\dot{x}_i)_0 = 0$ for $i = k+1, k+2, \dots, n$. This implies that the dynamical trajectories are contained in

that part of E_n described by

$$(23.3) \qquad x_i = (x_i)_0 ,$$

for $i = k+1, k+2, \ldots, n$.

These equations define a k-flat with direction μ_k which is contained in a (k+1) flat parallel to the direction μ_k .

<u>Case 2.</u> <u>Suppose $\dot{x}_0$ is not parallel to the direction μ_k.</u> Then $\dot{x}_0$ can be expressed in terms of its components in the μ_k direction and a direction perpendicular to μ_k . By a rotation of the axes holding x_i fixed for $1 \leq i \leq k$ this perpendicular direction can be made to coinside with that of the x_{k+1} axis.

Under this assumption it is found that the dynamical trajectories are contained in that part of E_n described by

$$(23.4) \qquad x_i = (x_i)_0 ,$$

for $i = k+2, k+3, \ldots, n$, which defines a (k+1) flat parallel to μ_k .

Thus Case 1 and Case 2 both lead to the described conclusion.

24. <u>An additional property of motion in a k-dimensional parallel positional field of force</u>. The results obtained in this section do not relate directly to Halphen's theorem. They were developed and are presented here to emphasize the validity of Definition 23.1.

<u>Definition 24.1</u>. Let P_1 and P_2 be two distinct points in E_n and through P_1 and P_2 construct k-flats each with direction μ_k. The minimum distance between these two k-flats is defined to be the <u>distance from P_1 to P_2 relative to the direction μ_k</u>.

In the following theorem it will be supposed that P_1 and P_2 do not lie on a straight line parallel to the fixed direction μ_k.

<u>Theorem 24.1</u>. <u>In a k-dimensional parallel field of force in E_n, the time required by a particle in going from P_1 to P_2 along a fixed dynamical trajectory is proportional to the distance from P_1 to P_2 relative to the direction of the force field</u>.

By the previous theorem, every dynamical trajectory in such a field of force is contained in a (k+1) flat parallel to the direction μ_k of the field of force.

Impose a rectangular coordinate system on each of these (k+1) flats such that a point is represented by

$$(24.1) \qquad (x,y) = (x,y_1,y_2,\ldots,y_k) ,$$

where the k-flat defined by $x = 0$ is parallel to the direction μ_k. The force vector in such a $(k+1)$ flat can be expressed as

$$(24.2) \qquad F = (0,F_2,F_3,\ldots,F_{k+1}) ,$$

where F_i is a function of (x,y). The dynamical trajectories of a particle of mass m in such a $(k+1)$ flat are described by

$$(24.3) \qquad \begin{aligned} \ddot{x} &= 0 \\ m\ddot{y}_i &= F_{i+1} , \end{aligned}$$

for $i = 1,2,\ldots,k$.

Suppose that at $t = 0$ the particle is at $[(x)_0,(y_1)_0,(y_2)_0,\ldots,(y_k)_0]$ with velocity

$$[(\dot{x})_0,(\dot{y}_1)_0,\ldots,(\dot{y}_k)_0] .$$

Under these conditions, it is readily seen that the x-component of a point on the dynamical trajectory is given by

$$(24.4) \qquad x = (\dot{x})_0 t + (x)_0 .$$

The distance between points P_1 and P_2 in the $(k+1)$ flat relative to the direction μ_k is, by definition, the difference between the x-components of the two points.

The desired result follows immediately from equation (24.4).

25. A characterization of k-dimensional parallel fields of force. A converse of Theorem 24.1 is established in the next theorem.

Theorem 25.1. Consider a positional field of force in E_n such that

(a) each dynamical trajectory is contained in some (k+1)-flat, and

(b) each such (k+1)-flat is parallel to the k-flats with a fixed direction μ_k .

If the time required in moving on a fixed dynamical trajectory from a point P to a point Q is directly proportional to the distance from P to Q relative to the fixed direction μ_k , then the field of force is k-parallel.

Suppose without loss in generality that the fixed k-dimensional direction μ_k , is that of the k-flat defined by

$$x_i = 0 , \tag{25.1}$$

for k+1, k+2,..., n .

Thus by hypothesis each dynamical trajectory can be described by an equation of the form

$$\vec{r} = (x_1, x_2, \ldots, x_k; y) , \tag{25.2}$$

where y is the component in a direction perpendicular to $\ddot{\mu}_k$.

