PROPERTIES OF
COMPOSITE SAMPLING PROCEDURES

by

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TABLE OF CONTENTS

TABLE OF CONTENTS (cont.)

I. INTRODUCTION

In this dissertation we discuss the problems of evaluating and selecting composite sampling procedures. A typical composite sampling procedure consists of the following steps: (1) take samples of material (increments) from a lot, (2) partition the set of increments into subsets of equal size, (3) mix the increments in each subset to form composite samples, (4) take subsamples from each composite sample, and (5) analyze each subsample. Composite sampling procedures are characterized by the presence of step (3), the physical mixing of sets of increments to form composite samples.

Composite sampling procedures used to estimate lot means (the most common use) are chosen to minimize either the variance of the estimator of the lot mean or a cost function that depends on this variance. Though the variance of the estimator is assumed constant from lot to lot, it is desirable to choose a procedure with which this assumption can be checked. Choosing a procedure is a matter of deciding what composite size and how many composites, subsamples, and tests to use.

Composite sampling procedures are ordinarily employed with bulk materials such as oil, fertilizer, corn, coal, and steel. Measuring the properties of interest in such

materials often entails expensive laboratory tests; there is strong incentive in sampling these materials to select procedures that require as few tests as possible.

The alternative to a composite sampling procedure is a procedure that tests increments individually and estimates the lot mean with the arithmetic average of the test results. This alternative is often impractical due to high testing costs. The physical averaging that occurs in a composite sampling procedure when increments are mixed to form composites permits estimates of required precision with fewer tests, and greater economy, than would be possible using the alternative procedure. One price of this greater economy is that composite sampling procedures are more difficult to evaluate than the alternative procedures.

The customary approach to evaluating a composite sampling procedure, discussed by Duncan in [9, 10], is to identify the sources of variability in the procedure, to set up a random effects linear model that includes each source of variability, and to use this model to find the variance of the estimator of the lot mean. Examples of this approach are given in [8, 9, 10].

In 1972 Brown and Fisher [6] developed a model for composite sampling procedures that is less empirical than the Duncan model. The basis of the Brown-Fisher model is

the fact that a subsample value is a randomly weighted average of the values associated with increments in a composite.

Brown and Fisher used their model to evaluate composite sampling procedures for segmented lots (e.g., lots consisting of bales of wool, or bags of fertilizer). Their results [6] assume that there is only one composite and there is no testing error. Brown and Fisher did not discuss either the differences between their model and Duncan's model or the problems of inference connected with their model. Rohde [17] extended the results in [6] to permit unequal numbers of increments from sampled segments.

In 1976 Rohde [17] used the Brown-Fisher model to evaluate composite sampling procedures for nonsegmented lots (e.g., a tank of oil or a lake). He found the variance of the sample mean and the expected values of the between and within composite mean squares for such procedures, assuming an infinite composite and no testing error. Rohde also found general formulas for variances and covariances of bilinear forms that greatly facilitate application of the Brown-Fisher approach. He did not compare his results with results from the customary model.

There have been two obstacles to the widespread application of Brown-Fisher models. First, these models require much more matrix manipulation to develop than

the customary models. Second, and more important, previous papers on Brown-Fisher models [6, 17] have concentrated on the mechanics of developing these models, without showing their practical importance. To remove these obstacles, Brown-Fisher models are developed here in enough generality to cover the common cases for stationary lots, and the practical need for these models is emphasized.

The remainder of this dissertation is divided into seven chapters. In the next three chapters we discuss in detail balanced composite sampling procedures for nonsegmented lots. The basic Brown-Fisher model for such procedures is derived in Chapter II, allowing for finite composites and for testing error. Results from this Brown-Fisher model are compared with results from the customary model. In Chapter III we discuss questions of inference that arise from the basic Brown-Fisher model. This model is extended in Chapter IV to include two subsampling stages or within-increment variability, complications that are frequently found in actual composite sampling procedures.

Chapter V consists of a discussion in less detail of procedures for balanced composite sampling of segmented lots, allowing for several composites and for testing error. This chapter follows the same course as the previous three chapters, progressing from a derivation of

the basic Brown-Fisher model through a discussion of questions of inference to extensions of the basic model. Results in this chapter are compared with results from the usual model and with results from the Brown-Fisher model for nonsegmented lots.

Problems of general interest for either segmented or nonsegmented lots are discussed in Chapters VI and VII. In Chapter VI we illustrate by example how to use results in earlier chapters to choose a composite sampling procedure. Conclusions based on the Brown-Fisher approach are compared with conclusions drawn from the Duncan approach. In Chapter VII we discuss distributional models for subsampling proportions. After a review of models suggested by Brown and Fisher (hypergeometric) and Rohde (Dirichlet), we propose the singular multivariate normal model. The distribution of subsample values is derived, assuming that subsampling proportions and increment values are normally distributed, and the asymptotic distribution of subsample values is investigated.

The dissertation concludes with a summary of key results in Chapter VIII.

II. NONSEGMENTED LOTS – THE BASIC MODEL

2.1 Introduction

Suppose we take rn increments from a large nonsegmented lot. The increments are fixed-sized portions of material drawn from randomly selected locations in the lot. We form r composites by randomly partitioning the set of rn increments into r subsets of n increments each and physically mixing the increments in each subset. We randomly select s subsamples from each composite and run t analyses on each subsample. It is assumed that each composite contains enough material for S subsamples, with S an integer. Figure 1 illustrates the sampling procedure.

Because of its simplicity, the above procedure is a convenient vehicle for introducing Brown-Fisher models. In succeeding chapters we discuss extensions of this basic procedure that yield models more useful in practical applications of composite sampling. However, the simplicity of the above procedure facilitates both the derivation of fundamental properties of composite sampling procedures and the comparison of Brown-Fisher models with customary models for such procedures; thus we devote Chapters II and III to a discussion of this procedure.

Rohde [17] develops a Brown-Fisher model for the above

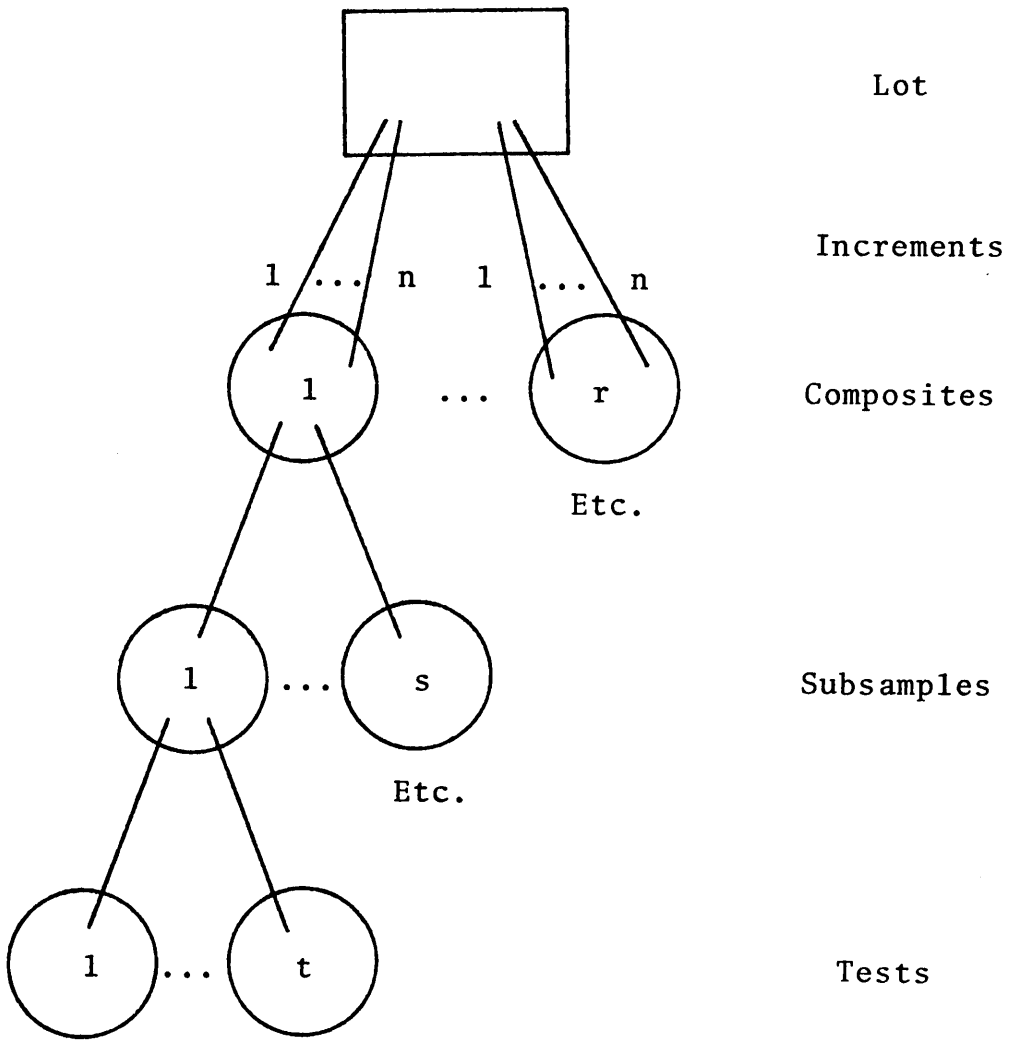

Figure 1

Diagram of Composite Sampling

Procedure for Nonsegmented Lots

procedure and derives formulas for moments of bilinear forms that are useful in general in deriving Brown-Fisher models. In this chapter we review Rohde's results and extend them to allow for measurement error and for procedures with $r>1$, $t>1$, and $S<\infty$. We show that the assumptions on which Rohde's moment formulas are based can be significantly weakened.

Our main contributions in this chapter (aside from the extensions of Rohde's work) are our discussion of the rationale for assumptions in the Brown-Fisher model, our comparison of the Brown-Fisher model with the customary model, and our interpretation of the results of the Brown-Fisher model. We show that subsample values are unbiased estimators of the lot mean and that there is a relative bound on the variance of the estimator of the lot mean from a composite sampling procedure.

2.2 Notation, Assumptions, and Examples

Let $X_{i\ell}$ represent the value associated with any portion of the ℓth increment in the ith composite (we assume in this chapter that there is no within-increment variability). Then write $\underline{X}_i'=(X_{i1},\ldots,X_{in})$ for $i=1,\ldots,r$ and $\underline{X}'=(\underline{X}_1',\ldots,\underline{X}_r')$, and assume that

$$E(\underline{X}) = \mu_X \underline{1} \tag{2.1}$$

and

$$Var(\underline{X}) = \sigma_X^2 I \ , \tag{2.2}$$

where $\underline{1}$ is a column vector of 1's and I is the identity matrix. The lot mean is μ_X, the parameter we want to estimate.

Let g and G represent the amount (e.g., volume) of material in each increment and each subsample, respectively, so that

$$ng = SG \ . \tag{2.3}$$

Let $g_{ij\ell}$, a random variable, be the amount of material from the ℓth increment in the ith composite that appears in the jth subsample from that composite. Note that for every $i=1,\ldots,r$

$$\sum_{\ell=1}^{n} g_{ij\ell} = G \tag{2.4}$$

for $j=1,\ldots,S$ and

$$\sum_{j=1}^{S} g_{ij\ell} = g \tag{2.5}$$

for $\ell=1,\ldots,n$.

Now define

$$\alpha_{ij\ell} = \frac{g_{ij\ell}}{G} \tag{2.6}$$

and write $\underline{\alpha}'_{ij}=(\alpha_{ij1},\ldots,\alpha_{ijn})$ for $i=1,\ldots,r$ and $j=1,\ldots,S$. The random variable $\alpha_{ij\ell}$ is the proportion of the jth subsample from the ith composite that comes from the ℓth increment in that composite. Definition (2.6) and restrictions (2.4) and (2.5) impose the constraints

$$\sum_{\ell=1}^{n} \alpha_{ij\ell} = 1 \tag{2.7}$$

and

$$\sum_{j=1}^{S} \alpha_{ij\ell} = \frac{g}{G} . \tag{2.8}$$

These constraints are used in the next section to find moments of the α's.

It is assumed that there is randomization at each stage of the composite sampling procedure discussed in this chapter. That is, increments are taken from randomly selected locations in the lot, the set of increments is randomly partitioned into composites, the increments in a composite are placed in the blending-subsampling apparatus

in random order, and subsamples are taken from random locations in the composite. The assumption that there is randomization at each stage of the procedure ensures that all increments receive the same treatment in a procedure, which is intuitively appealing and which allows us to assume a simple moment structure for the α's in Section 2.3. The assumption also justifies our use of arbitrary indices for increments, composites, and subsamples.

At this point we introduce the basic idea of Brown and Fisher – that the value associated with the jth subsample from the ith composite is the bilinear form

$$y_{ij} = \underline{\alpha}'_{ij} \underline{X}_i = \sum_{\ell=1}^{n} \alpha_{ij\ell} X_{i\ell} . \tag{2.9}$$

Equation (2.9) indicates that y_{ij} is a randomly weighted average of the values associated with increments in composite i. After the first and second moments of α's are developed in Section 2.3, formula (2.9) is used in Section 2.4 to find the first and second moments of subsample values.

To illustrate the concepts that have been introduced and to show the need for models that are more general than

the one discussed in this chapter, we conclude this section with examples of composite sampling procedures. Rohde [17] describes a procedure for estimating the density of phytoplankton, zooplankton or pollen grains in large bodies of water. Samples of water from random locations in a lake are mixed together to form a composite, then the density for the lake is estimated using the density measurement on a subsample from the composite. Obtaining a density measurement is "costly and time consuming." The sizes of increments and subsamples in this example are determined by volume. The X's and y's in the model correspond to densities of pollen grains in particular increments and subsamples. The subsampling proportion $\alpha_{ij\ell}$ represents the proportion (by volume) of water in the jth subsample from the ith composite that comes from the ℓth increment in that composite. If (as Rohde assumed) there is no variability in density within an increment, the model in this chapter is appropriate for his example.

An example of a composite sampling procedure for which the model discussed in this chapter is inadequate is the U. S. Department of Agriculture procedure for estimating the percent damaged kernels in a barge-load of corn or wheat. A single composite is formed from increments taken by sampling tube (probe) from random locations in the barge, then the composite is mixed and subsampled by passing it

through a riffle. The percent damaged kernels in the barge is estimated by the percent damaged kernels (by weight) in the subsample. Since in general not all kernels in an increment are damaged, there is within-increment variability in this example which is assumed not to exist in the basic Brown-Fisher model. In Chapter IV we extend the basic model to allow for such within-increment variability.

Duncan [9] gives the following description of a procedure for estimating the percent nitrogen in a lot of bagged fertilizer. "... the usual procedure for sampling fertilizer in bags consists in selecting a sample of bags from the lot, taking a small quantity of fertilizer from each bag by means of a sampling tube, mixing these sample increments to form a single composite sample, reducing this composite sample by riffling or hand quartering to a much smaller laboratory sample which is ground to pass a 20 or 40 mesh and bottled for analysis. The laboratory runs nitrogen tests on, say, two small portions from the bottle and reports their mean. This is the estimate of the mean nitrogen content of the lot." This example shows the need for models that allow for two subsampling stages or for segmented lots. The extension of the basic model for nonsegmented lots to allow for two subsampling stages is discussed in Chapter IV. Chapter V discusses Brown-Fisher models for segmented lot procedures, including those

with two subsampling stages.

2.3 Properties of Subsampling Proportions

The first step in developing a Brown-Fisher model is to find the first and second moments of subsampling proportions. The constraints on sums of α's in (2.7) and (2.8) imply that

$$\sum_{\ell=1}^{n} E(\alpha_{ij\ell}) = 1 , \tag{2.10}$$

$$\sum_{\ell=1}^{n} [\alpha_{ij\ell} - E(\alpha_{ij\ell})] = 0 , \tag{2.11}$$

and

$$\sum_{j=1}^{S} [\alpha_{ij\ell} - E(\alpha_{ij\ell})] = 0 . \tag{2.12}$$

As did Brown and Fisher [6] and Rohde [17], we assume that

$$E(\alpha_{ij\ell}) = \mu_{\alpha} \tag{2.13}$$

and

$$Var(\alpha_{ij\ell}) = \sigma_{\alpha}^2 \tag{2.14}$$

for all i, j, and ℓ. We also follow [6] and [17] in assuming that

$$\mathrm{Cov}(\alpha_{ij\ell},\alpha_{ij\ell'}) = \mathrm{Cov}(\alpha_{ij\ell},\alpha_{ij\ell''}) \tag{2.15}$$

for any i, j, and $\ell \neq \ell'$ or ℓ''; that

$$\mathrm{Cov}(\alpha_{ij\ell},\alpha_{ij'\ell}) = \mathrm{Cov}(\alpha_{ij\ell},\alpha_{ij''\ell}) \tag{2.16}$$

for any i, ℓ, and $j \neq j'$ or j''; that

$$\mathrm{Cov}(\alpha_{ij\ell},\alpha_{ij'\ell'}) = \mathrm{Cov}(\alpha_{ij\ell},\alpha_{ij''\ell''}) \tag{2.17}$$

for any i, $j \neq j'$ or j'' and $\ell \neq \ell'$ or ℓ''; and that

$$\mathrm{Cov}(\alpha_{ij\ell},\alpha_{i'j'\ell'}) = 0 \tag{2.18}$$

for any j, ℓ, and $i \neq i'$. Joint moments of X's and α's are discussed in Section 2.4.

Assumptions (2.13) through (2.17) result from the requirement of randomization at each stage of the composite sampling procedure (Brown and Fisher [6] and Rohde [17] justify the above moment structure "by symmetry"). Assumption (2.18) is reasonable whenever the mixing-subsampling processes for different composites are independent, which

requires, for example, thorough cleaning of the mixing-subsampling apparatus after each use.

The physical significance of σ_α^2 is essential to understanding how composite sampling procedures work. A value of σ_α^2 greater than zero indicates that a subsampling procedure differs from the ideal procedure, one in which each subsample consists of equal proportions of material from every increment in the composite. In the ideal procedure $\sigma_\alpha^2=0$, and each subsample value equals the arithmetic average of the increment values in the composite (as a consequence of (2.9) and (2.19), assuming that there is no within-increment variability). The larger σ_α^2, the worse the subsampling procedure; however, there is an upper bound on σ_α^2 so there is a limit to how bad a subsampling procedure can be.

The parameter σ_α^2, unlike the variance components in an ordinary linear model, is not constant; its value depends on n, the number of increments in a composite. We shall show in Theorum 1 that $\sigma_\alpha^2>0$ must eventually decrease as n increases. However, the behavior of σ_α^2 in any finite n-interval is not clear because of two conflicting tendencies, which affect σ_α^2 to different degrees depending on the application. First, a composite is physically harder to mix the larger it is, so σ_α^2 tends to <u>increase</u> with n. Second, a particular increment constitutes a

smaller proportion of a composite the larger the composite is, so σ_α^2 tends to decrease with n (this form of relationship is illustrated in the distributional models for subsampling proportions discussed in Chapter VII). Because the relationship between σ_α^2 and n is generally unknown, choosing an appropriate composite sampling procedure is more difficult than is indicated by an analysis based on the customary linear model.

Let us now return to the problem of finding first and second moments of α's. As shown by Brown and Fisher [6], these moments can be expressed in terms of n, S, and σ_α^2 by combining the Brown-Fisher assumptions with constraints (2.10) to (2.12). Equations (2.10) and (2.13) give

$$E(\alpha_{ij\ell}) = \frac{1}{n} . \tag{2.19}$$

Multiplying (2.11) by $\alpha_{ij\ell'} - E(\alpha_{ij\ell'})$, taking expected values, and using (2.15) gives

$$Cov(\alpha_{ij\ell},\alpha_{ij\ell'}) = \frac{-\sigma_\alpha^2}{(n-1)} . \tag{2.20}$$

Similarly, (2.12) and (2.16) give

$$Cov(\alpha_{ij\ell},\alpha_{ij'\ell}) = \frac{-\sigma_\alpha^2}{(S-1)} , \tag{2.21}$$

and (2.11) and (2.17) give

$$\mathrm{Cov}(\alpha_{ijl},\alpha_{ij'l'}) = \frac{\sigma_\alpha^2}{(n-1)(S-1)} . \qquad (2.22)$$

Summarizing the moments of α's in matrix notation, we have

$$E(\underline{\alpha}_{ij}) = \frac{1}{n}\underline{1} , \qquad (2.23)$$

$$\mathrm{Var}(\underline{\alpha}_{ij}) = \frac{\sigma_\alpha^2}{(n-1)}[nI - J] , \qquad (2.24)$$

and

$$\mathrm{Cov}(\underline{\alpha}_{ij},\underline{\alpha}_{i'j'}) = \begin{cases} \dfrac{-\sigma_\alpha^2}{(n-1)(S-1)}[nI - J] & i=i',\ j\neq j' \\ \\ 0 & i\neq i' \end{cases} \qquad (2.25)$$

(the matrix J in (2.24) and (2.25) is an n x n matrix of 1's).

There are physical meanings and rationales for the properties summarized in (2.23) to (2.25). In equation (2.23) we require that the subsampling procedure be unbiased; that is, that on the average subsamples consist of equal proportions of material from all increments in

the composite. This property is desirable because it means that on the average the weights used in (2.9) are the weights one would use in taking the arithmetic average of the X values.

Equations (2.24) and (2.25) imply that the α's have equal variances and that they can be divided into four categories in terms of covariances: within the same subsample (2.15), within the same increment (2.16), in different increments and subsamples within a composite (2.17), or in different composites (2.18). Covariances between α's in any one category are assumed equal.

The primary reason for assuming the variance-covariance structure given by (2.24) and (2.25) is that it implies that all increments receive the same treatment in the compositing-subsampling process. This variance-covariance structure yields simple and meaningful final results from the Brown-Fisher model. If (2.23) is true, the distributional models for α's to be discussed in Chapter VII also require this structure.

This section concludes with an elementary theorem that provides an upper bound on σ_α^2. This theorem will be used later to find a relative bound on the variance of the estimator of μ_X.

Theorem 1.

For any i, j, and ℓ,

$$\mathrm{Var}(\alpha_{ij\ell}) \leq E(\alpha_{ij\ell})\ [1 - E(\alpha_{ij\ell})] .$$

In particular if $E(\alpha_{ij\ell}) = \frac{1}{n}$,

$$\sigma_\alpha^2 \leq \frac{n-1}{n^2} .$$

Proof:

Since $0 \leq \alpha_{ij\ell} \leq 1$, $\alpha_{ij\ell}^2 \leq \alpha_{ij\ell}$. Thus $E(\alpha_{ij\ell}^2) \leq E(\alpha_{ij\ell})$, and

$$\mathrm{Var}(\alpha_{ij\ell}) = E(\alpha_{ij\ell}^2) - E(\alpha_{ij\ell})^2$$

$$\leq E(\alpha_{ij\ell}) - E(\alpha_{ij\ell})^2 .$$

Substituting 1/n for $E(\alpha_{ij\ell})$ gives

$$\sigma_\alpha^2 \leq \frac{n-1}{n^2} .$$

2.4 Properties of Subsample Values

We have assumed (2.9) that subsample values can be written as bilinear forms. Rohde [17] gives general formulas for first and second moments of bilinear forms consisting of stochastically independent vectors. We begin this section by showing that Rohde's formulas can be obtained with a weaker form of independence defined below. We then apply these formulas to find first and second moments of subsample values.

In deriving formulas for moments of bilinear forms we use the concepts of expectation-, variance-, and covariance-independence discussed by Bohrnstedt and Goldberger [4]. We say that $\underline{w}$ is expectation-independent of $\underline{u}$ if and only if the conditional expectation of $\underline{w}$, $E(\underline{w}|\underline{u})$, is constant for all $\underline{u}$. Similarly, we say that $\underline{w}$ is variance-independent of $\underline{u}$ if and only if $\mathrm{Var}(\underline{w}|\underline{u})$ is constant for all $\underline{u}$. And finally, we say that $\underline{w}_1$ and $\underline{w}_2$ are covariance-independent of $\underline{u}$ if and only if $\mathrm{Cov}(\underline{w}_1,\underline{w}_2|\underline{u})$ is constant for all $\underline{u}$. These forms of independence are considerably weaker than stochastic independence, which requires that all conditional moments of both variables be constant.

Bohrnstedt and Goldberger show that if $\underline{w}$ is expectation- and variance-independent of $\underline{u}$, then $E(\underline{w}|\underline{u}) = E(\underline{w})$

and $\mathrm{Var}(\underline{w}|\underline{u}) = \mathrm{Var}(\underline{w})$. It can also be shown that if $\underline{w}_1$ and $\underline{w}_2$ are covariance-independent of $\underline{u}$, then $\mathrm{Cov}(\underline{w}_1,\underline{w}_2|\underline{u}) = \mathrm{Cov}(\underline{w}_1,\underline{w}_2)$.

