Optimal Filters For Deconvolution Of Transient Signals In The Presence Of Noise

by

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(ABSTRACT)

This dissertation presents different methods for the deconvolution of time domain signals. The techniques developed in this work are frequency domain filtering techniques, and are suitable for the type of deconvolution problems encountered in time domain reflectometry (TDR). They include a smoothing technique that is a variant of the well known lowpass filter. This technique is parameter dependent in order to allow for adequate choice of cutoff frequency. Another more powerful method developed is an adaptive smoothing (regularization) technique, which is both frequency dependent and input-signal dependent as well. Thus, it is an adaptive technique whose performance depends on a parameter associated with its smoothing constraint.

These frequency domain techniques and their variants are parameter dependent; hence a parameter optimization criterion must be included. However, in deriving an optimization criterion, great importance must be given to its adequacy in the determination of the appropriate parameter value as well its time efficiency. A parameter optimization method that fulfills those two requirements is also developed. The method is fully implemented in the frequency domain in which the filtering techniques are used.

The techniques developed are derived with a magnitude component only, i.e., non-causal. The limited derivation is due to the fact that we are usually interested in reducing only the noise level from the magnitude point of view. However, if we consider time domain meas-
urements as an example, physical pulses and transients are causal functions of time, i.e., their values are zero before \( t = 0 \), the time at which they begin. Their measured waveform data are also causal. When deconvolution processing is applied to remove instrumentation errors and/or suppress the effects of noise, non-causal deconvolution methods, that were mentioned previously, may introduce unacceptable errors. The conventional deconvolution is modified to ensure that causality is maintained in the deconvolution result.

The impulse response of an unknown system is recovered from time domain reflectometry data by implementing a method based on the homomorphic deconvolution technique. In time domain reflectometry, the reflected waveform by a line with several discontinuities is represented as the convolution of the reflection coefficient of the line and the input excitation of the line source. The reflection coefficient is generally a train of spikes (delta functions) when the discontinuities are resistive. However, this is not the case when the discontinuities are capacitive in nature. In this work, we will attempt to show that the conventional frequency domain deconvolution techniques fail to provide good estimates when the waveform contains certain amounts of noise. Since it has been shown that homomorphic systems are useful in separating signals which have combined through convolution, homomorphic filtering can then be applied to recover either the input excitation or the impulse response (reflection coefficient) of the network.
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CHAPTER I

INTRODUCTION

The characterization of materials usually involves the use of harmonics that cover a wide range of frequencies. Different types of electrical devices are made out of these materials. Measurements are made by exciting them with a sweep of frequencies, obtaining the response to each frequency. By simply applying some mathematical formulas to these results, the characterization of the material is complete.

Frequency domain measurements are performed at a single frequency rate, allowing for the determination of the amplitude and the phase of the quantity being measured for that specific frequency. The system being measured must be assumed to be linear and time invariant. In order to evaluate the frequency dependence of the measured quantity, a frequency sweep over the band of interest must be performed. The amplitudes and phases obtained for each frequency must be recorded. However, certain physical quantities cannot be measured using frequency domain measurements, for example, the maximum amplitude of the system's impulse response or its duration. Such quantities can only be measured through time domain measurements. The characterization of the material is simply achieved through the measurement of the scattering parameters. These parameters are determined using identification or deconvolution techniques that are implemented to remedy the distortions that occur during the measurement process. In the area of time domain measurements, linearity of the system being measured (under test) is always assumed.
The theory of linear systems involves two types: those that can be described by parametric models, and others that can be described by non-parametric models. In the area of control, the first type of systems dominate the research studies. However, in areas such as time domain the second type is the most frequently encountered. To study such systems one seeks knowledge beyond what he sees directly. It involves the understanding of the relation between the excitation applied to the system and its corresponding response. While parametric system models consist of differential equations of predetermined form and order, non-parametric models consist of impulse responses.

For a linear system, the output signal is the convolution of its impulse response and the input signal applied to it. The convolution equation for a linear measurement system is

\[ y(t) = \int_{-\infty}^{+\infty} x(t - \tau)h(\tau)d\tau \]  \hspace{1cm} (1.1a)

where the waveforms \( x(t) \), \( h(t) \), and \( y(t) \) are the measurement system input, impulse response, and output waveforms, respectively. This equation is known as the Fredholm integral of the second kind, and is commonly expressed as

\[ y(t) = x(t)^*h(t) \]  \hspace{1cm} (1.1b)

where \( ^* \) denotes the convolution operation. When one of the integrand waveforms is unknown, while the other two waveforms are known, the equation becomes the deconvolution integral equation for the unknown waveform. The problem of solving Eq. (1.1a) analytically for either \( x(t) \) or \( h(t) \), is commonly known as deconvolution. In general, it has no analytical solution.

Deconvolution, as defined, is an inverse mathematical operation; it it pertains to the recovery of a signal degraded by the response of a linear system.

The deconvolution problem has been widely studied, from seismic exploration to time-domain measurements, and has been capturing the attention of researchers. Signals combined through convolution are encountered in various areas of physics, image processing, and spectroscopy, etc. Deconvolution is encountered in electromagnetic measurements while
measuring the impulse response (scattering parameters) of a system, network, or device. Basically, the problem is concerned with the separation of two convolved signals. The de-convolution problem is concerned with solving the integral Eq. (1.1a) in order to determine either the input signal \( x(t) \) or the impulse response \( h(t) \) provided that one of the two is known in addition to the output signal \( y(t) \).

The choice of the test signal is very important; since it must have a frequency domain bandwidth that would cover the bandwidth of the unknown signal for complete estimation. If the test signal acts as a lowpass filter with a certain cutoff frequency, the unknown signal can be theoretically estimated outside the passband by processes of analytic continuation of the spectrum from within the frequency band where it can be estimated. In addition, in an experimental situation, the imperfections of the data acquisition device and the measurement system noise add to the distortion and error corruption of the known waveforms. Thus, an ill-posed deconvolution problem arises. To circumvent the difficulties of the ill-posedness, different methods that account for the noise corruption must be sought.

There exist a number of approaches that perform signals deconvolution. One of the most widely used approaches is the linear inverse filtering technique that involves the use of the Fourier transform domain. The filtering of time series or signals consists basically of three parts: interpolation, smoothing and extrapolation. The interpolation is simply an estimation of the behavior of the signal \( x(t_0) \) over the interval \([0, t_0]\), while smoothing (noise reduction or elimination) is the estimation of \( x(t) \) for \( t < t_0 \) over the interval \([0, t_0]\); extrapolation (or prediction) means the estimation of \( x(t) \) for \( t > t_0 \) over the interval \([0, t_0]\). In the work being presented in this dissertation, the term filtering (or filter) that will be used very often will simply refer to the second kind of filtering (smoothing). Generally, noise occupies the high frequency bands of the measured data. As a consequence, the recorded data could be lowpass filtered prior to any type of processing.

This dissertation is divided into several chapters. The first chapter is simply a review of some deconvolution techniques that include time domain as well as frequency domain techniques, in addition to techniques based on homomorphic systems and cepstrum theory. The
time domain techniques are two types: those which are based on straightforward successive divisions or matrix inversion and techniques that use time domain iterative approaches. The frequency domain techniques benefit from the application of frequency domain filters. The homomorphic deconvolution techniques are a very important signal deconvolution tool for the class of systems where only the output signal is known.

In chapter two, we discuss the derivation of two frequency domain deconvolution techniques based on the Wiener filtering theory. These techniques use parametric filters that attempt to reduce the noise that contaminates certain frequency bands. Given the parametric nature of these frequency domain techniques, the optimal result can only be achieved through the implementation of a quality criterion that decide the appropriate (optimal) filter. It will be seen that, depending on the value of the parameter associated with the filters, different results are obtained.

In chapter three, the parametric optimization of the filters that are derived in chapter two is discussed. Two time domain based optimization criteria are reviewed in light of their applications. We also develop a new technique that is fully implemented in the frequency domain. The new technique is compared to the time domain techniques. Results that attest to the success of the new optimization technique are presented.

The fourth chapter presents a study of the causality of the frequency domain filters described in previous chapters. It is shown that the type of filters developed satisfy the Paley-Wiener criterion for causal functions. Knowing that the filters used are magnitude ones, a phase component is associated with the magnitude component. Results that are consistent with the nature of the signals being deconvolved are achieved.

Chapter five presents the application of the homomorphic deconvolution technique to time domain reflectometry. It is shown that signals combined through convolution are successfully separated and recovered from the output alone. The success of the cepstrum techniques is studied when noisy signals are used and new results are reported. Finally, chapter six presents a summary of the deconvolution methods and the major contributions of the work presented in this dissertation. Also, some concluding remarks about the solution to the de-
convolution problem are presented along with an identification of future directions of research in the area.
CHAPTER II

REVIEW OF DECONVOLUTION TECHNIQUES

2.1 INTRODUCTION

The deconvolution problem is widely studied; from seismic exploration to time-domain measurements, the problem has captured the attention of researchers. We encounter signals combined through convolution in areas such as physics, image processing and spectroscopy to name just a few. Deconvolution is encountered in electromagnetic measurements while measuring the impulse response (scattering parameters) of a system, network, or device. Basically, the problem is concerned with the separation of two convolved signals. However, the problem does not have a unique solution, and thus, qualifies as an ill-posed problem as explained by Tikhonov et al [1]. Over the years, many deconvolution techniques have been developed for identification purposes of linear systems. Some of the techniques were developed based on a particular area of interest and were successfully used; however, few of those techniques were generalized to deal with all types of deconvolution applications. In the following section we will attempt to make a non-exhaustive review of some of the techniques that have found wide areas of application. Several studies of different deconvolution techniques are found in the literature. While some authors emphasize frequency domain techniques [2], others deal with techniques that have found wide application fields [3,4,5]. Other studies of
deconvolution techniques were limited to techniques applied in a particular area of interest, such as seismology [6].

We find that "linear" deconvolution methods have dominated deconvolution research and applications. They were developed with special characteristics to suit them to specific deconvolution problems. Among these techniques, we find the most intuitive method which is the direct approach that involves solving for each point step by step [5]. Another variant of this approach is a method that stems from a basic matrix formulation of the problem, and involves matrix inversion. However, the approach does not take into account the noise effect and the computation errors due to ill-conditioned data, which will cause oscillatory fluctuations in the estimate. On the other hand, iterative techniques that are due to Van Cittert have the advantage of allowing control of spurious fluctuations by interaction with the solution as it evolves. Van Cittert's method is somewhat intuitive, and this fact may account for its wide use and apparently independent reinvention by workers following Van Cittert [4]. The basic method has numerous variants and has engendered a number of constrained nonlinear methods that exhibit significantly better performance [7]. It has been firmly identified with both inverse filtering and the solution of linear equations by relaxation. Some of its variants are concerned with reducing and damping the noise that grows with each iteration when the method is applied to real data [8,9]. The method, as originally proposed, has utility where only modest correction is required.

Other techniques were developed using frequency domain inverse filtering techniques [10]. Using Wiener theory [11] on filters based on reducing the minimum mean squared error, different filters were derived by simply adding new constraints. Wiener filtering, for example, presupposes knowledge of the signal and noise spectra as well as the system response in order to specify a filter which minimizes the mean square error between the original signal and the estimate obtained. Most often, knowledge of the noise is not available, and therefore the method must be modified, as we will attempt to show, to compensate for that. In the area of time domain measurements, the signals being identified are usually frequency band-limited. These types of signals enjoy a particular interest in other fields [12,13], however, in deconvol-
olution problems their importance lies in their realizability. Using the concept of band-limitation, different techniques based on Wiener theory were developed to perform the deconvolution using frequency domain filters [14,15,16,17]. In general, the filters are parameter dependent and their performance depends on such parameters. Thus, optimization criteria must be implemented to find adequate parameter values [16,18].

The choice of an appropriate measurement method to characterize the transmission properties of a system is a delicate one. The choice has to be made between time domain and frequency domain techniques. If the impulse response of the system under test is known to be time limited and, not frequency band-limited, then the deconvolution should be performed in the time domain. In addition, if the DC term is important, then only time domain techniques can achieve the recovery of the DC term [19].

Generally, “nonlinear” systems such as convolutive or multiplicative systems are relatively difficult to analyze and to characterize mathematically. Some studies of this type of system exist, but are almost always reduced to very particular cases. Homomorphic processing while limited in regard to the enormous number of nonlinear systems, has the advantage of being easy to analyze mathematically and having important practical applications of general interest. Homomorphic processing stems from a generalization of linear system theory. It is based on the principle of generalized superposition, [20]. The importance of homomorphic processing lies in the separation of signals that are combined through multiplication or convolution, where only limited information is known about both of them. Conventional deconvolution techniques are helpless in such situations. In the past, linear “filtering” techniques, which are generally simple, were commonly used. However, they cannot be used if the signal is not combined with the others by algebraic addition, as in the case of signals combined by multiplication of convolution. Homomorphic deconvolution is a technique for decomposing a composite signal of unknown multiple wavelets overlapping in time. The technique offers the advantage that no prior assumption about the nature of the impulse response of the system. The complex cepstrum filtering presupposes that the measured signal can be mapped into ‘quefrency’ domain, wherein convolved functions occupy different re-
regions and can thus be separated from each other. Although the method could find a wide use in time domain measurements, it has been limited to few applications.

2.2 TIME DOMAIN TECHNIQUES

2.2.1a The Direct deconvolution

Time domain deconvolution methods operate directly on the time domain data rather than on transform data (such as Fourier transform). Since the physical manifestation of the signal occurs with the evolution of time, there are certain advantages in using time domain operations, simply because time is the physical domain of the signal where the causal and stable nature of the signal is readily recognized [21]. Direct operations on a physical signal involve operations on real quantities, and the properties are preserved under the operations applied to the signal. However, we always deal with discrete variables or sequences. There is always an error due to representing a continuous function by a sequence but it is not considered in this work. However, such an error (aliasing) can be very significant when spectral information is desired [22]. A sequence of N-values for \( x, x(0), x(1), \ldots, x(N - 1) \), is denoted as \( x(k) \). For discrete variables (i.e. sequences), the convolution operation corresponding to the integral in Eq. (2.1a) is the convolution sum of two sequences \( h(k) \) and \( x(k) \) which is given by:

\[
y(k) = x(k) * h(k) \tag{2.2.1a}
\]

\[
y(k) = \sum_{l=0}^{k} x(l) h(k - l) \quad k = 0, 1, 2, \ldots, N - 1 \tag{2.2.1b}
\]

For the causal signals, \( h(k) \) and \( x(k) \) are both equal to zero for \( k < 0 \). In this case, convolution equation can be written as set of equations as:
\[ y(0) = h(0) x(0) \]
\[ y(0) = h(1) x(0) + h(0) x(1) \]
\[ \ldots = \ldots + \ldots + \ldots \]
\[ y(N + 1) = h(N - 1) x(0) + h(N - 2) x(1) + \ldots + h(0) x(N - 1) \quad (2.2.2) \]

If we assume that \( x(0) \neq 0 \), then we may write the estimate of the first impulse response value as

\[
\hat{h}(0) = \frac{y(0)}{x(0)}
\]
\[ \hat{h}(1) = \frac{1}{x(0)} [y(1) - \hat{h}(0) x(1)] \]
\[ \ldots = \ldots + \ldots + \ldots \]
\[ \hat{h}(k) = \frac{1}{x(0)} [y(k) - \sum_{i=1}^{k-1} \hat{h}(i) x(k + 1 - i)] \quad (2.2.3) \]

In the previous equations, the values of \( \hat{h}(k) \) is a function of the estimated impulse response values \( \hat{h}(i) \) for \( i < k \). However, it is possible to express the estimated value \( \hat{h}(k) \) in terms of \( x(i) \) and \( y(i) \) for \( i = 1,k \) exclusively. In the practical application of the direct time domain method to the deconvolution problem, the choice of the first term in the input sequence, \( x(0) \), is critical. If the term \( x(0) \) is ill chosen the resultant deconvolved result for \( \hat{h}(k) \) will not converge.

The effect of errors or noise on the deconvolution process can be seen by considering Eq. (2.2.1b). The noise-free signals \( x(k) \) and \( y(k) \) were used to derive \( y(k) \). The sequence \( y(k) \) was consistent with \( x(k) \) and \( h(k) \) through the convolution equation. Now let us assume that only the first values of the sequence \( y(k) \) is noise contaminated as follows:

\[ y_n(0) = y(0) + n_y(0) \quad (2.2.4) \]

then the first two equation of the deconvolution would be rewritten as,
\[
\hat{h}_n(0) = \frac{1}{x(0)} \left[ y(0) + n_y(0) \right] \\
\hat{h}_n(1) = \frac{1}{x(0)} \left[ y(1) - x(1)\hat{h}(0) \right] + \frac{1}{x^2(0)} x(1)n_y(0)
\] (2.2.5)

The direct approach is of little use when dealing with data acquired from real measurement observations. Even for good signal-to-noise ratios, the estimated impulse response could suffer considerably from the fluctuations. The approach is very intuitive and does not provide any means to explicitly account for the effects of noise.

2.2.1b The Matrix formulation

In the formulation of the convolution equation, it was assumed that the type of convolution used is linear. However, if the convolution is circular, the equations would be formulated differently and the direct approach described in the previous section would be useless. Based on the equations, which represent a set of linear equations in unknowns \(h(i)\) \((i = 1,n)\), a matrix formulation to the problem can be sought. However, in the solution of linear equations, small changes in the values of the elements in \(x(k)\) can cause very large changes in the solution \(h(k)\). When this occurs, the matrix \(X(k)\) is said to be ill-conditioned. The preceding set of equations can be written in a matrix form as

\[
Y(k) = X(k) H(k)
\] (2.2.6)

where

\[
Y(k) = [y(0) \ y(1) ... y(N - 1)]^T
\]

\[
H(k) = [h(0) \ h(1) ... h(N - 1)]^T
\]
\[
\mathbf{X}(k) = \begin{bmatrix}
    x(0) & x(N-1) & x(N-2) & \ldots & x(1) \\
    x(1) & x(0) & x(N-1) & \ldots & x(2) \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    x(N-1) & x(N-2) & x(N-3) & \ldots & x(0)
\end{bmatrix}
\]

An intuitive solution to Eq. (2.2.6) would be

\[
\mathbf{H}(k) = \mathbf{X}^{-1}(k) \mathbf{Y}(k)
\]

The matrix has a certain property of symmetry. Each row is the same as the row above it except that it is shifted one element to the right. Usually, the indices are chosen so that the largest element is on the diagonal. The direct solution of the set of linear equations involves the matrix inversion of \(\mathbf{X}(k)\). If the matrix is very large, computational problems occur due to limitation in computer time and storage. However, this problem is not as important as the difficulty usually encountered with this type of matrix. The rows are similar to a certain degree, and each new row brings little added information. This special symmetry property constitutes the main obstacle in the implementation of the matrix inversion. Usually, if the matrix is not singular, it is at best ill-conditioned. And in all of these difficulties, we still did not account for the effects of noise that occur in real data.

It is well known that large systems of linear equations can be solved iteratively provided that the diagonal element is larger than any of its neighbors in the same row. For special cases of data, the Gauss-Seidel fixed-point iteration can be used to estimate the vector [4,23]. Gauss-Seidel iteration converges if the coefficient matrix \(\mathbf{X}(k)\) is strictly (row) diagonally dominant. It also converges if \(\mathbf{X}(k)\) is positive definite, i.e., if \(\mathbf{X}(k)\) is real symmetric and for all nonzero vectors.

The solution of the linear system defined in Eq. (2.2.6) can be achieved by fixed-point iteration. The iteration scheme is based on the notion of approximate inverse matrix \(\mathbf{C}(k)\) for which
\[ \| I - C(k)X(k) \| < 1 \] (2.2.8)

and the fixed-point iteration would be given by

\[ h_{m+1}(k) = h_m(k) + C(k)(Y(k) - X(k)h_m(k)) \quad m = 0, 1, 2, \ldots. \] (2.2.9)

If the matrix \( X(k) \) is row diagonally dominant, then we let \( D(k) = \text{diag}(X(k)) \) be the diagonal of \( X(k) \). Then the corresponding iteration scheme, called Jacobi iteration, becomes,

\[ h_{m+1}(k) = h_m(k) + D^{-1}(k)(Y(k) - X(k)h_m(k)) \quad m = 0, 1, 2, \ldots. \] (2.2.10)

If the matrix \( C(k) \) is chosen as the inverse of a triangular matrix, the Gauss-Seidel iteration technique is derived. This leads to a faster convergence rate than the Jacobi iteration. As pointed out earlier, iterative methods are usually applied to large systems. They are less vulnerable to the growth of round-off error. However, they will not always converge, and even when they do converge, they may require large numbers of iterations.

In an experimental situation, the imperfections of the acquisition device and the measurement system noise which is added to the signal make the available data not representative of the original continuous waveform but of something close to it. The convolution relationship holds; however, the deconvolution becomes unstable. Furthermore, an exact solution does not exist and must be replaced by an estimate. The problem is to find an approximate solution that is stable under small changes in the initial data. To do that one uses a Wiener filter type of solution, where the quality of the solution is controlled through the use of some specific parameters. The result depends on the characteristics of the processed waveforms and on the method used to perform the deconvolution that involves the mentioned parameters.
2.2.1c The Van Cittert technique

The Van Cittert technique is an iterative time domain technique that has a certain similarity to the Gauss-Seidel iteration technique. This technique consists of forming successive approximations of the unknown system impulse response \( h(t) \), using the convolution equation,

\[
y(t) = h(t) * x(t)
\] (2.2.11)

For physically realizable signals the duration of \( x(t) \) is less than \( y(t) \). The initial approximation to \( h(t) \), \( h_1(t) \), is chosen to be \( y(t) \)

\[
h_1(t) = y(t);
\] (2.2.12)

then using \( h_1(t) \) in the convolution yields \( y_1(t) \).

\[
y_1(t) = h_1(t) * x(t);
\] (2.2.13)

however, \( y_1(t) \) differs from the true result \( y(t) \) by the error, \( e_y(t) \).

\[
e_y(t) = y(t) - y_1(t)
\] (2.2.14)

In the Van Cittert technique the term \( h_1(t) \) is corrected by adding an error term correction to yield the second approximation to \( h(t) \), \( h_2(t) \).

\[
h_2(t) = h_1(t) + \left[ y(t) - h_1(t) * x(t) \right]
\] (2.2.15)

Consequently, the \((i+1)\)th approximation is given by the following relation:

\[
h_{i+1}(t) = h_i(t) + \left[ y(t) - h_i(t) * x(t) \right]
\] (2.2.16)

The iteration is ended whenever

\[
h_{i+1}(t) * x(t) = y(t) \quad \text{then} \quad h_i(t) = h_{i+1}(t) = h(t)
\] (2.2.17)
If the Fourier transform is used, we get

\[ H_0(j\omega) = Y(j\omega) \]  
(2.2.18)

\[ H_{n+1}(j\omega) = H_n(j\omega) + \left[ Y(j\omega) - H_n(j\omega) \cdot X(j\omega) \right] \]  
(2.2.19)

Successive substitution gives the equation

\[ H_n(j\omega) = \left\{ 1 + \left[ 1 - X(j\omega) \right]^2 + \ldots + \left[ 1 - X(j\omega) \right]^n \right\} Y(j\omega) \]  
(2.2.20)

The series in braces consists of the first \( n \) terms of the binomial expansion of \( \left\{ 1 - [1 - X(j\omega)] \right\}^{-1} \). Hence the sequence of \( H_n(j\omega) \) converges \( Y(j\omega)/X(j\omega) \) when \( |1 - X(j\omega)| < 1 \). By the convolution theorem, this sequence limit is the transform of a solution to Eq. (2.2.11). For values of \( \omega \) such that \( X(j\omega) = 0 \) (as for frequencies greater than the cutoff frequency) it must be when the noise is absent that \( Y(j\omega) = 0 \); therefore \( H_n(j\omega) = 0 \) at these frequencies. Consequently, there is convergence if and only if

\[ |1 - X(j\omega)| < 1 \quad \{ \omega : X(j\omega) \neq 0 \} \]  
(2.2.21)

\[ X(j\omega) = 0 \quad \{ \omega : X(j\omega) = 0 \} \]  
(2.2.22)

The conditions for convergence of the Van Cittert techniques given in Eqs. (2.2.21) and (2.2.22) were first given by Bracewell et al [24]. A study of the convergence conditions of the technique was also reported [25]. It was shown that the technique can not be used with some types of impulse responses. Some of the restrictions are apparent from the shape of the impulse response.

The Van Cittert technique can also be considered as a Wiener type filtering technique as it will be shown. Equation (2.2.20) can be written in the following form:

\[ H_n(j\omega) = Y(j\omega) \sum_{k=0}^{n} [1 - X(j\omega)]^k \]  
(2.2.23)
If we use the following identity Eq. (2.2.24), for which \( \eta = 1 - X(j\omega) \)

\[
(1 - \eta) \sum_{k=0}^{j} \eta^k = 1 - \eta^{j+1}
\]  

(2.2.24)

Then, Eq. (2.2.20) can be expressed as a filtering type estimate;

\[
H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} B(j\omega)
\]  

(2.2.25)

where

\[
B(j\omega) = 1 - [1 - X(j\omega)]^{j+1}
\]

For \( i \to \infty \), the term \( B(j\omega) \) approaches unity, provided that \( |1 - X(j\omega)| < 1 \). In this case, \( H_j(j\omega) \) represents the estimated transfer function obtained by inverse filtering. For \( k \) finite, the inverse filter is modified by a factor that suppresses frequencies for which \( X(j\omega) \) is small. For larger \( k \), we would obtain less suppression. For a typical transfer function \( X(j\omega) \) that suppresses the high frequencies, the factor \( B(j\omega) \) controls the high-frequency content of \( h_i(t) \).

Bracewell's convergence conditions showed that Van Cittert's method and its variants only converge for \( |1 - X(j\omega)| < 1 \). But, the restriction is alleviated by introducing some type of parameter scaling [7]. The scaling is achieved by simply replacing \( Y(j\omega) \) and \( X(j\omega) \) with \( CY(j\omega) \) and \( CX(j\omega) \) in the convolution equation. Thus, the new condition for the convergence of the method becomes \( |1 - CX(j\omega)| < 1 \). Although the non-negativity of the transfer function does not ensure the convergence of the method, it represents a necessary condition to achieve convergence. Thus, a reblurring concept was introduced to guarantee that the transfer function does not have negative values [26,27]. The reblurring concept is based on the convolution of a reversed version of the input waveform with it and with the observed output waveform. The obtained function cannot have negative values.
In all the previous discussion, not once did we mention the effects of noise on the convergence of the technique. Suppose that the observed response is known with a certain error \( n_y(t) \), then

\[
y_n(t) = y(t) + n_y(t). \tag{2.2.26}
\]

Now, if we substitute \( y_n(t) \) for \( y(t) \) in Eq. (2.2.23), we obtain the following relation:

\[
H_{i+1}(j\omega) = H(i\omega) \sum_{k=0}^{i+1} \left(1 - X(j\omega)\right)^k + (i + 1)N_y(j\omega). \tag{2.2.27}
\]

From Eq. (2.2.27), it is easy to see that the noise content grows linearly with each iteration [3]. Unless some other types of filtering techniques are used, to reject frequencies that \( X(j\omega) \) is incapable of transmitting, the solution obtained is distorted by noise. Some modifications of Van Cittert’s method were developed to attenuate the noise effects. A method that was successful [8,9], used a relaxation parameter that depends on the estimate in the iterative process. The modified iteration equation is given by

\[
h_{i+1}(t) = h_i(t) + r[h_i(t)[y(t) - h_i(t)x(t)]. \tag{2.2.28}
\]

where \( r[h_i(t)] \) was arbitrarily defined by Jansson [28] as:

\[
r[h_i(t)] = r_0[1 - 2|h_i(t) - 0.5|] \tag{2.2.29}
\]

The practical application of the new constrained Van Cittert’s method yielded a high resolution enhancement of spectra [8]. The relaxation function defined in Eq. (2.2.29) is not unique [28]. Considerable imagination could be used to devise different relaxation functions, however, inferior results are obtained if the function does not ensure a gradual correction to the error in the result as the iterative process proceeds. The value of \( r_0 \) is arbitrary and is established either by trial and error, or by investigations of the eigenvalues of the convolution matrix, which could be a difficult task to achieve. Different types of relaxation functions were studied.
results using quadratic relaxation function and its generalization to other powers were reported. Discussion of the reblurrring method, and a study of a generalized class of iterative deconvolution algorithms are found elsewhere [27,29].

A broad class of iterative signal restorations techniques which can be applied to remove the effects of many different types of distortions are described in great detail by Schaeffer et al [30]. Those techniques allow for the incorporation of prior knowledge of the signal in terms of the specification of the constraint operator. Conditions for convergence of the iteration under various combinations of distortions and constraints are also explored. In another paper [31], an accelerated iterative deconvolution algorithm is developed based on these iterative restoration techniques. The algorithm was based on the idea of updating the observation equation after each iteration. It is shown that with this approach the modified iterative algorithm achieves a quadratic rate of convergence instead of only a linear one. Another iterative method for optimum correction of the convolutional distortion was formulated entirely in the time domain [32,33]. The method uses system compensation to correct the convolutional distortions associated with linear physical systems. The correction is based on the minimization of a time domain error criterion that is subject to physical realizability constraints.