The time required in moving on such a dynamical trajectory from a point P to a point Q is directly proportional to the distance from P to Q relative to μ_k . This implies that

$$\dot{y} = \text{constant} \ , \text{ and}$$
$$(25.3) \qquad \ddot{y} = 0 \ .$$

This is equivalent to the condition that the force vector at each point in such a force field has a zero component in every direction perpendicular to μ_k . That is, the force field is k-parallel.

26. <u>Positional fields of force that are (k-1) dimensional central</u>. Consider a positional field of force in E_n in which there exists at least k force vectors which are linearly independent where $1 \leq k \leq n-1$.

<u>Definition 26.1</u>. If the straight lines determined by the force vectors operating at their points of definition always intersect a given (k-1)-flat, L^0_{k-1} , and k is the smallest such integer, then the force field is defined to be <u>(k-1)-central with center L^0_{k-1}</u> .

This concept is an extension of the central field of force that is discussed in most classical analytical dynamics texts. (See [12] for example.)

Without loss in generality, it can be assumed that the central flat L^0_{k-1} is that described by

$$x_i = 0 , \tag{26.1}$$

for $i = k, k+1, \ldots, n$.

A force vector in such a situation has the form

$$\vec{F} = (F_1, F_2, \ldots, F_{k-1} ; x_k f(x), x_{k+1} f(x), \ldots, x_n f(x)) \, 0, \tag{26.2}$$

where $f(x)$ is a scalar function that is not identically zero in the region of definition.

<u>Theorem 26.1.</u> <u>In a (k-1) dimensional central positional field of force in E_n with the (k-1) flat L^0_{k-1} as center, each dynamical trajectory is contained in a (k+1) dimensional flat passing through the center L^0_{k-1}</u> .

The dynamical trajectories under these conditions and the assumption of constant mass m are described by

$$\begin{aligned} m\ddot{x}_i &= F_i && \text{for } i = 1,2,\ldots,k-1 \\ m\ddot{x}_i &= x_i f(x) && \text{for } i = k,k+1,\ldots,n . \end{aligned} \tag{26.3}$$

In discussing these equations further, an extension of the method of Appell which was discussed in Chapter II is employed.

To the system (26.3) apply the transformation

$$y_i = \frac{x_i}{x_k} \qquad \text{for } 1 \leq i \leq n \text{ but } i \neq k$$

(26.4)

$$y_k = \frac{1}{x_k} \ ,$$

together with the change in time $dt_1 = \dfrac{dt}{x_k x_k}$.

Note that if the (n-1) flat defined by $x_k = 0$ is not considered, this transformation is a one to one correspondence of the set of remaining points into itself. For the remaining discussion it will supposed that E_n has been extended to a projective space, S_n , by the introduction of an (n-1) flat at infinity. The transformation is then a one to one transformation of S_n into itself.

In what follows the dots over the y's will represent differentiation with respect to t_1 while dots over the x's will represent differentiation with respect to the old time t . It is shown in Chapter II that under this transformation, the differential equations describing the set of dynamical trajectories are of the form

$$(26.5) \qquad m\ddot{y}_i = Y_i$$

for $i = 1,2,\ldots,n$, where Y_i is a function of $y_1, y_2, \ldots, y_n$.

Differentiate equations (26.4) twice with respect to time t_1 to obtain

$$\ddot{y}_i = x_k x_k (x_k \ddot{x}_i - x_i \ddot{x}_k) \text{ for } 1 \leq i \leq n,\ i \neq k$$

$$(26.6) \quad \ddot{y}_k = - x_k x_k \ddot{x}_k \ .$$

Thus, the dynamical trajectories are represented by

$$m\ddot{y}_i = x_k x_k x_k (F_i - x_i f(x)), \text{ for } 1 \leq i \leq k-1 \ ,$$

$$(26.7) \quad m\ddot{y}_k = -x_k x_k x_k f(x) \ ,$$

$$\ddot{y}_j = 0 \ , \qquad \text{for } k+1 \leq j \leq n \ .$$

These are recognized as the differential equations representing the dynamical trajectories in a k-dimensional parallel field of force with direction μ_k defined by

$$(26.8) \qquad y_j = 0 \ ,$$

for $k+1 \leq j \leq n$.

Note that under the discussed transformation the (k-1) dimensional center, L^0_{k-1} , corresponds to the direction μ_k . In other words, every flat L_p passing through the center would after the transformation be a flat of the same dimension parallel to the direction μ_k and conversely.