The following Lemma expresses first and second moments of bilinear forms in terms of first and second moments of their constituent vectors.

Lemma.

Suppose that the pair $\underline{w}_1,\underline{w}_2$ is expectation-, variance-, and covariance-independent of the pair $\underline{u}_1,\underline{u}_2$, and that $E(\underline{w}_i) = \underline{\mu}_w$, $\mathrm{Var}(\underline{w}_i) = V_w$, $\mathrm{Cov}(\underline{w}_1,\underline{w}_2) = C_w$, $E(\underline{u}_i) = \underline{\mu}_u$, $\mathrm{Var}(\underline{u}_i) = V_u$, and $\mathrm{Cov}(\underline{u}_1,\underline{u}_2) = C_u$ for $i=1,2$. Define $v_1 = \underline{w}_1'\underline{u}_1$, $v_2 = \underline{w}_2'\underline{u}_1$, and $v_3 = \underline{w}_2'\underline{u}_2$. Then

$$E(v_i) = \underline{\mu}_w'\underline{\mu}_u \ , \tag{2.26}$$

$$\mathrm{Var}(v_i) = \underline{\mu}_w'V_u\underline{\mu}_w + \underline{\mu}_u V_w\underline{\mu}_u + \mathrm{tr}(V_uV_w) \ , \tag{2.27}$$

$$\mathrm{Cov}(v_1,v_2) = \underline{\mu}_w'V_u\underline{\mu}_w + \underline{\mu}_u C_w\underline{\mu}_u + \mathrm{tr}(V_uC_w) \ , \tag{2.28}$$

and

$$\mathrm{Cov}(v_1,v_3) = \underline{\mu}_w'C_u\underline{\mu}_w + \underline{\mu}_u C_w\underline{\mu}_u + \mathrm{tr}(C_uC_w) \tag{2.29}$$

for $i=1,2$.

Proof:

Formula (2.26) follows directly from the assumption that $\underline{w}_i$ is expectation-independent of $\underline{u}_j$ for $i=1,2$ and $j=1,2$.

To prove formula (2.29) – proofs for (2.27) and (2.28) are similar – we use a well-known result concerning conditional expectation to write

$$\mathrm{Cov}(v_1,v_3) = E[\mathrm{Cov}(\underline{w}_1'\underline{u}_1, \underline{w}_2'\underline{u}_2 | \underline{u}_1, \underline{u}_2)]$$

$$+ \mathrm{Cov}[E(\underline{w}_1'\underline{u}_1 | \underline{u}_1, \underline{u}_2), E(\underline{w}_2'\underline{u}_2 | \underline{u}_1, \underline{u}_2)].$$

The assumption that $\underline{w}_1, \underline{w}_2$ is expectation- and covariance-independent of $\underline{u}_1, \underline{u}_2$ then gives

$$\mathrm{Cov}(v_1,v_3) = E(\underline{u}_1'C_w\underline{u}_2) + \mathrm{Cov}(\underline{\mu}_w'\underline{u}_1, \underline{\mu}_w'\underline{u}_2).$$

Finally, Searle's formula for the expected value of a quadratic form (Theorem 1s, [19, p. 67]) yields the required result.

Replacing Rohde's assumption that w's and u's are stochastically independent with the assumption that $\underline{w}_1, \underline{w}_2$ is expectation-, variance-, and covariance-independent of $\underline{u}_1, \underline{u}_2$ involves an important weakening of assumptions. In

using the Lemma to find moments of subsample values, we shall see that by limiting our assumptions to conditional first and second moments of subsampling proportions (α's) we broaden the applicability of Brown-Fisher models.

If $\underline{\alpha}_{ij}$ is expectation-independent of $\underline{X}_i$, the mean of a subsample value can be obtained by putting (2.1) and (2.23) into (2.26) to get

$$E(y_{ij}) = \mu_x \ , \tag{2.30}$$

with y_{ij} defined by (2.9).

An interesting new result is that (2.30) can be obtained without assuming (2.23); that is, subsample values are unbiased estimators of μ_x whether or not the subsampling procedure is unbiased. To see this, suppose that

$$E(\alpha_{ij\ell}) = \frac{1}{n} + d_{ij\ell}$$

for all i, j, and ℓ, where $d_{ij\ell}$ is a constant satisfying the constraint required by (2.10); namely,

$$\sum_{\ell=1}^{n} d_{ij\ell} = 0 \ .$$

Then as long as $\underline{\alpha}_{ij}$ is expectation-independent of $\underline{X}_i$,

(2.26) yields (2.30), and there is no need to check whether subsample values are unbiased.

Variances and covariances of subsample values can be found using the other formulas in the Lemma. Assuming that $\underline{\alpha}_{ij}$ is expectation- and variance-independent of $\underline{X}_i$, we obtain

$$\mathrm{Var}(y_{ij}) = \frac{\sigma_x^2}{n} + n\sigma_\alpha^2\sigma_x^2 \qquad (2.31)$$

by putting (2.1), (2.2), (2.23), and (2.24) into (2.27). Similarly, if $\underline{\alpha}_{ij}$ and $\underline{\alpha}_{ij}'$ are expectation- and covariance-independent of $\underline{X}_i$, we can show using (2.28) that

$$\mathrm{Cov}(y_{ij}\ y_{ij}') = \frac{\sigma_x^2}{n} - \frac{n\sigma_\alpha^2\sigma_x^2}{(S-1)} . \qquad (2.32)$$

Using (2.29) and the assumption that $\underline{\alpha}_{ij}$ and $\underline{\alpha}_{i'j'}$ are expectation- and covariance-independent of $\underline{X}_i$, we can show that

$$\mathrm{Cov}(y_{ij}, y_{i'j'}) = 0 \qquad (2.33)$$

for any $i \neq i'$. Each of the above properties holds for the population of subsample values ($i=1,\ldots,r$ and $j=1,\ldots,S$).

To establish the properties of the mean of a sample

from this population, suppose we take a random sample of $s \leq S$ subsamples from each composite and denote the resulting subsample values by $\underline{y}' = (y_{11},\ldots,y_{1s};\ldots;y_{r1},\ldots,y_{rs})$. Suppose further that subsample values can be measured without error (an assumption relaxed in the next section), so that only one measurement per subsample is needed. Then the mean of the rs observations from a lot is

$$\bar{y} = \frac{1}{rs}\,\underline{1}'\underline{y}\ , \tag{2.34}$$

with

$$E(\bar{y}) = \mu_x \tag{2.35}$$

and

$$\mathrm{Var}(\bar{y}) = \frac{\sigma_x^2}{rn} + \left(\frac{S-s}{S-1}\right)\frac{n\sigma_\alpha^2\sigma_x^2}{rs}\ . \tag{2.36}$$

Rohde [17] found the analog of (2.36) for r=1 and infinite S.

2.5 Testing Error

Since in general there will be testing error associated with observing y_{ij}, suppose that instead of y_{ij} we observe

$$z_{ijk} = y_{ij} + e_{ijk} , \tag{2.37}$$

where e_{ijk} is the error in the kth analysis of the jth subsample from the ith composite, for $k=1,\ldots,t$. Write $\underline{e}' = (e_{111},\ldots,e_{11t};\ldots;e_{rs1},\ldots,e_{rst})$ and assume that

$$E(\underline{e}) = \underline{0} ,$$

$$Var(\underline{e}) = \sigma_e^2 I ,$$

and that e's are not correlated with y's. With these assumptions the vector $\underline{z}' = (z_{111},\ldots,z_{11t};\ldots;z_{rs1},\ldots,z_{rst})$ of rst observations from a lot has moments

$$E(\underline{z}) = \mu_x \underline{1} \tag{2.38}$$

and

$$Var(\underline{z}) = diag\ (V,\ldots,V) , \tag{2.39}$$

with

$$V_{st \times st} = c_1 I + c_2 U_{(t)} + c_3 J .$$

The matrix $U_{(t)}$ is defined by

$$U_{(t)} = \text{diag } (J,\ldots,J) , \qquad (2.40)$$

where J is a t x t matrix of 1's. The constants in V are

$$c_1 = \sigma_e^2 ,$$

$$c_2 = \left(\frac{S}{S-1}\right) n\sigma_\alpha^2\sigma_x^2 , \quad \text{and}$$

$$c_3 = \frac{\sigma_x^2}{n} - \frac{n\sigma_\alpha^2\sigma_x^2}{(S-1)} .$$

Using (2.38) and (2.39) it can be shown that the mean and variance of the average of the rst observations from a lot,

$$\bar{z} = \frac{1}{rst} \underline{1}' \underline{z} , \qquad (2.41)$$

are

$$E(\bar{z}) = \mu_x \tag{2.42}$$

and

$$\mathrm{Var}(\bar{z}) = \frac{\sigma_x^2}{rn} + \left(\frac{S-s}{S-1}\right)\frac{n\sigma_\alpha^2\sigma_x^2}{rs} + \frac{\sigma_e^2}{rst} . \tag{2.43}$$

Equation (2.42) shows that $\bar{z}$ is an unbiased estimator of the lot mean. Equation (2.43) gives the variance of $\bar{z}$ in terms of the parameters of the model for a given number of increments, n, and given sampling allocations r, s, and t. If one can estimate the components of (2.43) for given n, one can use (2.43) to find an approximate confidence interval for μ_x or to choose sampling allocations to minimize $\mathrm{Var}(\bar{z})$. These problems are discussed in later chapters. The problem of choosing n is also discussed later.

An important and newly discovered consequence of viewing the composite sampling problem through the Brown-Fisher model is the realization that there are bounds on $\mathrm{Var}(\bar{z})$ relative to comparable noncompositing procedures. Theorem 1 and (2.43) show that under the Brown-Fisher model,

$$\frac{\sigma_x^2}{rn} + \frac{\sigma_e^2}{rst} \leq \mathrm{Var}(\bar{z}) \leq \frac{\sigma_x^2}{rn} + \left(\frac{S-s}{S-1}\right)\left(\frac{n-1}{n}\right)\frac{\sigma_x^2}{rs} + \frac{\sigma_e^2}{rst} . \tag{2.44}$$

In particular, if s=1, then (2.44) reduces to

$$\frac{\sigma_x^2}{rn} + \frac{\sigma_e^2}{rt} \leqq \mathrm{Var}(\bar{z}) \leqq \frac{\sigma_x^2}{r} + \frac{\sigma_e^2}{rt} . \qquad (2.45)$$

The upper bound in (2.45) equals the variance of the arithmetic average of the rt observations obtained by running t tests on each of r uncomposited increments (assuming we can test individual increments with variance σ_e^2). Thus (2.45) shows that in terms of the variance of the estimator of μ_x, a composite sampling procedure with s=1 can be no worse than the comparable alternative procedure in which one tests increments individually. The lower bound in (2.45) indicates how much better the variance for the composite sampling procedure can be. If s>1 and n is large, (2.44) gives

$$\frac{\sigma_e^2}{rst} \leqq \mathrm{Var}(\bar{z}) \leqq \frac{\sigma_x^2}{rs} + \frac{\sigma_e^2}{rst} . \qquad (2.46)$$

The bounds in (2.46) have interpretations similar to those for the bounds in (2.45). Once again, the variance of the estimator of μ_x is not increased by compositing.

2.6 The Usual Linear Model

The customary approach to developing a model for composite sampling from a nonsegmented stationary lot is to assume that the observations z_{ijk} (i=1,...,r; j=1,...,s; k=1,...,t) have the form

$$z_{ijk} = \mu_x + c_i + d_{ij} + e_{ijk} , \tag{2.47}$$

where the composite effect c_i has mean zero and variance σ_c^2, the subsample effect d_{ij} has mean zero and variance σ_d^2, the testing effect e_{ijk} has mean zero and variance σ_e^2, and all these random variables are uncorrelated. This is the usual balanced three stage nested random effects linear model. Since each composite consists of n increments whose values satisfy (2.2), we must have $\sigma_c^2 = \sigma_x^2/n$. It is easily shown that for this model

$$E(\bar{z}) = \mu_x \tag{2.48}$$

and

$$\mathrm{Var}(\bar{z}) = \frac{\sigma_x^2}{rn} + \frac{\sigma_d^2}{rs} + \frac{\sigma_e^2}{rst} , \tag{2.49}$$

with $\bar{z}$ defined by (2.41). Duncan [9, 10] is the principal proponent of this type of model.

The variance given by (2.49) is comparable to the variance given by (2.43) for the Brown-Fisher model. The three terms in (2.43) and (2.49) can be attributed to composites, subsamples, and tests, in that order. Only the subsampling terms differ in the two formulas. Equating the two expressions for $\mathrm{Var}(\bar{z})$ gives

$$\sigma_d^2 = \left(\frac{S-s}{S-1}\right) n\sigma_\alpha^2\sigma_x^2 \ . \qquad (2.50)$$

Thus the parameter σ_d^2 of the usual model is replaced by the expression in (2.50) in the Brown-Fisher model.

The Brown-Fisher model has more statistical appeal than the customary linear model because it reflects the physical process of subsampling a composite more closely than the usual model and it yields a formula for $\mathrm{Var}(\bar{z})$ that reveals a number of properties of composite sampling procedures not suggested by (2.49). We noted in the previous section that the Brown-Fisher model gives relative bounds on $\mathrm{Var}(\bar{z})$. The usual model gives the same lower bound as the Brown-Fisher model, but it gives no upper bound on $\mathrm{Var}(\bar{z})$. The expression for $\mathrm{Var}(\bar{z})$ in (2.43) shows that the size of the subsampling term depends on the variability of increment values (σ_x^2) and on how well composites are subsampled (σ_α^2); when there is perfect sub-

sampling ($\sigma_\alpha^2=0$) or the whole composite is subsampled (s=S), the subsampling term is zero. Equation (2.43) also shows that the subsampling component of $Var(\bar{z})$ is a function of the composite size (n). Perhaps all these properties are recognized and subjectively taken into account by experienced users of the customary model. While this may be so, the Brown-Fisher formulation gives a result that is useful because it summarizes the properties of the variance of the estimator of μ_x in an explicit formula that one can use objectively to evaluate a composite sampling procedure.

The practical consequences of ignoring the information provided by the Brown-Fisher formulation discussed above depend on the situation. For example, omitting the finite composite correction factor (S-s)/(S-1) generally causes one to overestimate $Var(\bar{z})$. However, if s=1 (a common case), then the correction factor is one and no error is made. Ignoring the fact that the subsampling term is a function of σ_x^2 causes one to underestimate the effect of changes in σ_x^2 on $Var(\bar{z})$. The magnitude of the error depends on the size of the coefficient of σ_x^2 in the second term of (2.43). The relationship of $Var(\bar{z})$ to n is not clear in (2.43) because, as noted in Section 2.3, σ_α^2 is an unknown function of n. In choosing a value of n it is important to know whether $Var(\bar{z})$ decreases or increases as n increases. The problem of determining whether $Var(\bar{z})$

decreases with n is discussed in the next chapter, as is the problem of estimating $\mathrm{Var}(\bar{z})$ for given n.

III. NONSEGMENTED LOTS - INFERENCE

3.1 Introduction

In this chapter we discuss problems of inference associated with the composite sampling procedure described in Chapter II; namely, estimating $\mathrm{Var}(\bar{z})$, testing whether $\mathrm{Var}(\bar{z})$ decreases with n, and estimating σ_α^2. The problems of estimating $\mathrm{Var}(\bar{z})$ and testing whether $\mathrm{Var}(\bar{z})$ decreases with n are discussed in Sections 3.2 and 3.3, respectively. These problems are practically important; we nearly always encounter them in evaluating composite sampling procedures. The estimation of σ_α^2 is discussed in Section 3.5. This problem is of more theoretical than practical interest because we can evaluate a composite sampling procedure without estimating σ_α^2.

In Section 3.2 we show how to estimate $\mathrm{Var}(\bar{z})$ using results from an ordinary compositing experiment — one consisting of r composites of n increments each, s subsamples per composite, and t tests per subsample. The form of the estimator of $\mathrm{Var}(\bar{z})$ depends on whether the numbers of subsamples and tests in the procedure being evaluated are the same as the corresponding numbers in the experiment (s and t). If these numbers are different, we must estimate σ_e^2, $\sigma_\alpha^2\sigma_x^2$, and σ_x^2 in order to estimate $\mathrm{Var}(\bar{z})$. If these numbers

are the same, we can estimate $\mathrm{Var}(\bar{z})$ directly. In this case it is shown that r must be large for the experiment to yield a reasonable estimate of $\mathrm{Var}(\bar{z})$.

In Section 3.4 we derive an expression for the probability of obtaining positive variance component estimates in a twofold nested classification. The results of this section indicate that we must take r large in the ordinary compositing experiment to obtain useful (positive) estimates of the components σ_e^2, $\sigma_\alpha^2\sigma_x^2$, and σ_x^2.

We show in Section 3.3 how to test whether $\mathrm{Var}(\bar{z})$ decreases with increasing n using two ordinary compositing experiments (with a different n-value in each). For this test to be reasonably powerful, r must be large.

Though we find in Sections 3.2 through 3.4 that the problems of inference crucial to evaluating composite sampling procedures can be solved using information from ordinary compositing experiments, we show in Section 3.5 that such experiments may not provide information with which to solve the theoretically interesting problem of estimating σ_α^2. The moments of the natural estimator of σ_α^2 from such an experiment do not exist. An alternative experimental procedure for estimating σ_α^2 (suggested by Brown and Fisher [6]) is investigated.

The results of this chapter support Duncan's statement [9] that "extensive research" is required to evaluate a

composite sampling procedure. These results show that whether the Duncan or the Brown-Fisher model is postulated, large experiments are necessary to effectively evaluate these procedures.

3.2 Estimating Var($\bar{z}$)

The objective of this section is to find the best unbiased estimator of the variance of the estimator of μ_x from a composite sampling procedure with n increments per composite and sampling allocations r', s', and t'. It is assumed that we have run an experiment consisting of r composites of n increments each, with s subsamples per composite and t tests per subsample, and with (r, s, t) not necessarily equal to (r', s', t'). It is also assumed that the vector $\underline{z}$ of observations from the experiment, defined as in Section 2.5, has a multivariate normal distribution with mean vector and covariance matrix given by (2.38) and (2.39). This assumption is justified by the central limit property of subsample values discussed in [5, 17] and in Chapter VII and the definition of z_{ijk} given in (2.37). The required estimator is developed by finding the complete sufficient statistic ($\underline{T}$) for the parameters of the model, then finding a function of $\underline{T}$ that is an

unbiased estimator of the quantity of interest.

By Theorem 3.8 in Graybill [12] the density function of $\underline{z}$ is

$$g(\underline{z}) = (2\pi)^{-\frac{rst}{2}} |V|^{-\frac{r}{2}} \exp\left[-\frac{1}{2}\sum_{i=1}^{r}(\underline{z}_i-\mu_x\underline{1})'V^{-1}(\underline{z}_i-\mu_x\underline{1})\right] \tag{3.1}$$

with $\underline{z}_i'=(z_{i11},\ldots,z_{i1t};\ldots;z_{is1},\ldots,z_{ist})$. A more convenient expression for $g(\underline{z})$ is found with the aid of explicit forms of $|V|$ and V^{-1}. By the method of proof employed in Theorem 8.3.4 of Graybill [13] it can be shown that

$$|V| = w_3^{s(t-1)} w_2^{s-1} w_1 , \tag{3.2}$$

with

$$w_3 = c_1$$

$$= \sigma_e^2 ,$$

$$w_2 = c_1 + tc_2$$

$$= \sigma_e^2 + t\left(\frac{S}{S-1}\right)n\sigma_\alpha^2\sigma_x^2 , \tag{3.3}$$

and

$$w_1 = c_1 + tc_2 + stc_3$$

$$= \sigma_e^2 + t\left(\frac{S-s}{S-1}\right)n\sigma_\alpha^2\sigma_x^2 + \frac{st\sigma_x^2}{n}$$

(c_1, c_2, and c_3 are defined in Section 2.5). Note that $|V| \neq 0$ if $\sigma_e^2 \neq 0$. It is easy to verify that

$$V^{-1} = \frac{1}{w_3} I + \frac{1}{t}\left(\frac{1}{w_2} - \frac{1}{w_3}\right) U_{(t)} + \frac{1}{st}\left(\frac{1}{w_1} - \frac{1}{w_2}\right) J \ . \tag{3.4}$$

Putting (3.2) and (3.4) into (3.1) and simplifying yields

$$g(\underline{z}) = (2\pi)^{-\frac{rst}{2}} w_3^{-\frac{rs(t-1)}{2}} w_2^{-\frac{r(s-1)}{2}} w_1^{-\frac{r}{2}} \tag{3.5}$$

$$\times \exp\left[-\frac{1}{2}\left(\frac{rst(\bar{z}-\mu_x)^2}{w_1} + \frac{SS_B}{w_1} + \frac{SS_W}{w_2} + \frac{SS_E}{w_3}\right)\right]$$

with $\bar{z}$ defined by (2.41),

$$SS_B = \underline{z}' A_B \underline{z}$$

$$= \frac{1}{st} \sum_{i=1}^{r} \left(\sum_{j=1}^{s} \sum_{k=1}^{t} z_{ijk} \right)^2 - rst\, \bar{z}^2 ,$$

$$SS_W = \underline{z}' A_W \underline{z}$$

$$= \frac{1}{t} \sum_{i=1}^{r} \sum_{j=1}^{s} \left(\sum_{k=1}^{t} z_{ijk} \right)^2 \qquad (3.6)$$

$$- \frac{1}{st} \sum_{i=1}^{r} \left(\sum_{j=1}^{s} \sum_{k=1}^{t} z_{ijk} \right)^2 , \quad \text{and}$$

$$SS_E = \underline{z}' A_E \underline{z}$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{s} \sum_{k=1}^{t} z_{ijk}^2$$

$$- \frac{1}{t} \sum_{i=1}^{r} \sum_{j=1}^{s} \left(\sum_{k=1}^{t} z_{ijk} \right)^2 .$$

The matrices of the quadratic forms in (3.6) are

$$A_B = \frac{1}{st} U_{(st)} - \frac{1}{rst} J ,$$

$$A_W = \frac{1}{t} U_{(t)} - \frac{1}{st} U_{(st)} , \quad \text{and} \tag{3.7}$$

$$A_E = I - \frac{1}{t} U_{(t)} .$$

These quadratic forms are the usual sums of squares for the balanced twofold nested classification. SS_B, SS_W, and SS_E are the sums of squares between composites, between subsamples within composites, and between tests within subsamples, respectively.

The expression for $g(\underline{z})$ in (3.5) satisfies the factorization criterion discussed in Volume 2 of Kendall and Stuart [14, p. 27], so the components of $\underline{T}' = (\bar{z}, SS_B, SS_W, SS_E)$ are jointly sufficient statistics for the parameters of the model. Next we seek the joint distribution of the components of $\underline{T}$.

By Theorem 3.6 in Graybill [12] $\bar{z}$ is normally distributed with mean and variance given by (2.42) and (2.43). By Theorem 3 in Searle [19] $\bar{z}$ is independent of SS_B, SS_W, and SS_E, since

$$\underline{1}'\mathrm{Var}(\underline{z})A_B = \underline{1}'\mathrm{Var}(\underline{z})A_W = \underline{1}'\mathrm{Var}(\underline{z})A_E = \underline{0}' \ .$$

Theorem 5 in Searle [19] is used to find the joint distribution of SS_B, SS_W, and SS_E. It can be shown by matrix multiplication that $A_B\mathrm{Var}(\underline{z})/w_1$, $A_W\mathrm{Var}(\underline{z})/w_2$, and $A_E\mathrm{Var}(\underline{z})/w_3$ are (symmetric) idempotent matrices, so Theorem 1.63 in Graybill [12] gives $\mathrm{rank}(A_B) = r-1$, $\mathrm{rank}(A_W) = r(s-1)$, and $\mathrm{rank}(A_E) = rs(t-1)$. More matrix multiplication shows that

$$A_B\mathrm{Var}(\underline{z})A_W = A_B\mathrm{Var}(\underline{z})A_E = A_W\mathrm{Var}(\underline{z})A_E = 0$$

and

$$\underline{1}'A_B\underline{1} = \underline{1}'A_W\underline{1} = \underline{1}'A_E\underline{1} = 0 \ .$$

Thus (a) and (b) of part 1 of Searle's theorem are true, and we conclude that SS_B, SS_W, and SS_E are independent with central chi-square distributions:

$$\begin{aligned} SS_B/w_1 &\sim \chi^2_{r-1} \ , \\ SS_W/w_2 &\sim \chi^2_{r(s-1)} \ , \quad \text{and} \\ SS_E/w_3 &\sim \chi^2_{rs(t-1)} \ . \end{aligned} \tag{3.8}$$

The expected mean squares are (by Theorem 1 in Searle [19])

$$E(MS_B) = w_1 ,$$

$$E(MS_W) = w_2 , \quad \text{and} \tag{3.9}$$

$$E(MS_E) = w_3 .$$

The properties of $\underline{T}$ just developed indicate that the pdf of $\underline{T}$ has the form specified by Gautschi [11], so by his lemma $\underline{T}$ is a complete sufficient statistic. Now

$$\hat{\mu}_x = \bar{z} ,$$

$$\hat{\sigma}_e^2 = MS_E ,$$

$$\widehat{\sigma_\alpha^2\sigma_x^2} = \left(\frac{S-1}{ntS}\right)\left[MS_W - MS_E\right] , \quad \text{and} \tag{3.10}$$

$$\hat{\sigma}_x^2 = \frac{n}{st}\left[MS_B - \left(\frac{S-s}{S}\right) MS_W - \left(\frac{s}{S}\right) MS_E\right]$$

are unbiased estimators of μ_x, σ_e^2, $\sigma_\alpha^2\sigma_x^2$, and σ_x^2, respectively, so the estimators in (3.10) are best (minimum variance) unbiased estimators of their expected values.