In order to solve Fredholm's integral equations of the first kind, i.e., the convolution integral, Philips [34] and Twomey [35,36] separately developed a numerical formulation of the problem. Using a matrix approximation to the integral equations, they replaced them by a linear system. Philips noticed that the oscillations in the estimated \( \hat{h}(k) \) lead to high values for second-differences \( \hat{h}(k + 1) - 2\hat{h}(k) + \hat{h}(k - 1) \). Thus, it seemed reasonable to seek a solution that has small second-differences values since such a solution tends to be more accurate. In a trade-off of consistency for smoothness, he selected to seek a solution that satisfies both using an arbitrary parameter to determine the extent of the smoothness being applied. Subsequent work based on this technique was also reported [37,38]. Some iterative deconvolution techniques based upon the non-negativity of the measured quantity [3,42], were revealed to be very successful. However, their implementation would require too much time, and their convergence is not always possible.
2.3 FREQUENCY DOMAIN TECHNIQUES

Several frequency domain deconvolution techniques have been developed over the years for use in linear system identification. While linear systems are defined by the convolution operation, the deconvolution is basically the determination or recovery of the impulse response of such systems. Therefore, for a linear system having an impulse response \( h(t) \), and an input signal \( x(t) \), the output signal is given by the integral relation:

\[
y(t) = \int_{-\infty}^{\infty} x(t - \tau) h(\tau) d\tau \quad (2.3.1a)
\]

which is commonly expressed as:

\[
y(t) = x(t) * h(t) \quad (2.3.1b)
\]

Where \( * \) denotes the convolution. In the frequency-domain, relation (2.3.1) is expressed by the following product:

\[
Y(j\omega) = X(j\omega) \cdot H(j\omega) \quad (2.3.2)
\]

Where \( Y(j\omega) \), \( X(j\omega) \) and \( H(j\omega) \) are the Fourier transforms of \( y(t) \), \( x(t) \) and \( h(t) \) respectively.

Typically, \( X(j\omega) \) is small for \( \omega \) large, and if the data has appreciable noise content at those frequencies, it is certain that the restored result will show a noticeable noise presence. However, it is not possible to restore frequencies beyond the band limit \( \Omega \) by the method of inverse filtering when such limit exists. When the frequencies are strongly suppressed, the signal-to-noise ratio is poor, and the filter function \( 1/X(j\omega) \) will amplify mainly the noise, producing a noisy estimate.

To remedy the problem of noise corruption of the result, Wiener devised his well-known smoothing filter. The filter was derived based on the satisfaction of the minimum mean-square error between the estimate and the true result. So, if the convolution operation is
formulated in a realistic way that accounts for the nature of the additive noise \( n(t) \) that is contained in the observed response waveform, then we get,

\[
y(t) = x(t)^* h(t) + n(t)
\]

(2.3.3)

Wiener studied the problem of noise corruption by itself (i.e., \( x(t) = \delta(t) \)) and devised his smoothing filter [11]. The original solution of Wiener was carried out in the time domain rather than in the frequency domain. The filter is found to be function of the auto-correlation function of the signal being filtered and the auto-correlation function of the additive noise. If these quantities are known for a noisy signal then the Wiener filter is more than enough to eliminate or reduce the noise that corrupts it. The Wiener filtering theory was adapted to the deconvolution problem and the derived filter was found to be function of the power spectra functions of the signal being estimated and the noise, respectively. Other modifications are possible to this approach of seeking a filter that is optimum in the sense of least mean-square error. Thus, if the derivation of the Wiener smoothing filter is combined with the minimum mean-square error (MMSE) criterion, and applied to the case of the convolution Eq. (2.3.2), different variants are obtained [39]. Provided that the noise and impulse response power spectra are known and that the noise is additive and Gaussian distributed, then this type of filters are more than adequate. However, the power spectra are not always easy to obtain. Other researchers [40] considered the deconvolution problem using constraints. By adding a sharpness criterion to the MMSE quality criterion, a criterion that is a sum of the MMSE criterion and the sharpness constraint. The filter derived was similar to previous Wiener type filters, but contains a sharpness controlling factor that either enhances or suppresses the high frequencies. Another variant of the Wiener filter [41] was suggested to minimize the response noise and the response departure of signal restoration from the true estimate. As a measure of the response noise, the total mean-square response noise was used. And a convenient measure for the departure that was adopted, was the mean-square departure of the filter output from a required windowed estimate. The Wiener filter and successive variants that
were derived were intended for noise reduction only. They were optimum filters for noise reduction provided that the noise is additive and Gaussian distributed.

Other types of filtering methods were derived from a convolution operation formulation that does not account for the effects of system distortion explicitly. These methods used constraints that compensate for the absence of knowledge about the noise and signal power spectra. They have been used to solve for $H(j\omega)$ by multiplying $Y(j\omega)$ by an appropriate filter function $C(j\omega)$ to yield an adequate estimate of the frequency response:

$$H_s(j\omega) = Y(j\omega)C(j\omega)$$

(2.3.4)

In these methods, the researchers took a Wiener filter approach and sought a restoring filter $C(j\omega)$ that obeys a minimum mean-square error (MMSE) quality technique. In order to limit the noise in the estimate $H_s(j\omega)$, these researchers [1,5,43] added an extra constraint on the filter. For example, in the case of the optimal compensation [15], the compensating filter function $C(j\omega)$ was constrained to be a bounded function. So, in addition to the MMSE quality criterion, a second criterion was added to keep the energy finite. However, in the case of the regularization technique [16,17] a smoothness criterion was added. In both cases, the filter was derived with the following form:

$$C(j\omega) = \frac{X^*(j\omega)}{[|X(j\omega)|^2 + F(\lambda, |X(j\omega)|)]}$$

(2.3.8)

where the superscript (*) denotes the complex conjugate. We find that $F(\lambda, |X(j\omega)|) = \lambda$ for the case of the optimal compensation, and $F(\lambda, |X(j\omega)|) = \lambda \sin^{4}(\omega |\Omega|)$, where $\Omega$ is the folding frequency, for the case of the regularization technique. Equation (2.3.8) shows that the compensating filter function $C(j\omega)$ depends on the nature of the input test signal and the constraint controlling parameter. Both filters have been successfully and widely used in the area of time domain reflectometry [5,15,16].

CHAPTER II

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2.4 HOMOMORPHIC DECONVOLUTION TECHNIQUE

Homomorphic processing, while limited in regard to the enormous number of "nonlinear" systems, has the advantage of being easy to analyze mathematically and having important practical applications of general interest. It was initially developed by Oppenheim [44]. In the beginning, it was a theoretical study based on linear algebra, but now it is great importance due to the increasing number of applications which have been implemented since its first introduction [45]. In many applications of signal processing, the problem of extracting signals combined through the convolution (or multiplication) operation arises. Such problems have to be dealt with according to the specific needs of the problem. For example, a signal transmitted by using amplitude modulation is multiplied by a high frequency carrier signal, which must be eliminated by the receiver. This is the case when a multiplication operation is used. For the case of convolution, several examples can be listed. A natural signal observed through a linear system, which may be the measurement device itself (distortion through measurement), a signal recorded in a reverberating environment which is altered by unexpected echoes, etc.

The importance of the homomorphic deconvolution lies in areas of convolution problems where only the output signal is fully known. In the area of seismic data processing, the wavelet is usually unknown and only the detected (recorded) data is available. It was found that the power cepstrum is efficient in recognizing wavelet arrival times and amplitudes while the complex cepstrum is invaluable in estimating the form of the basic wavelet and its echoes. The solution to the decomposition problem is severely complicated by the presence of noise. In the previous section, we described the method of inverse filtering, which is widely used. In the case where a useful signal convolved with another must be isolated, the composite signal is filtered with a filter having a frequency response which is the inverse of the Fourier transform of the disturbing signal. However, this requires a prior knowledge as accurate and detailed as possible of the disturbing signal, which is not always the case. Another possible method is the homomorphic deconvolution technique. Bogert et al [46] introduced a
homomorphic deconvolution system, in which the Fourier transform of the logarithm of the power spectrum of the input signal is proposed for detecting echoes, and they defined the function cepstrum.

A variant technique based on the homomorphic deconvolution was also suggested [47], but has not been widely used. The system developed in this technique is similar to the homomorphic system developed by Oppenheim, except for the logarithm and exponential operations which are replaced by the \( \gamma \)th and \( 1/\gamma \)th powers operations. With proper choice of \( \gamma \) the technique was found to yield better results than conventional homomorphic deconvolution. However, this technique assumes that one of the components is a train of pulses. In addition, an a priori information on the approximate number of poles and zeros is required.

The homomorphic deconvolution technique is established as a powerful for the deconvolution of signals when none of the component signals is fully known. The technique found a wide field of applications that is increasing as confirmed by the increasing number of publications on the subject reported Childers et al [48]. The method was applied to the problem of echo removal in speech studies. In seismic studies, the homomorphic deconvolution was applied to the recovery of seismic wavelet from time series formed by the convolution of this wavelet with an impulse train [49]. In time domain reflectometry (TDR) [50,51], the homomorphic transformation was used to separate a TDR signal into its rapidly and slowly varying components, respectively. The technique was shown to be successful in the case where the multiple reflections can not be viewed in non-overlapping time windows. However, the technique was tested on deterministic noise-free data only. The use of the homomorphic deconvolution (cepstrum) was shown to simplify cable discontinuity identification as compared to time domain reflectometry [52]. A thorough study of signal detection and extraction by cepstrum techniques was also reported [54,55]. It was found that the power cepstrum is most efficient in recognizing wavelet arrival times and amplitudes while the complex cepstrum is invaluable in estimating the form of the basic wavelet and its echoes, even if the latter are distorted [53]. One of the goals of the work being carried out is to increase the use of the

CHAPTER II
homomorphic transformation in the area of time domain measurements such as modeling active transmission lines components, and the detection of discontinuities.
CHAPTER III

FREQUENCY DOMAIN DECONVOLUTION TECHNIQUES

3.1 INTRODUCTION

In the preceding chapter, we reviewed different deconvolution techniques that were developed over the years for use in linear systems identification. Generally, these techniques were developed for some particular areas of interest, and are suitable to these areas. However, few of them can be generalized to include the different deconvolution problems. For example, in the area of time domain reflectometry we deal with waveforms of 512 or twice data points. Thus, the deconvolution process must take into account the computer time needed to reach an acceptable estimate. Iterative techniques such as the Van Cittert technique may not be suitable for TDR deconvolution problems. However, frequency domain techniques tend to be more appropriate to use in such identification problems. However, the main obstacle in performing such identification is the lack of information over an infinite frequency band. Finite bandwidth test signals would yield adequate identification of the system within the signal bandwidth. However, at frequencies outside the signal bandwidth, an indetermination of the system transfer function occurs, resulting in noise-like errors in the computed results. When transformed to the time-domain, to compute the system's impulse response, the noise-like errors appear to dominate the entire waveform. Without proper filtration, such a result becomes inadequate for identification of the system's temporal characteristics.
Different frequency-domain techniques were developed to reduce the noise that contaminates the transfer function through the use of special filters. These types of filters are developed based on Wiener filtering theory, and are designed to minimize the mean-squared error in the computed estimate for the transfer function. In this chapter, we will derive two frequency domain filters based on Wiener theory and implement them into the deconvolution process. The first filter is a smoothing filter and is derived by adding an additional smoothness constraint to the MMSE quality criterion. The constraint attempts to reduce the oscillations of the second derivative of the computed estimate. The second is an adaptive smoothing filter since it is derived dependent on the nature of the excitation being applied to the system. Both of these filters attenuate the high frequency terms because they are frequency dependent. The higher the frequency at which the filter is operating, the stronger is the attenuation. Thus, the attenuation rate of the noise that contaminates the high frequency components is considerably fast. In general, a design parameter is included in both filters to control the balance between smoothness and accuracy of the estimate. The parameter has also the property of allowing the filter's performance optimization.

This chapter is divided into several sections. In the next section, a review of certain characteristics and properties of transient signals is given. The review is important in order to be able to proceed into the section where the frequency domain deconvolution filters are derived. The filters are derived for the deterministic and statistical cases as it will be shown. Then, two examples of deconvolution problems, one simulated and one experimental, are given to show the applicability of the derived filters to noise elimination in deconvolution. A comparative study of the different frequency domain deconvolution filters is provided to identify the limitations of such filters. Finally, a conclusion summarizes the main findings of this chapter.
3.2 REVIEW OF BAND LIMITED FUNCTIONS

3.2.1 Band limited functions

We say that a function $x(t)$ is frequency band-limited, if its energy $E$ is finite and its Fourier transform $X(j\omega)$ is zero outside a finite interval;

$$X(j\omega) = 0 \quad \text{for} \quad |\omega| > \Omega \quad (3.2.1)$$

The energy $E$ of a transient signal $x(t)$ is given by the following integral relation:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt \quad (3.2.2)$$

When the signal is real and represents a voltage at the terminal of a resistance of 1 ohm, then $E$ is the energy dissipated by the signal in the resistance.

A band-limited function is analytic in the entire $t$-axis, hence its Taylor series expansion yields $x(t)$ for every $t$. A finite duration function cannot be band-limited, and inversely a band-limited function cannot be finite duration. The type of time-domain signals that we deal with, are usually frequency band-limited. In general, we define a positive frequency $\Omega$ such that the modulus of the complex spectrum is practically negligible for frequencies satisfying $|\omega| \geq \Omega$. The frequency $\Omega$ is referred to as the upper cutoff frequency. Thus, a band-limited signal $X(j\omega)$ contains no spectral components beyond a certain limit $\Omega$. This property has important consequences on the possibilities of signal recovery on one hand, and the practical representation of such a signal by a sequence of discrete values on the other hand. The latter property is expressed in Sec. (3.2.3). However, concerning the first property, we need to consider an example of a convolution problem where the signal $x(t)$ represents the input to a linear system, while $y(t)$ and $h(t)$ represent the output signal and the unknown system’s im-
pulse response. Then, if the convolution equation is expressed in the frequency domain, the Fourier transforms of \( x(t) \), \( h(t) \), and \( y(t) \) are related through the following relation:

\[
Y(j\omega) = H(j\omega)X(j\omega)
\]

\( X(j\omega) = 0 \quad \text{for} \quad |\omega| > \Omega \) \hspace{1cm} (3.2.3)

which leads to

\[
H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \quad |\omega| \leq \Omega
\]

\( = 0 \quad |\omega| > \Omega \) \hspace{1cm} (3.2.4)

Therefore, the recovery of \( h(t) \) is only valid within the frequencies such that \( |\omega| < \Omega \). Any signal having identical information as \( h(t) \) within these frequencies would satisfy the convolution equation no matter what its spectral information beyond the limit \( \Omega \) is. The whole property leads to the following corollary.

### 3.2.2 Corollary of the convolution

The following corollary of the convolution theorem shows what happens to a signal if the high frequency components of its spectrum are eliminated [60,61]. Suppose that the Fourier transform \( F(j\omega) \) of a function \( f(t) \) is truncated above \( |\omega| = \Omega \), then the following band-limited function results:

\[
F_{\Omega}(j\omega) = \begin{cases} 
F(j\omega) & |\omega| \leq \Omega \\
0 & |\omega| > \Omega 
\end{cases} = F(j\omega)P_{\Omega}(j\omega)
\]

\hspace{1cm} (3.2.5)

where \( P_{\Omega} \) is a rectangular pulse \( (= 1 \quad |\omega| \leq \Omega \quad \text{and} \quad = 0 \quad |\omega| > \Omega) \). Such an operation takes place, for example, if \( f(t) \) is the input to an ideal low-pass filter. It is of interest to determine the relationship between the inverse transform \( f_{\Omega}(t) \) of \( F_{\Omega}(j\omega) \) and the given function \( f(t) \). Since the inverse Fourier transform of the pulse \( P_{\Omega}(j\omega) \) equals \( \frac{\sin \Omega t}{\pi t} \), we conclude that
\[ f_\Omega(t) = \int_{-\infty}^{\infty} f(\tau) \frac{\sin \Omega(t - \tau)}{\pi(t - \tau)} d\tau \]  
\[ f(t) \frac{\sin \Omega t}{\pi t} \xrightarrow{\cdot} F(j\omega) \frac{\sin \Omega \omega}{\pi \omega} \]  

\[ (3.2.6) \]

\[ (3.2.7) \]

3.2.3 Sampling of time signals

A band limited signal which has no spectral components above the cut-off frequency \( \omega_m \) rad/sec can be uniquely represented by its sampled values spaced at uniform intervals that are not more than \( 1/2f_m \) seconds apart (where \( \omega_m = 2\pi f_m \)).

3.2.4 Transition duration, bandwidth of a signal

In many applications, it is necessary to describe a signal without using its exact function time dependence. Consequently, terms such as rise time, transition duration, bandwidth, etc, are commonly used to describe them. In this section, we wish to give some relations and definitions that are associated with signals commonly encountered in time domain reflectometry.

The bandwidth is a number we attach to an amplitude spectrum to characterize the significant band of frequencies in a signal spectrum or transfer function. There are several ways of defining the bandwidth of a system. For example, one definition that is often used is that the bandwidth of the system \( H(j\omega) \) is the interval of frequencies for which \(|H(j\omega)|\) remains within \( 1/\sqrt{2} \) (within 3 db) of its maximum value. Another approach is to define the bandwidth of a spectrum \( H(j\omega) \) by the following:
\[ 2Bw = \frac{1}{|H(j\omega)|_{\text{max}}} \int_{-\infty}^{\infty} |H(j\omega)| \, d\omega \]  \hspace{1cm} (3.2.8)

The above equation is equivalent to replacing the amplitude spectrum by a rectangle with area equal to the area under the curve \( |H(j\omega)| \) and height equal to \( |H(j\omega)|_{\text{max}} \). In all our references to the bandwidth we will use the first definition. Usually, the bandwidth of a step waveform is always related to its transition time by the following equation:

\[(\text{Bandwidth}) \times (\text{transition time}) = \text{constant}\]

The speed of response for a linear system is the transition duration required for the step response to move from the 10% level to the 90% level.

For example, in the case of an RC circuit, i.e., a single constant lowpass, the transition duration is related to the system time constant \( \tau (= RC) \) by the following relation:

\[ t_d = 2.2 \times \tau \]

And a relationship between the transition duration and the bandwidth can be derived as [59],

\[ f_0T_r = 0.3467 \]  \hspace{1cm} (3.2.9)

where \( f_0 \) represents the 0.707 response frequency (cutoff frequency), which defines the bandwidth of the signal.

The preceding two relations are valid for the case of the lowpass system only. The lowpass is physically realizable with causal systems of finite extent. However, for other type of signals such as the gaussian system different relations are derived [59].
3.2.5 Root mean square (RMS) minimization

Consider the class of functions $h_\varepsilon(t)$ whose Fourier transform $H_\varepsilon(j\omega)$ is band-limited in the sense

$$H_\varepsilon(j\omega) = 0 \quad \text{for } |\omega| > \Omega$$  \hspace{1cm} (3.2.10)

If the function $h(t)$ is approximated by a function of this class, an RMS (root mean squared) error results and is defined as:

$$E = \int_{-\infty}^{\infty} |h(t) - h_\varepsilon(t)|^2 dt$$  \hspace{1cm} (3.2.11)

It can be proven that this error is minimum if the Fourier transform $H(j\omega)$ of $h(t)$ equals the Fourier transform $H_\varepsilon(j\omega)$ of $h_\varepsilon(t)$ in the interval $(-\Omega, \Omega)$:

$$H(j\omega) = H_\varepsilon(j\omega) \quad \text{for } |\omega| > \Omega$$  \hspace{1cm} (3.2.12)

Expanding the square in the error equation, we obtain

$$E = \int_{-\infty}^{\infty} h_\varepsilon(t)^2 dt + \int_{-\infty}^{\infty} h(t)^2 dt - 2\int_{-\infty}^{\infty} h_\varepsilon(t)h(t)dt$$  \hspace{1cm} (3.2.13)

using Parseval's formulas

$$2\pi \int_{-\infty}^{\infty} h_\varepsilon(t)^2 dt = \int_{-\infty}^{\infty} |H_\varepsilon(j\omega)|^2 d\omega$$  \hspace{1cm} (3.2.14)
we obtain the following expression for the error

\[ 2\pi E = \int_{-\Omega}^{\Omega} |H_\ell(\omega)|^2 d\omega + \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega \]

\[ - \int_{-\Omega}^{\Omega} H_\ell(-j\omega)H(j\omega)d\omega - \int_{-\Omega}^{\Omega} H_\ell(j\omega)H(-j\omega)d\omega \]  

(3.2.16)

Using the assumption of Eq. (3.2.12), we write

\[ 2\pi E = \int_{-\Omega}^{\Omega} H(j\omega)H(-j\omega)d\omega + \int_{-\infty}^{\infty} |H(\omega)|^2 d\omega \]

\[ - \int_{-\Omega}^{\Omega} H(-j\omega)H(j\omega)d\omega - \int_{-\Omega}^{\Omega} H(j\omega)H(-j\omega)d\omega \]

\[ + \int_{-\infty}^{\infty} |H_\ell(\omega) - H(\omega)|^2 d\omega \]

(3.2.17)

The first two integrals above are independent of \( h_\ell(t) \) and the second is non-negative because the integrand equals \( |H_\ell(j\omega) - H(j\omega)|^2 \). Therefore \( E \) is minimum if Eq. (3.2.12) is true because only then the last integral equals zero. The minimum value of \( E \) is given by
\[ \varepsilon_{\text{min}} = \left( \frac{1}{2\pi} \right) \int_{-\infty}^{-\Omega} |H(j\omega)|^2 d\omega + \left( \frac{1}{2\pi} \right) \int_{\Omega}^{\infty} |H(j\omega)|^2 d\omega \]  \hspace{1cm} (3.2.18)

clearly the optimum function \( h_*(t) \) equals the function \( h_\Omega(t) \) given by

\[ h_\Omega(t) = \left( \frac{1}{2\pi} \right) \int_{-\Omega}^{\Omega} H(j\omega)e^{j\omega t} d\omega \]  \hspace{1cm} (3.2.19)

The function \( h_\Omega(t) \) is optimum in the sense of minimizing the RMS error \( \varepsilon \)

\[ \lim_{\Omega \to \infty} h_\Omega(t) = h(t) \]  \hspace{1cm} (3.2.20)

These RMS minimization relations are very important in deriving techniques that would yield good estimates of unknown waveforms. The minimization criterion has been used successfully in the derivation of several filters used in deconvolution techniques [2,5]. The success of deconvolution techniques is very much linked to the nature of the signal being identified, and band-limited signals represent the best cases where deconvolution techniques are highly successful.

### 3.3 FREQUENCY-DOMAIN DECONVOLUTION

For a linear system having an impulse response \( h(t) \), and an input signal \( x(t) \), the output signal is given by Fredholm's integral equation of the second kind,

\[ y(t) = \int_{-\infty}^{\infty} x(t - \tau)h(\tau)d\tau \]  \hspace{1cm} (3.3.1)

which is commonly expressed as,
\[ y(t) = x(t) \ast h(t) \]  \hspace{1cm} (3.3.2)

Where \(( \ast )\) denotes the convolution. In the frequency-domain, relation (3.3.1) is expressed by the following product:

\[ Y(j\omega) = X(j\omega) \cdot H(j\omega) \]  \hspace{1cm} (3.3.3)

Where \(Y(j\omega), X(j\omega)\) and \(H(j\omega)\) are the Fourier transforms of \(y(t), x(t)\) and \(h(t)\) respectively.

In chapter two, we reviewed some Wiener type filtering methods used in noise reduction. These methods have been used to solve for \(H(j\omega)\) by multiplying \(Y(j\omega)\) by an appropriate filter function \(C(j\omega)\) to yield an adequate estimate of the frequency response:

\[ H_e(j\omega) = Y(j\omega).C(j\omega) \]  \hspace{1cm} (3.3.4)

An approach similar to the one taken by Wiener in deriving his smoothing filter approach will be used. However, the additive noise that corrupts the signal is not taken into consideration explicitly since no information about it is available. Seeking a restoring filter \(C(j\omega)\) that obeys a minimum mean-squared error (MMSE) quality technique, we define the following error function

\[ E_e = \int_{-\Omega}^{\Omega} |H(j\omega) - H_e(j\omega)|^2 d\omega = \text{minimum} \]  \hspace{1cm} (3.3.5)

where

\[ H_e(j\omega) = \begin{cases} Y(j\omega).C(j\omega) & \text{for } |\omega| \leq \Omega \\ 0 & \text{for } |\omega| > \Omega \end{cases} \]

is the estimate of the transfer function \(H(j\omega)\) which is assumed to be frequency-band limited and having no significant information at frequencies \(|\omega| > \Omega\). Consider the integrand quantity in Eq. (3.3.5),

CHAPTER III
\[ Q = |H(j\omega) - H_e(j\omega)|^2 \]
\[ = |H(j\omega) - H(j\omega)X(j\omega)C(j\omega)|^2 \]
\[ = |H(j\omega)|^2 |1 - X(j\omega)C(j\omega)|^2 \]
\[ = |H(j\omega)|^2 Q_s \]

where

\[ Q_s = |1 - X(j\omega)C(j\omega)|^2 \]

The minimization of the quantity \( E_s \) is simply equivalent to the minimization of the quantity \( Q_s \), i.e.,

\[ \frac{\partial E_s}{\partial C(j\omega)} = 0 \quad \Leftrightarrow \quad \frac{\partial Q_s}{\partial C(j\omega)} = 0 \]  
(3.3.6)

Using the complex forms

\[ X(j\omega) = X_R(j\omega) + jX_I(j\omega) \]  
(3.3.7a)

\[ C(j\omega) = C_R(j\omega) + jC_I(j\omega) \]  
(3.3.7b)

where the subscript \( R \) and \( I \) indicate the real and imaginary parts respectively, Eq. (3.3.5) yields

\[ Q_s = |X(j\omega)|^2 \times [C_R(j\omega)^2 + C_I(j\omega)^2] \]
\[ - 2X_R(j\omega)C_R(j\omega) + 2X_I(j\omega)C_I(j\omega) \]

To minimize \( Q \), the partial derivatives with respect to \( C_R \) and \( C_I \) are set to zero:

\[ 0 = \frac{\partial Q_s}{\partial C_R(j\omega)} = C_R(j\omega) |X(j\omega)|^2 - X_R(j\omega) \]  
(3.3.8a)

and
\[ 0 = \frac{\partial Q_s}{\partial C(i\omega)} = C(i\omega) |X(i\omega)|^2 + X(i\omega) \] (3.3.8b)

Using equations (3.25b), (3.26a) and (3.26b), it can be shown that

\[ C(i\omega) = \frac{X^*(i\omega)}{|X(i\omega)|^2} = \frac{1}{X(i\omega)} \] (3.3.9)

where the superscript (*) denotes the complex conjugate. All the steps that were taken led us simply to a result that we knew all along. However, it was intended to show that without additional constraints on the deconvolution approach we always obtain the same deconvolution result, and the noise error is not eliminated.

The solution of the above equations is therefore derived as \( C(i\omega) = 1/X(i\omega) \). However, this is valid for an ideal case only where the unknown \( H(i\omega) \) is obtained by using frequency-domain division, and then using the inverse Fourier transform to obtain \( h(t) \). However, this is not always the case. In general, the poles and zeros of \( X(i\omega) \) are also the poles and zeros of \( Y(i\omega) \). At these poles and zeros, the straightforward division leads to an indetermination, leading to noise-like errors in the division operation. In order to limit the noise in the estimate \( H_s(i\omega) \), the deconvolution operation needs to be modified beyond the simple division to yield a low-noise good estimate. Numerous techniques have been developed in the literature [2,5,16,17,18]. One thing that all these approaches have in common is the use of iterations to design an optimal filter to reduce the noise in the deconvolved result and yield the best estimate. The optimal filter interpolates \( H_s(i\omega) \) at the poles and zeros. Thus, it is possible to take a slightly different approach by adding another constraint on the filter performance or the estimate. The use of such constraints is meant to compensate for the lack of knowledge about the signals and noise power spectra.

CHAPTER III
3.3.1 Deterministic constrained methods

3.3.1a Smoothing (Lowpass) deconvolution filter

For example, in the case of the smoothing deconvolution filter, the additional constraint is on the total sharpness in the restoration. When direct deconvolution methods are applied the solution exhibits sharp oscillations. Therefore, in order to reduce the magnitude of such oscillations a minimization constraint must be imposed on the secondary derivatives of the estimated solution. Thus, we seek the minimization of the following power function:

\[ S \equiv \int_{-\infty}^{\infty} \left| \frac{d^2 h_e(t)}{dt^2} \right|^2 \]  

(3.3.10)

Using Parseval's theorem, the above relation can be written as:

\[ S \equiv \int_{-\Omega}^{\Omega} d\omega \omega^4 |H_e(j\omega)|^2 \]  

(3.3.11)

The sharpness \( S \) is seen to measure the total edge gradient content in output \( h_e(t) \). So, in addition to the MMSE quality criterion, a smoothness criterion is added. By grouping the two design requirements into one, the new criterion becomes the minimization of the energy \( \hat{E} \) defined as

\[ \hat{E} = E_e + \lambda S, \quad \lambda > 0 \]  

(3.3.12)

Where \( \lambda \) is simply a parameter that controls the balance between the degree of noise reduction and the errors introduced by the filter. For example, a small \( \lambda \) value produces an \( H(j\omega) \) which is very close to the ratio \( Y(j\omega)/X(j\omega) \) with very small degree of noise reduction. While a large \( \lambda \) will greatly reduce the noise content with large deviation from the ratio.
In this case \( \lambda \) is the filter iteration parameter which is to be "optimized" to achieve the optimal balance between noise reduction and filter errors. The constant \( \lambda \) is an optimization parameter whose significance can be demonstrated by considering the two following extreme cases:

- \( \lambda = 0 \), this results in a minimization of \( E_s \) with no constraints on \( C(j\omega) \).
- \( \lambda = \infty \), this results in a strong emphasis on the smoothness factor.

