THREE-LOOP RENORMALIZATION OF YANG-MILLS THEORY IN BACKGROUND FIELD GAUGE

by

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(ABSTRACT)

Quantization and renormalization of non-Abelian gauge fields is studied. Yang-Mills theory is renormalized up to two-loops using the background field method retaining arbitrary value of the gauge parameter. The result confirms the expectations for calculations performed in background field gauge. Namely, the counter-term depending on the background field only is a renormalization constant times the square of the field strength of the background field, and the constant upon renormalization of the gauge parameter is independent of the gauge parameter. Finally, the three-loop contribution to the renormalization group β function of pure Yang-Mills theory is calculated in the background field gauge.

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Chapter 1

Introduction

In the 1960's the usage of gauge groups to describe the elementary particle spectrum was astonishingly successful. One of the most appealing predictions of particle physics — the existence of the Ω particle — is based upon pure group theoretic considerations. However, apart from the simplest gauge group, the Abelian U(1) group, for a long time one encountered great difficulties in promoting the idea of using non-Abelian gauge groups to describe particle interactions to a quantum field theory. The problem arising in canonical quantization was bypassed in 1967 when Faddeev and Popov showed an elegant way to use the Feynman path integral formalism to quantize non-Abelian gauge theories. In order to get rid of the infinities obtained by integrating over gauge equivalent field configurations, they introduced a gauge fixing term in the integration measure which they successfully could rewrite as a Lagrangian term of fictitious fields (ghosts) added to the original Lagrangian. There is an obvious drawback to this method. The choice of a particular gauge makes the explicit calculations very tedious, similar in nature to the complicated calculations in QED in the non-Lorentz covariant formalism as compared to Feynman's covariant method.

It was not until the early 1970's that non-Abelian gauge theories were accepted as possible candidates for describing the fundamental interactions in particle physics. Until that time the quantized versions of gauge theories had been plauged by infinities in perturbation theory which made their use in practice very limited. By the invention of a new regularization procedure, 't Hooft and Veltman could give a renormalization scheme which respected the gauge symmetry, making it possible to prove the renormalizability of non-Abelian gauge theories. Since then the use of non-Abelian gauge theories in particle physics has been an undisputed success — at least phenomenologically. During the golden years of the seventies the concept of asymptotic freedom in non-Abelian gauge field theories was established, which gives meaning to perturbative calculations at least above a certain energy scale. Non-Abelian gauge theories provided the first phenomenologically successful unified theory of particle interactions. By the early 1980's, many conceptual problems had been clarified and it was time for precision measurements. Up to now, these precision measurements are in complete agreement with the theory.

This ultimate success of the usage of gauge theories in elementary particle physics provides the rationale for further investigations of the conceptual problems and precision predictions. The aim of the present work is to take a minor step in this direction. I shall investigate the possibility of quantization, while retaining full gauge invariance. Also, the Superconducting Super Collider is planned to be built in order to reach energy and luminosity regions which may provide experimental results that either give further confirmation of gauge field theories or show some signs of new physics. In either case the more precisely we know the present physics the better chance we have to find something interesting in new experiments.

The three-loop calculation of the Yang-Mills β function is intended to improve our knowledge in this regard.

Chapter 2

Gauge Invariance and the Background Field Method for Multi-loop calculations

Introduction

The background field method is a technique for quantizing locally symmetric field theories without losing explicit covariance of the symmetry. For one-loop calculations the method was introduced by DeWitt [9] . To include multi-loop calculations, the method has to be reformulated. This generalization was first discussed by Kluberg-Stern and Zuber in [21] where they have shown the validity of the renormalization procedure in the background field gauge to any order in perturbation theory, by heavy use of Slavnov-Taylor identities derived from supergauge transformations. Their approach is based on conventional functional methods, but they did not give explicit rules useful for calculations. A different approach was taken by 't Hooft [15] to obtain explicit two-loop results. His method was worked out in detail by van Damme [8] discussing the two-loop renormalization of the gauge coupling of an arbitrary renormalizible field theory and by Ichinose and Omote [18] applying the method to gravity interacting with a scalar field. In 't Hooft's approach, one introduces unconventional vertices which did not help the theory to gain popularity. To avoid the use of these vertices, Abbott [1] worked out in detail the multi-loop generalization closely following the conventional functional approach. In all these works, the background field is treated perturbatively, i. e., the effective action is written as a perturbative expansion about zero background field. The advantage of this treatment is that the background field is arbitrary. In the third chapter I shall present some speculations about the advantages of treating the background field exactly.

The basic idea of the background field method is to write the field variable appearing in the classical action as a sum of a background field (A) which obeys the classical equations of motion, and a quantum field (Q) over which the functional integration is carried out. If one perturbs about zero background field, then the quantum inverse propagator is the same as in the conventional case, namely a projection operator, and so it cannot be inverted. In the works in which the method was developed, this difficulty was overcome by the introduction of a gauge-fixing term. It was shown that one can choose a gauge (the background field gauge) which fixes the gauge of the Q field, but retains the gauge invariance of the A field. Then one can assure full gauge invariance in terms of the A field provided that the external sources are coupled to the quantum field only. Of course, the introduction of the gauge-fixing term must be accompanied by the insertion of a Faddeev-Popov determinant. This procedure of background field quantization was elaborated upon by Abbott [1], who also computed the two-loop β function in renormalized Feynman gauge and found agreement with the result obtained by conventional method.

Abbott assumes implicitly the result of [21], namely that the counterterm depending on the background field only need be renormalized by a single multiplicative renormalization constant which upon additional renormalization of the gauge parameter becomes independent of the gauge parameter. In this chapter this claim is investigated and found correct in an explicit two-loop calculation in arbitrary gauge.

The background field method

For the sake of simplicity, I consider only pure Yang-Mills theory. This model shows the important features of the method, but does not have too many vertices to cast a shade on the essence. Also, it has practical importance as it is one of the basic ingredients of the Standard Model.

In the conventional functional approach to field theory, one defines the generating functional by

$$Z[J] = \int \delta Q \det \left[\frac{\delta G^a}{\delta \omega^b}\right] \exp i \int d^4x \left[\mathcal{L}(Q) + \mathcal{L}_{GF} + J^a_\mu Q^a_\mu\right], \tag{2.1}$$

where

$$\mathcal{L}(Q) = -\frac{1}{4}(F^a_{\mu\nu})^2 \tag{2.2}$$

is the Lagrangian, with

$$F^a_{\mu\nu} = \partial_{[\mu} Q^a_{\nu]} + g f^{abc} Q^b_\mu Q^c_\nu. \tag{2.3}$$

$\mathcal{L}_{GF} = -\frac{1}{2\alpha}(G^a)^2$ is the gauge-fixing term and $\delta G^a/\delta\omega^b$ is the derivative of the gauge condition corresponding to an infinitesimal gauge transformation

$$\delta Q^a_\mu = -f^{abc}\omega^b Q^c_\mu + \frac{1}{g}\partial_\mu \omega^a. \tag{2.4}$$

Differentiation of $Z[J]$ with respect to J gives the disconnected Green functions of the theory. Somewhat more useful is

$$W[J] = -i \ln Z[J] \tag{2.5}$$

which generates the connected Green functions. Finally, the effective action which is of most interest is defined by making the Legendre transformation

$$\Gamma[\bar{Q}] = W[J] - \int d^4x \, J^a_\mu \bar{Q}^a_\mu, \tag{2.6}$$

where

$$\bar{Q}^a_\mu = \frac{\delta W}{\delta J^a_\mu}. \tag{2.7}$$

The importance of this lies in the property that the derivatives of the effective action with respect to $\bar{Q}$ are the one-particle irreducible (1PI) Green functions.

The quantities analogous to Z, W, and Γ will be denoted by $\tilde{Z}$, $\tilde{W}$, and $\tilde{\Gamma}$ in the background field method. They are defined exactly like the conventional generating functionals except that the field in the classical Lagrangian is shifted by A, which is the background field. However, as in [1,15] the background field is not coupled to the source J. The generating functional is now a functional of both J and A:

$$\tilde{Z}[J, A] = \int \delta Q \det \left[\frac{\delta G^a}{\delta \omega^b}\right] \exp i \int d^4x \left[\mathcal{L}(A+Q) + \mathcal{L}_{GF} + J^a_\mu Q^a_\mu\right]. \tag{2.8}$$

Here one uses the background field gauge choice

$$G^a = D^{ac}_\mu [A^b] Q^c_\mu \equiv \partial_\mu Q^a_\mu + g f^{abc} A^b_\mu Q^c_\mu. \tag{2.9}$$

The advantage of this choice will become apparent soon. To compute the generating functional (2.8), one trades the Faddeev-Popov determinant for a ghost Lagrangian,

$\mathcal{L}_{ghost}$, in the exponent. This can be achieved by noticing that the background field generating functional is invariant under the infinitesimal transformations

$$\delta A^a_\mu = 0, \tag{2.10}$$

$$\delta Q^a_\mu = -f^{abc}\omega^b (A+Q)^c_\mu + \frac{1}{g}\partial_\mu \omega^a, \tag{2.11}$$

$$\delta J^a_\mu = -f^{abc}\omega^b J^c_\mu. \tag{2.12}$$

(Notice that the sum of (2.10) and (2.11) is just the infinitesimal gauge transformation of the total field $A+Q$ under which $\mathcal{L}(A+Q)$ is invariant.) As a consequence, one obtaines the ghost Lagrangian

$$\begin{aligned} L_{ghost} \quad &= \quad -\theta^*_a[\Box^2\delta^{ab} + gf^{abc}A^c_\mu\partial_\mu + g^2 f^{acx} f^{xdb} A^c_\mu (A^d_\mu + Q^d_\mu)]\theta_b \qquad (2.13) \\ &\quad +g(\partial_\mu\theta^*_a) f^{abc}(A^c_\mu + Q^c_\mu)\theta_b \end{aligned}$$

It is worth noticing that the quantum Lagrangian,

$$\mathcal{L}_Q = \mathcal{L}(A+Q) + \mathcal{L}_{GF} + \mathcal{L}_{ghost}, \tag{2.14}$$

with the background field gauge condition (2.9) is invariant under the following infinitesimal BRST [2] transformations (type II transformations):

$$\delta Q^a_\mu = -\delta\lambda D^{ab}_\mu [A^x + Q^x]\theta^b_2, \tag{2.15}$$

$$\delta A^a_\mu = 0, \tag{2.16}$$

$$\delta\theta^a_1 = i\delta\lambda\frac{1}{\alpha} D^{ab}_\mu[A^x] Q^b_\mu, \tag{2.17}$$

$$\delta\theta^a_2 = -\frac{1}{2}\delta\lambda g f^{abc}\theta^b_2\theta^c_2, \tag{2.18}$$

where the two real fields θ^a_1, θ^a_2 were introduced by

$$\theta^a_1 = (\theta^a + \theta^{a*})/\sqrt{2}, \tag{2.19}$$

$$\theta_2^a = (\theta^a - \theta^{a*})/i\sqrt{2}. \tag{2.20}$$

In terms of θ_1^a, θ_2^a the ghost Lagrangian is

$$\mathcal{L}_{ghost} = i(D_\mu^{ab}[A^x]\theta_1^b)(D_\mu^{ac}[A^y + Q^y]\theta_2^c). \tag{2.21}$$

(2.15, 2.16) are obtained from (2.10, 2.11) by using the following ansatz [2]

$$\omega^a = -g\delta\lambda\theta_2^a, \tag{2.22}$$

where $\delta\lambda$ is an infinitesimal Grassmann number independent of x and anticommutes with $\theta_2^a(x)$:

$$\{\delta\lambda, \theta_2^a(x)\} = 0. \tag{2.23}$$

The peculiarity of the background field gauge choice is that there is another set of BRST transformations (type I transformations) under which $\mathcal{L}_Q$ is invariant, namely

$$\delta Q_\mu^a = \delta\lambda g f^{abc}\theta_2^b Q_\mu^c \tag{2.24}$$

$$\delta A_\mu^a = -\delta\lambda D_\mu^{ab}[A^x]\theta_2^b, \tag{2.25}$$

$$\delta\theta_1^a = -\delta\lambda g f^{abc}\theta_2^b\theta_1^c, \tag{2.26}$$

$$\delta\theta_2^a = -\delta\lambda g f^{abc}\theta_2^b\theta_2^c. \tag{2.27}$$

If the source is assigned to the adjoint representation of the group, then the generating functional, and as consequence $\tilde{W}[J, A]$, defined as

$$\tilde{W}[J, A] = -i\ln\tilde{Z}[J, A], \tag{2.28}$$

are also invariant under the (2.24, 2.25, 2.12) gauge transformations. It then follows that the background field effective action,

$$\tilde{\Gamma}[\tilde{Q}, A] = \tilde{W}[J, A] - \int d^4x J_\mu^a \bar{Q}_\mu^a \tag{2.29}$$

is invariant under

$$\delta A_\mu^a = \frac{1}{g} D_\mu^{ab}[A^x]\omega_2^b, \tag{2.30}$$

$$\delta \tilde{Q}_\mu^a = -f^{abc}\omega^b \tilde{Q}_\mu^c, \tag{2.31}$$

where

$$\tilde{Q}_\mu^a = \frac{\delta \tilde{W}}{\delta J_\mu^a}. \tag{2.32}$$

This means that $\tilde{\Gamma}[0, A]$ must be a gauge-invariant functional of A itself. One can show (see [1]) that this gauge-invariant effective action is equal to the conventional effective action evaluated in an unusual gauge ($G^a = \partial_\mu Q_\mu^a - \partial_\mu A_\mu^a + g f^{abc} A_\mu^b Q_\mu^c$). It can thus be used in the usual manner to generate the S-matrix of the theory.

The important question is whether or not these nice invariance properties survive in the renormalized theory. The conclusion of [21] is that in perturbation theory to any order, the *renormalized* generating functional in the background field gauge is invariant under both type I and type II transformations. The content of the next section is to check this statement up to two-loops by performing the renormalization explicitly.

To perform the calculations, I adopt the Feynman rules given in [1] for an arbitrary gauge (see Appendix A). The gauge field and ghost propagators are exactly the same as they are in the conventional method, while all the other terms in the Lagrangian $\mathcal{L}(A + Q) + \mathcal{L}_{GF} + \mathcal{L}_{ghost}$ generate the various gauge-gauge and gauge-ghost vertices. The gauge-invariant effective action, $\tilde{\Gamma}[0, A]$, is computed by summing all 1PI diagrams with A fields on external legs (but not inside loops since the functional integration is only over Q) and Q fields inside loops (but not on external legs since $\tilde{Q} = 0$).

Two-loop renormalization of pure Yang-Mills theory

To test the gauge invariance and gauge parameter independence of the renormalized Lagrangian provided by this method, I computed the two-loop renormalization constants of pure Yang-Mills theories for an arbitrary α. One can show [1] that the renormalization of the quantum fields is irrelevant for the computation of the effective action because the quantum lines appear inside loops only. Indeed, suppose that we did renormalize the quantum gauge and ghost fields by

$$Q_0 = Z_Q^{1/2} Q, \; \theta_0 = Z_\theta^{1/2} \theta. \tag{2.33}$$

Then, one has a factor of $Z_Q^{1/2}$ at each end of the gauge propagator coming from the renormalization of the field at each vertex, and a factor Z_Q^{-1} coming from renormalizing the propagator. The two factors of $Z_Q^{1/2}$ and the Z_Q^{-1} associated with each propagator then cancel exactly (the same can be said about the ghost renormalization). Therefore, there are only two parameters to renormalize. These are the coupling constant and the background field itself:

$$g_0 = \mu^\epsilon Z_g g, \; A_0 = Z_A^{1/2} A. \tag{2.34}$$

The explicit gauge invariance of the effective action $\tilde{\Gamma}[0, A]$ ensures that it must take the gauge invariant form of a constant times $(F^a_{\mu\nu}[A])^2$; therefore, the renormalization factors Z_A and Z_g are related. According to (2.34), $F^a_{\mu\nu}[A]$ is renormalized by

$$(F^a_{\mu\nu}[A])_0 = Z_A^{1/2}[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g Z_g Z_A^{1/2} f^{abc} A^b_\mu A^c_\nu]. \tag{2.35}$$

This will only take on the gauge invariant form of a constant times $F^a_{\mu\nu}[A]$ if

$$Z_g = Z_A^{-1/2}. \tag{2.36}$$

As was emphasized in [1,21] these parameters may depend on the gauge-fixing parameter. However, the renormalization of the gauge-fixing parameter, i. e. the longitudinal part of the quantum gauge propagator has to remove this gauge parameter dependence. If this is true, then the gauge parameter renormalization can be avoided by the choice of Landau gauge ($\alpha = 0$). Unfortunately, this cannot be done from the beginning, for some of the vertices contain terms proportional to $1/\alpha$. In the case of covariant Lorentz gauge the longitudinal part of the gauge field propagator is free from higher order corrections, since the self-energy is transverse. The consequence of this is that the α parameter renormalization can be chosen the same as the gauge field renormalization and the whole propagator can be renormalized by one constant (see e. g., [23]). In our case the same is true except that, as we saw earlier, we do not have to renormalize the transverse part of the quantum gauge field. Therefore, the counter-term corresponding to the gauge-fixing parameter renormalization is of the form

$$\left(1 - \frac{1}{Z_\alpha}\right) \frac{1}{2\alpha_r} (\partial_\mu Q^a_\mu + g f^{abc} A^b_\mu Q^c_\mu)^2, \tag{2.37}$$

where

$$\begin{aligned} Z_\alpha = Z_Q &= 1 + \frac{1}{\varepsilon}\left(\frac{g_r}{4\pi}\right)^2 C_A \left(\frac{13}{3} - \alpha_r\right) \\ &+ \left(\frac{g_r}{4\pi}\right)^4 C_A^2 \left(\frac{1}{4\varepsilon^2}\left(-\frac{13}{2} - \frac{17}{6}\alpha_r + \alpha_r^2\right) + \frac{1}{8\varepsilon}\left(\frac{59}{2} - \frac{11}{2}\alpha_r - \alpha_r^2\right)\right). \end{aligned} \tag{2.38}$$

which is the standard value of the quantum gauge field renormalization constant derived in [10]. (The $O(g_r^4)$ term with $\alpha_r = 1$ will be used in the next chapter.) The

Feynman rules corresponding to these counterterms are given in Appendix A.

