TRACING PARAMETRIZED OPTIMA FOR INEQUALITY CONSTRAINED NONLINEAR MINIMIZATION PROBLEMS

by

Joanna Rakowska

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APPROVED:

Layne T. Watson
L. T. Watson, Chairman

R. T. Haftka, Cochairman

Kenneth B. Hannsgen
K. B. Hannsgen

Werner E. Kohler
W. E. Kohler

Ch. A. Beattie

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Blacksburg, Virginia

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Joanna Rakowska

Committee Chairman: Layne T. Watson
Computer Science

Cochairman: Raphael T. Haftka
Aerospace & Ocean Engineering

(ABSTRACT)

A general algorithm for tracing the path of optima of inequality constrained optimization problems as a function of a parameter was developed in this research. The algorithm is an active set algorithm using a homotopy method to trace the path. A new feature of the algorithm is a capability of handling the transition points between segments in a routine way. The algorithm locates the transition points, and finds an active set for the next segment by considering all possible sets of active constraints. The nonoptimal sets are eliminated on the basis of the Lagrange multipliers and the derivatives of the optimal solutions with respect to the parameter.

The algorithm was implemented for three different problems. The first application, a spring-mass problem, was used to illustrate various kinds of transition events between segments. The second application, a well known ten-bar truss structural optimization problem, was used to validate the algorithm, since the numerical results for this problem have been obtained by other methods. The third application, bi-objective control-structure optimization, had an important engineering application. The numerical results obtained in this application could be used in the design process – they allowed selection of the best designs and provided some insight into behavior of the structure.

The sufficient conditions for persistence of the minima were given using the results of the stability theory, and the connection was shown between these results and classical optimization theory. For the standard nonlinear programming problem these conditions are equivalent to the Mangasarian-Fromovitz criterion and the standard second order sufficient optimality condition.

A method for computational verification of the conditions for persistence of the minima was proposed. By using Motzkin's Transposition Theorem, the Mangasarian-Fromovitz criterion can be reduced to the problem of finding a feasible point for a linear optimization problem. In this form it can be easily solved by Phase I of the simplex method. It was shown that the linear programming problem can be transformed to the form of general nonlinear problem in such a way that the regularity of the constraints is preserved. Therefore for linear problems necessary and sufficient condition for solvability of the perturbed system can be verified computationally.

The results of bifurcation theory were used to characterize the possible points of discontinuity of the path. The necessary conditions for discontinuity of the path of optima can be checked by the developed algorithm with no extra work. The results of bifurcation theory were also used to describe the possible singularities of the path of optima and the behavior of the path near singular points.

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Introduction

Optimization problems often depend on parameters that define constraint boundaries or objective function properties. These parameters are kept constant during optimization, but it is often necessary to know what is the effect of a change in some parameter on the optimum solution.

Traditionally, the effects of parameter variatiations have been studied via sensitivity analysis, where a parameter represents a small perturbation, often a random error in the problem data (e.g., Fiacco [5], Fiacco [6]). In mathematical programming sensitivity analysis was often used in convergence and rate of convergence proofs, which, in turn, lead to acceleration of algorithms, and sometimes resulted in new algorithms (e.g., Robinson [30]).

More recently, there has been an effort to develop an approach to tracing the family of optima obtained by varying a parameter over an extended, user specified range. This application can be helpful in the modeling process, since singularities in the behavior of the system can be revealed in this way (Rao and Papalambros [27]). These singularities are not necessarily revealed when the parameter is treated as a design variable.

Another motivation for parametric programming is multi-objective optimization where two or more objectives are combined into the multi-objective cost function (e.g., [17], [9], [24], [38]). The optimal solutions to the problem of minimizing the bi-objective cost function $f = (f_1, f_2)$ can be found by optimizing the convex combination $f = (1-\alpha)f_1 + \alpha f_2$ of two objectives f_1 and f_2. All solutions for α between 0 and 1 give the efficient curve, which is a trade-off curve for the two objectives.

There have been recent attempts to construct algorithms for tracing a path of optimal solutions. Almost all algorithms used for this purpose are predictor-corrector algorithms. The predicted next point on the path can be obtained by polynomial or Hermite extrapolation

(Watson et al. [44]). Other predictors use the tangent vector to the curve at the point (Watson et al. [44]) or a combination of this and previously computed tangents (Lundberg and Poore [19]) to get a predicted point. In the correction phase a point on the zero curve is found by solving the system of equations represented by the curve. Methods such as Newton, Newton-like, or quasi-Newton and others can be applied to solve this system of nonlinear equations. The predicted point is an initial approximation of the solution for these methods. Some predictors use augmented system of equations, where the equation appended to the system confines the correction iterates in some way, for example to a hyperplane orthogonal to the predictor direction (Lundberg and Poore [20]), or to one of the unit basis vectors in R^{n+1} (Rheinboldt [29]), or in some other way (Watson et al. [44]).

A few examples of the algorithms for tracing the path of parametrized optima are given next.

Rao and Papalambros [28] use simple continuation to find the family of parametrized optima. The predicted point lies on the tangent to the curve at a given point. In the correction phase they use an NLP solver based on the Sequential Quadratic Programming Method to get to the zero curve along a vertical direction. This algorithm can trace the path as long as no singularities (loss of strict complementarity, violation of linear independence constraint qualification, or failure of the second order sufficient condition) are encountered. The program does not have a capability of automatic jumping over singularities, and needs to be restarted at a value of the parameter beyond the singularity.

Lundberg and Poore [19], [20], use a sophisticated predictor-corrector homotopy curve tracking algorithm to investigate the dependence of the solution on a parameter and to locate bifurcations and points of extreme sensitivity. They use Adams-Bashforth predictors that compute the predicted point taking into account current and previous tangents to the curve. The correction to the path is a Newton-like iteration in the hyperplane orthogonal to the prediction direction, based on Keller's bordering algorithm [13]. Their algorithm detects singularities along the path, but they do not provide the mechanism for handling such singularities.

Shin et al. [39] developed an algorithm capable of making the transition from one segment to another for the special case of the design of a beam with a given weight so as to maximize the buckling load. The algorithm uses a homotopy method to trace the segments where the active set is fixed.

The continuation algorithms can be used only if the zero curve is continuous. Therefore an important question is what the conditions are for continuity of this curve. This question is related to the problem of the solvability of arbitrarily perturbed systems of equalities.

The problem of solvability of perturbed optimization problems has been examined by a number of authors. The problem considered has the form

$$
P(\alpha) \qquad \begin{aligned} &\underset{x}{\text{minimize}}\ \ f(x,\alpha) \\ &\text{subject to } g_i(x,\alpha) = 0, \qquad (i = 1,\ldots,l), \\ &\qquad\qquad\ \ g_i(x,\alpha) \leqq 0, \qquad (i = l+1,\ldots,l+k). \end{aligned}
$$

McCormick and Fiacco [4] proved the existence of a differentiable function $x(\alpha)$ that solves $P(\alpha)$ in a particular case when the dependence on α is linear. This result was obtained under the assumption of strict complementarity, independence of the gradients of active constraints, and the second order sufficient optimality condition.

Fiacco [3] extended this result for general perturbations α. Besides the three assumptions given above, Fiacco assumed that the functions defining $P(\alpha)$ are twice continuously differentiable in x and that their gradients with respect to x are once continuously differentiable in α in a neighborhood of the optimal solution $(x^*, 0)$. With these hypotheses he proved the existence and uniqueness of a local minimum $(x(\alpha), \lambda(\alpha), \mu(\alpha))$ for α in a neighborhood of zero.

Robinson [30] obtained a similar result under a weaker assumption on differentiability. He assumed only that the second order partial derivatives of the problem with respect to x are continuous in (x, α).

Later Robinson [35] proved local existence and uniqueness of the solution ($x(\alpha)$, $\lambda(\alpha)$, $\mu(\alpha)$) to the problem $P(\alpha)$ without the requirement of strict complementarity. He assumed linear independence of the gradients of the active constraints and a strengthened form of the standard second order condition. He strengthened the standard second order condition by requiring that the Hessian of the Lagrangian be positive definite for all directions tangent to equality constraints and active inequality constraints with positive Lagrange multipliers, whereas in the standard second order sufficient optimality condition these directions are limited to those which form nonacute angles with the gradients of active inequality constraints with zero Lagrange multipliers.

A similar result was also obtained by Jittorntrum [11], who also proved the existence of the directional derivative of the optimal solution with respect to the perturbation.

Kojima [15] relaxed the hypotheses further by dropping the constraint linear independence requirement. Instead he required that the Mangasarian-Fromovitz criterion and the strengthened form of the second order sufficient condition hold. With these assumptions he proved the existence and uniqueness of a stationary point of $P(\alpha)$, and he also showed that the stationary points are continuous under small perturbations.

Robinson [36] proved the persistence of minimizers under small perturbations assuming a second order condition and a constraint qualification, which for the standard nonlinear problem $P(\alpha)$ are equivalent to the standard second order sufficient optimality condition and the Mangasarian-Fromovitz constraint qualification.

A more numerically oriented approach to the problem of the persistence of minima was taken in bifurcation theory. Bifurcation techniques together with continuation methods were used to characterize the dependence of the minimum on a parameter rather than on an arbitrary perturbation. Continuation methods for solving nonlinear equations $F = 0$ are based on the Implicit Function Theorem, which requires the nonsingularity of the Jacobian matrix for the function F. Therefore singularities of the Jacobian DF and the phenomena that can occur at the singularities are of much interest in bifurcation theory.

Kojima and Hirabashi [16] investigated the behavior of the Kuhn-Tucker points when the minimization problem is continuously deformed. They showed that the set of all points (x, α) such that x is a stationary point with a parameter α is a disjoint union of paths and loops. They characterized how the number of local minima, saddle points and local maxima changes along a path or loop, and related these changes to the instability of the solutions and the orientation of the path or loop.

Related research was done by Jongen, Jonker and Twilt [12]. They introduced the concept of a "generalized critical point", which is a point where the gradients of the objective function and the gradients of active constraints are linearly dependent. Then they divided the set of generalized critical points (x, α) into five types: (a) nondegenerate critical points, i.e., points where strict complementarity, linear independence of the active constraints, and nonsingularity of the Hessian of the Lagrangian on the space tangent to active constraints hold, and the points where: (b) strict complementarity does not hold, (c) one of the eigenvalues of the Hessian of the Lagrangian on the space tangent to active constraints is zero, (d) the gradients of active constraints are linearly dependent and the number of active constraints is less than $(n + 1)$ (where n is the number of variables), (e) points where the gradients of active constraints are linearly dependent and the number of active constraints equals $n + 1$. They examined the behavior of the set of generalized critical points, and its Kuhn-Tucker subset.

A similar approach was taken by Poore and Tiahrt [40], [25]. They characterized the necessary and sufficient conditions for singularities of the path of parametrized optima, and they subdivided all singularities into seven types. Then they examined the behavior of the path near each type under the assumption of a low codimension singularity. Poore and Tiahrt were very much interested in the numerical aspect of their results. Their theory will be presented in more detail in Chapter 4.

There were two objectives of the research presented in this dissertation. The first one was to develop a general algorithm for tracing the optima of an inequality constrained optimization problems as a function of a parameter. The developed algorithm is an active set algorithm using a homotopy method to trace the path. **A new feature of the algorithm is a capability of handling the transition points between segments in a routine way**, which differentiates this algorithm from the algorithms for tracing the path of parametrized optima discussed before.

The second objective was to determine the conditions for continuity of the path of stationary points and optima for parametrized problems. This is important since the algorithm can be used only for tracing the path of stationary points as long as the path is continuous. There exist theoretical results that address this problem, but many of them are too abstract and general to be of immediate use. There are also theories, which are motivated by different problems, but whose results can be applied for our purpose (for example, bifurcation theory). This work provides a synthesis of existing theories, in particular a general stability theory and bifurcation theory useful in determination of the continuity of the path of optima.

The dissertation is organized as follows. In Chapter 2 we described an active set homotopy algorithm for tracing the path of parametrized optima. In Chapter 3 we give three applications of the algorithm. In Chapter 4 we characterize the possible points of discontinuity of the path of optima using the results of bifurcation theory. In Chapter 5 we give the sufficient conditions for continuation of the path of optima using the results of the stability theory by Robinson [36], and show the connection of these results with the classical optimization theory. Implications of bifurcation theory and stability theory for continuity of the path are given in Chapter 6. In this chapter we also discuss the possibility of computational verification of these theoretical conditions. Chapter 7 is an application of the theory for analysis of the path discontinuities for one of the applications given in Chapter 3.

2. An active set algorithm for tracing parametrized optima

In this chapter we will present an algorithm for tracking paths of optimal solutions of inequality constrained nonlinear programming problems as a function of a parameter. The proposed algorithm employs homotopy zero-curve tracing techniques to track segments where the set of active constraints is unchanged. The transition between segments is handled by considering all possible sets of active constraints and eliminating nonoptimal ones based on the signs of the Lagrange multipliers and the derivatives of the optimal solutions with respect to the parameter.

2.1 Problem statement

We want to have an algorithm that would minimize a cost function $f(x,\alpha)$ subject to constraints $g_j(x,\alpha) \leq 0$, $j = 1,\ldots,n_2$, where x is a n_1-vector of design variables subject to the minimum value constraints $x_i \geq x_{0i}$ and α is a parameter. The minimization problem is formulated as

$$\text{minimize } f(x,\alpha) \tag{2.1}$$

$$\text{subject to } g_i = x_{0i} - x_i \leqq 0, \qquad i = 1,\ldots,n_1, \tag{2.2}$$

$$g_{j+n_1}(x,\alpha) \leqq 0, \qquad j = 1,\ldots,n_2. \tag{2.3}$$

The solution should be obtained for a specified range of α, say $\alpha_a \leqq \alpha \leqq \alpha_b$. The Lagrangian function and Kuhn-Tucker conditions for this problem are:

$$\mathcal{L}(x,\lambda,\alpha) = f(x,\alpha) + \sum_{i=1}^{n_1} \lambda_i(x_{0i} - x_i) + \sum_{j=n_1+1}^{n_1+n_2} \lambda_j g_j(x,\alpha) \tag{2.4}$$

$$\frac{\partial f}{\partial x_i} + \sum_{j=n_1+1}^{n_1+n_2} \lambda_j \frac{\partial g_j}{\partial x_i} - \lambda_i = 0, \qquad i = 1,\ldots,n_1, \tag{2.5}$$

$$g_j \lambda_j = 0, \qquad j = 1,\ldots,n_1+n_2, \tag{2.6}$$

$$\lambda_j \geqq 0, \qquad j = 1,\ldots,n_1+n_2, \tag{2.7}$$

$$g_j \leqq 0, \qquad j = 1,\ldots,n_1+n_2. \tag{2.8}$$

Equations (2.5)–(2.6) form a system of nonlinear equations to be solved for the design variables x_i and for the Lagrange multipliers λ_j associated with active constraints of the form (2.3) and with the bounds for design variables (2.2). The solution of these equations is a function of α. As the value of α increases the solution of the Kuhn-Tucker conditions follows a path that consists of several smooth segments, each segment characterized by a different set of active constraints.

2.2 Homotopy method

The system of nonlinear equations (2.5)–(2.6) is solved by a homotopy method. The homotopy method uses the fact that if the solution to the system of equations

$$F(z,\alpha) = 0 \tag{2.9}$$

is known at some point (z_0, α_0), and the Jacobian matrix $DF(z_0, \alpha_0)$ of the function F at (z_0, α_0) has full rank, then there is some neighborhood U of (z_0, α_0) such that there is a unique curve of zeros of $F(z, \alpha)$ in U passing through (z_0, α_0). According to the theory by Watson [42], [43], this full rank of the Jacobian matrix implies that the zero set of equations (2.9) contains a smooth curve Γ in $(N+1)$-dimensional (z,α) space, which has no bifurcations and is disjoint from other zeros of (2.9). The curve Γ can be parametrized by the arc length s as

$$z = z(s), \qquad \alpha = \alpha(s). \tag{2.10}$$

Taking the derivative of (2.9) with respect to arc length, the nonlinear system of equations is transformed into a set of ordinary differential equations

$$\left[F_z(z(s), \alpha(s)), \quad F_\alpha(z(s), \alpha(s)) \right] \begin{pmatrix} \dfrac{dz}{ds} \\ \dfrac{d\alpha}{ds} \end{pmatrix} = 0, \tag{2.11}$$

and

$$\left\| \begin{pmatrix} \dfrac{dz}{ds} \\ \dfrac{d\alpha}{ds} \end{pmatrix} \right\| = 1, \tag{2.12}$$

where F_z and F_α denote the partial derivatives of F with respect to z and α, respectively. With the initial conditions at $s = 0$,

$$z(0) = z_0, \qquad \alpha(0) = \alpha_a, \tag{2.13}$$

(2.11)–(2.13) can be treated as an initial value problem. Its trajectory is the path Γ of optimal solutions $Z(s) = (z(s), \alpha(s))$.

A software package HOMPACK by Watson [43] which implements several homotopy algorithms can be used to track the zero curve Γ. The HOMPACK algorithms take steps along the zero curve using prediction and correction to find the next point. In the prediction phase a Hermite cubic $p(s)$ is constructed which interpolates the zero curve Γ at two known points $Z(s_1)$ and $Z(s_2)$.

The predicted next point is

$$Z^{(0)} = p(s_2 + h), \tag{2.14}$$

where $p(s)$ is the Hermite cubic, and h is an estimate of the optimal step (in arc length) to take along Γ.

The corrector iteration is

$$Z^{(k+1)} = Z^{(k)} - [DF(Z^{(k)})]^{+} F(Z^{(k)}), \qquad k = 0, 1, \ldots$$

where $[DF(Z^{(k)})]^{+}$ is the Moore-Penrose pseudoinverse of the $n \times (n+1)$ Jacobian matrix DF. In practice this pseudoinverse is not calculated explicitly; see Watson [43] for the details of the Hermite cubic interpolant construction and the corrector iteration.

The optimal step size h is chosen to prevent the correction iteration from being too costly. The user can specify nondefault values used in determining the step size such as, for example, the maximum and minimum allowed step size. The parameter α in equations (2.11)–(2.13) is a dependent variable, which distinguishes homotopy methods from standard continuation, imbedding, or incremental methods. The homotopy approach is also different from initial value or differentiation methods, since the controlling variable is arc length s, rather than α.

2.3 Solution along a segment

Since the active constraints in a segment are fixed they can be treated as equality constraints. Furthermore along each segment some design variables are fixed at their lower bound. The vector of these inactive (passive) variables is denoted x_p and need not be considered as design variables for that segment. The vector of active design variables x_i ($i \in \mathcal{I}_a$) is denoted as x_a. Along each segment the Kuhn-Tucker conditions are solved for the active design variables x_i ($i \in \mathcal{I}_a$) and for the Lagrange multipliers λ_g associated with the active constraints of the form (2.3) (λ_j, $j \in \mathcal{I}_g$). For each segment there are 2 types of equations:

$$V1 : g_j(x, \alpha) = 0, \qquad j \in \mathcal{I}_g, \tag{2.15}$$

$$V2 : \frac{\partial f}{\partial x_i} + \sum_{j \in \mathcal{I}_g} \lambda_j \frac{\partial g_j}{\partial x_i} = 0, \qquad i \in \mathcal{I}_a. \tag{2.16}$$

The active design variables and the Lagrange multipliers associated with active constraints (2.3) are the unknowns in these equations. For the homotopy solution we need the Jacobian matrix of functions V1 and V2 with respect to α, x_a, and λ_g. The Jacobian matrix has components of the following form:

$$\frac{\partial V1}{\partial \alpha} = \frac{\partial g_j}{\partial \alpha}, \qquad \frac{\partial V1}{\partial x_m} = \frac{\partial g_j}{\partial x_m}, \qquad \frac{\partial V1}{\partial \lambda_t} = 0, \tag{2.17}$$

$$\frac{\partial V2}{\partial \alpha} = \frac{\partial^2 f}{\partial \alpha \partial x_i} + \sum_{j \in \mathcal{I}_g} \lambda_j \frac{\partial^2 g_j}{\partial \alpha \partial x_i}, \quad \frac{\partial V2}{\partial x_m} = \frac{\partial^2 f}{\partial x_i \partial x_m} + \sum_{j \in \mathcal{I}_g} \lambda_j \frac{\partial^2 g_j}{\partial x_i \partial x_m}, \quad \frac{\partial V2}{\partial \lambda_j} = \frac{\partial g_j}{\partial x_i},$$

where $i, m \in \mathcal{I}_a$ and $t, j \in \mathcal{I}_g$. The derivatives with respect to x_i and x_m denote the derivatives with respect to all active design variables, and the derivatives with respect to λ_t and λ_j denote the derivatives with respect to all Lagrange multipliers associated with active constraints of the form (2.3).

For example, for $n_1 = 5$ with active constraint g_{n_1+2} of the form (2.3) and active design variables x_1, x_5 the system of equations is:

$$\frac{\partial f}{\partial x_1} + \lambda_7 \frac{\partial g_7}{\partial x_1} = 0, \tag{2.18}$$

$$\frac{\partial f}{\partial x_5} + \lambda_7 \frac{\partial g_7}{\partial x_5} = 0, \tag{2.19}$$

$$g_7(x_1, x_5, \alpha) = 0. \tag{2.20}$$

The set of unknown variables is ordered as $(\alpha, x_1, x_5, \lambda_7)$ and the corresponding Jacobian matrix is:

$$\begin{pmatrix} \frac{\partial^2 f}{\partial \alpha \partial x_1} + \lambda_7 \frac{\partial^2 g_7}{\partial \alpha \partial x_1} & \frac{\partial^2 f}{\partial x_1^2} + \lambda_7 \frac{\partial^2 g_7}{\partial^2 x_1} & \frac{\partial^2 f}{\partial x_1 \partial x_5} + \lambda_7 \frac{\partial^2 g_7}{\partial x_1 \partial x_5} & \frac{\partial g_7}{\partial x_1} \\ \frac{\partial^2 f}{\partial \alpha \partial x_5} + \lambda_7 \frac{\partial^2 g_7}{\partial \alpha \partial x_5} & \frac{\partial^2 f}{\partial x_1 \partial x_5} + \lambda_7 \frac{\partial^2 g_7}{\partial x_1 \partial x_5} & \frac{\partial^2 f}{\partial^2 x_5} + \lambda_7 \frac{\partial^2 g_7}{\partial^2 x_5} & \frac{\partial g_7}{\partial x_5} \\ \frac{\partial g_7}{\partial \alpha} & \frac{\partial g_7}{\partial x_1} & \frac{\partial g_7}{\partial x_5} & 0 \end{pmatrix}. \tag{2.21}$$

The variables of the system of equations are stored in the vector of homotopy variables $y = \begin{pmatrix} \alpha \\ x_a \\ \lambda_g \end{pmatrix}$.

At the start of a segment the set of active design variables and active constraints for this segment has to be found, so that the vector y be defined. A set of equations is then generated, with the type of each variable determining the form of the equation appended to the system of equations. For the Lagrange multiplier associated with an active constraint of the form (2.3) the equation has the form (2.15), and for an active design variable the equation has the form (2.16). The Jacobian matrix is calculated according to (2.17) and the system of equations for the segment is solved using the homotopy technique. Then the

Lagrange multipliers for inactive design variables are calculated according to (2.5). In these equations the Lagrange multipliers associated with active constraints of the form (2.3) have been computed by the homotopy method, and the Lagrange multipliers associated with inactive constraints (2.3) are known to be zero.

2.4 Segment termination and transition to the next segment

At each point of a segment the Lagrange multipliers associated with the lower bound of the inactive design variables or the active constraints of the form (2.3) in the segment should be nonnegative, the value of each g_j, $j = n_1, \ldots, n_1 + n_2$ should be less than or equal to zero, and all design variables should be larger than or equal to their lower bound. If any of the above conditions is not satisfied the segment is terminated and a new one is started. The transition point to a new segment is called here a *switching point.* (It is assumed throughout that a switching point is not also a turning point of the path of optima.) Depending on the type of termination, the switching point is the point where

1) one of the positive Lagrange multipliers becomes equal to zero, or
2) a previously negative g_j of the form (2.3) becomes equal to zero, or
3) an active design variable x_i ($i \in \mathcal{I}_a$) becomes inactive (equal to x_{0i}).

At the beginning of each segment the system of linear equations (2.5) is solved for $\lambda_1, \ldots, \lambda_m$, $m = n_1 + n_2$, to check which design variables and constraints are active and to find the initial values of the Lagrange multipliers for the segment. First the Lagrange multipliers for inactive constraints are set to zero so that we consider Lagrange multipliers only for potentially active constraints (those equal to zero).

Since some of the constraints (2.3) may be inactive (their values at the switching point are less than zero), or the derivatives of the constraints (2.3) with respect to the design variables can assume values for which some columns or rows in the coefficient matrix of the system (2.5) are linearly dependent, the rank of this matrix can be less than n_2. The rank of the coefficient matrix for the system (2.5) determines the number of the constraints (2.3) which are assumed to be active in the next segment.

The QR factorization with column pivoting is used to find the rank (r) of the coefficient matrix. Next the system (2.5) is solved for all subsets of r columns which are linearly independent assuming that the Lagrange multipliers for the constraints (2.3) corresponding to the remaining columns are zero. To get the solution for each subset at least r design variables are assumed to be active (the corresponding Lagrange multipliers are set to zero). For each subset of r columns (corresponding to r constraints) all combinations of r out of n_1 design variables are assumed to be active. The system is solved in turn for each combination to find all sets of active design variables and active constraints (2.3) such that the Lagrange multipliers are nonnegative.

For each transition point there are at least two active sets with nonnegative Lagrange multipliers: a set corresponding to the last segment and a set corresponding to a new segment. Sometimes there are more than two solutions satisfying the condition that all the Lagrange multipliers be nonnegative. Then for each solution the derivatives of the design variables with respect to the arc length s are calculated as described below. A set of active constraints (2.3) and active design variables is accepted when the values of these derivatives indicate that no active constraint will be violated for increasing values of s.