Both extremes are undesirable, and an optimum value \( \lambda_{opt} \) is expected to yield an optimum compensator \( C_{opt}(j\omega) \), for which \( H_{opt}(j\omega) \) is the best estimate for the unknown input function \( H(j\omega) \). Using Eq. (3.3.4) in Eq. (3.3.12), \( \hat{E} \) takes the form:

\[
\hat{E} = \int_{-\Omega}^{\Omega} d\omega |H(j\omega) - C(j\omega)Y(j\omega)|^2 + \lambda \int_{-\Omega}^{\Omega} d\omega \omega^4 |C(j\omega)Y(j\omega)|^2 = \text{minimum}
\]

We see that more or less emphasis on sharpness control relative to expected error (or, data consistency) is expected by choice of the parameter \( \lambda \). We get,

\[
\hat{E} = \int_{-\Omega}^{\Omega} d\omega \{|H(j\omega) - C(j\omega)Y(j\omega)|^2 + \lambda \omega^4 |C(j\omega)Y(j\omega)|^2\} = \text{minimum}
\]

The problem is recognizable as the Euler-Lagrange type, where the square bracketed quantity is Lagrangian \( \mathscr{J} \) and is not a function of \( \frac{\partial C}{\partial \omega} \) or \( \frac{\partial C}{\partial \omega^*} \). The solution can be achieved by setting the partial derivatives with respect to \( C(j\omega) \) to zero.

\[
\frac{\partial \mathscr{J}}{\partial C} = \frac{\partial \mathscr{J}}{\partial C^*} = 0
\]

where
\[ \mathcal{F} = |H(j\omega) - C(j\omega)Y(j\omega)|^2 + \lambda\omega^4 |C(j\omega)Y(j\omega)|^2 \]
\[ \mathcal{F} = |H(j\omega)|^2 \{ |1 - C(j\omega)X(j\omega)|^2 + \lambda\omega^4 |C(j\omega)X(j\omega)|^2 \} \]
\[ \mathcal{F} = |H(j\omega)|^2 \mathcal{F}_s \]

where

\[ \mathcal{F}_s = \{ |1 - C(j\omega)X(j\omega)|^2 + \lambda\omega^4 |C(j\omega)X(j\omega)|^2 \} \]

The minimization of the quantity \( \mathcal{F} \) is simply equivalent to the minimization of the quantity \( \mathcal{F}_s \), i.e.,

\[ \frac{\partial \mathcal{F}}{\partial C(j\omega)} = 0 \quad \iff \quad \frac{\partial \mathcal{F}_s}{\partial C(j\omega)} = 0 \quad (3.3.13) \]

Using the complex forms given in Eqs. (3.3.6) and (3.3.7) yields,

\[ \mathcal{F}_s = |X(j\omega)|^2 (1 + \lambda\omega^4) \times [C_R(j\omega)^2 + C_I(j\omega)^2] \]
\[ - 2X_R(j\omega)C_R(j\omega) + 2X_I(j\omega)C_I(j\omega) \]

To minimize \( \mathcal{F}_s \), the partial derivatives with respect to \( C_R \) and \( C_I \) are set to zero:

\[ 0 = \frac{\partial \mathcal{F}_s}{\partial C_R(j\omega)} = C_R(j\omega) \times |X(j\omega)|^2 (1 + \lambda\omega^4) - X_R(j\omega) \quad (3.3.14a) \]

and

\[ 0 = \frac{\partial \mathcal{F}_s}{\partial C_I(j\omega)} = C_I(j\omega) \times |X(j\omega)|^2 (1 + \lambda\omega^4) + X_I(j\omega) \quad (3.3.14b) \]

so, the filter that we seek has the following form:

\[ C(j\omega) = \frac{X^*(j\omega)}{|X(j\omega)|^2 + |X(j\omega)|^2 \lambda\omega^4} \]
\[ = \frac{1}{X(j\omega)} \frac{1}{1 + \lambda\omega^4} \quad (3.3.15) \]
We simply obtained a lowpass type filter that attenuates the high frequencies components. It is clear that for $\omega \to 0$, we have $C(j\omega) = 1/X(j\omega)$. Thus, no signal attenuation occurs at low frequencies, the deconvolution result remains unchanged. However, for $\omega \to \infty$, $C(j\omega) = 0$, the high frequencies components are practically eliminated. The importance of the parameter $\lambda$ lies in the fact that it controls the cutoff frequency of the filter. The filter is a butterworth type of filter, and therefore, its practical implementation is possible. The properties of the Butterworth or maximally flat lowpass filters are well documented in the literature. Its use in deconvolution is linked to the optimization method developed in chapter four, which permits its implementation.

3.3.1b Adaptive smoothing (regularization) deconvolution filter

In the following technique, we take an approach similar to the one taken by other researchers [1,5,16] in developing the regularization technique. The technique itself has different variants as found in the literature. The problem of deconvolution is approached in a slightly different manner than the usual approach used previously. When direct division deconvolution methods are used they lead to noisy errors in the estimate. The noise-free transfer function is simply given by:

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \quad (3.3.16)$$

However, the transfer function is noisy, and must be multiplied by a chosen filter function $R(j\omega)$ to reduce the noise level and yield an appropriate estimate.

$$H_e(j\omega) = H(j\omega)R(j\omega) = \frac{Y(j\omega)}{X(j\omega)} R(j\omega) \quad (3.3.17)$$

An error function is defined to monitor the quality of the estimate in the following manner:
\[ e(t) = y(t) - x(t)^*h_e(t) \]  
\[ (3.3.18) \]

In the frequency domain the error function is simply given by:

\[ E(j\omega) = Y(j\omega) - X(j\omega)H_e(j\omega) \]  
\[ (3.3.19) \]

We then define an energy error \( \bar{E} \) function as follows:

\[ \bar{E} = \int_{-\infty}^{\infty} |e(t)|^2 dt \]  
\[ (3.3.20) \]

Using Parseval's theorem leads to the following relation:

\[ \bar{E} = \int_{-\infty}^{\infty} |Y(j\omega) - X(j\omega)H_e(j\omega)|^2 d\omega \]  
\[ (3.3.21) \]

Instead of applying directly the MMSE quality criterion, it is preferable to add another constraint on the estimate such as smoothness in the restoration. By using the constraint we attempt to reduce the oscillations in the \( i^{th} \) derivative of the estimate. In this particular case, the use of the first derivative seems very adequate. We define the following power function:

\[ S \equiv \int_{-\infty}^{\infty} dt \left\| \frac{dh_e(t)}{dt} \right\|^2 \]  
\[ (3.3.22) \]

Using Parseval's theorem, the above relation can be written

\[ S \equiv \int_{-\infty}^{\infty} d\omega \omega^2 \left| H_e(j\omega) \right|^2 \]  
\[ (3.3.23) \]

In their method, Nahman and Guillaume [5,16] used a similar approach with discrete variables as opposed to continuous. Their additive constraint was to eliminate the oscillations in the estimate. They achieved that by minimizing the second-difference expression
\( h_n(k + 1) - 2h_n(k) + h_n(k - 1) \) in the same manner as Phillips [34] and twomey [35] did in their method. However, for our case we attempt to reduce the oscillations in the estimate by minimizing the power of the first derivative. The result that we obtain as we will see is slightly different than that obtained by Nahman and Guillaume which is given by Eq. (1.3.5). The new criterion becomes the minimization of the following expression \((\bar{E} + \lambda S)\):

\[
\int_{-\infty}^{\infty} d\omega |Y(\omega) - X(\omega)H_e(\omega)|^2 + \lambda \int_{-\infty}^{\infty} d\omega \omega^2 |\dot{H}(\omega)|^2 = \text{minimum}
\]

\[
\int_{-\infty}^{\infty} d\omega \left( |Y(\omega) - Y(\omega)R(\omega)|^2 + \lambda \omega^2 \frac{Y(\omega)R(\omega)}{X(\omega)} \right)^2 = \text{minimum}
\]

Let us represent the square bracketed quantity by \( \mathcal{S} \)

\[
\mathcal{S} = \{ |Y(\omega) - Y(\omega)R(\omega)|^2 + \lambda \omega^2 |R(\omega)|^2 \}
\]

\[
\mathcal{S} = Y(\omega)^2 \{ |1 - R(\omega)|^2 + \lambda \omega^2 \frac{R(\omega)}{|X(\omega)|^2} \}
\]

\[
\mathcal{S} = Y(\omega)^2 \mathcal{S}_e \quad (3.3.24)
\]

where the expression \( \mathcal{S}_e \) is given by

\[
\mathcal{S}_e = |1 - R(\omega)|^2 + \frac{\lambda \omega^2}{|X(\omega)|^2} |R(\omega)|^2 \quad (3.3.25)
\]

The minimization is simply achieved by simply setting the partial derivatives with respect to \( R(\omega) \) to zero

\[
\frac{\partial \mathcal{S}}{\partial R} = \frac{\partial \mathcal{S}_e}{\partial R} = 0 \quad (3.3.26)
\]

We see that the minimization of \( \mathcal{S} \) is equivalent to the minimization of \( \mathcal{S}_e \).
\[
\frac{\partial J}{\partial R(j\omega)} = 0 \quad \Leftrightarrow \quad \frac{\partial J_e}{\partial R^*(j\omega)} = 0
\]  
(3.3.27)

Taking either of these derivatives produces the same error, we find that

\[
J_e = 1 + R(j\omega)R^*(j\omega) - (R(j\omega) + R^*(j\omega)) + \frac{\lambda \omega^2}{|X(j\omega)|^2} R(j\omega)R^*(j\omega)
\]  
(3.3.28)

with \( \frac{\partial J_e}{\partial R^*(j\omega)} = 0 \), we get

\[
0 = R(j\omega) - 1 + \frac{\lambda \omega^2}{|X(j\omega)|^2} R(j\omega)
\]  
(3.3.29)

\[
R(j\omega) = \frac{1}{1 + \lambda \omega^2/|X(j\omega)|^2}
\]  
(3.3.30)

The filter being sought using the regularization approach is an adaptive type filter. It is function of the type of input signal used to excite the signal as well as the frequency. It is also function of the parameter \( \lambda \) which controls the noise reduction in the spectrum of the impulse response. The filter emphasizes the low frequency components and attenuates the high frequency components.

3.3.2 Statistical (Generalized) methods

3.3.2a Smoothing (Lowpass) deconvolution filter

The convolution equation is modified to take into account the nature of the additive additive that corrupts the observed response waveform, and the statistical properties of the signals involved in the convolution operation. It will be assumed that the output sequence is

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corrupted by an additive uncorrelated noise with zero-mean [62]. Thus, a more realistic formulation of the convolution operation is given by:

$$y(t) = x(t)^\ast h(t) + n(t)$$  \hspace{1cm} (3.3.31)

and therefore, in the frequency domain we obtain,

$$Y(j\omega) = X(j\omega)H(j\omega) + N(j\omega)$$  \hspace{1cm} (3.3.32)

Hence, the frequency domain that were derived must be modified to include the noise component and the statistical properties of all the signals involved in the convolution operation.

The zero-mean noise is assumed to be of gaussian distribution.

For example, in the case of the smoothing deconvolution filter, the additional constraint is on the total sharpness in the restoration which is given by,

$$S \equiv \int_{-\Omega}^{\Omega} d\omega \omega^4 |H_e(j\omega)|^2$$  \hspace{1cm} (3.3.33)

So, in addition to the MMSE quality criterion, a second criterion was added as mentioned previously.

$$\hat{E} = \int_{-\Omega}^{\Omega} d\omega \left\{ |H(j\omega) - C(j\omega)Y(j\omega)|^2 + \lambda\omega^4 |C(j\omega)Y(j\omega)|^2 \right\} = \text{minimum}$$

Substituting for $Y(j\omega)$ in the preceding equation and taking the statistical average, we get

$$\hat{E} = \langle \int_{-\Omega}^{\Omega} d\omega \left\{ |H(j\omega) - C(j\omega)X(j\omega)H(j\omega) - C(j\omega)N(j\omega)|^2 \\
+ \lambda\omega^2 |C(j\omega)Y(j\omega)|^2 \right\} \rangle = \text{minimum}$$

where $\langle \rangle$ represents the statistical average [63].

Using the statistical independence relations between $H(j\omega)$ and $N(j\omega)$, and denoting
\[ \phi_H(\omega) = E\{H(j\omega)H^*(j\omega)\} \]

\[ \phi_N(\omega) = E\{N(j\omega)N^*(j\omega)\} \]

where the expression $E\{\ \}$ represents the mean value of the expression between brackets.

The statistical independence leads to

\[ E\{N(j\omega)H(j\omega)\} = 0 \quad E\{N(j\omega)H^*(j\omega)\} = 0 \]
\[ E\{N^*(j\omega)H(j\omega)\} = 0 \quad E\{N^*(j\omega)H^*(j\omega)\} = 0 \]

The terms containing the product of the noise spectrum with any other spectrum term are equal to zero,

\[
\hat{E} = \int_{-\Omega}^{\Omega} d\omega \left\{ \phi_H(\omega) - C^*(j\omega)X^*(j\omega)\phi_H(\omega) - C(j\omega)X(j\omega)\phi_H(\omega) \right. \\
+ \left. C^*(j\omega)C(j\omega)(1 + \lambda \omega^4)\left[ X(j\omega)X^*(j\omega)\phi_H(\omega) + \phi_N(\omega) \right] \right\} = \text{minimum}
\]

In order to minimize the preceding expression, we need only to minimize the expression:

\[
\mathcal{J} = \phi_H(\omega) - C^*(j\omega)X^*(j\omega)\phi_H(\omega) - C(j\omega)X(j\omega)\phi_H(\omega) \\
+ C^*(j\omega)C(j\omega)(1 + \lambda \omega^4)\left[ X(j\omega)X^*(j\omega)\phi_H(\omega) + \phi_N(\omega) \right]
\]

The minimization is simply achieved by simply setting the partial derivatives with respect to $C(j\omega)$ to zero,

\[
\frac{\partial \mathcal{J}}{\partial C} = \frac{\partial \mathcal{J}}{\partial C^*} = 0 \quad (3.3.34)
\]

which leads to the following result

\[ 0 = -X^*(j\omega)\phi_H(\omega) + C(j\omega)(1 + \lambda \omega^4)\left[ X(j\omega)X^*(j\omega)\phi_H(\omega) + \phi_N(\omega) \right] \]

Finally, we obtain the following expression for the compensating filter,

CHAPTER III
\begin{align*}
C(j\omega) &= \frac{X(j\omega)\phi_H(\omega)}{|X(j\omega)|^2\phi_H(\omega) + \phi_N(\omega)} \times \frac{1}{(1 + \lambda \omega^4)} \\
C(j\omega) &= \frac{1}{X(j\omega)} \times \frac{1}{1 + \phi_N(\omega)(|X(j\omega)|^2\phi_H(\omega))} \times \frac{1}{(1 + \lambda \omega^4)} 
\end{align*}

(3.3.35)

The final result is a combination of the smoothing (lowpass) filter and an optimized optimal compensation filter, Eq. (2.3.8). For example, if \( \lambda = 0 \), we are left with \( \frac{1}{1 + \phi_N(\omega)(|X(j\omega)|^2\phi_H(\omega))} \) only, which represents the optimal compensating filter when \( \phi_N(\omega) \) and \( \phi_H(\omega) \) are known. Usually, the quantities \( \phi_N(\omega) \) and \( \phi_H(\omega) \) are not known.

### 3.3.2b Adaptive smoothing (regularizator) deconvolution filter

In the case of the regularization technique, the derivation is performed in a very different manner. The additional constraint is similar to the case of the smoothing filter. The total sharpness is given by the following expression,

\[
S = \int_{-\Omega}^{\Omega} d\omega \ \omega^2 |H_e(j\omega)|^2 
\]

(3.3.36)

The deconvolved impulse response is assumed to be corrupted by an additive noise such that:

\[
h(t) = h_c(t) + n(t)
\]

(3.3.37)

where \( h_c(t) \) is simply the correct estimate which is noise free, and its Fourier transform is given by

\[
H_c(j\omega) = Y(j\omega)/X(j\omega)
\]

(3.3.38)

Therefore,
\[ H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} + N(j\omega) \]

However, the estimated transfer function is obtained by application of a regularization function \( R(j\omega) \) in the following manner,

\[ H_e(j\omega) = Y(j\omega)R(j\omega) \] \hspace{1cm} (3.3.39)

When the sharpness constraint is combined with the MMSE criterion, the new criterion becomes the minimization of the following expression

\[ \hat{E} = \int_{-\Omega}^{\Omega} d\omega \{ |Y(j\omega) - X(j\omega)H_e(j\omega)|^2 + \lambda \omega^2 |C(j\omega)Y(j\omega)|^2 \} = \text{minimum} \]

Substituting for \( H_e(j\omega) \) in the preceding equation, we get

\[ \hat{E} = < \int_{-\Omega}^{\Omega} d\omega \{ \left| Y(j\omega) - Y(j\omega)R(j\omega) - X(j\omega)R(j\omega)N(j\omega) \right|^2 + \lambda \omega^2 \left| \frac{Y(j\omega)}{X(j\omega)} \right|^2 R(j\omega) + R(j\omega)N(j\omega) \} > = \text{minimum} \]

Using the statistical independence relations between \( Y(j\omega) \) and \( N(j\omega) \), and denoting

\[ \phi_Y(\omega) = E\{Y(j\omega)Y^*(j\omega)\}, \]

the terms containing the product of the noise spectrum with any other spectrum term are zero.

\[ \hat{E} = < \int_{-\Omega}^{\Omega} d\omega \{ \phi_Y(\omega) - R^*(j\omega)\phi_Y(\omega) - R(j\omega)\phi_Y(\omega) \\
+ R(j\omega)R^*(j\omega)(\phi_Y(\omega) + |X(j\omega)|^2\phi_N(\omega))(1 + \frac{\lambda \omega^2}{|X(j\omega)|^2}) \} > \\
= \text{minimum} \]

In order to minimize the preceding expression, we need only to minimize the expression:
\[ \mathcal{J} = \phi_Y(\omega) - R^*(j\omega)\phi_Y(\omega) - R(j\omega)\phi_Y(\omega) + R(j\omega)R^*(j\omega)(\phi_Y(\omega) + |X(j\omega)|^2\phi_N(\omega))(1 + \frac{\lambda\omega^2}{|X(j\omega)|^2}) \]

The minimization is simply achieved by simply setting the partial derivatives with respect to \( R(j\omega) \) to zero

\[ \frac{\partial \mathcal{J}}{\partial R} = \frac{\partial \mathcal{J}}{\partial R^*} = 0 \]  \hspace{1cm} (3.3.40)

which leads to the following result

\[ 0 = -\phi_Y(\omega)R(j\omega)(\phi_Y(\omega) + |X(j\omega)|^2\phi_N(\omega))(1 + \frac{\lambda\omega^2}{|X(j\omega)|^2}) \]

and finally, we obtain the following expression for the compensating filter,

\[ R(j\omega) = \frac{1}{1 + \phi_N(\omega)(|X(j\omega)|^2\phi_Y(\omega))} \times \frac{1}{(1 + \lambda\omega^2/|X(j\omega)|^2)} \]  \hspace{1cm} (3.3.41)

The same remarks that we made concerning the statistical smoothing filter can be made about this filter. Thus, if \( \phi_N(\omega) \) and \( \phi_Y(\omega) \) are known, the part depending on the parameter \( \lambda \) can be dropped.

### 3.3.2c Optimal compensation filter

In the optimal compensation technique, a filter function \( C(j\omega) \) is sought as an appropriate function by which \( Y(j\omega) \) is multiplied to yield an adequate estimate of the frequency response. Therefore,

\[ H_s(j\omega) = Y(j\omega) \times C(j\omega) \]  \hspace{1cm} (3.3.42)
where \( H_e(j\omega) \) designates the estimate of \( H(j\omega) \). In the case of the optimal compensation, a restoring filter \( C(j\omega) \) that is constrained to be bounded and obeys a MMSE quality criterion is being sought. The MMSE quality criterion is given by:

\[
E_e = \int_{-\Omega}^{\Omega} |H(j\omega) - H_e(j\omega)|^2 d\omega = \text{minimum} 
\]  

(3.3.43)

The estimate \( H_e(j\omega) \) is assumed to be frequency band-limited and having no significant information at frequencies \(|\omega| > \Omega\).

In order to limit the noise in the estimate \( H_e(j\omega) \), some researchers [2,14,15] took a slightly different approach by adding another constraint on the filter. And in the case of the optimal compensation, the compensating filter \( C(j\omega) \) was constrained to be a bounded function.

\[
E_c = \int_{-\Omega}^{\Omega} |H(j\omega)C(j\omega)|^2 d\omega 
\]  

(3.3.44)

Now, using a realistic formulation of the convolution operation, we intend to derive a generalized form of the optimal compensation technique that takes into consideration the nature of the additive noise that corrupts the observed response.

\[
y(t) = x(t)h(t) + n(t) 
\]  

(3.3.45a)

and in the frequency domain

\[
Y(j\omega) = X(j\omega)H(j\omega) + N(j\omega) 
\]  

(3.3.45b)

By grouping the two design requirements into one, the new criterion becomes the minimization of the energy \( \hat{E} \) defined as:

\[
\hat{E} = E_e + \lambda E_c, \quad \lambda > 0 
\]  

(3.3.46)
\[
\hat{E} = \left< \int_{-\Omega}^{\Omega} \left\{ |H(j\omega) - H_0(j\omega)|^2 + \lambda |H_s(j\omega)|^2 \right\} d\omega \right> = \text{minimum}
\]

\[
\hat{E} = \left< \int_{-\Omega}^{\Omega} \left\{ |H(j\omega) - Y(j\omega)C(j\omega)|^2
+ \lambda |H(j\omega)C(j\omega)|^2 \right\} d\omega \right> = \text{minimum}
\]

\[
\hat{E} = \left< \int_{-\Omega}^{\Omega} \left\{ |H(j\omega) - X(j\omega)H(j\omega)C(j\omega) - N(j\omega)C(j\omega)|^2
+ \lambda |H(j\omega)C(j\omega)|^2 \right\} d\omega \right> = \text{minimum}
\]

Using the statistical independence relations between \(Y(j\omega)\) and \(N(j\omega)\), we eliminate the terms that are zero,

\[
\hat{E} = \left< \int_{-\Omega}^{\Omega} \left\{ \phi_H(\omega) - X^*(j\omega)\phi_H(\omega)C^*(j\omega) - X(j\omega)\phi_H(\omega)C(j\omega)
+ \lambda \phi_H(\omega)C^*(j\omega)C(j\omega) \right\} d\omega \right> = \text{minimum}
\]

the integrand expression is to be minimized

\[
\mathcal{F} = \phi_H(\omega) - X^*(j\omega)\phi_H(\omega)C^*(j\omega) - X(j\omega)\phi_H(\omega)C(j\omega)
+ C^*(j\omega)C(j\omega)\left[ \phi_H(\omega) |X(j\omega)|^2 + \phi_N(\omega) + \lambda \phi_H(\omega) \right] = \text{minimum}
\]

Setting the partial derivative \(\partial \mathcal{F} / \partial C^*(j\omega) = 0\), we obtain

\[
0 = -X^*(j\omega)\phi_H(\omega) + C(j\omega) \times \left[ \phi_H(\omega) |X(j\omega)|^2 + \phi_N(\omega) + \lambda \phi_H(\omega) \right]
\]

Finally, when the additive output noise is taken into consideration, the expression of the filter becomes

\[
C(j\omega) = \frac{X^*(j\omega)\phi_H(\omega)}{\phi_H(\omega) |X(j\omega)|^2 + \phi_N(\omega) + \lambda \phi_H(\omega)}
\]
\[ C(j\omega) = \frac{X'(j\omega)}{|X(j\omega)|^2 + \lambda + \phi_N(\omega)/\phi_H(\omega)} \quad (3.3.47) \]

Where \( \lambda \) is simply a parameter that controls the balance between the degree of noise reduction and the errors introduced by the filter. It is clear from the expression of the filter in (3.3.47) that for \( \lambda = 0 \), the term \( \phi_N(\omega)/\phi_H(\omega) \) would yield the appropriate result if both \( \phi_N(\omega) \) and \( \phi_H(\omega) \) are known. For that case, the use of the ratio \( \phi_N(\omega)/\phi_H(\omega) \) would be equivalent to the use of the optimal value \( \lambda_{\text{opt}} \) in the deterministic optimal compensation filter. Thus, the knowledge of \( \phi_N(\omega) \) and \( \phi_H(\omega) \) would eliminate the use of any optimization criterion. Because the use of their ratio would yield the best estimate possible, as demonstrated through the derivation process.

In all the statistical methods that we derived in previous sections, we used second-order statistics quantities such as the second moment. The importance of these quantities is linked with the fact that most optimal filter design criteria require knowledge only of second-order statistics and do not require more detailed knowledge, such as of probability densities. It is of primary importance, then, to be able to extract such quantities from the actual measured data. Different techniques for spectral estimation exist in the literature [70-71]. In this work, we do not intend on developing or deriving spectral estimation techniques. The techniques that are available do fill the requirements of all types of measured data.

### 3.3.3 Comparison of frequency domain filters

The different frequency domain deconvolution techniques achieve noise reduction in the deconvolved result by using different adaptive filters. When the excitation is a step-like or lowpass type waveform, all the deconvolution filters that we presented can be successfully used. In this case, the excitation is band-limited having no significant spectral components beyond a frequency \( \Omega \). Thus, for frequencies greater than \( \Omega \), the spectrum of the output (re-
response) signal would also be identically zero. Therefore, when direct frequency domain division is applied for deconvolution, it leads to an indetermination, and the noise contained in these frequencies is greatly amplified. The deconvolution filters acting as lowpass filters would successfully eliminate such noise effects in the high frequencies. This is illustrated in Figure (3.1) where the excitation is a step-like waveform that has a fast decaying magnitude. We notice that all the filters achieve the same level of attenuation. In the figure, we made sure that all the filters have the same 3 dB cutoff frequency. We see that they are all almost identical, except that some of them emphasize the smoothness of the result. This is seen in the case of the lowpass smoothing filter for different orders, i.e., $n=2$ and $3$. However, the lowpass smoothing filter and its variants is limited to systems excited by step-like waveforms and lowpass type (or band-limited) signals. In the figure, the notations used are to describe the following filters: 1) Opt comp: the optimal compensation filter, Eq. (2.3.8), 2) Dis reg: the discrete regularization filter, Eq. (2.3.8), 3) LP: the smoothing lowpass filter with $n=2,3$, 4) Cont reg: the continuous regularization filter, Eq. (3.3.15).