The estimators in (3.10) are used if we run an experiment with certain values of r, s, and t, but want to estimate $\mathrm{Var}(\bar{z})$ for a procedure with different values, say r', s', and t'. The best unbiased estimator of $\mathrm{Var}(\bar{z})$ in this case is

$$\widehat{\mathrm{Var}(\bar{z})} = \frac{\hat{\sigma}_x^2}{r' n} + \left(\frac{S-s'}{S-1}\right)\frac{n\hat{\sigma}_\alpha^2\hat{\sigma}_x^2}{r' s'} + \frac{\hat{\sigma}_e^2}{r' s' t'} . \qquad (3.11)$$

Normally $s=s'=t=t'=1$, but even if $s=s'$ and $t=t'$, (3.11) reduces to

$$\widehat{\mathrm{Var}(\bar{z})} = \frac{MS_B}{r' st} . \qquad (3.12)$$

By (3.8), the standard deviation of the estimator in (3.12) is

$$\sqrt{\frac{2}{r-1}}\ \mathrm{Var}(\bar{z}) . \qquad (3.13)$$

Unless r is large, (3.13) indicates that $\widehat{\mathrm{Var}(\bar{z})}$ defined by (3.12) is not a useful estimator of $\mathrm{Var}(\bar{z})$.

If we run the same experiment but use the customary model (the one discussed in Section 2.6) for analysis rather than the Brown-Fisher model, the best unbiased

estimator of $\mathrm{Var}(\bar{z})$ for the case corresponding to (3.11) is

$$\widehat{\mathrm{Var}(\bar{z})} = \frac{\hat{\sigma}_x^2}{r\ n} + \frac{\hat{\sigma}_d^2}{r\ s} + \frac{\hat{\sigma}_e^2}{r\ s\ t}\ , \qquad (3.14)$$

with

$$\hat{\sigma}_e^2 = MS_E\ ,$$

$$\hat{\sigma}_d^2 = \frac{1}{t}\left[MS_W - MS_E\right]\ , \quad \text{and} \qquad (3.15)$$

$$\hat{\sigma}_x^2 = \frac{n}{st}\left[MS_B - MS_W\right]\ .$$

If $s=s'$ and $t=t'$, (3.14) reduces to (3.12). Thus whether the Duncan or the Brown-Fisher approach is followed in this case, the best unbiased estimator of $\mathrm{Var}(\bar{z})$ is given by (3.12), and this estimator is not a useful one unless r is large.

3.3 Testing Whether Var($\bar{z}$) Decreases With Increasing n

The formula for Var($\bar{z}$) derived in Section 2.6 from the usual model indicates that Var($\bar{z}$) decreases as n increases, so it appears that there is no need for the test discussed in this section. However, we know from the Brown-Fisher formulation that Var($\bar{z}$) can decrease or increase with n, depending on how σ_α^2 varies with n. For this reason, a test of whether Var($\bar{z}$) decreases with n is needed.

Figures 2 and 3 illustrate the range of possible relationships between Var($\bar{z}$) and n. Consider a composite sampling procedure with s=t=1 and with values of g and G (defined in Section 2.2) such that n can be any integer 2 or greater. For σ_α^2 constant – σ_α^2 can be constant as long as it satisfies $\sigma_\alpha^2 \leq (n-1)/n^2$– Figure 2 shows $rVar(\bar{z})/\sigma_e^2$ versus n for $\sigma_x^2/\sigma_e^2 = 0, 1, 4$. For

$$\sigma_\alpha^2 = \frac{(n-1)}{n^2(nv+1)}$$

(this formula for σ_α^2 comes from the Dirichlet model for $\underline{\alpha}_{ij}$ discussed in Chapter VII), Figure 3 shows $rVar(\bar{z})/\sigma_e^2$ versus n for $\sigma_x^2/\sigma_e^2 = 0, 1, 4$ and $v = 0, 1/4, \infty$. We do not claim that either figure represents the true relationship between Var($\bar{z}$) and n for a particular application,

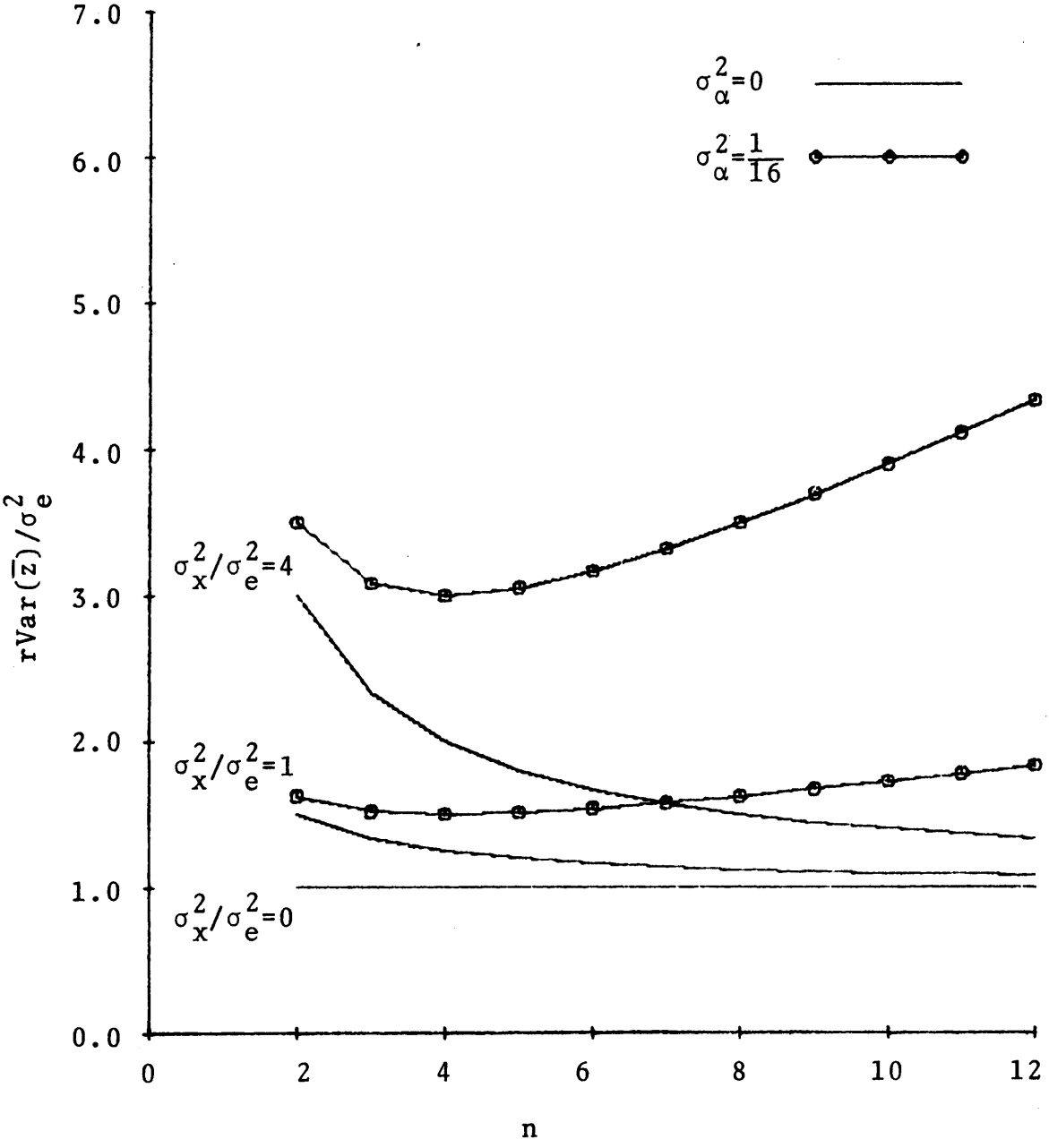

Figure 2. Plot of $rVar(\bar{z})/\sigma_e^2$ Versus n for Constant σ_α^2 and s=t=1

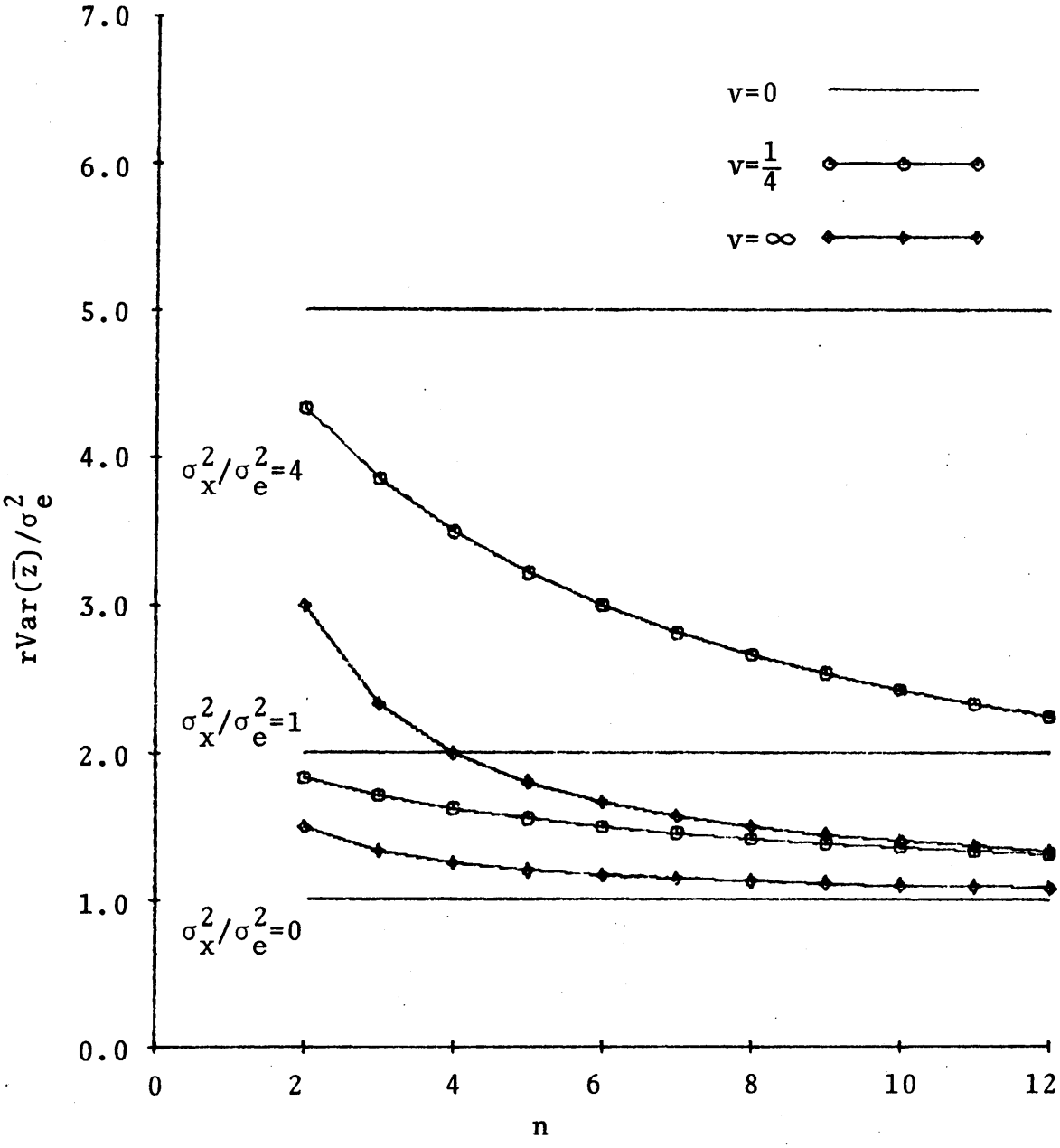

Figure 3. Plot of $rVar(\bar{z})/\sigma_e^2$ Versus n for s=t=1 and σ_α^2 from the Dirichlet Model

but since the illustrated relationships are possible, we should be wary of assuming that $\mathrm{Var}(\bar{z})$ decreases as n increases. This assumption, if incorrect, could lead to wasted effort and poor estimates of μ_x.

Suppose we run two experiments on a large lot, one with r composites of size n_1 and the other with r composites of size n_2 $(n_1 < n_2)$. We want to test

$$H: \quad \mathrm{Var}(\bar{z}_1) \geqq \mathrm{Var}(\bar{z}_2)$$

versus

$$A: \quad \mathrm{Var}(\bar{z}_1) < \mathrm{Var}(\bar{z}_2) \ ,$$

where $\mathrm{Var}(\bar{z}_i)$ is the variance of the mean of the rst observations from experiment i $(i = 1, 2)$. Let $w_{11} = \mathrm{rstVar}(\bar{z}_1)$ and $w_{12} = \mathrm{rstVar}(\bar{z}_2)$. Then H is equivalent to

$$H': \quad \frac{w_{12}}{w_{11}} \leqq 1$$

and to

$$H'': \quad \theta \leqq 0$$

with

$$\theta = \frac{-1}{2w_{12}} + \frac{1}{2w_{11}} \ .$$

We use Theorem 3 in Lehmann [15, p. 136] to find the uniformly most powerful unbiased (UMPU) level α test of H. First we note that the joint density of the observations can be written in the form

$$C(\theta, \underline{I}) \exp\left[\theta u + \sum_{i=1}^{8} I_i t_i\right]$$

with $u = SS_{B_2}$, $I_1 = rst\mu_x/w_{11}$, $I_2 = rst\mu_x/w_{12}$, $I_3 = -rst/2w_{11}$, $I_4 = -rst/2w_{12}$, $I_5 = -1/2w_{11}$, $I_6 = -1/2w_{21}$, $I_7 = -1/2w_{22}$, $I_8 = -1/2w_3$, $t_1 = \bar{z}_1$, $t_2 = \bar{z}_2$, $t_3 = \bar{z}_1^2$, $t_4 = \bar{z}_2^2$, $t_5 = SS_{B_1} + SS_{B_2}$, $t_6 = SS_{W_1}$, $t_7 = SS_{W_2}$, and $t_8 = SS_{E_1} + SS_{E_2}$. The w's are defined as in (3.3), with the second subscript denoting the experiment number. The sums of squares are defined as in (3.6); the subscript again denotes the experiment number.

Now since H' is equivalent to H'', there exists a UMPU test of H' by Lehmann's Theorem 3. When $w_{12} = w_{11}$, the distribution of

$$V = \frac{SS_{B_2}/w_{12}}{SS_{B_1}/w_{11}} = \frac{SS_{B_2}}{SS_{B_1}}$$

does not depend on $\underline{I}$, so by Corollary 1 [15, p. 162] the distribution of V is independent of $\underline{t}$. The UMPU level α

test of H is therefore given by

$$\phi(V) = \begin{cases} 1 & \text{if } V \geq C \text{ (reject)} \\ 0 & \text{if } V < C \text{ (accept)} \end{cases}$$

with C determined by $E[\phi(V)] = \alpha$ when $w_{11} = w_{12}$. The test statistic MS_{B_2}/MS_{B_1} is distributed as $F_{r-1,\ r-1}$ when $w_{11} = w_{12}$, so we reject H when

$$\frac{MS_{B_2}}{MS_{B_1}} \geq F_{r-1,\ r-1,\ 1-\alpha} \ .$$

Suppose $w_{12}/w_{11} = \lambda^2 > 1$ (H is false). Then

$$\frac{MS_{B_2}/w_{12}}{MS_{B_1}/w_{11}} = \frac{MS_{B_2}}{\lambda^2 MS_{B_1}} \sim F_{r-1,\ r-1}$$

and the power of the test is

$$\Pr\left[\text{ reject H } \mid w_{12} = \lambda^2 w_{11} \right]$$

$$= \Pr\left[F_{r-1,\ r-1} \geq \frac{1}{\lambda^2} F_{r-1,\ r-1,\ 1-\alpha} \right].$$

OC curves for this test for particular values of r and α are found in [3, p. 324]. These curves show, for example, that if $\alpha = 0.05$, $r = 15$, and $\lambda = 1.5$, the probability of rejecting H is about 0.42. Figures 2 and 3 indicate that there are cases in which we would like to detect changes in $\text{Var}(\bar{z})$ of size $\lambda = 1.5$ or smaller. The example shows that one would have to run a very large experiment to have a good chance of detecting such differences.

3.4 Probability of Positive Variance Component Estimates

The possibility of nonpositive estimates of positive parameters is a common problem in variance component estimation. Wang [21] gives an expression for the probability of a negative estimate of the between group variance in a balanced one-way classification with even degrees of freedom. He shows that the probability of a negative estimate increases as $E(MS_B)$ approaches $E(MS_W)$, where MS_B and MS_W are the between and within group mean squares.

In this section we extend Wang's result to the balanced twofold nested classification. The probability that the variance component estimators in such a model are simultaneously positive is a property of general interest, because knowing how this probability depends on the expe-

riment size is helpful in designing an experiment. The relationship of experiment size to the probability that $\hat{\sigma}_x^2$ and $\widehat{\sigma_\alpha^2\sigma_x^2}$ defined in (3.10) are both positive is of particular interest here because these estimators are sometimes required in estimating $\mathrm{Var}(\bar{z})$. This relationship is also of interest because the natural estimator of σ_α^2 discussed in the next section is nonzero only when $\hat{\sigma}_x^2$ and $\widehat{\sigma_\alpha^2\sigma_x^2}$ are both positive.

An expression for the probability of positive variance component estimates in a balanced twofold nested classification with even degrees of freedom is given in the following theorem.

Theorem 2.

Suppose that $X_1/w_1 \sim \chi^2_{2m_1}$, $X_2/w_2 \sim \chi^2_{2m_2}$, and $X_3/w_3 \sim \chi^2_{2m_3}$ are independent and define

$$Y_1 = c_1X_1 - c_2X_2 - c_3X_3 ,$$

$$Y_2 = c_4X_2 - c_5X_3 , \quad \text{and} \tag{3.16}$$

$$Y_3 = c_1X_1 ,$$

where $w_1 \geq w_2 \geq w_3$, c_1, c_2, and c_3 are arbitrary nonnegative constants. Then

$$\Pr\left[Y_1 > 0, Y_2 > 0 \right] =$$

$$\left[\frac{c_1 c_4 w_1}{(c_2 c_5 + c_3 c_4)(c_1 w_1 + c_2 w_2)} \right]^{m_2} \left[\frac{c_2 c_5 + c_3 c_4}{c_4} \right] p_3^{m_3}$$

$$\times \sum_{h=0}^{m_1-1} (1-p_2)^h \sum_{i=0}^{m_2-1} \binom{h+i}{h} \left(\frac{c_2 c_5}{c_4} \right)^i c_3^{m_2-i-1} \tag{3.17}$$

$$\times \sum_{j=0}^{h+i} \binom{m_2+h-j-1}{h+i-j} \binom{m_3+j-1}{j}$$

$$\times \left[\frac{(c_2 c_5 + c_3 c_4)(c_1 w_1 + c_2 w_2) w_3}{c_2 [c_1 c_4 w_1 w_2 + c_1 c_5 w_1 w_3 + (c_2 c_5 + c_3 c_4) w_2 w_3]} \right]^j$$

with

$$p_2 = \frac{c_1 w_1}{c_1 w_1 + c_2 w_2}$$

and (3.18)

$$p_3 = \frac{c_1 c_4 w_1 w_2}{c_1 c_4 w_1 w_2 + c_1 c_5 w_1 w_3 + (c_2 c_5 + c_3 c_4) w_2 w_3} .$$

Proof:

The definitions of the Y's and the distributions of the X's given in the theorem imply that the joint pdf of the Y's is

$$g(y_1,y_2,y_3) = k_0 y_3^{m_1-1}\left(y_3-y_1+\frac{c_3}{c_5}y_2\right)^{m_2-1}\left(y_3-y_1-\frac{c_2}{c_4}y_2\right)^{m_3-1} \tag{3.19}$$

$$\times \exp\left(\frac{k_1 y_1}{2} + \frac{k_2 y_2}{2} - \frac{k_3 y_3}{2}\right)$$

with

$$k_0 = \frac{2^{-(m_1+m_2+m_3)}}{(m_1-1)!(m_2-1)!(m_3-1)!(c_1w_1)^{m_1}}\left[\frac{c_5}{w_2(c_2c_5+c_3c_4)}\right]^{m_2}$$

$$\times\left[\frac{c_4}{w_3(c_2c_5+c_3c_4)}\right]^{m_3}\left[\frac{c_2c_5+c_3c_4}{c_4c_5}\right],$$

$$k_1 = \frac{c_4w_2+c_5w_3}{(c_2c_5+c_3c_4)w_2w_3},$$

$$k_2 = \frac{c_2w_2 - c_3w_3}{(c_2c_5+c_3c_4)w_2w_3}, \quad \text{and}$$

$$k_3 = \frac{c_1c_4w_1w_2 + c_1c_5w_1w_3 + (c_2c_5+c_3c_4)w_2w_3}{c_1(c_2c_5+c_3c_4)w_1w_2w_3}.$$

We integrate out y_3 to get the joint pdf of Y_1 and Y_2. When $Y_1>0$ and $Y_2>0$, the conditional pdf of Y_1 and Y_2 is proportional to the function

$$g_{++}(y_1,y_2) = \int_{y_1+\frac{c_2}{c_4}y_2}^{\infty} g(y_1,y_2,y_3)dy_3 .$$

Let $u = y_3 - y_1 - c_2y_2/c_4$, expand the integrand using the Binomial Theorem (the assumption that the degrees of freedom are even is needed here), and use Euler's integral [1],

$$\int_0^{\infty} t^{x-1} e^{-kt} dt = \frac{\Gamma(x)}{k^x},$$

to get

$$g_{++}(y_1,y_2) = k_0 e^{-\frac{y_1}{2c_1w_1} - \left(\frac{c_1w_1+c_2w_2}{2c_1c_4w_1w_2}\right)y_2}$$

$$\times \sum_{h=0}^{m_1-1} \binom{m_1-1}{h} y_1^{m_1-h-1} \sum_{i=0}^{m_2-1} \binom{m_2-1}{i} \left(\frac{c_3y_2}{c_5}\right)^{m_2-i-1}$$

$$\times \sum_{j=0}^{h+i} \binom{h+i}{j} \left(\frac{c_2y_2}{c_4}\right)^{h+i-j} (m_3+j-1)! \left(\frac{2}{k_3}\right)^{m_3+j} . \qquad (3.20)$$

The symbol $\binom{n}{k}$ denotes the binomial coefficient. Integrating (3.20) over the region $Y_1>0$, $Y_2>0$ yields the result in the theorem. Formula (3.20) will be used in the next section to investigate moments of the natural estimator of σ_α^2.

Corollary.

If $c_3=0$ in (3.16), then Y_1 and Y_2 have the same form as the estimators in the usual random effects model for the balanced twofold nested classification, and

$$\Pr\left[Y_1 > 0,\ Y_2 > 0\right] =$$

$$p_2^{m_2}\, p_3^{m_3} \sum_{h=0}^{m_1-1} \binom{m_2+h-1}{h} (1-p_2)^h \sum_{j=0}^{m_2+h-1} \binom{m_3+j-1}{j} (1-p_3)^j$$

with p_2 and p_3 given in (3.18).