To calculate the values of the derivatives of the design variables with respect to s we first calculate their derivatives with respect to α. The Kuhn–Tucker conditions (2.5)–(2.6) are differentiated with respect to α. Thus we obtain:

$$(A+Z)\frac{\partial x_a}{\partial \alpha} + N\frac{\partial \lambda_g}{\partial \alpha} + \frac{\partial(\nabla f)}{\partial \alpha} + (\frac{\partial N}{\partial \alpha})\lambda_g = 0, \tag{2.22}$$

$$N^T\frac{\partial x_a}{\partial \alpha} + \frac{\partial g_a}{\partial \alpha} = 0, \tag{2.23}$$

where x_a, λ_g are a vector of design variables and a vector of the Lagrange multipliers for active g_j, g_a is a vector of active constraints g_j, $j \in \mathcal{I}_g$, N has components $n_{ij} = \dfrac{\partial g_j}{\partial x_i}$,

$(j \in \mathcal{I}_g,\ i \in \mathcal{I}_a)$, A is the Hessian of the objective function f, $a_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$, and Z is a matrix with elements $z_{il} = \sum_{j \in \mathcal{I}_g} \dfrac{\partial^2 g_j}{\partial x_i \partial x_l} \lambda_j$. After equations (2.22) and (2.23) are solved, derivatives of each g_j corresponding to an active constraint (2.3) with respect to α are calculated according to

$$\frac{\partial g_j}{\partial \alpha} = \sum_{i \in \mathcal{I}_a} \frac{\partial g_j}{\partial x_i} \frac{\partial x_i}{\partial \alpha}, \qquad j \in \mathcal{I}_g. \tag{2.24}$$

For each candidate solution satisfying the Kuhn-Tucker conditions, derivatives with respect to arc length s are then calculated by multiplication by $d\alpha/ds$ (taken from the previously calculated point on the last segment–all that matters is the sign of $d\alpha/ds$). We calculate the derivatives with respect to arc length s of design variables, Lagrange multipliers and g_j's corresponding to active constraints. A solution is accepted if the derivatives with respect to s of active design variables that are at their lower bound are nonnegative, the derivatives with respect to s of zero Lagrange multipliers that correspond to active constraints (2.3) are nonnegative and the derivatives of g_j's that are equal to zero are nonpositive. The case of more than two acceptable sets of active constraints and active design variables at the same transition point is signaled by a flag. A flowchart of the algorithm is shown in Figure 2.1 and a more detailed description of this segment switching algorithm is given in the next section.

Second order optimality conditions (see Haftka et al., [8]) are checked to verify that the stationary points found by solving the Kuhn-Tucker conditions are indeed minima. Second order necessary conditions are

$$d^t \left[\nabla^2_{x_a} \mathcal{L}\right] d \geqq 0 \quad \text{for every } d \text{ such that } (\nabla g_j)^t d = 0 \quad \forall j \in I_g,$$

where $\left[\nabla^2_{x_a} \mathcal{L}\right]_{lm} = \dfrac{\partial^2 \mathcal{L}}{\partial x_l \partial x_m}$, $l, m \in \mathcal{I}_a$.

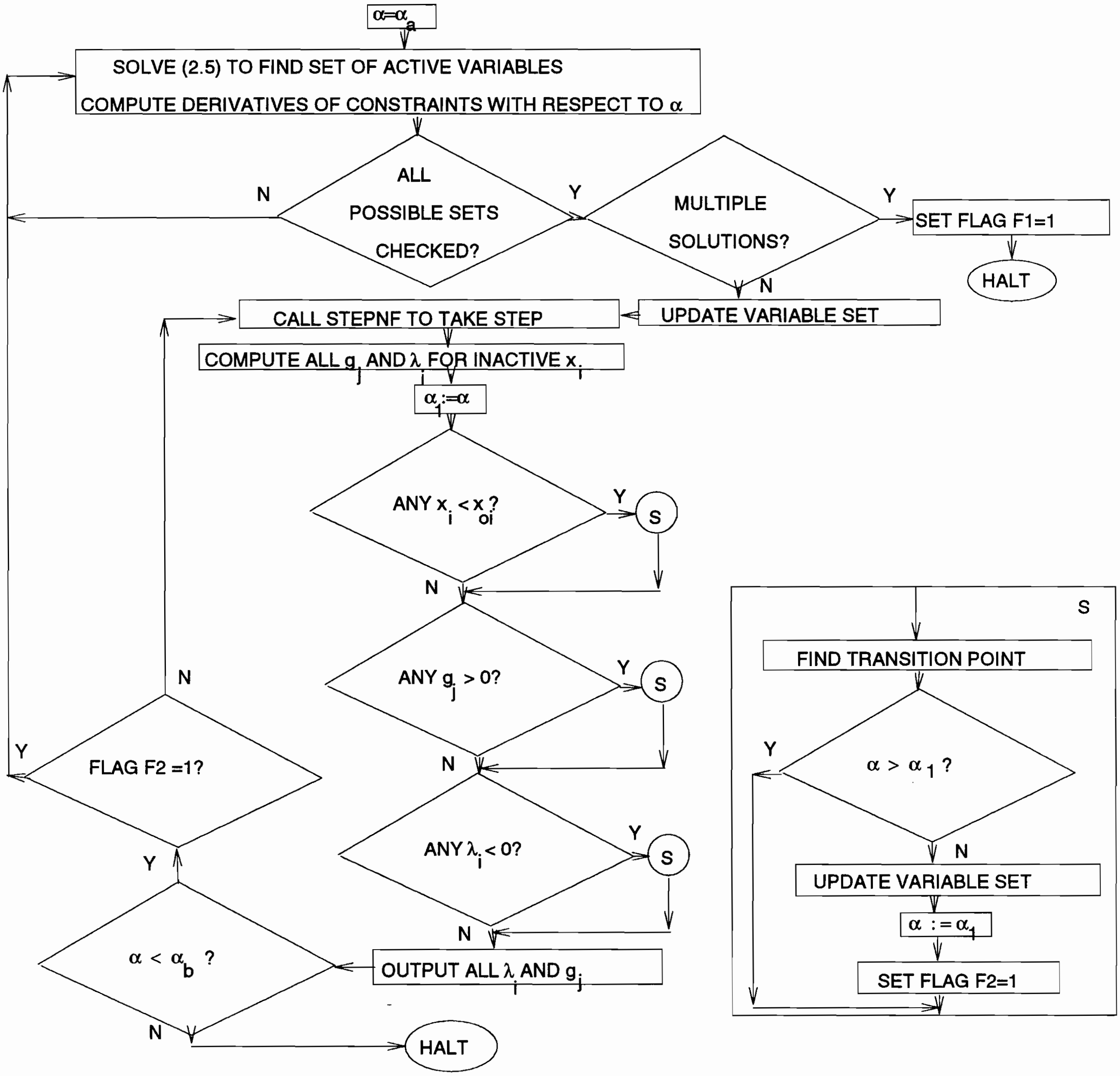

Figure 2.1. Flowchart for algorithm.

2.5. Pseudo-code

The subroutine STEPNF from HOMPACK is used to track the zero curve Γ of the system of equations (2.5)–(2.6) in (α, x, λ) space. The subroutine takes one step at a time along Γ, choosing the optimal size of the step.

A switching point is localized using Hermite cubic interpolation and the secant method (subroutine ROOTNF in HOMPACK). The accuracy of tracking the zero curve Γ and of finding the switching point is set to 10^{-4}.

The variables used by the program are:

LPAR: identity vector of homotopy variables, IPAR, IVAR: work identity vectors of homotopy variables, y: vector of values of homotopy variables, w: work vector of values of homotopy variables, y(1): value of the homotopy parameter α, α_1: temporary value of the homotopy parameter α, *flag*: a flag set true for a switching point;

1. $flag := true$; $\alpha := \alpha_a$; y:= solution at α_a.
2. If $flag = true$ then
   3. set initial values for STEPNF;
   4. do 5.–7. over all possible variable sets:
   5. solve the system of equations (2.5) to check which design variables become inactive and to find the values of the Lagrange multipliers for active constraints of the form (2.3);
   6. if a set of active design variables and active constraints (2.3) with positive Lagrange multipliers is found, solve the system of equations (2.22)–(2.23) to check that no active constraint will be violated for increasing s;
   7. if any active constraint will be violated for increasing s go to 5 to find the next set;
   8. if there are multiple (more that two) valid subsets of variable, set a flag and halt;

9. if no set of active design variables and active constraints (2.3) can be found for the next segment set an error flag and terminate the computations;

10. set initial values for the new variables;

11. $flag := false$; endif.

12. Call STEPNF to take the next step (calculating the new set of variables).

13. Compute the constraints (2.3) and the Lagrange multipliers for inactive design variables.

14. Save the current vector of variables in LPAR.

15. IVAR:= LPAR.

16. w:=y, α_1:= y(1).

17. If any design variable became less than x_{0i} then

18. IPAR:= LPAR;

19. choose the design variable with largest violation of its lower bound;

20. use ROOTNF to find the point where that design variable is equal to x_{0i};

21. if any other design variable is less than x_{0i} go to 19;

22. $flag := true$;

23. w:=y;

24. IVAR:= IPAR;

25. use the current values of the design variables to calculate the g_j's of the form (2.3);

26. use the current values of the Lagrange multipliers for active constraints of the form (2.3) to find the Lagrange multipliers for inactive design variables from equations (2.5);

27. α_1:= y(1); endif.

28. If the value of any g_j of the form (2.3) becomes greater than 0 then

29. IPAR:= LPAR;

30. choose the greatest g_j of the form (2.3) with inactive constraint;

31. use ROOTNF to find the point where the value of this g_j is equal to 0;

32. if any g_j of the form (2.3) with inactive constraint is greater than 0 go to 30;

33. if α_1 < y(1) go to 43;

34. if α_1 = y(1) IPAR:= IVAR;

35. add the Lagrange multiplier associated with the constraint for g_j to the set of variables IPAR;

36. if any other g_j with inactive constraint of the form (2.3) is equal to 0 add the Lagrange multipliers associated with this constraint to the set of variables IPAR;

37. $flag := true$;

38. w:= y;

39. IVAR:= IPAR;

40. use the current values of the design variables to calculate the g_j's of the form (2.3);

41. use the current values of the Lagrange multipliers for active constraints (2.3) to find the Lagrange multipliers for inactive design variables from equations (2.5);

42. α_1:= y(1); endif.

43. if any Lagrange multiplier for inactive design variable is less than 0 then

44. IPAR:=LPAR;

45. choose the most negative Lagrange multipliers associated with an inactive design variable;

46. use ROOTNF to find the point where the Lagrange multiplier is equal to 0;

47. if any other Lagrange multipliers associated with inactive design variables are negative go to 45;

48. if α_1 < y(1) go to 55;

49. if α_1 = y(1) IPAR:= IVAR;

50. $flag := true$;

51. w:=y

52. IVAR:=IPAR

53. use the current values of the design variables to calculate the g_j's of the form (2.3);

54. use the current values of the Lagrange multipliers for active constraints of the form (2.3) to find the values of the Lagrange multipliers for inactive design variables from equations (2.5); endif.

55. if any Lagrange multiplier for active constraint of the form (2.3) is less than 0 then

56. IPAR:=LPAR;

57. choose the most negative Lagrange multiplier associated with an active constraint of the form (2.3);

58. use ROOTNF to find the point where the Lagrange multiplier is equal to 0;

59. if any other Lagrange multipliers associated with active constraints of the form (2.3) are negative go to 57;

60. if $\alpha_1 <$ y(1) go to 69;

61. if $\alpha_1 =$ y(1) IPAR:= IVAR;

62. remove the Lagrange multiplier from the set of variables;

63. if any other Lagrange multipliers associated with active constraints of the form (2.3) are equal to 0 remove these Lagrange multipliers from the set of variables;

64. $flag := true$;

65. w:=y

66. IVAR:=IPAR

67. use the current values of the design variables to calculate the g_j's of the form (2.3);

68. use the current values of the Lagrange multipliers for active constraints of the form (2.3) to find the values of the Lagrange multipliers for inactive design variables from equations (2.5); endif.

69. LPAR:=IVAR.

70. y:=w.

71. Output the values of the Lagrange multipliers, g_j's of the form (2.3) and design variables.

72. If $\alpha < \alpha_b$ then

    73. if $flag = true$ go to 2;

    74. if $flag = false$ go to 12; endif.

75. Halt.

3. Applications

In this chapter we present three applications of the algorithm for tracing the parametrized optima described in Chapter 2. The first application, a spring-mass problem, was used to illustrate various kinds of transition events between segments. The second application, a well known ten-bar truss structural optimization problem, was used to validate the algorithm, since the numerical results for this problem have been obtained by other methods. The third application, bi-objective control-structure optimization, had an important engineering application. The numerical results obtained in this application could be used in the design process – they allowed selection of the best designs and provided insight into behavior of the structure.

3.1 Spring-mass system example

Consider the mass-spring system in Figure 3.1.1 including 5 springs and 3 masses. Let $c_1, \ldots, c_5 > 0$ be costs associated with the springs, u_1, u_2, u_3 be the displacements of the masses and $k_1, \ldots, k_5$ the spring constants. The objective is to vary the spring constants so as to minimize the cost of the springs subject to the condition that displacements are bounded in magnitude by u_a. We want to find the dependence of the optimum solution on the displacement limit u_a. This simple problem permits us to generate a variety of segment transition scenarios by varying the spring constants and applied forces. The problem is formulated as

(3.1.1) $\quad$ minimize $c(k) = \mathrm{c}_1\mathrm{k}_1 + \mathrm{c}_2\mathrm{k}_2 + \mathrm{c}_3\mathrm{k}_3 + \mathrm{c}_4\mathrm{k}_4 + \mathrm{c}_5\mathrm{k}_5$

subject to

(3.1.2) $\quad g_i = 1 - k_i \leqq 0, \quad i = 1, \ldots, 5,$

(3.1.3) $\quad g_6 = -u_a + u_1(k,F) \leqq 0, \quad g_7 = -u_a + u_2(k,F) \leqq 0, \quad g_8 = -u_a + u_3(k,F) \leqq 0,$

(3.1.4) $\quad g_9 = -u_a - u_1(k,F) \leqq 0, \quad g_{10} = -u_a - u_2(k,F) \leqq 0, \quad g_{11} = -u_a - u_3(k,F) \leqq 0,$

where F is the force vector and k is the vector of spring constants.

The displacements u_i are obtained by solving the equilibrium equations

$$Ku = F,$$

where K is the stiffness matrix related to the spring constants:

$$K = \begin{pmatrix} k_2 + k_1 + k_3 & -k_2 & -k_3 \\ -k_2 & k_2 + k_4 & -k_4 \\ -k_3 & -k_4 & k_5 + k_3 + k_4 \end{pmatrix}.$$

The solution needs to be obtained for all values of allowable displacement u_a. The homotopy parameter is taken as $\alpha = 1/u_a$, $(0 < \alpha < \infty)$. The Lagrangian function and Kuhn-Tucker conditions for this problem are:

$$\mathcal{L} = \sum_{i=1}^{5} c_i k_i + \sum_{i=1}^{5} \lambda_i (1 - k_i) + \sum_{j=6}^{8} \lambda_j (u_{j-5} - 1/\alpha) + \sum_{l=9}^{11} \lambda_l (-u_{l-8} - 1/\alpha),$$

$$c_i + \sum_{j=6}^{8} \lambda_j \frac{\partial u_{j-5}}{\partial k_i} - \sum_{j=9}^{11} \lambda_j \frac{\partial u_{j-8}}{\partial k_i} - \lambda_i = 0, \qquad i = 1, \ldots, 5,$$

$$g_j \lambda_j = 0, \qquad j = 1, \ldots, 11,$$

$$\lambda_j \geqq 0, \qquad j = 1, \ldots, 11,$$

$$g_j \leqq 0, \qquad j = 1, \ldots, 11.$$

Paths of optimal solutions for the spring problem for different cost coefficients and sets of forces selected so as to bring about different switching point scenarios are described in Tables 3.1.1–3.1.7. For small values of α (large values of the allowable displacement u_a) none of the displacement constraints are active. At the beginning of the first segment all design variables are assumed to be at their lower bound ($k_0 = 1.0$) and the greatest displacement is assumed to be active. The reciprocal of the magnitude of the greatest displacement is the starting value for the homotopy parameter α. Next the value of the α is increased and the

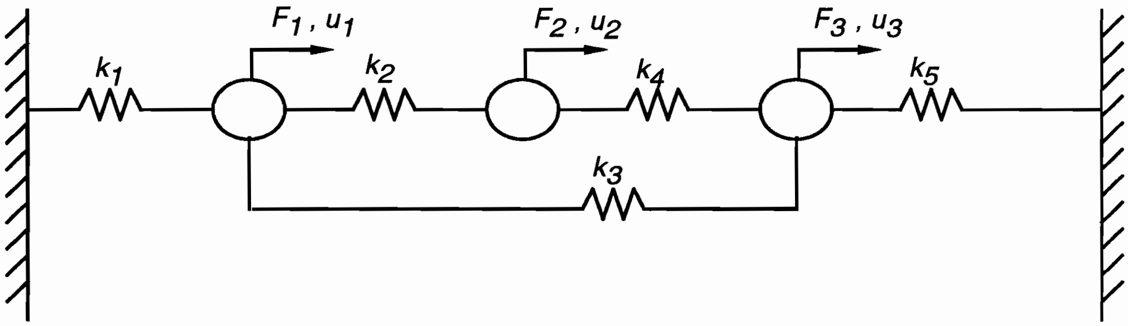

Figure 3.1.1. Spring-mass system.

optimal values of the design variables and the Lagrange multipliers associated with active displacements are computed using the homotopy method.

The path shown in Table 3.1.1 consists of five segments. Segments were terminated when a design variable became active or when a constraint for a displacement became active. The design variable k_1, which was decreasing in Segments 2 and 3, starts from its lower bound value in Segment 4. The last segment of this path (and also in all the other cases except for Table 3.1.6) was terminated arbitrarily at some point of the segment.

The path in Table 3.1.2 contains three segments, shown in Figure 3.1.2 for the displacement u_1 (the large solid dots mark the transition points). For both switching points in this table a design variable and a constraint for a displacement became active simultaneously. The new design variable in the new segment was chosen by considering all possible sets of active design variables according to the procedure described in Section 2.4. Note that the initial value of the Lagrange multiplier λ_8 (for constraint on u_3) in Segment 2 differs from its end value in the previous segment.

The path in Table 3.1.3 consists of four segments. The cost coefficients in Table 3.1.3 have been chosen to get a switching point where two variables (k_4 and k_5) become active at the same time (Segment 1 – Segment 2). Two other switching points are the points where the constraints for a displacement became active.

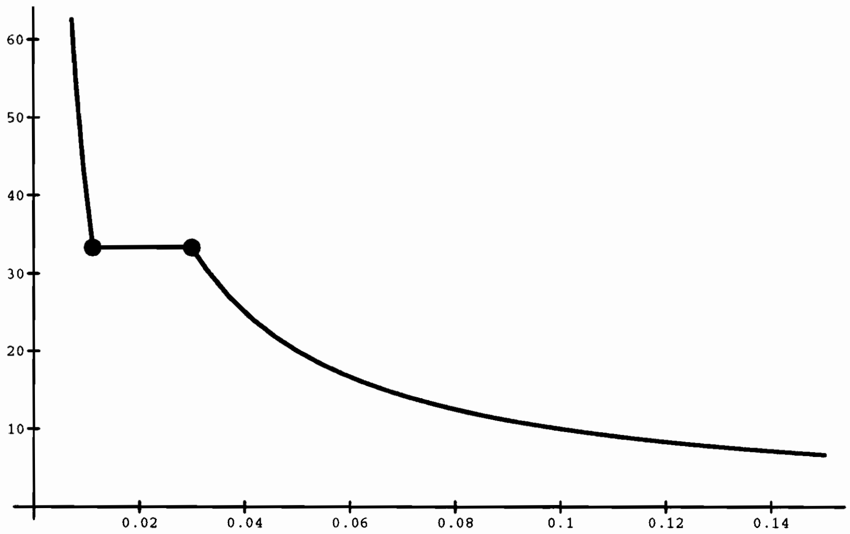

FIGURE 3.1.2. *Displacement $u_1(\alpha)$ from Table 3.1.2.*

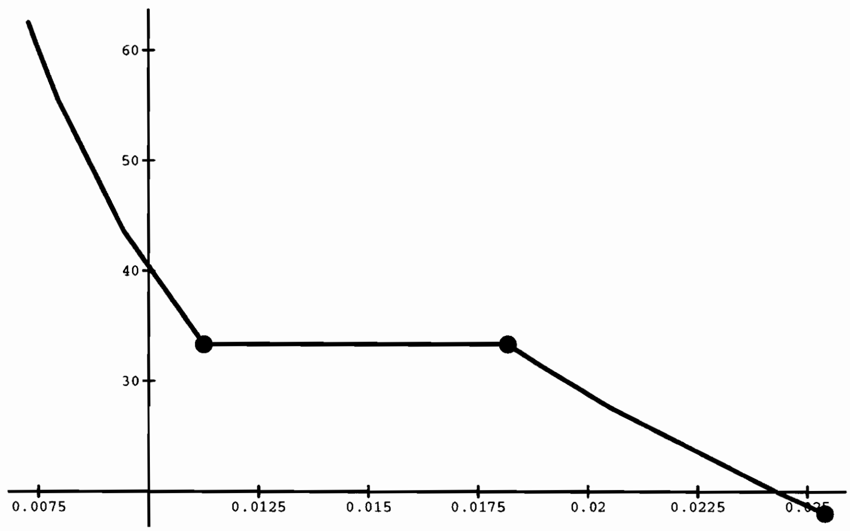

FIGURE 3.1.3. *Displacement $u_1(\alpha)$ from Table 3.1.6.*

In Table 3.1.4 all three displacements and three spring constants become active at the starting point. The path of optimal solutions contains only one segment. The active design variables in this segment have been found by considering all possible sets of active variables.

The path in Table 3.1.5 consists of two segments. At the switching point two spring constants and two displacements become active simultaneously.

In Table 3.1.6 the cost coefficient for spring 5 is a strongly decreasing function of α

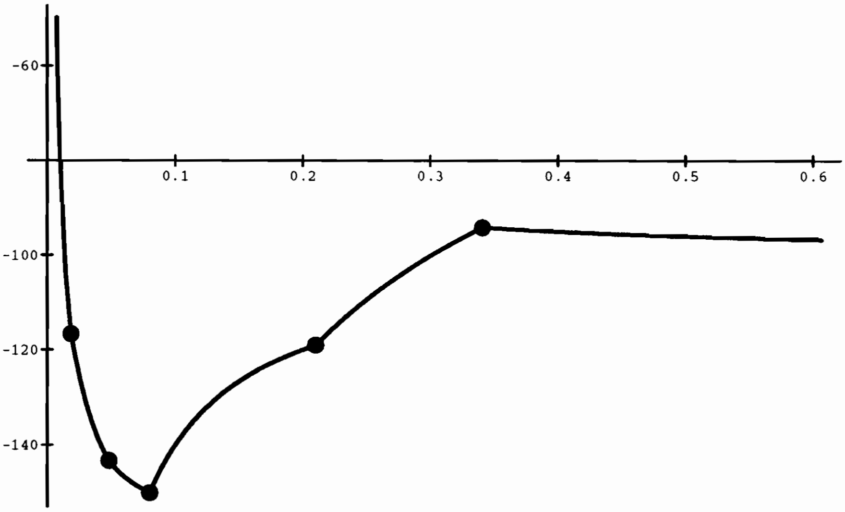

FIGURE 3.1.4. *Displacement $u_2(\alpha)$ from Table 3.1.7.*

and becomes negative in Segment 3. The path (see Figure 3.1.3) contains three segments. For the first switching point a design variable (k_4) and a constraint for displacement u_2 become active simultaneously. At the next switching point another spring constant (k_2) becomes active. For $\alpha > 0.0254071$ the problem becomes unbounded and the cost function could be decreased indefinitely for increasing values of k_2, k_4 and k_5.

In Table 3.1.7 the constraints on u_2 were changed to depend on the parameter α in a different way than given in (3.1.3) and (3.1.4). The path (see Figure 3.1.4) consists of six segments. At the first switching point the lower bound for displacement u_2 and the spring constant k_4 become active. The second segment is terminated when the constraint for displacement u_2 and the spring constant k_4 become inactive. At the next switching point the lower constraint for the displacement u_1 and the spring constant k_4 become active. Later the constraint for displacement u_1 becomes inactive (Segment 5). In Segment 6 the upper constraint for the displacement u_1 becomes active.

Table 3.1.1. Spring example with cost function $c(k) = k_1 + 2k_2 + 3k_3 + 4k_4 + 5k_5$.

Forces: $F_1 = 100$, $F_2 = -300$, $F_3 = 400$.

segment	variable	start value	end value	displacement	start value	end value
1.	α	0.00727	0.00768	u_3	137.50	130.10
	k_1	1.00000	1.39271	u_2	−50.00	−59.85
	λ_8	0.04266	0.06618	u_1	62.50	50.18
	Lagrange multiplier for k_5 lower bound becomes equal to 0					
2.	α	0.00768	0.01118	u_3	130.10	89.43
	k_1	1.39271	1.15654	u_2	−59.85	−89.43
	λ_8	0.06618	0.13861	u_1	50.18	31.68
	k_5	1.00000	1.82648			
	Constraint on u_2 becomes active					
3.	α	0.01118	0.01125	u_3	89.43	88.88
	k_1	1.15654	1.00000	u_2	−89.43	−88.88
	λ_8	0.13861	0.15421	u_1	31.68	33.33
	k_5	1.82648	1.87500			
	λ_{10}	0.00000	0.01875			
	Lagrange multiplier for k_4 lower bound becomes equal to 0 and k_1 hits its lower bound					
4.	α	0.01125	0.08000	u_3	88.88	12.50
	λ_8	0.15421	6.08000	u_2	−88.88	−12.50
	k_5	1.87500	10.00000	u_1	33.33	12.50
	λ_{10}	0.01875	1.60000			
	k_4	1.00000	11.00000			
	k_1	1.00000	6.00000			
	Constraint on u_1 becomes active					
5.	α	0.08000	0.11764	u_3	12.50	8.49
	λ_8	6.08000	12.48470	u_2	−12.50	−8.49
	k_5	10.00000	13.76484	u_1	12.50	8.49
	λ_{10}	1.60000	3.68175			
	k_4	11.00000	16.64726			
	k_1	6.00000	9.76484			
	λ_6	0.00000	0.44292			

All Lagrange multipliers are positive. They increase or remain at the same values. Program was terminated at this point.

Table 3.1.2. Spring example with cost function $c(k) = k_1 + 2k_2 + 3k_3 + 4k_4 + k_5$.

Forces: $F_1 = 100$, $F_2 = -300$, $F_3 = 400$.

segment	variable	start value	end value	displacement	start value	end value
1.	α	0.00727	0.01125	u_3	137.50	88.88
	k_5	1.00000	1.87500	u_2	−50.00	−88.88
	λ_8	0.01163	0.02784	u_1	62.50	33.33
	Lagrange multiplier for k_4 lower bound becomes equal to 0 and constraint on u_2 becomes active					
2.	α	0.01125	0.03000	u_3	88.88	33.33
	k_5	1.87500	5.00000	u_2	−88.88	−33.33
	λ_8	0.05484	0.38999	u_1	33.33	33.33
	λ_{10}	0.03374	0.23999			
	k_4	1.00000	3.50000			
	Lagrange multiplier for k_2 lower bound becomes equal to 0 and constraint on u_1 becomes active					
3.	α	0.03000	0.14410	u_3	33.33	6.93
	k_5	5.00000	27.82023	u_2	−33.33	−6.93
	λ_8	0.38999	8.45016	u_1	33.33	6.93
	λ_{10}	0.23999	5.26333			
	k_4	3.50000	14.91011			
	k_2	1.00000	6.70505			
	λ_6	0.00000	0.82210			

The Lagrange multipliers associated with inactive spring constants remain at the same values ($\lambda_1 = 1.0$, $\lambda_3 = 3.0$) as segment continues. Program was terminated at this point.

Table 3.1.3. Spring example with cost function $c(k) = k_1 + 2k_2 + 3k_3 + 1.75734k_4 + 5k_5$.