Since the transformed force field is k-kimensional parallel, an appeal to Theorem 23.1 finds that all the dynamical trajectories lie in a (k+1) flat parallel to the direction, μ_k . Now apply the inverse of the discussed transformation which is itself. This transformation as observed, sends each (k+1) flat parallel to the direction μ_k into a (k+1) flat passing through the center L^0_{k-1} . Thus the result is established.

27. An extension of Kepler's second law of motion. Since the extended form of Kepler's second law of motion is a projective result, it will be necessary to define two additional geometrical concepts.

Definition 27.1. Suppose L_q is the flat of highest dimension that is parallel to both L_k and L_m , where $q \leq k$ and $q \leq m$. Suppose L_q is generated by the orthogonal set of vectors $\{u_1,u_2,\ldots,u_q\}$. Extend this set to an orthogonal basis $\{u_1,u_2,\ldots,u_q, V_{q+1},\ldots,V_k\}$ for L_k and to an orthogonal basis $\{u_1,u_2,\ldots,u_q, W_{q+1},\ldots,W_m\}$ for L_m . L_k is orthogonal to L_m if and only if every V_i for $q+1 \leq i \leq k$, is orthogonal to every W_j for $q+1 \leq j \leq m$.

Impose a rectangular coordinate system on one of the (k+1)-flats described in Theorem 26.1 such that a point is represented by

$$(x_1,x_2,\ldots,x_{k-1};\ x,y) ,$$

where the central flat L^0_{k-1} is given by $x = 0$, $y = 0$.

Definition 27.2. Consider any point P in such a (k+1)-flat. Its projection P_0 , into a 2-flat L_2 , orthogonal to L^0_{k-1} is defined to be the <u>orthogonal projection relative to L^0_{k-1}</u> of the point P on L_2 .

In what follows it will be assumed that the projection plane L_2 is the (x,y) plane.

<u>Theorem 27.1.</u> <u>In a (k-1) dimensional central field of force in E_n with the (k-1) dimensional flat L^0_{k-1} as center the time required in going on a fixed dynamical trajectory from point P to point Q is proportional to the vectorial area O P_0 Q_0 swept out by the radius vector of the orthogonal projection of the dynamical trajectory in the (x,y) plane. The O is the projection of the central flat L^0_{k-1} in the (x,y) plane.</u>

From Theorem 26.1 it is seen that a dynamical trajectory passing through both P and Q must be contained in a (k+1) flat passing through the center L^0_{k-1} . In discussing the dynamical trajectories in such a situation, it will be convenient to use "cylindrical" coordinates, i.e.,

$$x_i = x_i \text{ , for } i = 1,2,\ldots,k-1 \text{ ,}$$

$$(27.1) \qquad x = \rho \cos \Theta \text{ ,}$$

$$y = \rho \sin \Theta \text{ ,}$$

where $\rho = (x^2 + y^2)^{1/2}$ and $\Theta = \tan^{-1}(y/x)$.

In such a (k+1)-flat consider a vector $\lambda = (\lambda_{x_1}, \lambda_{x_2}, \ldots, \lambda_{x_{k-1}}; \lambda_x, \lambda_y)$ from point R to point S where R and S do not lie in a (k-1) flat parallel to L^0_{k-1}. The projection of this vector into the (x,y) plane is the vector $\lambda_0 = (\lambda_x, \lambda_y)$ from R_0 to S_0. By trigonometric techniques and Figure 1, it is easily shown that

$$\begin{aligned} \lambda_x &= \lambda_\rho \cos\Theta - \lambda_\Theta \sin\Theta\,, \\ \lambda_y &= \lambda_\rho \sin\Theta + \lambda_\Theta \cos\Theta\,, \end{aligned} \tag{27.2}$$

where λ_ρ and λ_Θ are the polar components of the vector λ_0.