To illustrate the significance of the theorem, let $m_1 = (r-1)/2$, $m_2 = r(s-1)/2$, $m_3 = rs(t-1)/2$, $c_1 = 1/st(r-1)$, $c_2 = (S-s)/Srst(s-1)$, $c_3 = 1/Srst(t-1)$, $c_4 = 1/rt(s-1)$, $c_5 = 1/rst(t-1)$,

$$w_1 = \sigma_e^2 + t\left(\frac{S-s}{S}\right)\sigma_d^2 + st\sigma_c^2 ,$$

$$w_2 = \sigma_e^2 + t\sigma_d^2 , \quad \text{and} \tag{3.21}$$

$$w_3 = \sigma_e^2 .$$

In this case Y_1 and Y_2 have the form of the variance component estimators in the finite population model for the balanced twofold nested classification [7]. Figures 4, 5, and 6 show the effect of σ_c^2/σ_e^2, σ_d^2/σ_e^2, r, and S on the probability of positive variance component estimates (P)

in this model. For convenience $\sigma_e^2=1$ was assumed in all computations, so $\sigma_c^2/\sigma_e^2 = \sigma_c^2$ and $\sigma_d^2/\sigma_e^2 = \sigma_d^2$ hereafter.

Figure 4 shows the effect of population size (S) on P for different values of σ_c^2 and σ_d^2 in experiments in which r=s=t=3. The difference in results for S=4 and S=∞ is largest when σ_c^2 and σ_e^2 are approximately the same size. There is little difference in the curves for S=4 and S=∞ when σ_c^2 is small, or large, compared to σ_e^2.

Figure 5 shows the effect of σ_c^2 on P for different σ_d^2 values in experiments in which S=∞ and r=s=t=3. For fixed σ_d^2, P increases as σ_c^2 increases. When σ_c^2 is small, P is small. For fixed $0 < \sigma_c^2 < \infty$, P increases to a maximum, then decreases, as σ_d^2 increases. It is easy to show that for fixed $\sigma_c^2 < \infty$,

$$\lim_{\sigma_d^2 \to \infty} P = p_2^{m_2} \sum_{h=0}^{m_1-1} \binom{m_2+h-1}{h} (1-p_2)^h , \qquad (3.22)$$

with $p_2 = r(s-1)/(rs-1)$. We can also show that for fixed $\sigma_d^2 < \infty$,

$$\lim_{\sigma_c^2 \to \infty} P = p_3^{m_3} \sum_{j=0}^{m_2-1} \binom{m_3+j-1}{j} (1-p_3)^j \qquad (3.23)$$

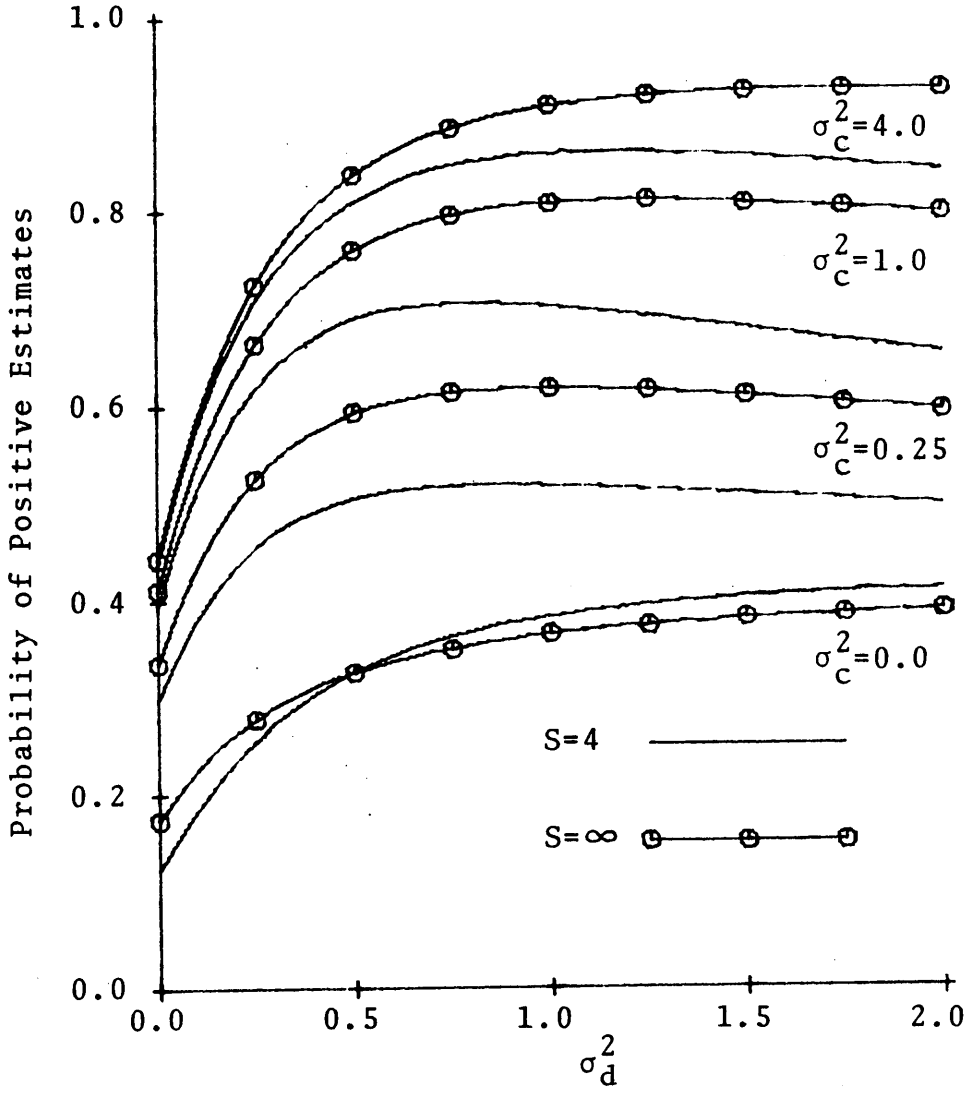

Figure 4. Effect of Population Size (S) on Probability of Positive Variance Component Estimates in Twofold Nested Classification (r=s=t=3)

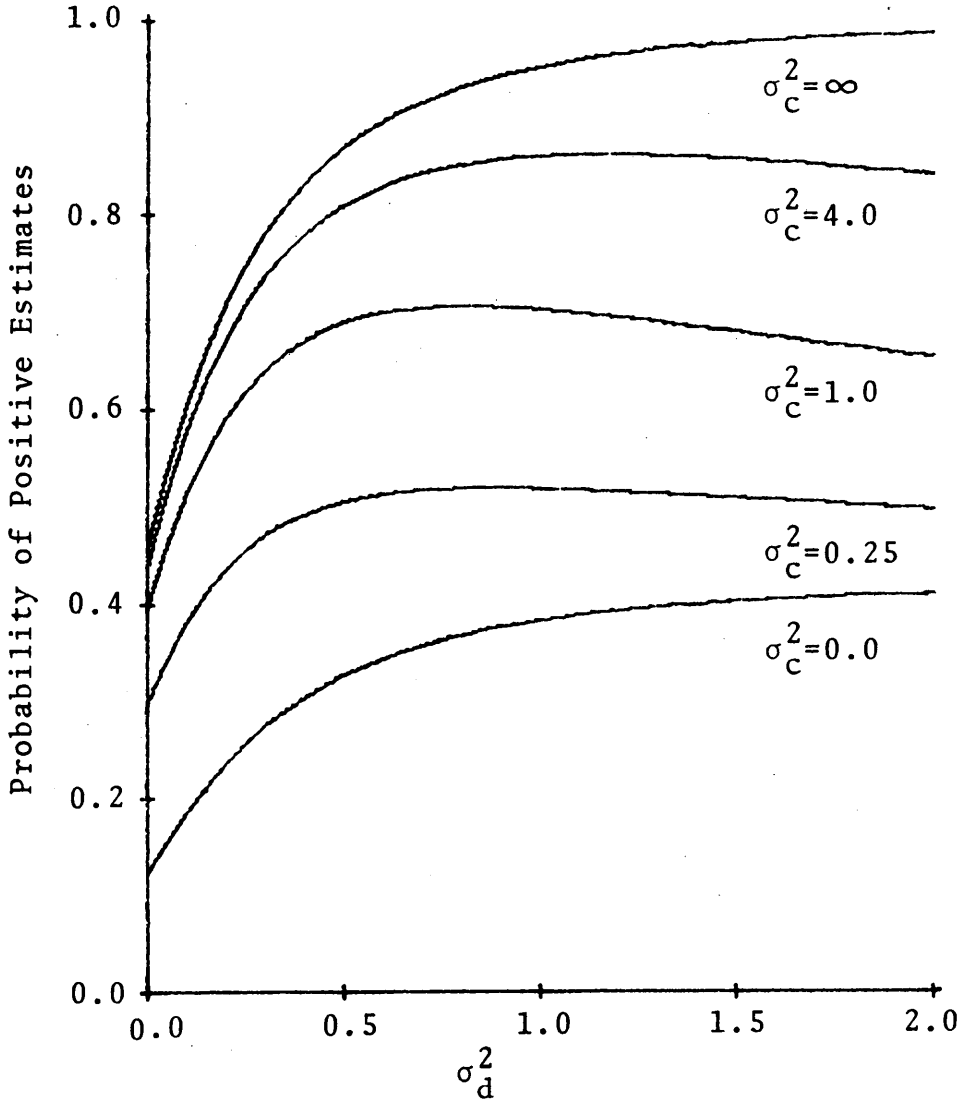

Figure 5. Effect of σ_c^2 on Probability of Positive Variance Component Estimates in Twofold Nested Classification ($r=s=t=3$, $S=\infty$)

with $p_3 = s(t-1)w_2/[s(t-1)w_2+(s-1)w_3]$. The results in (3.22) and (3.23) are, as one would expect, expressions for the probability that the estimate of the fixed variance component is positive (see Wang [21]).

Figure 6 shows the effect of r on P in experiments with $S=\infty$ and s=t=3. An increase in r yields the largest increase in P when σ_c^2 and σ_d^2 are moderately small compared to σ_e^2.

The estimators of components of $\mathrm{Var}(\bar{z})$ derived in Section 3.2 for the Duncan and Brown-Fisher models are special cases of the estimators for the finite population model just discussed. If $\sigma_c^2=\sigma_x^2/n$ and $S=\infty$, then nY_1 and Y_2 are the estimators of σ_x^2 and σ_d^2 given in (3.15) for the Duncan model. If $\sigma_c^2=\sigma_x^2/n$ and

$$\sigma_d^2 = \left(\frac{S}{S-1}\right) n\sigma_\alpha^2\sigma_x^2 ,$$

then nY_1 and $(S-1)Y_2/nS$ are the estimators of σ_x^2 and $\sigma_\alpha^2\sigma_x^2$ given in (3.10) for the Brown-Fisher model. Therefore, the results in the figures apply to experiments run to estimate the components of $\mathrm{Var}(\bar{z})$, whether the Duncan or the Brown-Fisher model is postulated.

All three figures show that P is small when σ_c^2 is near zero. For composite sampling procedures with large

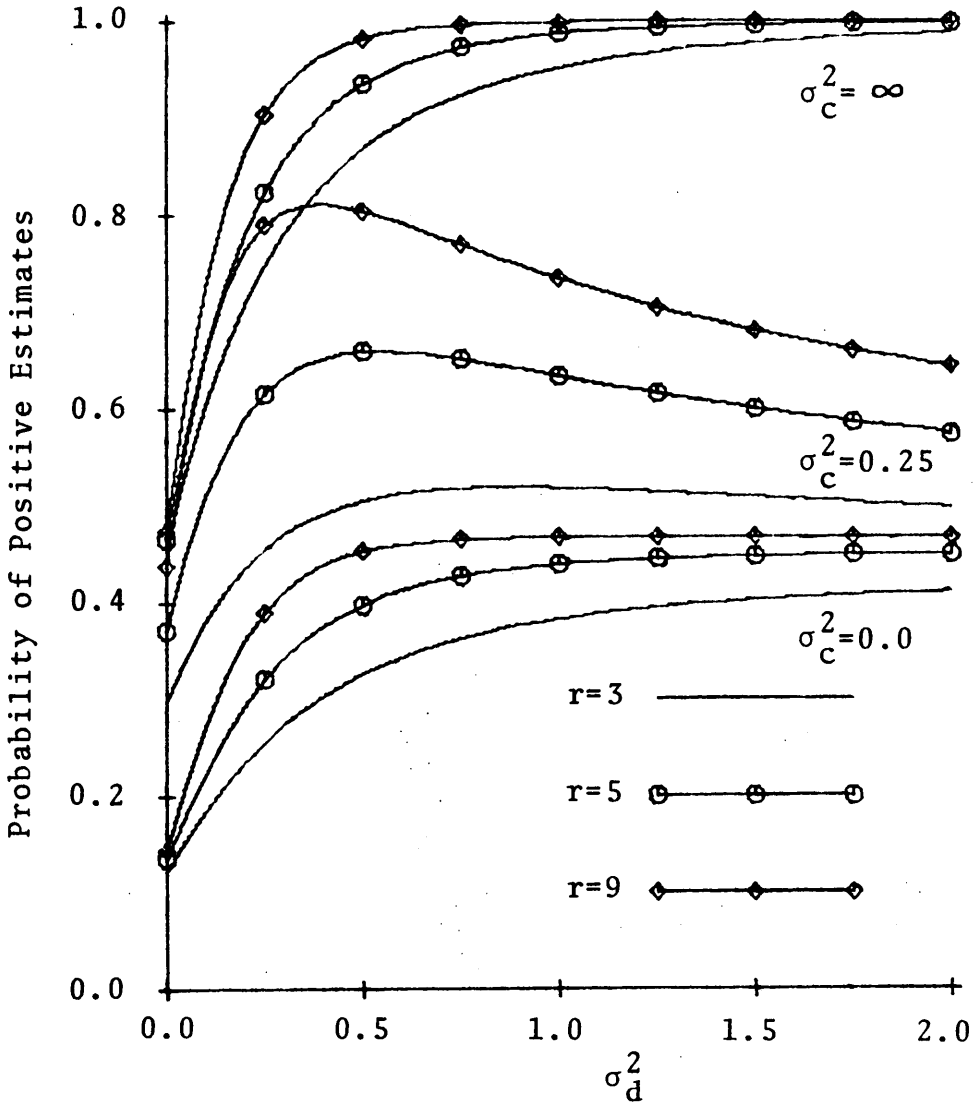

Figure 6. Effect of r on Probability of Positive Variance Component Estimates in Twofold Nested Classification ($r=s=t=3$, $S=\infty$)

n, $\sigma_c^2=\sigma_x^2/n$ is near zero. Thus for large n it is unlikely that an ordinary compositing experiment will give useful (positive) estimates of the components of $\mathrm{Var}(\bar{z})$ whether the Duncan or the Brown-Fisher model holds. One can improve the chance of getting positive estimates by increasing the size of the experiment (taking r large). However, a more practical approach may be to estimate σ_x^2 from a different kind of experiment, one in which increments are analyzed separately (not composited).

3.5 Estimating σ_α^2

Though we can evaluate composite sampling procedures without estimating σ_α^2 (using the results of Sections 3.2 and 3.3), we discuss the problem of estimating σ_α^2 for two reasons. First, estimating σ_α^2 is a theoretically interesting problem because of the way σ_α^2 and σ_x^2 appear together in the expected mean squares for the ordinary compositing experiment. Second, our results indicate that estimating σ_α^2 is a practically difficult task; anyone planning to estimate σ_α^2 should consider these results.

We first investigate the moments of the natural estimator of σ_α^2 from an ordinary compositing experiment with even degrees of freedom. After showing that the

moments of this estimator do not exist, we investigate an alternative experiment suggested by Brown and Fisher [6] for determining properties of α's.

The natural estimator of σ_α^2 using results from an ordinary compositing experiment is

$$\sigma_\alpha^2 = \begin{cases} \left(\frac{n-1}{n^2}\right) & \text{if } 0 < \left(\frac{n-1}{n^2}\right)\hat{\sigma}_x^2 \leq \widehat{\sigma_\alpha^2\sigma_x^2} \\ \dfrac{\widehat{\sigma_\alpha^2\sigma_x^2}}{\hat{\sigma}_x^2} & \text{if } 0 < \widehat{\sigma_\alpha^2\sigma_x^2} < \left(\frac{n-1}{n^2}\right)\hat{\sigma}_x^2 \\ 0 & \text{otherwise} \end{cases} \tag{3.24}$$

with $\hat{\sigma}_x^2$ and $\widehat{\sigma_\alpha^2\sigma_x^2}$ given by (3.10). It can be shown that $\hat{\sigma}_\alpha^2$ is the maximum likelihood estimator (MLE) of σ_α^2, except that MS_B replaces SS_B/r and $\hat{\sigma}_\alpha^2=0$ in the case that the MLE of σ_α^2 is not uniquely determined ($\hat{\sigma}_{x=}^2 \leq 0$).

To find the moments of $\hat{\sigma}_\alpha^2$, note that $\hat{\sigma}_x^2$ and $\widehat{\sigma_\alpha^2\sigma_x^2}$ have the same form as Y_1 and Y_2 in Theorem 2. Thus <u>for even degrees of freedom</u>, we can find the moments of $\hat{\sigma}_\alpha^2$ using (3.20). The vth moment of $\hat{\sigma}_\alpha^2$ is

$$E[(\hat{\sigma}_\alpha^2)^v] = 0^v \cdot [1 - Pr(Y_1 > 0, Y_2 > 0)]$$

$$+ \int_0^\infty \int_0^{\left(\frac{n-1}{n^2}\right)y_1} \left(\frac{y_2}{y_1}\right)^v g_{++}(y_1, y_2)\, dy_2 dy_1$$

$$+ \left(\frac{n-1}{n^2}\right)^v Pr\left[Y_2 > \left(\frac{n-1}{n^2}\right)Y_1, Y_1 > 0\right] .$$

$E[(\sigma_\alpha^2)^v]$ is a weighted sum of double integrals, which for $h = m_1 - 1$ have the form

$$\int_0^\infty \frac{e^{-b_1 y_1}}{y_1^v}\, \gamma(b_2, b_3, b_4 y_1)\, dy_1 , \qquad (3.25)$$

with $b_1 = 1/2c_1w_1$, $b_2 = m_1+m_2+v-j-1$, $b_4 = (n-1)/n^2$, $b_3 = (c_1w_1+c_2w_2)/2c_1c_4w_1w_2$, and

$$\gamma(b_2, b_3, b_4 y_1) = \int_0^{b_4 y_1} e^{-b_3 y_2}\, y_2^{b_2-1}\, dy_2 .$$

Let

$$f(y_1) = \frac{e^{-b_1 y_1}}{y_1^{v}} \gamma(b_2, b_3, b_4 y_1)$$

and

$$g(y_1) = \frac{e^{-b_1 y_1}}{y_1^{v}} .$$

Then Theorem III in Taylor [20, p. 638] shows that (3.25) does not converge for v = 1, 2, 3,..., since

$$\int_0^{\infty} g(y_1)\,dy_1$$

does not converge for v = 1, 2, 3,... [1, p. 255] and

$$\lim_{y_1 \to \infty} \frac{f(y_1)}{g(y_1)} = \frac{\Gamma(b_2)}{(b_3)^{b_2}}$$

is finite and nonzero. This proves that moments of $\hat{\sigma}_{\alpha}^2$ do not exist for even degrees of freedom.

The results of Section 3.4 show that the probability

of obtaining a value of $\widehat{\sigma_\alpha^2\sigma_x^2}/\hat{\sigma}_x^2$ in the interval $(0, (n-1)/n^2)$ is small unless r is large; thus, unless we run a large experiment, it is likely that $\hat{\sigma}_\alpha^2$ will equal one of the two endpoint values, zero or $(n-1)/n^2$. For this reason and because of the nonexistence of moments of $\hat{\sigma}_\alpha^2$ discovered above, we investigate an alternative method of estimating σ_α^2.

The alternative suggested by Brown and Fisher [6] is direct estimation of σ_α^2 using a tracer method. Details of such a procedure were not given in [6], so we present them here. We find that large experiments must be run to obtain precise estimates of σ_α^2. Since tracer experiments are usually sophisticated, expensive experiments, the estimator described here will rarely be of practical use.

Suppose we form a composite of n increments, one of which is tagged in some manner so that we can determine what proportion of each subsample comes from that increment. Let α_j represent the proportion of the jth subsample that comes from the tagged increment and write $\underline{\alpha}' = (\alpha_1,\ldots,\alpha_S)$. Define

$$u = \frac{1}{S}\left[\sum_{j=1}^{S} \alpha_j^2 - \frac{1}{S}\left(\sum_{j=1}^{S} \alpha_j\right)^2\right] \tag{3.26}$$

$$= \frac{1}{S}\,\underline{\alpha}'\,A\,\underline{\alpha}$$

with

$$A = I - \frac{1}{S}\,J\ .$$

Note that u is (S-1)/S times the usual sample standard deviation. Suppose

$$E(\underline{\alpha}) = \frac{1}{n}\,\underline{1} \tag{3.27}$$

and

$$Var(\underline{\alpha}) = \frac{\sigma_\alpha^2}{(S-1)}\,[SI - J]\ . \tag{3.28}$$

Then Theorem 1s in Searle [19] shows that

$$E(u) = \sigma_\alpha^2\ . \tag{3.29}$$

If we assume that $\underline{\alpha}$ has a singular multivariate normal distribution (justification for this assumption is given in Chapter VII), then it can be shown using Searle's Theorem 1s [19] that the standard deviation of u is

$$\sqrt{\frac{2}{(S-1)}}\ \sigma_\alpha^2 \ . \tag{3.30}$$

We cannot usually increase S arbitrarily to improve the precision of this estimator of σ_α^2 since S is related to n by the equation ng = SG.

Suppose we repeat the tracer experiment r times (form r composites of n increments each). Let u_i be the estimator of σ_α^2 from composite i and define

$$\bar{u} = \frac{1}{r} \sum_{i=1}^{r} u_i \ .$$

Since the u's are independent,

$$E(\bar{u}) = \sigma_\alpha^2 \tag{3.31}$$

and

$$\sigma_{\bar{u}} = \sqrt{\frac{2}{r(S-1)}}\ \sigma_\alpha^2 \tag{3.32}$$

(assuming α's are normally distributed). Equation (3.31) shows that $\bar{u}$ is an unbiased estimator of σ_α^2. Equation (3.32) shows that the estimator $\bar{u}$ can be made as precise as desired by taking r sufficiently large. Unfortunately, taking r large will be impractical in most situations.

One advantage of the tracer experiment is that it allows one to check assumptions (3.27) and (3.28). Suppose the set of S-component observation vectors $\underline{\alpha}_1,\ldots,\underline{\alpha}_r$ comes from a singular multivariate normal distribution. If

$$\underline{\alpha}_i^* = (\alpha_{i1},\ldots,\alpha_{i,S-1})' , \tag{3.33}$$

then the hypothesis that all components of $\underline{\alpha}_i$ have the same mean, $E(\alpha_{ij}) = 1/n$, is equivalent to

$$H_1 : E(\underline{\alpha}_i^*) = \frac{1}{n}\underline{1} , \tag{3.34}$$

since

$$\sum_{j=1}^{S} \alpha_{ij} = \frac{S}{n} .$$

The likelihood ratio test of H_1 is given by

$$T^2 = r(r-1)(\underline{\bar{\alpha}}^* - \frac{1}{n}\underline{1})'A^{-1}(\underline{\bar{\alpha}}^* - \frac{1}{n}\underline{1}) ,$$

with

$$\underline{\bar{\alpha}}^* = \frac{1}{r}\sum_{i=1}^{r} \underline{\alpha}_i^*$$

and

$$A = \sum_{i=1}^{r} (\underline{\alpha}_i^* - \underline{\bar{\alpha}}^*)(\underline{\alpha}_i^* - \underline{\bar{\alpha}}^*)' .$$

For a level α test we reject H_1 when [2]

$$T^2 > \frac{(r-1)(S-1)}{(r-S+1)} F_{S-1,\ r-S+1,\ 1-\alpha} . \qquad (3.35)$$

The power of this test depends on the degrees of freedom, which are functions of r and S. We can improve the power of the test by increasing r (S is fixed for a given composite sampling procedure).