Forces: $F_1 = 100$, $F_2 = -300$, $F_3 = 400$.

segment	variable	start value	end value	displacement	start value	end value
1.	α	0.00727	0.00768	u_3	137.50	130.10
	k_1	1.00000	1.39271	u_2	−50.00	−59.85
	λ_8	0.00426	0.06618	u_1	62.50	50.18
Lagrange multipliers for k_4 and k_5 lower bounds become equal to 0 simultaneously						
2.	α	0.00768	0.01992	u_3	130.10	50.17
	λ_8	0.06618	0.38823	u_2	−59.85	−50.17
	k_1	1.39271	2.26352	u_1	50.18	23.45
	k_4	1.00000	2.25560			
	k_5	1.00000	2.92772			
Constraint on u_2 becomes active						
3.	α	0.01992	0.09121	u_3	50.17	10.96
	λ_8	0.38823	6.45331	u_2	−50.17	−10.96
	λ_{10}	0.00000	0.73171	u_1	23.45	10.96
	k_1	2.26352	7.12131			
	k_4	2.25560	12.68197			
	k_5	2.92772	11.12131			
Constraint on u_1 becomes active						
4.	α	0.09121	0.12734	u_3	10.96	7.85
	λ_8	6.45331	11.91582	u_2	−10.96	−7.85
	λ_{10}	0.73171	1.62785	u_1	10.96	7.85
	λ_6	0.00000	0.46004			
	k_1	7.12131	10.73401			
	k_4	12.68197	18.10101			
	k_5	11.12131	14.73401			

The Lagrange multipliers associated with the active spring constants remain at the same values or increase ($\lambda_3 = 3.0$, $\lambda_2 = 8.24$). Program was terminated at this point.

Table 3.1.4. Spring example with cost function $c(k) = k_1 + 2k_2 + 3k_3 + 4k_4 + 5k_5$.

Forces: $F_1 = 100$, $F_2 = -133.33$, $F_3 = 100$.

segment	variable	start value	end value	displacement	start value	end value
1.	α	0.03000	0.40465	u_3	33.33	2.47
	k_1	1.00000	25.97681	u_2	−33.33	−2.47
	λ_8	0.15000	17.18364	u_1	33.33	2.47
	λ_{10}	0.09000	18.90107			
	λ_6	0.03000	13.03830			
	k_4	1.00000	19.73261			
	k_2	1.00000	7.24420			

The values of the Lagrange multipliers for inactive spring constants k_3 and k_5 change very slowly ($\lambda_3 = 3.0$, $\lambda_5 = 3.0$). Program was terminated at this point.

Table 3.1.5. Spring example with cost function $c(k) = k_1 + 2k_2 + 3k_3 + 4k_4 + 5k_5$.

Forces: F_1=103.33, F_2=−133.33, F_3=100.

segment	variable	start value	end value	displacement	start value	end value
1.	α	0.02823	0.03000	u_3	34.58	33.33
	k_1	1.00000	1.10000	u_2	−31.66	−33.33
	λ_6	0.04517	0.05100	u_1	35.41	33.33
	Lagrange multipliers for k_4 and k_2 lower bounds become equal to 0, constraints on u_3 and u_2 become active					
2.	α	0.03000	0.39936	u_3	33.33	2.50
	λ_8	0.14999	16.74794	u_2	−33.33	−2.50
	λ_{10}	0.09000	18.40774	u_1	33.33	2.50
	λ_6	0.03299	13.03830			
	k_1	1.10000	26.95553			
	k_2	1.00000	7.15607			
	k_4	1.00000	19.46823			

The values of the Lagrange multipliers for inactive spring constants k_3 and k_5 change very slowly ($\lambda_3 = 3.0$, $\lambda_5 = 3.0$). Program was terminated at this point.

Table 3.1.6. Spring example with cost function
$c(k) = k_1 + 2k_2 + 3k_3 + 4k_4 + \left(2 - e^{(\alpha/0.01171-1)}\right)k_5$.

Forces: $F_1 = 100$, $F_2 = -300$, $F_3 = 400$.

segment	variable	start value	end value	displacement	start value	end value
1.	α	0.00727	0.01125	u_3	137.50	88.88
	k_5	1.00000	1.87500	u_2	−50.00	−88.88
	λ_8	0.01530	0.02893	u_1	62.50	33.33
	Lagrange multiplier for k_4 lower bound becomes equal to 0 and constraint on u_2 becomes active					
2.	α	0.01125	0.01818	u_3	88.88	55.00
	k_5	1.87500	3.03011	u_2	−88.88	−55.00
	λ_8	0.05581	0.09823	u_1	33.33	33.33
	λ_{10}	0.03360	0.09260			
	k_4	1.00000	1.92409			
	Lagrange multiplier for k_2 lower bound becomes equal to 0					
3.	α	0.01818	0.02540	u_3	55.00	39.35
	k_5	3.03011	4.62390	u_2	−55.00	−39.35
	λ_8	0.09823	0.00000	u_1	33.33	18.00
	λ_{10}	0.09260	0.18975			
	k_4	1.92409	2.49822			
	k_2	1.00000	1.80146			

For $\alpha = 0.02540$ the path terminates with no neighboring solutions. The problem becomes unbounded ($c(k) \to -\infty$).

Table 3.1.7. Spring example with cost function $c(k) = k_1 + 2k_2 + 3k_3 + 4k_4 + k_5$.

Forces: $F_1 = 100, \quad F_2 = -300, \quad F_3 = 400.$

$$g_7 = u_2 - 1/\alpha - 0.009(1/\alpha - 137.5)^2,$$

$$g_{10} = -u_2 - 1/\alpha - 0.009(1/\alpha - 137.5)^2.$$

segment	variable	start value	end value	displacement	start value	end value
1.	α	0.00727	0.01846	u_3	137.50	54.16
	k_5	1.00000	3.46153	u_2	−50.00	−116.66
	λ_8	0.01163	0.07498	u_1	62.50	12.50
	Lagrange multiplier for k_4 lower bound becomes equal to 0 and constraint on u_2 becomes active					
2.	α	0.01846	0.04800	u_3	54.16	20.83
	k_5	3.46153	9.96000	u_2	−116.66	−143.33
	λ_8	0.10127	0.52656	u_1	12.50	−7.50
	λ_{10}	0.03287	0.02460			
	k_4	1.00000	1.00000			
	k_4 and the constraint for u_2 become inactive					
3.	α	0.04800	0.08000	u_3	20.83	12.50
	k_5	9.96000	17.00000	u_2	−143.33	−150.00
	λ_8	0.50688	1.40800	u_1	−7.50	−12.50
	Lagrange multiplier for k_4 lower bound becomes equal to 0 and constraint on u_1 becomes active					
4.	α	0.08000	0.21000	u_3	12.50	4.76
	k_5	17.00000	43.00000	u_2	−150.00	−119.04
	k_4	1.00000	1.50000	u_1	−12.50	−4.76
	λ_8	1.43384	9.15923			
	λ_9	0.04307	0.00000			
	Constraint on u_1 becomes inactive					
5.	α	0.21000	0.34088	u_3	4.76	2.93
	k_5	43.00000	67.17601	u_2	−119.04	−94.13
	λ_8	9.15923	23.11248	u_1	−4.76	2.93
	k_4	1.50000	2.09066			
	Constraint on u_1 becomes active					
6.	α	0.34088	0.60946	u_3	2.93	1.64
	k_5	67.17601	120.89250	u_2	−94.13	−96.71
	λ_8	23.11248	73.88685	u_1	2.93	1.64
	k_4	2.09066	2.05004			
	λ_6	0.00000	0.27801			

The Lagrange multipliers associated with inactive spring constants k_1, k_2, k_3 change very slowly. Program was terminated at this point.

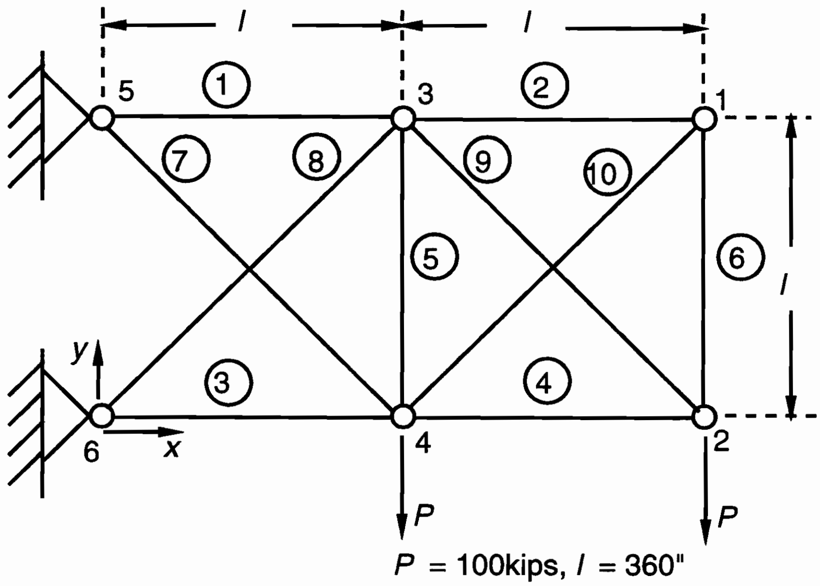

FIGURE 3.2.1. *Ten-bar truss configuration.*

3.2. Ten-bar truss example.

The ten–bar truss shown in Figure 3.2.1 is a well known structural optimization problem. The cross-sectional areas of the truss members, a_i, are designed for minimum weight subject to the condition that stresses in bars do not exceed the allowable stresses $\sigma_{mi}(\alpha)$, $i = 1, \ldots, 10$. The lower bound on the cross-sectional areas is $0.1 in^2$.

The problem is formulated as

$$\text{minimize } w(a) = \sum_{i=1}^{10} \rho l_i a_i,$$

subject to

$$g_i = a_0 - a_i \leqq 0, \qquad i = 1, \ldots, 10,$$

$$g_j = -\sigma_{m(j-10)}(\alpha) + \sigma_{j-10}(a, F) \leqq 0, \qquad j = 11, \ldots, 20,$$

$$g_j = -\sigma_{m(j-20)}(\alpha) - \sigma_{j-20}(a, F) \leqq 0, \qquad j = 21, \ldots, 30,$$

where F is the force vector, l is a bar-length vector and a is the design variable vector consisting of cross-sectional areas. The solution needs to be obtained for $\alpha \in (\alpha_a, \alpha_b)$, where α is the homotopy parameter described later.

The Lagrangian function and Kuhn-Tucker conditions for this problem are:

$$\mathcal{L} = \sum_{i=1}^{10} \rho l_i a_i + \sum_{i=1}^{10} \lambda_i (0.1 - a_i) + \sum_{j=11}^{20} \lambda_j (\sigma_{j-10} - \sigma_{m(j-10)}) + \sum_{l=21}^{30} \lambda_l (-\sigma_{l-20} - \sigma_{m(l-20)}),$$

$$\rho l_i + \sum_{j=11}^{20} \lambda_j \frac{\partial \sigma_{j-10}}{\partial a_i} - \sum_{j=21}^{30} \lambda_j \frac{\partial \sigma_{j-20}}{\partial a_i} - \lambda_i = 0, \qquad i = 1, \ldots, 10,$$

$$g_j \lambda_j = 0, \qquad j = 1, \ldots, 30,$$

$$\lambda_j \geqq 0, \qquad j = 1, \ldots, 30,$$

$$g_j \leqq 0, \qquad j = 1, \ldots, 30.$$

First all the allowable stresses for all bars were set equal in magnitude to the reciprocal of the homotopy parameter α (that α is the reciprocal of the magnitude of the stress allowable). The results are presented in Table 3.2.1 and Figure 3.2.2.

For small values of α the allowable stress is very large and all the design variables are assumed to be at their lower bound ($a_0 = 0.1$ in^2). When α exceeds the reciprocal of the greatest stress magnitude for this minimum gage design some areas must increase. The value of α is increased and the optimal design variables and the Lagrange multipliers associated with active constraints for stresses are computed. The reason for terminating each segment is given in the table, except for the last segment which was terminated arbitrarily. The classical solution for this problem is obtained when the allowable stress is 25 ksi ($\alpha = 0.04$). For this value of the allowable stress truss members 2, 5, 6 and 10 are at their lower bound, and all the other truss members are fully stressed. This solution corresponds to the value $\alpha = 0.04$ which occurs in Segment 5, Table 3.2.1. The values of the cross-sectional areas for the truss members agree with the values obtained by other authors (for $\alpha = 0.03959$ the values for the cross-sectional areas are different from the values given by Kirsch for

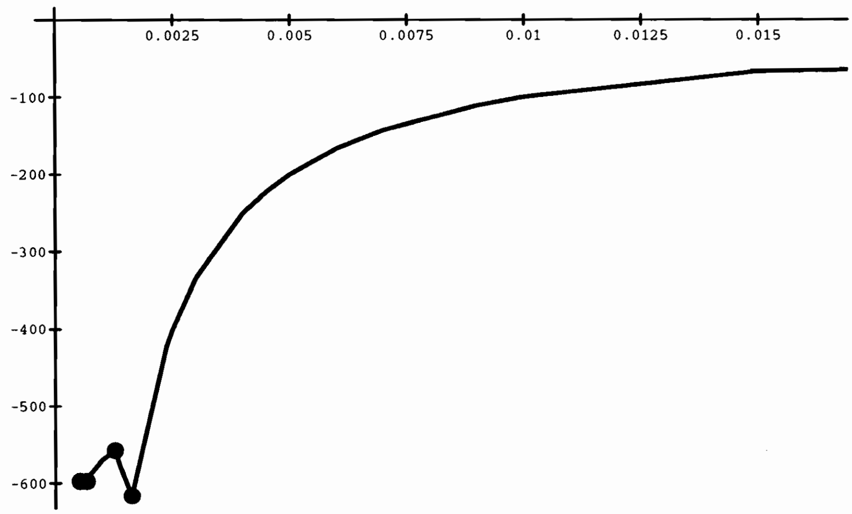

FIGURE 3.2.2. *Stress $\sigma_4(\alpha)$ from Table 3.2.1.*

$\alpha = 0.04$ [14, pp. 97] by about 1%). Segments 1 – 5 show for which values of the allowable stress the particular truss members became fully stressed.

In Table 3.2.2 the allowable stresses for bars $1, \ldots, 8, 10$ are fixed and equal to 25 ksi whereas the allowable stress for bar 9 is an increasing function of α $(-21 - 100\alpha \leq \sigma_{m9} \leq 21 + 100\alpha)$. It is known (e.g., Haftka et al., [8], pp. 238) that when the allowable stress is larger than 37.5 ksi the optimal design is no longer fully stressed, as member 9 is neither at the allowable maximum stress nor at minimum gage. The first segment starts at the optimum point for all allowable stresses equal to 25 ksi. Next the allowable stress for bar 9 is increased. The path of optimal points consists of three segments. For $\alpha = 0.09177669$ ($\sigma_{m9} = 30.17$ksi) the cross-sectional area of bar 10 becomes an active design variable and the stress in that bar assumes the maximum allowable value (constraint on σ_{10} becomes active). For $\alpha = 0.16500000$ ($\sigma_9 = 37.5$ ksi) design variables a_2 and a_6 become active, stresses in these bars attain the maximum allowable value (constraints for σ_2 and σ_6 become active), and the constraint for the stress in bar 9 becomes inactive. For increasing values of α all design variables and all the Lagrange multipliers remain at the same value. For

α=0.165 (σ_9 = 37.5 ksi) the values for the cross-sectional areas agree with the values given by Kirsch [14, pp. 97].

Table 3.2.1. Ten-bar truss example with uniform stress limits:
$-1/\alpha \leq \sigma_i \leq 1/\alpha, \qquad i = 1, \ldots, 10$, areas given in in^2.

segment	variable	start value	end value	stress (ksi)	start value	end value
1.	α	0.00048	0.00051	σ_1	1953.65	1939.50
	a_3	0.10000	0.10623	σ_2	401.25	402.71
	λ_{23}	0.00005	0.00006	σ_3	−2046.35	−1939.50
				σ_4	−598.75	−597.29
				σ_5	354.90	342.21
				σ_6	401.25	402.71
				σ_7	1479.76	1499.76
				σ_8	1348.67	1328.65
				σ_9	846.77	844.68
				σ_{10}	567.45	569.51

Lagrange multiplier for a_1 lower bound becomes equal to 0,
constraint on σ_1 becomes active

2.	α	0.00051	0.00066	σ_1	1939.50	1499.76
	a_1	0.10000	0.12932	σ_2	402.71	402.71
	a_3	0.10623	0.13738	σ_3	−1939.50	−1499.76
	λ_{23}	0.00005	0.00009	σ_4	−597.29	−597.28
	λ_{11}	0.00051	0.00008	σ_5	342.21	342.21
				σ_6	402.71	402.71
				σ_7	1499.76	1499.76
				σ_8	1328.65	1328.65
				σ_9	844.68	844.68
				σ_{10}	569.51	569.51

Lagrange multiplier for a_7 lower bound becomes equal to 0,
constraint on σ_7 becomes active

3.	α	0.00066	0.00126	σ_1	1499.76	788.78
	a_1	0.12932	0.19748	σ_2	402.71	442.24
	a_3	0.13738	0.30961	σ_3	−1499.76	−788.79
	a_7	0.10000	0.25857	σ_4	−597.28	−557.76
	λ_{23}	0.00012	0.00046	σ_5	342.21	0.00
	λ_{17}	0.00016	0.00059	σ_6	402.71	442.24
	λ_{11}	0.00005	0.00018	σ_7	1499.76	788.79
				σ_8	1328.65	788.79
				σ_9	844.68	788.79
				σ_{10}	569.51	625.43

Lagrange multipliers for a_8, a_9 lower bounds become equal to 0, constraints on σ_8, σ_9 become active

4.	α	0.00126	0.00162	σ_1	788.78	616.78
	a_1	0.19748	0.26213	σ_2	442.24	383.22
	a_3	0.30961	0.38639	σ_3	−788.79	−616.78
	a_7	0.25857	0.31715	σ_4	−557.76	−616.78
	a_8	0.10000	0.14142	σ_5	0.00	0.00
	a_9	0.10000	0.14142	σ_6	442.24	383.22
	λ_{23}	0.00034	0.00057	σ_7	788.79	616.78
	λ_{19}	0.00026	0.00043	σ_8	788.79	616.78
	λ_{18}	0.00026	0.00043	σ_9	788.79	616.78
	λ_{17}	0.00037	0.00061	σ_{10}	625.43	542.95
	λ_{11}	0.00029	0.00048			

Lagrange multiplier for a_4 lower bound becomes equal to 0,
constraint on σ_4 becomes active

5.	α	0.00162	0.03959	σ_1	616.78	25.25
	a_1	0.26213	7.85717	σ_2	383.22	15.69
	a_3	0.38639	7.98143	σ_3	−616.78	−25.25
	a_4	0.10000	3.89752	σ_4	−616.78	−25.25
	a_7	0.31715	5.68766	σ_5	0.00	0.0
	a_8	0.14142	5.51192	σ_6	383.22	15.69
	a_9	0.14142	5.51192	σ_7	616.78	25.25
	λ_{23}	0.00057	0.31357	σ_8	616.78	25.25
	λ_{19}	0.00043	0.31357	σ_9	616.78	25.25
	λ_{18}	0.00043	0.31357	σ_{10}	542.95	22.19
	λ_{17}	0.00061	0.31357			
	λ_{11}	0.00048	0.31357			
	λ_{24}	0.00000	0.15678			

Program was terminated at this point.

Table 3.2.2. Ten-bar truss example with variable allowable stress for member 9:
$-25.0 \leq \sigma_i \leq 25.0, \qquad i = 1, \ldots, 8, 10,$
$-21.0 - 100\alpha \leq \sigma_9 \leq 21.0 + 100\alpha$, areas given in in^2.

segment	variable	start value	end value	stress (ksi)	start value	end value
1.	α	0.04	0.09177	σ_1	25.00	25.00
	a_1	7.94000	7.93000	σ_2	15.52	17.67
	a_3	8.06000	8.07071	σ_3	−25.00	−25.00
	a_4	3.94000	3.93000	σ_4	−25.00	−25.00
	a_7	5.74000	5.75685	σ_5	0.05	0.0
	a_8	5.57000	5.55685	σ_6	15.52	17.67
	a_9	5.57000	4.60344	σ_7	25.00	25.00
	λ_{23}	0.31357	0.32062	σ_8	25.00	25.00
	λ_{19}	0.31357	0.22013	σ_9	25.00	30.17
	λ_{18}	0.31357	0.31874	σ_{10}	21.95	25.00
	λ_{17}	0.31357	0.32125			
	λ_{11}	0.31357	0.31937			
	λ_{24}	0.15678	0.15937			
	Lagrange multipliers for a_{10} lower bound becomes equal to 0, constraint on σ_{10} becomes active					
2.	α	0.09177	0.16500	σ_1	25.00	25.00
	a_1	7.93000	7.90000	σ_2	17.67	25.00
	a_3	8.07071	8.10000	σ_3	−25.00	−25.00
	a_4	3.93000	3.90000	σ_4	−25.00	−25.00
	a_7	5.75685	5.79827	σ_5	0.0	0.0
	a_8	5.55685	5.51543	σ_6	17.67	25.00
	a_9	4.60344	3.67695	σ_7	25.00	25.00
	a_{10}	0.10000	0.14213	σ_8	25.00	25.00
	λ_{23}	0.31400	0.32333	σ_9	30.17	37.50
	λ_{19}	0.21300	0.13999	σ_{10}	25.00	25.00
	λ_{18}	0.31000	0.31333			
	λ_{17}	0.31600	0.32666			
	λ_{11}	0.31100	0.31666			
	λ_{24}	0.15500	0.15666			
	λ_{20}	0.00290	0.00666			
	Lagrange multipliers for a_6, a_2 bounds become equal to 0, constraints on σ_6, σ_2 become active, constraint on σ_9 becomes inactive					
3.	α	0.16500	15.96428	σ_1	25.00	25.00
	a_1	7.90000	7.90000	σ_2	25.00	25.00

a_3	8.10000	8.10000	σ_3	-25.00	-25.00
a_4	3.90000	3.90000	σ_4	-25.00	-25.00
a_7	5.79827	5.79827	σ_5	0.0	0.0
a_8	5.51543	5.51543	σ_6	25.00	25.00
a_9	3.67695	3.67695	σ_7	25.00	25.00
a_{10}	0.14213	0.14213	σ_8	25.00	25.00
λ_{23}	0.39333	0.39333	σ_9	37.50	62.50
λ_{20}	0.14666	0.14666	σ_{10}	25.00	25.00
λ_{18}	0.17333	0.17333			
λ_{17}	0.46666	0.46666			
λ_{11}	0.24666	0.24666			
λ_{24}	0.08666	0.08666			
λ_{16}	0.07333	0.07333			
λ_{12}	0.06666	0.06666			
a_2	0.10000	0.10000			
a_6	0.10000	0.10000			

Program was terminated at this point.

3.3 Bi-objective control-structure optimization

The problem of simultaneous structure-control optimization is formulated as the minimization of the structural weight W and maximum control force $F_{\max}$ subject to constraints on the damping ratios ζ_i of the first n_m vibration modes of the structure.

The equations of motion of the structure controlled by n_c collocated sensors and actuators are written as

$$M\ddot{u} + D_0\dot{u} + Ku = F,$$

where M, D_0 and K are the mass, structural damping and stiffness matrices respectively, u is the displacement vector, F is the applied control force vector, and a dot denotes differentiation with respect to time. A simple direct-rate feedback control law (see Martinovic et al. [23]) is used for the actuator force vector F given as

$$F = -D_c\dot{u},$$

where D_c is the control matrix which has nonzero rows and columns at positions corresponding to components of $\dot{u}$ measured by the sensors. Assuming that there is no structural damping ($D_0 = 0$), the structure is described by the system

$$M\ddot{u} + D_c\dot{u} + Ku = 0$$

with the general solution $u = u_0 e^{\mu t}$. The stability of the system is controlled by the real parts of the eigenvalues μ_i. The stability margins are characterized by the damping ratios ζ_i defined as

$$\zeta_i = \frac{-\sigma_i}{\sqrt{\sigma_i^2 + \omega_i^2}},$$

where σ_i and ω_i are the real and imaginary parts of μ_i.

We assume that the matrix D_c is positive semidefinite so that the closed loop system has at least the same stability as the open loop system. Following Martinovic et al. [23], the goal is to have a control system which minimizes the maximum control forces for a given velocity bound $\|\dot{u}\|_\infty \le U$. The maximum control force applied by the actuators is

$$F_{\max} = \max \frac{\|F\|_\infty}{\|\dot{u}\|_\infty} = \|D_c\|_\infty = \max_i \sum_j |d_{ij}|,$$

where the d_{ij} are the elements of the control matrix D_c.