Calculation of the β function

In what follows (and later in Chapter 3 also), I use dimensional regularization [16] in $4-2\varepsilon$ dimensions and the minimal subtraction scheme [17] in which Z_A is written as

$$Z_A = 1 + \sum_{i=1}^{n} \frac{Z_A^{(i)}}{\varepsilon^i} \tag{2.39}$$

in n-th order perturbation theory. The renormalization group β function is defined as

$$\beta(g_r, \varepsilon) = \left(\mu \frac{\partial g_r}{\partial \mu}\right)_{g_0, \varepsilon} . \tag{2.40}$$

which is assumed to be analytic in ε:

$$\beta(g_r, \varepsilon) = \sum_{i=0}^{n} \beta_i(g_r) \varepsilon^i. \tag{2.41}$$

By using the chain rule of differentiation on eq. (2.34) and the Ward identity (2.36) on can derive the following important equation

$$\beta(g_r, \varepsilon)\left(1 + \varepsilon g_r - \frac{1}{2} g_r \frac{\partial}{\partial g_r}\right) Z_A = 0. \tag{2.42}$$

Upon substitution of (2.39) into this equation, one obtains

$$\varepsilon g_r + \beta(g_r, \varepsilon) + g_r Z_A^{(1)} + \sum_{i=1}^{n} \frac{1}{\varepsilon^i}\left(\beta(g_r, \varepsilon)\left[1 - \frac{1}{2} g_r \frac{\partial}{\partial g_r} Z_A^i\right] + g_r Z_A^{i+1}\right) = 0. \tag{2.43}$$

By using (2.41) and collecting the coefficients of various ε powers, one finds

$$\beta(g_r, \varepsilon) = -\varepsilon g_r + \beta, \tag{2.44}$$

$$\beta\left(2 - g_r \frac{\partial}{\partial g_r}\right) Z_A^{(i)} = -g_r^2 \frac{\partial}{\partial g_r} Z_A^{(i+1)}. \tag{2.45}$$

Here $\beta = \beta_0(g_r)$ is the quantity which is commonly referred as "the β" function and is the objective of the present calculation. Recalling that $Z_A^{(0)} = 1$, the recursion relation (2.45) gives

$$\beta = -\frac{1}{2} g_r^2 \frac{\partial}{\partial g_r} Z_A^{(1)}. \tag{2.46}$$

If we expand the β function in g_r as

$$\beta = -g_r \left[\beta_0 \left(\frac{g_r}{4\pi} \right)^2 + \beta_1 \left(\frac{g_r}{4\pi} \right)^4 \right] \tag{2.47}$$

and notice that for the piece of $Z_A^{(1)}$ proportional to g_r^2, $(2 - g_r \partial/\partial g_r) Z_A^{(1)} = 0$, then up to two loops Z_A must be

$$Z_A = 1 + \frac{\beta_0}{\varepsilon} \left(\frac{g_r}{4\pi} \right)^2 + \frac{\beta_1}{2\varepsilon} \left(\frac{g_r}{4\pi} \right)^4 \tag{2.48}$$

(no second order pole appears at two loops).

The two-point graphs at the one-loop level are given in figure B. 1. The computation is fairly simple even in arbitrary gauge. One finds that both diagrams are independent of the gauge-fixing parameter α. The divergent contribution of figure B. 1a is

$$\frac{i g^2 C_A \delta^{ab}}{(4\pi)^2} \left(\frac{1}{3\varepsilon} \right) [g_{\mu\nu} - k_\mu k_\nu], \tag{2.49}$$

and that of figure B. 1b is

$$\frac{i g^2 C_A \delta^{ab}}{(4\pi)^2} \left(\frac{10}{3\varepsilon} \right) [g_{\mu\nu} k^2 - k_\mu k_\nu]. \tag{2.50}$$

The sum of the two diagrams determines Z_A and hence $\beta_0 = \frac{11}{3} C_A$.[24]

The two-loop graphs for computing Z_A are given in figure B. 2. The tensorial structure of the diagrams is such that all the divergent contributions are of the form

$$\frac{i g^4 C_A^2 \delta^{ab}}{(4\pi)^4} [A g_{\mu\nu} k^2 - B k_\mu k_\nu] \tag{2.51}$$

and we want to compute the two constants A and B. For arbitrary α the number of integrals to be performed is of the order of ten thousand. In order to compute these integrals, a program was developed in the *Mathematica* symbolic manipulation language.[29] It is convenient to define the integrals in terms of scalars. From (2.51) two equations were obtained for the two unknowns A and B by contracting (2.51) once with $g_{\mu\nu}$, once with $k_\mu k_\nu$. Both A and B are functions of α. The individual contributions to A and B are given in table 2.1 ($\Delta = 1 - 2\varepsilon(\gamma_E + \ln \frac{k^2}{4\pi\mu^2})$. The columns contain the coefficients of the various α powers. The total contribution is transverse as it should be because of gauge invariance and the fact that in the background field method the renormalization of the A-field alone renormalizes the theory. This transverse property is true in all orders of perturbation theory making the calculations somewhat easier in the sense that for the total contribution it is sufficient to compute one of the constants A and B. The $1/\varepsilon^2$ poles cancel in the total contribution and the coefficients of the $1/\alpha$ and $1/\alpha^2$ factors are zero, as is expected. The coefficient of the constant term reproduces the known $\beta_1 = \frac{34}{3}C_A^2$ result [20], showing that at this stage one can take $\alpha = 0$ and leave out the gauge-fixing parameter renormalization. The insertion diagrams are shown in figure B. 3, dots representing gauge fixing term insertions. The values of these diagrams are given in table 2. 2. Clearly, the inclusion of the gauge-fixing parameter renormalization cancels all the α dependence in Z_A, and, therefore, by (2.36) in Z_g, too.

The advantage of this gauge choice is obvious. In the case of covariant Lorentz gauge, after taking into account the various Slavnov-Taylor identities, there are still three independent renormalization constants in pure Yang-Mills theory. In the

Table 2.1: Contributions to the two-loop propagator from the graphs of figure B.2.

graph	*A*			
	1	α	α^2	α^3
a	$\frac{1}{4\varepsilon^2}(\Delta + \frac{31}{6}\varepsilon)$	$-\frac{1}{12\varepsilon^2}(\Delta + \frac{23}{6}\varepsilon)$	0	0
b	$\frac{275}{48\varepsilon^2}(\Delta + \frac{283}{110}\varepsilon)$	$-\frac{49}{24\varepsilon^2}(\Delta + \frac{5}{7}\varepsilon)$	$\frac{1}{16\varepsilon^2}(\Delta - \frac{5}{6}\varepsilon)$	$-\frac{3}{16\varepsilon}$
c	$\frac{11}{48\varepsilon^2}(\Delta + \frac{85}{22}\varepsilon)$	$\frac{1}{8\varepsilon^2}(\Delta + \frac{77}{18}\varepsilon)$	$\frac{1}{16\varepsilon^2}(\Delta + \frac{53}{6}\varepsilon)$	$\frac{1}{8\varepsilon}$
d	$\frac{1}{12\varepsilon}$	$\frac{1}{24\varepsilon}$	0	0
e	$-\frac{1}{2\varepsilon}$	$-\frac{15}{32\varepsilon}$	$-\frac{1}{16\varepsilon}$	$-\frac{3}{32\varepsilon}$
f	$-\frac{27}{8\varepsilon^2}(\Delta + \frac{7}{3}\varepsilon)$	$-\frac{9}{4\varepsilon^2}(\Delta + \frac{23}{6}\varepsilon)$	$-\frac{3}{8\varepsilon^2}(\Delta + \frac{25}{3}\varepsilon)$	$-\frac{3}{8\varepsilon}$
g+h	$-\frac{3}{16\varepsilon^2}(\Delta + \frac{31}{6}\varepsilon)$	$\frac{1}{48\varepsilon^2}(\Delta - \frac{1}{6}\varepsilon)$	0	0
i	$\frac{1}{12\varepsilon}$	$-\frac{7}{48\varepsilon^2}(\Delta + \frac{161}{42}\varepsilon)$	$-\frac{1}{16\varepsilon^2}(\Delta + \frac{23}{6}\varepsilon)$	0
j	$-\frac{75}{32\varepsilon^2}(\Delta + \frac{13}{6}\varepsilon)$	$\frac{15}{16\varepsilon^2}(\Delta + \frac{17}{15}\varepsilon)$	$\frac{9}{32\varepsilon^2}(\Delta + \frac{59}{18}\varepsilon)$	$\frac{9}{32\varepsilon}$
k	$\frac{1}{6\varepsilon}$	$\frac{1}{24\varepsilon^2}(\Delta + \frac{29}{6}\varepsilon)$	0	0
l	$-\frac{7}{24\varepsilon^2}(\Delta + \frac{23}{7}\varepsilon)$	$\frac{1}{24\varepsilon^2}(\Delta - \frac{1}{2}\varepsilon)$	$\frac{1}{48\varepsilon}$	0
m	$-\frac{1}{96\varepsilon^2}(\Delta - \frac{729}{2}\varepsilon)$	$\frac{161}{48\varepsilon^2}(\Delta + \frac{611}{322}\varepsilon)$	$\frac{1}{32\varepsilon^2}(\Delta + \frac{317}{6}\varepsilon)$	$\frac{1}{2\varepsilon}$
total	$\frac{17}{3\varepsilon}$	$-\frac{13}{4\varepsilon}$	$-\frac{1}{3\varepsilon}$	$\frac{1}{4\varepsilon}$
	B			
a	$\frac{1}{4\varepsilon^2}(\Delta + \frac{14}{3}\varepsilon)$	$-\frac{1}{12\varepsilon^2}(\Delta + \frac{10}{3}\varepsilon)$	0	0
b	$\frac{275}{48\varepsilon^2}(\Delta + \frac{1409}{550}\varepsilon)$	$-\frac{49}{24\varepsilon^2}(\Delta + \frac{43}{98}\varepsilon)$	$\frac{1}{16\varepsilon^2}(\Delta - \frac{17}{6}\varepsilon)$	$-\frac{3}{16\varepsilon}$
c	$\frac{11}{48\varepsilon^2}(\Delta + \frac{91}{22}\varepsilon)$	$\frac{1}{8\varepsilon^2}(\Delta + \frac{37}{9}\varepsilon)$	$\frac{1}{16\varepsilon^2}(\Delta + \frac{53}{6}\varepsilon)$	$\frac{1}{8\varepsilon}$
d	$-\frac{1}{24\varepsilon}$	$\frac{1}{24\varepsilon}$	0	0
e	$-\frac{1}{32\varepsilon}$	$\frac{3}{32\varepsilon}$	$\frac{1}{32\varepsilon}$	$-\frac{3}{32\varepsilon}$
f	$-\frac{27}{8\varepsilon^2}(\Delta + \frac{7}{3}\varepsilon)$	$-\frac{9}{4\varepsilon^2}(\Delta + \frac{23}{6}\varepsilon)$	$-\frac{3}{8\varepsilon^2}(\Delta + \frac{25}{3}\varepsilon)$	$-\frac{3}{8\varepsilon}$
g+h	$-\frac{3}{16\varepsilon^2}(\Delta + \frac{19}{6}\varepsilon)$	$\frac{1}{48\varepsilon^2}(\Delta - \frac{13}{6}\varepsilon)$	0	0
i	$-\frac{1}{24\varepsilon}$	$-\frac{7}{48\varepsilon^2}(\Delta + \frac{149}{42}\varepsilon)$	$-\frac{1}{16\varepsilon^2}(\Delta + \frac{23}{6}\varepsilon)$	0
j	$-\frac{75}{32\varepsilon^2}(\Delta + \frac{77}{30}\varepsilon)$	$\frac{15}{16\varepsilon^2}(\Delta - \frac{1}{15}\varepsilon)$	$\frac{9}{32\varepsilon^2}(\Delta + \frac{47}{18}\varepsilon)$	$\frac{9}{32\varepsilon}$
k	$\frac{5}{48\varepsilon}$	$\frac{1}{24\varepsilon^2}(\Delta + \frac{13}{3}\varepsilon)$	0	0
l	$-\frac{7}{24\varepsilon^2}(\Delta + \frac{23}{7}\varepsilon)$	$\frac{1}{24\varepsilon^2}(\Delta - \frac{1}{2}\varepsilon)$	$\frac{1}{48\varepsilon}$	0
m	$-\frac{1}{96\varepsilon^2}(\Delta - \frac{831}{2}\varepsilon)$	$\frac{161}{48\varepsilon^2}(\Delta + \frac{611}{322}\varepsilon)$	$\frac{1}{32\varepsilon^2}(\Delta + \frac{359}{6}\varepsilon)$	$\frac{1}{2\varepsilon}$
total	$\frac{17}{3\varepsilon}$	$-\frac{13}{4\varepsilon}$	$-\frac{1}{3\varepsilon}$	$\frac{1}{4\varepsilon}$

Table 2.2: Renormalization insertion contributions to the two-loop propagator from the graphs of figure B.3

graph	*A*			
	1	α	α^2	α^3
a	$\frac{91}{36\varepsilon^2}(\Delta + \frac{104}{91}\varepsilon)$	$\frac{1}{2\varepsilon^2}(\Delta + \frac{145}{18}\varepsilon)$	$-\frac{3}{12\varepsilon^2}\Delta$	$-\frac{1}{4\varepsilon}$
b	$-\frac{91}{36\varepsilon^2}(\Delta + \frac{104}{91}\varepsilon)$	$-\frac{1}{2\varepsilon^2}(\Delta + \frac{14}{9}\varepsilon)$	$\frac{3}{12\varepsilon^2}(\Delta + \frac{14}{3}\varepsilon)$	0
total	0	$\frac{13}{4\varepsilon}$	$\frac{1}{3\varepsilon}$	$-\frac{1}{4\varepsilon}$
	B			
	1	α	α^2	α^3
a	$\frac{91}{36\varepsilon^2}(\Delta + \frac{104}{91}\varepsilon)$	$\frac{1}{2\varepsilon^2}(\Delta + \frac{145}{18}\varepsilon)$	$-\frac{3}{12\varepsilon^2}\Delta$	$-\frac{1}{4\varepsilon}$
b	$-\frac{91}{36\varepsilon^2}(\Delta + \frac{104}{91}\varepsilon)$	$-\frac{1}{2\varepsilon^2}(\Delta + \frac{14}{9}\varepsilon)$	$\frac{3}{12\varepsilon^2}(\Delta + \frac{14}{3}\varepsilon)$	0
total	0	$\frac{13}{4\varepsilon}$	$\frac{1}{3\varepsilon}$	$-\frac{1}{4\varepsilon}$

case of background field gauge choice there is only one, namely Z_A which can be computed from the background field two-point function. Practice shows that the amount of algebra and analysis involved in the computation of two-point functions is considerably less then that of three- or more-point functions. The only drawback is that there are additional vertices involving background fields. The structure of these vertices is such that they give considerable surplus work in arbitrary gauge, but they become especially simple in Feynman gauge. As the final result is independent of the gauge parameter, the Feynman gauge — if possible — is an especially well-suited choice in background field calculations.

Conclusions

The gauge invariance and gauge parameter independence of the counter-terms for operators depending on the background field only was tested in the background field method by means of an explicit two-loop calculation. The result was found to be in full agreement with the claims of [1] and [21].

Chapter 3

Three-Loop renormalization of Yang-Mills Theory

Introduction

Renormalization group techniques are very useful to improve the results of perturbative calculations. This is important in QCD calculations where the coupling is somewhat large. However, for the same reason, the renormalization group coefficients also depend on the renormalized coupling strongly; therefore, one expects considerable dependence on higher order effects. Moreover, the next to leading order corrections have fairly large numeric coefficients [20] which leaves open the possibility that the higher order contributions will be important.

The three-loop corrections to the QCD β function were computed about ten years ago [26]. No details of the calculation were published, though there are considerable complications arising in the evaluation of essentially three-loop type integrals. Also, the calulation in [26] was performed in conventional Lorentz gauge. In the previous chapter we saw how a well-suited method is the background field method to perform such calculations. Therefore, I considered the three-loop renormalization of Yang-Mills theory in background field gauge a challenging work which provides an independent check of the important result of.[26]

The conceptual part of the calculation is straightforward after the discussion of the previous chapter. The main difficulty is to perform the essentially three-loop type integrals — integrals which cannot be computed by the recursive application of the well-known one-loop formula

$$\int \frac{d^{4-2\varepsilon}p}{(2\pi)^{4-2\varepsilon}} \frac{1}{(k-p)^{2\alpha}p^{2\beta}} = i\frac{1}{(4\pi)^{2-\varepsilon}} k^{2-\alpha-\beta-\varepsilon} \frac{\Gamma(\alpha+\beta-2+\varepsilon)}{\Gamma(\alpha)\Gamma(\beta)} B(2-\alpha-\varepsilon, 2-\beta-\varepsilon). \tag{3.1}$$

This part of the work contains a good number of subtleties. I, therefore, present a detailed description of the ideas involved in the next section.