The hypothesis that $\mathrm{Var}(\underset{\sim}{\alpha}_i)$ is given by (3.28) for $i = 1,\ldots,r$ is equivalent to

$$H_2 : \mathrm{Var}(\underset{\sim}{\alpha}_i^*) = \sigma_\alpha^2 \, V \tag{3.36}$$

with

$$\underset{(S-1)\times(S-1)}{V} = \frac{1}{(S-1)} [SI - J]$$

and σ_α^2 unknown. The likelihood ratio criterion for testing H_2 is [2]

$$\lambda^{2/r} = \frac{|AV^{-1}|}{\left[\mathrm{tr}\left(\frac{AV^{-1}}{S-1}\right)\right]^{(S-1)}} . \tag{3.37}$$

It can be easily verified that

$$V^{-1} = \left(\frac{S-1}{S}\right) [I + J] .$$

Under the null hypothesis, $-2 \ell n\lambda$ is asymptotically chi-square with $v = \frac{1}{2}S(S-1)-1$ degrees of freedom. Thus for

an approximate level α test of H_2, we reject H_2 when

$$-2\ \ell n\lambda > \chi^2_{v,\ 1-\alpha} \quad . \tag{3.38}$$

The power of this asymptotic test depends on the degrees of freedom, v, which is a function of S. Since S is fixed for a given composite sampling procedure, we cannot arbitrarily increase S to increase the power of the test.

In summary, the results of this section show that estimating σ^2_α is a difficult problem. If we can afford to run a large tracer experiment, however, we can obtain a precise estimate of σ^2_α as well as check some of the moment assumptions of the Brown-Fisher model.

IV. NONSEGMENTED LOTS – EXTENSIONS OF BASIC MODEL

4.1 Two Subsampling Stages

Since the customary models for composite sampling procedures (e.g., the model discussed in Section 2.6) show that $\mathrm{Var}(\bar{z})$ is a decreasing function of the number of increments per composite (n), there has been a natural tendency in practice to take n large. Unfortunately, in some applications the resulting composites are too large to be blended effectively.

A common response to the problem of large composites is to use composite sampling procedures with two subsampling stages – such as the procedure for sampling fertilizer described in Section 2.2. In the first subsampling stage, called the "reduction" stage by Duncan [9], the composite is passed through a device such as a riffle, which mixes the composite and obtains subsamples (reductions) that are small enough to be blended effectively. After each reduction is blended individually, it is subsampled, and tests are run on these second-stage subsamples.

The basis for including the reduction stage in a composite sampling procedure is that a small amount of material can be blended better than a larger amount. Those who use the reduction stage assume that the gain from

improved blending outweighs the loss due to added complexity and the introduction of another component of subsampling variability.

Our development of the Brown-Fisher model for a composite sampling procedure with two subsampling stages follows the course followed in Chapter II, including the derivation of properties of subsampling proportions, subsample values, and subsample values observed with measuring error. From these properties we obtain the formula for the variance of the mean of observations from such a procedure. This variance formula is compared to (2.43) to determine when two subsampling stages reduce the variance of the estimator of μ_x. The section concludes with a comparison of our results to those obtained from the usual random effects variance component model.

Suppose we randomly select qn increments from a large lot and form q composites of n increments each. We then take $r \leq R$ reductions of size SG from each composite, blend each reduction, and select $s \leq S$ subsamples of size G from each. Finally, we obtain a total of qrst observations by running t tests on each subsample. We assume that each composite contains enough material for RS subsamples of size G and that each increment has size g; thus

$$ng = RSG \ . \qquad (4.1)$$

Let $X_{h\ell}$ represent the value associated with the ℓth increment in the hth composite. Write $\underline{X}_h' = (X_{h1},\ldots,X_{hn})$ and $\underline{X}' = (\underline{X}_1',\ldots,\underline{X}_q')$, and assume that

$$E(\underline{X}) = \mu_x \underline{1}$$

and (4.2)

$$\mathrm{Var}(\underline{X}) = \sigma_x^2 I .$$

Let $g_{hij\ell}$ be the amount of the ℓth increment that appears in the jth subsample from the ith reduction from the hth composite and define

$$\alpha_{hij\ell} = \frac{g_{hij\ell}}{G} . \tag{4.3}$$

If we write $\underline{\alpha}_{hij}' = (\alpha_{hij1},\ldots,\alpha_{hijn})$, then the value associated with a particular subsample is

$$y_{hij} = \underline{\alpha}_{hij}' \underline{X}_h . \tag{4.4}$$

Suppose that

$$E(\alpha_{hij\ell}) = \frac{1}{n}$$

and (4.5)

$$Var(\alpha_{hij\ell}) = \sigma_\alpha^2 \ .$$

Then it can be shown by the methods of Section 2.3 that

$$Cov(\alpha_{hij\ell},\alpha_{hij\ell'}) = \frac{-\sigma_\alpha^2}{(n-1)} \qquad (4.6)$$

and

$$Cov(\alpha_{hij\ell},\alpha_{hij'\ell}) = -(n-1)Cov(\alpha_{hij\ell},\alpha_{hij'\ell'}) \ . \qquad (4.7)$$

The random variable $\beta_{hi\ell}$, defined by

$$\beta_{hi\ell} = \sum_{j=1}^{S} \frac{g_{hij\ell}}{SG} \qquad (4.8)$$

$$= \frac{1}{S} \sum_{j=1}^{S} \alpha_{hij\ell} \ ,$$

is the proportion of the ith reduction from the hth composite that comes from the ℓth increment in that composite. Suppose

$$E(\beta_{hi\ell}) = \frac{1}{n}$$

and (4.9)

$$Var(\beta_{hi\ell}) = \sigma_\beta^2 .$$

Then by the methods of Section 2.3 we can show that

$$Cov(\beta_{hi\ell},\beta_{hi\ell'}) = \frac{-\sigma_\beta^2}{(n-1)} , \tag{4.10}$$

$$Cov(\beta_{hi\ell},\beta_{hi'\ell}) = \frac{-\sigma_\beta^2}{(R-1)} , \tag{4.11}$$

and

$$Cov(\beta_{hi\ell},\beta_{hi'\ell'}) = \frac{\sigma_\beta^2}{(n-1)(R-1)} . \tag{4.12}$$

By (4.8) and (4.9)

$$\sigma_\beta^2 = \mathrm{Var}\left[\frac{1}{S}\sum_{j=1}^{S}\alpha_{hij\ell}\right]$$

$$= \frac{1}{S}\left[\sigma_\alpha^2 + (S-1)\mathrm{Cov}(\alpha_{hij\ell},\alpha_{hij'\ell})\right] ,$$

so

$$\mathrm{Cov}(\alpha_{hij\ell},\alpha_{hij'\ell}) = \frac{-\sigma_\alpha^2 + S\sigma_\beta^2}{(S-1)} . \qquad (4.13)$$

Putting (4.13) into (4.7) gives

$$\mathrm{Cov}(\alpha_{hij\ell},\alpha_{hij'\ell'}) = \frac{\sigma_\alpha^2 - S\sigma_\beta^2}{(n-1)(S-1)} .$$

The assumption that $\mathrm{Cov}(\alpha_{hij\ell},\alpha_{hi'j\ell}) = \mathrm{Cov}(\alpha_{hij\ell},\alpha_{hi'j'\ell})$ and result (4.11) give

$$\mathrm{Cov}(\alpha_{hij\ell},\alpha_{hi'j'\ell}) = \frac{-\sigma_\beta^2}{(R-1)}$$

for any j. The assumption that $\mathrm{Cov}(\alpha_{hij\ell},\alpha_{hi'j\ell'}) = \mathrm{Cov}(\alpha_{hij\ell},\alpha_{hi'j'\ell'})$ and equation (4.12) give

$$\mathrm{Cov}(\alpha_{hij\ell}, \alpha_{hi'j'\ell'}) = \frac{\sigma_\beta^2}{(n-1)(R-1)} .$$

Other pairs of α's (those with different h's) are assumed to be uncorrelated. The α's are assumed to be expectation-, variance-, and covariance-independent of the X's (these forms of independence are defined in Section 2.4).

The properties of α's are summarized below in matrix notation:

$$E(\underline{\alpha}_{hij}) = \frac{1}{n}\underline{1} , \tag{4.14}$$

$$\mathrm{Var}(\underline{\alpha}_{hij}) = \frac{\sigma_\alpha^2}{(n-1)}[nI - J] , \tag{4.15}$$

and

$$\mathrm{Cov}(\underline{\alpha}_{hij}, \underline{\alpha}_{h'i'j'}) = \tag{4.16}$$

$$\begin{cases} \left[\dfrac{-\sigma_\alpha^2 + S\sigma_\beta^2}{(n-1)(S-1)}\right][nI - J] & \text{for } h=h',\ i=i',\ j\neq j' \\ \left[\dfrac{-\sigma_\beta^2}{(n-1)(R-1)}\right][nI - J] & \text{for } h=h',\ i\neq i',\ \text{any } j \\ 0 & \text{for } h\neq h',\ \text{any } i,\ j. \end{cases}$$

Next we apply the Lemma given in Section 2.4 to find the means, variances, and covariances of subsample values. Write $\underline{y}' = (y_{111},\ldots,y_{11s};\ldots;y_{qr1},\ldots,y_{qrs})$. Then it can be shown that $\underline{y}$ has moments

$$E(\underline{y}) = \mu_x \underline{1}$$

and (4.17)

$$Var(\underline{y}) = diag(V_y,\ldots,V_y) ,$$

with

$$V_y = \frac{nS\sigma_x^2}{(S-1)} [\sigma_\alpha^2 - \sigma_\beta^2] I$$

$$+ \frac{n\sigma_x^2}{(S-1)} \left[\left(\frac{RS-1}{R-1}\right) \sigma_\beta^2 - \sigma_\alpha^2\right] U_{(s)}$$

$$+ \frac{\sigma_x^2}{n} \left[1 - \frac{n^2\sigma_\beta^2}{(R-1)}\right] J$$

(as in Chapter II, $U_{(s)} = diag(J_{s\times s},\ldots,J_{s\times s})$).

In general we observe

$$z_{hijk} = y_{hij} + e_{hijk} \qquad (4.18)$$

rather than y_{hij}, where e_{hijk} (k=1,...,t) is the error of measurement associated with the kth analysis on the jth subsample from the ith reduction of the hth composite. We assume that e's have the properties described in Section 2.5. With these assumptions the vector $\underline{z}' = (z_{1111},\ldots, z_{111t};\ldots;z_{qrs1},\ldots,z_{qrst})$ has moments

$$E(\underline{z}) = \mu_x \underline{1}$$

and (4.19)

$$\mathrm{Var}(\underline{z}) = \mathrm{diag}(V,\ldots,V) ,$$

where

$$V = c_1 I + c_2 U_{(t)} + c_3 U_{(st)} + c_4 J$$

with

$$c_1 = \sigma_e^2$$

$$c_2 = \frac{nS\sigma_x^2}{(S-1)} (\sigma_\alpha^2 - \sigma_\beta^2) ,$$

$$c_3 = \frac{n\sigma_x^2}{(S-1)} \left[\left(\frac{RS-1}{R-1} \right) \sigma_\beta^2 - \sigma_\alpha^2 \right] , \quad \text{and}$$

$$c_4 = \frac{\sigma_x^2}{n} - \frac{n\sigma_\beta^2\sigma_x^2}{(R-1)} .$$

It can be shown using (4.19) that the mean and variance of

$$\bar{z} = \frac{1}{qrst}\,\underline{1}'\underline{z}$$

are

$$E(\bar{z}) = \mu_x \tag{4.20}$$

and

$$\mathrm{Var}(\bar{z}) = \frac{\sigma_x^2}{qn} + \left(\frac{R-r}{R-1}\right)\frac{n\sigma_\beta^2\sigma_x^2}{qr} + \left(\frac{S-s}{S-1}\right)\frac{n(\sigma_\alpha^2-\sigma_\beta^2)\sigma_x^2}{qrs} + \frac{\sigma_e^2}{qrst}\,. \tag{4.21}$$

If $\sigma_\alpha^2 \geq \sigma_\beta^2$, we can identify the four terms in (4.21) with composites, reductions, subsamples, and tests (in that order). In deriving (4.13) we saw that

$$\frac{\sigma_\beta^2}{\sigma_\alpha^2} = \frac{1}{S}\left[1 + (S-1)\mathrm{Corr}(\alpha_{hij\ell},\alpha_{hij'\ell})\right]\,,$$

so $\sigma_\alpha^2 \geq \sigma_\beta^2$ whenever

$$\frac{1}{S}\left[1 + (S-1)\mathrm{Corr}(\alpha_{hij\ell},\alpha_{hij'\ell})\right] \leq 1\,.$$

Since this is equivalent to

$$\mathrm{Corr}(\alpha_{hij\ell}, \alpha_{hij'\ell}) \leq 1 \ ,$$

we always have $\sigma_\alpha^2 \geq \sigma_\beta^2$, and the terms in (4.21) can always be associated with sources of variation as described.

Equations (4.21) and (2.43) can be used to determine when it is worthwhile to employ two subsampling stages. Suppose we form q composites of size ng = RSG, take rs subsamples of size G from each composite, and run t analyses on each subsample. By (2.43) the variance of the mean of the qrst observations is

$$v_1 = \frac{\sigma_x^2}{qn} + \left(\frac{RS-rs}{RS-1}\right)\frac{n\sigma_\alpha^2\sigma_x^2}{qrs} + \frac{\sigma_e^2}{qrst} \ .$$

If we form q composites of size ng = RSG, take r reductions of size SG from each composite, take s subsamples of size G from each reduction, and run t analyses on each subsample, (4.21) shows that the variance of the mean of the qrst observations is

$$v_2 = \frac{\sigma_x^2}{qn} + \left(\frac{R-r}{R-1}\right)\frac{n\sigma_\beta^2\sigma_x^2}{qr} + \left(\frac{S-s}{S-1}\right)\frac{n(\sigma_\alpha^{2'}-\sigma_\beta^2)\sigma_x^2}{qrs} + \frac{\sigma_e^2}{qrst}$$

($\sigma_\alpha^{2'}$ has a prime to emphasize that different subsampling processes are used in the two procedures). These two procedures are comparable in that they require the same numbers of increments, composites, subsamples, and tests. The second procedure gives a more precise estimator of μ_x ($v_2 < v_1$) when

$$\left(\frac{R-r}{R-1}\right)\sigma_\beta^2 + \left(\frac{S-s}{S-1}\right)\frac{(\sigma_\alpha^{2'}-\sigma_\beta^2)}{s} < \left(\frac{RS-rs}{RS-1}\right)\frac{\sigma_\alpha^2}{s} . \qquad (4.22)$$

This is not an intuitively meaningful condition. Rather than try to use (4.22) to decide whether to include the reduction stage, we can test the hypothesis H: $v_2 \geq v_1$ versus the alternative A: $v_2 < v_1$. The test for H is similar to the test described in Section 3.3.

If r=s=1 (a common case), (4.22) reduces to

$$\sigma_\alpha^{2'} < \sigma_\alpha^2 . \qquad (4.23)$$

Condition (4.23) indicates that the procedure with a reduction stage gives a better estimator of μ_x ($v_2 < v_1$) when adding the reduction stage yields better subsamples ($\sigma_\alpha^{2'} < \sigma_\alpha^2$). We can check (4.23) using tracer experiments of the form discussed in Section 3.5.

In summary, if r=s=1, the Brown-Fisher model shows that two subsampling stages are better than one (in terms of $\mathrm{Var}(\bar{z})$) when small portions can be blended better than larger ones; in this case the condition favoring two subsampling stages (4.23) does not depend on σ_β^2. If either r>1 or s>1, the condition favoring two subsampling stages (4.22) is more complicated – it depends on the quality of first stage subsamples (σ_β^2).

If we restrict ourselves to the assumption that the usual random effects linear model for a balanced threefold nested classification holds for z_{hijk}, then it can be shown that

$$E(\bar{z}) = \mu_x$$

and (4.24)

$$\mathrm{Var}(\bar{z}) = \frac{\sigma_x^2}{qn} + \frac{\sigma_c^2}{qr} + \frac{\sigma_d^2}{qrs} + \frac{\sigma_e^2}{qrst} .$$

The parameters σ_c^2 and σ_d^2 in (4.24) are the variances of reduction and subsample effects, respectively. Comparing (4.24) to (4.21), we see that only the terms due to reductions and subsamples differ. The Brown-Fisher result (4.21) gives more detail as to how $\mathrm{Var}(\bar{z})$ varies with

r, R, s, S, n, and σ_x^2. In particular, note the finite composite and reduction correction factors, the parameter σ_x^2, and n in the reduction and subsampling terms of (4.21). Comments in Section 2.6 showing why the Brown-Fisher result is more useful than the usual result are applicable to the more complicated procedure discussed here.

4.2 Within-Increment Variability

The Brown-Fisher model discussed in Chapter II is suitable for materials (e.g., liquids) for which the assumption of within-increment homogeneity is reasonable. However, there are many cases in which the material within an increment is not homogeneous and cannot be made homogeneous. For example, each kernel in an increment is either damaged or undamaged in the U. S. Department of Agriculture example described in Section 2.2. Since the variable of interest in that procedure is the percent damaged kernels in the lot, each sampled kernel must be kept intact throughout the blending-subsampling process; there is no way to eliminate within-increment variability.

In other cases there is such extreme within-increment variability that homogeneity is unattainable even if composites are ground. For example, in a lot of peanuts

contaminated with aflatoxin (a potent toxin produced by the mold A. flavus) most kernels are uncontaminated; those that are contaminated may have amounts of aflatoxin ranging from one part per billion to over a million parts per billion.

In this section we extend the Chapter II model to include within-increment variability and examine the effect of this extension on $\mathrm{Var}(\bar{z})$.

Let $X_{ij\ell}$ represent the value associated with the portion of the ℓth increment that appears in the jth subsample from the ith composite (we assumed in Chapter II that $X_{ij\ell} = X_{ij'\ell}$ for all j). Let $g_{ij\ell}$ be the amount of material from the ℓth increment that appears in the jth subsample from the ith composite (as in Chapter II). Then $g_{ij\ell}/g$ is the proportion of the ℓth increment in the ith composite that appears in the jth subsample from that composite. With $\alpha_{ij\ell}$ defined by (2.6),

$$\frac{g_{ij\ell}}{g} = \frac{n}{S}\,\alpha_{ij\ell} \,. \tag{4.25}$$

If $X_{i\ell}$ is the value associated with the ℓth increment in the ith composite, then

$$X_{i\ell} = \frac{n}{S} \sum_{j=1}^{S} \alpha_{ij\ell} X_{ij\ell}$$

$$= \frac{n}{S} \underline{\alpha}'_{i\ell} \underline{X}_{i\ell} , \tag{4.26}$$

with $\underline{\alpha}'_{i\ell} = (\alpha_{i1\ell},\ldots,\alpha_{iS\ell})$ and $\underline{X}'_{i\ell} = (X_{i1\ell},\ldots,X_{iS\ell})$. Equation (4.26) suggests that X's and α's are correlated in this model. We must show that this is not so before we can use the Lemma in Section 2.4 to find properties of subsample values.

Suppose

$$E(X_{i\ell}) = E(X_{ij\ell}) = \mu_x$$

and $\tag{4.27}$

$$Var(X_{i\ell}) = \sigma_x^2$$

for all i, j, ℓ. Then equation (49) in Searle [19, p. 65], (4.26), and (4.27) give

$$E(X_{i\ell}) = \mu_x$$

$$= \frac{n}{S} \left[tr[Cov(\underline{\alpha}_{i\ell}, \underline{X}_{i\ell})] + E(\underline{\alpha}_{i\ell})' E(\underline{X}_{i\ell}) \right] .$$

If α's have the moments described in (2.23) to (2.25), this gives

$$\mathrm{tr}[\mathrm{Cov}(\underline{\alpha}_{i\ell}, \underline{X}_{i\ell})] = 0 \ .$$

Thus if diagonal elements of $\mathrm{Cov}(\underline{\alpha}_{i\ell}, \underline{X}_{i\ell})$ are equal, we must have

$$\mathrm{Cov}(\alpha_{ij\ell}, X_{ij\ell}) = 0 \qquad (4.28)$$

for all i, j, ℓ. Using (4.28) and constraint (2.7), we can show that

$$\mathrm{Cov}(\alpha_{ij\ell}, X_{ij\ell'}) = 0 \qquad (4.29)$$

for any i, j, and $\ell \neq \ell'$. Using (4.28), (4.29), and constraint (2.8), we can show that

$$\mathrm{Cov}(\alpha_{ij\ell}, X_{ij'\ell'}) = 0$$

for any i, ℓ, and $j \neq j'$. We assume that α's and X's from different composites are stochastically independent.

To summarize, we have found that

$$\mathrm{Cov}(\underline{\alpha}_{i\ell}, \underline{X}_{i'\ell'}) = 0 \qquad (4.30)$$

for any i and ℓ. Equation (4.30) does not in general guarantee that α's are expectation-, variance-, and covariance-independent of X's, as the Lemma in Section 2.4 requires; neither does (4.30) preclude these forms of independence. However, if α's and X's are normally distributed, (4.30) implies that they are stochastically independent, and we can use the Lemma. Justification for assuming that α's are normally distributed is given in Chapter VII; whether it is reasonable to assume that X's are normally distributed must be decided for each application.

Suppose that

$$\mathrm{Var}(\underline{X}_{i\ell}) = (\sigma_x^2+\sigma_w^2-c)I + cJ \ , \qquad (4.31)$$

where σ_x^2 and σ_w^2 are the between- and within-increment variances, respectively. The covariance c in (4.31) can be expressed in terms of σ_x^2 and σ_w^2 using the Lemma. By (2.27) and (4.26),

$$\sigma_x^2 = \frac{1}{S}\,[\sigma_x^2+\sigma_w^2+(S-1)c + n^2\sigma_\alpha^2(\sigma_x^2+\sigma_w^2-c)] \ .$$

Solving for c gives

$$c = \frac{(S-1)\sigma_x^2 - \sigma_w^2 - n^2\sigma_\alpha^2(\sigma_x^2+\sigma_w^2)}{(S-1) - n^2\sigma_\alpha^2} \ . \qquad (4.32)$$

Writing $\underline{X}'_{ij} = (X_{ij1},\ldots,X_{ijn})$, assumptions (4.27) and (4.31) give

$$E(\underline{X}_{ij}) = \mu_X \underline{1}$$

and (4.33)

$$\mathrm{Cov}(\underline{X}_{ij},\underline{X}_{i'j'}) = \begin{cases} (\sigma_X^2+\sigma_W^2)I & i=i',\ j=j' \\ cI & i=i',\ j\neq j' \\ 0 & i\neq i',\ \text{any } j. \end{cases}$$

This completes the task of finding first and second moments of X's in terms of μ_X, σ_X^2, and σ_W^2.

If we write $\underline{\alpha}'_{ij} = (\alpha_{ij1},\ldots,\alpha_{ijn})$, the value associated with the jth subsample from the ith composite is

$$y_{ij} = \underline{\alpha}'_{ij}\,\underline{X}_{ij} \tag{4.34}$$

for $i=1,\ldots,r$ and $j=1,\ldots,S$. Assuming that $\underline{\alpha}_{ij}$ has the properties discussed in Section 2.3, we can use the Lemma to find first and second moments of the y's. If we randomly select $s\leq S$ subsamples from each composite and denote the results by $\underline{y}' = (y_{11},\ldots,y_{1s};\ldots;y_{r1},\ldots,y_{rs})$, the Lemma and (4.33) give

$$E(\underline{y}) = \mu_x \underline{1}$$

and (4.35)

$$\mathrm{Var}(\underline{y}) = \mathrm{diag}(V_y,\ldots,V_y)$$

with

$$V_y = \left(\frac{S}{S-1}\right)\left[\frac{\sigma_w^2}{n} + n\sigma_\alpha^2(\sigma_x^2+\sigma_w^2)\right] I$$

$$+\left[\frac{\sigma_x^2}{n} - \frac{\sigma_w^2 + n^2\sigma_\alpha^2(\sigma_x^2+\sigma_w^2)}{n(S-1)}\right] J \ .$$

In general we observe

$$z_{ijk} = y_{ij} + e_{ijk} \tag{4.36}$$

instead of y_{ij}, where e_{ijk} (k=1,...,t) is the error of measurement associated with the kth measurement on the jth subsample from the ith composite. Assume that the e's have the usual properties and write $\underline{z}' = (z_{111},\ldots,z_{11t};\ldots;z_{rs1},\ldots,z_{rst})$. Then

$$E(\underline{z}) = \mu_x \underline{1}$$

and (4.37)

$$Var(\underline{z}) = diag(V,\ldots,V) ,$$

where

$$V = c_1 I + c_2 U_{(t)} + c_3 J$$

with

$$c_1 = \sigma_e^2 ,$$

$$c_2 = \left(\frac{S}{S-1}\right)\left[\frac{\sigma_w^2}{n} + n\sigma_\alpha^2(\sigma_x^2+\sigma_w^2)\right] ,$$

and

$$c_3 = \frac{\sigma_x^2}{n} - \frac{\sigma_w^2 + n^2\sigma_\alpha^2(\sigma_x^2+\sigma_w^2)}{n(S-1)} .$$

The mean and variance of

$$\bar{z} = \frac{1}{rst}\underline{1}'\underline{z}$$

are therefore

$$E(\bar{z}) = \mu_x \qquad (4.38)$$

and

$$\mathrm{Var}(\bar{z}) = \frac{\sigma_x^2}{rn} + \frac{1}{rs}\left(\frac{S-s}{S-1}\right)\left[\frac{\sigma_w^2}{n} + n\sigma_\alpha^2(\sigma_x^2+\sigma_w^2)\right] + \frac{\sigma_e^2}{rst} . \tag{4.39}$$

To see the effect on a composite sampling procedure of the presence of within-increment variability, we can compare the results above with results in Section 2.6. Result (4.38) shows that $\bar{z}$ is still an unbiased estimator of the lot mean, μ_x. In the common case s=1, (4.39) reduces to (2.43) with σ_x^2 replaced by $\sigma_x^2+\sigma_w^2$.