The problem of simultaneous control-structure optimization is the bi-objective optimization problem

$$\begin{aligned} &\text{minimize } (W(a), F_{\max}(a, D_c)) \\ &\text{such that } \sum_j |d_{ij}| \leqq F_{\max}, \\ &\zeta_i(a, D_c) \geqq \zeta_{0i} \quad \text{for} \quad i = 1, \dots, n_m, \\ &D_c \geqq 0, \quad (D_c \text{ positive semidefinite}), \\ &a_i \geqq a_{0i} \quad \text{for} \quad i = 1, \dots, n_s, \end{aligned} \tag{3.3.1}$$

where a is a vector of structural dimensions and $W(a)$ is the structure's weight. The curve of all efficient solutions (designs for which neither $W(a)$ nor $F_{\max}$ can be simultaneously improved) can be obtained by minimizing the combination $(1-\alpha)W + \alpha F_{\max}$ of the two objective functions for all values of α between 0 and 1. The problem can be rewritten as

$$\text{minimize } c(x, \alpha) = (1-\alpha)W + \alpha F_{\max} \tag{3.3.2}$$

$$\text{subject to } g_i(x) = x_{0i} - x_i \leqq 0, \qquad i = 1, \dots, n_1, \tag{3.3.3}$$

$$g_{j+n_1}(x) \leqq 0, \qquad j = 1, \dots, n_2, \tag{3.3.4}$$

where x is the n_1-vector of design variables including a structural size vector a, the nonzero elements of the matrix D_c, and $F_{\max}$. The design variables are subject to the minimum value constraints $x_i \geqq x_{0i}$; the constraints (3.3.4) correspond to the other constraints in

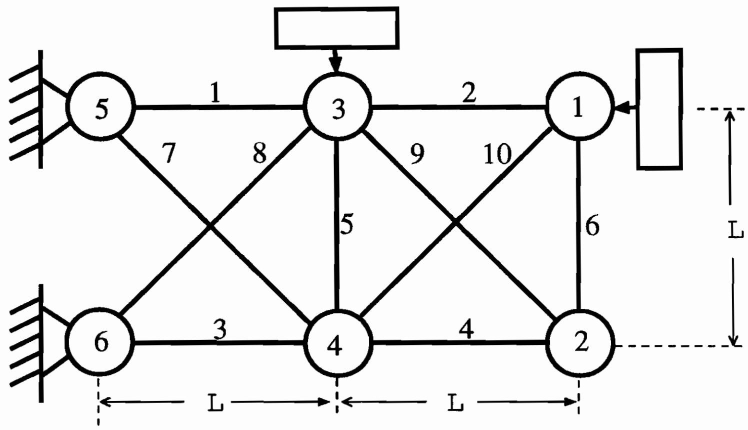

FIGURE 3.3.1. *Ten-bar truss with actuators.*

the problem (3.3.1); and α is the parameter assuming all values between 0 and 1. The Lagrangian function and Kuhn-Tucker conditions for this problem are:

$$\mathcal{L}(x,\lambda,\alpha) = c(x,\alpha) + \sum_{i=1}^{n_1} \lambda_i(x_{0i} - x_i) + \sum_{j=n_1+1}^{n_1+n_2} \lambda_j g_j(x), \tag{3.3.5}$$

$$\frac{\partial c}{\partial x_i} + \sum_{j=n_1+1}^{n_1+n_2} \lambda_j \frac{\partial g_j}{\partial x_i} - \lambda_i = 0, \qquad i = 1,\ldots,n_1, \tag{3.3.6}$$

$$g_j\lambda_j = 0, \qquad j = 1,\ldots,n_1+n_2, \tag{3.3.7}$$

$$\lambda_j \geqq 0, \qquad j = 1,\ldots,n_1+n_2, \tag{3.3.8}$$

$$g_j \leqq 0, \qquad j = 1,\ldots,n_1+n_2. \tag{3.3.9}$$

Numerical results are presented here for the ten-bar truss structure shown in Figure 3.3.1. Numbers in circles indicate joints and plain numbers label truss elements. The truss is controlled by two pairs of direct-rate feedback collocated sensors and actuators shown by boxes in the figure. The sensors measure velocities, and the actuators apply forces at the positions and directions indicated in Figure 3.3.1. The positions of the actuators have been obtained by an optimization that determined the most effective locations for controlling the first four modes. The sensor and actuator pairs are associated with the first (horizontal velocity at joint 1) and sixth (vertical velocity at joint 3) components of the velocity vector $\dot{u}$. The weight of the truss consists of the structural and nonstructural components. The structural weight of the truss is given by $\sum_{i=1}^{10} \rho a_i l_i$, where a_i and l_i are the cross-sectional area and length, respectively, of the i-th truss member and ρ is the weight density. The nonstructural weight is in the form of 4 concentrated masses located at nodes 1, 2, 3 and 4. The first four modes are required to have at least three percent damping ($\zeta_{0i} = 0.03$), $L = 354\,\text{in}$, and the minimum gage area for all truss members is $a_{0i} = 0.1085\,\text{in}^2$. The optimization problem (3.3.1) then becomes

$$
\begin{aligned}
\text{minimize } c(a,\alpha) &= (1-\alpha)k\sum_{i=1}^{10} \rho a_i l_i + \alpha F_{\max}, \\
\text{subject to } g_i &= a_{0i} - a_i \leqq 0, \qquad i = 1,\ldots,10, \\
g_{11} &= -F_{\max} \leqq 0, \\
g_{12} &= -d_{11} \leqq 0, \\
g_{13} &= -d_{66} \leqq 0, \\
g_{14} &= d_{16}^2 - d_{11}d_{66} \leqq 0, \\
g_{15} &= |d_{11}| + |d_{16}| - F_{\max} \leqq 0, \\
g_{16} &= |d_{16}| + |d_{66}| - F_{\max} \leqq 0, \\
g_{j+16} &= -0.03 + \zeta_j(a, d_{11}, d_{16}, d_{66}) \geqq 0, \qquad j = 1,\ldots,4,
\end{aligned}
$$

where a is a vector of truss element cross-sectional areas, d_{11}, d_{16}, d_{66} are the nonzero entries of the control matrix D_c, $F_{\max}$ is the control force applied by actuators, and k is a scaling constant taken here to be 0.0261. The design variables in this formulation include the cross-sectional area vector a, the gains d_{11}, d_{16}, d_{66} and $F_{\max}$. The constraint g_{14} is the positive semidefinite requirement. Since $F_{\max}$ is not a smooth function of the other design variables, adding it as a design variable removes discontinuities in the derivative of the objective function. Furthermore, the absolute value function $|d_{ij}|$ is not differentiable at zero and so is replaced by a quartic polynomial near zero:

$$|d_{ij}| = \frac{d_t}{2}\left[3\left(\frac{d_{ij}}{d_t}\right)^2 - \left(\frac{d_{ij}}{d_t}\right)^4\right] \qquad \text{for } |d_{ij}| \leqq d_t,$$

where d_t is taken to be 5% of a typical value for d_{ij}.

The results have been obtained for three values of the ratio of the nonstructural weight to structural weight: low (5.51 lb at each of nodes $1,\ldots,4$), medium (22.04 lb at each of nodes $1,\ldots,4$), and high (88.16 lb at each of nodes $1,\ldots,4$). These weights correspond to the ratios 0.4548, 1.8191 and 7.2765 of the nonstructural weight to the weight of the structure when all members are at minimum gage.

The switching points on the path of stationary points for the low weight ratio are shown in Table 3.3.1. For $\alpha = 0$ the weight is the only objective, hence the cost function is minimized when all the areas are at minimum gage. The values for d_{11}, d_{16}, d_{66} and $F_{\max}$ were obtained by minimizing the control objective with a standard sequential quadratic programming algorithm (VMCON). The same solution holds for small values of α. For $\alpha \geqq 0.11747$ the derivative of the objective function with respect to a_1 becomes negative and therefore the objective function can be reduced by using a_1 as an active design variable. The homotopy method is used to follow the path of stationary points starting with this value of α.

The path shown in Table 3.3.1 consists of 9 segments, with the second column in the table giving α at the beginning of the segment. The last column in the table describes the event that signaled the switching point at the beginning of the segment. Segments are terminated when a design variable or a constraint becomes active, or when an active design variable becomes inactive. Plots of $F_{\max}$ as a function of W, the structural weight W as a function of α, and the control force $F_{\max}$ as a function of α, are given in Figures 3.3.2, 3.3.3, and 3.3.4, respectively. The kink in one of the segments in the efficient curve for this

TABLE 3.3.1
Path of solutions for low nonstructural weight.

Segment	α	$F_{\max}$	W	c	Event
0.	0.00000	2.02826	48.46283	1.26477	$F_{\max}$, d_{11}, d_{16}, d_{66} and g_{15}, g_{16}, g_{20} are active
1.	0.11747	2.02826	48.46283	1.35446	a_1 becomes active
2.	0.18311	1.75177	50.28659	1.39282	Constraint on ζ_2 becomes active
3.	0.37383	1.75176	50.28673	1.47663	a_7 becomes active
4.	0.40406	1.69351	51.70699	1.48846	Constraint on ζ_1 becomes active
5.	0.59911	1.69351	51.70703	1.55557	a_4 becomes active
6.	0.77852	1.67529	53.34251	1.61257	a_6 becomes active
7.	0.89849	1.66523	55.31673	1.64274	a_7 becomes inactive
8.	0.92169	1.66484	55.46817	1.64783	a_3 becomes active
9.	1.00480	1.65938	59.60609	1.65988	α becomes greater than 1

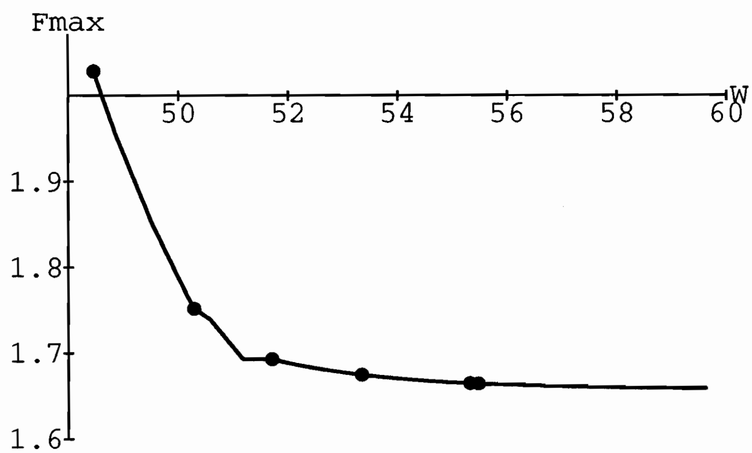

FIGURE 3.3.2. *$F_{\max}$ as a function of W for low nonstructural weight.*

case (Fig. 3.3.2) is caused by a small number of points found for this segment (the stepsize for tracing the curve is chosen automatically).

Plots of the weight and the maximum control force indicate that the best designs can be obtained for values of α near 0.4. For these values of α the maximum control force $F_{\max}$ is reduced by 92% of its maximum decrease (corresponding to α changing from 0 to

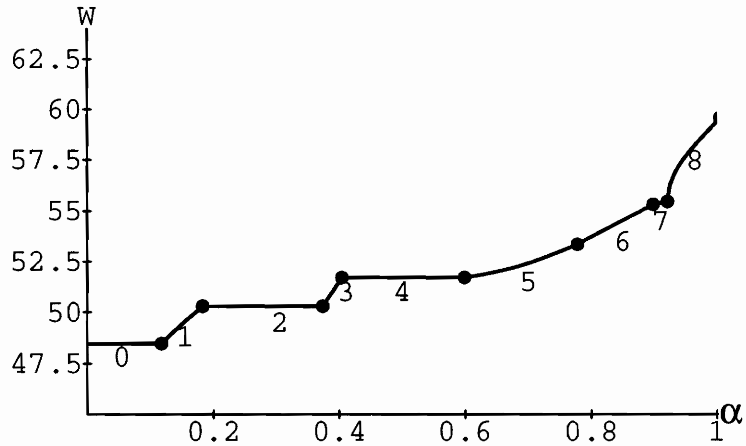

FIGURE 3.3.3. *Structural weight W (pounds) for low nonstructural weight.*

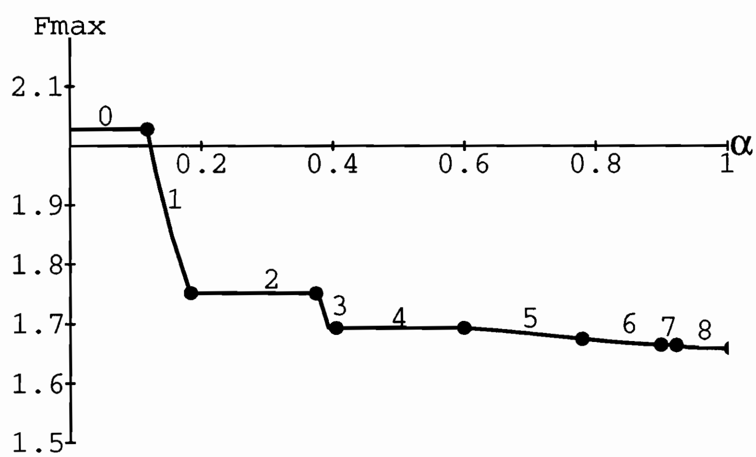

FIGURE 3.3.4. F_{max} *(pounds) for low nonstructural weight.*

1), whereas the weight is increased only by 29% of its maximum change. It is interesting to note that along some segments (0, 2, 4) the design is essentially frozen with only the Lagrange multipliers changing.

The path for the medium weight ratio is described in Table 3.3.2. Plots of F_{max} as a function of W and the two components of the objective function W and F_{max} are given in Figures 3.3.5, 3.3.6, and 3.3.7, respectively. In this case the best designs can be obtained for values of α near 0.8 (F_{max} reduced by 83% of its maximum change and the weight

TABLE 3.3.2
Path of solutions for
medium weight ratio of the nonstructural to the structural weight.

Segment	α	$F_{\max}$	W	c	Event
0.	0.00000	3.02251	48.46283	1.26477	$F_{\max}$, d_{11}, d_{16}, d_{66} and g_{15}, g_{16}, g_{20} are active
1.	0.10921	3.02251	48.46283	1.45673	a_1 becomes active
2.	0.16127	2.74944	50.15051	1.54114	Constraint on ζ_2 becomes active
3.	0.28693	2.74944	50.15056	1.72217	a_7 becomes active
4.	0.31255	2.65684	51.66604	1.75733	Constraint on ζ_1 becomes active
5.	0.83346	2.65684	51.66609	2.43893	a_4 becomes active
6.	0.86771	2.65520	52.02666	2.48356	a_6 becomes active
7.	0.73749	2.60349	58.87609	2.32340	a_7 becomes inactive
8.	0.87006	2.59907	59.62525	2.46354	Constraint on ζ_2 becomes inactive
9.	0.93036	2.54966	76.44878	2.51104	a_5 becomes active
10.	0.94390	2.53224	86.29556	2.51653	a_3 becomes active
11.	0.94940	2.52316	92.48853	2.51762	a_1 becomes inactive
12.	1.00183	2.51446	105.45971	2.51402	α becomes greater than 1

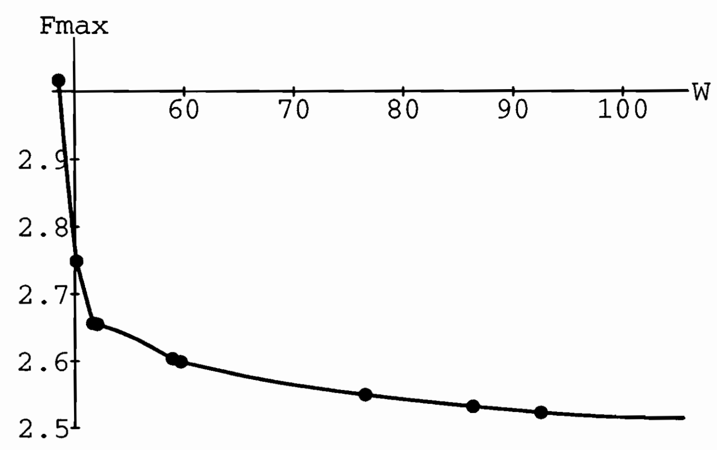

FIGURE 3.3.5. *$F_{\max}$ as a function of W for medium nonstructural weight.*

increased only by 20% of its maximum change).

Along Segments 2 and 4 the design variables again stay essentially at the same value.

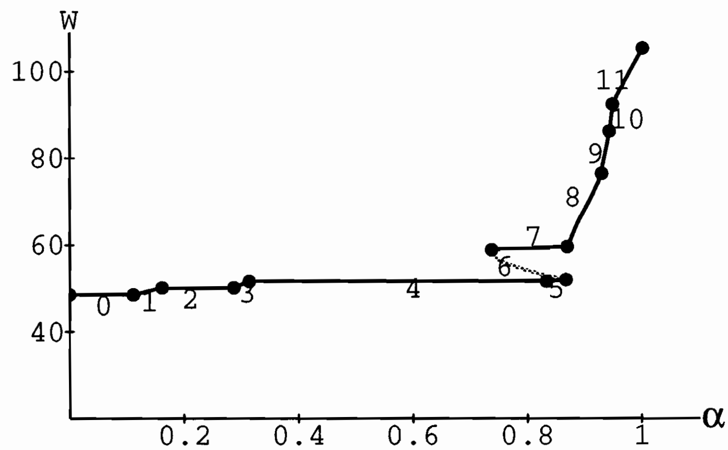

FIGURE 3.3.6. *Weight W (pounds) for the medium nonstructural weight (gray line denotes nonoptimal stationary points, black line denotes optimal points).*

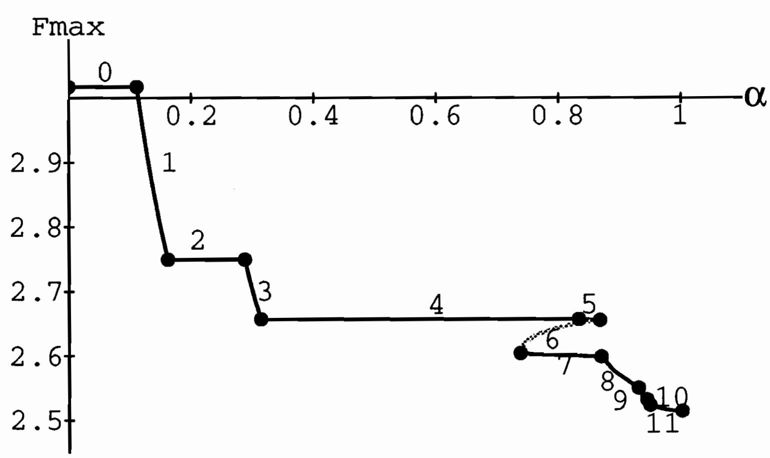

FIGURE 3.3.7. *$F_{\max}$ (pounds) for medium nonstructural weight (gray line denotes stationary nonoptimal points, black line denotes optimal points).*

This time these constant segments account for most of the range of α variation. At the end of Segment 5 no new segment for increasing α can be found. However it is possible to continue the path by decreasing α to obtain Segment 6. The second order necessary conditions are not satisfied along this segment, so points of Segment 6 are only stationary points for the problem. The path of optimal solutions is resumed in Segment 7. The plot of the objective function in Segments 5, 6, and 7 is magnified in Figure 3.3.8. The figure indicates that in the range of $0.738 \leq \alpha \leq 0.870$ there are at least two local minima. Up to about $\alpha = 0.78$, Segment 4 represents the better minimum, and then Segment 7 does.

At points of discontinuity of the path of optimal solutions a standard optimization program (e.g., VMCON) can be used to find a point where the solutions again become optimal. It can be also worthwhile to follow the path of nonoptimal stationary points until a new optimal point is encountered, if the nonoptimal segment is short or if it is difficult to find a point on another optimal branch using standard optimization. In this work the path of stationary points was followed even if they did not satisfy the necessary optimality conditions.

At the beginning of Segment 8 the path of the stationary points can again be tracked only by decreasing the parameter α along a nonoptimal segment. After α decreases from 0.8700583 to 0.8700568 the path of stationary points turns smoothly (F_x becomes singular) and continues for increasing values of α, becoming optimal again. The two components of the objective function, the structural weight W and the control force $F_{\max}$, at the beginning of Segment 8 are shown in Figures 3.3.9 and 3.3.10, respectively. The scale in Figures 3.3.9–3.3.10 indicates that the solution undergoes extreme changes in that region with the logarithmic derivative of the weight with respect to α (percent change in W divided by percent change in α) being of the order of 300. This requires tracing the curve with high accuracy.

A similar behavior of the objective function is observed at the beginning of Segment 9. The path of stationary points exists only for decreasing values of α. The path turns smoothly after α decreases by about 0.00013 and continues for increasing values of α. Points corresponding to decreasing values of α are again nonoptimal points satisfying only the first order necessary conditions.

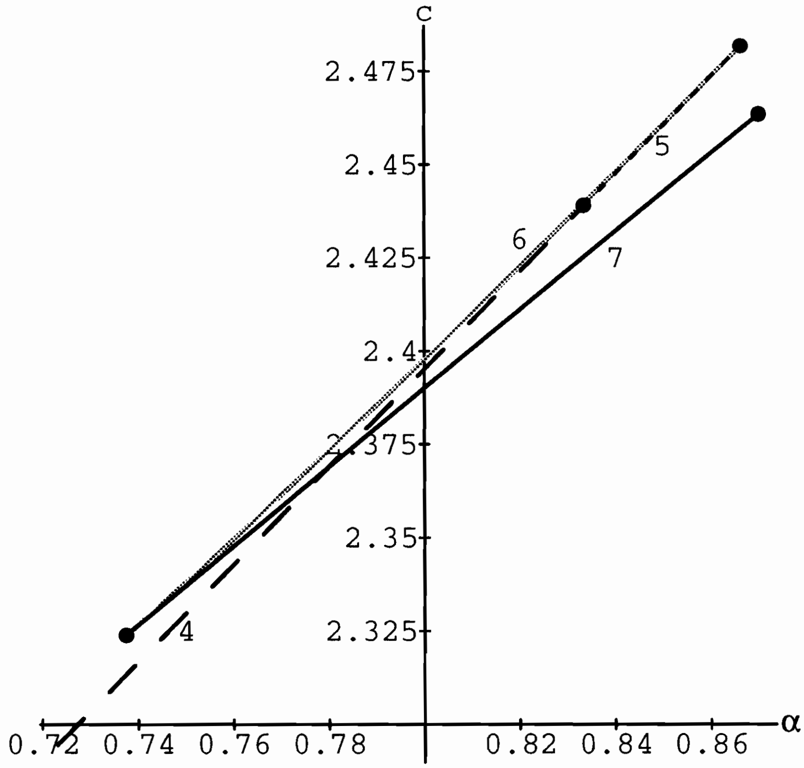

FIGURE 3.3.8. *Objective function c (medium nonstructural weight) along Segments 4–7; black lines (4: dashed, 5: dotted, 7: solid) denote optimal solutions, gray line (6) denotes nonoptimal stationary points.*

The path for the high weight ratio is given in Table 3.3.3. Plots of the objective function, F_{max} as a function of W, the control force $F_{\text{max}}(\alpha)$ and the structural weight $W(\alpha)$ are shown in Figures 3.3.11, 3.3.12, 3.3.13, and 3.3.14, respectively.

The path consists of three disconnected parts. Part 1 (Segments 0–16) starts at $\alpha = 0$. After α reaches 0.989 at the end of Segment 5, the path continues for decreasing values of α. The second order sufficient optimality conditions are not satisfied along Segment 3 and along Segments 6–16. Figures 3.3.11 and 3.3.13 show that Segment 3 and Segments beyond 5 are only saddle points. The program was terminated in Segment 16, due to numerical difficulties in calculating damping ratios.

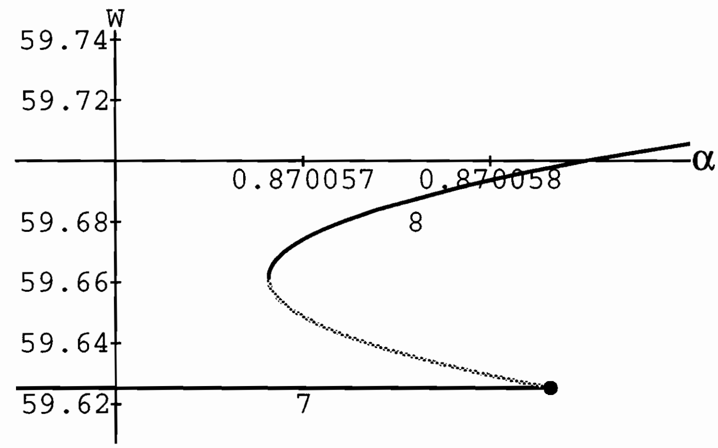

FIGURE 3.3.9. *Weight W (medium nonstructural weight) at the beginning of Segment 8 (black line denotes optimal solutions, gray line denotes stationary nonoptimal points).*

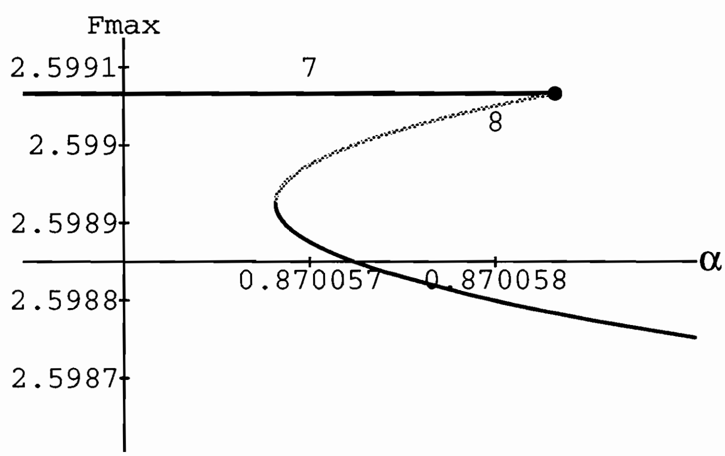

FIGURE 3.3.10. *$F_{\max}$ (medium nonstructural weight) at the beginning of Segment 8 (black line denotes optimal solutions, gray line denotes stationary nonoptimal points).*

Segment 3 corresponds to decreasing values of α. The plot of the objective function in Segments 2, 3 and 4 (Figure 3.3.16) indicates that the end of Segment 2 and the beginning

TABLE 3.3.3

Path of solutions for high weight ratio of the nonstructural to the structural weight.

Seg-ment	α	$F_{\max}$	W	c	Event
0.	0.00000	5.50241	48.46283	1.26477	$F_{\max}$, d_{11}, d_{16}, d_{66} and g_{15}, g_{16}, g_{20} are active
1.	0.07049	5.50241	48.46283	1.56347	a_1 becomes active
2.	0.11359	4.97967	50.43289	1.73231	Constraint on ζ_2 becomes active
3.	0.29756	4.97967	50.43290	2.40629	a_7 becomes active
4.	0.20892	4.87604	51.54865	2.08296	Constraint on ζ_1 becomes active
5.	0.74981	4.87604	51.54868	3.99269	a_6 becomes active
6.	0.98954	4.87275	52.62111	4.83614	a_{10} becomes active
7.	0.37866	4.76709	57.49997	2.73736	a_4 becomes active
8.	0.26466	4.69717	58.77131	2.37100	Constraint on ζ_1 becomes inactive
9.	0.21374	4.88150	56.56069	2.20399	a_7 becomes inactive
10.	0.17569	4.68685	58.35044	2.07872	Constraint on ζ_2 becomes inactive
11.	0.16092	4.69890	58.26212	2.03197	a_9 becomes active
12.	0.14978	5.12788	55.43446	1.99808	a_4 becomes inactive
13.	0.16106	5.39386	53.70674	2.04461	a_7 becomes active
14.	0.17442	5.32668	54.21070	2.09709	a_6 becomes inactive
15.	0.16120	5.40809	53.58279	2.04475	a_7 becomes inactive
16.	0.12435	5.54933	52.69812	1.89434	a_1 becomes inactive
	0.07236	5.32467	53.42554	1.67869	Program terminated
17.	1.00000	4.54845	211.67533	4.54845	F_{max}, d_{11}, d_{16}, d_{66}, a_3, a_4, a_5, a_6 and g_{15} g_{16}, g_{17}, g_{19}, g_{20} are active
18.	0.99379	4.54915	211.64788	4.55521	Constraint on ζ_3 becomes inactive
19.	0.96882	4.56388	181.89706	4.56959	a_3 becomes inactive
20.	0.95304	4.56442	181.38046	4.57237	a_1 becomes active
21.	0.94695	4.60025	155.98565	4.57216	a_5 becomes inactive
22.	0.88676	4.73886	94.31461	4.48096	Constraint on ζ_3 becomes active
23.	0.79872	4.74454	93.18657	4.27905	a_7 becomes active
	1.00286	4.88804	64.47497	4.89721	α greater than 1, program terminated

of Segment 4 are only local minima for the problem.

TABLE 3.3.3 (continued)
Path of solutions for high weight ratio of the nonstructural to the structural weight.