Evaluation of essentially three-loop type Feynman integrals

It is well-known that an integral corresponding to a Feynman graph is plauged by the presence of logarithms of the dimensionful parameters of the model. However, as was shown in [6] that the renormalized version of the same diagram, if it is regularized dimensionally, has a counter-term which contains pure poles in $\varepsilon = \frac{1}{2}(4-d)$ and its dependence on the dimensionful parameters — such as external momenta k_{ex}, masses m — is polynomial, and, therefore, on dimensional grounds, known beforehand. This means exactly what we understand under renormalization, namely by *local* counter-terms (which are polynomials in momentum space) the theory can be made finite.

More precisely, given an unrenormalized Feynman integral $I(k_{ex}, m, \mu^2, \varepsilon)$ corresponding to a Feynman graph G, its renormalized version can be written as [3,4]

$$I_r = RI = (1-K)R'I, \tag{3.2}$$

where R is the Bogoliubov-Parasiuk R-operation. R' is the incomplete R-operation which corresponds to subtraction of all the subdivergencies of G leaving the overall divergence only. We have the recursion relation for R':

$$R'G = G + \sum \left(\prod_{i=1}^{m} (-KR'G_i) G/(G_1 + \cdots + G_m) \right), \; m \geq 1, \tag{3.3}$$

where the sum is taken over all possible sets of nonintersecting strongly connected — i. e., connected by more then one line — subgraphs of G and $G/(G_1 + \cdots + G_m)$ is the diagram obtained from G by shrinking the subgraphs $G_1, \ldots, G_m$ to points. It was shown in [4] that R' removes the subdivegencies of each diagram separately (not only from the Lagrangian as a whole). The operator K separates the poles in ε:

$$K \sum_i c_i \varepsilon^i = \sum_{i \leq 0} c_i \varepsilon^i. \tag{3.4}$$

For renormalization group purposes, one needs the pole part of the diagrams. From this point of view, the operation KR' has the following important property.[6] If a diagram G does not contain any infrared singularity, then $KR'G$ is a polynomial in the external momenta and in the masses corresponding to its internal lines. The degree of the polynomial is determined by the dimensionality of G. In particular, for logarithmically divergent diagrams $KR'G$ does not depend on these parameters which, therefore, can be set to zero. However, a word of caution is in order here. By setting all these parameters to zero one introduces "spurious infrared divergencies" [28] which cancel the UV poles and obtains zero just as in the case of the tadpole integral:[16]

$$\int d^{4-2\epsilon} \frac{1}{k^{2\alpha}} = 0. \tag{3.5}$$

We may, however, change the masses and the external momenta in a totally arbitray fashion, as long as no IR divergence is introduced. Typically, all but one external parameters can be set to zero, making it possible to reduce the necessary integrals to the recursive use of the one-loop integral (3.1). It should be emphasized that it is the whole counter-term $KR'G$ that will undergo this IR rearrangement unchanged for it is $KR'G$ which is independent of dimensional parameters. Therefore, all the terms corresponding to subdivergences which are subtracted by R' must be modified in the same manner.

Linearly or quadratically divergent diagrams can be treated the same way after proper differentiation with respect to their external parameters, which reduces the degree of divergence to zero. Again, the the counter-term $KR'G$ as a whole should be differentiated. One subtlety emerges here. The massless tadpoles (3.5) are zero in dimensional regularization; therefore, they do not contribute to $KR'G$. However, they should be kept in the calculation when one reduces a quadratically divergent diagram to a logarithmically divergent one. The differentiation introduces spurious IR divergences which can be taken care of by infrared rearrengement which, however, should apply to all the diagrams in $KR'G$ irrespective of their dimensionally regularized value.

Having set the theoretical framework for the calculations, it is time to provide some examples to show how the concepts work in practice. For the sake of simplicity the integral measures will be denoted by dp, dq, dr in the following subsections.

Example of a logarithmically divergent essentially three-loop type integral

Consider

$$\int dp\, dq\, dr \frac{(p \cdot r)^2}{(k-p)^2(k-q)^2(k-r)^2(p-q)^2(q-r)^2 p^2 q^2 r^2}, \tag{3.6}$$

which is a logarithmically divergent integral containing two logarithmically divergent one-loop subintegrals (p and r loops). These are of the form

$$(4\pi)^2 K \int dp \frac{p^\mu p^\nu}{(k-p)^2(p-q)^2 p^2} = A\left(\frac{1}{\varepsilon}\right) g^{\mu\nu}. \tag{3.7}$$

Their pole part can be obtained by using the one-loop (3.1) formula after contraction with $g^{\mu\nu}$. One has

$$(4\pi)^2 K \int dp \frac{p^\mu p^\nu}{(k-p)^2(p-q)^2 p^2} = i\frac{1}{4\varepsilon} g^{\mu\nu}. \tag{3.8}$$

The KR' operation on (3.6) then gives

$$\begin{aligned}(4\pi)^6 K \Bigg\{ &\int dp\, dq\, dr \frac{(p \cdot r)^2}{(k-p)^2(k-q)^2(k-r)^2(p-q)^2(q-r)^2 p^2 q^2 r^2} + \\ 2i\left(-\frac{1}{4\varepsilon}\right) &\int dp\, dq \frac{g_{\mu\nu} p^\mu p^\nu}{(k-p)^2(k-q)^2(p-q)^2 p^2 q^2} + \left(-i\frac{1}{4\varepsilon}\right)^2 \int dq \frac{g_{\mu\nu} g^{\mu\nu}}{(k-q)^2 q^2} \Bigg\}.\end{aligned} \tag{3.9}$$

At this stage, we are free to do any kind of infrared rearrangement, so we set $k = 0$, but to avoid the introduction of spurious infrared divergencies, we introduce an auxiliary external momentum l in the q-loop.[1] After this infrared rearrangement the integrals are easily evaluated by (3.1) giving

[1] In [28], where this technique was developed, auxiliary masses were introduced rather then momentum. However, this requires the inclusion of the massive versions of the one-loop type (3.1) integrals, making the programming more complicated.

$$(4\pi)^6 K \left\{ \int dp\, dq\, dr \frac{(p\cdot r)^2}{(l-q)^2(p-q)^2(q-r)^2 p^4 q^2 r^4} + \right. \tag{3.10}$$
$$\left. 2i\left(-\frac{1}{4\varepsilon}\right) \int dp\, dq \frac{1}{(l-q)^2(p-q)^2 p^2 q^2} + \left(-i\frac{1}{4\varepsilon}\right)^2 \int dq \frac{4-2\varepsilon}{(l-q)^2 q^2} \right\} =$$
$$\frac{1}{12\varepsilon^3} - \frac{1}{6\varepsilon^2} + \frac{1}{12\varepsilon}.$$

Lo and behold, this result does not contain any logarithmic dependence on l. The subtracted integrals can be computed without the infrared rearrangement and can be added to this result to give the value for (3.6):

$$K \int dp\, dq\, dr \frac{(p\cdot r)^2}{(k-p)^2(k-q)^2(k-r)^2(p-q)^2(q-r)^2 p^2 q^2 r^2} = \tag{3.11}$$
$$\left(i\frac{k^{-2\varepsilon}}{16\pi^2}\right)^3 \left(\frac{1}{12\varepsilon^3} + \frac{17}{24\varepsilon^2} + \frac{49}{12\varepsilon}\right).$$

In this result the dependence on k can be found on dimensional grounds. To see why the terms $\log(4\pi)$, γ_E, $\zeta(2)$ disappeared, see the next section.

Example of a quadratically divergent essentially three-loop type integral

Consider

$$\int dp\, dq\, dr \frac{q^4}{(k-p)^2(k-q)^2(p-q)^2(p-r)^2(q-r)^2 p^2 r^2}, \tag{3.12}$$

which is overall quadratically divergent, containing a quadratically divergent one-loop subintegral (the q-loop), a quadratically divergent two-loop subintegral (the $q-r$-loop) and a logarithmically divergent two-loop subintegral (the $p-q$-loop). As for the latter one, it multiplies a tadpole which is independent of the external momentum ($\int dr 1/r^2$) and, therefore, is not included in any infrared rearrangement and can be dropped. The pole part of the one-loop subintegral is

$$(4\pi)^2 K \int dq \frac{q^4}{(k-q)^2(p-q)^2(q-r)^2} = \\ A\left(\frac{1}{\varepsilon}\right)(k^2+p^2+q^2) + B\left(\frac{1}{\varepsilon}\right)(k\cdot p + k\cdot r + p\cdot r). \tag{3.13}$$

Differentiation with respect to k twice gives an integral from which A can be computed, while differentiation with respect to k and p gives an integral from which B can be obtained. Using the relations

$$\frac{\partial^2}{\partial k_\mu \partial k_\mu}\frac{1}{(k-q)^2} = \frac{4\varepsilon}{(k-q)^4}, \quad \frac{\partial^2}{\partial k_\mu \partial p_\mu}\frac{1}{(k-q)^2(p-q)^2} = \frac{4(k-q)\cdot(p-q)}{(k-q)^4(p-q)^4}, \tag{3.14}$$

and setting all but one external momenta to zero — keep k, say —, we get

$$A = K\frac{1}{8-4\varepsilon}\int dq \frac{4\varepsilon}{(k-q)^2 q^2} = 0, \tag{3.15}$$

$$B = K\frac{1}{4-2\varepsilon}\int dq \frac{4}{(k-q)^2 q^2} = i\frac{1}{\varepsilon}. \tag{3.16}$$

The action of KR' on the $q-r$-loop is as follows:

$$(4\pi)^4 K\left\{\int\int dq\, dr \frac{q^4}{(k-q)^2(p-q)^2(p-r)^2(q-r)^2 r^2} - \\ i\frac{1}{\varepsilon}\int dr \frac{k\cdot p + k\cdot r + p\cdot r}{(p-r)^2 r^2}\right\} = A\left(\frac{1}{\varepsilon}\right)k^2 + B\left(\frac{1}{\varepsilon}\right)p^2 + C\left(\frac{1}{\varepsilon}\right)k\cdot p. \tag{3.17}$$

To calulate A (B) one differentiates with respect to k (p) twice, for C once with respect to k and once with respect to p. After proper infrared rearrangement the integrals can be evaluated by means of (3.1), the result of these operations is

$$A = i\frac{1}{4\varepsilon},\ B = -i\left(\frac{1}{4\varepsilon^2}+\frac{3}{8\varepsilon}\right),\ C = -i\left(\frac{3}{4\varepsilon^2}+\frac{7}{8\varepsilon}\right). \tag{3.18}$$

Finally, the action of KR' on (3.12) gives a quadratic polynomial in k:

$$(4\pi)^6 K \left\{ \int dp\, dq\, dr \frac{q^4}{(k-p)^2(k-q)^2(p-q)^2(p-r)^2(q-r)^2p^2r^2} - \right. \tag{3.19}$$
$$i\frac{1}{\varepsilon}\int dr \frac{k\cdot p + k\cdot r + p\cdot r}{(k-p)^2(p-r)^2p^2r^2} - i^2\frac{k^2}{4\varepsilon}\int dp \frac{1}{(k-p)^2p^2} -$$
$$\left. i^2\left(-\frac{1}{4\varepsilon^2}+\frac{3}{8\varepsilon}\right)\int dp \frac{1}{(k-p)^2} - i^2\left(-\frac{3}{4\varepsilon^2}+\frac{7}{8\varepsilon}\right)\int dp \frac{k\cdot p}{(k-p)^2p^2} \right\}.$$

Notice that the tadpole depending on the external momentum ($4th$ term) was not dropped, for differentiation with respect to k twice and infrared rearrangement will make a perfect one-loop integral of the type (3.1) out of it. The same procedure as before yields

$$(4\pi)^6 K R' \int dp\, dq\, dr \frac{q^4}{(k-p)^2(k-q)^2(p-q)^2(p-r)^2(q-r)^2p^2r^2} = \tag{3.20}$$
$$i^3 k^2 \left(\frac{1}{8\varepsilon^3} - \frac{23}{48\varepsilon^2} + \frac{9}{16\varepsilon}\right).$$

Then, adding back what we subtracted in (3.19) — those integrals are recursively of the one-loop type — one finds

$$K \int dp\, dq\, dr \frac{q^4}{(k-p)^2(k-q)^2(p-q)^2(p-r)^2(q-r)^2p^2r^2} = \tag{3.21}$$
$$k^2\left(i\frac{k^{-2\varepsilon}}{16\pi^2}\right)^3 \left(\frac{1}{8\varepsilon^3} + \frac{49}{48\varepsilon^2} + \frac{177}{32\varepsilon}\right).$$

Example of three-loop integral which contains essentially two-loop type subintegral

There is a class of integrals which contain the essentially two-loop type subintegral

$$\int dp\, dq\, dr \frac{1}{(k-p)^2(k-q)^2(p-q)^2p^2q^2}. \tag{3.22}$$

These integrals are readily evaluated by the method described above. However, I should like to point out that an analytic expression was derived in [5] by the Gegenbauer x-space technique which is more advantegeous for programming purposes.

These formulae are given in Appendix D. Also, there are three-loop integrals which contain a loop which can be evaluated by means of (3.1), leaving an essentially two-loop type integral as result. In this case, one of the propagators in (3.22) has ε in the exponent. The formulae in [5] were derived for integer values of the exponents. However, analytic continuation assures that one can use the expansion formulae for non-integer exponents, too. To check the results obtained by the expansion of the formulae given in [5], I computed these type of integrals by the method described in the previous subsections and found full agreement.

To conclude this section, I should like to mention that there are rare cases when the diagrams are UV finite but IR singular. They occur in massless theories such as the one considered here. It is well-known that dimensional regularization can be used in this case, too [12]. However, it is not clear how one can regularize an integral which is IR divergent but can not be computed by recursive use of (3.1). Consider, for example,

$$\int d^{(4-2\varepsilon)}p\, d^{(4-2\varepsilon)}q\, d^{(4-2\varepsilon)}r \frac{1}{(k-p)^2(k-q)^2(p-q)^2(p-r)^2(q-r)^2p^2q^2r^2}. \quad (3.23)$$

This integral is UV finite as can easily be seen by power counting, but IR divergent in the region when p, q, r approaches zero at the same rate. The cure here is to multiply the integral by k^2 and write this as $k^2 = (k-p)^2 + 2k\cdot p - p^2$. The last two terms give IR (and UV) finite integrals. The first one is still IR divergent, but we repeate the trick, this time with $k^2 = (k-q)^2 + 2k\cdot q - q^2$ obtaining UV divergent integrals which can be computed as described above. For massless theories one can always trade an IR divergence for a UV one what reflects the fact that there is no intrinsic scale (a mass) which would separate the IR and UV regions clearly.

Further details on the three-loop calculation

The G-scheme

By the expansion of (3.1) in ε, one can see that the usual measure $d^{(4-2\varepsilon)p}/(2\pi)^{(4-2\varepsilon)}$ leads to messy factors of $\log(4\pi)$, γ_E (Euler's constant) and $\zeta(2)$. These terms in the expansion can be removed by including an extra factor of $\Gamma(1-\varepsilon)/(4\pi)^{\varepsilon}$ in the integration measure.[2] We are allowed to do this as long as we are only interested in regularizing the four dimensional theory, but not if we want the results for arbitrary integer values of space time dimension n. It will be convenient to define

$$G(m,n,\alpha,\beta) = \frac{\Gamma(1-\varepsilon)\Gamma(\alpha+\beta-2+\varepsilon)}{\Gamma(\alpha)\Gamma(\beta)} B(m+2-\alpha-\varepsilon, n+2-\beta-\varepsilon). \quad (3.24)$$

In terms of G and with the new integration measure the one-loop integration formula (3.1) is simply $\frac{i}{(4\pi)^2} k^{(2-\alpha-\beta-\varepsilon)} G(0,0,\alpha,\beta)$, and the one-loop integrals which contain tensorial expressions in their numerator can be expressed in terms of $G(m,n,\alpha,\beta)$ (see Appendix D). Then one derives the recurrence relation for G from the recurrence relation of the Γ-function. The results are

$$G(m,n,\alpha,\beta) = \frac{n+1-\beta-\varepsilon}{m+n+3-\alpha-\beta-2\varepsilon} G(m,n-1,\alpha,\beta), \quad (3.25)$$

$$G(m,0,\alpha,\beta) = \frac{m+1-\alpha-\varepsilon}{m+3-\alpha-\beta-2\varepsilon} G(m-1,0,\alpha,\beta), \quad (3.26)$$

$$G(0,0,\alpha,\beta) = \frac{(\alpha+\beta-3+\varepsilon)(4-\alpha-\beta-2\varepsilon)}{(\beta-1)(2-\beta-\varepsilon)} G(0,0,\alpha,\beta-1), \quad (3.27)$$

$$G(0,0,\alpha,\beta) = G(0,0,\beta,\alpha). \quad (3.28)$$

The next step is to find the expansion formula. The Γ-functions are expanded in ε in the usual way, namely using [11]

$$\Gamma(1-z) = \exp\left\{\gamma z + \sum_{n=2}^{\infty} \frac{\zeta(n)}{n} z^n\right\}. \quad (3.29)$$

[2]This means nothing more than that I choose to use the $\overline{MS}$ renormalization scheme.

The expansion formula for G is as follows:

$$G(0,0,1+a\varepsilon,1+b\varepsilon) = G(0,0,1,1)\frac{1}{a+b+1}[1+(a+b)\varepsilon+(a+b)(a+b+2)\varepsilon^2]. \quad (3.30)$$

Practice shows that the use of this scheme speeds up the expansion of three-loop formulas 10 to 15 times, and this rate grows exponantially by the number of loops.