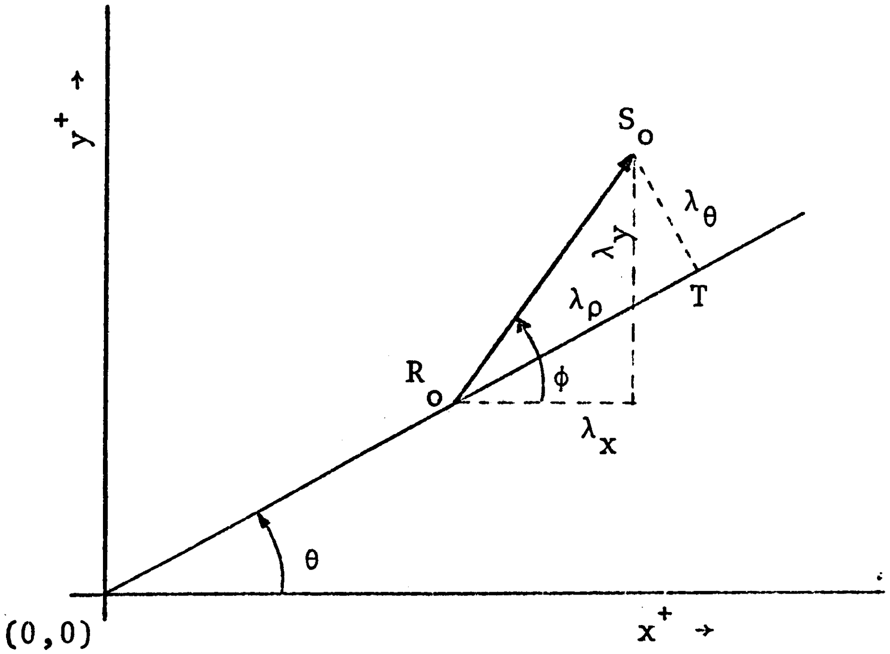

Figure 1

Differentiate $x = \rho \cos \Theta$ and $y = \rho \sin \Theta$ with respect to time t twice and it is found that

$$(27.3)\qquad \begin{aligned} \ddot{x} &= A_x = (\ddot{\rho} - \rho\dot{\Theta}^2)\cos \Theta - (\rho\ddot{\Theta} + 2\dot{\rho}\dot{\Theta})\sin \Theta \\ \ddot{y} &= A_y = (\ddot{\rho} - \rho\dot{\Theta}^2)\sin \Theta + (\rho\ddot{\Theta} + 2\dot{\rho}\dot{\Theta})\cos \Theta . \end{aligned}$$

Thus the polar components of acceleration in the (x,y) plane are given by

$$(27.4)\qquad \begin{aligned} A_\rho &= \ddot{\rho} - \rho\dot{\Theta}^2 \\ A_\Theta &= \rho\ddot{\Theta} + 2\dot{\rho}\dot{\Theta} . \end{aligned}$$

The force field is $(k-1)$ dimensional central with L^0_{k-1} as center and, therefore, A_Θ is zero.

Now $\rho\ddot{\Theta} + 2\dot{\rho}\dot{\Theta} = 0$ implies that

$$(27.4)\qquad \rho^2\dot{\Theta} = c ,$$

where c is a constant. This expression can be written as

$$(27.5)\qquad \rho^2 d\Theta = c\ dt$$

from which the conclusion of the theorem is immediate.

This is an important result in that it represents an extension of Kepler's second law of planetary motion.

28. <u>A characterization of (k-1) central positional fields of force</u>. A converse of Theorem 27.1 is established in the following theorem.

Theorem 28.1. If each dynamical trajectory in a positional field of force is E_n is contained in some (k+1)-flat, L_{k+1}, but not in a k-flat, and there exists a fixed (k-1) flat L^0_{k-1} in each L_{k+1} such that the time required in going on a fixed dynamical trajectory from a point P to a point Q is directly proportional to the vectorial area, $0\, P_0\, Q_0$, swept out by the orthogonal projection of the motion into a plane L_2 perpendicular to L^0_{k-1}, then the force field is (k-1) dimensional central with L^0_{k-1} as center. The point 0 in the plane L_2 is the image of the fixed (k-1) flat L^0_{k-1}.

Consider a dynamical trajectory in such a force field and suppose that the (k+1) flat in which it is contained has a rectangular coordinate system imposed such that the fixed L^0_{k-1} flat is described by

$$x_i = 0 \,, \tag{28.1}$$

for $i = 1,2,\ldots,k-1$.

Thus, the (x_k, x_{k+1}) plane could serve as the L_2 plane.