Segment	α	$F_{\max}$	W	c	Event
24.	1.00000	3.81418	90.56545	3.81418	F_{max}, d_{11}, d_{16}, d_{66}, a_3, a_4, a_6, a_9, a_{10} and g_{15}, g_{16}, g_{17}, g_{19}, g_{20} are active
25.	0.94858	3.82382	88.30493	3.74570	Constraint on ζ_3 becomes inactive
26.	0.95335	3.84368	81.18759	3.76321	a_3 becomes inactive
27.	0.92254	3.85194	79.43595	3.71414	Constraint on ζ_1 becomes inactive
	0.19586	4.25997	59.44636	2.08192	Program terminated

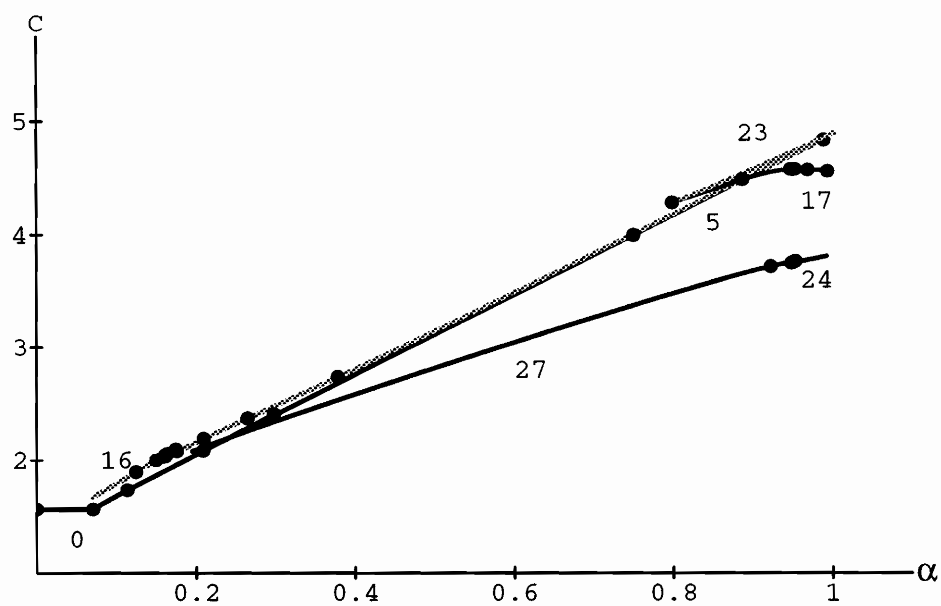

FIGURE 3.3.11. *Objective function c for high nonstructural weight.*

Because Part 1 of the curve never reached $\alpha = 1$, we used homotopy with nonstructural weight as the homotopy parameter starting from the point $\alpha = 1$ in the medium nonstructural weight case. Part 2 (Segments 17–23) starts therefore at $\alpha = 1$. After α becomes 0.79871 at the end of Segment 22 the path continues for increasing values of α and reaches the point $\alpha = 1$ with larger value of the cost function. The optimality conditions were not satisfied along Segment 23.

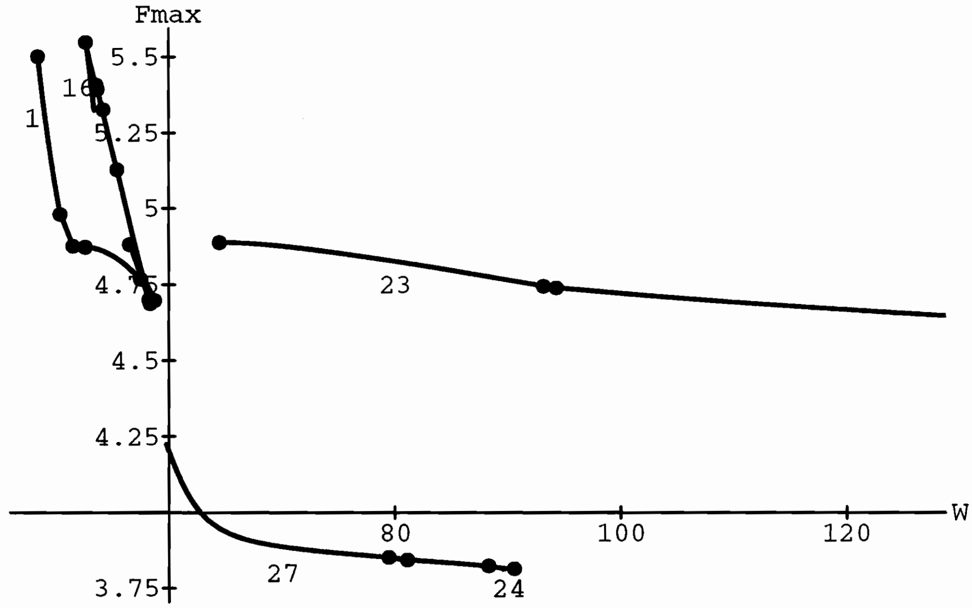

FIGURE 3.3.12. F_{max} *as a function of W for high nonstructural weight.*

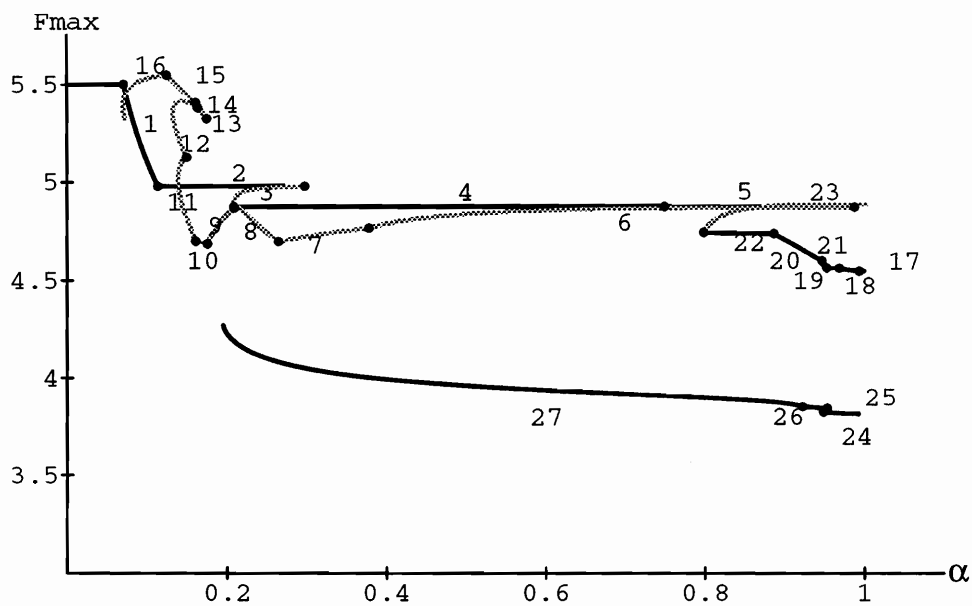

FIGURE 3.3.13. F_{max} *(pounds) for high nonstructural weight (gray line denotes stationary nonoptimal points, black line denotes optimal points).*

Another starting point at $\alpha = 1$ (for Segment 24) was found by standard optimization

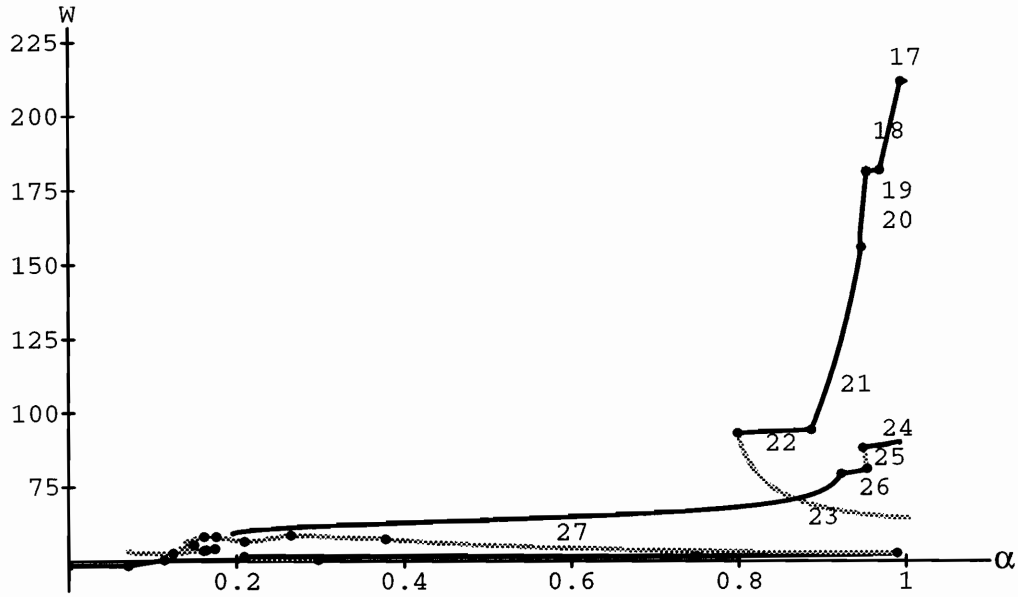

FIGURE 3.3.14. *Weight W (pounds) for high nonstructural weight (gray line denotes nonoptimal stationary points, black line denotes optimal points).*

program, MINOS. The cost function along this part of the path is significantly lower than the corresponding values for the two other parts of the path. After α decreases below 0.25 the cost function becomes larger than in Part 1 of the path which means that from that point on the path represents only the local minima. The program was terminated at α below 0.2 due to numerical difficulties (problem becomes singular when the eigenvalues μ_i are real). The path is nonoptimal along Segment 25. Segment 27 also appears to contain the best compromise design. At $\alpha = 0.39476$, the weight is at 65.04234 which is 39.38% of its total increase while F_{max} is already at 3.93567 which is 92.80% of its total decrease.

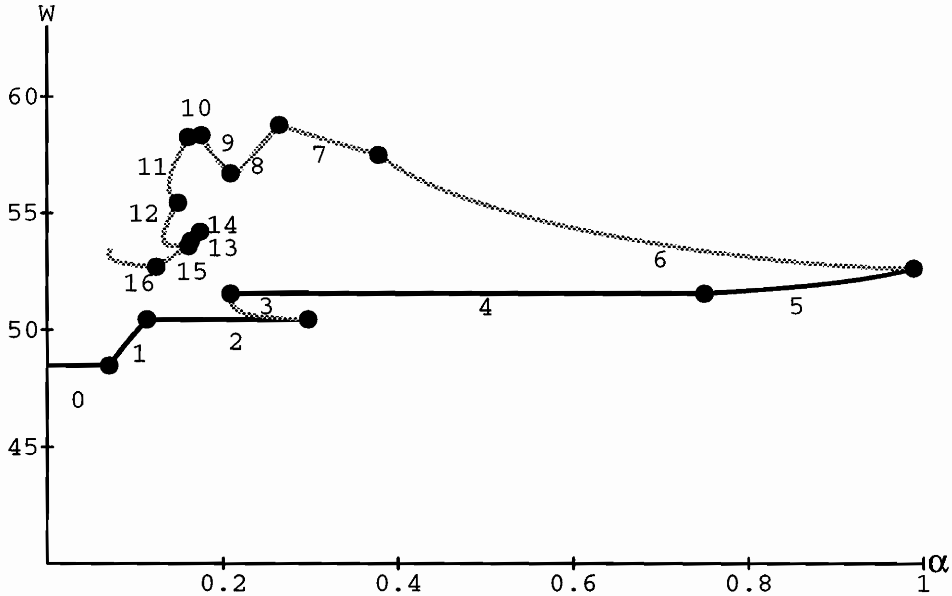

FIGURE 3.3.15. *Weight W (pounds), the lower part of the path for high nonstructural weight from Fig. 3.3.14 (gray line denotes nonoptimal stationary points, black line denotes optimal points).*

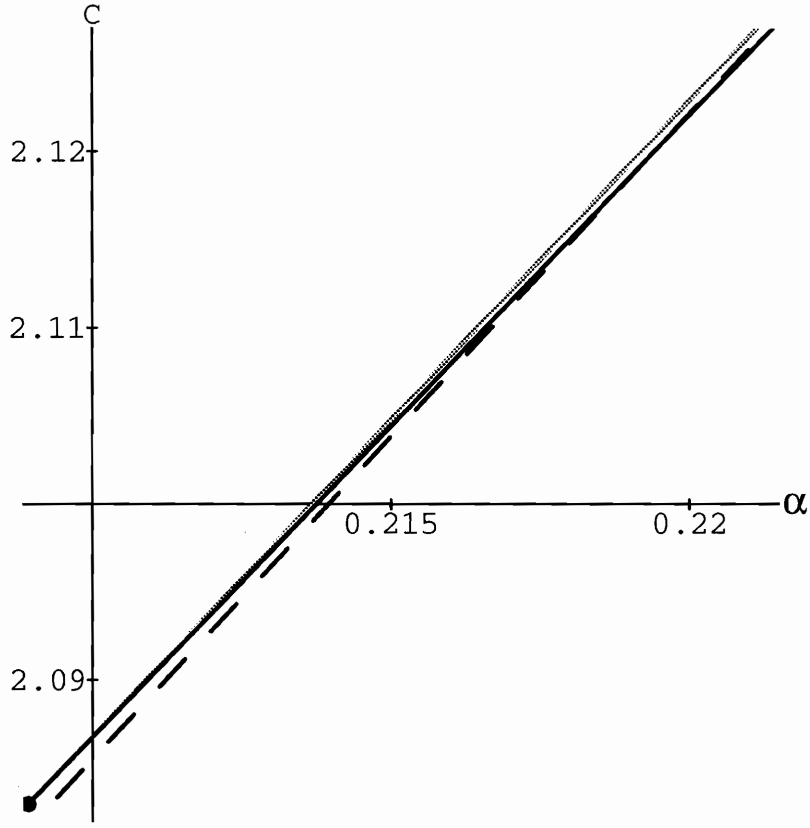

FIGURE 3.3.16. *Objective function in Segments 2–4 (2: dashed; 3: gray; 4: solid) for high nonstructural weight.*

4. The possible points of discontinuity of the path of minima

In this chapter we want to characterize the possible points of discontinuity of the path of minima. It is easy to show that the path of minima can be discontinuous only at the singularities of the path of solutions to the Fritz John conditions. The Fritz John conditions can be written as a system of equations $F = 0$. By the Implicit Function Theorem, a discontinuity of the path of solutions to these equations can happen only at a singularity of the Jacobian matrix DF. The Jacobian matrix DF becomes singular only if: (1) an active constraint has a zero Lagrange multiplier, or (2) gradients of active constraints are linearly dependent, or (3) the Hessian of the Lagrangian is singular on the tangent space to active constraints. Each of these cases will be discussed under the assumption of low codimension singularity of the Jacobian matrix, this is, when the dimension of the null space of DF is one or two.

Throughout this chapter we use the following notation. Let L: $R^n \to R^n$ be a linear operator and let M denote a subspace of R^n with corresponding orthogonal projection P_M. The projection of L onto M is denoted $P_M L|_M$. If L also denotes a matrix representation of L, and the columns of a matrix $V \in R^{n\times m}$ form an orthonormal basis for M, then the eigenvalues of $P_M L|_M$ are the eigenvalues of the matrix $V^T LV$ (e.g., Golub and Van Loan [7]). We also use the notation $D_x f = (\nabla_x f)^T$.

The standard nonlinear minimization problem can be written

$$\text{(4.1)} \qquad \begin{aligned} &\underset{x}{\text{minimize}}\ f(x,\alpha) \\ &\text{subject to } g_i(x,\alpha) = 0, \qquad (i = 1,\dots,l), \\ &\qquad\qquad g_i(x,\alpha) \leqq 0, \qquad (i = l+1,\dots,l+k), \end{aligned}$$

where $x \in R^n$ and $\alpha \in R$ is a parameter. The solutions to the nonlinear minimization problem (4.1) satisfy the Fritz John conditions

$$\text{(4.2)} \qquad \nabla_x \mathcal{L}(x,\mu,\lambda,\alpha) = 0,$$

$$(4.3) \qquad \mu_i g_i(x,\alpha) = 0, \qquad (i = l+1, \ldots, l+k),$$

$$(4.4) \qquad g_i(x,\alpha) = 0, \qquad (i = 1, \ldots, l),$$

$$g_i(x,\alpha) \leqq 0, \qquad (i = l+1, \ldots, l+k),$$

$$\mu_i \geqq 0, \qquad (i = l+1, \ldots, l+k+1),$$

where

$$\mathcal{L}(x,\mu,\lambda,\alpha) = \mu_{l+k+1} f(x,\alpha) + \sum_{j=1}^{l} \lambda_j g_j(x,\alpha) + \sum_{i=1}^{k} \mu_{l+i} g_{l+i}(x,\alpha)$$

is the Lagrangian. The multipliers μ and λ can be normalized so that

$$(4.5) \qquad \mu^T \mu + \lambda^T \lambda - 1 = 0.$$

The system of equations (4.2), (4.3), (4.4) and (4.5) can be represented as

$$(4.6) \qquad F(z,\alpha) = 0, \qquad z = \begin{pmatrix} x \\ \lambda \\ \mu \end{pmatrix}.$$

The solutions of (4.6) that satisfy $g_i(x,\alpha) \leqq 0, (l+1 \leqq i \leqq l+k)$ and $\mu_i \geqq 0, (l+1 \leqq i \leqq l+k+1)$ are the stationary points for problem (4.1). Thus, the solutions to the system $F = 0$ contain all stationary points, but also infeasible points and points with negative Lagrange multipliers. By the Implicit Function Theorem, if the Jacobian matrix $D_z F(z_0, \alpha_0)$ of the function F at (z_0, α_0) has full rank then there is some neighborhood $U(z_0, \alpha_0)$ such that there is a unique curve of zeros of $F(z,\alpha)$ in U passing through (z_0, α_0). Therefore the only points where the path of solutions to $F = 0$ can be discontinuous are the points of singularity of $D_z F(z,\alpha)$.

A necessary and sufficient condition that $D_z F(z_0, \alpha_0)$ be nonsingular is given in the next theorem due to Poore and Tiahrt [25].

THEOREM 4.1. *Let* (z_0, α_0) *be a solution of* $F(z,\alpha) = 0$, *i.e., a solution of equation* (4.6) *which combines* (4.2), (4.3), (4.4) *and* (4.5). *Assume* f *and* g *are twice continuously* *differentiable in a neighborhood of* (x_0, α_0) *and define two index sets* $\bar{\mathcal{A}}$ *and* $\mathcal{A}$ *and a corresponding tangent space* $T_{\bar{\mathcal{A}}}$ *by*

$$\bar{\mathcal{A}} = \{i : l+1 \leqq i \leqq l+k, g_i(x_0, \alpha_0) = 0\},$$

$$\mathcal{A} = \{i \in \bar{\mathcal{A}} : \mu_i \neq 0\},$$

$$\begin{aligned} T_{\bar{\mathcal{A}}} = \{y \in R^n : & D_x g_i(x_0, \alpha_0) y = 0 \quad (1 \leqq i \leqq l), \\ & D_x g_i(x_0, \alpha_0) y = 0 \quad (i \in \bar{\mathcal{A}})\}. \end{aligned}$$

Then a necessary and sufficient condition that $D_z F(z_0, \alpha_0)$ *be nonsingular is that each of the following three conditions hold:*

(a)$\bar{\mathcal{A}} = \mathcal{A}$*;*

(b)$\{\nabla_x g_i(x_0, \alpha_0)\}_{i \in \bar{\mathcal{A}}} \cup \{\nabla_x g_j(x_0, \alpha_0)\}_{j=1}^{l}$ *is a linearly independent collection of* $|\bar{\mathcal{A}}| + l$ *vectors where* $|\bar{\mathcal{A}}|$ *denotes the cardinality of* $\bar{\mathcal{A}}$*;*

(c) The Hessian of the Lagrangian $\nabla_x^2 \mathcal{L}$ *is nonsingular on the tangent space* $T_{\bar{\mathcal{A}}}$ *at* (z_0, α_0).

The condition (a) is called a strict complementarity condition, condition (b) will be referred to as the constraint qualification, and (c) is the second-order condition.

The points of singularity of $D_z F(z, \alpha)$ are also the only points of discontinuity of the path of stationary points, since (z, α) is a continuous solution of $F = 0$ and all functions g_i are continuous. Therefore if (z_0, α_0) is a stationary point, then all positive Lagrange multipliers stay positive near (z_0, α_0). The Lagrange multipliers equal to zero correspond to inactive constraints (strict complementarity condition holds if $D_z F(z_0, \alpha_0)$ is nonsingular). Since the g_i are continuous, all inactive constraints remain inactive near (z_0, α_0). Therefore the Fritz John conditions are satisfied, and the path of stationary points is continuous at (z_0, α_0).

These points are also the only possible points of discontinuity of the path of minima, since, by the second order necessary conditions, if (z_0, α_0) is a minimizer, then $P_{T_{\bar{A}}} \nabla_x^2 \mathcal{L}(z_0, \alpha_0)|_{T_{\bar{A}}} \geqq 0$. If the Hessian of the Lagrangian $\nabla_x^2 \mathcal{L}(z_0, \alpha_0)$ is positive definite on the tangent space to the active constraints $T_{\bar{A}}$, then it stays positive definite in some neighborhood of (z_0, α_0) because the eigenvalues of $P_{T_{\bar{A}}} \nabla_x^2 \mathcal{L}(z, \alpha)|_{T_{\bar{A}}}$ are continuous functions of z and α [40]. On the other hand, if $\nabla_x^2 \mathcal{L}(z_0, \alpha_0)$ is singular on $T_{\bar{A}}$, then, by Theorem 4.1, $D_z F(z_0, \alpha_0)$ is singular at (z_0, α_0).

The following theorems describe the behavior of the path of solutions to $F = 0$ in the case when any of the conditions (a), (b), or (c) in Theorem 4.1 is violated.

The next theorem due to Tiahrt and Poore [40] discusses the case of loss of strict complementarity.

THEOREM 4.2. *Let f and g be C^{m+1} $(m \geq 1)$ in some neighborhood of (x_0, α_0), and let $F(z_0, \alpha_0) = 0$. Suppose $\mathcal{A} \neq \bar{\mathcal{A}}$ and let I be an index set satisfying $\mathcal{A} \subseteq I \subseteq \bar{\mathcal{A}}$. If the Hessian of $\nabla_x^2 \mathcal{L}(z_0, \alpha_0)$ is nonsingular on the tangent space to active inequality constraints g_i, $i \in I$ and equality constraints and if $\nabla_x g_i(x_0, \alpha_0)$ $(1 \leqq i \leqq l)$ and $\nabla_x g_i(x_0, \alpha_0)$ $(i \in I)$ represent $l + |I|$ linearly independent vectors, then there is an m times continuously differentiable, local solution $(z^I(\alpha), \alpha)$ to $F = 0$ for which those inequality constraints indexed by I remain active and all other inequality constraints may be disregarded. If further, for all choices of I and each $i \in (\bar{\mathcal{A}} - I)$,*

$$\frac{d^s}{d\alpha^s} g_i(x^I(\alpha_0), \alpha_0) \neq 0 \tag{4.7}$$

for some s depending upon i and I, then each of the solutions $(z^I(\alpha), \alpha)$ is distinct. Finally, suppose $\nabla_x^2 \mathcal{L}(z_0, \alpha_0)$ is positive definite on

$$\begin{aligned} T_{\mathcal{A}}(x_0, \alpha_0) = \{y \in R^n : \; & D_x g_i(x_0, \alpha_0) y = 0 \quad (1 \leqq i \leqq l), \\ & D_x g_i(x_0, \alpha_0) y = 0 \quad (i \in \mathcal{A})\} \end{aligned}$$

and (4.7) holds with $s = 1$ for all I. If for each I and all $i \in I - \mathcal{A}$ we have $\frac{d}{d\alpha}\mu_i^I(\alpha_0) \neq 0$, then exactly one of these solutions consists of minima for $\alpha < \alpha_0$ and one consists of minima for $\alpha > \alpha_0$.

By the above theorem, the loss of strict complementarity results in a switching point. Since at the switching point the set of active constraints changes, the space tangent to active constraints also changes. Hence the Hessian $\nabla_x^2 \mathcal{L}(z_0, \alpha_0)$ projected on the new tangent space may not be positive semidefinite. As long as the Hessian $\nabla_x^2 \mathcal{L}(z_0, \alpha_0)$ is positive definite on the tangent space to the active constraints corresponding to the least constrained branch, the minima persist beyond the switching point. The theorem gives also the conditions for uniqueness of the path of minima.

Two examples will illustrate this theorem.

Example 1.

$$\text{minimize } (x_1 - \alpha)^2 + x_2$$

$$\text{subject to } -x_1 \leqq 0$$

$$-x_2 \leqq 0$$

The minimization problem and the solution $|z|$ as a function of the parameter α is shown in Figure 4.1 ($|z|$ denotes Euclidean norm of z). In this example we have two curves of solutions to (4.6): $(\alpha, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $(0, 0, -\frac{2\alpha}{\sqrt{2+4\alpha^2}}, \frac{1}{\sqrt{2+4\alpha^2}}, \frac{1}{\sqrt{2+4\alpha^2}})$. The first curve consists of local minimizers for $\alpha \geqq 0$ and of infeasible points for $\alpha < 0$. The second curve consists of local minimizers for $\alpha \leqq 0$, but not for $\alpha > 0$, since then one of the Lagrange multipliers becomes negative. At $\alpha = 0$ the first constraint becomes active, but the Lagrange multiplier associated with this constraint is equal to zero. Hence the strict complementarity condition does not hold, which causes the singularity of the Jacobian

$D_zF(z_0, \alpha_0)$. The Jacobian matrix in this example is

$$D_zF = \begin{pmatrix} 2\mu_3 & 0 & -1 & 0 & 2(x_1-\alpha) \\ 0 & 0 & 0 & -1 & 1 \\ \mu_1 & 0 & x_1 & 0 & 0 \\ 0 & \mu_2 & 0 & x_2 & 0 \\ 0 & 0 & 2\mu_1 & 2\mu_2 & 2\mu_3 \end{pmatrix},$$

and its rank at $z_0 = (0, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, $\alpha_0 = 0$ is 4. In this case $\bar{\mathcal{A}} = \{1, 2\}$, $\mathcal{A} = \{2\}$, and we have two choices for I: $I_1 = \{2\}$ and $I_2 = \{1, 2\}$. The assumptions of Theorem 4.2 are satisfied, since

(i) $\nabla_x g_1(x_0, \alpha_0) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $\nabla_x g_2(x_0, \alpha_0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ are linearly independent,

(ii) $\nabla_x^2 \mathcal{L}(z_0, \alpha_0) = \begin{pmatrix} 2\mu_3 & 0 \\ 0 & 0 \end{pmatrix}$, hence $P_{T_{I_1}} \nabla_x^2 \mathcal{L}|_{T_{I_1}}(z_0, \alpha_0) = 2\mu_3$ is nonsingular, and $P_{T_{I_2}} \nabla_x^2 \mathcal{L}|_{T_{I_2}}$ is nonsingular, since $T_{I_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (where T_{I_1} is defined as $T_{\mathcal{A}}$ with $\mathcal{A} = I_1$, and similarly T_{I_2}).