If there is no within-increment variability ($\sigma_w^2=0$), (4.39) reduces to (2.43); otherwise, the subsampling term of $\mathrm{Var}(\bar{z})$ is larger in (4.39) than in (2.43). This fact suggests that it may be possible to reduce $\mathrm{Var}(\bar{z})$ in some cases by mixing the material in individual increments before compositing – to decrease σ_w^2.

Examination of formula (4.39) provides further insight into how composite sampling procedures work. The formula shows how the two aspects of the blending process – the physical mixing of the material in a composite and the physical averaging of values associated with that material – are interrelated. The material in a composite is perfectly mixed when subsamples consist of equal proportions of each

increment ($\sigma_\alpha^2=0$). There is perfect physical averaging of the values associated with material in an increment when $\sigma_w^2=0$. A good blending process should yield small values of both σ_α^2 and σ_w^2.

The relationship of $\mathrm{Var}(\bar{z})$ to n is unclear in (4.39) because σ_α^2 and σ_w^2 may both change with n. Nevertheless, (4.39) indicates that the assumption in the usual model that the subsampling term is constant with respect to n is unwarranted. We can show that even when $\sigma_w^2>0$,

$$E\left[\frac{MS_B}{r' st}\right] = \mathrm{Var}(\bar{z}) \ ,$$

with MS_B and r' defined as in Section 3.2. This result is useful in testing whether $\mathrm{Var}(\bar{z})$ decreases with increasing n, and in estimating $\mathrm{Var}(\bar{z})$ for a procedure with given n.

V. SEGMENTED LOTS

5.1 The Basic Model

Suppose we want to estimate the mean of a lot that is stored in N segments (e.g., bins or bags). We randomly select $rn \leq N$ segments and partition them into r sets of n segments for compositing purposes. We take p increments per segment, so that each of the r composites consists of np increments. It is assumed that the amount of material in an increment is negligible compared to the amount of material in a segment, and that different segments contain equal amounts of material.

Let $X_{i\ell m}$ represent the value associated with the mth increment from the ℓth segment in the ith composite ($i=1,\ldots,r$; $\ell=1,\ldots,n$, and $m=1,\ldots,p$). Suppose

$$X_{i\ell m} = \mu_x + b_{i\ell} + c_{i\ell m} , \qquad (5.1)$$

where μ_x is the lot mean, $b_{i\ell}$ is the deviation from μ_x of the mean of the ℓth segment in the ith composite, and $c_{i\ell m}$ is the deviation from $b_{i\ell}$ of the mean of the mth increment from segment i, ℓ. For convenience we may re-index the b's, writing b_ℓ for $\ell=1,\ldots,N$. Then since μ_x is the

lot mean,

$$\sum_{\ell=1}^{N} b_\ell = 0 \ , \tag{5.2}$$

and we define

$$\kappa_b^2 = \frac{1}{N} \sum_{\ell=1}^{N} b_\ell^2 \ . \tag{5.3}$$

We assume that c's are uncorrelated with each other and that $E(c_{i\ell m}) = 0$ and $Var(c_{i\ell m}) = \sigma_c^2$. Write $\underline{X}'_{i\ell} = (X_{i\ell 1},\ldots,X_{i\ell p})$ and $\underline{X}'_{i} = (\underline{X}'_{i1},\ldots,\underline{X}'_{in})$. Then

$$E(\underline{X}_i) = \mu_x \underline{1} \ , \tag{5.4}$$

$$Var(\underline{X}_i) = \sigma_c^2 I + \frac{N\kappa_b^2}{(N-1)} U_{(p)} - \frac{\kappa_b^2}{(N-1)} J \ , \tag{5.5}$$

and

$$Cov(\underline{X}_i, \underline{X}'_{i'}) = \frac{-\kappa_b^2}{(N-1)} J \tag{5.6}$$

for $i \neq i'$ ($U_{(p)} = diag(J_{p\times p},\ldots,J_{p\times p})$).

As in the models for nonsegmented lots, the jth subsample value from the ith composite is the bilinear form

$$y_{ij} = \underline{\alpha}'_{ij}\,\underline{X}_i \tag{5.7}$$

for $i=1,\ldots,r$ and $j=1,\ldots,S$. Let $\underline{\alpha}'_{ij} = (\underline{\alpha}'_{ij1},\ldots,\underline{\alpha}'_{ijn})$, where $\underline{\alpha}'_{ij\ell} = (\alpha_{ij\ell 1},\ldots,\alpha_{ij\ell p})$ and $\alpha_{ij\ell m}$ is the proportion of the jth subsample from the ith composite that comes from the mth increment from the ℓth segment. By a development similar to the one in Section 2.3, details of which are given in [6], we can justify assuming that

$$E(\underline{\alpha}_{ij}) = \frac{1}{np}\,\underline{1}\ , \tag{5.8}$$

$$Var(\underline{\alpha}_{ij}) = \frac{\sigma_\alpha^2}{(np-1)}\,[npI - J]\ , \tag{5.9}$$

and

$$Cov(\underline{\alpha}_{ij},\underline{\alpha}_{i'j'}) = \begin{cases} \dfrac{-\sigma_\alpha^2}{(np-1)(S-1)}\,(npI - J) & i=i',\ j\neq j' \\ \\ 0 & i\neq i' \end{cases} \tag{5.10}$$

We also assume that α's and X's are expectation-, variance-, and covariance-independent.

Now suppose we form r composites and select s of the

S possible subsamples from each composite. As in Section 2.4, we can use the Lemma to find first and second moments of the resulting subsample values. Write $\underline{y}' = (y_{11},\ldots, y_{1s};\ldots;y_{r1},\ldots,y_{rs})$. Then

$$E(\underline{y}) = \mu_x \underline{1} \tag{5.11}$$

and

$$\begin{aligned} \mathrm{Var}(\underline{y}) = {} & \left(\frac{S}{S-1}\right) np\sigma_\alpha^2 \left[\sigma_c^2 + \frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\kappa_b^2\right] I \\ & + \left[\left(\frac{N}{N-1}\right)\frac{\kappa_b^2}{n} + \frac{\sigma_c^2}{np} - \frac{np\sigma_\alpha^2\sigma_c^2}{(S-1)}\right. \\ & \left. - \frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\frac{np\sigma_\alpha^2\kappa_b^2}{(S-1)}\right] U_{(s)} - \frac{\kappa_b^2}{(N-1)} J . \end{aligned} \tag{5.12}$$

If subsample values can be observed without measurement error, then

$$\bar{y} = \frac{1}{rs}\underline{1}'\underline{y}$$

has moments

$$E(\bar{y}) = \mu_x \tag{5.13}$$

and

$$\mathrm{Var}(\bar{y}) = \frac{1}{rn}\left[\left(\frac{N-rn}{N-1}\right)\kappa_b^2 + \frac{\sigma_c^2}{p}\right] \tag{5.14}$$

$$+ \frac{1}{rs}\left(\frac{S-s}{S-1}\right) np\sigma_\alpha^2 \left[\frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\kappa_b^2 + \sigma_c^2\right] .$$

With r=1 this result simplifies to that of Brown and Fisher [6]. If p=1 and N is large, (5.14) reduces to (2.36), the variance formula for nonsegmented lots, with $\sigma_x^2 = \kappa_b^2 + \sigma_c^2$.

It is more realistic to assume that we can only observe y_{ij} with error. Let

$$z_{ijk} = y_{ij} + e_{ijk} , \tag{5.15}$$

where e_{ijk} is the error made in the kth analysis of the jth subsample from the ith composite (k=1,...,t). Assuming that the e's have the usual properties (see Section 2.5), the vector of observations $\underline{z}' = (z_{111},\ldots,z_{11t};\ldots;z_{rs1},\ldots,z_{rst})$ has moments

$$E(\underline{z}) = \mu_x \underline{1} \tag{5.16}$$

and

$$\mathrm{Var}(\underline{z}) = d_1 I + d_2 U_{(t)} + d_3 U_{(st)} + d_4 J \ , \qquad (5.17)$$

with

$$d_1 = \sigma_e^2 \ ,$$

$$d_2 = \left(\frac{S}{S-1}\right) np\sigma_\alpha^2 \left[\sigma_c^2 + \frac{(n-1)p}{(np-1)} \left(\frac{N}{N-1}\right)\kappa_b^2\right] \ , \qquad (5.18)$$

$$d_3 = \left(\frac{N}{N-1}\right) \frac{\kappa_b^2}{n} + \frac{\sigma_c^2}{np} - \frac{np\sigma_\alpha^2}{(S-1)} \left[\sigma_c^2 + \frac{(n-1)p}{(np-1)} \left(\frac{N}{N-1}\right)\kappa_b^2\right] \ ,$$

and

$$d_4 = \frac{-\kappa_b^2}{(n-1)} \ .$$

Using (5.16) and (5.17) it can be shown that the mean and variance of

$$\bar{z} = \frac{1}{rst} \underline{1}' \underline{z}$$

are

$$E(\bar{z}) = \mu_x \qquad (5.19)$$

and

$$\mathrm{Var}(\bar{z}) = \frac{1}{rn}\left[\left(\frac{N-rn}{N-1}\right)\kappa_b^2 + \frac{\sigma_c^2}{p}\right] \tag{5.20}$$

$$+ \frac{1}{rs}\left(\frac{S-s}{S-1}\right)np\sigma_\alpha^2\left[\frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\kappa_b^2 + \sigma_c^2\right] + \frac{\sigma_e^2}{rst} .$$

For p=1 and N large (5.20) reduces to (2.43), with $\sigma_x^2 = \kappa_b^2 + \sigma_c^2$. If p>1, the subsampling portion (the second term) of (5.20) appears to be quite complicated. However, for large n and N,

$$\frac{(n-1)p}{(np-1)} \to 1 \quad \text{(from below)}$$

and

$$\frac{N}{N-1} \to 1 \quad \text{(from above)} .$$

In fact n and N need not be very large before these factors can be neglected for practical purposes. Thus in many cases we may use the approximate formula

$$\mathrm{Var}(\bar{z}) = \left(\frac{N-rn}{N-1}\right)\frac{\kappa_b^2}{rn} + \frac{\sigma_c^2}{rnp} + \left(\frac{S-s}{S-1}\right)\frac{np\sigma_\alpha^2}{rs}(\kappa_b^2+\sigma_c^2) + \frac{\sigma_e^2}{rst} .$$

(5.21)

The bounds on σ_α^2 in Theorem 1 yield relative bounds on $\mathrm{Var}(\bar{z})$. For example, in the common case p=s=1, (5.20) gives

$$\frac{1}{rn}\left[\left(\frac{N-rn}{N-1}\right)\kappa_b^2+\sigma_c^2\right]+\frac{\sigma_e^2}{rt} \tag{5.22}$$

$$\leq \mathrm{Var}(\bar{z}) \leq \frac{1}{r}\left[\left(\frac{N-r}{N-1}\right)\kappa_b^2+\sigma_c^2\right]+\frac{\sigma_e^2}{rt}\,.$$

The upper bound in (5.22) equals the variance of the mean of rt observations obtained by running t tests on individual increments from r randomly selected segments. Thus (5.22) shows that, in terms of the variance of the estimator of μ_x, the composite sampling procedure with p=s=1 can be no worse than the comparable noncompositing procedure. The lower bound shows how much better the composite sampling procedure can be.

5.2 The Usual Model

The customary analysis assumes that the observations z_{ijk} (i=1,...,r; j=1,...,s; k=1,...,t) come from the usual random effects linear model for the balanced twofold nested

classification. In [10] Duncan assumes that the variance of the composite effect is

$$\frac{\kappa_b^2 + \sigma_c^2/p}{n} .$$

With this assumption

$$E(\bar{z}) = \mu_x$$

and

$$\text{Var}(\bar{z}) = \frac{\kappa_b^2}{rn} + \frac{\sigma_c^2}{rnp} + \frac{\sigma_d^2}{rs} + \frac{\sigma_e^2}{rst} , \qquad (5.23)$$

where σ_d^2 is the variance of the subsample effect. This model is appropriate for large N. If N is not large, Duncan [9] includes the finite population correction factor in the between-segment term of $\text{Var}(\bar{z})$, giving

$$\text{Var}(\bar{z}) = \left(\frac{N-rn}{N}\right) \frac{\kappa_b^2}{rn} + \frac{\sigma_c^2}{rnp} + \frac{\sigma_d^2}{rs} + \frac{\sigma_e^2}{rst} . \qquad (5.24)$$

In comparing (5.20) and (5.24) we see that the principal differences are in the subsampling terms. The term σ_d^2/rs in (5.24) corresponds to the term with σ_α^2 as a

factor in the Brown-Fisher formula (5.20). The Brown-Fisher result indicates that the subsampling component of $\mathrm{Var}(\bar{z})$ depends on n, N, S, κ_b^2, σ_c^2, and σ_α^2, whereas the usual result suggests that the subsampling term does not depend on these parameters. As noted in Chapter II, the choice of model could affect which composite sampling procedure we choose in a given situation.

5.3 Inference

Suppose $\underline{z}$ from (5.15) has a multivariate normal distribution with mean and variance-covariance structure given by (5.16) and (5.17), so that the density function of $\underline{z}$ is

$$g(\underline{z}) = (2\pi)^{\frac{-rst}{2}} |\mathrm{Var}(\underline{z})|^{-\frac{1}{2}} \times \exp\left[-\frac{1}{2}(\underline{z} - \mu_x\underline{1})' \mathrm{Var}(\underline{z})^{-1} (\underline{z} - \mu_x\underline{1})\right]. \tag{5.25}$$

A more convenient expression for $g(\underline{z})$ can be found by noting that

$$|\mathrm{Var}(\underline{z})| = w_4^{rs(t-1)} w_3^{r(s-1)} w_2^{r-1} w_1 \tag{5.26}$$

and

$$\text{Var}(\underline{z})^{-1} = \frac{1}{w_4} I + \frac{1}{t}\left(\frac{1}{w_3} - \frac{1}{w_4}\right) U_{(t)} \tag{5.27}$$

$$+ \frac{1}{st}\left(\frac{1}{w_2} - \frac{1}{w_3}\right) U_{(st)} + \frac{1}{rst}\left(\frac{1}{w_1} - \frac{1}{w_2}\right) J \ ,$$

where

$$w_4 = d_1 \ ,$$

$$w_3 = d_1 + td_2 \ ,$$

$$w_2 = d_1 + td_2 + std_3 \ , \quad \text{and}$$

$$w_1 = d_1 + td_2 + std_3 + rstd_4$$

with d's given in (5.18). If $d_1 = \sigma_e^2 \neq 0$, $|\text{Var}(\underline{z})| \neq 0$. Putting (5.26) and (5.27) into (5.25) and simplifying yields

$$g(\underline{z}) = (2\pi)^{-\frac{rst}{2}} w_4^{-\frac{rs(t-1)}{2}} w_3^{-\frac{r(s-1)}{2}} w_2^{-\frac{r-1}{2}} w_1^{-\frac{1}{2}} \tag{5.28}$$

$$\times \exp\left\{-\frac{1}{2}\left[\frac{rst(\bar{z}-\mu_x)^2}{w_1} + \frac{SS_B}{w_2} + \frac{SS_W}{w_3} + \frac{SS_E}{w_4}\right]\right\}$$

with SS_B, SS_W, and SS_E defined as in (3.6). The density function (5.28) shows that $(\bar{z}, SS_B, SS_W, SS_E)$ is a sufficient statistic for the parameters of the model [14].

By Theorem 1 in Searle [19] the expected mean squares are

$$E(MS_E) = w_4 = \sigma_e^2 ,$$

$$E(MS_W) = w_3$$

$$= t\left(\frac{S}{S-1}\right) np\sigma_\alpha^2 \left[\frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\kappa_b^2 + \sigma_c^2\right]$$

$$+ \sigma_e^2 ,$$

and (5.29)

$$E(MS_B) = w_2$$

$$= \frac{st}{n}\left[\left(\frac{N}{N-1}\right)\kappa_b^2 + \frac{\sigma_c^2}{p}\right]$$

$$+ t\left(\frac{S-s}{S-1}\right) np\sigma_\alpha^2 \left[\frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\kappa_b^2 + \sigma_c^2\right]$$

$$+ \sigma_e^2 .$$

Comparing $E(MS_B)$ to $Var(\bar{z})$ in (5.20), we see that, unlike the case for nonsegmented lots, $Var(\bar{z}) \neq E(MS_B)/rst$. In fact, there is no linear combination of MS_B, MS_W, and MS_E that is an unbiased estimator of $Var(\bar{z})$.

If we can obtain an unbiased estimate $\hat{\kappa}_b^2$ of κ_b^2 from another experiment in which increments are not composited, then

$$\frac{MS_B}{rst} - \frac{\hat{\kappa}_b^2}{(N-1)} \tag{5.30}$$

is an unbiased estimator of $Var(\bar{z})$. From a compositing study with certain values of r, s, and t, we can estimate $Var(\bar{z})$ for different values, say r', s', and t'. Let

$$U_1 = \frac{1}{st}\left[MS_B - \left(\frac{S-s}{S}\right)MS_W - \left(\frac{s}{S}\right)MS_E\right] - \frac{r'\hat{\kappa}_b^2}{(N-1)},$$

$$U_2 = \frac{1}{t}\left[MS_W - MS_E\right], \tag{5.31}$$

and

$$U_3 = MS_E .$$

Then

$$\mathrm{Var}(\bar{z}) = \frac{1}{r} U_1 + \frac{1}{rs}\left(\frac{S-s'}{S}\right)U_2 + \frac{U_3}{rst} \qquad (5.32)$$

is an unbiased estimator of $\mathrm{Var}(\bar{z})$.

5.4 Extension to Two Stages of Subsampling

The reason for using a composite sampling procedure with two subsampling stages is discussed in Section 4.1. An example of such a procedure is the one for sampling bagged fertilizer described in Section 2.2.

Suppose we form q composites, each from a different set of n randomly chosen segments, taking p increments per segment. Suppose that each increment has size g (so each composite has size npg) and that each composite contains enough material to make RS subsamples of size G (so npg = RSG). We take $r \leq R$ first-stage subsamples (reductions) from each composite, blend each reduction, and select $s \leq S$ subsamples of size G from each. We run t analyses on each of these second-stage subsamples.

Let $X_{h\ell m}$ represent the value associated with the mth increment from the ℓth segment in the hth composite.

Write $\underline{X}_h' = (X_{h11},\ldots,X_{h1p};\ldots;X_{hn1},\ldots,X_{hnp})$ for the values in the hth composite and assume that

$$E(\underline{X}_h) = \mu_x \underline{1} ,$$

$$\mathrm{Var}(\underline{X}_h) = \sigma_c^2 I + \frac{N\kappa_b^2}{(N-1)} U_{(p)} - \frac{\kappa_b^2}{(N-1)} J , \qquad (5.33)$$

and

$$\mathrm{Cov}(\underline{X}_h, \underline{X}_h') = \frac{-\kappa_b^2}{(N-1)} J$$

as in Section 5.1.

Let $\alpha_{hij\ell m}$ represent the proportion of the jth subsample from the ith reduction of the hth composite that comes from the mth increment from the ℓth segment in that composite. Then as in models discussed earlier (see Sections 2.3 and 4.1) we can justify assuming that

$$E(\underline{\alpha}_{hij}) = \frac{1}{np} \underline{1} , \qquad (5.34)$$

$$\mathrm{Var}(\underline{\alpha}_{hij}) = \frac{\sigma_\alpha^2}{(np-1)} [np\ I - J] , \qquad (5.35)$$

$$\mathrm{Cov}(\underline{\alpha}_{hij},\underline{\alpha}_{h'i'j'}) = \begin{cases} \dfrac{-\sigma_\alpha^2+S\sigma_\beta^2}{(np-1)(S-1)}[np\ I - J] & h=h',\ i=i',\ j\neq j' \\ \dfrac{-\sigma_\beta^2}{(np-1)(R-1)}[np\ I - J] & h=h',\ i\neq i' \\ 0 & h\neq h' \end{cases} \tag{5.36}$$

with $\underline{\alpha}'_{hij} = (\alpha_{hij11},\ldots,\alpha_{hijnp})$ for h=1,...,q; i=1,...,R; and j=1,...,S. We also assume that α's and X's are expectation-, variance-, and covariance-independent. The parameter σ_β^2 in (5.36) is the variance of the proportion of the ith reduction of the hth composite that comes from the (ℓ,m)th increment in that composite.

We can now apply the Lemma in Chapter II to find properties of subsample values, $y_{hij} = \underline{\alpha}'_{hij}\underline{X}_h$. Write the sample of qrs observations (with $r\leq R$ and $s\leq S$) in the form $\underline{Y}' = (y_{111},\ldots,y_{11s};\ldots;y_{qr1},\ldots,y_{qrs})$. Then

$$E(\underline{Y}) = \mu_X\ \underline{1} \tag{5.37}$$

and

$$\mathrm{Var}(\underline{Y}) = d_1\ I + d_2\ U_{(s)} + d_3\ U_{(rs)} + d_4\ J \tag{5.38}$$

with

$$d_1 = \left(\frac{S}{S-1}\right) np(\sigma_\alpha^2-\sigma_\beta^2)\left[\frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\kappa_b^2 + \sigma_c^2\right],$$

$$d_2 = \frac{np}{(S-1)}\left[-\sigma_\alpha^2 + \left(\frac{RS-1}{R-1}\right)\sigma_\beta^2\right]\left[\frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\kappa_b^2 + \sigma_c^2\right], \tag{5.39}$$

$$d_3 = \left(\frac{N}{N-1}\right)\frac{\kappa_b^2}{n} + \frac{\sigma_c^2}{np} - \frac{np\sigma_\beta^2}{(R-1)}\left[\frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\kappa_b^2 + \sigma_c^2\right],$$

and

$$d_4 = \frac{-\kappa_b^2}{(N-1)}.$$

If subsample values are observed without measurement error, then

$$\bar{y} = \frac{1}{qrs}\underline{1}'\underline{y}$$

has moments

$$E(\bar{y}) = \mu_x$$

and

$$\mathrm{Var}(\bar{y}) = \frac{1}{qn}\left[\left(\frac{N-qn}{N-1}\right)\kappa_b^2 + \frac{\sigma_c^2}{p}\right] \tag{5.40}$$

$$+ \frac{1}{qr}\left(\frac{R-r}{R-1}\right) np\sigma_\beta^2\left[\frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\kappa_b^2 + \sigma_c^2\right]$$

$$+ \frac{1}{qrs}\left(\frac{S-s}{S-1}\right) np(\sigma_\alpha^2-\sigma_\beta^2)\left[\frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\kappa_b^2 + \sigma_c^2\right] .$$

Suppose that instead of y_{hij} we observe

$$z_{hijk} = y_{hij} + e_{hijk}$$

for $k=1,\ldots,t$ with e_{hijk} defined in the usual manner. Then the vector $\underline{z}' = (z_{1111},\ldots,z_{111t};\ldots;z_{qrs1},\ldots,z_{qrst})$ has moments

$$E(\underline{z}) = \mu_x \underline{1} \tag{5.41}$$

and

$$\mathrm{Var}(\underline{z}) = d_0 I + d_1 U_{(t)} + d_2 U_{(st)} + d_3 U_{(rst)} + d_4 J, \tag{5.42}$$

with d_1 through d_4 given by (5.39) and $d_0 = \sigma_e^2$. Using (5.41) and (5.42) we can show that the mean and variance of

$$\bar{z} = \frac{1}{qrst}\,\underline{1}'\,\underline{z}$$

are

$$E(\bar{z}) = \mu_x \tag{5.43}$$

and

$$\begin{aligned} Var(\bar{z}) = {} & \frac{1}{qn}\left[\left(\frac{N-qn}{N-1}\right)\kappa_b^2 + \frac{\sigma_c^2}{p}\right] \\ & + \frac{1}{qr}\left(\frac{R-r}{R-1}\right) np\sigma_\beta^2\left[\frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\kappa_b^2 + \sigma_c^2\right] \\ & + \frac{1}{qrs}\left(\frac{S-s}{S-1}\right) np(\sigma_\alpha^2-\sigma_\beta^2)\left[\frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\kappa_b^2 + \sigma_c^2\right] \\ & + \frac{\sigma_e^2}{qrst}\,. \end{aligned} \tag{5.44}$$

As in Section 4.1, it can be shown that the terms in (5.44) are nonnegative ($\sigma_\alpha^2 \geq \sigma_\beta^2$) and can be identified

with composites, reductions, subsamples, and tests (in that order). If N is large and p=1, (5.44) reduces to (4.21) with $\sigma_x^2 = \kappa_b^2 + \sigma_c^2$.