Now impose a polar coordinate system on the plane L_2 by letting

$$\begin{aligned} x_k &= \rho \cos \Theta \,, \\ x_{k+1} &= \rho \sin \Theta \,, \end{aligned} \tag{28.2}$$

where $\rho = (x_k^2 + x_{k+1}^2)^{1/2}$ and $\Theta = \tan^{-1}(x_{k+1}/x_k)$. The area in the L_2 plane is then known to satisfy

$$dA = \frac{\rho^2}{2}\, d\Theta \quad \text{or}$$

(28.3)
$$\dot{A} = \frac{\rho^2}{2}\, \dot{\Theta}$$

This can be differentiated again and combined with (27.4) to obtain

(28.4)
$$\ddot{A} = \frac{\rho}{2} A_\Theta$$

By hypothesis $t = CA$ where C is a constant. This implies that $\ddot{A} = 0$, which in turn implies that $A_\Theta = 0$.

Since the angular acceleration about the flat L_{k-1}^0 is zero, the line of action of the force vectors must pass through the flat L_{k-1}^0 . Thus, the field of force is (k-1) dimensional central with the fixed flat L_{k-1}^0 as center.

29. <u>An additional property of k-parallel and (k-1) central fields of force in E_n</u> . The property under consideration refers to the coefficient functions K_i that are defined in equations (2.3).

The force vector $\vec{F}$ acting at the point $x = (x_1, x_2, \ldots, x_n)$ is represented by $(F_1, F_2, \ldots, F_n)$. The F_i are the rectangular components of the force vector and are functions of $(x_1, x_2, \ldots, x_n)$.

The force vector is assumed to be not identically equal to the n-dimensional zero vector. Without loss in generality, it can be further assumed that F_1 is not zero. Under these conditions, a set of direction functions describing the direction of the Faraday lines of force is given by

$$(29.1) \qquad 1:\alpha_1:\alpha_2:\ldots:\alpha_{n-1} \,,$$

where

$$\alpha_i = \frac{F_{i+1}}{F_1} \,.$$

The complete set of dynamical trajectories in E_n are described by the equations of Theorem 2.1. Those equations in terms of the direction functions α_i are

$$(29.2) \qquad \begin{aligned} x_i'' &= K_i x_2'' \quad , \text{ for } i = 3,4,5,\ldots,n \\ x_2''' &= G x_2'' + H (x_2'')^2 \,, \end{aligned}$$

where

$$[F_2 - x_2' F_1] G = \sum_{j=1}^{n} x_j' \frac{\partial F_2}{\partial x_j} - x_2' \sum_{j=1}^{n} x_j' \frac{\partial F_1}{\partial x_j} \,,$$

$$H = \frac{-3}{\alpha_1 - x_2'} \,,$$

$$K_i = \frac{\alpha_{i-1} - x_i'}{\alpha_1 - x_2'}$$

and it is assumed that $\alpha_1 - x_2'$ is not zero.

Theorem 29.1. If a positional field of force in E_n is either

(a) k-parallel with k-dimensional direction parallel to that of some coordinate (k-1)-flat,

or

(b) (k-1)-central with center parallel to some coordinate (k-1)-flat,

then (n-k-1) of the (n-2) coefficient functions K_i relative to any dynamical trajectory have their derivatives with respect to x_1 identically equal to zero.

This result is established by considering the parallel and central cases separately.

Case 1. Suppose the force field is k-parallel with direction μ_k parallel to some coordinate (k-1)-flat. Without loss in generality it can be assumed that (n-k+1) of the n direction functions in (29.1) are constant functions. In particular then it can be assumed that

$$\alpha_j = C_j \text{ for } 1 \leq j \leq n-k ,$$

where the C_j are constants.

Then from the expression for K_i given in (29.2) it is noted that

$$(29.3) \qquad K_i = \frac{C_{i-1} - x_i'}{C_1 - x_2'}$$

where $3 \leq i \leq n-k+1$, and $x_2' \neq C_1$.

Differentiate (29.3) with respect to x_1 to obtain

$$(29.4) \quad K_i' = \frac{1}{C_1 - x_2'} [-x_i'' + K_i x_2''] ,$$

for $3 \leq i \leq n-k+1$.

From the first of (29.2) it is found that the right side of (29.4) is identically zero. This establishes the desired result for a k-parallel field of force.

<u>Case 2.</u> <u>Suppose the force field is (k-1) central with a coordinate (k-1) flat as center.</u> Then (n-k+1) of the components of the force vector $\vec{F}$ are such that

$$(29.5) \quad F_i = (x_i + C_i) f(x_i, x_2, \ldots, x_n) ,$$

where $f(x_1, x_2, \ldots, x_n)$ is a scalar function and the C_i are constants.