Algorithm to find all topologically different diagrams

To derive the expression contributing to an n-point function at a certain order in the perturbation parameter, one has to differentiate the generating functional with respect to the fields n-times. At higher orders this procedure becomes rather tedious. To simplify this part of the calculations, Feynman invented his rules. The integrals then are represented by diagrams out of which all the topologically different ones are to be found. At the one-loop level this is a trivial task, but at higher loops one needs a well-defined strategy to find all the topologically different diagrams. The algorithm used in this calculation was as follows.

For gauge theories one has to find the topologically different diagrams containing gauge fields only and then the total set of the diagrams can be obtained by substituting ghost loops for gauge field loops. At the one-loop level there is only one diagram depicted in figure B. 1. Then to proceed from the n-loop level to the $(n+1)$-loop level, one identifies all the topological objects of a diagram and connects these objects by propagator lines as permitted by the vertices of the model. This procedure yields a good number of new diagrams, a great fraction of which will be identical. To see an example, we derive the two-loop diagrams from the one-loop one.

The different topological objects of figure B. 1 are

- the two external legs,
- the two vertices and
- the two propagators.

These six objects can be connected by an additional propagator line in $5! = 120$ different ways, all permitted by the Feynman rules. It is easy to see then that out of these 120 diagrams, there are only 6 which are different.

A different method to obtain the possible diagrams which is not so well algorithmized is good though to check the result obtained by the procedure described above. One writes down the collection of possible vertex configurations and then makes all the different connections among the vertices. For instance, there are four different possible vertex configurations for the three-loop background field propagator. These are

- three four-point vertices,
- two four-point vertices and two three-point vertices,
- one four-point vertex and four three-point vertices and
- six three-point vertices.

The total number of topologically different three-loop diagrams (containing gauge lines only) is 49; they are given in Appendix C. Thirty diagrams can be grouped into 15 pairs such that one member of a pair can be obtained from the other by reflection with respect to a vertical axis going through the middle of the diagram. In other words this means that the members of a pair differ only in the interchange of the Lorentz and group indices of the external legs in which the propagators are symmetric. This means that one has to evaluate 34 diagrams.

Having derived all topologically different diagrams containing gauge fields, one finds all the diagrams by substituting ghost loops for gauge loops (with two different directions of the ghost lines).

Results and conclusions

The tensorial structure of the diagrams is the same as in the two-loop calculation (see (2.51)). For the sum of the diagrams one can derive constraints from renormalization group analysis just as in the two-loop case. By using (2.45) and the expansion

$$\beta = -g_r \left[\beta_0 \left(\frac{g_r}{4\pi}\right)^2 + \beta_1 \left(\frac{g_r}{4\pi}\right)^4 + \beta_2 \left(\frac{g_r}{4\pi}\right)^6\right], \tag{3.31}$$

we find that up to three loops Z_A must be

$$Z_A = 1 + \frac{\beta_0}{\varepsilon}\left(\frac{g_r}{4\pi}\right)^2 + \frac{\beta_1}{2\varepsilon}\left(\frac{g_r}{4\pi}\right)^4 + \frac{\beta_2}{3\varepsilon}\left(\frac{g_r}{4\pi}\right)^6 - \frac{\beta_0\beta_1}{6\varepsilon^2}\left(\frac{g_r}{4\pi}\right)^6, \tag{3.32}$$

i. e., there is no third order pole and the coefficient of the second order pole can be calculated from the lower order results.

There is an efficient way of calculating the structure constant traces [7] by hand. However, I found it safer to do it on computer by simple numeric multiplication of the matrices representing the structure constants. This would be an impossible task to perform in a reasonable time if the dependence on N (in $SU(N)$) were not trivial. Since all the three-loop diagrams are proportional to N^3, all we need to know is the proportionality factor. This can be obtained by numerical multiplication of Levi-Civita tensors — the structure constants of $SU(2)$ in its adjoint representation.

The sums of the three-loop diagrams, two-loop counter-term diagrams and one-loop counter-term diagrams are transverse separately in my calculation. Also, none of these contain third order poles. However, for the coefficient of the second order pole I have not obtained the correct answer yet; therefore, the result for β_2 cannot be accepted. The expression for the sum of the three-loop diagrams (together with the gauge parameter renormalization) that I obtained is

$$\frac{ig^6 C_A^3 \delta^{ab}}{(4\pi)^6}\left(\frac{44}{27\varepsilon^2}+\frac{8969}{81\varepsilon}-\frac{22}{\varepsilon}\zeta(3)\right)[g_{\mu\nu}k^2-k_\mu k_\nu]. \tag{3.33}$$

This expression is transverse and does not contain a $1/\varepsilon^3$ pole as expected. These are non-trivial cancellations, which assures that most of the calculation must be correct.

Chapter 4

A possibility to avoid gauge fixing in the quantization of non-Abelian gauge fields

Introduction

In the first two chapters I described the conventional usage of the background field method, namely when the background field is treated perturbatively. However, in principle it is possible to treat the background field exactly. In this case one must use the exact quantum field propagator in some specified background. The use of the exact quantum field propagator makes the gauge fixing unnecessary, for

- the transformation property of the quantum field is that of a matter field (2.24),
- the inverse propagator for the quantum field is not a projection operator,

so the two features of the conventional method which made the gauge fixing necessary are not valid any more. The search for the exact propagator is a formidable task except for some very simple background field configurations. For example, it was used to generate the effective potential for a scalar field theory [19] for constant background field, and was used to calculate the effective action for covariantly constant gauge fields [22]. One has to keep in mind that the background field must be a solution of the classical field equations. Of course, if one posesses the exact propagator on a fixed background, one can still use perturbation expansions to improve the calculations for arbitrary background field. The rationale for finding the exact propagator in some non-zero field is that one expects that the renormalization procedure does not depend on the background field, and so it can be carried out more easily. In the next section I describe briefly what could be a possible way of quantizing Yang-Mills theories in a fully gauge-covariant manner.

Quantization of non-Abelian gauge fields without gauge fixing

The generating functionals — $\tilde{Z}$, $\tilde{W}$, and $\tilde{\Gamma}$ — are defined as in the first chapter (see (2.8) except the gauge fixing term — $\mathcal{L}_{GF}$ — is missing:

$$\tilde{Z}[J,A] = \int \delta Q \exp i \int d^4x \left[\mathcal{L}(A+Q) + J^a_\mu Q^a_\mu\right]. \tag{4.1}$$

The gauge was not fixed in this definition, and there are no ghosts in the theory. This means that this background field generating functional is invariant under the infinitesimal transformations

$$\delta A^a_\mu = -f^{abc}\omega^b A^c_\mu + \frac{1}{g}\partial_\mu \omega^a, \tag{4.2}$$

$$\delta J^a_\mu = -f^{abc}\omega^b J^c_\mu \tag{4.3}$$

together with the change of integration variables

$$Q^a_\mu \to Q^a_\mu - f^{abc}\omega^b Q^c_\mu. \tag{4.4}$$

As a consequence, $\tilde{W}[J, A]$ is also invariant under the (4.2) – (4.4) gauge transformations. It then follows that the background field effective action is invariant under

$$\delta A^a_\mu = -f^{abc}\omega^b A^c_\mu + \frac{1}{g}\partial_\mu \omega^a, \tag{4.5}$$

$$\delta \tilde{Q}^a_\mu = -f^{abc}\omega^b \tilde{Q}^c_\mu, \tag{4.6}$$

i. e., $\tilde{\Gamma}[0, A]$ must be a gauge-invariant functional of A itself. To find the propagator we expand the Lagrangian $\mathcal{L}(A+Q)$ using

$$F^a_{\mu\nu}(A+Q) = F^a_{\mu\nu}(A) + D^{ac}_{[\mu}[A^b]Q^c_{\nu]} + gf^{abc}Q^b_\mu Q^c_\nu. \tag{4.7}$$

Then one obtaines

$$\begin{aligned} \mathcal{L}(A+Q) \;=\; & -\frac{1}{4}[(F^a_{\mu\nu}(A))^2 + 2D^{ac}_{[\mu}(A)Q^c_{\nu]}F^{a\mu\nu}(A) && (4.8) \\ & +(D^{ac}_{[\mu}(A)Q^c_{\nu]})^2 + 2gf^{abc}Q^b_\mu Q^c_\nu F^{a\mu\nu}(A) \\ & +2gD^{ac}_{[\mu}(A)Q^c_{\nu]}f^{abc'}Q^b_\mu Q^{c'}_\nu + g^2 f^{abc}f^{ab'c'}Q^b_\mu Q^{b'\mu}Q^c_\nu Q^{c'\nu}]. && (4.9) \end{aligned}$$

The first line of (4.9) does not contribute to $\tilde{\Gamma}[0, A]$. Now, as usual the propagator is the inverse of the operator which couples two quantum fields together; the rest is the perturbation. The background field being present, we have more terms quadratic in Q than usual. However, the terms that contain the background field also contain the coupling and so they appear to be interactions, and in fact were treated that way in previous chapters. Here we make the substitution

$$gA \rightarrow \mathcal{A}. \tag{4.10}$$

Upon this substitution the Lagrangian becomes

$$\begin{aligned}\mathcal{L}(\mathcal{A}+Q) &= -\frac{1}{4}[(\frac{1}{g^2}\mathcal{F}^a_{\mu\nu}(\mathcal{A}))^2 + 2\mathcal{D}^{ac}_{[\mu}(\mathcal{A})Q^c_{\nu]}\frac{1}{g}\mathcal{F}^{a\mu\nu}(\mathcal{A}) \\ &+(\mathcal{D}^{ac}_{[\mu}(\mathcal{A})Q^c_{\nu]})^2 + 2f^{abc}Q^b_\mu Q^c_\nu \mathcal{F}^{a\mu\nu}(\mathcal{A}) \\ &+2g\mathcal{D}^{ac}_{[\mu}(\mathcal{A})Q^c_{\nu]}f^{abc'}Q^b_\mu Q^{c'}_\nu + g^2 f^{abc}f^{ab'c'}Q^b_\mu Q^{b'\mu}Q^c_\nu Q^{c'\nu}],\end{aligned} \tag{4.11}$$

where

$$\mathcal{D}^{ac}_\mu(\mathcal{A}) = \partial_\mu \delta^{ac} + f^{abc}\mathcal{A}^b_\mu \tag{4.12}$$

and

$$\mathcal{F}^a_{\mu\nu}(\mathcal{A}) = \partial_{[\mu}\mathcal{A}^a_{\nu]} + f^{abc}\mathcal{A}^b_\mu \mathcal{A}^c_\nu. \tag{4.13}$$

The second line of (4.12) is quadratic in Q, and upon integration by parts becomes

$$-\frac{1}{2}Q^{a\mu}[(-\delta_{\mu\nu}\Box(\mathcal{A}) + \mathcal{D}_\nu(\mathcal{A})\mathcal{D}_\mu(\mathcal{A}))^{ab} + f^{abc}\mathcal{F}^c_{\mu\nu}(\mathcal{A})]Q^{b\nu}. \tag{4.14}$$

The quantity in the bracket is the inverse propagator. The last line in (4.9) produces three vertices.

Clearly, the number of vertices is reduced substantially. However, there is a price to pay for this reduction — the propagator becomes very complicated. A closed form for the propagator has yet to be found.

Chapter 5
Summary

In this work, I discussed some calculational aspects of the quantization and renormalization of non-Abelian gauge field theories. The background field method was described in detail. This method is a very convenient way of quantizing locally symmetric field theories as compared to other more conventional ways. Some speculations were given as in what direction one could step further in using the background field method.

I presented explicit two and three-loop calculations in Yang-Mills theory. The complexity of the calculations required computer programming of Lorentz algebra manipulations and Feynman integral evaluations. This computer program is a byproduct of the work. Just before the completion of this work, I found that similar computer packages exist for the SCHOONSCHIP system [13] as well as for the REDUCE symbolic manipulation system [25]. These programs use different algorithms for the essentially three-loop type integrals [27], which enable them to compute the finite parts as well (necessary for four-loop calculations). The two-loop calculation is a new result in the sense that it is the first calculation in background field gauge with arbitrary value of the gauge fixing parameter. The three-loop calculation is also the first in background field gauge. However, unfortunately it is not finished yet.

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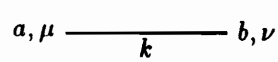
Appendix A

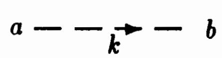
Feynman rules

Feynman rules for Yang-Mills theory in background field gauge

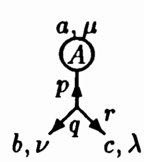
Solid lines are quantum gauge propagators, solid lines ending in a circle are external background fields and dashed lines are ghost propagators.



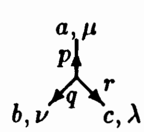
$$\frac{-i\delta^{ab}}{k^2+i\epsilon}\left[g_{\mu\nu} - (1-\alpha)\frac{k_\mu k_\nu}{k^2}\right]$$



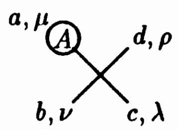
$$\frac{i\delta^{ab}}{k^2+i\epsilon}$$

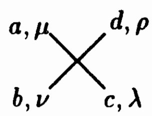


$$g f^{abc}\big[g_{\mu\lambda}(p - r - \tfrac{1}{\alpha}q)_\nu + g_{\nu\lambda}(r-q)_\mu$$
$$+ g_{\mu\nu}(q - p + \tfrac{1}{\alpha}r)_\lambda\big]$$

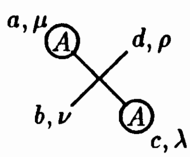


$$g f^{abc}\big[g_{\mu\lambda}(p - r)_\nu + g_{\nu\lambda}(r-q)_\mu$$
$$+ g_{\mu\nu}(q - p)_\lambda\big]$$

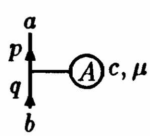




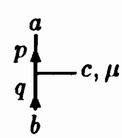
$$-ig^2[f^{abx}f^{xcd}(g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda})$$
$$+f^{adx}f^{xbc}(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho})$$
$$+f^{acx}f^{xdb}(g_{\mu\rho}g_{\nu\lambda}-g_{\mu\nu}g_{\lambda\rho})]$$



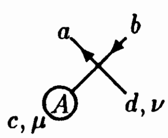
$$-ig^2[f^{abx}f^{xcd}(g_{\mu\lambda}g_{\nu\rho}-g_{\mu\rho}g_{\nu\lambda}+\tfrac{1}{\alpha}g_{\mu\nu}g_{\lambda\rho})$$
$$+f^{adx}f^{xbc}(g_{\mu\nu}g_{\lambda\rho}-g_{\mu\lambda}g_{\nu\rho}-\tfrac{1}{\alpha}g_{\mu\rho}g_{\nu\lambda})$$
$$+f^{acx}f^{xdb}(g_{\mu\rho}g_{\nu\lambda}-g_{\mu\nu}g_{\lambda\rho})]$$



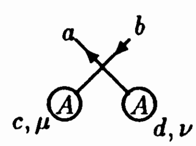
$$gf^{abc}(p+q)_\mu$$



$$gf^{abc}p_\mu$$

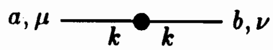


$$-ig^2f^{acx}f^{xdb}g_{\mu\nu}$$

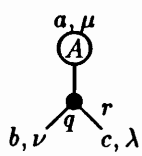


$$-ig^2(f^{acx}f^{xdb}+f^{adx}f^{xcb})g_{\mu\nu}$$

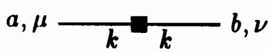
Counter-term vertices



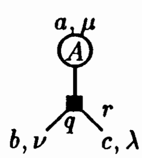
$$i\frac{1}{\epsilon}\left(\frac{g_r}{4\pi}\right)^2 C_A\left(\frac{13}{3}-\alpha_r\right)\delta^{ab}k_\mu k_\nu$$



$$-\frac{1}{\epsilon}\left(\frac{g_r}{4\pi}\right)^2 C_A\left(\frac{13}{3}-\alpha_r\right)$$
$$g f^{abc}[g_{\mu\nu}\frac{1}{\alpha}r_\lambda - g_{\mu\lambda}\frac{1}{\alpha}q_\nu]$$



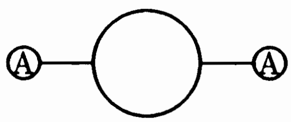
$$-i\left(\frac{g_r}{4\pi}\right)^4 C_A^2\left(\frac{1}{4\epsilon^2}\left(-\frac{13}{2}-\frac{17}{6}\alpha_r+\alpha_r^2\right)\right.$$
$$\left.+\frac{1}{8\epsilon}\left(\frac{59}{2}-\frac{11}{2}\alpha_r-\alpha_r^2\right)\right)\delta^{ab}k_\mu k_\nu$$



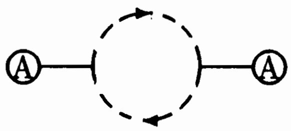
$$\left(\frac{g_r}{4\pi}\right)^4 C_A^2\left(\frac{1}{4\epsilon^2}\left(-\frac{13}{2}-\frac{17}{6}\alpha_r+\alpha_r^2\right)\right.$$
$$\left.+\frac{1}{8\epsilon}\left(\frac{59}{2}-\frac{11}{2}\alpha_r-\alpha_r^2\right)\right)$$
$$g f^{abc}[g_{\mu\nu}\frac{1}{\alpha}r_\lambda - g_{\mu\lambda}\frac{1}{\alpha}q_\nu]$$

Appendix B

Diagrams for calculation of the one and two-loop β functions



(a)



(b)

Figure B.1: Diagrams for one-loop calculation of the β function. Solid lines ending in a circle are background field lines.