The optimal compensation filter is very adaptive filter. It works with any type of excitation. The other filters are less adaptive because their dependence on the input excitation is overshadowed by their dependence on frequency. So, some filters such as the optimal compensation filter would adjust based on the magnitude of the input signal by providing a stop band whenever this signal is small. However, other types of filters such as regularizers would emphasize the low frequency components due to the frequency term; $\omega^2$ for the continuous regularizer, and $\sin^4(2\pi \omega/\Omega)$ with $\Omega$ being the folding frequency for the discrete regularizer, and attenuate the high frequency ones. The results concerning the adaptive nature of the different filters are illustrated in Figure (3.2) where the excitation is simply a linear combination of a wideband and a narrowband signals. We see that if the signal is small at low frequencies, only the optimal compensation filter is capable of providing a stopband around such frequencies.
Figure 3.1. Magnitude of deconvolution filters when the applied excitation is a step-like (lowpass) signal.
Figure (3.2). Magnitude of deconvolution filters when the applied excitation is a combination of wide-band and narrow-band signals.
3.4 SIMULATION EXAMPLES

In this section, the deconvolution techniques derived previously are tested using a simulated example of a deconvolution problem. The unknown signal was chosen to be a wideband type signal, since wideband microwave devices are frequently encountered in TDR measurement situations. To be able to simulate the wideband device, a bandpass network was used. The choice of the wideband signal was also meant to show that the assumption that noise occupies the high frequencies instead of the low frequencies is valid in most cases. We also wanted to show that the techniques are not limited to lowpass type (band-limited) signals that represent the best examples of the success of the proposed deconvolution techniques. A lossless bandpass network circuit is used to generate a bandpass signal. The bandpass filter has a passband extending from 30 to 40 GHz, and the lowest stopband frequency located at the 16 GHz point. In order to simulate a TDR example, a step waveform is generated and about 60 dB gaussian noise was added to it. The step reference has a transition duration of about 11.8 picoseconds, which yields a frequency bandwidth of about 29.38 GHz. The step waveform that is used to excite the bandpass network, and the obtained response (transmission) are shown in Figures. (3.3) and (3.4), respectively. Then, different noise levels were added to the response waveform. The noise that was added was a zero-mean type with gaussian distribution. The signal-to-noise ratio is defined as the ratio of the maximum value of the signal being corrupted to the variance of the noise expressed in dB's. The following noise levels yielding following signal-to-noise ratios 60, 40, and 20 SNR's were added. Both of the techniques that were developed in the previous section were used in order to test their performance. When the noise level is high, the flat portion (=0) of the signal will always contaminated by low noise oscillations. The recovery of the impulse response was successful for all the cases. However, only the results corresponding to the cases of 20 and 40 dB's SNR's are reported in this section. The transfer function magnitude for the 20 dB noise case, when a smoothing filter is used with different levels, is shown in Figure. (3.5). The corresponding impulse responses are shown in Figure. (3.6). The results corresponding to the 40
dB noise case are shown in Figures. (3.7) and (3.8), respectively. Similar results were also obtained when the adaptive smoothing filter was used. They are summarized in Figures. (3.9) and (3.10) for the case of 20 dB SNR, and in Figures. (3.11) and (3.12) for the 40 dB SNR case. We see that the impulse responses obtained using the two different methods are almost identical except for certain small details. These details are due to the difference in the filters used. However, both impulse responses represent the best estimates within the bandwidth of excitation signal.
Figure (3.3). Reference step waveform used to excite the bandpass network filter.
Figure (3.4). Bandpass response waveform without any additive noise.
Figure (3.5). Transfer function magnitude obtained using the direct frequency domain division for the case of 20 dB SNR.
Figure (3.6). Impulse response obtained using the direct frequency domain division for the case of 20 dB SNR.
Figure (3.7). Transfer function magnitude when different deconvolution filters are applied for the case of 20 dB SNR.
Figure (3.8). Impulse responses obtained when different deconvolution filters are applied for the case of 20 dB SNR.
Figure (3.9). Transfer function magnitude obtained using the direct frequency domain division for the case of 40 dB SNR.
Figure (3.10). Impulse response obtained using the direct frequency domain division for the case of 40 dB SNR.
Figure (3.11). Transfer function magnitude when different deconvolution filters are applied for the case of 40 dB SNR.
Figure (3.12). Impulse responses when different deconvolution filters are applied for the case of 40 dB SNR.
3.5 EXPERIMENTAL EXAMPLES

In this section, we consider a microwave device that has been measured for system’s impulse determination. The model represents a perfect example of a typical time domain reflectometry (TDR) deconvolution problem. The impulse response being estimated is valid within the TDR system bandwidth. The microwave device is excited by a TDR step waveform generated by the Hypres PSP-1000 (PicoSecond Signal Processor). A TDR response response waveform for the microwave device is acquired at its interface port to the coaxial line. A reference waveform is also acquired by replacing the device with a coaxial short network termination. The reference and response waveforms are shown in Figures. (3.13) and (3.14), respectively. The step reference has a 6 picoseconds transition duration, which yields a frequency bandwidth of about 80 GHz.

We apply the deconvolution techniques that were derived in Sec. 3.3. No information concerning the type of noise that corrupts the measured data is available, therefore, the deterministic techniques are used. Figure. (3.15) shows the deconvolved result obtained when the direct frequency domain division is used. In Figure. (3.16), we see the transfer function of the device when no filter is applied. It is clear that the result is noise contaminated, and we see that the noise occupies the high frequencies. However, when adequate deconvolution filters are applied, high frequencies noise is reduced drastically, and no loss of information occurred. This is confirmed by Figure. (3.17) where the impulse responses are shown for two filters: lowpass type filter and continuous regularizer. Figures. (3.18) and (3.19) show the corresponding the transfer functions. However, it is clear that the details of optimal impulse responses are different. The filters have different characteristics and both yield good estimates of the solution that is not unique. The filtered magnitudes using both filters are compared in Figure. (3.18). The results of the linear convolution of the optimally deconvolved results with the TDR reference waveform are shown in Figure. (3.18).
Figure (3.13). TDR reference $x(t)$ for the microwave device experiment.
Figure (3.14). TDR response $y(t)$ for the microwave device experiment.
Figure (3.15). The impulse response obtained using the direct division for the microwave device experiment.
Figure (3.16). Transfer function magnitude of the microwave device when different deconvolution filters are applied.
Figure (3.17). Impulse response of the microwave device when different deconvolution filters are applied.
Figure (3.18). Comparison of the response with the result of the linear convolution of the optimal estimate with the TDR reference.
3.6 CONCLUSION

In this chapter, we derived several frequency domain deconvolution techniques that could be applied to different types of deconvolution problems. These techniques were derived for the deterministic cases where the additive noise component is not included explicitly in the convolution integral. Then they were generalized to take into account the nature of the noise that corrupts the observed response waveform. All these filters are parameter dependent, and thus, their performance must be optimized in order to obtain the best result possible. The parameter optimization is achieved for the deterministic filters in chapter three, where a new technique is developed. However, the statistical methods used second-order statistics quantities such as the second moment. The importance of these quantities is linked with the fact that most optimal filter design criteria require knowledge only of second-order statistics and do not require more detailed knowledge, such as of probability densities. It is of primary importance, then, to be able to extract such quantities from the actual measured data. If the second-order statistics quantities were known, the filters parameter dependence is eliminated. However, the knowledge of such quantities is not always possible.

We also compared some of the widely used deconvolution filters. We showed that the optimal compensation filter, which is function of the excitation spectrum, is a very adaptive filter when applied to deconvolution. The other filters are less adaptive because their dependence on the input excitation is overshadowed by their dependence on frequency. So, some filters would adjust based on the magnitude of the input signal by providing a stop band whenever this signal is small. However, other types of filters would emphasize the low frequency components.

Both of the techniques that were derived were tested on two examples, simulated and experimental. For the simulated example, different noise levels were used and very good results were obtained even for a 20 dB signal-to-noise (SNR). The experimental case was a typical time domain reflectometry deconvolution problem. Successful results were achieved using both techniques.
CHAPTER IV

AN OPTIMIZATION TECHNIQUE FOR ITERATIVE FREQUENCY DOMAIN DECONVOLUTION

4.1 INTRODUCTION

In practical deconvolution as discussed in previous chapters, it is not possible to obtain an exact or even a unique solution. Only estimates and approximate solutions can be obtained [65,66]. Different methods have been used to achieve good estimates of the deconvolution result. However, the quality of these estimates can be measured through the implementation of qualitative error function criteria [67]. The use of these criteria allows the choice between several possible estimates. For a physical linear system with input signal $x(t)$, and output signal $y(t)$, if $h_e(t)$ is an estimate of the unknown impulse response, the corresponding error function is given by the following relation,

$$e(t) = y(t) - [h_e(t) * x(t)].$$

(4.1.1)

The definition of the error function in Eq. (4.1.1) is very simple, however, it may not provide a true indication of the quality of the estimates obtained. Different parameters must be defined to characterize the error function, such as the mean value, the standard deviation, or the maximum value. The minimization of these parameters could lead to good estimates,
however, this may not be true for all cases. In order to be able to use these parameters as indicators of the quality of the deconvolution estimate, one needs to define a reliable criterion that would decide the appropriate and adequate estimate.

Another quality criterion that has been used with iterative deconvolution techniques consists of using the root-mean-square value of the error. In practice, discrete data is used to describe waveforms, which are assumed to be causal. Thus, the error function definition can be slightly different that the definition given in (4.1.1). When iterative deconvolution techniques such as the Van Cittert technique are used, their convergence can be monitored by forming the root-mean square error of the current estimate.

\[
e_i = \left\{ \frac{1}{N} \sum_{k=0}^{N-1} \left[ y(k) - \sum_{j=0}^{L} h(j)x(k-j) \right]^2 \right\}^{1/2}
\]  

(4.1.2)

where \( h(i) \) designates the \( i^{th} \) iterative estimate, and \( N \) the number of data points. The decision to stop the iterative process is based on finding the lowest (minimum) error value. However, there is no defined pattern for the error to follow. Thus, it is subject to the shape of the waveforms which may yield several minima. The quantity \( e_i \) is not an infallible measure of convergence.

In chapter three, we covered some iterative frequency domain deconvolution techniques. These techniques involve the use of filters that include a parameter that decides their performance. The parameter is used to optimize the filter performance in terms of noise reduction while maintaining deconvolution accuracy. Previous research work led to the development of different techniques for the optimization of the iteration parameter. In all, there are two techniques that were developed for use in the area of time domain deconvolution. The two techniques involve the use of the time domain (impulse) response information in selecting the optimum performance parameter. One technique is limited to a certain class of impulse responses as it will be explained in the next section. And the other technique, is
subject to the randomness of the waveforms that are real. Because it attempts to reduce the imaginary part component in the estimate, which is supposed to be zero. However, the technique disregards the fact the imaginary part could be caused by computation errors.

In this chapter, we will present a new technique for iteration parameter optimization which can be fully implemented using the frequency domain transfer function data. By doing so, the time consuming inverse Fourier transformation required to compute the system's impulse response for each iteration is eliminated. Thus, resulting in a tremendous saving of computation time. In addition, the proposed technique yields an insight into the filter performance by monitoring the amount of noise reduction as well as filtration error introduced in the computed transfer function at various frequencies.

This chapter is divided into different sections. The first section includes separate reviews of two time domain optimization techniques. The second section presents the new frequency domain optimization technique. The third section includes the algorithms of the different techniques. Some simulated and experimental examples of deconvolution are also provided to illustrate the successful application of the proposed new technique. Finally, the chapter is ended by some concluding remarks.

4.2 TIME DOMAIN OPTIMIZATION TECHNIQUES

4.2.1 Introduction

An optimization criterion is required to make an appropriate selection of the iteration parameter as used in the iterative deconvolution techniques. The choice can be made based on the inspection of the the result of the criteria used, as much as it could be made through an implemented automated process. In the past, two optimization criteria have been applied in conjunction with the iterative deconvolution techniques, they are:
a) The minimization of the standard deviation of the imaginary part of the deconvolved impulse response.

b) The compromise between the noise and accuracy of the step response, obtained as the integral of the deconvolved impulse response.

4.2.2 Nahman-Guillaume technique [5]

In time domain measurements, as an example, the signals are real and causal. However, in deconvolving signals we use frequency domain filters, it was found that the recovered impulse response usually contains an imaginary part due to the non-uniformity of the noise. Based on these findings, an intuitive approach was developed to optimize the filtering technique. The approach simply assumes that the minimization of the imaginary part can be achieved by an adequate and appropriate filter. It involves the examination of the imaginary part of the deconvolved impulse response $h_e(t, \lambda)$. It was also noticed that the standard deviation of the imaginary part passed through a minimum with increasing parameter value, with an initial parameter value equal to zero. This is based on the fact that since both the input $x(t)$ and the output $y(t)$ are real, the impulse response response $h(t)$ must also be real. However, due to the presence of noise and computational errors, when deconvolving it is found that $h_e(t, \lambda)$ has an imaginary part.

$$h_e(t, \lambda) = h_e^R(t, \lambda) + jh_e^I(t, \lambda)$$  \hspace{1cm} (4.2.1)

Therefore, the power of the imaginary part can be used as a measure of the quality of the deconvolution result. Hence, it can be used in selecting the optimum iteration parameter.

$$\sigma_I(\lambda) = \sqrt{\frac{1}{N} \sum_{k=0}^{N-1} \left[ h_e^I(t, \lambda) - E[h_e^I(t, \lambda)] \right]^2}$$  \hspace{1cm} (4.2.2)
Where \( E[ . ] \) represents the mean value of the term between brackets. When using the minimization of the standard deviation of the imaginary part, it was observed that the resulting curve depends on the optimization parameter \( \lambda \). The curve exhibits a wide minimum with respect to \( \lambda \) in certain cases, which coincides with the values of \( \lambda \) yielding good estimates of the impulse response \( h(t) \). However, very often the curve exhibits an infinite decay towards zero, which makes difficult to decide the optimum parameter. There is no proof that the minimum exhibited by the standard deviation \( \sigma_i(\lambda) \) would yield the optimum value of \( \lambda \). The finding of this method was arrived at empirically and was not shown or proven to work in general. The imaginary part is dependent on the randomness of the signals used, and highly dependent on computation errors and on the software used as well as on the computer precision hardware.

4.2.3 Riad-Parruck technique [18]

Although developed in conjunction with the optimal compensation deconvolution technique, the second criterion can be used with other deconvolution techniques as well. The derivation of this criterion is based on the assumption that the integration of the impulse response will lead to a step response that is constant over a certain time interval. From different filters defined in the preceding section, it is clear that the accuracy of the deconvolution process deteriorates as \( \lambda \) increases. On the other hand, the result obtained with a very small \( \lambda \) is too noisy to reveal the information content of \( h_s(t) \). However, integrating \( h_s(t) \) in order to obtain \( w_s(t) \), yields a waveform with much less noise content. In addition, the step response as an integration of a duration limited impulse response, provides a nonzero ideally flat region which can be used to estimate the deconvolution accuracy as well as noise content. For a duration limited \( h(t) \), a time \( T_1 \) exists such that:

\[
h(t) = 0 \quad t > T_1
\]  

(4.2.3)

Hence, the step response \( w(t) \) must be flat for \( t > T_1 \).
The variations of \( w_*(t) \) in a region \( t > T_1 \), can be used to estimate the deconvolution accuracy and noise content.

\[
H_*(j\omega) = \frac{Y(j\omega)}{X(j\omega)} \frac{1}{1 + F(\lambda, |X(j\omega)|)}
\]  
(4.2.4)

From Eq. (4.2.4), increasing \( \lambda \) leads to a decrease in \( |H_*(j\omega)| \) and consequently the step level (the average of \( w_*(t) \) in the region) is expected to decrease. The magnitude of the step-level of \( w_*(t) \) can be used as an indicator of the deconvolution accuracy. In addition, the root-mean-square deviations of \( w_*(t) \) from its average (in the same region) is a measure of the noise content of the waveform. As \( \lambda \) increases, the response \( w_*(t) \) smoothen and leads to a very small noise content. Mathematically, the accuracy \( A(\lambda) \) and noise content \( N(\lambda) \) are defined as follows:

\[
A(\lambda) = \text{Average}[w_*(t)] \quad t > T_1
\]
(4.2.5)

\[
N(\lambda) = \text{RMS}[w_*(t) - A(\lambda)] \quad t > T_1
\]
(4.2.6)

From Eq. (4.2.3), it can be seen that a relatively small \( \lambda \) has little effect in frequency ranges where \( |X(j\omega)| \) is significant, but has a greater influence in the ranges where \( |X(j\omega)| \) is very small. Regions of small \( |X(j\omega)| \) contribute more to the deconvolution noise than to its information content. Therefore, it is expected that for small values of \( \lambda \), \( A(\lambda) \) will decrease at a much slower rate compared to that of \( N(\lambda) \). Consequently, it is expected that a range of \( \lambda \) exists for which \( A(\lambda) \) is close to \( A(\lambda = 0) \) while \( N(\lambda) \) is much reduced compared to \( N(\lambda = 0) \). The optimum value for the iteration parameter \( \lambda \) can then be defined such that:

\[
[A(0) - A(\lambda_{opt})]/A(0) \ll 1
\]
(4.2.7)

and

\[
N(\lambda_{opt}) \ll N(0)
\]
(4.2.8)
The above criterion is only applicable to cases where the impulse response is duration limited. Some impulse responses do not yield step-like waveforms after integration, that is constant over a certain interval. Hence, the technique is by then limited to the estimation of duration limited waveforms. For such deconvolution cases, it was shown to yield good optimal estimates [18].

4.3 FREQUENCY-DOMAIN OPTIMIZATION TECHNIQUE

The purpose in any of the frequency domain filtering techniques is to attenuate the noise that contaminates the signal, however, at the same time try to avoid any kind of distortion of the useful signal information. Typically, in the area of time domain measurements, the type of signals that are encountered are frequency band-limited, and it is found that the disturbing effect of the noise appears in the high frequency range. Hence, the useful signal information lies in the low frequencies terms, while the high frequencies terms carry most of the signal noise. Different frequency domain deconvolution techniques achieve the noise reduction by using different adaptive filters. Some of these filters would adjust based on the magnitude of the input signal by providing a stop band whenever this signal is small. However, other types of filter would emphasize the low-frequency components due to the $\omega^4$ and attenuate the high frequency ones. Based on all these observations, it is possible to separate the transfer function $H(j\omega)$ into several frequency intervals and detect how much attenuation of the signal is reached.

This section presents an optimization criterion for the selection of the iteration parameter for the iterative frequency domain deconvolutions that involve filtering techniques [85,86]. The optimization has the advantage of being performed in the frequency domain instead of the time domain, therefore the use of the inverse Fourier transformation at each iteration step is eliminated. The process is achieved by partitioning the transfer function bandwidth into several frequency intervals based on the degree of signal information contained in each interval.
Then, the level of signal attenuation is monitored in each interval for each value of the iteration parameter.

To illustrate the criterion a sample study is discussed in this section. Figure (4.1) shows the spectrum of an ideal lowpass signal, it can be seen that the spectrum can be divided into three main regions. The signal or information region that carries most of the useful signal information, the noise level within this region is expected to be insignificant. Then, the transition region that also carries a certain amount of useful signal information, but the noise level could be slightly higher depending on the bandwidth of the input signal. Therefore, the noise reduction has to be very moderate for it could lead to some loss of useful information. Finally, the noise region which carries very little useful information if none at all, has a very high noise content and for that reason the noise attenuation should be very strong.

In order to be able to monitor the degree of noise reduction reached in the different frequency regions, the frequency range from 0 to $\Omega$ is divided into several frequency intervals (not necessarily equal). In the case of Figure (4.1), the transfer function is divided into three intervals, one covers the information region, the second contains the transition region, and the third extends over the noise region.

Then, the function $\sigma(\lambda)$ is defined as the standard deviation of the transfer function obtained for a filter parameter $\lambda$ and defined over the $i^{th}$ interval from its value over the same interval obtained when no filtering is applied ($\lambda = 0$).

$$\sigma(\lambda) = \text{rms}_i \left\{ |H^i(j\omega) - H^0(j\omega)| \right\}$$  \hspace{1cm} (4.3.1)

In this definition, the abbreviation rms, refers to the root-mean-square operation performed on the quantity between brackets $\{ \}$ over the $i^{th}$ frequency interval. Also, $H^i(j\omega)$ is the transfer function obtained for a given $\lambda$, and $H^0(j\omega)$ is the transfer function obtained using the direct division (i.e., $\lambda = 0$). In practice, we deal with discrete data waveforms, and therefore, the discrete Fourier transform (DFT) is implemented using the FFT algorithm. Hence, the standard deviation computed for $i^{th}$ spectrum interval is given by the following relation:
Figure (4.1). Ideal LP transfer function as partitioned into three major regions.
\[ \sigma_i(\lambda) = \left\{ -\frac{1}{N_i} \sum_{i=k_i}^{k_{i+1}} \left[ H_R(i) - H_R(o) \right]^2 + \left[ H_I(i) - H_I(o) \right]^2 \right\}^{1/2} \] (4.3.2)

where \( H_R(i) \) and \( H_I(i) \) represent the real and imaginary parts of the discrete transfer function obtained when parameter value \( \lambda \) is applied with the filter. The same observations can be made for \( H_R(i) \) and \( H_I(i) \) except that \( (\lambda = 0) \), no filtering applied. The integers \( k_i \) and \( k_{i+1} \) designate the starting and ending points of the \( i^{th} \) interval, and \( N_i \) represents the number of points in the \( i^{th} \) interval.

In seeking an optimum filter that would attenuate the noise level and avoid any kind of signal distortion, our main concern was to be able to predict the proportion of noise reduction reached in each frequency range. The choice of the optimal value is based on a compromise that avoids signal distortion in the frequencies of interest and the attenuation of the noise level. In Figure. (4.2), three curves representing the transfer function of a noisy lowpass type signal for different values of the parameter used in the optimal compensation filter. It is shown that for very small parameter values, no noise is being reduced. However, for high parameter values, the signal is distorted. Between these two extreme cases there are some parameter values that would be considered to be optimum and the criterion that is proposed would give us those values. Figure. (4.3) shows the anticipated results of the frequency domain optimization technique. In the figure, the standard deviations \( \sigma_i (i=1,2,3) \) for the three intervals are given as functions of \( \lambda \). The \( \sigma_i \) values are normalized to a unity maximum, and the \( \lambda \) values are normalized with respect to the arbitrarily chosen root-mean-squared value of the transfer function magnitudes below the -20 dB level. When examining the results of this figure, the following observations can be stated,

1. \( \sigma_i \approx 0, \ i=1,2,3, \) for \( \lambda_n < 0.01 \), meaning that such values for \( \lambda_n \) produce negligible filtration.

CHAPTER IV
2. \( \sigma_n, i=1,2,3, \) for \( \lambda_n < 0.2 \), meaning that no significant signal distortion occurs in the useful information region with such \( \lambda_n \) values.

3. \( \lambda_n \approx 0.4 \) appears to be maximum safe value for no significant distortion in the first region. For this \( \lambda_n \), the noise reduction in the third region is not maximal, \( \sigma_n, i=0.6 \), and is moderate in the second region, \( \sigma_n, i=0.2 \).

From the above observations, it can be concluded that the range for optimum \( \lambda_n \) is \( \approx 0.2 - 0.4 \).

The preceding discussion is simply an illustration that is used to explain why and how the technique should be applied. Practical examples that include a simulated case and an experimental one, will provide a more complete and real test to the criterion. The two examples are discussed in the next sections. The criterion was defined using several regions, however, that can be sometimes confusing when deciding the minimum and maximum deviations allowed in each region. The criterion can be applied successfully using the main regions of concern, information and noise regions. We compute the \( \lambda \) values corresponding to the maximum and minimum deviations allowed in each region respectively. Then, the root-mean-square value of their product is taken to yield an approximate optimal value. However, a different combination of these values is always possible to yield an appropriate optimal parameter value.
Figure (4.2). The results of applying a frequency domain filter to the LP transfer function.
Figure (4.3). The anticipated results of the proposed frequency domain criterion.
4.4 DECONVOLUTION ALGORITHMS

4.4.1 Time-domain criteria

We wish to determine the optimum parameter value of the frequency domain filter used in the deconvolution process. The time domain optimization is an iterative technique that follows the following steps,

1. Acquisition of the input and output waveforms, \( x(t) \) and \( y(t) \) respectively.
2. Preprocessing of the waveforms, consisting of the use of the Gans-Nahman technique for the computation of the fast Fourier transform of step-like waveforms. The processing is applied whenever it is required [Appendix B].
3. Assign an arbitrary value (usually very small) to the parameter \( \lambda \) to start the iterative process.
4. Inverse Fourier transform the transfer function to obtain the impulse response \( h_\lambda(t) \).
5. Application of the time domain criteria:

4.4.1a Noise and Accuracy criterion,

- Integrate the real part of \( h_\lambda(t) \) to get the step response \( w_\lambda(t) \).
- Define \( t_1 \) and \( t_2 \) as the points where the computed step response \( w_\lambda(t) \) is expected to be flat.
- Calculate the accuracy \( A(\lambda) \) and the noise content \( N(\lambda) \) as defined in Eqs (4.2.4), (4.2.5).

4.4.1b Imaginary part standard deviation

- Obtain the imaginary part of the estimated impulse response \( h_\lambda(t) \).
• Calculate its rms (root mean square) value as defined:

\[ E(\lambda) = \text{RMS } \left[ \text{i.m}h_\lambda(t) \right] \]

Store \( \lambda \), \( A(\lambda) \) and \( N(\lambda) \), (or \( \lambda \) and \( E(\lambda) \) respectively). Choose the next value for \( \lambda \) and iterate by going to step 5 until satisfactory range of \( \lambda \) is obtained. Application of the criteria conditions for the parameter to satisfy to be the optimum value:

• Plot \( A(\lambda) \) and \( N(\lambda) \) together versus \( \lambda \) and determine the value of \( \lambda \) that satisfies the condition defined in (4.2.7) and (4.2.8)

• Plot \( E(\lambda) \) versus \( \lambda \) and then the value of \( \lambda \) corresponding to a minimum.

Use the optimum value of the parameter in the frequency domain filter to compute \( h_{\text{opt}}(t) \). The step by step procedure is illustrated in the block diagram shown in Figure. (4.4).

4.4.2 Frequency domain criterion

The frequency domain optimization technique is also an iterative technique and it follows the following steps,

1. Acquisition of the input and output waveforms, \( x(k) \) and \( y(k) \) respectively.
2. Preprocessing of the waveforms, consisting of the use of the Gans-Nahman technique for the computation of the fast Fourier transform of step-like waveforms. The processing is applied whenever it is required.
3. Assign an arbitrary value (usually very small) to the parameter \( \lambda \) to start the iterative process.
4. Obtain the transfer function using straightforward frequency domain division.
5. Define the frequency intervals and their respective boundaries.
6. Select the starting and ending values of \( \lambda \) and the increment of the iteration.
7. Use of the frequency domain criterion as defined in to derive the optimum value of the iterative parameter.

8. Plot the standard deviation waveforms for each interval, then derive the optimum value of the iterations parameter.

The procedure is illustrated in the block diagram shown in Figure. (4.5).

4.4.3 The automated algorithm used for deconvolution

The selection of the optimum values of the iteration parameters \( \lambda \) used in the preceding techniques is achieved through an implementation of a deconvolution Fortran program. The program reads the input and output waveforms that are used to perform the deconvolution. The Fourier transforms of the two waveforms are taken. Then, the straightforward frequency domain division is performed, and the computed transfer function data is stored. The boundaries of the selected frequency domain intervals are read to the program. The selected intervals are to be used in the proposed optimization technique. Usually, the intervals represent the noise region and the information region. Also, the iteration parameter starting and ending values, along with the incrementation step must be read to the program in order to start the optimization process. Once the different \( \sigma(\lambda) \) curves are obtained for the information and noise regions, the 10% and 90% deviation points are computed. The Corresponding \( \lambda \) values are derived and the root square of their product is taken. The final result is considered as a good estimate of the optimal parameter. The whole procedure is summarized in the Fortran program shown in Appendix A.
Figure (4.4). Algorithm of the time domain optimization techniques.
Figure (4.5). Algorithm of the frequency domain optimization technique.
4.5 A COMPUTER SIMULATION ILLUSTRATION

A simple example is given to illustrate the proposed technique. The deconvolution simulation involves the identification of a linear system through the knowledge of both the excitation and the response waveforms. The system was chosen to be a lossless lowpass filter shown in Figure. (4.6), for which the impulse response, $h(t)$, was computer generated and is shown in Figure. (4.7). The system excitation, $x(t)$ shown in Figure. (4.8), was chosen as the step-like waveform produced by the Hypres PSP-1000 TDR system. The resulting system response of the simulation is obtained and shown in Figure. (4.8), along with the excitation.

The compensation filter (3.3.8) is then applied to yield the estimate $h_n(t)$ of the system’s impulse response. The magnitude of the transfer function $|H(\omega)|$ is expected to be small over the noise (flat) region frequency interval. However, this is not the case as shown in Figure. (4.10). The magnitude is shown for two cases: 1) $\lambda = 0$, i.e. no filter is applied, and 2) $\lambda = 1.7 \times 10^{-2}$ which corresponds to the application of the optimum filter. The corresponding impulse responses are shown in Figure. (4.11).

The choice of $\lambda = 1.7 \times 10^{-2}$ was based on the results of the proposed technique that are summarized in Figure. (4.9). In this figure, the values of the different functions $B(\lambda)$ are plotted for the frequency intervals delimited by the following frequencies 0, 64, 128, 256, and 512 GHz.

Most of the useful signal information lies in the first interval, except for the second interval which has some useful signal information, the other intervals are mostly noise as shown on Figure. (4.10). A good observation of the waveform in Figure. (4.9), shows that a normalized values of $\lambda$ higher than $5 \times 10^{-2}$ no major signal distortion occurs. However, for values less than $5 \times 10^{-4}$ some high frequencies noise is reduced, but little noise is attenuated in the preceding frequency interval. Any value of $\lambda$ in the defined range would lead to a satisfactory impulse response estimate.

The choice of these intervals is based on the knowledge of the frequency spectrum of the input excitation, and their number is arbitrary depending on the various frequencies of interest. In order to find accurately a good parameter value, it is always possible to define different
other frequency intervals so as to facilitate the optimization and reduce the range of the parameter value.
Figure (4.6). Circuit diagram of the lossless lowpass filter used in the simulation.
Figure (4.7). Impulse response of the LP filter
Figure (4.8). Reference $x(t)$ and response $y(t)$ of the LP filter simulation.
Figure (4.9). Results of the frequency domain standard deviation criterion for the LP filter simulation.
Figure (4.10). Magnitude of the transfer function using the direct division and the optimal filter for the LP filter simulation.
Figure (4.11). The impulse responses obtained using the direct division and the optimum filter for the LP filter simulation.
4.6 AN EXPERIMENTAL APPLICATION

In this section, we apply the proposed technique to an experimental case where no a-priori knowledge of the system's impulse response is available. All the waveforms used were acquired using the Hypres PSP-1000 TDR system. The reference waveform in Figure (4.8) was acquired using a precision short at the end of a reference transmission line. A wideband coaxial device was chosen for the experiment, to be referred to here as the device under test (DUT). Next, the short-circuit was replaced by the DUT and the corresponding TDR response waveform was acquired; this waveform is shown in Figure (4.12).

The transfer function of the impulse response as shown in Figure (4.15) was divided in four frequency intervals having the end frequencies 0, 64, 128, 256, and 512 GHz. The step-like excitation used is the same as the one used in the preceding simulation. The bandwidth of this step-like excitation is about 64 GHz, hence, the choice of these intervals is appropriate based on the discussion in the previous section.