Without loss in generality, assume that (29.5) is true for $1 \leq i \leq n-k+1$. Then by definition

$$(29.6) \qquad \alpha_{i-1} = \frac{x_i + C_i}{x_1 + C_1} \, ,$$

for $2 \leq i \leq n-k+1$.

Differentiate the expression for K_i given in (29.2) to obtain

$$(29.7) \quad K_i' = \frac{1}{\alpha_1 - x_2'} \left[\alpha_{i-1}' - K_i \alpha_1'\right] \, ,$$

where $3 \leq i \leq n$.

Next, differentiate the expression for α_{i-1} in (29.6) and substitute in (29.7) to obtain

$$(29.8) \quad K_i' = \frac{(x_i' - K_i x_2') - (\alpha_{i-1} - K_i \alpha_1)}{(\alpha_1 - x_2')(x_1 + C_1)} \, ,$$

where $3 \leq i \leq n-k+1$.

The right side of (29.8) seen to be identically equal to zero by considering the expression for K_i given in (29.2). Thus, the first (n-k-1) of the K_i have their derivatives with respect to x_1 equal to zero.

This completes the proof of Theorem 29.1.

BIBLIOGRAPHY

1. E. Kasner, Differential Geometric Aspects of Dynamics, Princeton Colloquium Lectures, published by the American Mathematical Society, 1913, 1934.

2. E. Kasner and J. DeCicco, Generalized Dynamical Trajectories in Space, Duke Mathematical Journal, 10 (1943), pp.733-742.

3. A. Wintner, The Analytical Foundations of Celestial Mechanics, Princeton University Press (1947).

4. F. Brauer and J. Nohel, Ordinary Differential Equations, W. A. Benjamin, Inc. (1967).

5. M. Golomb and M. Shank, Elements of Ordinary Differential Equations, Second Edition McGraw-Hill, (1965).

6. E. Kasner, A Notation for Infinite Manifolds, American Mathematical Monthly, 49 (1942), pp.243-244.

7. E. Kasner and J. DeCicco, A Generalized Theory of Dynamical Trajectories, Bulletin of the American Mathematical Society, 48 (1942), p.833.

8. E. Kasner, The Trajectories of Dynamics, Transaction of the American Mathematical Society, 7 (1906), pp.401-424.

9. E. Kasner, Dynamical Trajectories: The Motion of a Particle in an Arbitrary Field of Force, Transactions of the American Mathematical Society, 8 (1907), pp.135-158.

10. D. M. V. Sommerville, An Introduction to the Geometry of N Dimensions, Dover (1958).

11. E. Sperner and O. Schrier, Modern Algebra and Matrix Theory, Chelsea (1959).

12. E. T. Whittaker, A Treatise on the Analytical Dynamics of Particles and Rigid Motions, Fourth Edition, Dover Publications (1944).

13. J. A. Todd, Projective and Analytical Geometry, Pittman Publishing Corporation (1946).

14.E. Kasner and J. DeCicco, Geometrical Properties of Physical Curves in Space of N Dimensions, Revista de Matematica y Fisica Teorica de la Universidad Nacional del Tucuman, (Argentina), 8 (1951), pp.127-137.

15.D. J. Struik, Analytic and Projective Geometry, Addison-Wesley (1953).

16.P. Appel, De L'Homographie en Mecanique, American Journal of Mathematics, 12 (1890), pp.103-114.

17.E. Kasner and J. DeCicco, Generalizations of Appell's Transformation, Journal of Mathematics and Physics, 27 (1949), pp.262-269.

18.G. E. Shilov, An Introduction to the Theory of Linear Spaces, Translated by R. A. Silverman, Prentice-Hall, Inc. (1961).

19.E. T. Browne, Introduction to the Theory of Determinants and Matrices, University of North Carolina Press (1958).

20.E. Kasner and J. DeCicco, The Geometry of Velocity Systems, Bulletin of the American Mathematical Society, 49 (1943), pp.236-245.

21.D. M. V. Sommerville, Analytical Geometry of Three Dimensions, Cambridge University Press (1959).

22.J. DeCicco, The Riemannian Geometry of Physical Systems of Curves, Annali Di Mathematica, Pura ed Applicata, Serie IV, Tomo LVII (1962), pp.339-403.

23.T. J. Willmore, An Introduction to Differential Geometry, Oxford University Press (1959).

24.W. Graustein, Differential Geometry, Dover (1966).