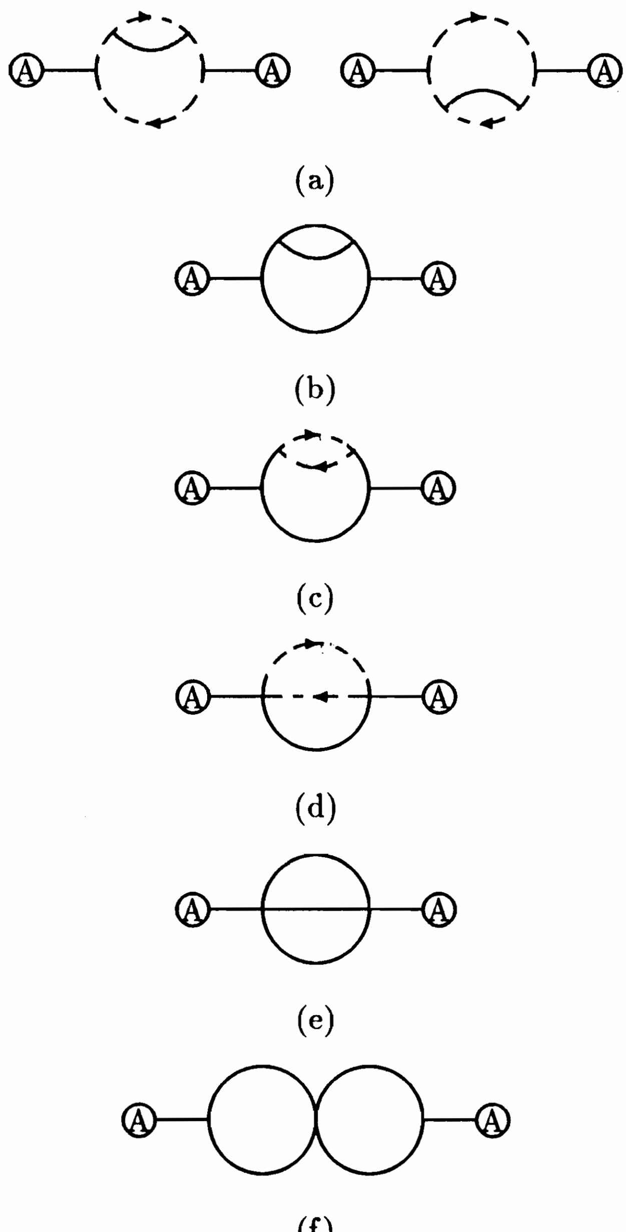
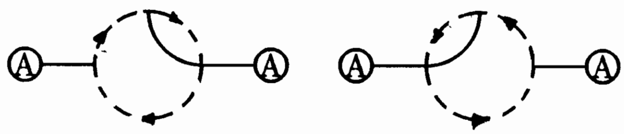
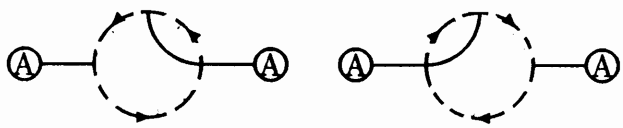


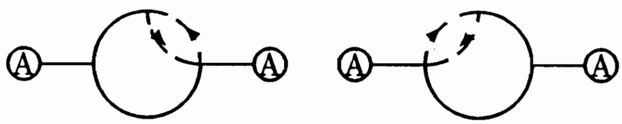
Figure B.2: Diagrams for two-loop calculation of the β function.



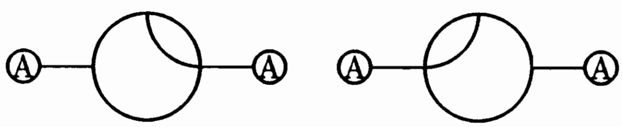
(g)



(h)



(i)



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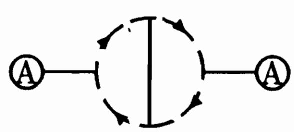

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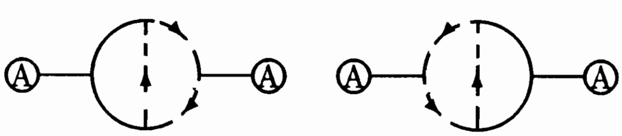
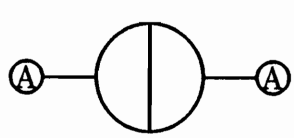
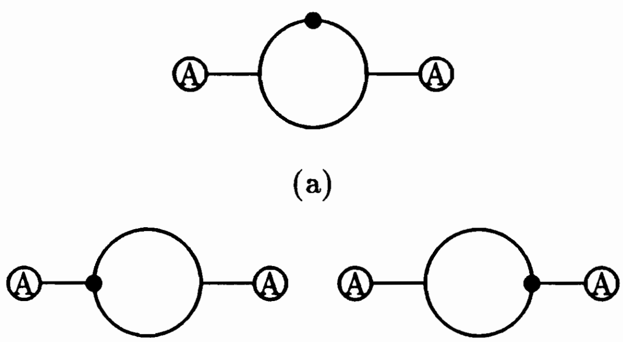


(l)



(m)

Figure B.2: Continued



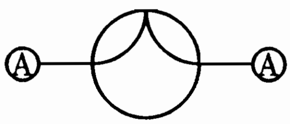
(a)

(b)

Figure B.3: Renormalization insertion diagrams to compute the contribution of the counter-term Lagrangian to the two-loop diagrams of figure B. 2.

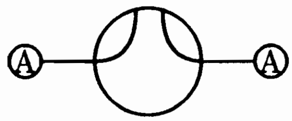
Appendix C

Diagrams for calculating the three-loop β function

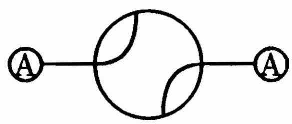


(a)

Figure C.1: Three-loop diagram with three four-gluon vertices.



(a)



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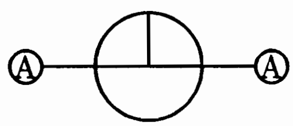

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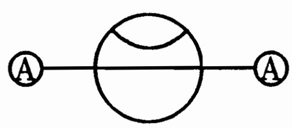

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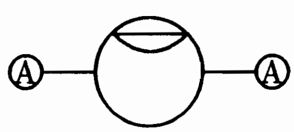
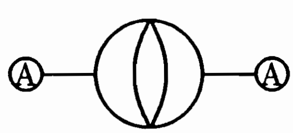
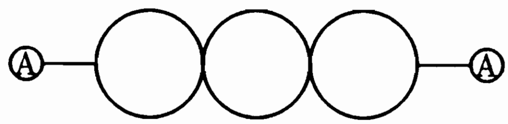


(e)



(f)

Figure C.2: Three-loop diagrams with two four-gluon vertices and two three-gluon vertices.



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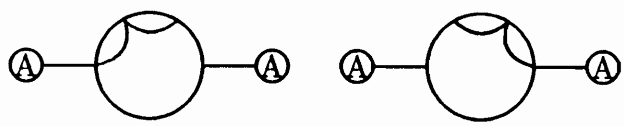

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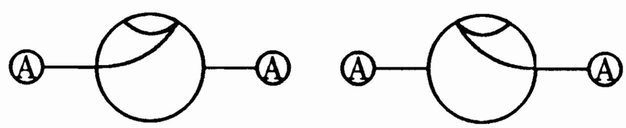
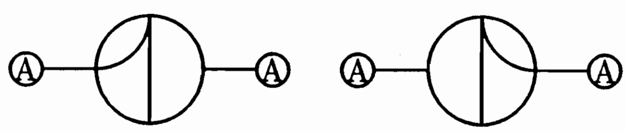


(i)



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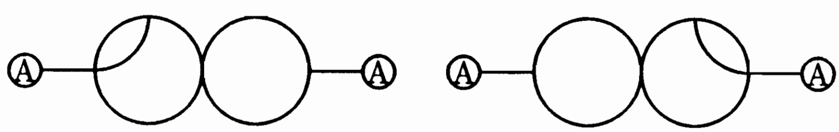


(k)

Figure C.2: Continued

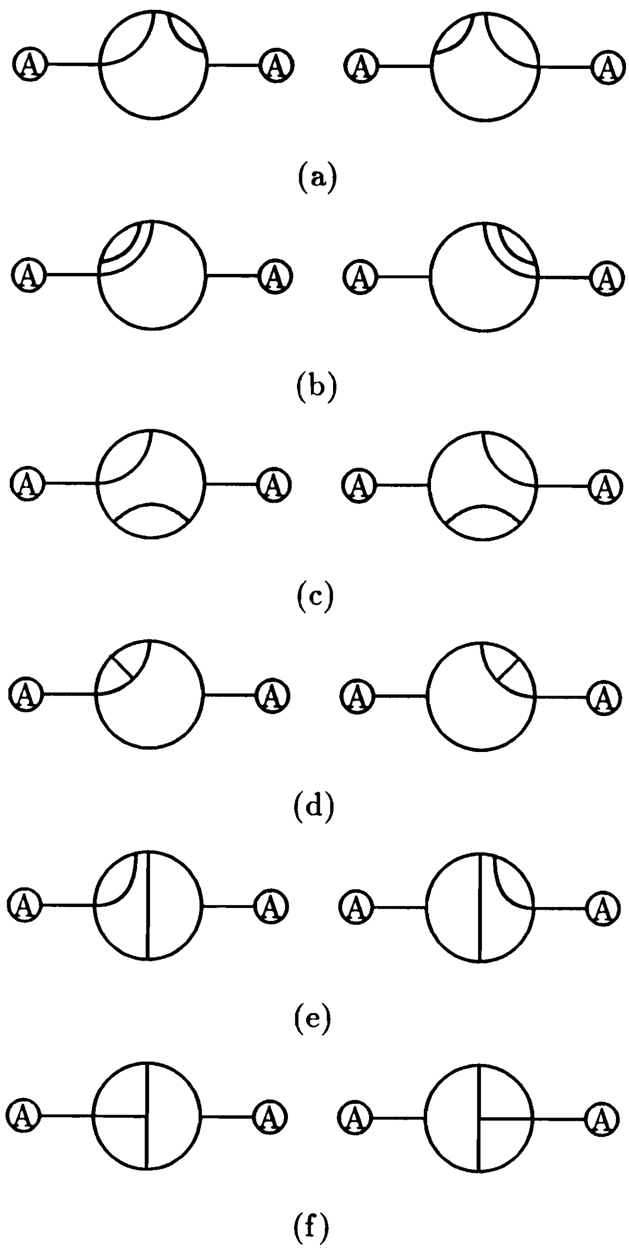
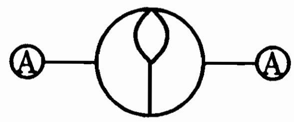
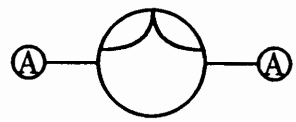


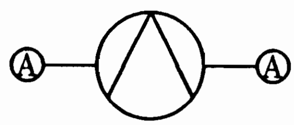
Figure C.3: Three-loop diagrams with one four-gluon vertex and four three-gluon vertices.



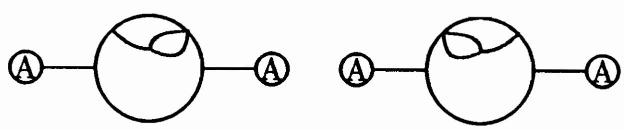
(g)



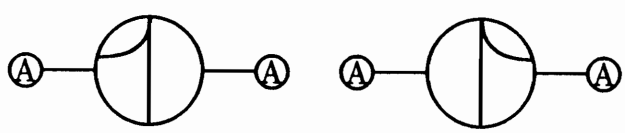
(h)



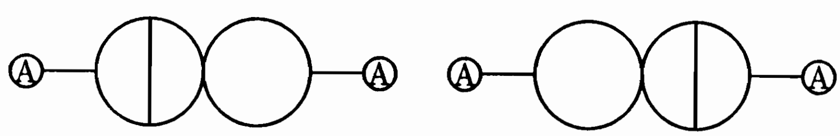
(i)



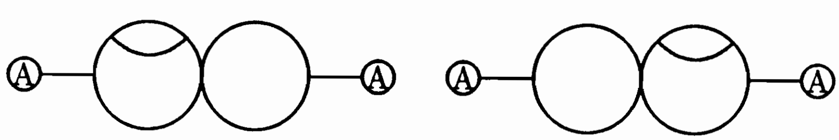
(j)



(k)



(l)



(m)

Figure C.3: Continued

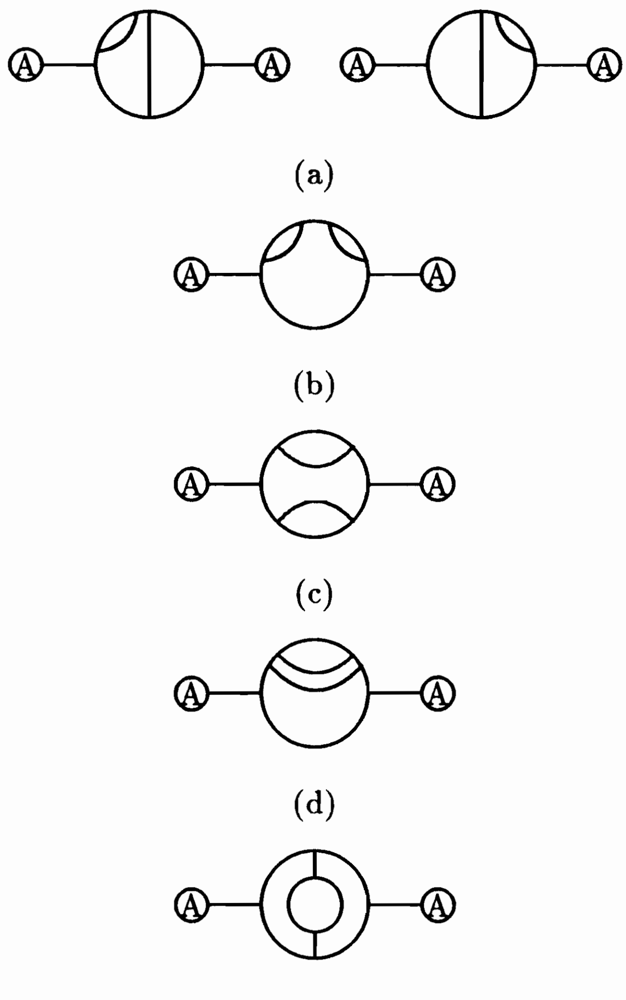
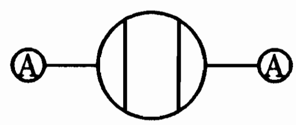
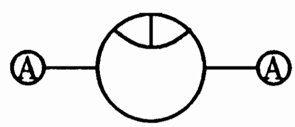


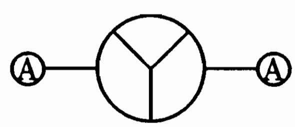
Figure C.4: Three-loop diagrams with six three-gluon vertices.



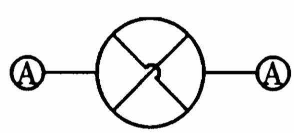
(f)



(g)

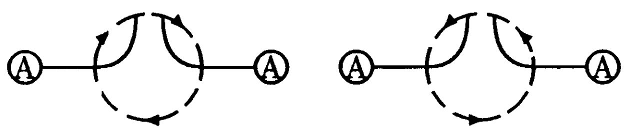


(h)

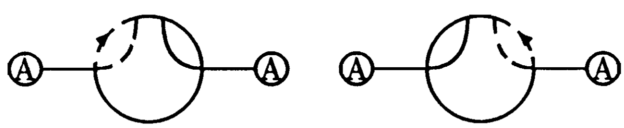


(i)

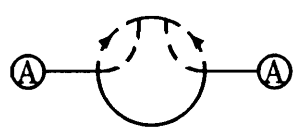
Figure C.4: Continued



(a)



(b)



(c)

Figure C.5: Ghost-loop insertion diagrams derived from diagram figure C. 2a.

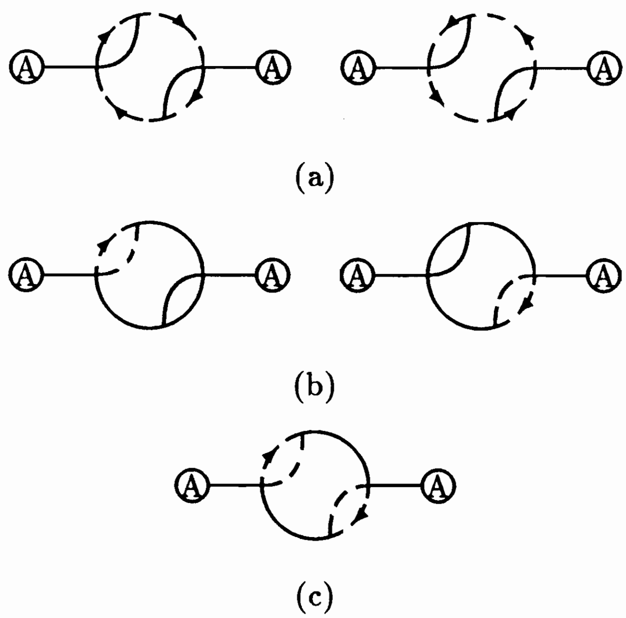


Figure C.6: Ghost-loop insertion diagrams derived from diagram figure C. 2b.