The results of the frequency domain standard deviation technique are shown in Figure (4.13). The curves show that normalized values in the interval $4 \times 10^{-2}$ to $4 \times 10^{-3}$ would be very appropriate to yield a very good estimate of the impulse response. Figure (4.14) shows the transfer function when no filter is applied in comparison with the case of a filter having a parameter value of $4 \times 10^{-3}$. The corresponding time-domain waveforms are shown in Figures (4.15) and (4.16).
Figure (4.12). Response $y(t)$ of the DUT experiment.
Figure (4.13). Results of the frequency domain standard deviation criterion for the DUT experiment.
Figure (4.14). Magnitude of the transfer function using the direct division and the optimal filter for the DUT experiment.
Figure (4.15). The impulse response obtained using the direct division for the DUT experiment.
Figure (4.16). The impulse response obtained using the optimum filter for the DUT experiment.
4.7 CONCLUDING REMARKS

In this chapter we presented a technique that would enable to measure the performance of frequency domain filtering techniques in the deconvolution of signals. The technique is very important in seeking out the right parameter value for these design parameter dependent filters.

The technique does not single out just one value or find accurately the optimum $\lambda$ value, however, it does provide an accurate range of values that could be considered as optimum. In addition, it clearly indicates the degree of noise reduction and the associated loss of information in the various frequency regions.

In addition to the computation time that the technique saves, it is found that it provides a reliable procedure to select an optimum parameter value. The choice of frequency intervals is based on the degree of noise and signal information in each interval.

For demonstration purposes the technique was applied to computer simulated and experimentally acquired data, and very successful results were obtained.

In this technique, the choice of the frequency intervals is based on the degree of noise content and signal information in each interval. The number of intervals that are needed is arbitrary, better results can be achieved by the selection of only two intervals, depending on the nature of the signal.
CHAPTER V

DECONVOLUTION OF CAUSAL AND PULSE DATA

5.1 INTRODUCTION

Physical pulses and transients are causal functions of time, i.e., their values are zero before t=0, the time at which they begin. Their measured waveform data are also causal. When deconvolution processing is applied to remove instrumentation errors and/or suppress the effects of noise, non-causal deconvolution methods may introduce unacceptable errors. The frequency domain filters used in the deconvolution techniques developed in chapter three are real. They represent a class of filters that operate only on the magnitude component of the deconvolved result. The inverse Fourier transforms of such filters are symmetric in the time domain, and thus non-causal. In this chapter, these filters are modified to ensure that causality is maintained in the deconvolution result [87].

An important problem in the theory of the Fourier integral and in the study of linear systems is the determination of the conditions that a given function

\[ R(j\omega) = R_R(\omega) + jR_I(\omega) = e^{a(\omega)} + j\beta(\omega) \] (5.1.1)

must satisfy in order to be the Fourier integral of a causal function \( r(t) \) which equals zero for negative \( t \).
\[ r(t) = 0 \quad t < 0 \] (5.1.2)

There is no complete general solution to this problem; however, if certain assumptions are made about \( R(j\omega) \), then one can establish necessary and sufficient conditions for its inverse Fourier transform to be zero for negative \( t \). A simple assumption that can be made is the condition that the spectrum magnitude \( A(\omega) = e^{1(\omega)} \) be square-integrable [60].

\[
\int_{-\infty}^{\infty} A^2(\omega) d\omega < \infty
\] (5.1.3)

i.e., the corresponding function \( r(t) \) has finite energy.

Another condition for \( R(j\omega) \) to be the Fourier transform of a causal function is given by the following theorem:

*Theorem* [60]: If \( R(s) \) is analytic for \( Re(s) \geq 0 \) and \( R(s) \to 0 \) for \( s \to \infty \) then \( R(j\omega) \) is the Fourier transform of a causal function.

*Paley-Wiener condition* [60, 77]: A necessary and sufficient condition for a square-integrable function \( A(\omega) \geq 0 \) to be the Fourier spectrum of a causal function is the convergence of the integral:

\[
\int_{-\infty}^{\infty} \frac{|\ln A(\omega)|}{1 + \omega^2} d\omega < \infty
\] (5.1.4)

Thus, the Paley-Wiener condition states that for a signal (or filter) having finite energy (i.e., its spectrum is square-integrable) to be causal, the attenuation \( \alpha(\omega) \) (log of the spectrum magnitude) of the spectrum must satisfy the preceding relation. Therefore, a linear increase (involving any finite slope) is an upper bound on the asymptotic behavior of any physically realizable attenuation function. No physical network can provide an attenuation which asymptotically increases faster than a constant times \( \omega \). For example, no attenuation can increase like \( \omega^{3/2} \) or \( \omega^2 \) for \( \omega \to \infty \) [78]. Hence, for the integral to converge to a finite value, \( \alpha(\omega) \) can increase no faster than \( \omega^m, 0 \leq m < 1 \) for increasing \( \omega \). Consequently, the attenuation function \( \alpha(\omega) \) cannot be infinite over a continuous band of frequencies whether finite or infinite.

CHAPTER V
Let us consider the case of the ideal lowpass filter as an example. The filter is defined such that its response is assumed to be unity over the range \( \Omega < \omega < \Omega \) and zero otherwise. Zero response simply means infinite attenuation, and therefore this property violates the Paley-Wiener criterion. Hence, the ideal lowpass filter is not realizable.

For a system, whose attenuation function satisfies the Paley-Wiener criterion, its attenuation and phase functions are shown to be related through a pair of relations known as the Hilbert transform relations. Over the years, Hilbert transforms [20,79] have played a useful role in signal and network theory in relating the real and imaginary components, and the magnitude and phase components of spectra. Systems for performing Hilbert transform operations have proved useful in diverse fields such as radar target indicators. In this chapter, the Hilbert transform relations are used to generate the phase of a spectrum given its amplitude.

5.2 NON-CAUSAL FILTERING TECHNIQUES

A class of filters, which have been used successfully in deconvolution techniques, were derived with the following form as shown in chapter two,

\[
R(\omega) = \frac{1}{1 + |K(\omega)/X(\omega)|^2} \tag{5.2.1}
\]

For example, in the case of the optimal compensation filter, \(|K(\omega)|^2 = \gamma\). However, in the case of the regularization filter, \(|K(\omega)|^2 = \gamma \omega^2\). Both of these filters and the several variants possess only a magnitude component. They are real filters that operate on the magnitude of the signal only with no alteration to the phase component. They are also parametric filters, and their performance depend on the parameter value \(\gamma\) as shown in chapter four. They work on the noise reduction to remove the errors caused by the deconvolution processing, however,
they are not causal. The absence of causality may be a source of errors in the deconvolution result. In this chapter, a study of the non-causality effects is done and results are reported.

By allowing the filter to possess a phase component, we can obtain causal systems with the same amplitude characteristic as the filter. It will be shown that this is impossible unless $A(\omega)$ satisfies the Paley-Wiener condition. It is easy to see that, among the type of filters widely used, only the Butterworth satisfies the above condition. Thus, if we associate the magnitude component

$$A(\omega) = \frac{A_0}{\sqrt{1 + (\omega/\alpha)^2}}$$

(5.2.2)

with a phase angle $\theta(\omega) = \tan^{-1}(\omega/\alpha)$, we yield the following system function:

$$H(j\omega) = \frac{A_0}{1 + j(\omega/\alpha)}.$$  

(5.2.3)

And using Laplace transform relations, we find that its inverse $h(t)$ is given by

$$h(t) = aA_0e^{-\alpha t}U(t)$$  

(5.2.4)

Thus, $h(t)$ is a causal function, and its corresponding step response $w(t)$ is also causal.

$$w(t) = A_0(1 - e^{-\alpha t})U(t)$$  

(5.2.5)

But now, consider the system function of a filter expressed in polar coordinates,

$$H(j\omega) = A(\omega)e^{-j\theta(\omega)}$$

(5.2.6)

Knowing that the spectrum magnitude of a real function is even, i.e., $A(\omega)$ is even. Then, the filter's impulse response can be evaluated as

$$h(t) = \frac{1}{\pi} \int_{0}^{\infty} A(\omega) \cos(\omega t - \theta(\omega))d\omega$$

(5.2.7)
We also know that the step response of the filter can be described as the convolution of its
impulse with a unit step function \( u(t) \), whose Fourier transform is \( \pi \delta(\omega) + 1/j\omega \). Thus, the
Fourier transform of the filter's step response \( w(t) \) would be given by:

\[
\left[ \pi \delta(\omega) + \frac{1}{j\omega} \right] A(\omega) e^{-j\theta(\omega)} = \pi A(0) \delta(\omega) + \frac{A(\omega)}{\omega} e^{-j\left[ \theta(\omega) + \frac{\pi}{2} \right]} \]  
\text{(5.2.8)}
\]

and therefore,

\[
w(t) = \frac{A(0)}{2} + \frac{1}{\pi} \int_0^\infty \frac{A(\omega)}{\omega} \sin[\omega t - \theta(\omega)] d\omega \]  
\text{(5.2.9)}
\]

However, since \( A(\omega) \) and \( \theta(\omega) \) are assumed independent of each other, the system under
consideration is not causal; thus \( h(t) \) does not equal zero for negative \( t \), and \( w(t) \) is related to
it by,

\[
w(t) = \int_{-\infty}^t h(\tau) d\tau \]  
\text{(5.2.10)}
\]

In the following section we will show that the filters described in chapter three and by the
general relation in Eq. (5.2.1) satisfy the Paley-Wiener condition. Hence, a phase component
could be added to the magnitude to yield causal deconvolution filters. This can only be
achieved through the use of the Hilbert transform relations, which relate the phase and mag-
nitude of causal filters.

5.3 HILBERT TRANSFORM RELATIONS

The Cauchy-Riemann conditions which assure the uniqueness of the derivative of a
complex function show that the real and imaginary parts of that function satisfy Laplace's
equation. The real and imaginary parts of the impedance function of a physical network or
of the logarithm of such functions (the attenuation and phase functions) are also related in the
same manner. The implications of this relationship show that the real part alone is sufficient
to characterize the complex function, or vice versa. Hence, such complex function can be

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characterized from either the real or imaginary part. The construction of the complex function can be accomplished using the property of stability, i.e., right half-plane analyticity, or using the property of causality in physical networks, i.e., the response is zero before the excitation is applied. However, when we deal with the logarithm of a transfer function, the transfer function is restricted to have neither zeros nor poles in the right half-plane, because the logarithm function becomes infinite for both cases. Since a transfer function having no right half-plane zeros is minimum-phase, we therefore see that this method to determine the phase from the attenuation, or vice versa, will automatically get a minimum-phase propagation function. The method for computing a real from an imaginary part, or vice versa, involves a pair of infinite integrals called the Hilbert transforms. There are two methods that yield Hilbert transform pairs. These two methods which are available in the literature [20,78] are discussed the following sections.

**Method 1: Real and Imaginary Parts**

There are different derivations of the Hilbert transforms, among which, a derivation using the convolution with the transform of a unit step based upon the causality property of physical systems. This method is based upon the fact that the time domain response of a physical system may be expressed as twice the even (or odd) part of this response for \( t > 0 \),

\[
x(t) = 2x_e(t) \quad t > 0
\]
\[
= 0 \quad t < 0
\]  

\[
x(t) = 2x_o(t) \quad t > 0
\]
\[
= 0 \quad t < 0
\]  

(5.3.1)  

(5.3.2)

where \( x(t) = x_e(t) + x_o(t) \), and hence, the even and odd parts may be expressed as,

\[
x_e(t) = (1/2)[x(t) + x(-t)]
\]
\[
x_o(t) = (1/2)[x(t) - x(-t)]
\]  

(5.3.3)
However, the relations in Eq. (5.3.3) are valid for $t \neq 0$ only. However, for $t = 0$, we know that $x(0) = x_e(0)$, and therefore Eqs. (5.3.1) and (5.3.2) can be written as:

$$x(t) = 2x_e(t) u(t)$$
$$= 2x_e(t) u(t) + x(0) \delta(t)$$

(5.3.4)

where $u(t)$ is the unit step function defined as

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

(5.3.5)

Knowing that the multiplication in the time domain of two signals transforms into $\pi/2$ times the convolution in the frequency domain of their respective Fourier transforms. Then, the Fourier transform of Eq. (5.3.4) yields the following:

$$X(j\omega) = \frac{1}{\pi}X_R(j\omega) * U(j\omega)$$
$$= \frac{j}{\pi}X_i(j\omega) * U(j\omega) + x(0)$$

(5.3.6)

where $U(j\omega) = \pi \delta(\omega) + (1/j\omega)$, and, $X_R(j\omega)$ and $jX_i(j\omega)$ are simply the Fourier transforms of $x_e(t)$ and $x_o(t)$, respectively. We also know that the real and imaginary part of the Fourier transform of a real function are simply the Fourier transforms of its even and odd parts, respectively. Using the definition of $U(j\omega)$, then Eq. (5.3.6) can be written as

$$X(j\omega) = X_R(j\omega) - \left(\frac{j}{\pi}\right) \int_{-\infty}^{\infty} \frac{X_R(j\xi)}{\omega - \xi} d\xi$$
$$= jX_i(j\omega) + (1/\pi) \int_{-\infty}^{\infty} \frac{X(j\xi)}{\omega - \xi} d\xi + x(0)$$
$$= X_R(j\omega) + jX_i(j\omega)$$

(5.3.7)

By inspection, we can determine that:

$$X_i(j\omega) = -(1/j\pi) \int_{-\infty}^{\infty} \frac{X_R(j\xi)}{\omega - \xi} d\xi$$

(5.3.8)

$$X_R(j\omega) = (1/\pi) \int_{-\infty}^{\infty} \frac{X(j\xi)}{\omega - \xi} d\xi + x(0)$$

(5.3.9)
It is of interest to remark that the functions $X_R(j\omega)$ and $X_I(j\omega)$ are not independent of each other but one of them can be uniquely determined in terms of the other. The relations defined in Eqs. (5.3.8), (5.3.9) are known as the Hilbert transforms for real and imaginary parts of the Fourier transform of a causal function.

**Method 2: Attenuation and Phase Components**

Another method of deriving the Hilbert transforms is by using a derivation based upon Cauchy's integral formula. This method has the advantage of relating the attenuation to the phase of a function. For any function $\gamma(s) = \alpha(s) + j\beta(s)$ that is analytic in the right half $s$-plane ($s = \sigma + j\omega$), it is shown that $\gamma(s)$ can be expressed in terms of $\alpha(s)$ or $\beta(s)$. Physically, $\gamma(s)$ may be a propagation function, or it may be a driving-point impedance, or a function characterizing a network in the frequency domain, or simply a frequency domain deconvolution filter. The details of the derivation of the Hilbert transforms via Cauchy's integral formula are omitted; they are given elsewhere [78]. Cauchy's integral formula can be applied to a region bounded by a contour which consists of the $j$-axis and a semi-circular arc of arbitrary large but finite radius that lies entirely in the right half-plane. If $s$ denotes an internal point, and $\epsilon$ a point on the boundary, Cauchy's formula reads

$$\gamma(s) = \frac{(1/2\pi j)}{s} \int_{\epsilon} \frac{\gamma(\epsilon)d\epsilon}{\epsilon - s}$$

(5.3.10)

however, for the external point $-s$, Cauchy's integral formula yields

$$0 = \frac{(1/2\pi j)}{s} \int_{\epsilon} \frac{\gamma(\epsilon)d\epsilon}{\epsilon + s}$$

(5.3.11)

By adding and subtracting Eqs. (5.3.10) and (5.3.11), we obtain

$$\gamma(s) = \frac{s}{\pi j} \int \frac{\gamma(\epsilon)d\epsilon}{\epsilon^2 - s^2}$$

(5.3.12a)
\[ \gamma(s) = (1/\pi j) \int_{-\infty}^{\infty} \frac{e\gamma(e)de}{e^2 - s^2} \quad (5.3.12b) \]

In either of these integrals, the path of integration may be simplified to the \( j \)-axis (\( \omega \)-axis) if the contribution from the arc is zero, which it is if the integrand for large values of \( e \) varies inversely as \( e \) to some power larger than unity. For that case, the contribution from the arc is proportional to

\[ \int_{\text{arc}} \frac{de}{e^{1+a}} \quad \text{where} \quad a > 0 \quad (5.3.13) \]

Then, if we let \( e = \rho e^{i\theta} \), the integral becomes

\[ j \int_{-\pi/2}^{\pi/2} e^{ia\theta} d\theta \quad (5.3.14) \]

which is smaller than an arbitrarily small but nonzero value for a sufficiently large but still finite value of the radius \( \rho \). If the asymptotic behavior of \( \gamma(e) \) is described by

\[ \gamma(e) \to e^n \quad \text{for} \quad e \to \infty \quad (5.3.15) \]

then in Eq. (5.3.12a) the contribution from the arc is zero for \( n < 1 \), and in Eq. (5.3.12b) if \( n < 0 \). Under these conditions

\[ \gamma(s) = (s/\pi j) \int_{-\infty}^{\infty} \frac{\gamma(e)de}{e^2 - s^2} \quad (5.3.16a) \]

\[ \gamma(s) = (1/\pi j) \int_{-\infty}^{\infty} \frac{e\gamma(e)de}{e^2 - s^2} \quad (5.3.16b) \]

where the change in algebraic sign of the integrand is the result of traversing the \( j \)-axis in the opposite direction that of the arc. With \( e \) on the \( j \)-axis, then let \( e = j\xi \), we obtain for \( \gamma(j\xi) = \alpha(\xi) + j\beta(\xi) \).
\[ \gamma(s) = \left( \frac{s}{\pi} \right) \int_{-\infty}^{\infty} \frac{\alpha(\xi)d\xi}{s^2 + \xi^2} \quad (5.3.17a) \]

\[ \gamma(s) = \left( \frac{-1}{\pi} \right) \int_{-\infty}^{\infty} \frac{\xi \beta(\xi)d\xi}{s^2 + \xi^2} \quad (5.3.17b) \]

The results given in Eq. (5.3.17) are generalized versions of the Hilbert transforms that permit the computation of the complex propagation function (or system function, if \( \alpha \) and \( \beta \) are its j-axis real and imaginary parts) for any right half-plane point, as well as for points on the j-axis, since \( s = \sigma + j\omega \) and \( \sigma \) may have finite value, as well as the value zero. When interpreting the relations in Eq. (5.3.17), we must not overlook the j-axis impulses that result when the integrands are evaluated on the j-axis [78]. The results of the integrands evaluation on the j-axis are summarized in the following relations

\[ \beta(\omega) = \frac{\omega}{\pi} \int_{-\infty}^{\infty} \frac{\alpha(\xi)}{\xi^2 - \omega^2} d\xi \quad (5.3.18) \]

\[ \alpha(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\xi \beta(\xi)}{(\xi^2 - \omega^2)} d\xi \quad (5.3.19) \]

This derivation may be considered implicitly restricted to the case of minimum-phase systems (transfer function) and not causal systems in general. However, it is a fact that any system can always be decomposed into a minimum-phase system in cascade with an all-pass system. Hence, any right half-plane zeros of a given causal system can be assigned to the all-pass component and left half-plane poles of the latter can be cancelled by zeros of the minimum-phase component. So, the attenuation of the system is fully described by that of its
minimum-phase component, and therefore behaves according to the asymptotic behavior implied by the Paley-Wiener criterion.

A question that was raised concerning the behavior of \( \beta(\xi) \) and the restriction to be applied to it if the integral in Eq. (5.3.19) is to have a finite value. The conditions for the asymptotic behavior of \( \beta(\xi) \) in Eq. (5.3.15)

\[
\beta(\xi) \to \xi^n \quad \text{for} \quad \xi \to \pm \infty \tag{5.3.20}
\]

require that \( n < 0 \). However, the conditions \( n < 1 \) for \( \alpha(\xi) \) and \( n < 0 \) for \( \beta(\xi) \) in Eqs. (5.3.17a) and (5.3.17b) are conditions assumed initially in order to eliminate the contributions from the semi-circular arc in the integrals in Eqs. (5.3.12a) and (5.3.12b), so that the Hilbert transforms can be derived \[78\. The restriction upon \( \alpha(\xi) \) agrees with the Paley-Wiener criterion, however, the one on \( \beta(\xi) \) is not physically consistent with the phase functions of minimum phase networks as reported by Guillemin \[78\. However, \( \alpha(\omega) \) is determined only within an arbitrary additive constant, and the value of this constant is sometimes infinite. It is infinite, for example, in any situation in which the phase \( \beta(\omega) \) approaches a constant asymptote. In such case \( \alpha(\omega) \) is computed from \( \beta(\omega) \) using the following expression derived by Guillemin \[78\] and also used by Papoulis \[60\]

\[
\alpha(\omega) - \alpha(0) = \frac{\omega^2}{\pi} \int_{-\infty}^{\infty} \frac{\beta(\xi)}{(\xi^2 - \omega^2)\xi} \, d\xi \tag{5.3.21}
\]

in which any additive constant in the function \( \alpha(\omega) \) drops out.

For causal functions, having the property of minimum-phase, the magnitude and the phase components are related through the Hilbert transform relations. And the phase of a physically realizable network can at most increase at an asymptotic rate that is limited by a quadratic parabola.
5.4 DECONVOLUTION FILTERS

In this section, we will show that some of the magnitude filters developed in chapter three satisfy the requirements of the Paley-Wiener criterion. Then, using Hilbert transforms, causal deconvolution filters are achieved. These filters are applied to cases of causal data deconvolution problems. The results are compared to the results obtained when non-causal filters are applied.

Smoothing deconvolution filter

The smoothing deconvolution filter derived in chapter three is simply a Butterworth lowpass type filter. It has the following expression:

\[
C(\omega) = \frac{1}{1 + \lambda \omega^{2n}}
\]  \hspace{1cm} (5.4.1)

where \( \lambda \) is a positive constant that determines the cutoff frequency of the filter, and \( n \) is a positive integer that defines the order of the filter. If we take the absolute value of the logarithm of the quantity \( C(\omega) \), we get

\[
| \ln C(\omega) | = | \ln(1 + \lambda \omega^{2n}) |
\]  \hspace{1cm} (5.4.2)

However, as \( \omega \to \infty \), then \( 1 + \lambda \omega^{2n} \to \lambda \omega^{2n} \), and hence \( \ln C(\omega) \approx 2n \ln \omega + \ln \approx 2n \ln \omega \)

Using the limiting identity that states that:

\[
\lim_{\omega \to \infty} x^{-\alpha} \ln x = 0
\]  \hspace{1cm} (5.4.3)

where \( \alpha \) is a constant and \( \Re(\alpha) > 0 \). Therefore it is obvious that:

\[
\frac{2n \ln \omega}{(1 + \omega^2)} \approx \frac{2n \ln \omega}{\omega^2} \approx 0 \quad \text{as} \quad \omega \to \infty.
\]  \hspace{1cm} (5.4.4)
We showed that the smoothing deconvolution filter defined in Eq. (5.4.1) satisfies the Paley-Wiener criterion, and therefore a minimum-phase component could be added to it to yield a minimum-phase causal deconvolution filter.

**Adaptive smoothing deconvolution filter**

Now, we consider the adaptive smoothing filter defined by Eq. (5.4.5), we will show that the regulator $|R(\omega)|$ satisfies the Paley-Wiener criteria in the same way as the smoothing filter. The regulator function is given by:

$$R(\omega) = \frac{1}{(1 + \lambda C(\omega)C^*(\omega))X(\omega)X^*(\omega)} \quad (5.4.5)$$

where $x(t)$ represents the known excitation (or system) signal, which is causal, and $c(t)$ is the $i$th derivative operator $\frac{d^i}{dt^i}$). Proceeding as we did previously, we define

$$|\ln R(\omega)| = |\ln X(\omega)X^*(\omega) - \ln (X(\omega)X^*(\omega) + \lambda C(\omega)C^*(\omega))| \quad (5.4.6)$$

Since $x(t)$ is causal, $X(\omega)X^*(\omega)$ will satisfy the Paley-Wiener criterion. The paley-Wiener integral can be expressed as follows:

$$\int_{-\infty}^{\infty} \frac{|\ln R(\omega)|}{1 + \omega^2} d\omega \leq \int_{-\infty}^{\infty} \frac{|\ln X(\omega)X^*(\omega)|}{1 + \omega^2} d\omega$$

$$+ \int_{-\infty}^{\infty} \frac{|\ln (X(\omega)X^*(\omega) + \lambda C(\omega)C^*(\omega))|}{1 + \omega^2} d\omega \quad (5.4.7)$$

otherwise,

$$\int_{-\infty}^{\infty} \frac{|\ln R(\omega)|}{1 + \omega^2} d\omega = l_1 + l_2 \quad (5.4.8)$$

where
\[
I_1 = \int_{-\infty}^{\infty} \frac{\ln |X(\omega)X^*(j\omega)|}{1 + \omega^2} \, d\omega \\
I_2 = \int_{-\infty}^{\infty} \frac{\ln |X(j\omega)X^*(j\omega) + \lambda C(j\omega)C^*(j\omega)|}{1 + \omega^2} \, d\omega
\]

(5.4.9)  
(5.4.10)

Using the fact that the signal \( x(t) \) is causal, hence it satisfies the Paley-Wiener criterion. Therefore the first integral \( I_1 \) is finite, however the second integral depends on the nature of the constraint \( C(j\omega) \).

\[
c(t) = \frac{d^{2i}}{dt^{2i}} \quad \text{i.e.,} \quad C(j\omega) = (-1)^i \times \omega^{2i}
\]

(5.4.11)

\[
C(j\omega)C^*(j\omega) = \omega^{4i}
\]

(5.4.12)

hence, for the second integral we have the following,

\[
I_2 = \int_{-\infty}^{\infty} \frac{\ln |X(j\omega)X^*(j\omega) + \lambda \omega^{4i}|}{1 + \omega^2} \, d\omega
\]

(5.4.13)

When \( \omega \to \infty, \lim_{\omega \to \infty} (X(j\omega)X^*(j\omega) + \lambda \omega^{4i}) \to \omega^4 \) with \( X(j\omega) \) being a bounded function. Then, we get

\[
\lim_{\omega \to \infty} \frac{\ln(X(j\omega)X^*(j\omega) + \lambda \omega^{4i})}{1 + \omega^2} \sim \lim_{\omega \to \infty} \frac{4\ln \omega}{1 + \omega^2} \sim 0
\]

(5.4.14)

\[
I_2 = \int_{-\infty}^{\infty} \frac{|\ln(X(j\omega)X^*(j\omega) + \lambda \omega^{4i})|}{1 + \omega^2} \, d\omega < \infty
\]

(5.4.15)

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\[ \int_{-\infty}^{\infty} \frac{\ln R(\omega)}{1 + \omega^2} d\omega = l_1 + l_2 < \infty \] (5.4.16)

The Paley-Wiener criterion is satisfied; consequently \( R(\omega) \) can be associated with a phase function to yield a causal filter.

As an illustration, we consider a simple example of a function that satisfies the Paley-Wiener criterion. A causal signal is used as an excitation to a linear system. Thus, the deconvolution filters are dependent on the nature of this signal. The signal is simply given by the following expression:

\[ x(t) = e^{-kt}u(t) \] (5.4.17)

where \( u(t) \) is zero for \( t < 0 \). The Fourier transform \( X(j\omega) \) of \( x(t) \) is simply given by

\[ X(j\omega) = \frac{1}{k + j\omega} \] (5.4.18)

and its spectrum is given by

\[ X(j\omega)X^*(j\omega) = \frac{1}{k^2 + \omega^2} \] (5.4.19)

If the constraint used is a smoothness criterion based on the continuous second derivative, then we get,

\[ C(j\omega)C^*(j\omega) = \omega^4 \] (5.4.20)

The deconvolution filter or regularizer will have the following form:
\[ R(\omega) = \frac{1}{1 + \lambda C(j\omega)C^*(j\omega)/X(j\omega)X^*(j\omega)} \]
\[ = \frac{1}{1 + \lambda \omega^4(k^2 + \omega^2)} \]
\[ = \frac{1}{1 + \lambda \omega^6 + \lambda k^2 \omega^4} \]
\[ = \frac{1}{\omega^6 + k^2 \omega^4 + 1/\lambda} \]  
(5.4.21)

\[ \ln R(\omega) = -\ln \lambda - \ln(\omega^6 + k^2 \omega^4 + 1/\lambda) \]  
(5.4.22)

As \( \omega \to \infty \), \( |\ln R(\omega)| \approx 6 \ln \omega \) and therefore
\[ \int_{-\infty}^{\infty} \frac{|\ln R(\omega)|}{1 + \omega^2} d\omega < \infty \]  
(5.4.23)

We simply see that the Paley-Wiener criterion is completely satisfied. The criterion applies to all types of magnitude deconvolution filters including the optimal compensation filter. When causal filters are applied to deconvolution problems, a delay component is introduced in the deconvolved result. However, this delay can be estimated through the deconvolution of the output and the result of the linear convolution of the input with the optimal deconvolved result. However, the delay is estimated with a certain error and could also be eliminated using time-shifting of these waveforms.