Therefore, by Theorem 4.2, there are two local solutions: $(z^{I_1}(\alpha), \alpha)$ and $(z^{I_2}(\alpha), \alpha)$. The first solution is $(\alpha, 0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and the second one is $(0, 0, \frac{-2\alpha}{\sqrt{2+4\alpha^2}}, \frac{1}{\sqrt{2+4\alpha^2}}, \frac{1}{\sqrt{2+4\alpha^2}})$. Furthermore, the condition $\frac{d^s}{d\alpha^s} g_i(x^I(\alpha_0), \alpha_0) \neq 0$ is satisfied with $s = 1$ for all choices of I and $i \in (\bar{\mathcal{A}} - I)$, since $\bar{\mathcal{A}} - I_1 = \{1\}$ and

$$\frac{d}{d\alpha} g_1(x^{I_1}(\alpha_0), \alpha_0) = \frac{d}{d\alpha}(-\alpha)|_{\alpha_0} = -1 \neq 0,$$

and $\bar{\mathcal{A}} - I_2 = \{\emptyset\}$. Therefore these two solutions are distinct.

Finally, $P_{T_{\mathcal{A}}} \nabla_x^2 \mathcal{L}(z_0, \alpha_0)|_{T_{\mathcal{A}}} = 2\mu_3$ is positive definite. Also, the condition $\frac{d}{d\alpha}\mu_i^I(\alpha_0) \neq 0$ for each I and $i \in I - \mathcal{A}$ is satisfied, since $I_1 - \mathcal{A} = \{\emptyset\}$, and for I_2 we have $I_2 - \mathcal{A} = \{1\}$ and $\frac{d}{d\alpha}\mu_1^{I_2}(\alpha_0) = \frac{d}{d\alpha}(\frac{-2\alpha}{\sqrt{2+4\alpha^2}})|_{\alpha=\alpha_0} = -\sqrt{2} \neq 0$. Therefore, by Theorem 4.2, exactly one solution consists of minima for $\alpha < \alpha_0$ and one consists of minima for $\alpha > \alpha_0$.

Example 2.

$$\text{minimize } x_1^2 - x_2^2$$

$$\text{subject to } -x_2 + \alpha \leqq 0$$

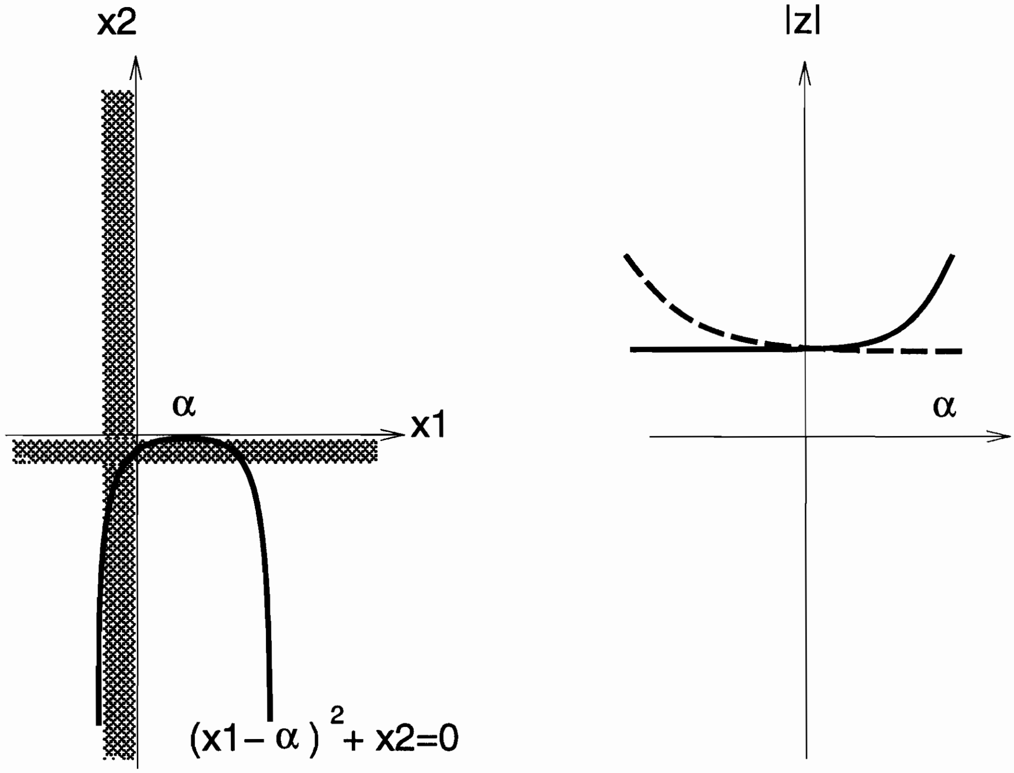

FIGURE 4.1. *Loss of strict complementarity – minima persist (solid line on the plot $|z|(\alpha)$ denotes minima, dashed line denotes infeasible points, or points with negative Lagrange multipliers).*

The minimization problem and the solution to system (4.6) $|z|$ as a function of the parameter α is shown in Figure 4.2. In this example there are two curves of solutions: $(0,0,0,1)$ and $(0,\alpha,\frac{-2\alpha}{\sqrt{1+4\alpha^2}},\frac{1}{\sqrt{1+4\alpha^2}})$. The first curve consists of saddle points for $\alpha \leqq 0$ and of infeasible points for $\alpha > 0$. The second curve consists of minimizers for $\alpha \leqq 0$, but not for $\alpha > 0$, since then one Lagrange multiplier becomes negative. At $\alpha = 0$ the strict complementarity condition is violated, but the Hessian of the Lagrangian projected on the tangent space to active constraints corresponding to the least constrained branch is

$$P_{T_A}\nabla_x^2\mathcal{L}(z_0,\alpha_0)|_{T_A} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix},$$

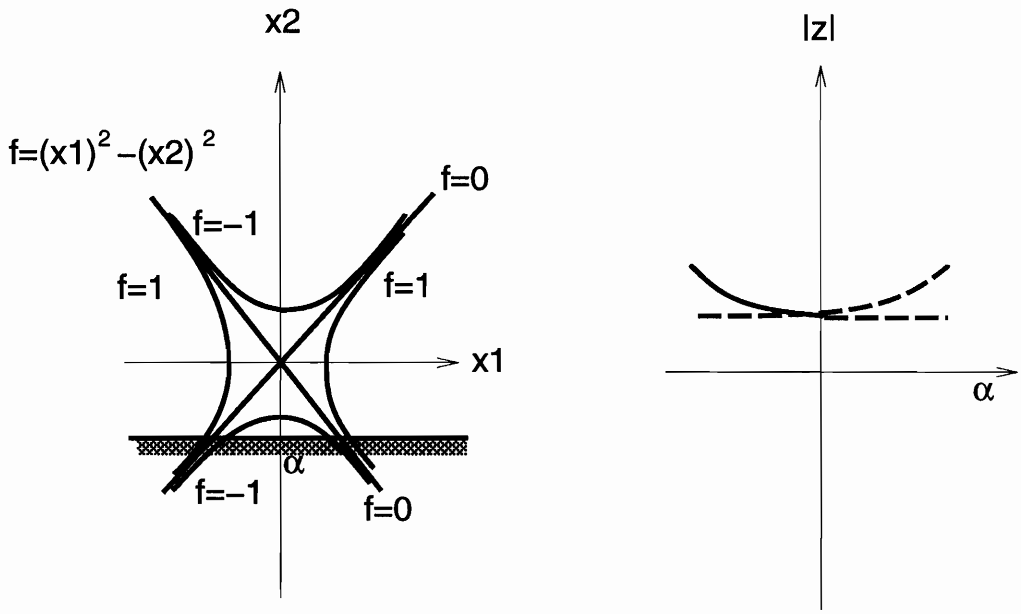

FIGURE 4.2. *Loss of strict complementarity – path of minima is discontinuous (solid line on the plot $|z|(\alpha)$ denotes minima, dashed line denotes infeasible, or saddle points, or points with negative Lagrange multipliers).*

hence is indefinite. Therefore the assumptions of the last part of Theorem 4.2 are not satisfied and the optima are not guaranteed to persist. Indeed, the path of optima is discontinuous at the singularity point $\alpha = 0$.

The next two theorems, due to Tiahrt and Poore [40], describe the path of solutions under the assumption of zero codimension singularities, i.e., singularities where the dimension of the null space of D_zF is one and $D_\alpha F$ is not in the range of D_zF. Throughout this chapter $\mathcal{N}$ and $\mathcal{R}$ will denote the null and range space, respectively. The first of these theorems analyses the case of constraint qualification violation.

THEOREM 4.3. *Let f and g be C^{m+1} $(m \geqq 3)$ in some neighborhood of (x_0, α_0) and let $F(z_0, \alpha_0) = 0$. Suppose that at the solution (z_0, α_0) the following hold:*

$$\mathcal{A} = \bar{\mathcal{A}},$$

$$P_{T_{\bar{A}}} \nabla_x^2 \mathcal{L}|_{T_{\bar{A}}} \text{ is nonsingular,}$$

$$\dim \text{ span } (\{\nabla_x g_i\}_{i=1}^{l} \cup \{\nabla_x g_i\}_{i \in \bar{A}}) = l + |\bar{A}| - 1,$$

where $|\cdot|$ *denotes the cardinality of the set,*

$$\begin{pmatrix} D_x g_i & D_\alpha g_i & (1 \leqq i \leqq l) \\ D_x g_i & D_\alpha g_i & (i \in \bar{A}) \end{pmatrix} \text{ has full rank,}$$

$$l + |\bar{A}| < n.$$

Then in an open neighborhood U *of* (z_0, α_0) *there exists a curve of solutions to* $F = 0$ *through* (z_0, α_0)*, and any other solution in* U *must lie on this curve.*

Furthermore, if f *and* g *are analytic functions of* x *and* α *and if* $\mu_{l+k+1} = 0$*, then there is a complete reversal of critical point type at the point of singularity of* $D_z F(z, \alpha)$*, that is, all Lagrange multipliers and all eigenvalues of* $\nabla_x^2 \mathcal{L}$ *on* $T_{\bar{A}}$ *change sign.*

According to this theorem, the path of stationary points can be continued beyond the point of violation of the constraint qualification. The path of minima, however, can be discontinuous at this point. If $\mu_{l+k+1} = 0$, then the path of minima turns into the path of maxima.

Example 3.

$$\begin{aligned} &\text{minimize } -x_1 \\ &\text{subject to } -x_1^2 + x_2 - 1 \leqq 0, \\ &\qquad -(x_1 - 1)^2 - x_2 + \alpha \leqq 0. \end{aligned}$$

The minimization problem and the solution $|z|$ as a function of the parameter α is shown in Figure 4.3. In this example we have two solution curves to (4.6): $(\frac{1 - \sqrt{2\alpha - 3}}{2}, \frac{1 + \alpha - \sqrt{2\alpha - 3}}{2}, \frac{1}{\sqrt{8\alpha - 10}}, \frac{1}{\sqrt{8\alpha - 10}}, \frac{\sqrt{8\alpha - 12}}{\sqrt{8\alpha - 10}})$ and $(\frac{1 + \sqrt{2\alpha - 3}}{2}, \frac{1 + \alpha + \sqrt{2\alpha - 3}}{2}, \frac{-1}{\sqrt{8\alpha - 10}}, \frac{-1}{\sqrt{8\alpha - 10}},$

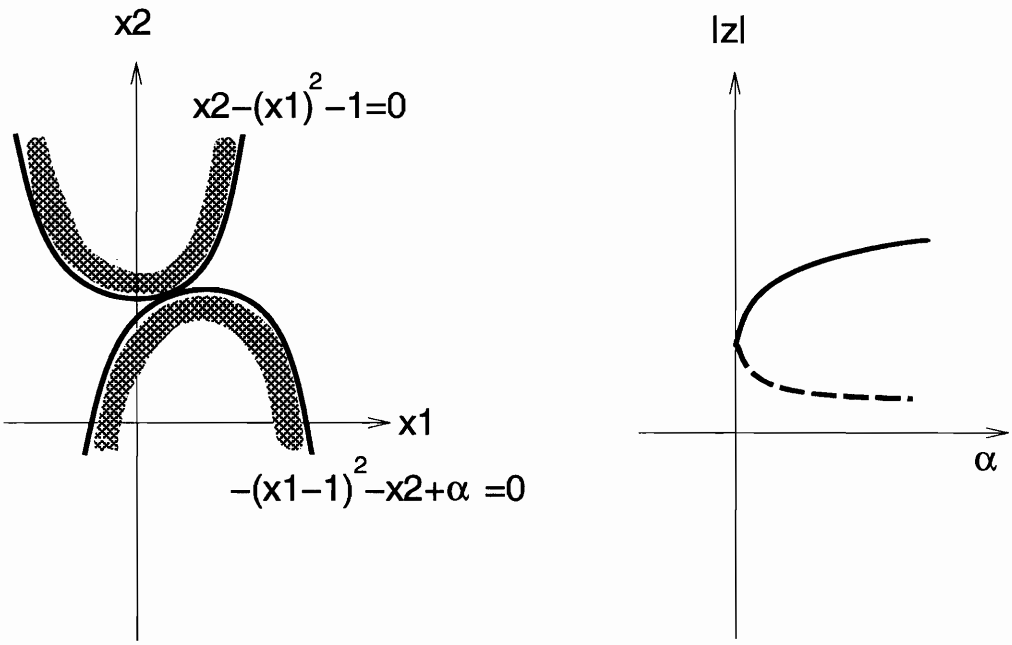

FIGURE 4.3. *Violation of constraint independence qualification – path of minima is discontinuous (solid line on the plot $|z|(\alpha)$ denotes maxima, dashed line denotes minima).*

$\frac{-\sqrt{8\alpha-12}}{\sqrt{8\alpha-10}}$). The first curve is the curve of local minimizers for $\alpha \geqq \frac{3}{2}$, and it does not exist for $\alpha < \frac{3}{2}$. The second curve consists of local maximizers for $\alpha \geqq \frac{3}{2}$, and it does not exists for $\alpha < \frac{3}{2}$. Both curves meet at the singular point $\alpha = \frac{3}{2}$ where the gradients of constraints become linearly dependent. Hence the singular point here is a turning point with the path of minima turning to the path of maxima.

The next theorem, due to Tiahrt and Poore [40], considers the case when the zero codimension singularity of the Jacobian matrix D_zF is caused by the singularity of the Hessian $P_{T_{\bar{\mathcal{A}}}} \nabla_x^2 \mathcal{L}|_{T_{\bar{\mathcal{A}}}}$.

THEOREM 4.4. *Let f and g be C^{m+1} $(m \geqq 3)$ in some neighborhood of (x_0, α_0), and let $F(z_0, \alpha_0) = 0$. Suppose (z_0, α_0) is a solution point to $F = 0$ at which*

$$\mathcal{A} = \bar{\mathcal{A}},$$

$$\dim \text{span } (\{\nabla_x g_i\}_{i=k+1}^{k+l} \cup \{\nabla_x g_i\}_{i\in\bar{\mathcal{A}}}) = l + |\bar{\mathcal{A}}|,$$

$$P_{T_{\bar{\mathcal{A}}}} \nabla_x^2 \mathcal{L}|_{T_{\bar{\mathcal{A}}}} \textit{ has a zero eigenvalue of algebraic multiplicity one,}$$

$$\text{orthogonal projection of } D_\alpha \nabla_x \mathcal{L} \text{ onto } T_{\bar{\mathcal{A}}} \textit{ is not in } \mathcal{R}(P_{T_{\bar{\mathcal{A}}}} \nabla_x^2 \mathcal{L}|_{T_{\bar{\mathcal{A}}}}).$$

Then in an open neighborhood U of (z_0, α_0), there exists a curve of solutions to $F = 0$ and any other solution in U must lie on this curve. Along this curve, at most one eigenvalue of $P_{T_{\bar{\mathcal{A}}}} \nabla_x^2 \mathcal{L}|_{T_{\bar{\mathcal{A}}}}$ can change sign, while no other aspect of the type of critical point changes.

This theorem says that the path of stationary points can be continued despite the singularity of $P_{T_{\bar{\mathcal{A}}}} \nabla_x^2 \mathcal{L}|_{T_{\bar{\mathcal{A}}}}$. The path of minima can be discontinuous at this point with path of minima becoming the path of saddle points.

Example 4.

$$\text{minimize } x_2$$

$$\text{subject to } \alpha x_1^2 - x_2 \leqq 0$$

The minimization problem and the solution $|z|$ as a function of the parameter α is shown in Figure 4.4. The solution to (4.6) is $(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ for $\alpha \neq 0$ and $(x_1, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, x_1 arbitrary, for $\alpha = 0$. The assumptions of Theorem 4.4 are satisfied at the singular point $\alpha = 0$, since

(a) the active constraint has the Lagrange multiplier different from zero ($\mathcal{A} = \bar{\mathcal{A}}$),

(b) dim span$(\{\nabla_x g_i\}_{i\in\bar{\mathcal{A}}})$=1 = $|\bar{\mathcal{A}}|$,

(c) $P_{T_{\bar{\mathcal{A}}}} \nabla_x^2 \mathcal{L}|_{T_{\bar{\mathcal{A}}}} = 2\mu_1 \alpha = 0$ has a zero eigenvalue of algebraic multiplicity one,

(d) the orthogonal projection of $D_\alpha \nabla_x \mathcal{L}$ onto $T_{\bar{\mathcal{A}}}$ is $2\mu_1 x_1$, (x_1 arbitrary), hence it is not in $\mathcal{R}(P_{T_{\bar{\mathcal{A}}}} \nabla_x^2 \mathcal{L}|_{T_{\bar{\mathcal{A}}}})$.

Since the assumptions of Theorem 4.4 are satisfied, there exists a curve of stationary points in the neighborhood of the singular point. One eigenvalue of $P_{T_{\bar{\mathcal{A}}}} \nabla_x^2 \mathcal{L}|_{T_{\bar{\mathcal{A}}}}$ changes

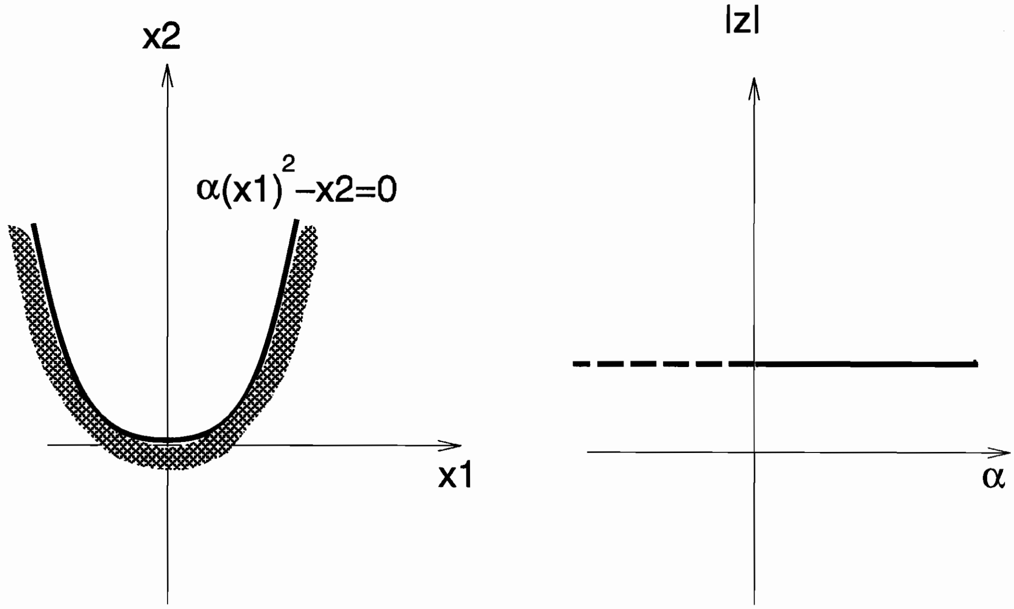

FIGURE 4.4. *Failure of the second order condition (solid line on the plot $|z|(\alpha)$ denotes minima, dashed line denotes saddle points).*

sign at α_0, hence the path of minima is discontinuous at $\alpha = 0$. The path of solutions to $F = 0$ consists of minima of $\alpha \geqq 0$ and of saddle points for $\alpha < 0$.

The theorems given in this chapter describe only possible points of discontinuity of the path. At any singular point the optima can can persist, or not. For instance, if we modify Example 4 so that we have

$$\text{minimize } x_2$$

$$\text{subject to } \alpha^2 x_1^2 - x_2 \leqq 0,$$

then we have again the solution $(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ for $\alpha \neq 0$ and $(x_1, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, x_1 arbitrary, for $\alpha = 0$, but this time the curve consists of minima for both $\alpha \geqq 0$ and $\alpha < 0$. Hence the optima persist beyond the point of singularity.

The same modification can be applied to any of the previous examples. For instance, we can replace α by $(-\beta^2)$ in Example 2 to get a continuous path of minimizers for both positive and negative values of the parameter β, despite the singularity of the Jacobian matrix D_zF. Similarly, we can use $\frac{3}{2}+\beta^2$ instead of α in Example 3 to get a continuous path of minimizers for all values of β.

Two other cases that are likely to happen are: (1) the dimension of the null space of D_zF is equal to one and $D_\alpha F(z_0,\alpha_0)$ in the range of D_zF, and (2) dim $\mathcal{N}(D_zF)=2$ and $D_\alpha F \notin \mathcal{R}(D_zF)$. These cases are described in the next theorem (see Tiahrt and Poore [40]). We use notation $D_z^2Fuu = (D_z[(D_zF)u])u$. The geometric multiplicity of the zero eigenvalue of L is the maximal number of linearly independent eigenvectors in the null space of L, and the algebraic multiplicity of the zero eigenvalue is the number of times $\lambda=0$ is repeated as a root of det $(\lambda I - L)$.

THEOREM 4.5. *Let* $X = R^{n+l+k+1}$ *and suppose* $F(z_0,\alpha_0)=0$ *and* $F\in C^m(U_1\times U_2, X)$ *where* $m \geqq 3$, U_1 *is an open set containing* $z_0\in X$, U_2 *is an open set containing* α_0. *Define* $L = D_zF(z_0,\alpha_0)$. *If dim* $\mathcal{N}(L)=k$ *and the geometric and algebraic multiplicities of the zero eigenvalue of* L *are equal, then there exist eigenvectors* $\psi_1,\ldots,\psi_k$, $\psi_1^*,\ldots,\psi_k^*$, *such that* $\mathcal{N}(L) = span\{\psi_i\}_{i=1}^k$, $\mathcal{N}(L^T)=span\{\psi_i^*\}_{i=1}^k$, $\langle\psi_i,\psi_j^*\rangle = \langle\psi_i^*,\psi_j^*\rangle = \delta_{ij}$, *the Kronecker delta. Furthermore,* P *and* E *defined by* $Pu=\sum_{i=1}^k\langle u,\psi_i^*\rangle\psi_i$ *and* $(I-E)u = \sum_{i=1}^k\langle u,\psi_i^*\rangle\psi_i^*$ *represent projections onto the null and range space of* L, *respectively. In case* $k=1$, *the subscript notation is dropped.*

(i) Let dim $\mathcal{N}(L)=1$ *and suppose* $D_\alpha F(z_0,\alpha_0)\in\mathcal{R}(L)$. *Define*

$$\mathcal{D} = \left\langle D_z^2F\psi w + D_\alpha D_zF\psi, \psi^*\right\rangle^2 - \left\langle D_z^2Fww + 2D_\alpha D_zFw + D_\alpha^2F, \psi^*\right\rangle\left\langle D_z^2F\psi\psi,\psi^*\right\rangle$$

where w *solves* $D_zFw = -D_\alpha F$, $\langle w,\psi^*\rangle = 0$, *and the derivatives are evaluated at* (z_0,α_0). *If* $\mathcal{D}>0$, *then there are exactly two curves with distinct tangents through* (z_0,α_0) *satisfying* $F=0$, *and any solution of* $F=0$ *in a sufficiently small neighborhood of* (z_0,α_0) *must lie on one of these curves. If* $\mathcal{D}<0$, *no smooth (real) solution exists through* (z_0,α_0).

(ii) Let dim $\mathcal{N}(L) = 2$, the algebraic and geometric multiplicities of the zero eigenvalue be equal, and $D_\alpha F \notin \mathcal{R}(L)$. Then $\langle D_\alpha F, \psi_i^* \rangle \neq 0$ for either $i = 1$ or 2. Assume the former without loss of generality and define $\mathcal{D} = b^2 - ac$ where

$$a = \langle D_z^2 F \psi_1 \psi_1, \psi_2^* \rangle - \frac{\langle D_\alpha F, \psi_2^* \rangle}{\langle D_\alpha F, \psi_1^* \rangle} \langle D_z^2 F \psi_1 \psi_1, \psi_1^* \rangle,$$

$$b = \langle D_z^2 F \psi_1 \psi_2, \psi_2^* \rangle - \frac{\langle D_\alpha F, \psi_2^* \rangle}{\langle D_\alpha F, \psi_1^* \rangle} \langle D_z^2 F \psi_1 \psi_2, \psi_1^* \rangle,$$

$$c = \langle D_z^2 F \psi_2 \psi_2, \psi_2^* \rangle - \frac{\langle D_\alpha F, \psi_2^* \rangle}{\langle D_\alpha F, \psi_1^* \rangle} \langle D_z^2 F \psi_2 \psi_2, \psi_1^* \rangle.$$

If $\mathcal{D} > 0$, then there exist exactly two curves with distinct tangents through (z_0, α_0) satisfying $F = 0$. If $\mathcal{D} < 0$, no smooth (real) curve of solutions exists through (z_0, α_0).

The above theorem is motivated by the assumption that although $L = D_z F(z_0, \alpha_0)$ is singular, we can still look for solutions that are smooth in some auxiliary parameter ϵ. We assume $z = z(\epsilon)$ and $\alpha = \alpha(\epsilon)$, and by successively differentiating $F(z(\epsilon), \alpha(\epsilon)) = 0$ with respect to ϵ and setting $\epsilon = 0$ we get:

$$F(z(0), \alpha(0)) = 0, \tag{4.8}$$

$$L z_1 = -\alpha_1 D_\alpha F, \tag{4.9}$$

$$L z_2 = -\alpha_2 D_\alpha F - D_z^2 F z_1 z_1 - 2 D_z D_\alpha F z_1 \alpha_1 - D_\alpha^2 F \alpha_1^2, \tag{4.10}$$

where $z_k = \dfrac{d^k z}{d\epsilon^k}(0)$, $\alpha_k = \dfrac{d^k \alpha}{d\epsilon^k}(0)$, all functions are evaluated at $\epsilon = 0$, and $L = D_z F(z(0), \alpha(0))$. Consider case (i) of the above theorem. Since $D_\alpha F \in \mathcal{R}(L)$, (4.9) is solvable with $z_1 = \gamma_1 \psi + \alpha_1 w$, where $Lw = -D_\alpha F$ and γ_1 and α_1 are parameters to be determined from solvability of (4.10). Since $L z_2 \in \mathcal{R}(L)$ and $-\alpha_2 D_\alpha F \in \mathcal{R}(L)$, then $D_z^2 F z_1 z_1 + 2 D_z D_\alpha F z_1 \alpha_1 + D_\alpha^2 F \alpha_1^2$ is also in $\mathcal{R}(L)$. This implies that

$$\psi^* (D_z^2 F z_1 z_1 + 2 D_z D_\alpha F z_1 \alpha_1 + D_\alpha^2 F \alpha_1^2) = 0.$$

Thus we get a quadratic

$$a\alpha_1^2 + 2b\alpha_1\gamma_1 + c\gamma_1^2 = 0, \tag{4.11}$$

where

$$a = \langle D_\alpha^2 F + 2D_z D_\alpha F w + D_z^2 F w w, \psi^* \rangle,$$

$$b = \langle D_z^2 F \psi w + D_z D_\alpha F w, \psi^* \rangle,$$

$$c = \langle D_z^2 F \psi\psi, \psi^* \rangle.$$

The quadratic (4.11) has two distinct solutions if $\mathcal{D} = b^2 - ac > 0$, and no real solution exists if $\mathcal{D} < 0$. Part (ii) of the theorem can be derived in a similar manner.