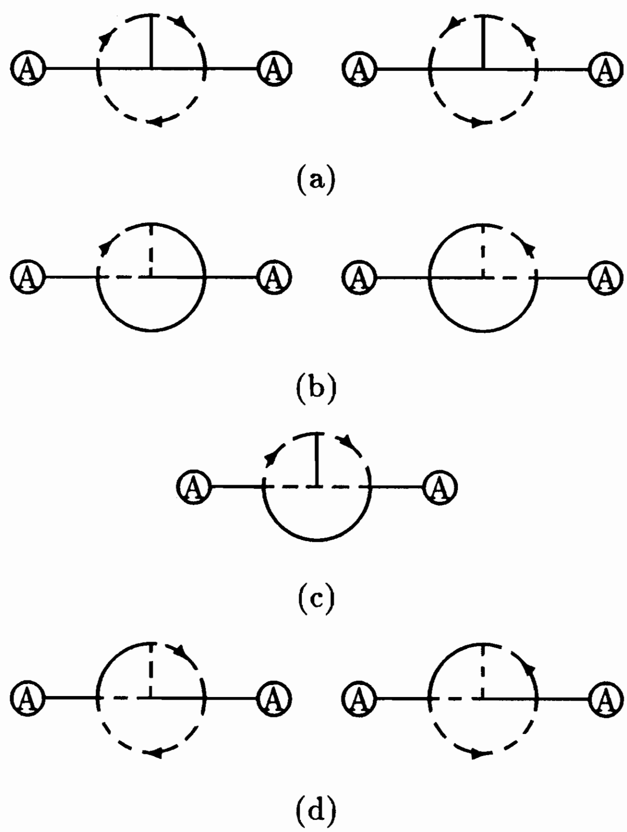
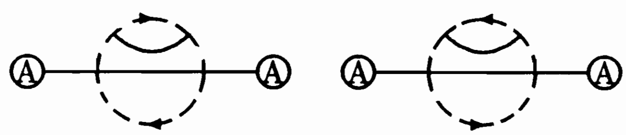
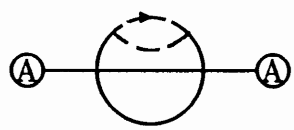


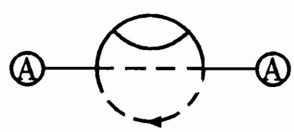
Figure C.7: Ghost-loop insertion diagrams derived from diagram figure C. 2c.



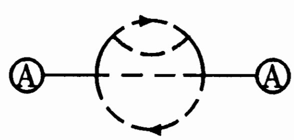
(a)



(b)

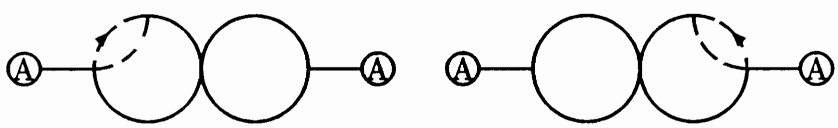


(c)



(d)

Figure C.8: Ghost-loop insertion diagrams derived from diagram figure C. 2d.



(a)

Figure C.9: Ghost-loop insertion diagrams derived from diagram figure C. 2k.

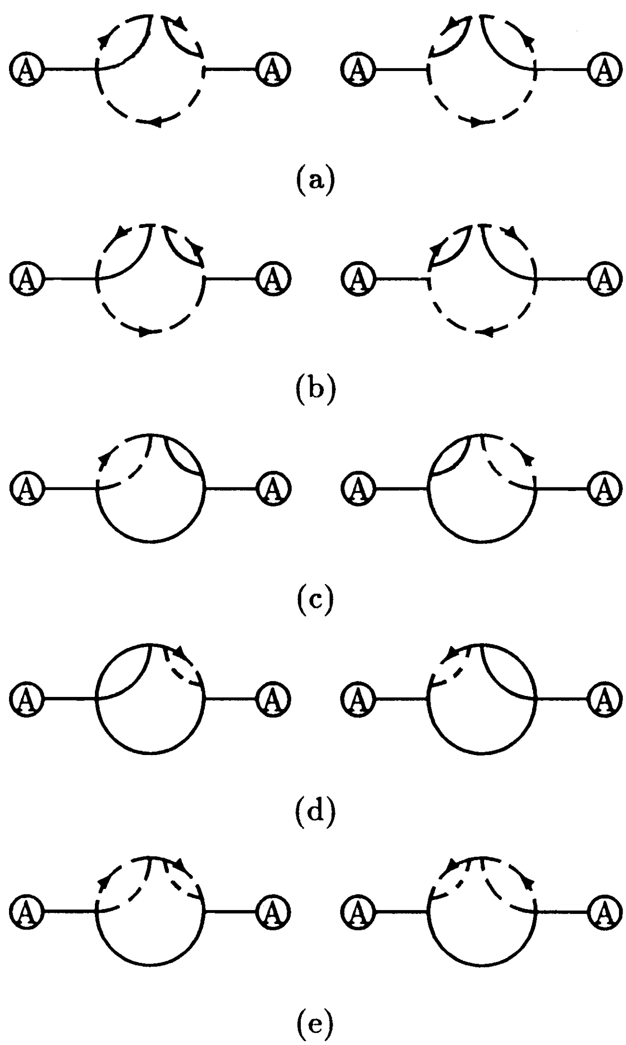


Figure C.10: Ghost-loop insertion diagrams derived from diagram figure C. 3a.

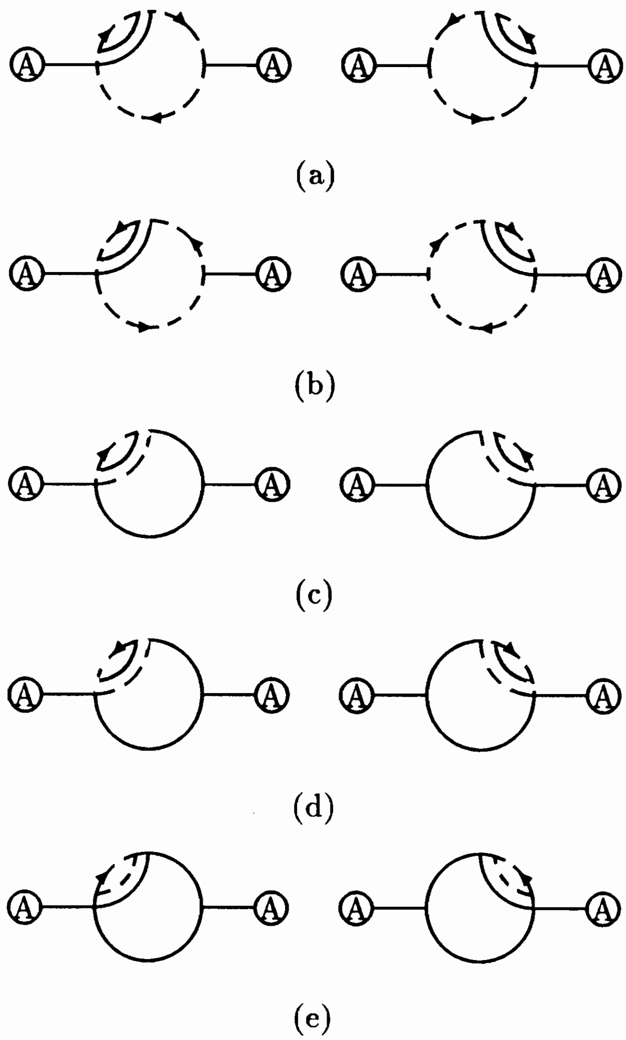
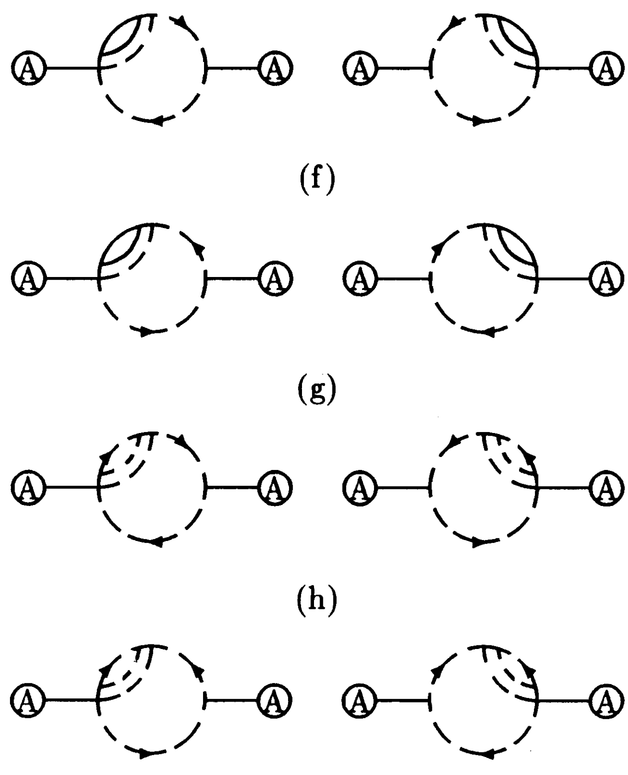


Figure C.11: Ghost-loop insertion diagrams derived from diagram figure C. 3b.



(f)

(g)

(h)

(i)

Figure C.11: Continued

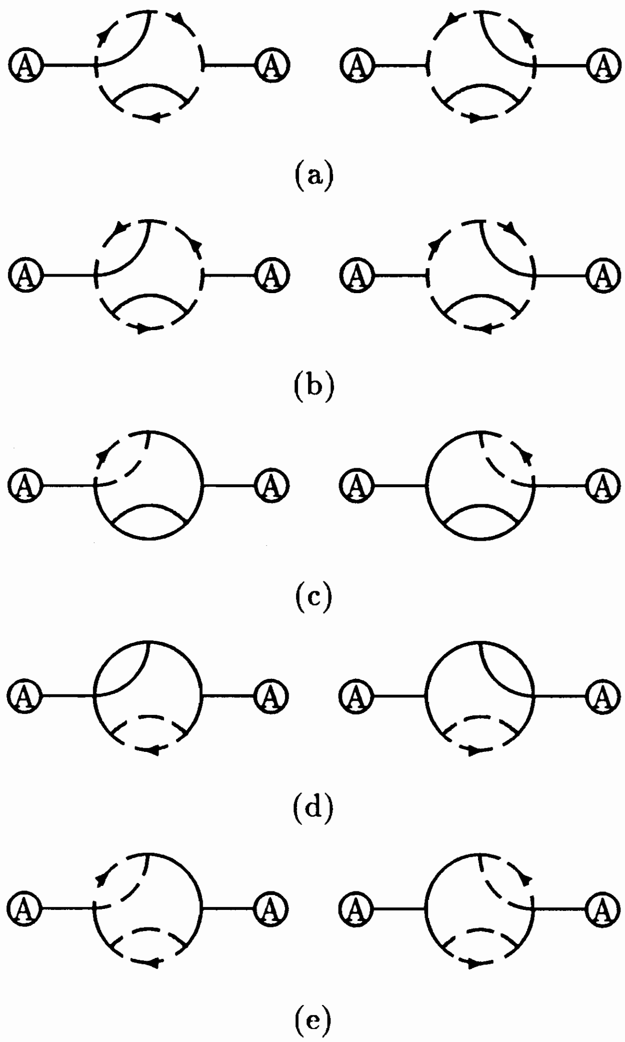


Figure C.12: Ghost-loop insertion diagrams derived from diagram figure C. 3c.

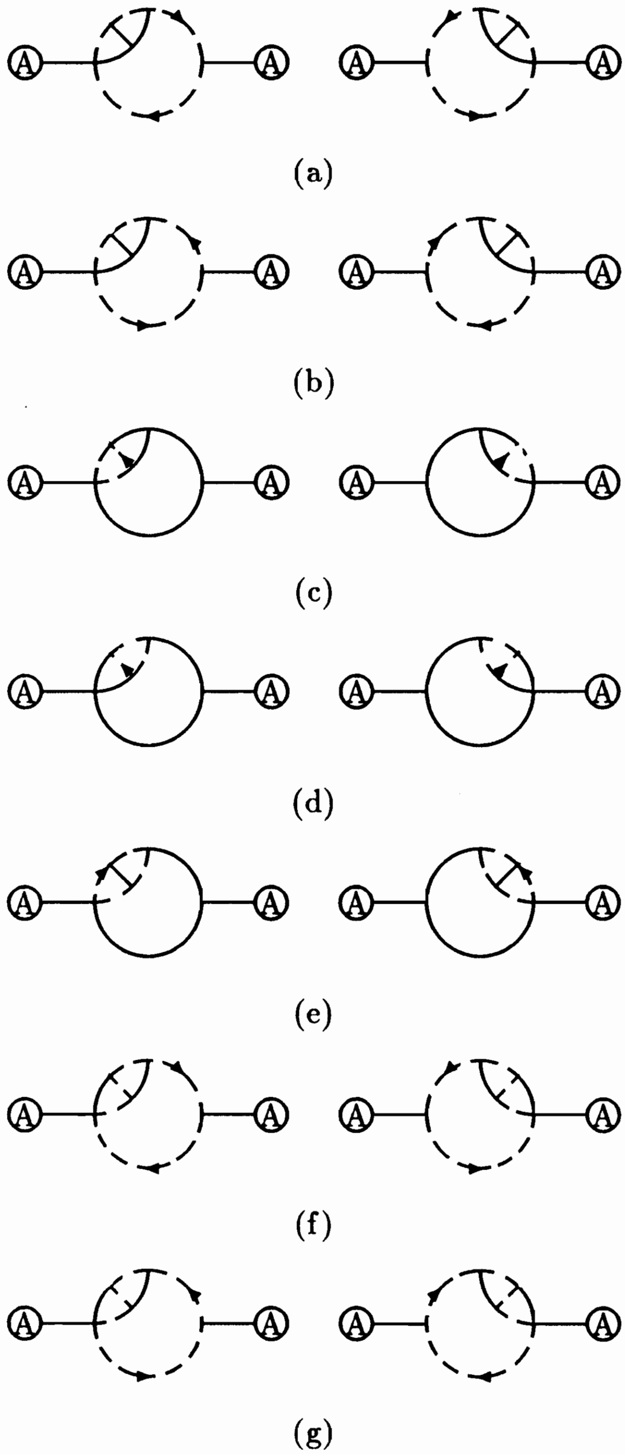


Figure C.13: Ghost-loop insertion diagrams derived from diagram figure C. 3d.

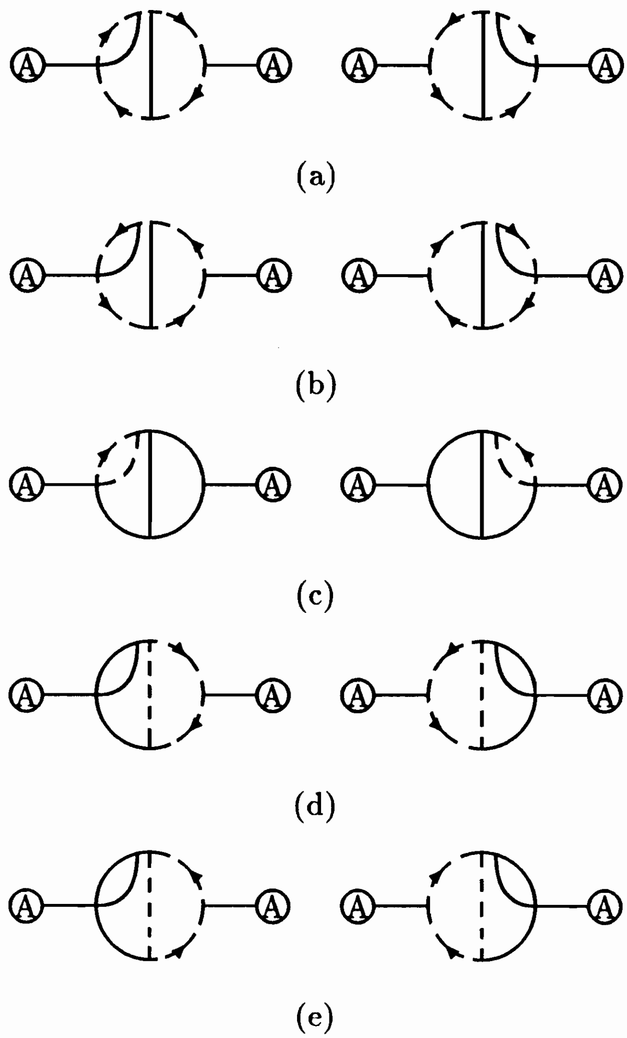


Figure C.14: Ghost-loop insertion diagrams derived from diagram figure C. 3e.