5.5 REALIZATION OF A CAUSAL FILTER

Consider a real, stable, and causal sequence \( r(n) \), whose Fourier transform is \( R(e^{j\omega}) \), representing the discrete impulse response of a deconvolution filter. Equivalently stated, this implies that the impulse response \( r(n) \) is equal to zero for \( n < 0 \). We take the complex logarithm of the Fourier transform \( R(e^{j\omega}) \) that is defined as
\[
\hat{R}(e^{j\omega}) = \text{Log}|R(e^{j\omega})| + j \arg[R(e^{j\omega})]
\] (5.5.1)

The formulation of the minimum-phase condition states that the function \( \hat{R}(e^{j\omega}) \) should not have any poles or zeros outside the unit circle. For that condition, the components \( \log |R(e^{j\omega})| \) and \( \arg[R(e^{j\omega})] \) are Hilbert transforms related through the relations (5.3.18) and (5.3.21). In this chapter, we deal with deconvolution filters that have a magnitude component only. Hence, we can only obtain the term \( \hat{R}_{\text{R}}(e^{j\omega}) = \log |R(e^{j\omega})| \) and, the phase term \( \arg[R(e^{j\omega})] \) is derived using Hilbert transforms, most often referred to as the minimum-phase condition, based on the assumption that the filter is real, stable, and causal [20]. The inverse Fourier transform of \( \hat{R}_{\text{R}}(e^{j\omega}) \) gives the even part of \( r(n) \), denoted here by \( c(n) \), since it is usually called the cepstrum of \( r(n) \) [20,48]. Hence,

\[
c(n) = \frac{\hat{r}(n) + \hat{r}(-n)}{2}
\] (5.5.2)

The phase and the log magnitude are related through Hilbert transform relations, which means they represent the Fourier transform of a minimum-phase sequence. Then the complex cepstrum (inverse Fourier transform of \( \hat{R}(e^{j\omega}) \)) is identically zero for \( n < 0 \). Then,

\[
r(n) = c(n)u_+(n)
\] (5.5.3)

where the sequence \( u_+(n) \) is defined as follows

\[
u_+(n) = \begin{cases} 
1 & n > 0 \\
2 & n = 0 \\
0 & n < 0
\end{cases}
\] (5.5.4)

To find the minimum-phase sequence \( r_{\text{min}}(n) \), the above procedure should be reversed. The Fourier transform of \( r(n) \) is calculated, then its complex exponential is taken, and finally the inverse Fourier transform of the result is computed to yield the minimum-phase sequence \( r_{\text{min}}(n) \) [20]. An infinite sequence \( r(n) \) is causal if \( r(n) = 0 \) for \( n < 0 \). However, a finite duration sequence of length \( N \) is causal if \( r(n) \) is zero in the latter half of the period 0,1,...,\( N - 1 \), i.e., for
\( n > N/2 \). So, the realization of a minimum-phase sequence can be slightly different from the above procedure, if the FFT is used to compute the Fourier transform. This is because the FFT assumes that it is dealing with a periodic sequence. If \( N \) is the number of samples, then, for a periodic sequence to be causal, the sequence should be zero for the second half of the period, i.e., for \( N/2 < n < N-1 \).

\[
\begin{align*}
    c_p(n) & \quad n = 0, n = N/2 \\
    r_{cp}(n) &= \begin{cases} 
        2c_p(n) & 1 \leq n < N/2 \\
        0 & N/2 < n < N - 1
    \end{cases}
\end{align*}
\]

(5.5.6)

where \( c_p(n) \) is the cepstrum of the periodic sequence. For our case the magnitude term is provided, its logarithm is taken and inverse Fourier transformed, and then multiplied by the function \( u_s(n) \) as shown in Eq. (5.5.6). The result is Fourier transformed, then its complex exponential is taken and finally it is inverse Fourier transformed again to yield the minimum-phase and causal sequence (filter). The procedure that is used for the realization of a causal filter is shown in Figure. (5.1).
Figure (5.1). Block diagram for the realization of a causal filter.
5.6 ILLUSTRATIVE EXAMPLE

An example of the deconvolution problem is considered in this section to illustrate the effects of the causality condition on the results of the deconvolution. We consider an example where the instrument impulse response waveform is known, and the observed response waveform is noise contaminated. However, the input waveform applied to the instrument is unknown and hence it is being sought. The objective of the problem is to obtain the waveform of the unknown input signal through the implementation of the deconvolution technique mentioned in the previous sections. The noise-free waveforms used in the following example are of an impulsive nature, and their corresponding step responses obtained through discrete integration are shown in Figure. (5.2). The step response $x_d(t)$ of the unknown input waveform has a 10-90% transition duration of 46.97 ps and the step response $y_d(t)$ of the observed waveform has a 10-90% transition duration of 136.99 ps, both of them are both shown in Figure. (5.2). In the same figure, the step response $h_d(t)$ of the instrument (oscilloscope) model is also shown and has a transition duration of 127.20 ps.

To proceed with the example, the observed waveform, $y'_o(t)$, is obtained and a computer generated random sequence (of zero mean) is added to it to yield a noisy signal having a signal to noise ratio of 40 dB. We denote this noisy impulsive waveform as $y'(t)$, Figure. (5.3). The voltage signal-to-noise ratio is equal to the maximum value of the waveform $y'_o(t)$ divided by the standard deviation of the instrument noise without signal expressed in decibels. In Figure. (5.3), the noisy observed response $y'(t)$ is shown to show the level of noise that it contains. The regularization deconvolution technique was used for the determination of the unknown input excitation $x'_o(t)$. The non-causal technique was applied first using the magnitude filter defined by the relation in Eq. (2.3.8). Then the filter was made causal through the implementation of the discrete Hilbert transform.

The step responses of the deconvolved result for the case of the non-causal filter are shown in Figure. (5.4) for different filter parameter ($\lambda$) values. It is seen that the waveforms contain undue oscillations before the rise of the step response and the shape of these oscil-
lations varies with the value of \( \lambda \). However, those oscillations are eliminated when a causal filter is applied in the technique, as shown in Figure. (5.5). Although a noticeable delay is introduced in the resulting step response, the actual delays are not shown in Figure. (5.5) in order to clearly show the oscillatory nature of the step-like responses. A comparison between the step response \( x_0(t) \) and the step response of the optimally deconvolved result is displayed in Figure. (5.6). In Figure. (5.7), we show the phase distortion that occurs when a causal filter is applied for deconvolution. A linear phase component is introduced by the minimum phase condition. Thus, the phase associated with the filter is dependent on the filter' magnitude alone. Its shape is function of the magnitude only, therefore the distortion that it may generate cannot be controlled. However, we see from the figure that no major distortion occurs in the phase. In the frequencies of interest, the phase is left unchanged. At higher frequencies, the distortion becomes apparent when the filtering is stronger.
Figure (5.2). Step responses of the instrument, input, and output waveforms.
Figure (5.3). The observed output waveform with a 40 dB signal-to-noise ratio.
Figure (5.4). Step responses of the deconvolved results obtained by applying a non-causal filter using different parameter values.
Figure (5.5). Step responses of the deconvolved results obtained by applying a causal filter using different parameter values.
Figure (5.6). Comparison between the step responses of the computed and the optimally deconvolved result obtained using a causal filter.
Figure (5.7). Phase distortion when using the minimum phase condition.
5.7 EXPERIMENTAL EXAMPLE

This section presents an experimental case of TDR measurements. A lowpass filter device is, which has a cutoff frequency of 12.5 GHz, is considered. Experimental measurements are performed on the device. Then, the regularization deconvolution technique is used to compute the device’s step responses when causal and non-causal deconvolution filters are applied. The experimental waveforms are acquired using the HP-54120A/54121A digitizing oscilloscope. The pulse source used was the 50 ohm TDR step-like pulse in channel 1 of the HP-54121A sampling head. The insertion reference waveform was acquired by connecting channel 1 to channel 4 using a cascade of two measurement-grade flexible coaxial cables. The insertion response waveform was acquired by inserting the device under test between the two coaxial cables. Both the reference and the response waveforms are shown in Figure (5.8). Next the reference and insertion waveforms were converted to duration-limited waveforms (appendix B), and their respective Fourier transforms (FFT’s) obtained. Upon performing the complex division of the transformed converted insertion response waveform by that of the reference waveform, the results shown in Figure (5.8). The results are the same for the magnitude component for causal and non-causal filters applied. Then, taking the IFFT’s of the data whose magnitude is given in Figure (5.9) for both filter cases yields the impulse responses of the device under test, Figures (5.10) and (5.11) respectively. Upon performing the discrete integration of the waveforms in Figures (5.10) and (5.11), the step responses of the device under test are obtained, non-causal case Figure (5.12), causal case Figure (5.13).

Notice that the non-causal filter produces a negative-going precursor which becomes larger as the filtering is made stronger, i.e., with increasing values of $\lambda$. The negative-going precursor is an artifact of the non-causal deconvolution filter. Also, negative-going precursors are seen in the non-causal impulse responses, Figure (5.10). In Figure (5.11), a limited frequency interval of the phase waveforms obtained when a causal filter is applied for deconvolution. The interval covers the frequency bandwidth of the excitation signal and extends over the noisy region where the filter’s attenuation is strong. The distortion of the phase is ap-
parent for frequencies higher than 80 GHz. The distortion is due to the introduction of a linear phase component by the minimum phase condition. However, we see from the figure that no major distortion occurs in the phase at the frequencies within the band of interest, the phase is left unchanged.
Figure (5.8). Reference $x(t)$ and response $y(t)$ of the 12.5 GHz lowpass filter.
Figure (5.9). Transfer function magnitude (in dB's) obtained applying causal and non-causal filters using different filtering levels.
Figure (5.10). Impulse responses of the deconvolved results obtained by applying a non-causal filter using different filtering levels.
Figure (5.11). Impulse responses of the deconvolved results obtained by applying a causal filter using different filtering levels.
Figure (5.12). Phase distortion when using the minimum phase condition.
5.8 CONCLUSION

In this chapter, we studied the effects of the use of non-causal deconvolution filters on the deconvolved result. We showed that when non-causal deconvolution filters are applied, they produce negative-going precursor which becomes larger as the filtering is made stronger, i.e., with increasing the filters parameters. We also showed that the magnitude filters derived in previous chapters satisfy the requirements of the Paley-Wiener criterion. Hence, using the property of minimum-phase which relates the attenuation function to the phase through a pair of Hilbert transform relations, a phase (minimum) component was computed from the magnitude (attenuation) component to yield a causal deconvolution filter. The whole procedure was achieved by implementing the discrete Hilbert transform as shown in section. The results of the study were summarized in two examples. One example with simulated data, and the other a typical TDR example of deconvolution. In both cases, it was shown that a negative-going precursor occurs when non-causal filters are used. However, the precursor is eliminated when a similar but causal filter is applied. We showed that phase distortion occurs when a causal filter is applied for deconvolution. A linear phase component is introduced by the minimum phase condition. Thus, the phase associated with the filter is dependent on the filter’s magnitude alone. Its shape is function of the magnitude only, therefore the distortion that it may generate cannot be controlled. However, we see from the figure that no major distortion occurs in the phase. In the frequencies of interest, the phase is left unchanged. At higher frequencies, the distortion becomes apparent when the filtering is stronger.
CHAPTER VI

HOMOMORPHIC DECONVOLUTION APPLIED TO TDR SIGNALS

6.1 INTRODUCTION

Homomorphic processing is based on the principle of generalized superposition, [20]. Its importance lies in the separation of signals that are combined through multiplication or convolution, where only limited information is known about both of them [49]. Conventional deconvolution techniques are helpless in such situations. In the past, linear "filtering" techniques, which are generally simple, were commonly used. However, they cannot be used if the signal is not combined with the others by algebraic addition, as in the case of signals combined by multiplication or convolution. Homomorphic processing transforms the convolution and multiplication operations into algebraic additions. Thus, linear filtering techniques can then be applied to separate the combined signals. The homomorphic deconvolution is a technique known for decomposing a composite signal of unknown multiple wavelets overlapping in time [51,55].

The complex cepstrum filtering presupposes that the measured signal can be mapped into 'quefrcency' (frequency paraphrased) domain [48], wherein convolved functions occupy different regions and can thus be separated from each other. In many applications of signal
processing, the problem of extracting signals combined through the convolution (or multiplication) operation arises. Such problems have to be dealt with according to the specific needs of the problem. The importance of the homomorphic deconvolution lies in areas of convolution problems where only the output signal is fully known [58]. The use of the homomorphic deconvolution (cepstrum) is shown to simplify cable discontinuity identification as compared to time domain reflectometry.

We present a simple case of a time domain reflectometry problem where the homomorphic deconvolution can be applied successfully to recover the excitation and the reflection coefficient of a network of transmission lines of different impedances (discontinuities).

6.2 THE HOMOMORPHIC DECONVOLUTION

The deconvolution is the process of separating convolved signals from each other. The homomorphic deconvolution is the use of a suitable homomorphic system (transformation) to separate (deconvolve) convolved signals. The input and output operations of a homomorphic deconvolution system are convolution operations. In building our homomorphic system, two useful mathematical operations are used:

- The z-transform (discrete Fourier transform) which converts the convolution into a product, i.e., the z-transform is a homomorphic transform. The convolution and the multiplication are its input and output operations, respectively.

- The complex logarithm converts a product into a sum, i.e., the log is a homomorphic transform. The multiplication and addition are its input and output operations, respectively.

The simplest form for the forward transformation can then be the cascade combination of the z-transform and the complex logarithm, as shown in Figure. (6.1), but if it is preferable in the
separation to use sequences rather than their z-transforms. The z-transform and its inverse are linear operations. Thus, when both transforms are applied in the homomorphic deconvolution process we get:

\[
Y(z) = z[y(k)]
\]
\[= z[x(k)h(k)]
\]
\[= z[x(k)].z[h(k)]
\]
\[= X(z).H(z) \quad (6.2.1)
\]

\[
\hat{Y}(z) = \log[Y(z)]
\]
\[= \log[X(z).H(z)]
\]
\[= \log[X(z)] + \log[H(z)]
\]
\[= \hat{x}(z) + \hat{h}(z) \quad (6.2.2)
\]

\[
\hat{y}(k) = z^{-1}[\hat{Y}(z)]
\]
\[= z^{-1}[\hat{x}(z) + \hat{h}(z)]
\]
\[= z^{-1}[\hat{x}(z)] + z^{-1}[\hat{h}(z)]
\]
\[= \hat{x}(\hat{k}) + \hat{h}(\hat{k}) \quad (6.2.3)
\]

The reason for denoting the sequence variable in Eq. (6.2.3) by \(\hat{k}\) is to distinguish the new domain from the k domain. The new \(\hat{k}\) domain is a non-physical domain (quefrency), it is a pure mathematical one called sometimes the hypothetical k domain. For example, if the \(k\) domain is the time domain, then \(\hat{k}\) domain is simply referred to as the hypothetical time domain. The signal \(\hat{y}(\hat{k})\) in the hypothetical time domain is called the cepstrum.

The separation process will be achieved using filters in the \(\hat{k}\) domain to split \(\hat{y}(\hat{k})\) into its components \(\hat{x}(\hat{k})\) and \(\hat{h}(\hat{k})\). The filtration problem will be greatly simplified if \(\hat{y}(\hat{k})\) is real. Assuming that both \(y(k)\) and \(\hat{y}(\hat{k})\) are real stable sequences is not in fact a restriction since these are the properties of the majority of signals encountered in time domain measurements deconvolution problems. In addition, simple mathematical operations can be used to secure
stability. This assumption implies that the regions of convergence of both \( Y(z) \) and \( \hat{Y}(z) \) must include the unit circle. On the unit circle \( z = e^{i\omega} \), \( \hat{Y}(z) \) can be expressed as \( \hat{Y}(e^{i\omega}) \),

\[
\hat{Y}(e^{j\omega}) = \hat{Y}_R(e^{j\omega}) + j\hat{Y}_I(e^{j\omega})
\]  

(6.2.4)

where the subscripts \( R \) and \( I \) refer to real and imaginary components of a complex quantity. Since \( \hat{y}(k) \) is assumed to be real, \( \hat{Y}_R(e^{i\omega}) \) and \( \hat{Y}_I(e^{i\omega}) \) must respectively be even and odd periodic functions of \( \omega \) with period \( 2\pi \). Both functions must be analytic (continuous functions of \( \omega \)) to meet the stability requirements. Since \( \hat{Y} = \log Y \) by virtue of equation, the analyticity of the complex logarithmic function is to be studied.

The complex logarithmic function \( \log \) is defined as

\[
\log Y = \log |Y| + j \arg[Y] \\
-\pi < \arg[Y] \leq +\pi
\]

(6.2.5)

Using Eq. (6.2.4), together with the above definition, the real and imaginary components of \( \hat{Y}(e^{i\omega}) \) are

\[
\hat{Y}_R(e^{j\omega}) = \log |Y(e^{j\omega})|
\]

(6.2.6)

and

\[
\hat{Y}_I(e^{j\omega}) = \arg[Y(e^{j\omega})]
\]

(6.2.7)

The real part, defined in Eq. (6.2.6), is analytic as long as the z-transform \( Y(z) \) does not have any zeros on the unit circle. However, the exponential weighting method can be used to move any zeros lying on the unit circle to make Eq. (6.2.6) analytic. The imaginary part is not a continuous function according to Eq. (6.2.7); this raises two undesired effects:

1. Eq. (6.2.2) will not always be true. Using Eq. (6.2.5), Eq. (6.2.2) becomes

\[
\log |Y| = \log |X| + \log |H|
\]

(6.2.8)
\[ \arg[Y] = \arg[X] + \arg[H] \]  \hspace{1cm} (6.2.9)

where the three arguments in Eq. (6.2.9) are all defined in the region \(-\pi\) to \(+\pi\). The relation of Eq. (6.2.8) is always true, but Eq. (5.2.9) is not because of the restrictions on the arguments.

- \( \hat{Y}(z) \) will not always be a valid z-transform because of the discontinuity of the imaginary component of \( \hat{Y} = \log Y \), thus invalidating Eq. (6.2.3).

The solution to the complex logarithm problem is simple; it is to replace the discontinuous complex logarithm by the continuous complex logarithm \( \text{clog} \) defined as

\[ \text{clog}Y = \log |Y| + j \text{carg}[Y] \]  \hspace{1cm} (6.2.10)

\[ \text{carg}[Y] = \arg[Y] + 2m\pi \]  \hspace{1cm} (6.2.11)

where \( \text{carg} \) is the continuous argument function. The new function is formed by reconstructing the argument function to be a continuous one by adding (or subtracting) integer multiples of \( 2\pi \). Different algorithms for phase unwrapping were developed and used successfully \([80,81]\).

### 6.3 REALIZATION OF THE HOMOMORPHIC DECONVOLUTION TRANSFORM

In most physical situations, the observed signals have to be dealt with numerically rather than analytically. The digital computer is often used to perform various kinds of numerical processing on such signals. For the separation of physical convolution signals, a digital computer version of the homomorphic deconvolution transform has to be derived. In a preceding section it was mentioned that the z-transforms have to include the unit circle in their
regions of convergence. Thus, the inverse z-transforms can be calculated using the unit circle as the integration contour. So, the evaluation of the z-transforms is only needed on the unit circle; i.e., it is employed to calculate the signals Fourier transforms rather than the z-transforms. Since we deal with discrete sequences, then the discrete Fourier transform \cite{83} must be implemented to be used instead of the z-transform.

\[ y(k) = Y(e^{j\omega}) \bigg|_{\omega = 2\pi k/N} = \sum_{n=0}^{N-1} y(n)e^{-j(2\pi k/N)n} \quad (6.3.1) \]

\[ \hat{y}(k) = clog[Y(k)] = \log |Y(k)| + jcarg[Y(k)] \quad (6.3.2) \]

\[ carg[Y(k)] = \arg[Y(k)] + 2m\pi \quad (6.3.3) \]

Here m may be zero or a positive or a negative integer, such that \( carg[Y(k)] \) is continuous.

\[ \hat{y}(\hat{k}) = \frac{1}{N} \sum_{n=0}^{N-1} \hat{y}(n)e^{-j(2\pi n/N)\hat{k}} \quad (6.3.4) \]

\[ \hat{\hat{y}}(\hat{k}) = L[\hat{y}(\hat{k})] \quad (6.3.5) \]

\[ \hat{\hat{y}}(n) = \frac{1}{N} \sum_{\hat{k}=0}^{N-1} \hat{\hat{y}}(\hat{k})e^{-j(2\pi n/N)\hat{k}} \quad (6.3.6) \]

\[ H(n) = \exp[\hat{\hat{y}}(n)] \quad (6.3.7) \]

\[ \text{CHAPTER VI} \]
\[ v(k) = \frac{1}{N} \sum_{n=0}^{N-1} V(n) e^{-j(2\pi n/N)k} \] (6.3.8)

It has to be noted that the input sequence \( y(k) \) is the convolution of two sequences \( x(k) \) and \( h(k) \) both having finite number of samples \( N \). Consequently, the cepstrum \( \hat{y}(\hat{k}) \) is the sum of two contributions \( \hat{x}(\hat{k}) \) and \( \hat{h}(\hat{k}) \) due to the terms \( x(k) \) and \( h(k) \), respectively. The output of the linear filter \( \hat{\nu}(\hat{k}) \) should be equal to either \( \hat{x}(\hat{k}) \) or \( \hat{h}(\hat{k}) \) to yield a system output \( v(k) \) which correspondingly equals \( x(k) \) or \( h(k) \). It is to be noted that all the sequences involved in the transform equations are each of a finite number of samples, and they satisfy the relation

\[ h(m) = h(m + rN), \quad r \text{ integer} \] (6.3.9)

i.e., all the computations need to be done only for sequences ranging between 0 and \( N-1 \).

### 6.4 LINEAR PHASE ELIMINATION

Let the sequence \( y(k) \), the input to a homomorphic deconvolution transform, be the result of delaying another sequence \( s(k) \) by \( k_0 \) samples; i.e.,

\[ y(k) = s(k - k_0) \] (6.4.1)

Following the transform Eqs. (6.3.1), (6.3.2), (6.3.3), and (6.3.4), it can be written that,

\[ Y(k) = S(k) e^{-j(2\pi k/N)k_0} \] (6.4.2)

\[ \hat{Y}(k) = \hat{S}(k) - j(2\pi k/N)k_0 \] (6.4.3)

and

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\[
\hat{y}(\hat{k}) = \hat{s}(\hat{k}) + (-1)^{\hat{k}} k_0/k
\]  

(6.4.4)

The second term in Eq. (6.4.4) having the form of a decaying oscillation [50]. It is called the linear phase term because it is due to the linear phase component in the Fourier transforms. Assuming that it is desired to perform linear filtering on \(\hat{s}(\hat{k})\), the linear phase term must be removed. The linear phase distorts the information contained in \(\hat{s}(\hat{k})\), therefore, it must be removed. The existence of such linear phase term also contradicts the requirements of having odd periodic continuous phase (argument) function of \(\omega\). The linear phase term must be eliminated, and the best place to perform the elimination is right after taking the continuous log (clog) of the discrete Fourier transform (DFT) of the signal.

### 6.5 Minimum Phase Condition

Suppose that \(H(z)\) is expressed in polar form as

\[
H(z) = |H(z)| e^{j \arg[H(z)]}
\]  

(5.5.1)

Then consider the complex logarithm of \(H(z)\) defined as

\[
\hat{H}(z) = \log |H(z)| + j \arg[H(z)]
\]  

(6.5.2)

If \(\hat{H}(z)\) is the z-transform of a sequence \(\hat{h}(\hat{k})\), then \(\log |H(z)|\) and \(\arg[H(z)]\) will be Hilbert transforms of each other if and only if \(\hat{h}(\hat{k})\) is a real, causal, and stable sequence. However, it must be noted that the logarithm of zero diverges. In addition, the definition of \(\arg[H(z)]\) is ambiguous since any multiple of \(2\pi\) can be added to the phase without affecting the value of \(H(z)\).
The argument of $H(z)$ is not generally unique, the ambiguity is resolved by the fact that the analyticity of $\hat{H}(z)$ implies that its real and imaginary parts must be continuous functions of $z$. Consequently, if $\hat{H}(z)$ is to be analytic, we must define $\arg[H(z)]$ to be a continuous function.

The requirement that $\log |H(z)|$ and $\arg[H(z)]$ be Hilbert transforms is referred to as the minimum phase condition. It corresponds to the requirement that the sequence $\hat{h}(k)$ is causal. But, it should be emphasized that a system (or sequence) can be causal but non-minimum phase. However, all stable, minimum phase systems (or sequences) are causal.

Let us consider a real, stable sequence $\hat{h}(k)$ whose z-transform is $\hat{H}(z)$. If $\hat{h}(k)$ is causal, then $\hat{H}(z)$ and consequently $H(z)$ can be recovered from $\hat{H}(e^{j\omega}) = \log |H(e^{j\omega})| \quad \text{or} \quad \hat{H}(e^{j\omega}) = \arg[H(e^{j\omega})]$. If $\hat{h}(k)$ is real, stable and causal, then

$$\log |H(e^{j\omega})| = \hat{h}(0) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \arg H(e^{j\omega}) \cot\left(\frac{\theta - \omega}{2}\right) d\theta \quad (6.5.3)$$

$$\arg[H(e^{j\omega})] = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |H(e^{j\omega})| \cot\left(\frac{\theta - \omega}{2}\right) d\theta \quad (6.5.4)$$

These relations are usually referred to as the minimum phase condition. In order that $\hat{h}(n)$ be a causal sequence, $\hat{H}(z)$ must be analytic in the region $|z| > R$, where $R < 1$; i.e., $\hat{H}(z)$ should not have any singularities outside the unit circle.

Among the properties of the cepstrum, we find that for some particular signals, the cepstrum can be computed analytically. Let us begin by considering a sequence that is a sum of delayed and scaled replicas of a sequence $x(n)$; i.e.,

$$y(n) = x(n) + \sum_{i=1}^{N} \alpha_i x(n - n_i) \quad (6.5.5)$$

where $0 < n_1 < n_2 < \ldots < n_N$. The above signal can be expressed as a convolution of:
\[ y(n) = x(n)h(n) \quad (6.5.6) \]

where

\[ h(n) = \sum_{i=1}^{N} a_i \delta(n - n_i) \quad (6.5.7) \]

If consider the case of a simple reflection; i.e.,

\[ h(n) = \delta(n) + a_1 \delta(n - n_1) \quad (6.5.8) \]

The Fourier transform of \( y(n) \) is given by

\[ Y(e^{j\omega}) = X(e^{j\omega}) \left[ 1 + a_1 e^{-j\omega n_1} \right] \quad (6.5.9) \]

Therefore, if the complex logarithm is taken, the contribution from the impulse train is

\[ \hat{h}(e^{j\omega}) = \log \left[ 1 + a_1 e^{-j\omega n_1} \right] \quad (6.5.10) \]

Thus, it is obvious that \( \hat{h}(e^{j\omega}) \) is periodic with period \( 2\pi/n_1 \), and therefore, at the integer multiples of \( n_1 \), only \( h(n) \) is nonzero. If \( |a_1| < 1 \), it can be shown that

\[ \hat{h}(n) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{a_1^k}{k} \delta(n - kn_1) \quad (6.5.11) \]

Thus, if \( \hat{X}(e^{j\omega}) \) is slowly varying relative to the variations of \( \hat{H}(e^{j\omega}) \), it is reasonable to separate these two components with a linear frequency invariant filter.

Now, let us consider the complete sequence of reflections that is characterized by the travel times \( k_1, k_2, k_3 \), etc, and by the reflection amplitudes \( h_1, h_2, h_3 \), etc. If every reflection
was received as a perfect impulse $\delta(\tau) = \delta(k - \tau)$, the received reflections train would be $h(t)$ given by:

$$h(k) = \sum_{j=1}^{\infty} h(k_j) \delta(k - k_j)$$  \hspace{1cm} (6.5.12)

However, instead of the impulse $\delta(\tau)$, a wavelet $x(\tau)$ for every reflection $h(k_j)$ is received. The received reflected waveform is of the form,

$$y(k) = \sum_{j=1}^{\infty} h(k_j) x(k - k_j)$$  \hspace{1cm} (6.5.13)

In the time domain reflectometry, the interested is more emphasized in the location of the reflections and their sequence $h(k)$ than in the wavelet $x(k)$. Ideally, we should receive the sequence uncontaminated by the wavelet $x(k)$. But, it is unlikely that the reflections are all separated sufficiently that the reflected waveforms do not overlap. It is not always possible to develop a filter that would filter out the wavelet $x(k)$. If it were possible, we would have a filter $f(k)$ whose output response is $\delta(k)$:

$$f(k) * x(k) = \delta(k)$$  \hspace{1cm} (6.5.14)

and therefore

$$f(k) * y(k) = f(k) * x(k) * h(k)$$
$$= \delta(k) * h(k)$$
$$= h(k)$$  \hspace{1cm} (6.5.15)
6.6 APPLICATION OF THE HOMOMORPHIC DECONVOLUTION TO TDR

In time domain reflectometry, the reflected waveform by a line with several discontinuities is represented as the convolution of the reflection coefficient of the line and the input excitation of the line source. The reflection coefficient is generally a train of spikes (delta functions) when the discontinuities are resistance. However, this is not the case when the discontinuities are of capacitive nature. In general, conventional frequency domain deconvolution techniques fail to provide good estimates when dealing with wideband noisy waveforms. Since it has been shown that homomorphic systems are useful in separating signals which are combined through convolution, homomorphic filtering can then be applied to recover either the input excitation or the impulse response (reflection coefficient) of the network. The reflection coefficient \( h(t) \) is given by the following equation,

\[
h(t) = \delta(t) + a_0 \delta(t-t_0) + a_1 \delta(t-t_1) + \ldots \ldots \tag{6.6.4}
\]

The inverse filtering technique fails to provide good estimates of \( h(t) \). The amplitude of the spikes given by \( a_i \) cannot be recovered correctly. The parameters \( a_i (i = 0, 1, \ldots) \) represent quantities that are function of the partial reflection coefficients at the different discontinuities of the line, and the times \( t_i (i = 0, 1, \ldots) \) are function of the length and the characteristics of the lines between discontinuities. The case of the impulse response in Eq. (6.6.4) represented by a train of ideal delta functions has been studied in details in [50,85]. In this work, \( h(t) \) is substituted by \( h_d(t) \) which has time variant constants given by Eq. (6.6.5),

\[
a_0(t) = \beta_0 e^{-t/\tau} \tag{6.6.5}
\]

The case of changing the values of the parameter \( \tau \) which defines the rise time of the pulse corresponding to the function \( e^{-t/\tau} \). It is obvious that as \( \tau \) tends towards zero, the function \( e^{-t/\tau} \) tends to ideal delta function. In this work, a threshold value of \( \tau \) that allows good estimate will be studied. The complex cepstrum \( y^*(t) \) contains the additive contributions of the
input excitation \( x'(t) \) and the impulse response \( h_y(t) \). These two components are separated by application of either a lowpass or a highpass filter. The success of the method is dependent on the partial knowledge of one of the convolution signals and the magnitude of the signal-to-noise ratio (SNR). The signal-to-noise ratio must be quite large to be able to achieve good signal recovery. Some of the examples in time domain convolution are: a) the reflected signal on a transmission line which is the convolution of the incident signal signal and the impulse response of the reflecting discontinuity and b) the observed signal using the oscilloscope which is the convolution of the measured signal at the input gate and the oscilloscope's impulse response. In order to separate the components of a convolution, a deconvolution process (separation) process is needed. Several deconvolution methods were developed for various convolution problems; they all required the knowledge of either one of the convolution components, and in some cases, information about the signals being analyzed. The information concerning the signals being analyzed is provided depending on the area of study, but for some cases little information is available. An example of TDR deconvolution problem is considered to illustrate the application of the homomorphic deconvolution. The method does not require the knowledge of any of the convolution components and suits the class of problems where the frequency domain forms of the convolution components have substantially different rates of variation.