We can use equations (5.44) and (5.20) to determine when two subsampling stages are worthwhile for segmented lots. Suppose we form q composites of size npg = RSG, take rs subsamples of size G from each composite, and run t analyses on each subsample. By (5.20) the variance of the mean of the resulting qrst observations is

$$v_1 = \frac{1}{qn}\left[\left(\frac{N-qn}{N-1}\right)\kappa_b^2 + \frac{\sigma_c^2}{p}\right]$$

$$+ \frac{1}{qrs}\left(\frac{RS-rs}{RS-1}\right) np\sigma_\alpha^2 \left[\frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\kappa_b^2 + \sigma_c^2\right]$$

$$+ \frac{\sigma_e^2}{qrst} .$$

If we form q composites of the same size but take r reductions of size SG from each composite, take s subsamples of size G from each reduction, and run t tests on each subsample, (5.44) shows that the variance of the mean of the qrst observations is

$$v_2 = \frac{1}{qn}\left[\left(\frac{N-qn}{N-1}\right)\kappa_b^2 + \frac{\sigma_c^2}{p}\right]$$

$$+ \frac{1}{qr}\left(\frac{R-r}{R-1}\right) np\sigma_\beta^2 \left[\frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\kappa_b^2 + \sigma_c^2\right]$$

$$+ \frac{1}{qrs}\left(\frac{S-s}{S-1}\right) np(\sigma_\alpha^{2'} - \sigma_\beta^2)\left[\frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\kappa_b^2 + \sigma_c^2\right]$$

$$+ \frac{\sigma_e^2}{qrst} .$$

The variances σ_α^2 and $\sigma_\alpha^{2'}$ differ because smaller amounts of material are blended in the second procedure. These two procedures are comparable in that they require the same numbers of increments, composites, subsamples, and tests.

Since $v_2 < v_1$ is equivalent to inequality (4.22) for nonsegmented lots, the conditions favoring two subsampling stages are identical for segmented and nonsegmented lots. In particular, in the common case r=s=1 the procedure with two subsampling stages gives a better estimator of μ_x when adding the reduction stage yields better subsamples ($\sigma_\alpha^{2'} < \sigma_\alpha^2$).

5.5 Extension to Include Within-Increment Variability

In this section we examine the effect of within-increment variability on the formula for $\mathrm{Var}(\bar{z})$ in the sampling procedure described in Section 5.1. The need for this extension of the basic model is illustrated by the two examples described in Section 4.2. Let $X_{ij\ell m}$ represent the value associated with the portion of the mth increment from the ℓth segment in the ith composite that appears in the jth subsample from that composite (in the basic model of Section 5.1 we assumed that $X_{ij\ell m} = X_{ij'\ell m}$ for all j, j'). It can be shown (see Section 4.2) that $\underline{X}'_{ij} = (X_{ij11},\ldots,X_{ij1p};\ldots;X_{ijn1},\ldots,X_{ijnp})$ has moments

$$E(\underline{X}_{ij}) = \mu_x \underline{1} , \tag{5.45}$$

$$\mathrm{Var}(\underline{X}_{ij}) = (\sigma_c^2+\sigma_w^2) I + \frac{N\kappa_b^2}{(N-1)} U_{(p)} - \frac{\kappa_b^2}{(N-1)} J \tag{5.46}$$

and

$$\mathrm{Cov}(\underline{X}_{ij},\underline{X}_{i'j'}) = \begin{cases} (c-\kappa_b^2) I + \dfrac{N\kappa_b^2}{(N-1)} U_{(p)} - \dfrac{\kappa_b^2}{(N-1)} J & i=i', j\neq j' \\ \dfrac{-\kappa_b^2}{(N-1)} J & i\neq i' \end{cases} \tag{5.47}$$

with

$$c = \frac{(S-1)(\kappa_b^2+\sigma_c^2) - \sigma_w^2 - (np)^2\sigma_\alpha^2(\kappa_b^2+\sigma_c^2+\sigma_w^2)}{(S-1) - (np)^2\sigma_\alpha^2} \tag{5.48}$$

The parameters κ_b^2 and σ_c^2 are defined in Section 5.1, and σ_w^2 is defined in Section 4.2. We assume that $\underline{\alpha}_{ij}$ has the moments given in (5.8) through (5.10). We also assume that α's and X's are expectation-, variance-, and covariance-independent, as they would be if they were normally distributed (see Section 4.2 for details).

We can now use the Lemma to show that if subsample values have the form $y_{ij} = \underline{\alpha}'_{ij}\,\underline{X}_{ij}$ for $i=1,\ldots,r$ and $j=1,\ldots,S$, then the rs sample observations denoted by $\underline{y}' = (y_{11},\ldots,y_{1s};\ldots;y_{r1},\ldots,y_{rs})$ have moments

$$E(\underline{y}) = \mu_x\,\underline{1} \tag{5.49}$$

and

$$Var(\underline{y}) = d_1\,I + d_2\,U_{(s)} + d_3\,J \tag{5.50}$$

with

$$d_1 = \frac{(\kappa_b^2+\sigma_c^2+\sigma_w^2)}{np} - \frac{c}{np(S-1)}\left[(S-1)-(np)^2\sigma_\alpha^2\right]$$

$$- \frac{np\sigma_\alpha^2\kappa_b^2}{(S-1)} + np\sigma_\alpha^2(\sigma_c^2+\sigma_w^2) + \frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\left(\frac{S}{S-1}\right)np\sigma_\alpha^2\kappa_b^2 ,$$

$$d_2 = \left(\frac{N}{N-1}\right)\frac{\kappa_b^2}{n} - \frac{\kappa_b^2}{np} + \frac{c}{np(S-1)}\left[(S-1)-(np)^2\sigma_\alpha^2\right] \qquad (5.51)$$

$$+ \frac{np\sigma_\alpha^2\kappa_b^2}{(S-1)} - \frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\frac{np\sigma_\alpha^2\kappa_b^2}{(S-1)} ,$$

and

$$d_3 = \frac{-\kappa_b^2}{(N-1)} .$$

Suppose that instead of y_{ij} we observe

$$z_{ijk} = y_{ij} + e_{ijk} ,$$

where e_{ijk} $(k=1,\ldots,t)$ is a measurement error with the usual properties. Then $\underline{z}' = (z_{111},\ldots,z_{11t};\ldots;z_{rs1},\ldots, z_{rst})$ has moments

$$E(\underline{z}) = \mu_x \underline{1} \tag{5.52}$$

and

$$\mathrm{Var}(\underline{z}) = d_0 I + d_1 U_{(t)} + d_2 U_{(st)} + d_3 J \tag{5.53}$$

with d_1, d_2, and d_3 defined in (5.51) and $d_0 = \sigma_e^2$. The mean and variance of

$$\bar{z} = \frac{1}{rst} \underline{1}' \underline{z}$$

are therefore

$$E(\bar{z}) = \mu_x \tag{5.54}$$

and

$$\mathrm{Var}(\bar{z}) = \frac{1}{rn}\left[\left(\frac{N-rn}{N-1}\right)\kappa_b^2 + \frac{\sigma_c^2}{p}\right]$$

$$+ \frac{1}{rs}\left(\frac{S-s}{S-1}\right)\left\{\frac{\sigma_w^2}{np} + np\sigma_\alpha^2\left[\frac{(n-1)p}{(np-1)}\left(\frac{N}{N-1}\right)\kappa_b^2+\sigma_c^2+\sigma_w^2\right]\right\}$$

$$+ \frac{\sigma_e^2}{rst} . \tag{5.55}$$

Equation (5.54) shows that $\bar{z}$ is still an unbiased estimator

of μ_x. Comparing (5.55) with (5.20) reveals the effect of within-increment variability on $Var(\bar{z})$ – only the subsampling terms differ in the two formulas. When $\sigma_w^2=0$ (there is no within-increment variability), (5.55) reduces to (5.20). The presence of within-increment variability increases the subsampling component of $Var(\bar{z})$. If s=1, (5.55) reduces to (5.20) except that σ_c^2 is replaced by $\sigma_c^2+\sigma_w^2$. If N is large and p=1, (5.55) reduces to (4.39) with $\sigma_x^2 = \kappa_b^2+\sigma_c^2$.

Formula (5.55) shows the interrelation between the physical mixing of the material in a composite and the physical averaging of values associated with that material. Perfect mixing ($\sigma_\alpha^2=0$) does not guarantee perfect physical averaging ($\sigma_w^2=0$). However, a good mixing procedure should result in small values of both σ_α^2 and σ_w^2.

It is not clear how $Var(\bar{z})$ changes with n in (5.55) because both σ_α^2 and σ_w^2 may vary with n. However, (5.55) does show that the implication of the usual model that the subsampling term of $Var(\bar{z})$ is constant with respect to n is not justified.

VI. CHOOSING COMPOSITE SAMPLING PROCEDURES

6.1 Introduction

In this chapter we discuss how to choose a composite sampling procedure to minimize either the cost of the procedure or $Var(\bar{z})$. Unfortunately, explicit answers cannot be found for most of the questions that arise here, because of lack of knowledge of how σ_α^2 varies with n in real situations. The objectives in this chapter, therefore, are to indicate the kinds of questions that arise in choosing a composite sampling procedure and to show where possible how to obtain answers to these questions. For simplicity the discussion concentrates on the choice of procedures for nonsegmented lots. Differences between segmented and nonsegmented lot cases are noted.

In selecting a composite sampling procedure for nonsegmented lots we must choose values of r, s, t, and n and decide whether to employ one or two subsampling stages. The question of how many subsampling stages to employ has already been discussed in Chapter IV. We show how to select r, s, and t in Section 6.2. Section 6.3 discusses the choice of n. In both the following sections Brown-Fisher results are compared with results from the usual models.

6.2 Choosing r, s, and t

The high testing costs associated with most composite sampling procedures require that the total number of tests on each lot, rst, be kept small. In fact, in many situations we would like to minimize testing costs by running just one test per lot, taking r=s=t=1. Other choices of rst should not be automatically ruled out, though, because as Duncan points out in [9], "Variance models based on empirically determined knowledge of basic variances ... are essential for the efficient design of sampling procedures. They do not, however, yield precise determinations of the precision of current mean estimates unless there is assurance that the previous estimates of basic variances ... continue to be valid. This means that current sampling procedures should be such as to yield continuing checks on the estimates of the basic parameters."

If we can afford to run two tests per lot (at least two tests are needed to check for changes in basic parameters), we must decide whether to take r, s, or t equal to two. Duncan says [9], "The need for control procedures emphasizes again the need to form more than one composite [emphasis added]. For only in this way can we get some check on all the basic variances involved." In other words, the usual model indicates that we must take r=2 and s=t=1

to be able to check for changes in all parameters when rst=2.

Brown-Fisher models indicate that we can check for changes in parameter values using either two composites (r=2 and s=t=1) or two subsamples (r=t=1 and s=2) when rst=2, since it can be shown for all procedures with one subsampling stage that $E(MS_W)$ is an increasing function of all basic parameters. This is an important difference from the result of the usual model because taking two composites is more expensive than taking two subsamples. With r=t=1 and s=2, MS_B is not defined and $K \cdot MS_W$ is not in general an unbiased estimator of $Var(\bar{z})$ for any constant K. However, as Duncan notes in [9], it is not necessary to have an unbiased estimator of $Var(\bar{z})$. All we need for control purposes is an estimator for which the expected value increases with increasing parameter values. Thus in the case of nonsegmented or segmented lots, we can run a control chart on MS_B (if r=2) or on MS_W (if s=2) to check for changes in parameter values.

In deciding whether to use two composites or two subsamples we are interested in both the cost of the procedure and the variance of the estimator of μ_x. Suppose we form one composite of $n_1=2n_2$ increments (n_2 an integer), take two subsamples from the composite, and analyze each subsample once. Formula (4.39) shows that the variance of the

mean of the two observations from this procedure is

$$v_1 = \frac{\sigma_x^2}{n_1} + \frac{1}{2}\left(\frac{S-2}{S-1}\right)\left[\frac{\sigma_w^2}{n_1} + n_1\sigma_\alpha^2(\sigma_x^2+\sigma_w^2)\right] + \frac{\sigma_e^2}{2} .$$

Now consider the alternative procedure in which we form two composites of n_2 increments each, take one subsample from each composite, and analyze each subsample once. Formula (4.39) shows that the variance of the mean of the two observations from this procedure is

$$v_2 = \frac{\sigma_x^2}{n_1} + \frac{1}{2}\left[\frac{\sigma_w^{2'}}{n_2} + n_2\sigma_\alpha^{2'}(\sigma_x^2+\sigma_w^{2'})\right] + \frac{\sigma_e^2}{2}$$

(as noted in Chapters II and IV, σ_α^2 and σ_w^2 may vary with the number of increments in the composite). The two procedures are comparable in that they require the same numbers of increments, subsamples, and tests. The variance for the first procedure is greater than the variance for the second procedure when

$$v_1 - v_2 = \frac{1}{2}\left(\frac{S-2}{S-1}\right)\left[\frac{\sigma_w^2}{n_1} + n_1\sigma_\alpha^2(\sigma_x^2+\sigma_w^2)\right]$$

$$- \frac{1}{2}\left[\frac{\sigma_w^{2'}}{n_2} + n_2\sigma_\alpha^{2'}(\sigma_x^2+\sigma_w^{2'})\right]$$

$$> 0 \; . \qquad (6.1)$$

It can be shown that (6.1) can be either true or false depending on the values of σ_x^2, σ_α^2, $\sigma_\alpha^{2'}$, σ_w^2, and $\sigma_w^{2'}$. That is, neither procedure is uniformly better than the other in terms of the variance of the estimator of μ_x.

Condition (6.1) is too complicated to be of practical use in determining whether $v_1 > v_2$. To decide whether v_1 or v_2 is smaller in a given situation, we run two experiments. In the first experiment we form r_1 composites of $n_1=2n_2$ increments each, take two subsamples from each composite, and run one test per subsample. Then we compute MS_{B_1} as shown in (3.6). In the second experiment we form r_2 composites of n_2 increments each, take one subsample from each composite, and run one test on each subsample. We compute MS_{B_2} as indicated in (3.6). Since it can be shown that

$$v_1 - v_2 = \frac{1}{2}\left[E(MS_{B_1}) - E(MS_{B_2})\right] , \qquad (6.2)$$

we can test H: $v_1 \leq v_2$ versus A: $v_1 > v_2$ using the test statistic MS_{B_1}/MS_{B_2}. The first experiment requires twice as many tests as the second, so it is convenient to take $r_1 < r_2$. The power of the level α test of H equals

$$\Pr\left[F_{r_1-1,\, r_2-1} > \frac{v_2}{v_1} F_{r_1-1,\, r_2-1,\, 1-\alpha}\right] , \qquad (6.3)$$

a probability that can be read from any table of the cumulative F-distribution. It can be shown that (6.2) holds for segmented lots, so the same test can be used in the segmented lot case. The power of the test for segmented lots is given by (6.3) with v_1 and v_2 replaced by the appropriate expected mean squares (e.g., from (5.29)).

6.3 Choosing n

In this section we assume that r, s, and t are fixed and discuss how to choose n, the number of increments per composite. One criterion for choosing n is to find the n in the interval of feasible n values that minimizes

$\mathrm{Var}(\bar{z})$. The interval of feasible values, $[n_m, n_M]$, is determined in practice by the capacity of the blending-subsampling apparatus, the time and money available for sampling, the difficulty of getting increments, etc. Another criterion for selecting n is to find the n value that minimizes a cost function that depends on $\mathrm{Var}(\bar{z})$. We saw in Chapter III that over a finite interval of n values $\mathrm{Var}(\bar{z})$ may or may not decrease with n, so that results discussed here are not as simple as results from the usual model, which assumes that $\mathrm{Var}(\bar{z})$ always decreases with n.

Consider first the problem of choosing $n \in [n_m, n_M]$ to minimize $\mathrm{Var}(\bar{z})$. The usual model indicates that we should take $n = n_M$. The Brown-Fisher model indicates that the choice is not so obvious; that the optimum n can be any integer in the interval of feasible values (see Figure 2).

The obvious approaches to choosing n to minimize $\mathrm{Var}(\bar{z})$ are not practical in most cases. Response surface optimum-seeking methods cannot be used because they require fairly precise estimates of $\mathrm{Var}(\bar{z})$ for different n values. Obtaining such estimates would be prohibitively expensive in all but the rarest applications of composite sampling procedures. The alternative of directly estimating how σ_α^2 varies with n is also impractical. An

experimental procedure for estimating σ_α^2 for given n was described in Chapter III, but one could rarely afford to run enough such experiments to get a good estimate of the functional relationship between n and σ_α^2.

Having pointed out the impracticability of the obvious approaches to choosing n to minimize $\mathrm{Var}(\bar{z})$, we shall now describe what can be done to choose n in ordinary composite sampling situations. The subsampling models discussed in Chapter VII yield formulas for σ_α^2 that depend on n in such a way that $\mathrm{Var}(\bar{z})$ is a decreasing function of n for these models. Thus there is some evidence that $\mathrm{Var}(\bar{z})$ may decrease with n in ordinary cases. We can check this conclusion in a particular situation by identifying the interval of feasible n values as described above, running experiments at $n = n_m$ and $n = n_M$, and performing the test of hypothesis described in Section 3.3. If the hypothesis that $\mathrm{Var}(\bar{z})$ is smaller at $n = n_M$ than at $n = n_m$ is not rejected, then we use the procedure with $n = n_M$. Otherwise, we use $n = n_m$. The test of hypothesis gives some protection against the worst cases illustrated in Figure 2.

Duncan [9, p. 335] discusses the problem of choosing n to minimize a cost function that depends on $\mathrm{Var}(\bar{z})$. He considers a situation in which there is an acceptance limit on the mean estimate for each lot equal to a nominal

value minus a constant times the standard error of the estimate. To maintain a given producer's risk in this situation, the production target must be varied linearly with the standard deviation of $\bar{z}$. Thus in the case of nonsegmented lots the cost function has the form

$$C = c_0 + c_1 rn + c_2 \, \mathrm{Var}(\bar{z})^{\frac{1}{2}} , \qquad (6.4)$$

where c_0 is a fixed cost, c_1 is the cost of taking an increment, and c_2 is the cost of setting the production target a fixed number of standard errors below the nominal value.

If we put a Brown-Fisher formula for $\mathrm{Var}(\bar{z})$ (e.g., formula (4.39)) into (6.4), there appears to be no practical way to find the n that minimizes C. This is in sharp contrast to the usual result. Duncan [9] indicates that if we put the formula for $\mathrm{Var}(\bar{z})$ from the usual model into (6.4), we can find the optimum n simply by plotting C versus n for given values of the parameters σ_x^2, σ_d^2, and σ_e^2. Of course, the optimum found in this manner is the true optimum only if the subsampling component of $\mathrm{Var}(\bar{z})$ (σ_d^2/rs in equation (2.49)) is constant with respect to n. Since the Brown-Fisher models developed in previous chapters indicate that the subsampling component of $\mathrm{Var}(\bar{z})$

is not constant with respect to n, there is reason to question the usefulness of the solution found by Duncan's method.

Unfortunately, if we attempt to find the minimum cost n using Brown-Fisher results, we encounter the same difficulty encountered in seeking the n that minimizes $Var(\bar{z})$ — the model does not show explicitly how $Var(\bar{z})$ changes with n. Although the expression for C in (6.4) suggests that there will usually be a unique optimum n, in most cases we will be unable to afford to run enough experiments to find the optimum value.

VII. DISTRIBUTIONAL MODELS FOR SUBSAMPLING PROPORTIONS

7.1 Introduction

With a distributional model for the subsampling proportions (α's) in a Brown-Fisher model, we can obtain formulas for the variances and covariances of the α's. We showed in Section 6.3 that it is important in choosing n to know how σ_α^2 varies with n. If we can find an appropriate distributional model for the α's in a given application, we may be able to derive σ_α^2 as a function of n for that application.

Given a distribution of subsampling proportions, we can derive the distribution of subsample values (y's). We can then examine the small-sample properties of this y-distribution; for example, investigating how well it is approximated by a normal distribution. We can also examine the y-distribution to gain further insight into the inadequacies of ordinary linear models for analyzing composite sampling procedures.

The model discussed in Section 4.2, which incorporates within-increment variability, is appropriate if (but not only if) subsampling proportions and increment values (X's) are normally distributed. Thus in an

application in which there is within-increment variability, we have reason to check whether the distribution of the α's has the central limit property.

Our investigation of distributional models for subsampling proportions begins with a discussion of the models proposed by Brown and Fisher [6] and Rohde [17]. We show that the standardized forms of these distributions are both asymptotically singular multivariate normal, which leads us to propose the singular multivariate normal model for the distribution of α's. Then having assumed that both α's and X's are normally distributed, we find the distribution of subsample values (y's) and examine its properties.

7.2 The Hypergeometric Model

The multivariate hypergeometric distribution was proposed as a model for the α's by Brown and Fisher [6], who applied it to a procedure for sampling baled wool. This model also appears to be appropriate for the α's in the grain-sampling example given in Section 2.2.

Suppose each of the n increments in a composite is divided into g equal-sized pieces, and each subsample is a random sample of G of the ng = SG pieces in the

composite. If $g_{j\ell}$ is the number of pieces from increment ℓ in subsample j, the hypergeometric model can be represented as a contingency table with fixed margins, as shown in Table 1. This model is discussed by Plackett [16], who gives the following expressions for moments of g's:

$$E(g_{j\ell}) = \frac{G}{n}$$

and (7.1)

$$Cov(g_{j\ell}, g_{j'\ell'}) = \frac{G(\delta_{j'j}ng-G)(\delta_{\ell'\ell}n-1)}{n^2(ng-1)}$$

with

$$\delta_{i'i} = \begin{cases} 1 & \text{if } i=i' \\ 0 & \text{if } i\neq i' \end{cases} .$$

If we define subsampling proportions in the usual manner ($\alpha_{j\ell} = g_{j\ell}/G$), then (7.1) yields the moments of the α's given in (2.23) through (2.25) for the nonsegmented lot case, with

$$\sigma_\alpha^2 = \frac{(n-1)(ng-G)}{n^2G(ng-1)} . \qquad (7.2)$$

Brown and Fisher [6] found the formula corresponding to (7.2) for the segmented lot case.

Table 1. Arrangement of Subsampling Counts (g_{ij}'s) as Contingency Table

Subsample	Increment 1	2	...	n	
1	g_{11}	g_{12}	...	g_{1n}	G
2	g_{21}	g_{22}		g_{2n}	G
⋮	⋮			⋮	⋮
S	g_{S1}	g_{S2}	...	g_{Sn}	G
	g	g	...	g	

Formula (7.2) shows that for the hypergeometric model σ_α^2 is a decreasing function of n for given g and G. Putting (7.2) into (2.43), it can be shown that when the hypergeometric model is appropriate, $\mathrm{Var}(\bar{z})$ is a decreasing function of n. Thus when the hypergeometric distribution can be assumed for the subsampling proportions, we can minimize $\mathrm{Var}(\bar{z})$ by forming composites that contain the largest feasible number of increments (n).

The asymptotic distribution of the standardized α's is found by letting $g \to \infty$, holding n and S constant, with $ng = SG$. Then $ng \to \infty$, $G \to \infty$, and $G/ng = 1/S$ (constant). Under these conditions Plackett [16] shows that the asymptotic distribution of the standardized g's is singular multivariate normal. Thus if we write $\underline{\alpha}_j' = (\alpha_{j1},\ldots,\alpha_{jn})$, the asymptotic distribution of standardized $\underline{\alpha}_j$ is $SN(\underline{0},V)$, with

$$V = \frac{1}{(n-1)}\,[nI - J] \,. \qquad (7.3)$$

That is, the distribution of $\underline{\alpha}_j$ has the central limit property.