The case $\mathcal{D} = 0$ can result in cusp points or points of higher contact. By the above theorem, the higher codimension singularities can cause discontinuity in the path of stationary points in the case when $\mathcal{D} < 0$.

In this chapter we characterized the possible points of discontinuity of the path of minima. The discontinuity can be caused by the loss of strict complementarity (the Lagrange multiplier for an active constraint becomes equal to zero), by violation of the constraint linear independence qualification, or by failure of the second order condition. Singularity of the Jacobian matrix DF, however, does not necessarily result in discontinuity of the path of minima.

5. Necessary and sufficient conditions for continuity of the parametrized optima.

The conditions for continuity of the path of parametrized optima are related to the problem of the solvability and stability of the arbitrarily perturbed optimization problems, since a small change of the parameter can be viewed as a perturbation. Therefore the solvability of the system with respect to arbitrary perturbations is a sufficient condition for the continuity of the path. Moreover, the path of parametrized optima can be traced only if the arbitrarily perturbed system is solvable, since arbitrary perturbations always occur during computations. Therefore, from the numerical point of view, the solvability of the perturbed problem is a necessary condition for persistence of minima. In this chapter we will give sufficient conditions for persistence of minima for general optimization problems. In the simpler case of linear programming problems it is also possible to give necessary conditions. These results, due to Robinson [31], [34], have been proved using topological arguments, but they are shown in Section 5.2 to be equivalent to the other conditions widely used in optimization practice.

5.1 Linear programming problems

In this section we consider linear programming problems. The necessary and sufficient conditions for solvability of perturbed linear problems are shown to be equivalent to any of two other conditions: (1) the boundedness of the primal and dual solution sets, (2) certain constraint regularity. The constraint regularity qualification assumed here is weaker than the constraint linear independence qualification. It is further characterized by theorems that allow determination of the constraint regularity in two special cases: (i) there are only equality constraints, (ii) the constraints can be written in the form where some of the inequality constraints are inactive for some x_0 in the feasible set and the others are active for all feasible x. Finally, an example is given to show that the constraints may be

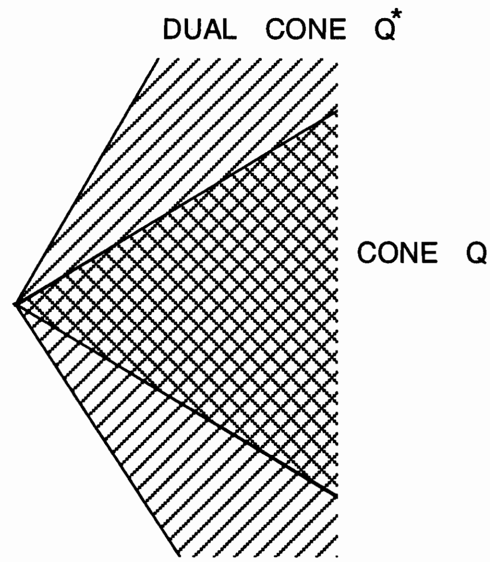

FIGURE 5.1.1. *Dual cone.*

made singular by inappropriate formulation, or they may be regularized by reformulation. A theorem is given describing when regularization is possible.

The following definitions will be used further on.

DEFINITION. A *cone* $P \subset R^n$ is a set of points such that if $x \in P$, then so is every non-negative scalar multiple of x, i.e., $x \in P$, $\lambda \geqq 0 \Rightarrow \lambda x \in P$.

A convex cone is a cone which is convex.

DEFINITION. A convex cone is a *convex polyhedral cone* if it consists of the set of all non-negative linear combinations of a finite set of vectors, i.e., for points $x_j \in R^n, j = 1, \ldots, m$,

$$P = \{x \mid x = \sum_{j=1}^{m} \lambda_j x_j, \lambda_j \geqq 0\}.$$

DEFINITION. If P is a cone then the *dual cone* $P^* = \{z \in R^n \mid \langle z, x\rangle \geq 0$ for each $x \in P\}$ (see Figure 5.1.1).

DEFINITION. A subset M of R^n is called an *affine set* if $(1-\lambda)x + \lambda y \in M$ for every $x \in M$, $y \in M$ and $\lambda \in R$.

DEFINITION. The smallest affine set containing S (the intersection of the collection of affine sets M such that $M \supset S$) is called the *affine hull* of S (denoted aff S).

Let B be the Euclidean unit ball in R^n:

$$B = \{x \mid |x| \leqq 1\} = \{x \mid d(x,0) \leqq 1\}.$$

For any $a \in R^n$, the ball with radius $\epsilon > 0$ and center a is given by

$$\{x \mid d(x,a) \leqq \epsilon\} = \{a + y \mid |y| \leqq \epsilon\} = a + \epsilon B.$$

For any set C in R^n, the set of points x whose distance from C does not exceed ϵ is

$$\{x \mid \exists y \in C : d(x,y) \leqq \epsilon\} = \cup\{y + \epsilon B \mid y \in C\} = C + \epsilon B.$$

The interior int C can be expressed as

$$\text{int } C = \{x \mid \exists \epsilon > 0 : x + \epsilon B \subset C\}.$$

DEFINITION. The *relative interior* of a convex set C in R^n is defined

$$\text{ri } C = \{x \in \text{aff } C \mid \exists \epsilon > 0 : (x + \epsilon B) \cap (\text{aff } C) \subset C\}.$$

The dimension of a nonempty affine set is defined as the dimension of the subspace parallel to it. A set M is said to be parallel to a set L if there exists a vector a such that $L = M + a$. The dimension of a convex set C is the dimension of the affine hull of C. For example, a disk is a two-dimensional set regardless of the dimension of the space. For an n-dimensional convex set in R^n, aff $C = R^n$ by definition, so ri C = int C. One face of a cube in R^3 does not have interior (no three-dimensional ball can be contained in a plane), but it has a nonempty relative interior (a face of the cube without edges). The last example also shows that the affine hull of the convex set can have an empty interior even though it always has a nonempty relative interior.

The affine subsets of R^n can be written as the solution sets to systems of simultaneous linear equations in n variables (see Rockafellar [37, THM. 1.4]):

THEOREM 5.1.1. *Given $e \in R^m$ and an $m \times n$ real matrix E, the set*

$$M = \{x \in R^n \mid Ex = e\}$$

is an affine set in R^n. Moreover, every affine set may be represented in this way.

A pair of dual linear programming problems can be written as

$$\begin{aligned} &\text{minimize } \langle c, x\rangle \\ &\text{subject to } A_1 x - b_1 \geqq 0, \\ &\qquad A_2 x - b_2 = 0, \\ &\qquad x \geqq 0, \end{aligned}$$

and

$$\begin{aligned} &\text{maximize } \langle u, b\rangle \\ &\text{subject to } c - uA \geqq 0, \\ &\qquad u_1 \geqq 0, \\ &\qquad u_2 \text{ free}, \end{aligned}$$

where $x \in R^n$, $A_1 \in R^{k\times n}$, $A_2 \in R^{l\times n}$, $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$, $b_1 \in R^k$, $b_2 \in R^l$, $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $u_1 \in R^k$, $u_2 \in R^l$, and $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

Equivalently, this problem can be written as

$$\begin{aligned} &\text{minimize } \langle c, x\rangle \\ \text{(P)} \qquad &\text{subject to } Ax - b \in Q^*, \qquad x \in P, \end{aligned}$$

and

$$\begin{aligned} &\text{maximize } \langle u, b\rangle \\ \text{(D)} \qquad &\text{subject to } c - uA \in P^*, \qquad u \in Q, \end{aligned}$$

where $P = R^n_+$, $P^* = R^n_+$, $Q = R^k_+ \times R^l$, and $Q^* = R^k_+ \times \{0\}^l$ (R^n_- and R^n_+ denote the non-positive and non-negative orthant in R^n, respectively, and $\{0\}^l$ is the origin in R^l).

The use of cones P and Q simplifies the notation because it is not necessary to distinguish between inequalities and equations and between nonnegative and unconstrained variables.

In the further analysis it is convenient to assume the following definition of the constraint regularity by Robinson [34]:

DEFINITION. The constraints of (P) are *regular* if $b \in \operatorname{int}(A(P) - Q^*) = \operatorname{int}\{Ax - q^* \mid x \in P, q^* \in Q^*\}$ and those of (D) are regular if $c \in \operatorname{int}((Q)A + P^*) = \operatorname{int}\{qA + p^* \mid q \in Q, p^* \in P^*\}$.

If the constraints are not regular we call them singular.

This definition of regularity is different from the requirement of the linear independence of the gradients of the active constraints. For example, the system

$$\begin{aligned} \text{minimize } & -x_1 - x_2 \\ \text{subject to } & x_1 \leqq 1 \\ & x_2 \leqq 2 \\ & 2x_1 - x_2 = 0 \\ & x_1 \geqq 0 \\ & x_2 \geqq 0 \end{aligned}$$

can be written as $Ax - b \in Q^*$, $x \in P$ where $P = R^2_+$, $Q = R^2_+ \times R$, $Q^* = R^2_- \times \{0\}$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, and R^2_+ (R^2_-) denotes the nonnegative (nonpositive) orthant in R^2. The constraints are regular, since $b \in \operatorname{int}(A(R^2_+) - R^2_- \times \{0\})$, as we show next. Let $\tilde{b}$ be any vector in a neighborhood of $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$. Let us write $\tilde{b}$ as $\begin{pmatrix} 1+\delta_1 \\ 2+\delta_2 \\ \delta_3 \end{pmatrix}$ where $\delta_1, \delta_2, \delta_3$ are small real numbers. Then $\tilde{b} \in (A(R^2_+) - R^2_- \times \{0\})$ if

$$1 + \delta_1 = r_1 + r_4$$

$$2 + \delta_2 = r_2 + r_3$$

$$\delta_3 = 2r_1 - r_2,$$

for some positive numbers r_1, r_2, r_3, r_4. Take two negative real numbers $-1 < \tilde{\delta}_1 < -|\delta_1|$ and $-2 < \tilde{\delta}_2 < -|\delta_2|$ such that $\delta_3 = 2\tilde{\delta}_1 - \tilde{\delta}_2$ (possible if δ_3 is small). Choose r_4 such that $\delta_1 - r_4 = \tilde{\delta}_1$. Choose r_3 such that $\delta_2 - r_3 = \tilde{\delta}_2$. Take $r_1 = 1 + \tilde{\delta}_1$ and $r_2 = 2 + \tilde{\delta}_2$. Then $\tilde{b} \in (A(R^2_+) - R^2_- \times \{0\})$. Therefore $b \in \operatorname{int}(A(R^2_+) - R^2_- \times \{0\})$. Yet the gradients of the active constraints (the first three constraints) are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$, hence they are linearly dependent.

Some properties of regular systems are given in the next two theorems.

If all the constraints in (P) are equality constraints, then $Q^* = \{0\}$ and the criterion for regularity of the constraints is given in the following theorem due to Robinson [34]:

THEOREM 5.1.2. *Let* aff $P = \{x \mid Ex = e\}$ *where the matrix* $(E \quad e)$ *has full row rank (if* int $P \neq \{\emptyset\}$ *then* E *and* e *do not appear in what follows). Then the system*

$$Ax - b \in \{0\}, \qquad x \in P$$

is regular if and only if (i) the matrix $\begin{pmatrix} E & e \\ A & b \end{pmatrix}$ *has full row rank, and (ii) there exists a point* $x_0 \in \operatorname{ri} P$ *with* $Ax_0 = b$.

By Theorem 5.1.2, if a standard linear programming problem, i.e., problem formulated as

$$\begin{aligned} &\text{minimize } c^T x \\ &\text{subject to } Ax = b \\ &\qquad\qquad x \geqq 0, \end{aligned}$$

where $A \in R^{m\times n}$, $c \in R^n$, $x \in R^n$, $b \in R^m$, has full row rank and is nondegenerate, then it must be regular. This is so because for a nondegenerate system every basic optimal variable is strictly positive. Therefore we can perturb the nonbasic variables to make them positive while keeping the basic variables positive. Hence both conditions of Theorem 5.1.2 are satisfied.

In the case when the linear programming problem contains equality and inequality constraints, the problem can be written in the form where some of the inequality constraints (G) are inactive for some x_0 in the feasible set and the others (H) are active for all feasible x:

$$Gx - g \in J, \qquad Hx - h \in \{0\}, \qquad x \in P, \tag{5.1.1}$$

where J is a nonempty closed convex cone. In this case the system (5.1.1) is regular if the smaller system corresponding to equalities and to always active inequalities

$$Hx - h \in \{0\}, \qquad x \in P \tag{5.1.2}$$

is regular. This result is stated in the next theorem due to Robinson [34].

THEOREM 5.1.3. *Suppose* $\operatorname{int} J \neq \emptyset$. *Then the system* (5.1.1) *is regular if and only if (i)* (5.1.2) *is regular, and (ii) there exist* $x_0 \in P$ *such that* $Gx_0 - g \in \operatorname{int} J$ *and* $Hx_0 - h \in \{0\}$.

By the last theorem, the constraints are also regular when there are no equality constraints present and the feasible region has an interior, since then H is vacuous.

Let the systems

$$(P') \qquad \begin{array}{c} \text{minimize } \langle c',x\rangle \\ \text{subject to } A'x - b' \in Q^*, \qquad x \in P, \end{array}$$

and

$$(D') \qquad \begin{array}{c} \text{maximize } \langle u,b'\rangle \\ \text{subject to } c' - uA' \in P^*, \qquad u \in Q, \end{array}$$

be perturbed systems with $\|A - A'\|, \|b - b'\|$ and $\|c - c'\|$ sufficiently small. Then the necessary and sufficient condition for the systems (P'), (D') to be solvable is given in the next theorem due to Robinson [31].

THEOREM 5.1.4. *The following are equivalent:*

(a) The constraints of (P) and (D) are regular.

(b) The sets of optimal solutions of (P) and (D) are nonempty and bounded.

(c) There exists an $\epsilon_0 > 0$ *such that for any* A', b' *and* c' *with* $\epsilon' = \max\{\|A - A'\|, \|b - b'\|, \|c - c'\|\} < \epsilon_0$ *the two dual problems*

$$(P') \qquad \begin{array}{c} \textit{minimize } \langle c',x\rangle \\ \textit{subject to } A'x - b' \in Q^*, \qquad x \in P, \end{array}$$

and

$$(D') \qquad \begin{array}{c} \textit{maximize } \langle u,b'\rangle \\ \textit{subject to } c' - uA' \in P^*, \qquad u \in Q, \end{array}$$

are solvable.

If these conditions are satisfied, then there exist constants $\epsilon_1 \in (0, \epsilon_0]$ *and* γ *such that for any* A', b', *and* c' *with* $\epsilon' < \epsilon_1$, *any* x' *solving* (P'), *and any* u' *solving* (D'), *one has*

$d[(x', u'), S_P \times S_D] \leqq \gamma\epsilon'$, where S_P and S_D are the sets of optimal solutions for (P) and (D), respectively.

The condition (b) of Theorem 5.1.4 is a useful test for regularity of the constraints.

By the above theorem the constraint regularity is a necessary and sufficient condition for solvability of perturbed linear programming problems. This theorem also says that if the constrains are regular, then the set of optimal solutions changes gradually in the sense that the distance of the solution to a perturbed system from the set of solutions to the unperturbed problem can be bounded by some constant multiple of the size of the perturbation.

Some examples of problems with regular and singular constraints are given below.

Example 1.

$$\text{minimize } x_2$$

$$\text{subject to } x_1 + x_2 \geqq 1$$

$$-x_1 + x_2 \geqq 0$$

$$x_2 = \theta$$

$$x_1 \geqq 0$$

$$x_2 \geqq 0$$

The corresponding dual problem is

$$\text{maximize } y_1 + \theta y_3$$

$$\text{subject to } y_1 - y_2 \leqq 0$$

$$y_1 + y_2 + y_3 \leqq 1$$

$$y_1 \geqq 0$$

$$y_2 \geqq 0$$

$$y_3 \text{ free}$$

For θ=0.5, the solution is (0.5, 0.5) for the primal problem and (0,0,1)+$\alpha(0.5, 0.5, -1)$, $\alpha \in R_+$ for the dual problem. The dual optimal set is is unbounded, and, indeed, the primal problem is unsolvable for any $\theta < 0.5$.

Example 2.

$$\text{minimize } x_2$$

$$\text{subject to } x_1 + x_2 \geqq 1$$

$$-x_1 + x_2 \geqq 0$$

$$x_2 \geqq \theta$$

$$x_1 \geqq 0$$

$$x_2 \geqq 0$$

The corresponding dual problem is

$$\text{maximize } y_1 + \theta y_3$$

$$\text{subject to } y_1 - y_2 \leqq 0$$

$$y_1 + y_2 + y_3 \leqq 1$$

$$y_1 \geqq 0$$

$$y_2 \geqq 0$$

$$y_3 \geqq 0$$

For $\theta = 0.5$ the solutions for primal and dual problems are (0.5, 0.5) and (0, 0, 1) + α $(0.5, 0.5, -1)$, $\alpha \in [0, 1]$, respectively. Both sets of optimal solutions are nonempty and bounded. It is easy to see that the problem is solvable for any perturbation of θ.

Example 3.

$$\text{minimize } x_1 + x_2$$

$$\text{subject to } x_1 = 1$$

$$-x_2 \geqq -2$$

$$x_1 \geqq 0$$

$$x_2 \geqq 0$$

The dual problem is

$$\text{maximize } y_1 - 2y_2$$

$$\text{subject to } -y_2 \leqq 1$$

$$y_1 \leqq 1$$

$$y_1 \text{ free}$$

$$y_2 \geqq 0$$

The solutions of (P) and (D) are (1,0) and (1,0). They are both bounded and the problem is solvable for any perturbation of the right hand vector b.

Example 4.

$$\text{minimize } x_1 + x_2$$

$$\text{subject to } x_1 \geqq 1$$

$$-x_1 \geqq -1$$

$$-x_2 \geqq -2$$

$$x_1 \geqq 0$$

$$x_2 \geqq 0$$

The dual problem is

$$\begin{aligned} \text{maximize } & y_1 - y_2 - 2y_3 \\ \text{subject to } & -y_3 \leqq 1 \\ & y_1 - y_2 \leqq 1 \\ & y_1 \geqq 0 \\ & y_2 \geqq 0 \\ & y_3 \geqq 0 \end{aligned}$$

The solutions for (P) and (D) are (1,0) and $(\alpha, \alpha - 1, 0)$, $\alpha \geqq 1$, respectively. The solution set for (D) is unbounded. It is easy to see that the primal problem is unsolvable for any perturbation of the right hand vector such that the first and the second component are both increased.

Note that Examples 3 and 4 represent two formulations of an identical problem. So these examples show that the system may be made singular by inappropriate formulation. Therefore, conversely, in some cases the system can be regularized by reformulating. The next theorem due to Robinson [34] describes when regularization is possible.

THEOREM 5.1.5. *The necessary and sufficient conditions that the system*

$$Gx \leqq g, \qquad Hx = h, \qquad x \in P \tag{5.1.3}$$

be representable as a regular system of inequalities and equations with the same solution set F is that $F \cap \operatorname{ri} P \neq \{\emptyset\}$. If this condition is satisfied, then (5.1.3) may be made regular by (i) changing certain inequalities to equations, and (ii) deleting certain redundant equations.

In this chapter we gave the necessary and sufficient conditions for persistence of minima for linear programming problems. The perturbed problem is solvable and the set of solutions changes gradually with a perturbation if the constraints satisfy certain regularity criterion.

This criterion is equivalent to the boundedness of the primal and dual solution sets. It was also pointed out that the formulation of the problem is essential, since the constraints can be made singular by inappropriate formulation.

5.2. Nonlinear programming problems.

In this chapter we will give the sufficient conditions for persistence of the minima for general nonlinear programming problems. These conditions require that the constraints are regular and that the second order sufficient conditions hold. For the standard nonlinear programming problems (defined below) the constraint regularity qualification is shown to be equivalent to Mangasarian-Fromovitz criterion, which is a weaker condition than the constraint linear independence qualification.

The following definitions will be needed further on.

DEFINITION. The *polar cone* Q^0 consists of vectors making non-acute angle with the vectors of a cone Q, i.e., $Q^0 = \{y \in R^m \mid \langle q, y\rangle \leqq 0 \text{ for each } q \in Q\}$ (see Fig. 5.2.1).

For example, if $Q = R_+ \times R$ then $Q^0 = R_- \times \{0\}$.

DEFINITION. The *normal cone* for a convex set $C \in R^n$ at a point x

$$\partial\Psi_C(x) = \begin{cases} \{y \in R^n \mid \langle y, c - x\rangle \leqq 0 \quad \forall c \in C\}, & \text{if } x \in C \\ \{\emptyset\}, & \text{if } x \notin C. \end{cases}$$

The normal cone to a convex set C at x is the set of all vectors y normal to C at x. A vector y is said to be normal to a convex set C at a point $x \in C$ if y does not make an acute angle with any line segment in C with x as endpoint.

For example, if a convex set C is defined as $C = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid -1 \leqq x_1 \leqq 1, -2 \leqq x_2 \leqq 2 \right\}$, then the normal cone to C at x is

$$\partial\Psi_C(x) = \begin{cases} \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}, & \text{if } x \text{ is an interior point of } C, \\ R_+ \times \{0\}, & \text{if } x = \begin{pmatrix}1\\a\end{pmatrix}, -2 < a < 2, \\ R_- \times \{0\}, & \text{if } x = \begin{pmatrix}-1\\a\end{pmatrix}, -2 < a < 2, \\ \{0\} \times R_+, & \text{if } x = \begin{pmatrix}a\\2\end{pmatrix}, -1 < a < 1, \\ \{0\} \times R_-, & \text{if } x = \begin{pmatrix}a\\-2\end{pmatrix}, -1 < a < 1, \\ R_+ \times R_+, & \text{if } x = \begin{pmatrix}1\\2\end{pmatrix}, \\ R_+ \times R_-, & \text{if } x = \begin{pmatrix}1\\-2\end{pmatrix}, \\ R_- \times R_-, & \text{if } x = \begin{pmatrix}-1\\-2\end{pmatrix}, \\ R_- \times R_+, & \text{if } x = \begin{pmatrix}-1\\2\end{pmatrix}, \\ \{\emptyset\}, & \text{if } x \notin C \end{cases}$$

(see Fig. 5.2.2).

For a convex cone Q at a point x

$$\partial\Psi_Q(x) = \begin{cases} \{y \in Q^0 \mid \langle y, x\rangle = 0\}, & \text{if } x \in Q, \\ \{\emptyset\}, & \text{if } x \notin Q, \end{cases}$$

since in this case the normal cone is a zero vector at an interior point of the cone, the polar cone at a vertex of the cone (since x=0 at the vertex), and at a point on the edge of the cone the perpendicular edge of the corresponding polar cone (see Fig. 5.2.3).

For example, if $Q = R_+ \times R$, then the normal cone

$$\partial\Psi_Q(x) = \begin{cases} \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}, & \text{if } x \text{ is an interior point of } Q, \\ R_- \times \{0\}, & \text{if } x = \begin{pmatrix}0\\r\end{pmatrix}, r \in R. \end{cases}$$

DEFINITION. The *tangent cone* is the polar of the normal cone.

For example, if $Q = R_+ \times R$, then the tangent cone for Q at x is written as

$$T_Q(x) = \begin{cases} R \times R, & \text{if } x \text{ is an interior point of } Q, \\ R_+ \times R, & \text{if } x = \begin{pmatrix} 0 \\ r \end{pmatrix}, r \in R. \end{cases}$$

For the convex set C defined as $C = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mid -1 \leqq x_1 \leqq 1, -2 \leqq x_2 \leqq 2 \right\}$, the tangent cone to C at x is

$$T_C(x) = \begin{cases} R \times R, & \text{if } x \text{ is an interior point of } C, \\ R_- \times R, & \text{if } x = \begin{pmatrix} 1 \\ a \end{pmatrix}, -2 < a < 2, \\ R_+ \times R, & \text{if } x = \begin{pmatrix} -1 \\ a \end{pmatrix}, -2 < a < 2, \\ R \times R_-, & \text{if } x = \begin{pmatrix} a \\ 2 \end{pmatrix}, -1 < a < 1, \\ R \times R_+, & \text{if } x = \begin{pmatrix} a \\ -2 \end{pmatrix}, -1 < a < 1, \\ R_- \times R_-, & \text{if } x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \\ R_- \times R_+, & \text{if } x = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \\ R_+ \times R_+, & \text{if } x = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \\ R_+ \times R_-, & \text{if } x = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \end{cases}$$

(see Fig. 5.2.4).

DEFINITION. We say that f is *Frechet differentiable* at x if there exists a vector x' such that

$$\lim_{y \to x} \frac{f(y) - f(x) - \langle x', y - x \rangle}{||y - x||} = 0,$$

where $\langle x', y - x \rangle$ denotes the inner product of vectors x', $y - x$ and $|| \cdot ||$ is the Euclidean norm.

We henceforth assume that all functions are Frechet differentiable.

The nonlinear programming problem can be formulated as follows:

$$\text{minimize } f(x)$$

(5.2.1) $$\text{subject to } g(x) \in Q^0, \qquad x \in C,$$

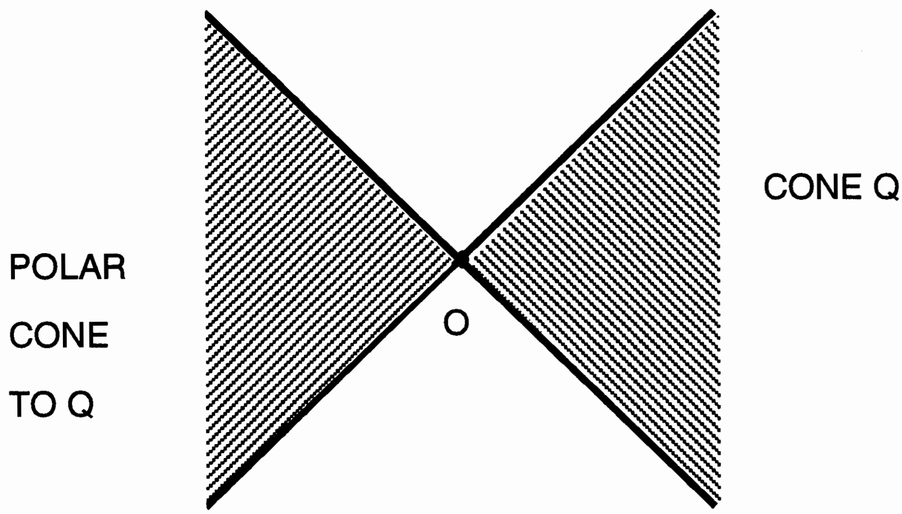

FIGURE 5.2.1. *Polar cone.*

where Q^0 denotes the polar cone of a closed convex cone Q, and C is a closed convex set.