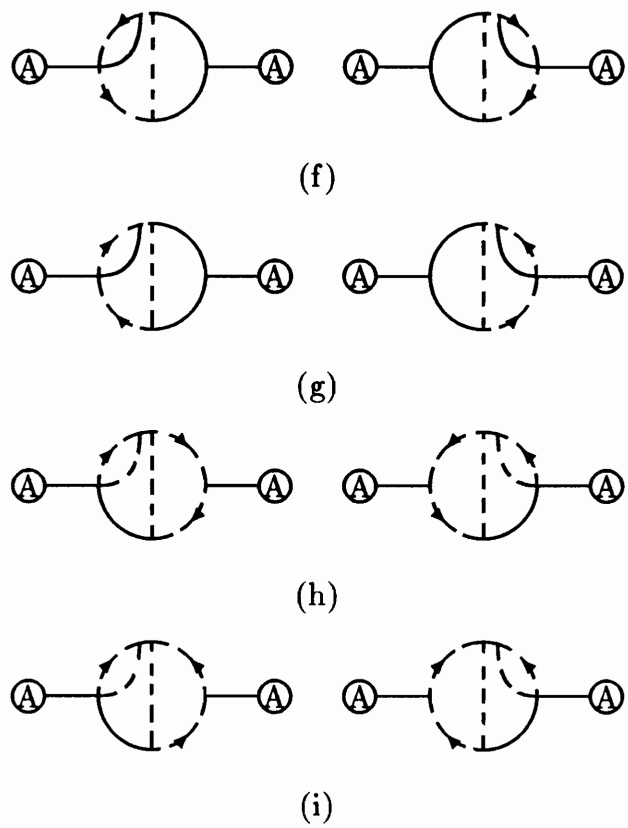
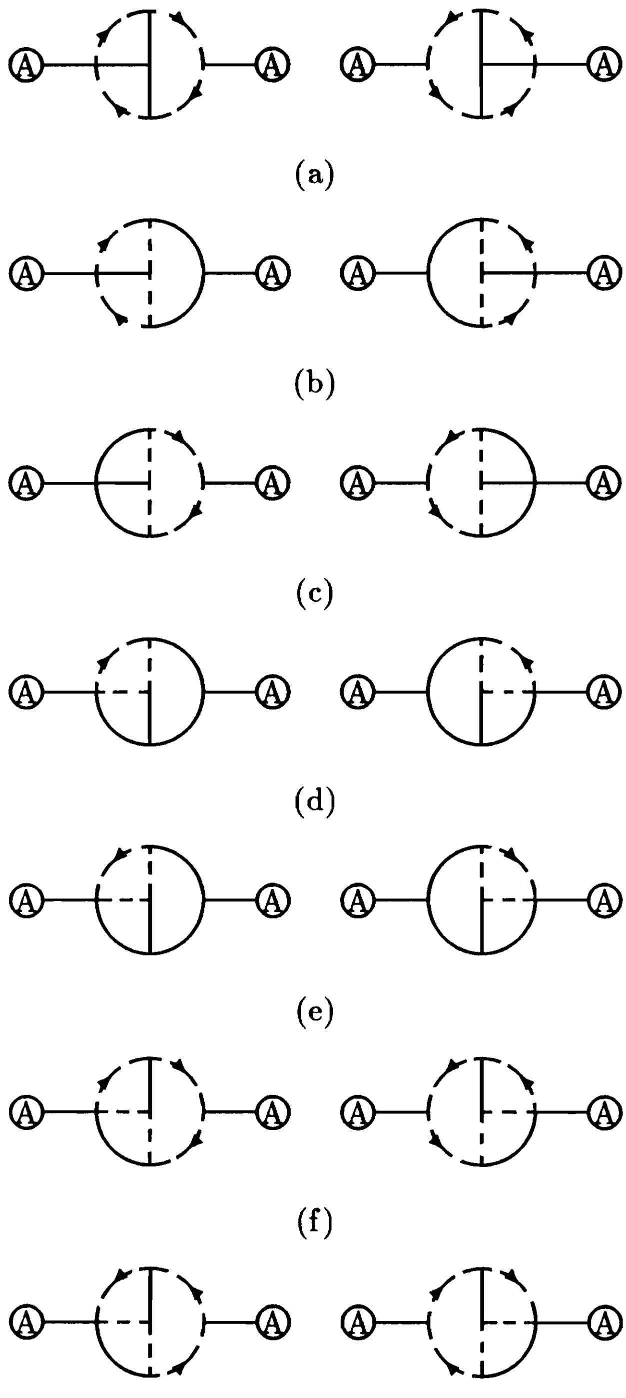


Figure C.14: Continued



(a)

(b)

(c)

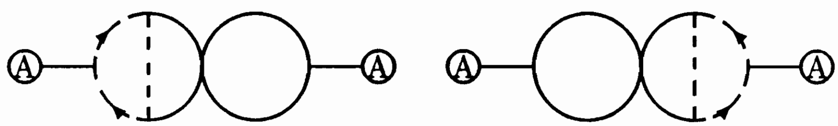
(d)

(e)

(f)

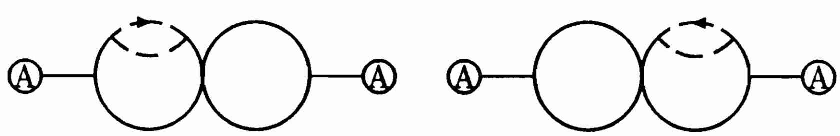
(g)

Figure C.15: Ghost-loop insertion diagrams derived from diagram figure C. 3f.



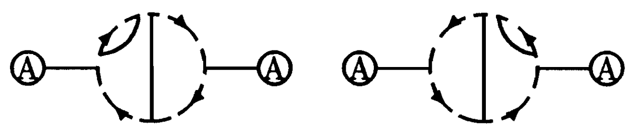
(a)

Figure C.16: Ghost-loop insertion diagrams derived from diagram figure C. 3l.

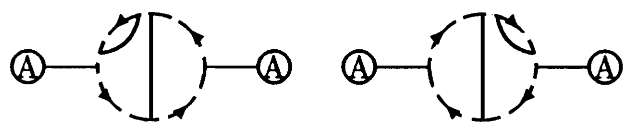


(a)

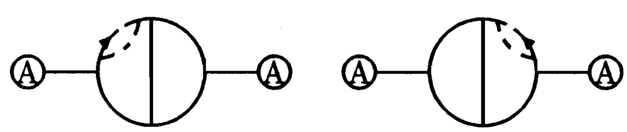
Figure C.17: Ghost-loop insertion diagrams derived from diagram figure C. 3m.



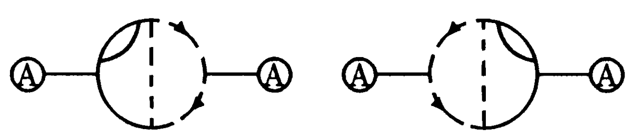
(a)



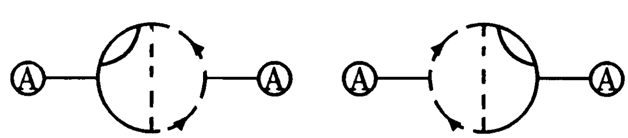
(b)



(c)

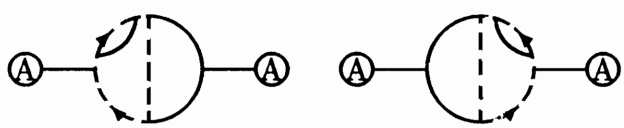


(d)

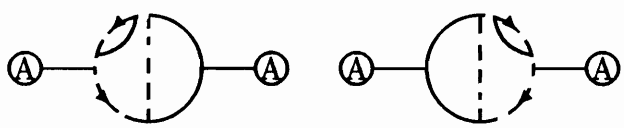


(e)

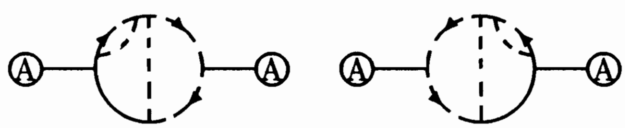
Figure C.18: Ghost-loop insertion diagrams derived from diagram figure C. 4a.



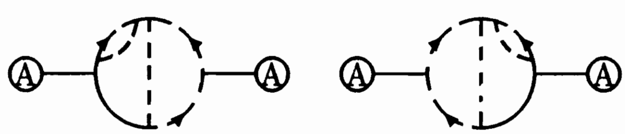
(f)



(g)



(h)



(i)

Figure C.18: Continued

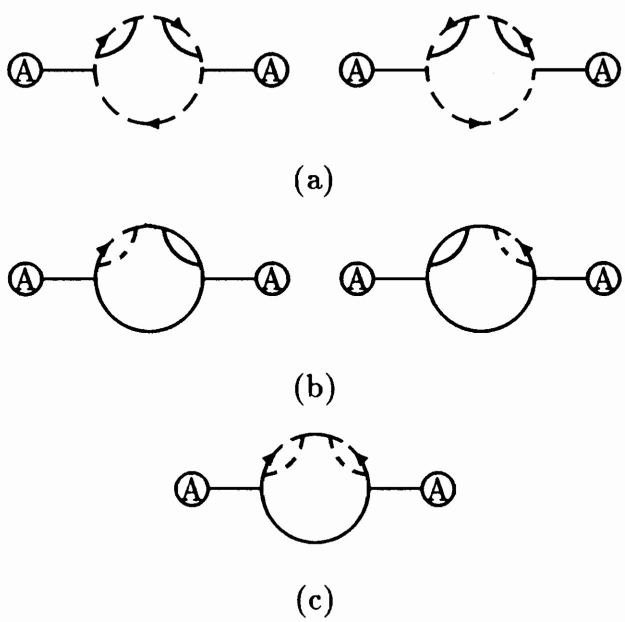


Figure C.19: Ghost-loop insertion diagrams derived from diagram figure C. 4b.

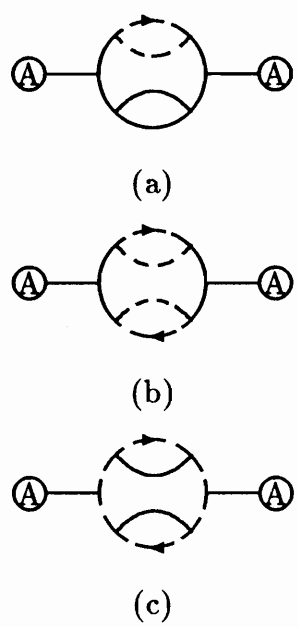
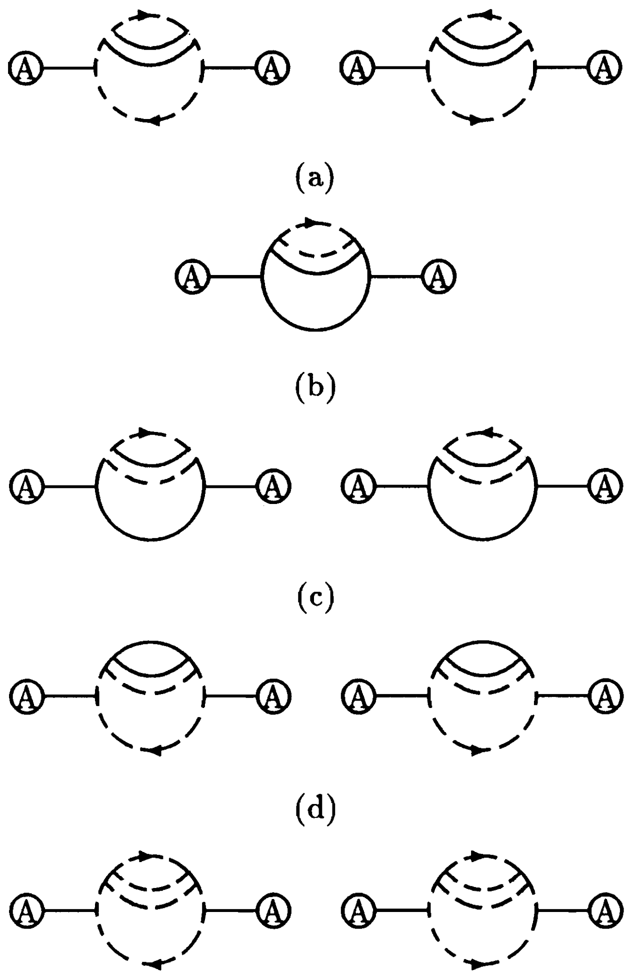


Figure C.20: Ghost-loop insertion diagrams derived from diagram figure C.4c.



(a)

(b)

(c)

(d)

(e)

Figure C.21: Ghost-loop insertion diagrams derived from diagram figure C. 4d.

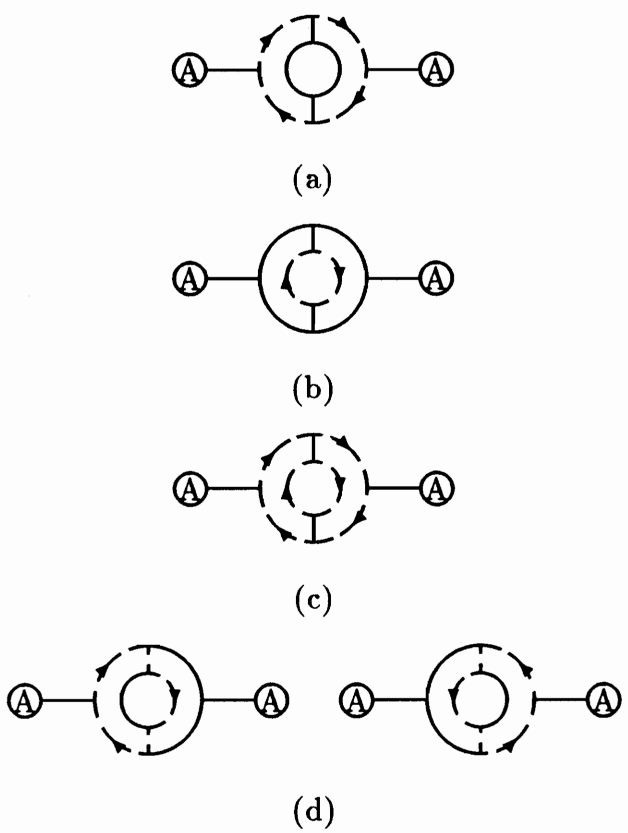
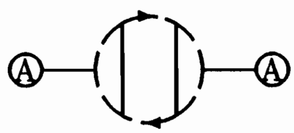
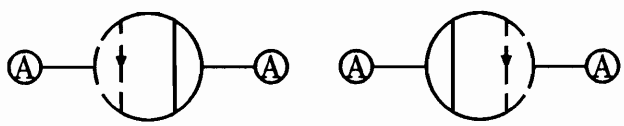


Figure C.22: Ghost-loop insertion diagrams derived from diagram figure C. 4e.



(a)



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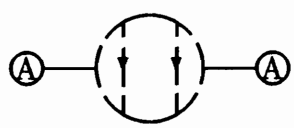
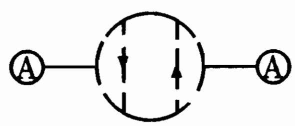
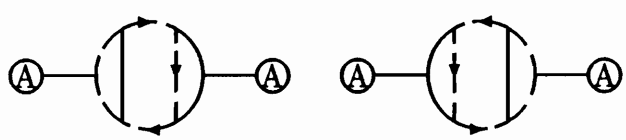


(c)



(d)



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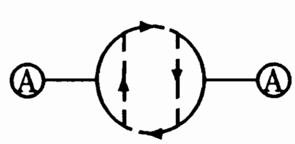
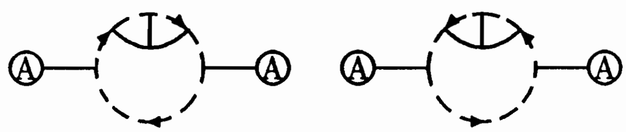
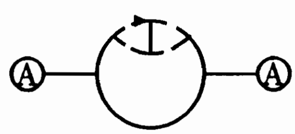


(f)

Figure C.23: Ghost-loop insertion diagrams derived from diagram figure C. 4f.



(a)



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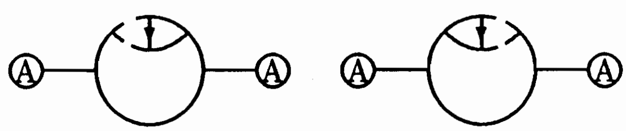
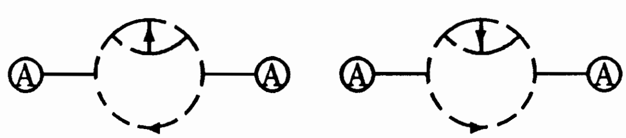


(c)



(d)

Figure C.24: Ghost-loop insertion diagrams derived from diagram figure C. 4g.

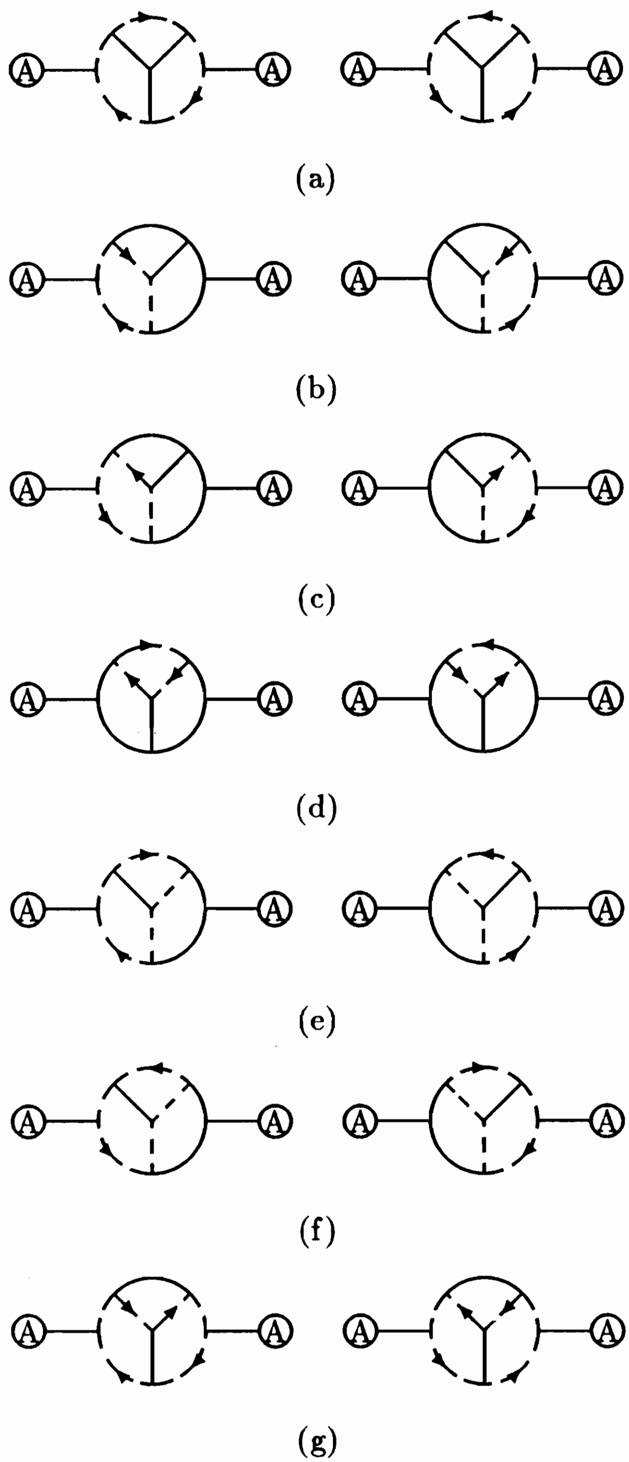


Figure C.25: Ghost-loop insertion diagrams derived from diagram figure C. 4h.