6.6 COMPUTER SIMULATION AND RESULTS

An example of a test signal reflected of capative and resistive discontinuities in a TDR measurement setup is considered. For the case of resistive discontinuities, results of the application of the homomorphic deconvolution for noisefree data are reported by Riad et al [50,51]. In this work, we follow up on the same problem using capacitances in addition to resistances. The network of lossless transmission lines being considered is shown in Figure (6.2). The following numerical values are used for the different parameters:
\[ R_0 = 50\Omega \quad R_1 = 2R_0 \quad R_2 = R_0/7 \]

\[ \tau_1 = 1.5\tau \quad \tau_2 = \tau \quad \text{and} \quad \tau = 84.67\,\mu\text{s} \]

Two different sets of values are used for the capacitances. For the first case: \( C_1 = 0.05\,\mu\text{F} \) and \( C_2 = 0.5\,\mu\text{F} \) yielding a frequency bandwidth of about 60 GHz. For the second case: \( C_1 = 0.25\,\mu\text{F} \) and \( C_2 = 2.5\,\mu\text{F} \) yielding a frequency bandwidth of about 20 GHz. The TDR waveforms used in this simulation, \( x(t) \), and \( h(t) = \rho(t) \) for both cases are shown in Figures (6.3). The waveform of the reflection coefficient for the second case was time shifted in order to show the difference between the two reflection coefficients. Random zero-mean gaussian noise was added to \( y(t) \) to produce a noisy case. The study was achieved for three cases of signal-to-noise ratios, SNR, of 65, 45, and 35 dB, however, it seemed that the 60 dB case was enough to summarize the noise effects. The Homomorphic deconvolution technique is then applied to the noisy \( y(t) \). A typical cepstrum is shown in Figure (6.4) for the first case, for both noise-free and noisy data with a 60 dB SNR. The separation gates are also shown in Figure (6.4). Similar waveforms corresponding to the second case are shown in Figure (6.5). The separated (recovered) \( x(t) \) and \( h(t) \) using gates 1 and 2, respectively, are shown in Figures (6.5), (6.7). We observe that the cepstrum operations amplify the noise contained in the signal used. This is shown in Figure (6.4) for the noisy (60 dB SNR) first case. However, the effects of the impulses bandwidth combined with noise lead to noisy and inaccurate cepstrum results as shown in Figure (6.5).

In Figure (6.6), we see that when the bandwidth of the reflection coefficient impulses is large, the recovery of the excitation is possible without any major distortion. However, distortion occurs when noise is added to the output response. We also see that the impulse train is more affected by the noise addition than the excitation waveform. For SNR's less than 55 dB, it becomes impossible to recover the excitation accurately. While it was observed that the homomorphic deconvolution technique can be successfully applied to separate the convolution components \( x(t) \) and \( h(t) \) from low noise TDR waveforms, when \( h(t) \) is a train of scaled and delayed delta (Dirac) functions. For cases of 65 dB and 45 dB SNR, \( x(t) \) is adequately

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recovered. However, recovery of \( h(t) \) starts to become difficult when the SNR drops below 35 dB \( x(t) \) is seen to be very noisy and loses amplitude for SNR of 35 dB or less. In the TDR example that we used, we found that for discontinuities of capacitive nature, the separation (recovery) is limited by the bandwidth of the impedances forming such discontinuities.
Figure 6.1. Homomorphic deconvolution block diagram.
Figure 6.2. Transmission line with two capacitive discontinuities.
Figure 6.3. TDR reference $x(t)$ and reflection coefficient $h(t)$ for the two cases of capacitances.
Figure 6.4. Cepstrum for case #1 for noise-free and noisy data, in addition to the typical separation gates.
Figure 6.5. Cepstrum for case #1 for noise-free and noisy data.
Figure 6.6. Recovered $x(t)$ for both cases using noise-free and noisy data.
Figure 6.7. Recovered h(t) for both cases using noise-free and noisy data.
6.7 CONCLUSIONS

In this chapter, we studied the homomorphic deconvolution technique, and showed that it can be applied to the separation of signals combined through convolution. The technique is applicable to the special case where a train of delta pulses represents one of the combined signals. The accuracy of the determination of the delay times between successive delta pulses is higher than that of any other technique. However, the Homomorphic deconvolution technique is fairly sensitive to the noise content in the TDR waveform. We presented an example of reflections occurring at the discontinuities of a capacitive network of transmission lines. The capacitive discontinuities consist of RC circuits. Therefore, from the time constants of these circuits we can derive the network bandwidth. And in this work, we studied the effects of the bandwidth on the success of the cepstrum technique. We found that as the bandwidth decreases, the cepstrum result deteriorates, and the separation becomes impossible. It was also found that for lower impulse bandwidth, the noise tolerance is reduced to no less than 55 dB SNR. For higher SNR's and lower impulse bandwidth, the use of the cepstrum becomes inadequate. It was noticed that when the homomorphic deconvolution (cepstrum) was used for uniquely resistive discontinuities, the noise tolerance was around 40 SNR. For SNR less than 40 dB the cepstrum becomes very noisy and the separation of the convolution components is not practical. Noise reduction techniques are needed to improve the SNR, thus enabling the use of the Homomorphic deconvolution technique.
CHAPTER VII

SUMMARY AND CONCLUSIONS

In this work, we attempted to derive some frequency domain filters for deconvolution of transient signals in the presence of noise. The techniques that are derived in chapter two are based on the minimization of the mean-squared error in the estimate, as developed in Wiener filtering theory. However, Wiener theory requires the maximum a-priori information concerning the processes (signals and noise) in terms of their power spectra. Since it is not always possible to have such information, the techniques are modified to involve some constraints to compensate for the absence of the spectral information. Different constraints may be applied and different filters result depending on the practical applications of such constraints. The constraint chosen for our case is to reduce the oscillations in the second derivative of the estimate. The techniques that we derived are suitable for the type of deconvolution problems encountered in time domain reflectometry (TDR). We showed that the first technique uses a lowpass type filter as a smoothing filter that greatly reduces the noise level in the deconvolved result. The filter emphasizes the low frequency components and attenuates the high frequency components. The second method is derived using the same type of constraint, but a different formulation of the deconvolution problem. The result is an adaptive smoothing filter that is dependent on the nature of the excitation being applied to the system. Both of these filters attenuate the high frequency terms because they are frequency dependent, the higher
the frequency at which the filter is operating the higher the attenuation is. Thus, the attenuation rate of the noise that contaminates the high frequency components is considerably fast.

It is shown that the techniques could easily be modified to account for the spectral properties of the signals involved in the convolution process. However, the statistical methods used second-order statistics quantities such as the second moment. The importance of these quantities is linked with the fact that most optimal filter design criteria require knowledge only of second-order statistics and do not require more detailed knowledge, such as of probability densities. It is of primary importance, then, to be able to extract such quantities from the actual measured data. If the second-order statistics quantities were known, the filters parameter dependence could be eliminated. However, the knowledge of such quantities is not always possible.

The derived filters are parameter dependent. A design parameter is included in both filters to control the balance between smoothness and accuracy of the estimate. The parameter has the advantage of allowing optimization of the filter performance. The performance of the filter depends on the parameter value used, hence a parameter optimization criterion must be included to determine the optimum parameter value. However, in deriving an optimization criterion, great importance must be given to its adequacy in the determination of the appropriate parameter value as well as its time efficiency.

A parameter optimization method that fulfills those two requirements is developed in chapter three. The method is fully implemented in the frequency domain in which the filtering techniques are used. It is very important in seeking out optimum parameter values for these design parameter dependent filters. The technique does not single out just one value or accurately find an optimum parameter value, however, it does provide an accurate range of values that could be considered as optimum. In addition, it clearly indicates the degree of noise reduction and the associated loss of information in the various frequency regions.

In addition to the computation time that the technique saves, it is found that it provides a reliable procedure to select an optimum parameter value. In this technique, the choice of the frequency intervals is based on the noise content and signal information in each interval.
The number of intervals that are needed is arbitrary, in most cases, typically baseband signals, adequate results can be achieved using only two intervals.

In chapter five, we studied the effects of the non-causality of the filters on the deconvolved result. The techniques that are developed are derived with a magnitude component only, i.e., non-causal. The limited derivation is due to the fact that we are usually interested in reducing only the noise level from the magnitude point of view. However, if we consider time domain measurements as an example, physical pulses and transients are causal functions of time, i.e., their values are zero before \( t = 0 \), the time at which they begin. Their measured waveform data are also causal. When deconvolution processing is applied to remove instrumentation errors, and/or suppress the effects of noise, non-causal deconvolution methods may introduce unacceptable errors. The conventional deconvolution methods are modified to ensure that causality is maintained in the deconvolution result. We showed that when non-causal deconvolution filters are applied, they produce negative-going precursor which becomes larger as the filtering is made stronger, i.e., with increasing the filters parameters. The precursor is due to the oscillations of the non-causal filter. We also showed that the magnitude filters derived in previous chapters satisfy the requirements of the Paley-Wiener criterion. Hence, using the property of minimum-phase which relates the attenuation function to the phase through the Hilbert transform relations, a phase (minimum) component was computed from the magnitude (attenuation) component to yield a causal deconvolution filter. It is shown that the negative-going precursor that occurs when non-causal filters are used, is eliminated when similar but causal filter are applied.

We showed that phase distortion occurs when a causal filter is applied for deconvolution. A phase component is introduced by the minimum phase condition. Thus, the phase associated with the filter is dependent on the filter's magnitude alone. Its shape is a function of the magnitude only, therefore the distortion that it may generate cannot be controlled. However, we see from the figure that no major distortion occurs in the phase. At the frequencies of interest, the phase is left unchanged. At higher frequencies, the distortion becomes apparent when the filtering is stronger.
In chapter six, it is shown that the impulse response of an unknown system is recovered from time domain reflectometry data by implementing a method based on the homomorphic deconvolution technique. In time domain reflectometry, the reflected waveform by a line with several discontinuities is represented as the convolution of the reflection coefficient of the line and the input excitation of the line source. The reflection coefficient is generally a train of spikes (delta functions) when the discontinuities are resistive. However, this is not the case when the discontinuities are of capacitive in nature. In this work, an attempt to show that the conventional frequency domain deconvolution techniques fail to provide good estimates when the waveform contain certain amounts of noise is presented. Since it has been shown that homomorphic systems are useful in separating signals which have combined through convolution, homomorphic filtering was applied to recover either the input excitation or the impulse response (reflection coefficient) of the network.

Future research in the area of deconvolution may involve a study of the errors introduced by the use of the deconvolution filters to the deconvolved result. Since different filters achieve different optimal results, a need for an investigation of the loss of information due to filtering would be justified. In chapter five, we studied the effects of the use of non-causal filters on the deconvolved result. However, the phase correction used could also be a source of error. This is seen in the phase distortion that occurs when using the minimum phase. We limited our case to causal minimum phase filters, but causal non-minimum phase filters could have also been used. More work needs to be done concerning this subject. If possible an analytic derivation of the phase component associated with deconvolution filters should be sought. In the area of homomorphic processing, a complete study of the noise effects on the success of the technique needs to be done. Finally, future research work could also include the hardware implementation of the different deconvolution filters.
APPENDIX A

This Fortran program performs the frequency domain deconvolution using the optimal compensation technique and the discrete regularization technique. The automated parameter optimization technique is also included. Most of the results reported in this work were obtained using either this program or a modification that includes the techniques described in chapter three.

CC******************************************************************************
CC PROGRAM FOR DECONVOLUTION OF OUTPUT AND INPUT USING :
CC * THE OPTIMUM COMPENSATION DECONVOLUTION
CC * THE REGULARIZATION TECHNIQUE
CC******************************************************************************
CC******************************************************************************
CC LE PROGRAMME UTILISE LA METHODE DE GANS-NAMAN POUR LA
CC DETERMINATION DE LA TRANSFORMEE DE FOURIER
CC******************************************************************************
IMPLICIT REAL*8(A-H,O-Z)
INTEGER KBAND(10)
CC**
CC**
DIMENSION XINPUT(2048),OUTPUT(2048),C(1028),HR(1028),HI(1028),
1HM(1028),HM0(1028),HR0(1028),XM(514),XT(100,6),TR(100),TI(50),
2HI0(1028),G(100),XY1(100),XY2(100),XL(100),YY(100)
CC**
COMPLEX*16 X(1028),Y(1028),H0(1028),H(1028),X1(1028),Y1(1028)
CC******************************************************************************
CC******************************************************************************
CC OPEN(1,FILE='REFEREN.INP',STATUS='OLD')
CC OPEN(2,FILE='OUTPUT.DAT',STATUS='OLD')
CC**
CC* READ INPUT AND OUTPUT WAVEFORMS
  WRITE(*,*)'ENTER THE NUMBER OF POINTS'
  READ(*,*) N
  WRITE(*,*)'ENTER THE INPUT SIGNAL FILE #1'
  READ(1,*) (XINPUT(I),I=1,N)
WRITE(*,*)'ENTER THE OUTPUT SIGNAL FILE #2'
READ(2,*) (OUTPUT(I),I=1,N)

CC***************************

CC**MAKE SURE THE WAVEFORMS START FROM ZERO
    AIN=XINPUT(1)
    AOU=OUTPUT(1)
    DO 21 I=1,N
        XINPUT(I)=XINPUT(I)-AIN
        OUTPUT(I)=OUTPUT(I)-AOU
    21 CONTINUE
    NN=2*N

CC**CONVERT THESE DURATION LIMITED FORMS
    XD=XINPUT(N)
    YD=OUTPUT(N)
    DO 22 I=1,N
        XINPUT(N+I)=XD-XINPUT(I)
        OUTPUT(N+I)=YD-OUTPUT(I)
    22 CONTINUE

CC**

CC**
    DO 23 I=1,N

CC**GO TO THE COMPLEX DOMAIN
    X1(I)=DCMPLX(XINPUT(I),0.D0)
    Y1(I)=DCMPLX(OUTPUT(I),0.D0)

    23 CONTINUE

CC**

CC**

CC**FREQUENCY DOMAIN
    CALL FFT(NN,X1,I)
    CALL FFT(NN,Y1,1)

CC**

CC**GET THE MAGNITUDE SQUARE FOR THE INPUT
    DO 24 I=1,N

CC**ELIMINATION OF EVEN HARMONICS
    II=2*I
    XR=DREAL(X1(II))
    XI=DIMAG(X1(II))
    X(I)=DCMPLX(XR,XI)
    YR=DREAL(Y1(II))
    YI=DIMAG(Y1(II))
    Y(I)=DCMPLX(YR,YI)

CC**

CC**GET THE MAGNITUDE SQUARE FOR THE INPUT
    XM(I)=CDABS(X(I))
    XM(I)=XM(I)*XM(I)

    24 CONTINUE

CC***************************

CC*** FIRST STEP USING STRAIGHTFORWARD DIVISION AND INVERSE-FFT ***
TO DEFINE CERTAIN PARAMETERS TO BE USED

FIND THE TRANSFER FUNCTION FOR GAMMA=0

DO 25 I=1,N
   HO(2*I)=Y(I)/X(I)
   NO(2*I-1)=0.0D+00
   HM0(I)=CDABS(HO(2*I))
   HRO(I)=DREAL(HO(2*I))
   HI0(I)=DIMAG(HO(2*I))
25 CONTINUE

DEFINING THE FREQUENCY REGIONS (2 INTERVALS)
FOR THIS PROGRAM YOU NEED TO DEFINE: THE POINT THAT GIVES THE CUT-OFF
FREQUENCY AND THE POINT THAT DEFINES THE FOLDING FREQUENCY
NBAND=2
WRITE(*,*)'DEFINE THE REGIONS FROM STARTING TO ENDING POINTS
X BY ENTERING (NBAND+1) INTEGERS FOR DEFINITION ON THE SAME LINE'
READ(*,*) (KBAND(I),I=1,NBAND+1)

START ALL OPERATIONS

START OF THE ITERATIVE PROCESS TO DETERMINE
THE APPROPRIATE VALUE OF GAMMA (OR LAMBDA)

WRITE(*,*)'WHAT METHOD YOU WANT TO USE: IF OPT COMP TYPE 1,
X IF REGULARIZATION TYPE 2'
READ(*,*) ICS
WRITE(*,*)'ENTER THE INITIAL VALUE OF THE PARAMETER (>0)'
READ(*,*) XPAR

THE INCREMENTATION IS ACHIEVED BY SUCCESSIVE MULTIPLICATION

WRITE(*,*)'ENTER THE MULTIPLICATIVE INCREMENT VALUE'
READ(*,*) XINC
WRITE(*,*)'ENTER THE NUMBER OF PARAMETER VALUES THAT YOU WANT'
READ(*,*) NUM

INITIALIZATION OF THE GAMMA OR LAMBDA

DO 20 J=1,NUM

FIND THE TRANSFER FUNCTION
 IF(ICS.EQ.1) GO TO 103
 DO 28 I=1,N
 NAHMAN-GUILLAUME TECHNIQUE
 PI=DATAN(-1.0D+00)
 C(I)=(DSIN((I-1)*PI/(N-1)))**4
 H(2*I)=Y(I)*DCONJG(X(I))/(XM(I)+XPAR*C(I))
 H(2*I-1)=0.0D+00
 HM(I)=CDABS(H(2*I))
28 CONTINUE
GO TO 102
CC****
CC****
103 DO 29 I=1,N
CC** OPTIMAL COMPENSATION APPROACH
CC**
H(2*I)=Y(I)*DCONJG(X(I))/(XM(I)+XPAR)
H(2*I-1)=0.D+00
HM(I)=CDABS(H(2*I))
29 CONTINUE
102 DO 30 I=1,N
HR(I)=DREAL(H(2*I))
HI(I)=DIMAG(H(2*I))
30 CONTINUE
CC**
CC**
104 DO 32 IK=1,NBAND
K1=KBAND(IK)
K2=KBAND(IK+1)
TR(IK)=XRMS(N,HR,HRO,K1,K2)
TI(IK)=XRMS(N,HI,HIO,K1,K2)
32 CONTINUE
DO 33 IK=1,NBAND
T1=TR(IK)
T2=TI(IK)
XT(J,IK)=SQRT(T1+T2)
33 CONTINUE
CC** INCREMENTING GAMMA (INCREMENT=3.2)
CC**
G(J)=XPAR
XPAR=XPAR*XINC
20 CONTINUE
CC**
CC** END OF OPERATIONS
CC**************************************************************
CC**
CC** NORMALIZATION OF THE QUANTITIES TO THEIR RESPECTIVE MAXIMUMS
CC**
DO 55 II=1,NBAND
DO 53 J=1,NUM
YY(J)=XT(J,II)
53 CONTINUE
CALL NORM(YY,NUM)
DO 54 J=1,NUM
XT(J,II)=YY(J)
54 CONTINUE
55 CONTINUE
CC**************************************************************
CC**************************************************************
CC** USING THE DEVIATIONS IN THE TWO REGIONS TO COMPUTE THE
CC** APPROPRIATE PARAMETER VALUE
     DO 56 J=1,NUM
     XL(J)=G(J)
     XY1(J)=XT(J,1)
     XY2(J)=XT(J,2)
 56 CONTINUE
     CALL XOPT(NUM,XY1,XY2,XL,XLO)
     WRITE(*,*) XLO
     READ(*,*)
CC**************************************
CC**************************************
     DO 57 J=1,NUM
CC**
         WRITE(7,301) G(J),XT(J,1),XT(J,2)
 301   FORMAT(1X,1P3E16.8)
CC**
     57 CONTINUE
     STOP
     END
CC******************************************************************************
CC CE SOUS-PROGRAMME DETERMINE LA DFT UTILISANT L'ALGORITHME DE *
CC COOLEY-TUKEY.
CC******************************************************************************
SUBROUTINE FFT(N,FF,IFLAG)
IMPLICIT REAL*8(A-H,O-Z)
COMPLEX*16 FF(2264),TERM,T
IF(IFLAG.LT.0) THEN
   DO 30 I=1,N
      FF(I)=DCONJG(FF(I))/N
 30    CONTINUE
ENDIF
PI=DACOS(-1.DO)
NU=ALOG10(N*1.)/ALOG10(2.)+.5
N2=N/2
NU1=NU-1
K=0
DO 20 I=1,NU
DO 21 I=1,N2
M=K/(2**NU1)
CALL BITREV(M,NU,IP)
P=IP
VAL=2.*PI*P/FLOAT(N)
T1=DCOS(VAL)
T2=DSIN(VAL)
K1=K+1
TERM=DCMPLX(T1,T2)*FF(K1+N2)
FF(K1+N2)=FF(K1)-TERM
FF(K1)=FF(K1)+TERM

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K=K+1
22 CONTINUE
K=K+N2
IF(K.LT.N) GO TO 21
K=0
NU1=NU1-1
N2=N2/2
20 CONTINUE
DO 23 K=1,N
KK=K-1
CALL BITREV(KK,NU,II)
I=II+1
IF(I.LE.K) GO TO 23
T=FF(K)
FF(K)=FF(I)
FF(I)=T
23 CONTINUE
RETURN
END

CC******************************************************************************
CC THIS SUBROUTINE PERFORMS THE BIT REVERSAL OPERATION *
CC******************************************************************************
SUBROUTINE BITREV(M,NU,IP)
IP=0
DO 30 I=1,NU
M1=M/2
IP=IP*2+(M-2*M1)
M=M1
30 CONTINUE
RETURN
END

CC******************************************************************************
CC SUBROUTINES USED TO COMPUTE THE MEAN,RMS,NOR, ETC *
CC******************************************************************************
FUNCTION AMEAN(X,N)
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION X(N)
AMean=0.D0
DO 31 I=1,N
AMean=AMean+X(I)
31 CONTINUE
AMean=AMean/N
RETURN
END

FUNCTION RMS(X,N)
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION X(N)
XMS=0.D0
AMEA=AMEAN(X,N)
DO 32 I=1,N
XMS=XMS+(X(I)-AMEA)**2
32 CONTINUE
XMS=XMS/N
RMS=DSQRT(XMS)
RETURN
END

FUNCTION XRMS(N,X,Y,N1,N2)
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION X(N),Y(N)
KK=N2-N1+1
XMS=0.DO
DO 32 I=N1,N2
XMS=XMS+(X(I)-Y(I))**2
32 CONTINUE
XRMS=XMS/KK
RETURN
END

SUBROUTINE INT(A,N)
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION A(N)
DO 33 I=2,N
A(I)=A(I)+A(I-1)
33 CONTINUE
RETURN
END

SUBROUTINE NORM(A,N)

NORMALIZATION OF AN ARRAY TO ITS MAXIMUM
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION A(N)
TMAX=AMAX(A,N)
DO 35 I=1,N
A(I)=A(I)/TMAX
35 CONTINUE
RETURN
END

FUNCTION AMAX(A,N)

IMPLICIT REAL*8(A-H,O-Z)

FINDING THE MAXIMUM ELEMENT OF AN ARRAY
DIMENSION A(N)
AMAX=A(1)
DO 36 I=2,N
IF(A(I).GT.AMAX) AMAX=A(I)
36 CONTINUE
RETURN
END

FUNCTION AMEAN1(A,N,N1,N2)

IMPLICIT REAL*8(A-H,O-Z)

MEAN VALUE OF A(N1:N2)
DIMENSION A(N),B(130)
N3=N2-N1
DO 37 I=1,N3
B(I)=A(I+N1-1)
37 CONTINUE
AMEAN1=AMEAN(B,N3)
RETURN
END

FUNCTION RMS1(A,N,N1,N2)

IMPLICIT REAL*8(A-H,O-Z)

RMS VALUE OF A(N1:N2)
DIMENSION A(N),B(130)
N3=N2-N1
DO 38 I=1,N3
B(I)=A(I+N1-1)
38 CONTINUE
RMS1=RMS(B,N3)
RETURN
END

SUBROUTINE NORAMP(A,N,N1,N2)

IMPLICIT REAL*8(A-H,O-Z)

REMOVAL OF THE RAMPING FROM THE STEP FUNCTION
DIMENSION A(N)
AMEA=AMEAN1(A,N,N1,N2)
DO 39 I=1,N
A(I)=A(I)-AMEA
39 CONTINUE
RETURN
END

THIS SUBROUTINE COMPUTES THE OPTIMAL PARAMETER

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SUBROUTINE XOPT(M,XY1,XY2,XL,XLO)
IMPLICIT REAL*8(A-H,O-Z)
DIMENSION XY1(M),XY2(M),XL(M)
CC** THE VECTOR XY1 CONTAINS THE DEVIATION IN THE REGION OF INTEREST
CC** THE VECTOR XY2 CONTAINS THE DEVIATION IN THE NOISY REGION
CC**
CC**
DO 30 I=1,M
   T1=0.10
   T2=0.05
   XX=XY1(I)
   IF(XX.LT.T1) GOTO 101
   GOTO 30
101 IF(XX.GT.T2) GOTO 102
   GOTO 30
102 XL1=XL(I)
   GOTO 103
30 CONTINUE
CC*******
CC*******
103 DO 31 I=1,M
   T1=0.95
   T2=0.90
   XX=XY2(I)
   IF(XX.LT.T1) GOTO 104
   GOTO 31
104 IF(XX.GT.T2) GOTO 105
   GOTO 31
105 XL2=XL(I)
   GOTO 106
31 CONTINUE
CC*******
CC** COMPUTATION OF THE OPTIMAL PARAMETER VALUE
CC*******
CC106 XLO=DSQRT(XL1*XL2)
106 XL1=0.5*DLOG10(XL1)
   XL2=0.5*DLOG10(XL2)
   XLO=XL1+XL2
   XLO=10**XLO
RETURN
END
APPENDIX B

FAST FOURIER TRANSFORM OF A STEP-LIKE SIGNAL

B.1 INTRODUCTION

Typical waveforms used in time domain measurements include impulses (reflected response waveforms) and step functions (incident reference waveforms). In practice, a finite number of discrete sampled data is used to represent a physical signal. The Fourier transform assumes that \( v(t) \) is a continuous signal. To deal with this type of data obtained, usually from a sampling oscilloscope, discrete Fourier transform (DFT) is used. The DFT of \( v(nT) \) is defined as:

\[
V(k\Omega) = \sum_{n=0}^{N-1} v(nT) \exp(-j\Omega kTn) \tag{B.1.1}
\]

and the inverse DFT (IDFT) is given by,

\[
v(nT) = \frac{1}{N} \sum_{k=0}^{N-1} V(n\Omega) \exp(+j\Omega kTn) \tag{B.1.2}
\]

where \( T \) is the sampling interval, \( N \) is the number of samples taken in a time window \( T_N \), and \( \Omega \) is the fundamental frequency given by the relation \( \Omega = 2\pi/T_N \).
The DFT is a discrete Fourier series and it assumes that it is dealing with a periodic waveform \( v_p(t) \), where \( v_p(nT) = v(nT) \) only in the interval \( 0 \leq t < T_N \). The DFT of the sampled signal is usually computed using what is known as the fast Fourier transform (FFT), which is the same as the DFT. However, the FFT has the restriction that the number of the sample points is an integer power of 2. Hence, an N point DFT requires a number of operations that is approximately proportional to \( N^2 \), while the FFT requires a number that is roughly proportional to \( N \log(N) \) operations, which reduces greatly the computer time required to evaluate the DFT of a signal.