25.H. Guggenheimer, Differential Geometry, McGraw-Hill (1963).

26.B O'Neill, Elementary Differential Geometry, Academic Press (1966).

27.E. Kreyszig, Differential Geometry, University of Toronto Press (1959).

28.A. Fialkow, Initial Motion in Fields of Force, Transactions of the American Mathematical Society, 40 (1936), pp.495-501.'

29.E. Kasner and D. Mittleman, Extended Theorems in Dynamics, Science, 95 (1942), pp.249-250.

30.E. Kasner and D. Mittleman, A General Theorem of the Initial Curvature of Dynamical Trajectories, Proceedings of the National Academy of Sciences, 28 (1942), pp.48-52.

31.J. DeCicco, Extensions of Certain Dynamical Theorems of Halphen and Kasner, Bulletin of the American Mathematical Society, 49 (1943), pp.736-744.

32.J. DeCicco, Dynamical Trajectories of the Curvature Type, Proceedings of the National Academy of Sciences, 29 (1943), pp.268-270.

33.E. Kasner, Dynamical Trajectories and Curvature Trajectories, Bulletin of the American Mathematical Society, 40 (1934), pp.449-455.

34.J. DeCicco, Dynamical and Curature Trajectories in Space, Transactions of the American Mathematical Society, 57 (1945), pp.270-286.

35.O. Veblen and J. Young, Projective Geometry, Volume 1, Ginn and Company (1910).

36.E. Kasner and J. DeCicco, Transformation Theory of Physical Curves, Proceedings of the National Academy of Sciences, 33 (1947), pp.338-342.

37.R. C. Buck, Advanced Calculus, Second Edition McGraw-Hill, (1965).

38.E. Kasner and A. Fialkow, Geometry of Dynamical Trajectories at a Point of Equilibrium, Transactions of the American Mathematical Society, 41 (1937), pp.314-320.

39.A. Dresden, Solid Analytical Geometry and Determinants, John Wiley and Sons (1946).

40.E. Kasner, Differential Equations of the Type; $y''' = Gy'' + Hy''^2$, Proceedings of the National Academy of Sciences, 28 (1942), pp.333-338.

The vita has been removed from the scanned document

HALPHEN'S THEOREM AND RELATED RESULTS

George Edward Culbertson

Abstract

Halphen's Theorem states that, "A necessary and sufficient condition for every dynamical trajectory in a positional field of force in E_3 to be planar is that the field of force is either parallel or central." This result has been known for some time, however only the sufficiency part of the theorem is widely documented.

A new analytic proof of the necessity part of Halphen's Theorem was developed. The details of this proof motivated the new concepts of a flat point in a field of force and a flat point on a dynamical trajectory in a positional field of force. Several new results related to Halphen's Theorem were then obtained. One such result is a less restrictive version of Halphen's Theorem which states that, "A positional field of force in E_3 is either parallel or central if and only if at each point P there exists six distinct directions, not all of which are on a quadric cone with vertex at P, such that each dynamical trajectory that passes through P in one of these six directions has P as a flat point."

Those positional fields of force that generate helical paths were studied and the new concepts of a helical point of a field of force and a helical point on a dynamical trajectory were defined. One of the new results obtained states that, "If a point P is not a helical point of a positional field of force in E_3, then in each direction through P there passes at most two dynamical trajectories that have P as a helical point."

The definitions of a k-parallel field of force and a (k-1)-central field of force were structured with the goal of extending Halphen's Theorem to higher dimensional space. A k-parallel field of force is one in which the lines of action of the force vectors at their points of definition are all parallel to a given k-flat and k is the smallest such integer. A (k-1)-central field of force is one in which the lines of action of the force vectors at their point of definition all intersects a given (k-1)-flat and k is the smallest such integer.

An extension of Halphen's Theorem to a Euclidean space of dimension four was obtained. That result states that, "Every dynamical trajectory in a positional field of force in E_4 is contained in some k-flat, with $k = 2$ or 3, if and only if the field of force is either (k-1)-parallel or (k-2)-central.

A partial extension of the theorem to Euclidean spaces E_n with $n>4$, was developed. This result states that, "Every dynamical trajectory in a k-parallel or a (k-1)-central field of force in E_n with $k\leq n-1$, is contained in some (k+1)-flat."

Other classical dynamical results in E_3 were extended to E_n through the use of the new concepts of a k-parallel and a (k-1)-central field of force.