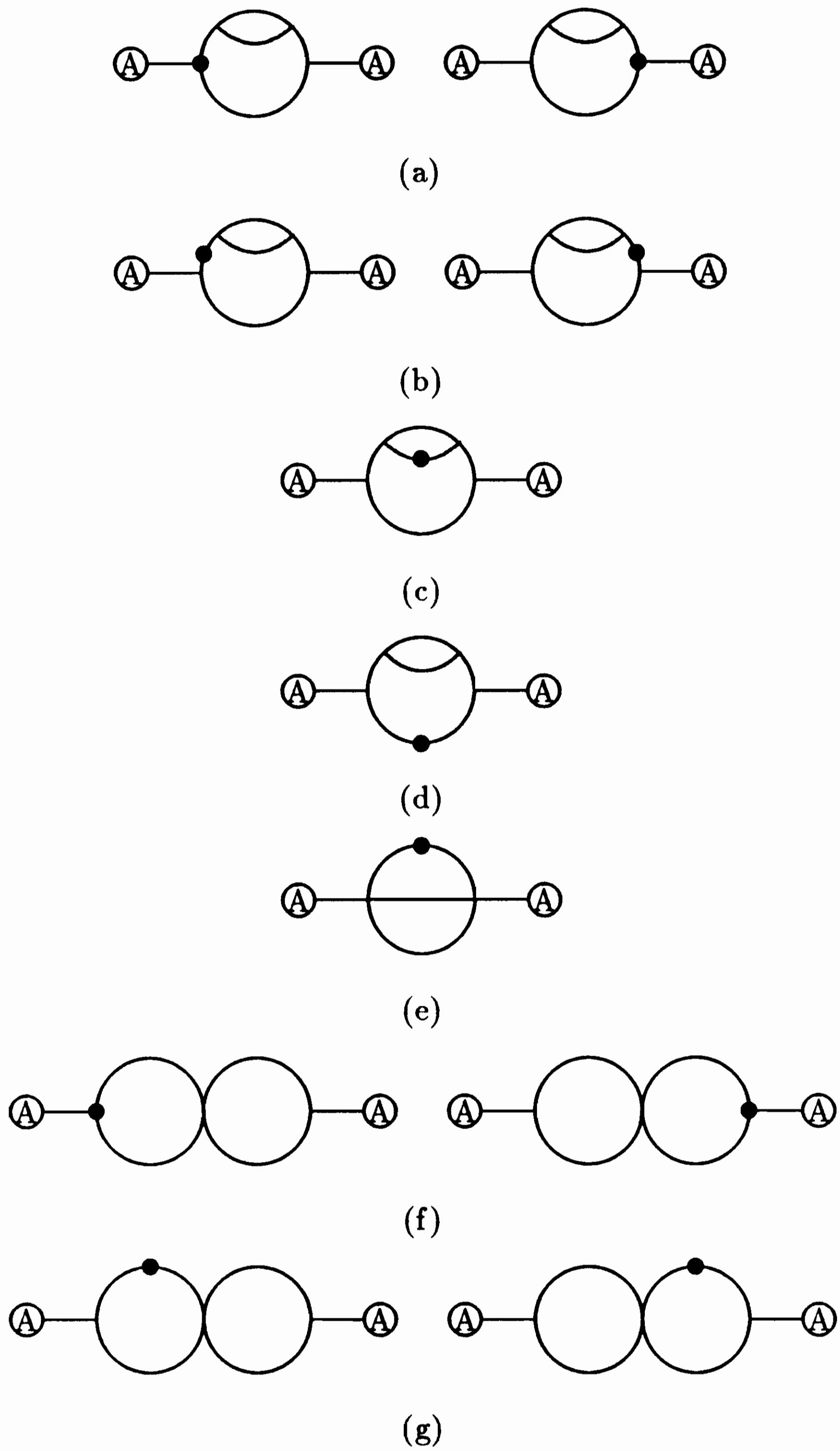
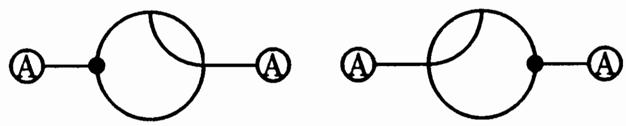
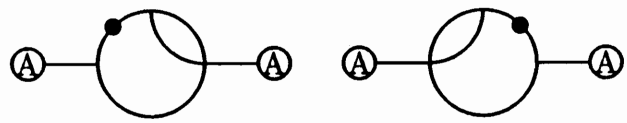


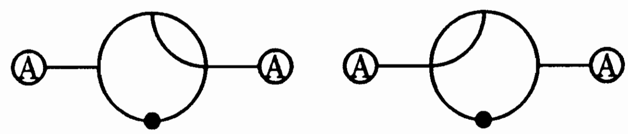
Figure C.26: Renormalization insertion diagrams to compute the contribution of the counter-term Lagrangian to the three-loop diagrams. Ghost-loop insertion diagrams are not depicted. Their total number is twenty.



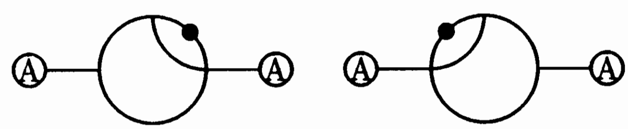
(h)



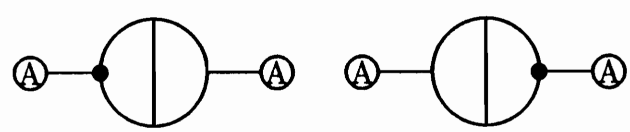
(i)



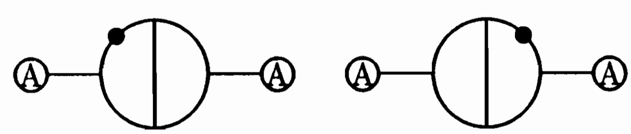
(j)



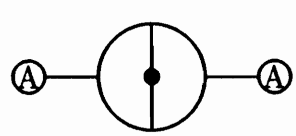
(k)



(l)

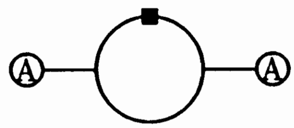


(m)

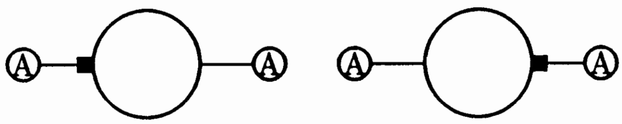


(n)

Figure C.26: Continued



(o)



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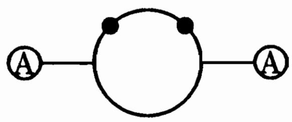
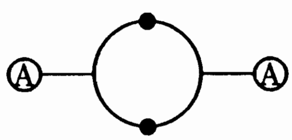


(q)



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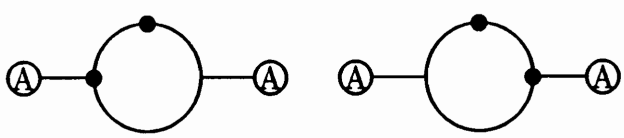

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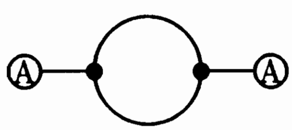

(t)

Figure C.26: Continued

Appendix D

Integral formulae in dimensional regularization

One-loop integration formulae as expressed in terms of G-functions

The integral measure is

$$dp = \frac{\Gamma(1-\varepsilon)}{(4\pi)^\varepsilon} \frac{d^{4-2\varepsilon}p}{(2\pi)^{4-2\varepsilon}} \tag{D.1}$$

The integral formulae with this normalization:

$$\int dp \frac{1}{(k-p)^{2\alpha}} = 0, \quad \text{for any } \alpha, \tag{D.2}$$

$$\int dp \frac{1}{p^{2\alpha}(k-p)^{2\beta}} = \frac{i}{4\pi^2} k^{2-\alpha-\beta-\varepsilon} G(0,0,\alpha,\beta), \tag{D.3}$$

$$\int dp \frac{p^\mu}{p^{2\alpha}(k-p)^{2\beta}} = k^\mu \frac{i}{4\pi^2} k^{2-\alpha-\beta-\varepsilon} G(1,0,\alpha,\beta), \tag{D.4}$$

$$\int dp \frac{p^\mu p^\nu}{p^{2\alpha}(k-p)^{2\beta}} = \tag{D.5}$$

$$\frac{i}{4\pi^2} k^{2-\alpha-\beta-\varepsilon} \left(k^2 g^{\mu\nu} \frac{G(1,1,\alpha,\beta)}{2(\alpha+\beta-3+\varepsilon)} + k^\mu k^\nu G(2,0,\alpha,\beta) \right)$$

$$\int dp \frac{p^\mu p^\nu p^\lambda}{p^{2\alpha}(k-p)^{2\beta}} = \frac{i}{4\pi^2} k^{2-\alpha-\beta-\varepsilon} \tag{D.6}$$

$$\left(k^2 (g^{\mu\nu} k^\lambda + g^{\nu\lambda} k^\mu + g^{\lambda\mu} k^\nu) \frac{G(2,1,\alpha,\beta)}{2(\alpha+\beta-3+\varepsilon)} + k^\mu k^\nu k^\lambda G(3,0,\alpha,\beta) \right).$$

Two-loop integration formulae

Denote

$$F(\alpha,\beta,\gamma,\delta) = -(4\pi)^4 (k^2)^{\alpha+\beta+\gamma+\delta-4+2\varepsilon} \int dp dq \frac{1}{p^{2\alpha}(k-p)^{2\beta}(p-q)^{2\gamma}q^{2\delta}(k-q)^2}. \tag{D.7}$$

Two general formulae were derived for F in [5]. These are

$$F(\alpha,\beta,1,1) = \tag{D.8}$$

$$\frac{\Gamma^3(1-\varepsilon)\Gamma(-1+2\varepsilon)\Gamma(1-\alpha-\varepsilon)\Gamma(1-\beta-\varepsilon)\Gamma(\alpha+\beta-2+2\varepsilon)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(3-\alpha-\beta-3\varepsilon)} \tag{D.9}$$

$$\left\{\frac{\Gamma(3-\alpha-\beta-3\varepsilon)}{\Gamma(2-\alpha-\beta-\varepsilon)} - \frac{\Gamma(\alpha+\beta-1+\varepsilon)}{\Gamma(\alpha+\beta-2+3\varepsilon)} + \frac{\Gamma(\alpha)}{\Gamma(\alpha-1+2\varepsilon)} + \frac{\Gamma(\beta)}{\Gamma(\beta-1+2\varepsilon)} - \frac{\Gamma(2-\alpha-2\varepsilon)}{\Gamma(1-\alpha)} - \frac{\Gamma(2-\beta-2\varepsilon)}{\Gamma(1-\beta)}\right\}$$

and

$$F(\alpha,1,\gamma,\delta) = \tag{D.10}$$

$$\frac{\Gamma^3(1-\varepsilon)\Gamma(2-\alpha-\varepsilon)\Gamma(2-\gamma-\varepsilon)\Gamma(2-\delta-\varepsilon)}{\Gamma(\alpha)\Gamma(\gamma)\Gamma(\delta)\Gamma(2-2\varepsilon)}$$

$$\sum_{m,n=0}^{\infty} \frac{(-)^m\Gamma(n+2-2\varepsilon)\Gamma(m+n+\alpha+\gamma+\delta-2+2\varepsilon)}{m!n!(n+1-\varepsilon)\Gamma(4-m-\alpha-\gamma-\delta-3\varepsilon)\Gamma(m+n+2-\varepsilon)}$$

$$\left\{\frac{1}{(n+\delta)(m+n+\alpha+\delta-1+\varepsilon)} + \frac{1}{(n+\delta)(m+n+\gamma+\delta-1+\varepsilon)} + \frac{1}{(m+n+\alpha)(m+n+\alpha+\delta-1+\varepsilon)} + \frac{1}{(m+n+\gamma)(m+n+\gamma+\delta-1+\varepsilon)} + \frac{1}{(m+n+\alpha)(n+2-\delta-2\varepsilon)} + \frac{1}{(m+n+\gamma)(n+2-\delta-2\varepsilon)}\right\}.$$

I give explicit Laurent expansion in terms ε of these formulae for those values of exponents which are necessary for the caclulation of the three-loop β function in the background field method. These are

$$F(1,1,1,1) = 6\zeta(3) \tag{D.11}$$

$$F(-1+\varepsilon,1,1,1) = \frac{1}{12\varepsilon^2} + \frac{13}{24\varepsilon} + \frac{143}{48} \tag{D.12}$$

$$F(\varepsilon,1,1,1) = \frac{1}{6\varepsilon^2} + \frac{7}{6\varepsilon} + \frac{37}{6} \tag{D.13}$$

$$F(1+\varepsilon,1,1,1) = 6\zeta(3) \tag{D.14}$$

$$F(2+\varepsilon,1,1,1) = \frac{1}{6\varepsilon^2} - \frac{1}{2\varepsilon} + \frac{7}{6} \tag{D.15}$$

$$F(1,1,-1+\varepsilon,1) = -\frac{1}{6\varepsilon^2} - \frac{3}{4\varepsilon} - \frac{49}{24} \tag{D.16}$$

$$F(1,1,\varepsilon,1) = \frac{1}{3\varepsilon^2} + \frac{5}{3\varepsilon} + \frac{17}{3} \tag{D.17}$$

$$F(1,1,1+\varepsilon,1) = 6\zeta(3) \tag{D.18}$$

There is one more integral which appears at intermediate stages ($F(1,1,2+\varepsilon,1)$); however, the various contributions from different integrals cancel this term.

Essentially three-loop integrals

In this final section, for the sake of completeness, I present the pole part of essentially three-loop type integrals of the form

$$i(4\pi)^3(k^2)^{3\epsilon-1}\int\frac{dp\,dq\,dr\,H(k,p,q,r)}{(k-p)^2(k-q)^2(k-r)^2(p-q)^2(p-r)^2(q-r)^2p^2q^2r^2} \tag{D.19}$$

Numerator(H)	integral
$(p-q)^8$	$-\frac{2}{3\epsilon^3}-\frac{61}{18\epsilon^2}-\frac{877}{108\epsilon}+\frac{4}{\epsilon}\zeta(3)$
$(p-q)^6k^2$	$\frac{1}{\epsilon^3}+\frac{41}{6\epsilon^2}-\frac{31}{\epsilon}-\frac{6}{\epsilon}\zeta(3)$
$(p-q)^4k^4$	$\frac{12}{\epsilon}\zeta(3)$
$(k-p)^8$	$\frac{2}{3\epsilon^2}+\frac{49}{6\epsilon}+\frac{4}{\epsilon}\zeta(3)$
$(k-p)^6k^2$	$\frac{1}{3\epsilon^2}+\frac{4}{\epsilon}+\frac{4}{\epsilon}\zeta(3)$
$(k-p)^4k^4$	$\frac{4}{\epsilon}\zeta(3)$
$(k-p)^2k^6$	$-\frac{2}{\epsilon}\zeta(3)$
$(k-p)^6(q-r)^2$	$\frac{5}{12\epsilon^3}+\frac{73}{24\epsilon^2}+\frac{661}{48\epsilon}$
$(k-p)^4(q-r)^4$	$\frac{1}{2\epsilon^2}+\frac{17}{3\epsilon}$
$(k-p)^2(q-r)^6$	$-\frac{1}{4\epsilon^3}+\frac{65}{24\epsilon^2}+\frac{865}{48\epsilon}$
$(k-p)^4(q-r)^2k^2$	$\frac{1}{3\epsilon^3}+\frac{7}{3\epsilon^2}+\frac{31}{3\epsilon}$
$(k-p)^2(q-r)^4k^2$	$\frac{1}{3\epsilon^3}+\frac{3}{\epsilon^2}+\frac{53}{3\epsilon}$
$(k-p)^4q^4$	$\frac{1}{6\epsilon^3}+\frac{17}{12\epsilon^2}+\frac{199}{24\epsilon}$
$(k-p)^6q^2$	$\frac{1}{8\epsilon^3}+\frac{49}{48\epsilon^2}+\frac{531}{96\epsilon}$
$(k-p)^4q^2k^2$	$\frac{1}{6\epsilon^3}+\frac{3}{2\epsilon^2}+\frac{55}{6\epsilon}$

This is the same set of integrals which is given in [26], and my results are in complete agreement with those. All other three-loop integrals can be evaluated by means of the formulae given in the previous two subsections.

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