In the discrete-time representation a waveform is represented by a sequence of data points. Given a waveform represented by the sequence of data points \( f(k) \) comprised of N values over a time window \( T_N \) as shown in Figure. (B.1a). Successive application of the DFT and the inverse DFT (IDFT) yield \( F(n) \) and \( f_p(k) \) respectively, which are periodic of period \( N \). In Figures. (B.1b) and (B.1c), are shown the magnitude of the DFT \( F(n) \) of the original sequence \( f(k) \) and the IDFT \( f_p(k) \) of its discrete Fourier transform. The resultant IDFT \( f_p(k) \) is periodic of period \( T_N \), and each one period is identical to the original sequence \( f(k) \).

The fast Fourier transform (FFT) works fine with duration limited waveforms such as impulses as in the case of the waveform shown in Figure. (B.1a). However, with step-like functions severe problems occur, producing the well known truncation error. When a function is sampled, it must have the same value at the two boundaries before the FFT is performed, otherwise oscillations are introduced by the calculation. Consider a causal step-like waveform varying in a time window of \( T_N \) in which it attains its final value \( V_0 \); i.e. \( v(t) = 0 \) for \( t \leq 0 \) and \( V_0 = v(T_N) \) for \( t > T_N \). If we select N discrete sampled data points over the time window \( T_N \) to represent such waveform as shown in Figure. (B.2a), we can apply the DFT to it. Actually, the step-like waveform is N points of a sequence that starts at \( k=0 \) and continues to \( k = \infty \), remaining constant at the value \( V_0 \) for \( k < k_0 \) where \( k_0 \leq N - 1 \). Since the DFT will use only N points, the infinite sequence \( v_i(k) \) is abruptly truncated at \( k = N - 1 \). The corresponding spectrum \( V_f(n) \) as shown in Figure. (B.2b) is not the intended spectrum \( V_i(n) \). The IDFT applied to \( V_f(n) \) shows a periodic sequence \( v_i^*(k) \) as shown in Figure. (B.2c), which does
not correspond to the sequence \( v(k) \), however each cycle of it is identical to the truncation of the sequence \( v(k) \) at the point \( k = N-1 \). The abrupt truncation in \( v(k) \) introduces the spectral components of an abrupt transition (step function). The DFT will assume to be dealing with a periodic signal as the form shown in Figure. (B.2c), where the signal is truncated abruptly at \( t = T_N \). Now, if we simply take the Fourier transform of this function over the period \( 0 \leq t < T_N \), then we obtain.

\[
V_f(j\omega) = \int_0^{T_N} v(t) \exp(-j\omega t) dt \\
= \int_0^{\infty} v(t) \exp(-j\omega t) dt - V_0 \int_{T_N}^{\infty} \exp(-j\omega t) dt \\
= V_f(j\omega) - V_0 \left[ \pi \delta(\omega) + \frac{1}{j\omega} \right] \exp(-j\omega T_N) \\
= V_f(j\omega) - E(j\omega)
\]

where \( E(j\omega) \) is an error spectrum introduced by the truncation. At the discrete frequency values \( \omega = k\Omega \), with \( \Omega = 2\pi/T_N \)

\[
E(jk\Omega) = \frac{V_0 \exp(-j2\pi k)}{jk\Omega} , \quad k \neq 0
\]

To deal with this truncation error several methods have been developed. In the first method by Samulon [90,92], the derivative of a step-like waveform is used to create an impulse-like waveform, however its major disadvantage is the unavoidable reduction of the signal to noise ratio. In the second method Nicolson [88] subtracts a linear sampled ramp from the step-like waveform prior to performing the FFT. A more practical method is the Gans-Nahman technique [89] which uses a window of period \( 2T_N \) to convert acquired waveform into a symmetrical square wave. For this method, we have to perform a double-length FFT, and due to the symmetry of the waveform the FFT gives only the odd harmonics of \( V_f(j\omega) \), while the even harmonics are equal to zero. A method [93] that involves combining the results
of the latter two methods, produces a complete-frequency transform which includes equally spaced harmonics. The method is referred to it as the complete FFT. The last three methods are the most commonly used in time-domain measurements, therefore they are discussed in detail in the following sections.

**B.2 THE RAMP TECHNIQUE [88]**

The ramp technique method depends on subtracting a ramp from a step-like waveform in order to make it duration-limited. Consider the ramp waveform shown in Figure. (B.3b), with a peak value \( V_0 = v_n(T_n) \). Its Fourier transform is given by:

\[
R(j\omega) = V_0 \left[ \pi \delta(\omega) + 2 \frac{\sin(\omega T_n/2)}{jT_n\omega^2} \right] \exp(-j\omega T_n/2) \quad (B.2.1)
\]

Suppose that we subtract \( r(t) \) from \( v(t) \) to form a duration-limited waveform as shown in Figure. (B.3c), which can be Fourier-transformed. For a continuous time signal the resulting impulse-like function \( \tilde{v}(t) \) is given by,

\[
\tilde{v}(t) = v(t) - r(t) \quad (B.2.2)
\]

then, we have

\[
\tilde{V}(j\omega) = V(j\omega) - R(j\omega)
= V(j\omega) - V_0 \left[ \pi \delta(\omega) + 2 \frac{\sin(\omega T_n/2)}{jT_n\omega^2} \right] \exp(-j\omega T_n/2) \quad (B.2.3)
\]

for \( \omega = k\Omega, \ k \neq 0 \) with \( \Omega = 2\pi/T_n \)

\[
\tilde{V}(j\omega) = V(jk\omega_0) - V_0 \left[ \frac{\sin(k\pi)}{jT_n\omega_0^2} \right] \exp(-jk\pi) \quad (B.2.4)
\]

\[
= V(jk\Omega)
\]
In this method the error caused by the abrupt truncation is removed at the discrete frequencies \( \omega = k \Omega \), \( \Omega = \frac{2\pi}{T_N} \) and \( k = 1,2,3,... \), simply by subtracting a ramp from the step response. However, this is not true for \( \omega \neq k \Omega \). In practice the discrete Fourier transform is used. Therefore, discrete waveforms having \( N \) sampled data points over the window \( 0 < t < T_N \) are used. The discrete ramp waveform can be written as:

\[
r(nT) = \frac{n}{N} V_0
\]  

(B.2.5)

hence,

\[
\tilde{v}_r(nT) = v_r(nT) - \frac{n}{N} V_0
\]  

(B.2.6)

Finding the DFT of the sequence \( \tilde{v}_r(nT) \)

\[
\tilde{V}_r(k \Omega_r) = \sum_{n=0}^{N-1} \left[ v_r(nT) - \frac{n}{N} V_0 \right] \exp(-j \Omega_r n T_N)
\]  

(B.2.7)

for \( k = 0,1,2,\ldots,N-1 \); where \( \Omega_r = 2\pi/T_N \) where the subscript \( r \) indicates the frequencies related to the ramp technique.

On the other hand \( v_r(nT) \) is not limited to \( N \) samples but is assumed to have a value of \( v_r(nT) = v_r(T_N) \) for \( t > T_N \). Then the Fourier transform of such a sequence will be given by:

\[
V_r(k \Omega_r) = \sum_{n=0}^{\infty} v_r(nT) \exp(-j \Omega_r n T_N)
\]  

(B.2.8)

If the Fourier transform is formed for a waveform such as that in Figure (B.2a), a large error results, because of the truncation at the \( N \)th sample. The missing terms are:

\[
\sum_{n=N}^{\infty} V_0 \exp(-j \Omega_r n T_N) = \frac{V_0}{1 - \exp(-j \Omega_r T_N)}
\]  

(B.2.9)
If we subtract the sampled ramp \( r(nT) \) given in Eq. (B.2.5) to \( v_s(nT) \) before taking the DFT, we eliminate the error since the DFT of the subtracted ramp \( r(nT) \) supplies the missing component, which is

\[
\sum_{n=0}^{N-1} r(nT) \exp(-j\Omega_n T nk) = \frac{V_0}{N} \sum_{n=0}^{N-1} n \exp(-j\Omega_n T nk)
\]

\[= -\frac{V_0}{[ 1 - \exp(-j\Omega_n kT) ]} \tag{B.2.10}
\]

From Eq. (B.2.10), it is clear that the DFT of the infinite train of samples truncated out is the same as that of a finite sampled ramp \( r(nT) \) with negative slope, in such a way, that there are no errors at the discrete angular frequencies \( \omega_k = k \Omega_n \), although it is not the case for \( \omega_k \neq k \Omega_n \).

Errors due to improper choice of the base line and assuming that neither \( v_s(nT) \) nor \( r(nT) \) starts at zero were pointed out in different papers [90-92].

**B.3 GANS-NAHMAN'S TECHNIQUE [89]**

In this method, to avoid the error due to the truncation \( v_s(nT) \) at the \( N^{th} \) sample, \( v_s(nT) \) is turned off in a practical physical manner. A rectangular-pulse waveform having a time window \( 2T_N \) twice that of the truncated step, is obtained as shown in Figure. (B.4c). It uses \( 2N \) points to provide \( N \) spectral values for the spectrum of the original discrete step-like waveform. The conversion of the waveform into a duration-limited is performed as follows,

\[
\tilde{v}(t) = v_s(t) - v_s(t - T_N) u(t - T_N)
\tag{B.3.1}
\]

Here, it is assumed that the step-like signal \( v_s(t) \) is causal and that it reached its final value \( V_0 \) at \( t = T_N \). The Fourier transform of \( v_s(t) \) is given by,
\[ \tilde{V}(j\omega) = \int_{-\infty}^{\infty} \tilde{v}(t) \exp(-j\omega t) \, dt \]

\[ = \int_{0}^{2T_N} \tilde{v}(t) \exp(-j\omega t) \, dt \]

\[ = V(j\omega) \left[ 1 - \exp(-j\omega T_N) \right] \]

\[ = V(j\omega) \left[ 1 - \cos(\omega T_N) - j \sin(\omega T_N) \right] \]

For the discrete angular frequencies, \( \omega_k = k\pi/T_N \), the time window being \( 2T_N \)

\[ \tilde{V}(j\omega_k) = V(j\omega_k) \left[ 1 - \cos(k\pi) - j \sin(k\pi) \right] \] (B.3.3)

\( \sin(k\pi) \) is always zero, while \( \cos(k\pi) \) can be either +1 or -1, depending on whether \( k \) is even or odd, respectively. Therefore \( \tilde{V}(j\omega_k) \) can be given by

\[ \tilde{V}(j\omega_k) = 2V(j\omega_k) \quad \text{for odd } k \]

\[ = 0 \quad \text{for even } k \] (B.3.4)

The method preserves the spectrum of the step-like waveform only at specific discrete frequencies. For a discrete waveform having \( N \) sampled points, the properly converted waveform can be expressed as,

\[ \tilde{v}(nT) = v(nT) - v((nT - nT_N) \]

\[ \text{the DFT of this sequence is} \]

\[ \tilde{v}(k\Omega_g) = \sum_{n=0}^{2N-1} \left[ v((nT_N) - v((nT - nT_N)) \right] \exp(-j\Omega_g T nk) \] (B.3.6)
For \( k = 0, 1, 2, \ldots, 2N-1 \), and with \( \Omega_g = \pi/T_N \), the subscript \( g \) indicating the frequency interval related to the Gans-Nahman technique. This equation can be written as

\[
\tilde{V}(k\Omega_g) = \sum_{n=0}^{\infty} \left[ v_i(T_N) - v_i(nT - nT_N) \right] \exp(-j\Omega_g T_n k)
- \sum_{n=2N}^{\infty} \left[ v_i(T_N) - v_i(nT - nT_N) \right] \exp(-j\Omega_g T_n k)
\]

\[
(B.3.7)
\]

Since it is assumed that \( v_i(nT) \) has attained its final value \( v_i(T_N) \) for \( t \geq T_N \), then the second term in the above expression will be identically zero. Hence,

\[
\tilde{V}(k\Omega_g) = v_i(k\Omega_g) - \exp(-j\Omega_g T_N) \cdot \tilde{V}(k\Omega_g)
= v_i(k\Omega_g) \left[ 1 - \exp(-j\Omega_g T_N) \right]
\]

\[
(B.3.8)
\]

The term \( \left[ 1 - \exp(-j\Omega_g T_N) \right] \) corresponds to the term \( \left[ 1 - \exp(-j\omega T_N) \right] \) in the case of a continuous waveform, where for \( \Omega_g = \pi/T_N \)

\[
\tilde{V}(k\Omega_g) = v_i(k\Omega_g) \left[ 1 - \exp(-j\pi k) \right]
\]

\[
(B.3.9)
\]

The term \( \left[ 1 - \exp(-j\pi k) \right] \) is zero if \( k \) is even and equals to 2 for odd values of \( k \). Consequently,

\[
\tilde{V}(jk\Omega_g) = 2v_i(jk\Omega_g) \quad \text{for odd } k
= 0 \quad \text{for even } k
\]

\[
(B.3.10)
\]

For \( v_i(t) \) defined over a time window \( T_N \), its DFT and the DFT of \( \tilde{V}_i(t) \) are simply related. \( \tilde{V}_i(k\Omega_g) \) is twice the value of \( V_i(k\Omega_g) \) for \( m = 2n-1 \). Thus, the spectrum of \( \tilde{V}_i(t) \) has:

- The same number of non-zero values \( N \) as that of \( v_i(t) \), but are graphically spaced twice apart.

- The magnitude twice that of \( v_i(t) \) at the non-zero values.
The method discussed in this section is the most practical for use in frequency-domain de-convolution techniques. It is possible to recover a time-domain waveform from its spectrum computed used this method.

**B.4 COMPLETE FFT [93]**

It was shown in the previous sections that the ramp technique uses a time window of $T_n$, while in Gans-Nahman's method the time window used was $2T_n$. The first method provides spectrum points at the frequencies $2k\pi/T_n$, and the second method at the frequencies $k\pi/T_n$. However in the second method, for $k$ even the spectrum points are forced to be zero. Hence, the two methods can be combined to complement each other as shown in Figures. (B.5a) and (B.5b), the zero points in the second method are replaced by the points in the first method. This can be seen by comparing the frequency values for the non-zero points in each method, where $\Omega_r = 2\pi/kT_n$ and $\Omega_g = \pi/kT_n$. From which it is clear that for the ramp method the values of the DFT are found at the following angular frequencies,

$$k\Omega_r = 2\pi \frac{k}{T_n} \quad (B.4.1)$$

for $k = 0, 1, 2, \ldots, N$

While for Gans-Nahman's technique the non-zero values correspond to angular frequencies of the values

$$k\Omega_g = \pi \frac{k}{T_n} \quad (B.4.2)$$

for $k = 1, 3, 5, \ldots, 2N-1$

By comparing Eqs. (B.4.1), (B.4.2), it can be seen that the ramp method is giving the even harmonics of the DFT of $V_1(nT)$, while the Gans-Nahman' method is giving the odd harmonics. Combining these two methods the frequency interval will be $(-\pi/kT_n)$, and the frequency resol-
ution can be doubled. The method is very recommended for use in S-parameters measurements. However, it is not possible to recover directly a time-domain waveform from its spectrum computed using the complete fast Fourier transform.
Figure (B.1). The process of successive application of DFT and IDFT on a simple band-limited signal.
Figure (B.2). The process of successive application of DFT and IDFT on a step-like signal.
Figure (B.3). Nicolson's technique of subtracting a linear ramp from the step.
Figure (B.4). Gans-Nahman approach to the DFT of step-like waveforms.
Figure (B.5). The result of combining both techniques (complete FFT)
APPENDIX C

THE DISCRETE HILBERT TRANSFORM

C.1 INTRODUCTION

In the following discussion the concept of causality will be used to relate the even and odd parts of a sequence, and thus, the real and imaginary parts of its z-transform (or Fourier transform). So, let us consider a sequence \( h(n) \) representing a causal system, for which the causality implies that,

\[
h(n) = 0, \quad \text{for } n < 0
\]  \hspace{1cm} (C.1.1)

Using the causality property, the Fourier transform pair of the sequence can be given by the following relations:

\[
h(n) = \frac{1}{2\pi} \int_{0}^{\infty} H(e^{j\omega}) \exp(j\omega n) d\omega
\]  \hspace{1cm} (C.1.2)

\[
H(e^{j\omega}) = \sum_{n=0}^{\infty} h(n) \exp(-j\omega n)
\]  \hspace{1cm} (C.1.3)

In general, \( H(e^{j\omega}) \) is a complex quantity that can be expressed as

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\[ H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega}) \]  \hspace{1cm} (C.1.4)

From Eq. (C.1.3), the real and imaginary components of \(H(e^{j\omega})\) can be written in the following series form,

\[ H_R(e^{j\omega}) = \sum_{n=0}^{\infty} h(n) \cos(\omega n) \]  \hspace{1cm} (C.1.5)

\[ H_I(e^{j\omega}) = \sum_{n=0}^{\infty} h(n) \sin(\omega n) \]  \hspace{1cm} (C.1.6)

Taking the inverse Fourier transform of the real component of \(H(e^{j\omega})\), yields the following relation:

\[ h_e(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(j\omega n)H_R(e^{j\omega})d\omega \]

\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(j\omega) \sum_{m=0}^{\infty} h(m) \cos(\omega m)d\omega \]  \hspace{1cm} (C.1.7)

Interchanging the order of the summation and the integration in the above equation

\[ h_e(n) = \frac{1}{2} \sum_{m=0}^{\infty} h(m) \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(j\omega(n + m))d\omega \]

\[ + \frac{1}{2} \sum_{m=0}^{\infty} h(m) \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(j\omega(n - m))d\omega \]  \hspace{1cm} (C.1.8)

using the following relation:

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(j\omega n)d\omega = \frac{\sin(n\pi)}{n\pi} \]  \hspace{1cm} (C.1.9)

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Eq. (C.1.8) can be rewritten as

\[
h_e(n) = \frac{1}{2} \sum_{m=0}^{\infty} h(m) \left\{ \frac{\sin((n + m)\pi)}{(n + m)\pi} + \frac{\sin((n - m)\pi)}{(n - m)\pi} \right\} \quad (C.1.10)
\]

since the term \(\sin(n\pi)/n\pi\) is always zero for all values of \(n\), except at \(n = 0\), then it is easy to see that:

\[
h_e(n) = \frac{[h(-n) + h(n)]}{2} \quad (C.1.11)
\]

The above equation shows that \(h_e(n)\) is just the even part of the sequence \(h(n)\). From Eq. (C.1.11), it can be seen that the inverse Fourier transform of the real component of \(H(e^{j\omega})\) is equal to \(h_e(n)\). In the same manner, it can be proven that the odd part of \(h(n)\) can be defined as:

\[
h_o(n) = \frac{[h(-n) - h(n)]}{2} \quad (C.1.12)
\]

and its Fourier transform gives the imaginary part of \(H(e^{j\omega})\), which is given by:

\[
h_o(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(j\omega n)H(e^{j\omega})d\omega \quad (C.1.13)
\]

Now, let us consider the z-transform of the causal sequence \(h(n)\). It is defined as

\[
H(z) = \sum_{n=0}^{\infty} h(n)z^{-n} \quad (C.1.14)
\]

It can be shown [20] that a real, stable, and causal sequence is completely known if we know either the real part \(H_r(e^{j\omega})\) or the imaginary part \(H_i(e^{j\omega})\) and \(h(0)\). Moreover, \(H(z)\) can also be determined anywhere in the region outside the unit circle (i.e., \(|z| > 1\)) as it will be shown.
Consider Eqs. (C.1.11) and (C.1.12), it can be seen that \( h(n) \) can be expressed in terms of its even components as follows,

\[
h(n) = h_e(n)u_+(n)
\]

(C.1.15)

where \( u_+(n) \) is defined as

\[
u_+(n) = \begin{cases} 
2 & n > 0 \\
1 & n = 0 \\
0 & n < 0
\end{cases}
\]

(C.1.16)

rewriting Eq. (C.1.14) using the definition of \( h(n) \) given in (C.1.15),

\[
H(z) = \sum_{n=0}^{\infty} h_e(n)u_+(n)z^{-n}
\]

(C.1.17)

This can be interpreted as the z-transform of the product of \( h_e(n) \) and \( u_+(n) \). Therefore, it can be obtained as the convolution of their respective z-transforms (or Fourier transforms). \( H_n(z) \) is the z-transform of \( h_e(n) \), and for \(|z| > 1\), the z-transform of \( u_+(n) \) is given by

\[
\sum_{n=0}^{\infty} u_+(n)z^{-n} = \frac{[1 + z^{-1}]}{[1 - z^{-1}]}
\]

(C.1.18)

The complex convolution theorem can be used to express \( H(z) \) of Eq. (C.1.17) in another form. The theorem can be simply expressed as follows,

\[
\sum_{n=0}^{\infty} x(n)y(n)z^{-n} = \frac{1}{2\pi j} \oint C X(n) \frac{Y(v)}{v} \frac{dv}{v}
\]

(C.1.19)

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where $C$ is any contour lying in the region of convergence, $X(z)$ and $Y(z)$ are the $z$-transforms of the sequences $x(n)$ and $y(n)$, respectively. Using this theorem together with Eq. (C.1.17), $H(z)$ can be related to $H_R(z)$ as follows,

$$H(z) = \frac{1}{2\pi j} \oint_C H_R(v) \left( \frac{z - v}{z - v} \right) dv$$

(C.1.20)

Since we are considering $H(z)$ outside the unit circle, let us define $z$ as $z = re^{j\theta}$ with $r > 1$, and set $v = e^{j\phi}$, then Eq. (C.1.20) becomes

$$H(re^{j\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_R(e^{j\phi}) P_r(\theta - \omega) d\theta$$

$$+ j \frac{1}{2\pi} \int_{-\pi}^{\pi} H_R(e^{j\phi}) Q_r(\theta - \omega) d\theta$$

(C.1.21)

with

$$P_r(\theta) = \Re \left( \frac{1 + r^{-1} e^{j\theta}}{1 - r^{-1} e^{j\theta}} \right)$$

$$= \frac{[1 - r^{-2}]}{[1 - 2r^{-1} \cos \theta + r^{-2}]}$$

(C.1.22)

and,

$$Q_r(\theta) = \Im \left( \frac{1 + r^{-1} e^{j\theta}}{1 - r^{-1} e^{j\theta}} \right)$$

$$= \frac{[2r^{-1} \sin \theta]}{[1 - 2r^{-1} \cos \theta + r^{-2}]}$$

(C.1.23)

$P_r(\theta)$ and $Q_r(\theta)$ are known as the Poisson kernel and the conjugate Poisson kernel, respectively. Equating the imaginary parts in (C.1.21), $H_R(e^{j\omega})$ can be expressed in terms of $H_R(e^{j\omega})$ as follows,
\[ H_r(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_R(e^{j\theta})Q_r(\theta - \omega) d\theta \]  

(C.1.24)

Similarly, starting with the relation

\[ h(n) = h_0(n)u_4(n) + h(0)\delta(n) \]  

(C.1.25)

\( H_r(e^{j\omega}) \) can be expressed in terms of \( H_r(e^{j\omega}) \) and \( h(0) \), where

\[ H_r(e^{j\omega}) = h(0) - \frac{1}{2\pi} \int_{-\pi}^{\pi} H_R(e^{j\theta})Q_r(\theta - \omega) d\theta \]  

(C.1.26)

To obtain a relation between the real and imaginary parts of \( H(z) \) on the unit circle, the limit as \( r \) approaches unity should be taken. But to do this, the integration should be performed first, or the limit taken first and then the integral would be evaluated carefully at the vicinity of the singularity at \( (\theta - \omega) = 0 \), since

\[ \lim_{r \to 1} Q_r(\theta - \omega) = \frac{[2 \sin(\theta - \omega)]}{[2(1 - \cos(\theta - \omega))]} = \frac{\cot\left(\frac{\theta - \omega}{2}\right)}{2} \]  

(C.1.27)

which is singular at \( \theta - \omega = 0 \). This can be done by interpreting the integrals as Cauchy principal values [20]. Hence, Eqs. (C.1.24) and (C.1.26) could be rewritten as

\[ H_r(e^{j\omega}) = \frac{1}{2\pi} P \int_{-\pi}^{\pi} H_R(e^{j\theta}) \cot\left(\frac{\theta - \omega}{2}\right) d\theta \]  

(C.1.28)
\[ H_P(e^{j\omega}) = h(0) - \frac{1}{2\pi} P \int_{-\pi}^{\pi} H(e^{j\theta}) \cot \left( \frac{\theta - \omega}{2} \right) d\theta \]  \quad (C.1.29)

The P denotes Cauchy principal value. This relation between the real and imaginary components of the Fourier transform of real, stable and causal sequence is known as the Hilbert transform.

**C.2 MINIMUM PHASE FUNCTIONS**

In the previous section, it was shown that for a real, stable and causal sequence, the real and imaginary parts of its Fourier transform are related by the Hilbert transform [79]. Now, suppose that \( H(e^{j\omega}) \) is expressed in terms of its magnitude \( |H(e^{j\omega})| \), and its phase \( \text{arg}[H(e^{j\omega})] \) as follows:

\[
H(e^{j\omega}) = |H(e^{j\omega})| \exp\left\{ \text{arg}[H(e^{j\omega})]\right\} \quad (C.2.1)
\]

taking the complex logarithm of this expression,

\[
\hat{H}(e^{j\omega}) = \log H(e^{j\omega}) = \log |H(e^{j\omega})| + j \text{arg}[H(e^{j\omega})] \quad (C.2.2)
\]

So, if \( \hat{H}(z) \) is the z-transform of a causal, real and stable sequence \( \hat{h}(n) \), then \( \log |H(e^{j\omega})| \) and \( \text{arg}[H(e^{j\omega})] \) can be related through the following Hilbert transform pair,

\[
\log |H(e^{j\omega})| = \hat{h}(0) - \frac{1}{2\pi} P \int_{-\pi}^{\pi} \text{arg}[H(e^{j\theta})] \cot \left( \frac{\theta - \omega}{2} \right) d\theta \quad (C.2.3)
\]
\[
\arg[H(e^{j\omega})] = \frac{1}{2\pi} P \int_{-\pi}^{\pi} \log |H(e^{j\theta})| \cot(\frac{\theta - \omega}{2}) d\theta
\] (C.2.4)

These relations are usually referred to as the minimum phase condition. In order that \(\hat{\hat{h}}(n)\) be a causal sequence, \(\hat{\hat{H}}(z)\) must be analytic in the region \(|z| > R\), where \(R < 1\); i.e., \(\hat{\hat{H}}(z)\) should not have any singularities outside the unit circle. To see what the relation between the causality of \(\hat{\hat{h}}(n)\) and the singularities of \(H(z)\) implies; let us consider the z-transform of \(n\hat{\hat{h}}(n)\).

\[
\sum_{n=0}^{\infty} n\hat{\hat{h}}(n)z^{-n} = -z \left[ \frac{d\hat{\hat{H}}(z)}{dz} \right]
\] (C.2.5)

where

\[
\hat{\hat{H}}(z) = \sum_{n=0}^{\infty} \hat{\hat{h}}(n)z^{-n}
\] (C.2.6)

The right hand side of Eq. (C.2.5) can be rewritten as

\[
-z \left[ \frac{d\hat{\hat{H}}(z)}{dz} \right] = -z \left[ \frac{d\log(\hat{H}(z))}{dz} \right]
= - \left[ \frac{z}{\hat{H}(z)} \right] \left[ \frac{d\hat{H}(z)}{dz} \right]
\] (C.2.7)

since \(H(z)\) is a rational function of \(z\), it can be written as a ratio of two polynomials, \(H(z) = P(z)/Q(z)\), substituting this ratio in Eq. (C.2.7), we obtain
\[-z \left[ \frac{d\hat{H}(z)}{dz} \right] = -z \left[ \frac{Q(z)}{P(z)} \frac{dP(z)}{dz} - P(z) \frac{dQ(z)}{dz} \right] \frac{1}{P(z)Q(z)} \]  

(E.2.8)

From Eq. (E.2.8), it is clear that although \( H(z) \) is a rational function, \( \hat{H}(z) \) is not rational, but its derivative is. On the other hand the poles of the derivatives of \( \hat{H}(z) \) are the roots of \( P(z)Q(z) \), i.e., they are the poles and zeros of \( H(z) \). The unit circle should lie within the region of convergence, in order for the system to be causal and stable. Hence, \( \hat{h}(n) \) and consequently \( \hat{H}(n) \) will be causal if and only if all poles and zeros of \( H(z) \) are inside the unit circle in the \( z \)-plane. This means that \( H(z) \) represents a minimum phase system if its poles and zeros are inside the unit circle. In general, a stable and causal system has all its poles inside the unit circle. However, starting from a non-minimum phase, causal, and stable system, a minimum phase could be constructed by reflecting inside the unit circle those zeros lying outside it. This is done by decomposing \( H(z) \) into a minimum phase function \( H_{\text{min}}(z) \), and an allpass function \( H_{\text{ap}}(z) \), where \( H(z) \) can be expressed as,

\[ \hat{H}(z) = H_{\text{min}}(z)H_{\text{ap}}(z) \]  

(E.2.9)

An allpass system function consists of a cascade of factors of the form

\[ \frac{z^{-1} - z_i}{1 - z^{-1}z_i^*} \]  

(E.2.10)

where the zeros lying outside the unit circle are located at \( z = 1/z_i \), \( |z_i| < 1 \). The allpass function has a magnitude of one, hence, \( H_{\text{min}}(z) \) differs from \( H(z) \) in that the zeros of \( H(z) \) lying outside the unit circle at \( z = 1/z_i \) are reflected inside the unit circle at \( z = z_i^* \). This reflection of the zeros inside the unit circle affects only the phase, where a linear phase component will be eliminated from the original phase. The resultant is the minimum phase lag between the incident and reflected signals.


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