Suppose that each increment consists of a fixed volume of solid material. Then since g is the number of pieces into which each increment is divided, increasing g

is equivalent physically to dividing each increment into more pieces of smaller size. In applications in which composites are ground, the hypergeometric model shows that the normal approximation to the distribution of the α's is better the more finely the material is ground. In applications in which the material consists of pre-existing equal-sized particles (e.g., kernels of grain), the normal approximation to the distribution of the α's is better the more particles there are per fixed-volume increment. Thus, for example, assuming increments of equal volume are taken in both cases, the normal approximation to the distribution of the α's is better when sampling wheat than when sampling corn.

7.3 The Dirichlet Model

The pdf of the Dirichlet distribution can be written [22]

$$f(u_1,\ldots,u_{n-1};u_n) \tag{7.4}$$

$$= \frac{\Gamma(v_1+\ldots+v_n)}{\Gamma(v_1)\ldots\Gamma(v_n)} u_1^{v_1-1} \ldots u_{n-1}^{v_{n-1}-1} \left(1-\sum_{i=1}^{n-1} u_i\right)^{v_n-1}$$

with $u_i \geq 0$ and $v_i > 0$ for all i and

$$\sum_{i=1}^{n} u_i = 1 \ . \tag{7.5}$$

The notation $f(u_1,\ldots,u_{n-1};u_n)$ is used to indicate that given n-1 components of $\underline{u}' = (u_1,\ldots,u_n)$, the other component is uniquely determined by (7.5). That is, the Dirichlet distribution with pdf (7.4) is essentially an (n-1)-variate distribution.

Rohde [17] proposes the Dirichlet distribution with $v_i = v$ for $i = 1,\ldots,n$ as a distributional model for subsampling proportions. He assumes that the $\underline{\alpha}_j' = (\alpha_{j1},\ldots,\alpha_{jn})$ for $j = 1,\ldots,s$ are independent and identically distributed, which is a reasonable assumption as long as S (the number of possible subsamples per composite) is large. If S is not large, however, (2.25) shows that the assumption that the $\underline{\alpha}_j$'s are independent is unwarranted, and the symmetric Dirichlet model is not appropriate for procedures with more than one subsample per composite.

If $\underline{\alpha}' = (\alpha_1,\ldots,\alpha_n)$ has a symmetric Dirichlet distribution, its moments are [17]

$$E(\underline{\alpha}) = \frac{1}{n}\underline{1}$$

and

$$\mathrm{Var}(\underline{\alpha}) = \frac{\sigma_\alpha^2}{(n-1)} [nI - J]$$

with

$$\sigma_\alpha^2 = \frac{(n-1)}{n^2(nv+1)} \tag{7.6}$$

(there is an error in Rohde's formula for σ_α^2 in [17]). Putting (7.6) into (2.43) with either s=1 or S= ∞ , we can show that $\mathrm{Var}(\bar{z})$ decreases as n increases if we assume the symmetric Dirichlet model for α's.

We now assume that $\underline{\alpha}$ has a symmetric Dirichlet distribution and show that the asymptotic distribution of standardized $\underline{\alpha}$ as $v \to \infty$ is singular multivariate normal. Define

$$\underline{w} = (\underline{\alpha} - \frac{1}{n}\underline{1})/\sigma_\alpha \ . \tag{7.7}$$

Then the pdf of $\underline{w}$ is

$$f(w_1,\ldots,w_{n-1};w_n) = \frac{\Gamma(nv)}{\Gamma(v)^n} \left(\frac{1}{n}\right)^{n(v-1)} \sigma_\alpha^{n-1} \tag{7.8}$$

$$\times (1-n\sigma_\alpha \sum_{j=1}^{n-1} w_j)^{v-1} \prod_{i=1}^{n-1} (1+n\sigma_\alpha w_i)^{v-1}$$

with $\sum_{i=1}^{n} w_i = 0$. Stirling's formula and (7.6) show that

$$\lim_{v \to \infty} \frac{\Gamma(nv)}{\Gamma(v)^n} \left(\frac{1}{n}\right)^{n(v-1)} \sigma_\alpha^{n-1}$$

$$= \left(\frac{n-1}{2\pi}\right)^{\frac{n-1}{2}} \left(\frac{1}{n}\right)^{\frac{n-2}{2}} . \tag{7.9}$$

Next we find the asymptotic value of the natural log of the other factor in (7.8),

$$\ell n \left[(1-n\sigma_\alpha \sum_{j=1}^{n-1} w_j)^{v-1} \prod_{i=1}^{n-1} (1+n\sigma_\alpha w_i)^{v-1}\right] \tag{7.10}$$

$$= (v-1)\left[\sum_{i=1}^{n-1} \ell n(1+n\sigma_\alpha w_i) + \ell n(1-n\sigma_\alpha \sum_{j=1}^{n-1} w_j)\right] .$$

The series expansions of $\ell n(1+n\sigma_\alpha w_i)$ and $\ell n(1-n\sigma_\alpha \sum w_j)$ are valid since for any given $\underline{w}' = (w_1,\ldots,w_n)$, $|n\sigma_\alpha w_i|<1$ and $|n\sigma_\alpha \sum w_j|<1$ for all sufficiently large v ($\sigma_\alpha \to 0$ as $v\to\infty$). Thus (7.10) can be written in the form

$$(v-1)\left(\sum_{i=1}^{n-1} \left[(n\sigma_\alpha w_i) - \frac{1}{2} (n\sigma_\alpha w_i)^2 + \ldots \right] \right.$$

$$\left. - \left[(n\sigma_\alpha \textstyle\sum w_j) + \frac{1}{2} (n\sigma_\alpha \textstyle\sum w_j)^2 + \ldots \right] \right) \qquad (7.11)$$

$$= - \frac{(v-1)}{2} n^2 \sigma_\alpha^2 \left[\sum_{i=1}^{n-1} w_i^2 + \left(\sum_{i=1}^{n-1} w_i \right)^2 \right] + r_v(\underline{w}) \ ,$$

where $r_v(\underline{w})$ tends to zero for fixed $\underline{w}$ as v goes to infinity. The limit of (7.11) as $v \to \infty$ therefore equals

$$- \frac{1}{2} \left(\frac{n-1}{n} \right) \left[\sum_{i=1}^{n-1} w_i^2 + \left(\sum_{i=1}^{n-1} w_i \right)^2 \right] \qquad (7.12)$$

using (7.6). Putting (7.9) and (7.12) together, we get

$$\lim_{v \to \infty} f_v(w_1, \ldots, w_{n-1}; w_n) = f(w_1, \ldots, w_{n-1}; w_n) \qquad (7.13)$$

$$= \left(\frac{n-1}{2\pi} \right)^{\frac{n-1}{2}} \left(\frac{1}{n} \right)^{\frac{n-2}{2}} \exp\left\{ - \frac{1}{2} \frac{n-1}{n} \left[\sum_{i=1}^{n-1} w_i^2 + \left(\sum_{i=1}^{n-1} w_i \right)^2 \right] \right\} .$$

Since $f(w_1, \ldots, w_{n-1}; w_n)$ is a multivariate normal pdf,

Scheffe's Theorem [18] says that $(w_1,\ldots,w_{n-1})$ is asymptotically multivariate normal with the (n-1)-dimensional variance-covariance matrix

$$\frac{1}{(n-1)}\,[nI - J]\ .$$

This and the condition that $\underline{1}'\underline{w} = 0$ imply that $\underline{w}$ is asymptotically $SN(\underline{0},V)$, with the singular n-dimensional variance-covariance matrix V given by (7.3).

7.4 The Singular Multivariate Normal Model

We have seen that the distributional models suggested in [6] and [17] lead asymptotically to the singular multivariate normal model. In fact, one would expect the singular multivariate normal model to be a reasonable approximation in many cases because of the physical averaging that occurs in blending a composite. Therefore, in this section we assume that

$$\underline{\alpha} \sim SN(\tfrac{1}{n}\,\underline{1},V) \qquad (7.14)$$

with V given by (2.24). This model is useful for finding

the distribution of subsample values, but it does not show how σ_α^2 varies with n.

Let $Y = \underline{\alpha}' \underline{X}$ and suppose that

$$\underline{X} \sim N(\underline{0}, \sigma_x^2 I) \tag{7.15}$$

and that

$$\mathrm{Cov}(\underline{\alpha}, \underline{X}) = 0$$

(we can assume without loss of generality that $\mu_x = 0$). We now find the moment generating function (mgf) of Y using a method demonstrated by Searle [19]. Since V has dimension n × n and rank n-1, we can write

$$\underline{\alpha} = \frac{1}{n} \underline{1} + L \underline{u} \tag{7.16}$$

where

$$\underline{u} \sim N(\underline{0}, I_{n-1}) \tag{7.17}$$

and

$$LL' = V \tag{7.18}$$

(L is n by n-1 with rank n-1). Then

$$M_Y(t) = (2\pi)^{-n} \sigma_x^{-n}$$

$$\times \int \exp\left[-\frac{1}{2} \underline{u}'\underline{u} - \frac{1}{2\sigma_x^2} \underline{x}'\underline{x} + t\underline{\alpha}'\underline{x} \right] d\underline{x}\, d\underline{u} \,. \tag{7.19}$$

Rewriting the exponent and integrating out the x's gives

$$M_Y(t) = (2\pi)^{-\frac{n}{2}}$$

$$\times \int \exp\left[-\frac{1}{2} \underline{u}'\underline{u} + \frac{t^2\sigma_x^2}{2} \left(\frac{1}{n}\underline{1} + L\,\underline{u}\right)' \left(\frac{1}{n}\underline{1} + L\,\underline{u}\right) \right] d\underline{u} \,. \tag{7.20}$$

Next rewrite the exponent in (7.20) in the form

$$-\frac{1}{2}\left[(\underline{u} - \underline{c})'(I - t^2\sigma_x^2 L'L)(\underline{u} - \underline{c}) \right]$$

$$+ \frac{t^2\sigma_x^2}{2} \left(\frac{1}{n}\underline{1}\right)' \left[I + t^2\sigma_x^2 L\,(I - t^2\sigma_x^2 L'L)^{-1} L' \right] \left(\frac{1}{n}\underline{1}\right) , \tag{7.21}$$

with

$$\underline{c}' = t^2\sigma_x^2 \left(\frac{1}{n}\underline{1}\right)' L(I - t^2\sigma_x^2 L'L)^{-1} \,.$$

Putting (7.21) in (7.20) and integrating out the u's gives

$$M_Y(t) = |I - t^2\sigma_x^2 L'L|^{-\frac{1}{2}}$$

$$\times \exp\left\{ \frac{t^2\sigma_x^2}{2} \left(\frac{1}{n}\underline{1}'\right)\left[I + t^2\sigma_x^2 L(I - t^2\sigma_x^2 L'L)^{-1}L'\right]\left(\frac{1}{n}\underline{1}\right)\right\} . \tag{7.22}$$

To simplify (7.22), we need an explicit expression for $L'L$. The eigenvalues of V are solutions to

$$0 = |V - \lambda I|$$

$$= \left[\frac{n\sigma_\alpha^2}{(n-1)} - \lambda\right]^{n-1} (-\lambda)$$

[13]. Thus eigenvalues of V are zero (multiplicity one) and $n\sigma_\alpha^2/(n-1)$ (multiplicity n-1). Now since V is symmetric, there is an orthogonal matrix P such that

$$P\,V\,P' = \begin{bmatrix} D_{n-1}^2 & \underline{0} \\ \underline{0}' & 0 \end{bmatrix} \tag{7.23}$$

with

$$D_{n-1}^2 = \frac{n\sigma_\alpha^2}{(n-1)} I ,$$

by Theorem 1.31 in Graybill [12]. Equation (7.23) is equivalent to

$$V = P' \begin{bmatrix} D_{n-1}^2 & \underline{0} \\ \underline{0}' & 0 \end{bmatrix} P ,$$

and this together with (7.18) gives

$$L = P' \begin{bmatrix} D_{n-1} \\ \underline{0}' \end{bmatrix} . \tag{7.24}$$

Therefore,

$$L'L = D_{n-1}^2 = \frac{n\sigma_\alpha^2}{(n-1)} I . \tag{7.25}$$

Putting (7.25) into (7.22) and simplifying gives the desired result, the mgf of Y:

$$M_Y(t) = e^{\frac{t^2\sigma_x^2}{2n}} \left[1 - \frac{nt^2\sigma_x^2\sigma_\alpha^2}{(n-1)} \right]^{-(\frac{n-1}{2})} . \tag{7.26}$$

As a check on (7.26), we find using this mgf that

$$E(Y) = 0$$

and

$$\text{Var}(Y) = \sigma_y^2 = \frac{\sigma_x^2}{n} + n\sigma_\alpha^2\sigma_x^2 \ .$$

These results agree with the results of Chapter II when $\mu_x = 0$.

The form of the mgf of Y reveals several properties of the distribution of a subsample value. Since $M_Y(it)$ is real (for $i = \sqrt{-1}$), the distribution of Y is symmetric. The mgf of Y is a product of two moment generating functions, the mgf of a $N(0,\sigma_x^2/n)$ random variable and the mgf of a random variable with a Bessell function distribution [23]. Thus Y can be written

$$Y = \bar{X} + d \ , \qquad (7.27)$$

where $\bar{X}$ is the mean of the n increment values in the composite and d is a random variable independent of $\bar{X}$ representing the error due to subsampling the composite. Relation (7.27) shows that the ordinary linear model for Y is adequate in that it has composite and subsample effects that are independent. However, the ordinary linear model for Y does not show the relationship of the variance σ_d^2 to σ_x^2 and n.

In using a composite sampling procedure to obtain a confidence interval on μ_x it is often assumed that subsample values are normally distributed. To check whether this assumption is reasonable, we define

$$w = Y/\sigma_y \tag{7.28}$$

and find the asymptotic distribution of w as $n \to \infty$. Equations (7.26) and (7.28) give

$$M_w(t) = \exp\left[\frac{t^2}{2(1+n^2\sigma_\alpha^2)}\right] \times \left[1 - \frac{t^2}{(n-1)(1+1/n^2\sigma_\alpha^2)}\right]^{-(\frac{n-1}{2})}, \tag{7.29}$$

or

$$\ell n\ M_w(t) = \frac{t^2}{2(1+n^2\sigma_\alpha^2)} - \left(\frac{n-1}{2}\right)\ell n\left[1 - \frac{t^2}{(n-1)(1+1/n^2\sigma_\alpha^2)}\right].$$

The series expansion for

$$\ell n\left[1 - \frac{t^2}{(n-1)(1+1/n^2\sigma_\alpha^2)}\right]$$

is valid since for any given t,

$$\frac{t^2}{(n-1)(1+1/n^2\sigma_\alpha^2)} < 1$$

for all sufficiently large n (recall that $n^2\sigma_\alpha^2 \leq (n-1)$ by Theorem 1). Thus

$$\ell n\ M_w(t) = \frac{t^2}{2(1+n^2\sigma_\alpha^2)}$$

$$-\left(\frac{n-1}{2}\right)\left[-\frac{t^2}{(n-1)(1+1/n^2\sigma_\alpha^2)} - \frac{t^4}{2(n-1)^2(1+1/n^2\sigma_\alpha^2)^2} - \cdots\right]$$

$$= \frac{t^2}{2} + r_n(t)$$

where the remainder $r_n(t)$ goes to zero for every t as $n \to \infty$. Thus

$$\lim_{n \to \infty} M_w(t) = e^{\frac{t^2}{2}},$$

and w is asymptotically N(0,1). Sufficient conditions for the asymptotic normality of bilinear forms in general are given in [5].

The skewness and kurtosis of the Y-distribution give an indication as to how well this distribution is approxi-

mated by a normal distribution for finite n. We can show using (7.26) that

$$E(Y^3) = 0$$

and

$$E(Y^4) = \frac{3\sigma_x^4}{n^2}\left[1 + 2n^2\sigma_\alpha^2 + \frac{n^4(n+1)\sigma_\alpha^4}{(n-1)}\right].$$

Thus the skewness and kurtosis of the Y-distribution are [14]

$$\gamma_1 = \frac{E(Y^3)}{\sigma_Y^3} = 0$$

and

$$\gamma_2 = \frac{E(Y^4)}{\sigma_Y^4} - 3$$

$$= \frac{6}{(n-1)}\left[\frac{n^4\sigma_\alpha^4}{(n^2\sigma_\alpha^2+1)^2}\right], \qquad (7.30)$$

respectively ($\gamma_1 = \gamma_2 = 0$ for a normal distribution). To evaluate γ_2 for given n, we must know σ_α^2 for that n. However, using Theorem 1 we can show that

$$\gamma_2 \leq \frac{6(n-1)}{n^2} . \qquad (7.31)$$

Thus as n increases, γ_2 approaches zero at least as rapidly as n^{-1}.

In this section we have derived the distribution of a subsample value, assuming that subsampling proportions and increment values are normally distributed. We have shown that the distribution of a subsample value has the central limit property and have indicated how well this distribution is approximated by a normal distribution for finite n.

VIII. SUMMARY

This dissertation is a study of composite sampling procedures used to estimate the mean (μ_x) of each of a sequence of stationary lots. Such procedures are commonly used in cases in which high testing costs preclude estimation of μ_x by separately testing several samples from a lot and computing the arithmetic average of the test results.

The use of ordinary linear models to describe composite sampling procedures is discussed by Duncan [9]. A more realistic approach to modeling these procedures, first applied by Brown and Fisher to a simplified sampling situation in [6], is developed in this dissertation.

Our objectives are (1) to develop Brown-Fisher models for the composite sampling procedures more commonly used with segmented or nonsegmented stationary lots, (2) to compare the Brown-Fisher results with results from the customary linear models for these procedures, (3) to describe the properties of composite sampling procedures revealed by Brown-Fisher models, and (4) to show where possible how to choose a composite sampling procedure. The final result of each Brown-Fisher model developed under the first objective is a formula that expresses the variance of the estimator of μ_x in terms of parameters of the model.

The remaining objectives are addressed by examining these formulas for different procedures and comparing them with corresponding formulas from the customary models.

Brown-Fisher models for nonsegmented lot procedures are presented in Chapters II and IV. The basic model in Chapter II extends results of Rohde [17] to allow for finite composites and testing error. It is shown that the mean of observations from the basic procedure, $\bar{z}$, is an unbiased estimator of μ_x. Formulas for $\text{Var}(\bar{z})$ derived from the Brown-Fisher model and from the usual linear model are given in (2.43) and (2.49), respectively. In Chapter IV the model for the basic procedure is extended to allow for two subsampling stages or within-increment variability. It is shown that $\bar{z}$ remains an unbiased estimator of μ_x in these more general models. The effects of the extensions on the formula for $\text{Var}(\bar{z})$ are shown in (4.21) and (4.39).

Problems of inference for the basic composite sampling procedure for nonsegmented lots are discussed in Chapter III. The results of Chapter III indicate the futility of running a small experiment (one with few composites) to estimate either $\text{Var}(\bar{z})$ or the components of $\text{Var}(\bar{z})$. Best unbiased estimators of components of $\text{Var}(\bar{z})$ are found, assuming that observations from the composite sampling experiment are normally distributed. It is shown that the probability of obtaining positive estimates of these components is small

unless the number of composites (r) in the experiment is large. The best unbiased estimator of $\mathrm{Var}(\bar{z})$ is shown to have large variance unless r is large. A tracer experiment for estimating the variance of subsampling proportions and testing assumptions concerning moments of these proportions is discussed.

Brown-Fisher models for segmented lots are discussed in Chapter V. For all the models investigated it is shown that $\bar{z}$ is an unbiased estimator of μ_x. The formulas for $\mathrm{Var}(\bar{z})$ from the basic Brown-Fisher model and from the usual linear model are given in (5.20) and (5.24), respectively. The effects on $\mathrm{Var}(\bar{z})$ of two subsampling stages or within-increment variability are shown in (5.44) and (5.55). It is shown that unlike the nonsegmented lot case there is no unbiased estimator of $\mathrm{Var}(\bar{z})$ from the usual compositing experiment.

The results of Chapters II through V show that Brown-Fisher formulas for $\mathrm{Var}(\bar{z})$ differ in important ways from corresponding formulas derived from the customary linear models. Brown-Fisher results indicate that the subsampling component of $\mathrm{Var}(\bar{z})$ is a function of the number of increments in a composite (n), the variance of subsampling proportions (σ_α^2), and the between and within-increment variances, so estimating $\mathrm{Var}(\bar{z})$ and choosing n are more difficult than is indicated by an analysis based on the

customary model. The dependence of the subsampling component on σ_α^2 and the upper bound on σ_α^2 given in Theorem 1 show that there are upper bounds on $\mathrm{Var}(\bar{z})$ relative to the variance of the estimator of μ_x from a comparable noncompositing procedure. The customary linear models for composite sampling procedures do not reveal this important property.

Problems of choosing composite sampling procedures are discussed in Chapters IV, V, and VI. In Chapters IV and V formulas for $\mathrm{Var}(\bar{z})$ are compared for procedures with one or two subsampling stages; results of these chapters show that conditions favoring two subsampling stages are the same for segmented and nonsegmented lots. If we take one subsample at each stage, a two-stage procedure gives a more precise estimator of μ_x than a comparable one-stage procedure if better blending is possible with smaller amounts of material. If we take more than one subsample at either stage, the condition favoring a two-stage procedure is more complicated.

Deciding how many composites (r), subsamples (s), and tests (t) to use in a procedure with one subsampling stage is discussed in Chapter VI. If we can afford to run two tests per lot (rst=2), it is shown that we can check for changes in basic parameters from lot to lot using either two composites (r=2) or two subsamples (s=2). This conclusion

differs from the result of the customary analysis, which shows that we must run more than one composite to check for changes in basic parameters.

Choosing the number of increments per composite (n) to use in a procedure with one subsampling stage is also discussed in Chapter VI. The relationship between $\mathrm{Var}(\bar{z})$ and n revealed by a Brown-Fisher model is more complicated than that suggested by the usual linear model, which suggests that $\mathrm{Var}(\bar{z})$ always decreases with n. Thus choosing n is a more difficult problem than an analysis based on the customary model indicates. An approach to choosing n from an interval of feasible values is proposed.

To show how the variance of subsampling proportions (σ_α^2) depends on n and to find the distribution of subsample values in certain applications, we investigate distributional models for subsampling proportions in Chapter VII. The hypergeometric model proposed by Brown and Fisher [6] and the Dirichlet model suggested by Rohde [17] are reviewed — both models support the contention that σ_α^2 changes with n. It is shown that standardized forms of both these distributions are asymptotically singular multivariate normal. The distribution of subsample values is found, assuming that increment values and subsampling proportions are normally distributed.

We have shown that Brown-Fisher models are flexible

enough to cover commonly used composite sampling procedures. Of course, the credibility of the results of these models depends on the credibility of the assumptions made in constructing the models. It is certainly reasonable to assume that a subsample value is a randomly weighted average of values associated with the increments in a composite. If we randomize at each stage of a composite sampling procedure (as discussed in Chapter II), the moment structures for increment values and subsampling proportions used in developing the models also seem reasonable.

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PROPERTIES OF
COMPOSITE SAMPLING PROCEDURES

by

Robert S. Elder

(ABSTRACT)

In a composite sampling procedure initial samples (increments) are drawn from a lot and physically mixed to form composite samples. Subsamples are then taken from these composite samples and tested to determine the lot quality, usually the lot mean, μ_x.

Composite sampling procedures typically are employed with bulk materials, for which high testing costs preclude estimation of μ_x using the arithmetic average of values from several individually tested increments. Because of the physical averaging that occurs when increments are mixed to form composite samples, it is possible to estimate μ_x with specified precision with greater economy using a composite sampling procedure than using a noncompositing procedure.

This dissertation extends and interprets the work of Brown and Fisher on modeling procedures that involve

subsampling mixtures of sampled material. Models are developed for sampling from segmented or nonsegmented lots, allowing for more than one finite composite, testing error, within-increment variability, or two subsampling stages. The result of each model is a formula expressing the variance of the estimator of μ_x in terms of model parameters. Each such formula is contrasted with the corresponding formula derived from the customarily employed random effects linear model.

Among the significant properties of composite sampling procedures derived are the following: (1) there is a (relative) upper bound on the variance of the estimator of μ_x in a composite sampling procedure; (2) the variance of the estimator of μ_x does not necessarily decrease as the number of increments per composite increases; (3) it is not necessary to form more than one composite to check for changes in parameter values from lot to lot. The impact of these and other properties on the process of selecting composite sampling procedures is discussed.