For instance, in the case of the standard nonlinear programming problem

$$\text{minimize } f(x)$$

$$\text{subject to } g_i(x) = 0, \qquad (i = 1, \ldots, l)$$

$$g_i(x) \leqq 0, \qquad (i = l+1, \ldots, l+k)$$

we have $C = R^n$, $Q = R^l \times R^k_+$, and $Q^0 = \{0\}^l \times R^k_-$ (where R^k_+ is the non-negative orthant in R^k, R^k_- is the non-positive orthant in R^k, and $\{0\}^l$ is the origin in R^l).

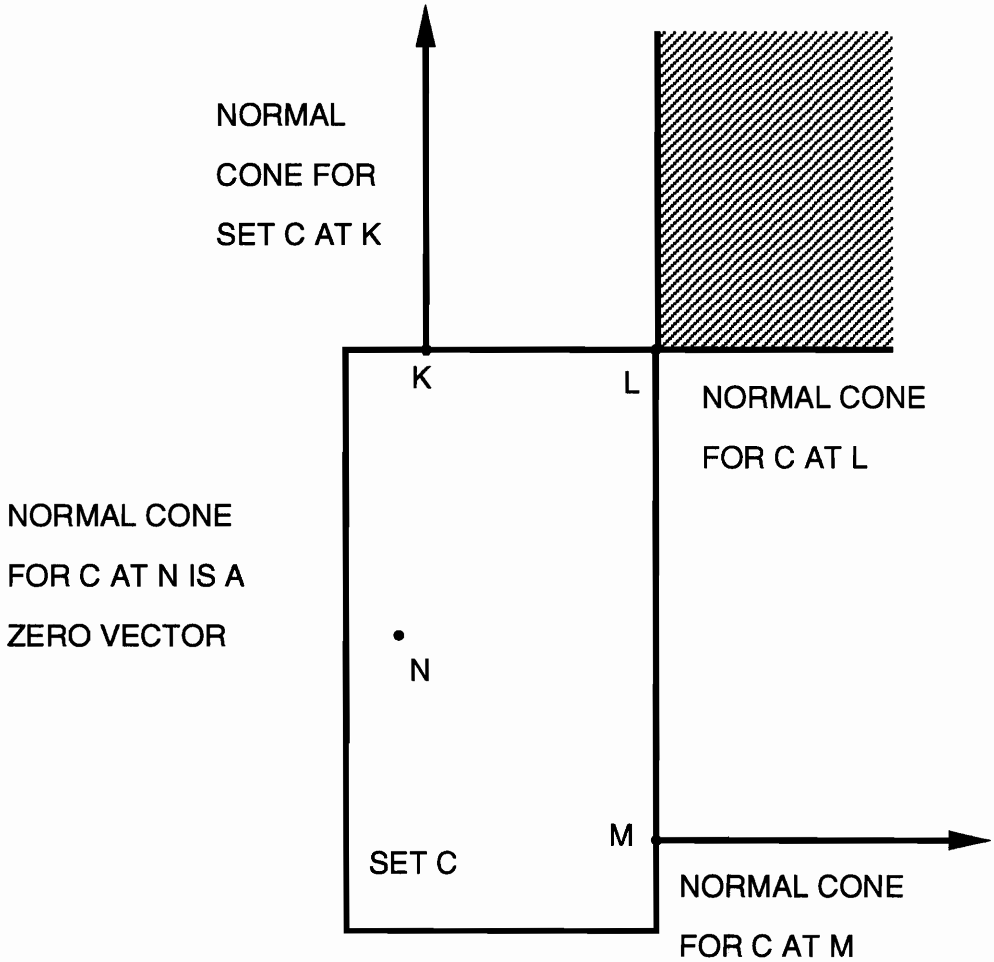

FIGURE 5.2.2. *Normal cone for a convex set C.*

Local minimizers for problem (5.2.1) satisfy a minimum-principle necessary optimality criterion (see Mangasarian [22], pp. 168), given next. The closure of a set X is denoted by $\bar{X}$.

THEOREM 5.2.1. *Let C be a convex set in R^n with a nonempty interior. Let f and g be differentiable on C. If $x_0 \in C$ is a solution of*

$$\textit{minimize } f(x)$$

$$\textit{subject to } g_i(x) = 0, \qquad (i = 1, \ldots, l)$$

$$g_i(x) \leqq 0, \qquad (i = l+1, \ldots, l+k), \qquad x \in C,$$

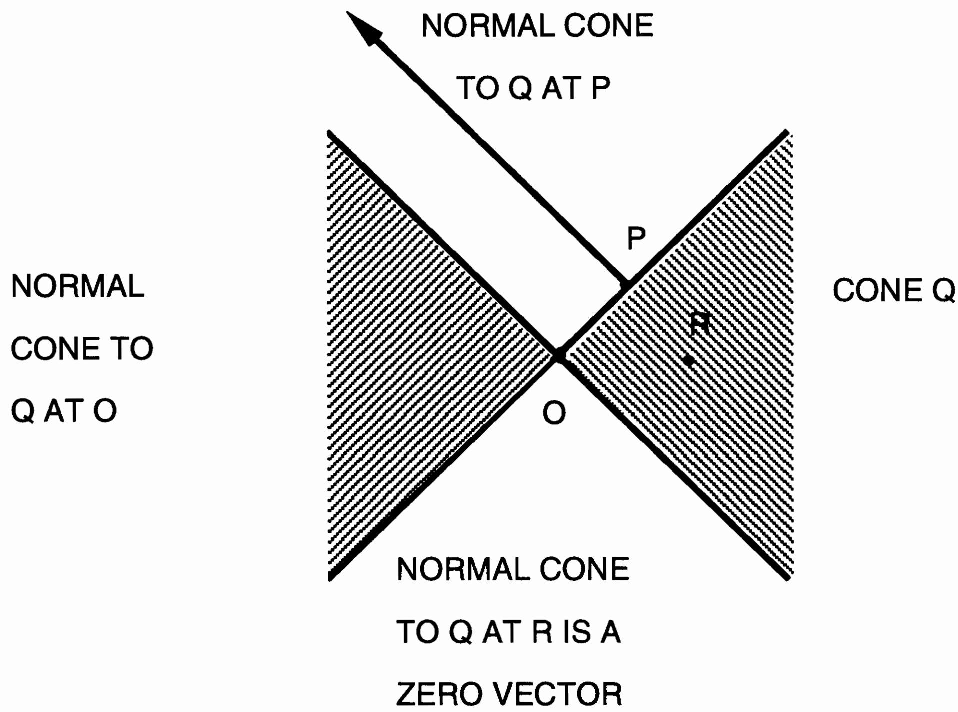

FIGURE 5.2.3. *Normal cone for a convex cone Q.*

then there exist multipliers μ_i $(l+1 \leqq i \leqq l+k+1)$ *and* λ_i $(1 \leqq i \leqq l)$ *(not all equal to zero) such that*

$$\Big(\mu_{l+k+1}\nabla_x f(x_0) + \sum_{j=1}^{l} \lambda_j \nabla_x g_j(x_0) + \sum_{i=1}^{k} \mu_{l+i}\nabla_x g_{l+i}(x_0)\Big)(x - x_0) \geqq 0 \quad \forall x \in \bar{C},$$

$$\mu_i g_i(x_0) = 0, \qquad (i = l+1, \ldots, l+k),$$

$$g_i(x_0) = 0, \qquad (i = 1, \ldots, l),$$

$$g_i(x_0) \leqq 0, \qquad (i = l+1, \ldots, l+k),$$

$$\mu_i \geqq 0, \qquad (i = l+1, \ldots, l+k+1).$$

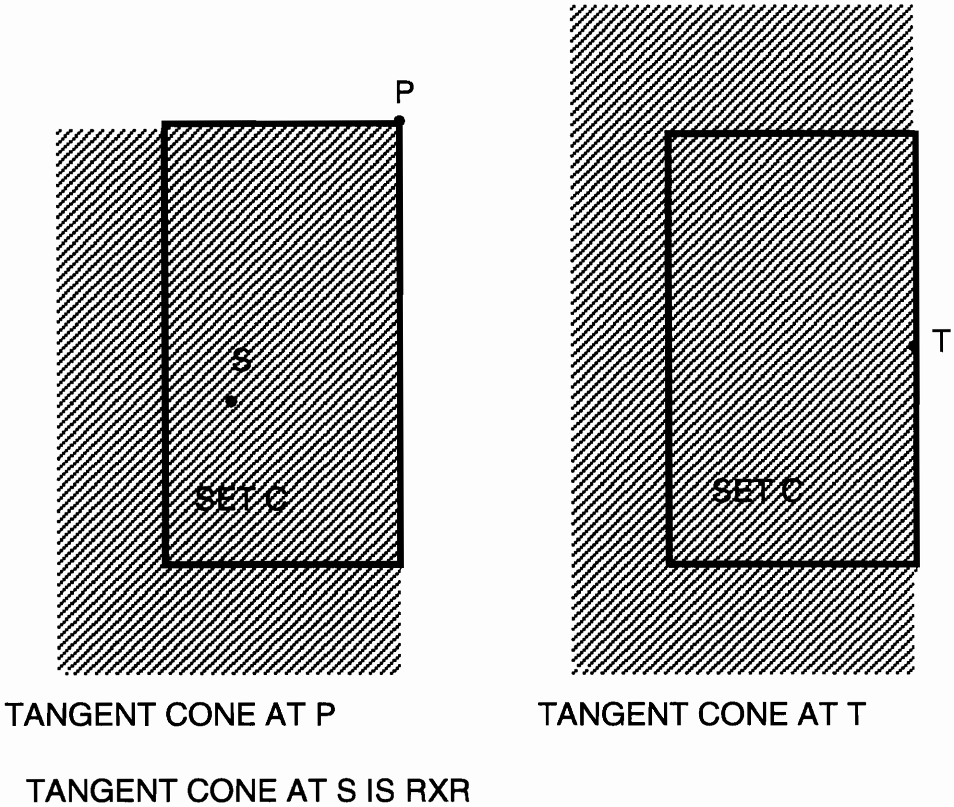

FIGURE 5.2.4. *Tangent cone for a convex set C.*

The minimum principle necessary optimality criterion also holds when C has an empty interior (which is allowed in minimization problem (5.2.1)), since a convex set without an interior is equivalent to the intersection of a convex set with a nonempty interior and a linear manifold $\{x|x \in R^n, g(x) = 0\}$.

The minimum principle condition is weaker than the Fritz John criterion which requires that

$$\mu_{l+k+1}\nabla_x f(x_0) + \sum_{j=1}^{l} \lambda_j \nabla_x g_j(x_0) + \sum_{i=1}^{k} \mu_{l+i} \nabla_x g_{l+i}(x_0) = 0, \qquad (\lambda, \mu) \neq 0.$$

The latter condition follows from the first one when the feasible region C is open, but does not hold in general.

If a certain regularity qualification (which will be discussed further on) holds, then it can be proved (see Robinson [33]) that $\mu_{l+k+1} > 0$. Then μ_{l+k+1} can be assumed to be equal to one, and the minimum principle necessary condition can be written as

$$-\mathcal{L}_x(x_0, u_0) \in \partial\Psi_C(x_0),$$

$$\mathcal{L}_u(x_0, u_0) \in \partial\Psi_Q(u_0),$$

where $u_0 = (\lambda_1, \ldots, \lambda_l, \mu_{l+1}, \ldots, \mu_{l+k}) \in Q$, $\mathcal{L}(x, u) = f(x) + \langle u, g(x)\rangle$ is the Lagrangian, the subscripts x and u denote partial differentiation with respect to x and u, respectively, and $\partial\Psi$ denotes the normal cone at a point. The last condition can be written equivalently

$$\begin{aligned} &0 \in \mathcal{L}_x(x, u) + \partial\Psi_C(x), \\ &0 \in -\mathcal{L}_u(x, u) + \partial\Psi_Q(u). \end{aligned} \tag{5.2.2}$$

The equations (5.2.2) can be linearized at (x_0, u_0):

$$0 \in \begin{pmatrix} \mathcal{L}_x(x_0, u_0) \\ -\mathcal{L}_u(x_0, u_0) \end{pmatrix} + \begin{pmatrix} \mathcal{L}_{xx}(x_0, u_0) & \mathcal{L}_{xu}(x_0, u_0) \\ -\mathcal{L}_{ux}(x_0, u_0) & -\mathcal{L}_{uu}(x_0, u_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ u - u_0 \end{pmatrix} + \partial\Psi_{C\times Q}(x, u),$$

where $\partial\Psi_{C\times Q}(x, u) = \partial\Psi_C(x) \times \partial\Psi_Q(u)$. Substituting for the derivatives of the Lagrangian, we get

$$0 \in \begin{pmatrix} \mathcal{L}'(x_0, u_0) \\ -g(x_0) \end{pmatrix} + \begin{pmatrix} \mathcal{L}''(x_0, u_0) & g'(x_0)^* \\ -g'(x_0) & 0 \end{pmatrix} \begin{pmatrix} x - x_0 \\ u - u_0 \end{pmatrix} + \partial\Psi_{C\times Q}(x, u),$$

where asterisk denotes transpose and prime denotes partial differentiation with respect to x. The following second-order sufficient condition will be used in the further analysis:

DEFINITION. Suppose (x_0, u_0) is a point satisfying the first order necessary optimality conditions (5.2.2). The *second-order sufficient condition* holds at (x_0, u_0) with modulus $\mu > 0$ if for each $h \in T_C(x_0)$ with

$$g'(x_0)h \in T_{Q^\circ}(g(x_0)), \qquad f'(x_0)h = 0,$$

one has $\langle h, \mathcal{L}''(x_0, u_0)h\rangle \geqq \mu||h||^2$, where $T_C(x_0)$ denotes the tangent cone to C at x_0.

The conditions $g'(x_0)h \in T_{Q^0}(g(x_0))$, $h \in T_C(x_0)$ denote that we consider only feasible directions, this is, directions such that $x_0 + h \in C$ and $g(x_0) + g'(x_0)h \in Q^0$. The condition $f'(x_0)h = 0$ in the above definition can be written as $f'(x_0)h \leqq 0$, which can be shown as follows. Since h belongs to the tangent cone to C, and, by the first order necessary conditions, $-[f'(x_0) + g'(x_0)^* u_0]$ belongs to the normal cone of C at x_0 (and a tangent cone is the polar to the normal cone),

$$0 \leqq \langle f'(x_0) + g'(x_0)^* u_0, h\rangle = f'(x_0)h + \langle u_0, g'(x_0)h\rangle.$$

Since, by the first order necessary conditions, $g(x_0)$ belongs to the normal cone to Q at u_0, u_0 also belongs to the normal cone to Q^0 at $g(x_0)$ (see Rockafellar [37]). As $g'(x_0)h$ belongs to the corresponding tangent cone, $\langle u_0, g'(x_0)h\rangle \leqq 0$. Hence $f'(x_0)h \geqq 0$, and since $f'(x_0)h \leqq 0$, $f'(x_0)h = 0$. Therefore the condition $f'(x_0)h = 0$ in the above definition can be written as $f'(x_0)h \leqq 0$.

For the standard nonlinear programming problems (i.e., when $C = R^n$) the above definition is equivalent to the standard second order sufficient condition (given below). The proof, due to Han and Mangasarian [10], is given next. Let us assume that $C = R^n$ and that the constraint regularity qualification holds. Then the minimum principle necessary conditions (5.2.2) reduce to the Kuhn-Tucker conditions. The condition $h \in T_C(x_0)$ is now always satisfied, since $T_C(x_0) = R^n$. If $h = 0$, the above condition is trivially satisfied, so we only need to check other admissible directions h. The condition $g'(x_0)h \in T_{Q^0}(g(x_0))$ means that we consider only h satisfying $\nabla g_i(x_0)h \leqq 0$ if g_i is an active inequality constraint, and $\nabla g_i(x_0)h = 0$ if g_i is an equality constraint.

Therefore the second order sufficient condition given in the above definition holds at the Kuhn-Tucker point (x_0, u_0) if the following implication is true:

$$\nabla f(x_0)h \leqq 0,$$

$$
\begin{aligned}
\nabla g_i(x_0)h = 0, &\qquad i = 1, \ldots, l \\
\nabla g_i(x_0)h \leqq 0, &\qquad \forall i \in \{l+1 \leqq i \leqq l+k \mid g_i(x_0) = 0\} \\
h \neq 0, &
\end{aligned}
$$

implies

$$
h^T \nabla_x^2 \mathcal{L}(x_0, u_0)h > 0.
$$

The standard second order sufficient condition holds at the Kuhn-Tucker point (x_0, u_0) if the following implication is true

$$
\begin{aligned}
\nabla g_i(x_0)h = 0, &\qquad i = 1, \ldots, l, \\
\nabla g_i(x_0)h = 0, &\qquad \forall i \in J = \{l+1 \leqq i \leqq l+k \mid g_i(x_0) = 0, u_{0i} > 0\}, \\
\nabla g_i(x_0)h \leqq 0, &\qquad \forall i \in K = \{l+1 \leqq i \leqq l+k \mid g_i(x_0) = 0, u_{0i} = 0\}, \\
h \neq 0, &
\end{aligned}
$$

implies

$$
h^T \nabla_x^2 \mathcal{L}(x_0, u_0)h > 0.
$$

To prove that these implications are equivalent we want to show that the sets T and S of directions h satisfying the hypothesis of the implication in the above given definition and in the standard definition of the second order sufficient conditions, respectively, are equal. First we show that $T \subset S$. If T is empty then, trivially, $T \subset S$. Suppose that $T \neq \{\emptyset\}$. We need to show that $\nabla g_j(x_0)h = 0$ if $j \in J$. By the Kuhn-Tucker conditions we have that

$$
\nabla f(x_0)h + \sum_{j \in J \cup K} u_{0j} \nabla g_j(x_0)h + \sum_{i=1}^{l} u_{0i} \nabla g_i(x_0)h = 0.
$$

By assumption, $\nabla f(x_0)h = 0$, $\nabla g_j(x_0)h = 0$ for $j = 1, \ldots, l$ and $u_{0j} = 0$ for $j \in K$, so we have

$$
\sum_{j \in J} u_{0j} \nabla g_j(x_0)h = 0.
$$

Since $u_{0j} > 0$ for $j \in J$,

$$\nabla g_j(x_0)h = 0 \qquad \forall j \in J.$$

To prove that $S \subset T$ we assume that S is nonempty. It is enough to show that $\nabla f(x_0)h \leqq 0$. By the Kuhn-Tucker conditions we have

$$\nabla f(x_0)h + \sum_{j \in J \cup K} u_{0j} \nabla g_j(x_0)h + \sum_{j=1}^{l} u_{0j} \nabla g_j(x_0)h = 0.$$

Then $\nabla f(x_0)h = 0$ because all the other terms are zero. This completes the proof.

We consider the perturbed optimization problem

$$\begin{aligned} &\underset{x}{\text{minimize}}\ f(x,\alpha) \\ &\text{subject to } g(x,\alpha) \in Q^0, \qquad x \in C, \end{aligned} \tag{5.2.3}$$

where α is a perturbation parameter, and x is the variable in which the minimization is done.

The first order necessary conditions for problem (5.2.3) are

$$\begin{aligned} &0 \in f'(x,\alpha) + g'(x,\alpha)^* u + \partial \Psi_C(x) \\ &0 \in -g(x,\alpha) + \partial \Psi_Q(u) \end{aligned} \tag{5.2.4}$$

The following constraint regularity qualification by Robinson [32] will be used in the further analysis (the dependence on the perturbation α will be dropped when defining constraint regularity):

DEFINITION. Let $g(x)$ be continuously Frechet differentiable at a point x_0. We say $g(x)$ is *regular* at x_0 if $0 \in \operatorname{int}\{g(x_0) + g'(x_0)(C - x_0) - Q^0\}$.

If we can write $g(x) = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}$, so that the constraints for problem (5.2.1) are

$$g_1(x) = 0, \qquad g_2(x) \leqq 0, \qquad x \in C, \tag{5.2.5}$$

then the regularity condition at some point x_0 satisfying (5.2.5) can be written

$$0 \in \operatorname{int}\left\{ \begin{pmatrix} g_1(x_0) \\ g_2(x_0) \end{pmatrix} + \begin{pmatrix} g_1{}'(x_0) \\ g_2{}'(x_0) \end{pmatrix} (C - x_0) + \begin{pmatrix} \{0\}^l \\ R_+^k \end{pmatrix} \right\}. \tag{5.2.6}$$

Note that the set on the right side of (5.2.6) can have a nonempty interior even if C does not have interior.

Next we can write $g_2(x) = \begin{pmatrix} g_{2A}(x) \\ g_{2I}(x) \end{pmatrix}$ with $g_{2A}(x_0) = 0$ and $g_{2I}(x_0) < 0$. We can write the affine hull of C as aff $C = \{x \mid Ex = e\}$, where E has full row rank (for aff $C = R^n$, E and e are vacuous). The equivalent form of the regularity condition (5.2.6) is given in the next theorem due to Robinson [32]:

THEOREM 5.2.2. *Let x_0 be any point solving (5.2.5). Then (5.2.6) is equivalent to the condition that the matrix $\begin{pmatrix} g_1'(x_0) \\ E \end{pmatrix}$ has full row rank and there exists a $z \in$ ri C with $g_{2A}'(x_0)(z - x_0) < 0$ and $g_1'(x_0)(z - x_0) = 0$.*

For the standard nonlinear programming problem the condition given in the above theorem corresponds to the well-known constraint qualification given by Mangasarian and Fromovitz [21]:

THEOREM 5.2.3. *Let $x_0 \in R^n$ be a solution of the minimization problem*

$$\begin{aligned} &\textit{minimize } f(x) \\ &\textit{subject to } g_i(x) = 0, \qquad 1 \leqq i \leqq l, \\ &\qquad\qquad g_i(x) \leqq 0, \qquad l + 1 \leqq i \leqq l + k \end{aligned}$$

where $f(x)$ and $g_i(x)$ have continuous first partial derivatives on R^n.

Then the constraints are regular at x_0 if there exists a vector $\bar{y} \in R^n$ such that

$$\nabla g_i(x_0)\bar{y} < 0, \qquad \forall i : g_i(x_0) = 0, l + 1 \leqq i \leqq l + k, \tag{5.2.7}$$

$$\nabla g_i(x_0)\bar{y} = 0, \qquad i = 1, \ldots, l, \tag{5.2.8}$$

and that

$$(5.2.9) \qquad \nabla g_i(x_0), \qquad i = 1, \ldots, l,$$

are linearly independent. For the case when no inequality constraints are active, condition (5.2.9) *alone is a sufficient constraint qualification.*

Geometrically, conditions (5.2.7) and (5.2.8) correspond to the condition that the gradients of the active inequality constraints at x_0 form a pointed cone and there is a vector in the polar of this cone that is tangent to the surface formed by the equality constraints (a cone is pointed if there exists a vector which makes an acute angle with all the vectors of the cone).

Define $U_0 = \{u \mid (x_0, u)$ satisfies (5.2.4) for $\alpha = \alpha_0\}$. The sufficient conditions for the solvability of a perturbed nonlinear programming problem are given in the following theorem due to Robinson [36]:

THEOREM 5.2.4. *Suppose that for $\alpha = \alpha_0$,* (5.2.3) *satisfies the second-order sufficient condition (defined in this section) at x_0 and some $u_0 \in U_0$ and its constraints are regular at x_0. Then for each neighborhood M of x_0 there is a neighborhood N of α_0 such that if $\alpha \in N$, then* (5.2.3) *has a local minimizer in M.*

In this chapter we gave the sufficient conditions for solvability of the perturbed systems. They require that the constraint regularity qualification (equivalent to the Mangasarian-Fromovitz criterion when $x \in R^n$) and the second order sufficient condition are satisfied. These conditions are only sufficient conditions unlike to the linear programming problems where the constraint regularity is the necessary and sufficient condition for solvability of the perturbed problems.

6. Implications of bifurcation theory and stability theory for continuity of the path of optima – synthesis.

In this chapter we want to show the implications of the bifurcation theory and the stability theory for continuation of the minima path. We will also consider the possibility of computational verification of the conditions for persistence of minima.

The results of bifurcation theory presented in Chapter 4 can be used to characterize the possible points of discontinuity of the path of minima. Since the conditions given there are necessary conditions for discontinuity of the path of minima, their negation provides sufficient conditions for continuity of the path. Thus the sufficient conditions for continuity of the optima path obtained by Poore (Theorem 4.1) require that all three of the following conditions hold:

(1) no active constraints have zero Lagrange multipliers;

(2) all gradients of active inequality constraints and equality constraints are linearly independent;

(3) the Hessian of the Lagrangian projected on the tangent space of active inequality constraints and equality constraints is nonsingular.

Condition (1) is weakened by his additional result (Theorem 4.2) that the minima persist even if an active constraint has a zero Lagrange multiplier provided that the Hessian of the Lagrangian is positive definite on the space tangent to active constraints with positive Lagrange multipliers. The basis for the proof of these sufficient conditions is the Implicit Function Theorem.

The assumptions for sufficient conditions for persistence of the minima provided by Robinson are weaker. He dropped the strict complementarity condition (1) and replaced the constraint qualification (2) by a weaker requirement equivalent (in the case of the standard nonlinear programming problem) to the Mangasarian Fromovitz criterion. Robinson's requirement that the second order sufficient condition holds is the same as condition (3) above, since we consider the path of minima (hence the projected Lagrangian is positive semidefinite).

If the strict complementarity condition is dropped, then the standard implicit function theorem condition cannot be used to prove the existence of solutions to the Fritz John conditions. To get around this difficulty Robinson expressed the first order optimality conditions in the form of generalized equations. A generalized equation is an inclusion of the form

$$0 \in F(z) + \partial\Psi_C(z), \tag{6.1}$$

where $F : R^k \to R^k$, C is a nonempty closed convex set and $\partial\Psi_C$ is the normal cone for set C at z. Then he was able to use some facts about generalized equations developed in [35], in particular the result that the linearization of (6.1) about z_0

$$0 \in F(z_0) + F'(z_0)(z - z_0) + \partial\Psi_C(z) \tag{6.2}$$

contains enough information to analyze the local behavior of the solutions to (6.1) with respect to the small perturbations. This allowed him to formulate the second order sufficient condition and the constraint regularity that together ensure the persistence of the minima for perturbed problems.

The constraint qualification assumed in Robinson's sufficient conditions is weaker than the corresponding assumption in Poore's conditions, since linear independence of the gradients implies the Mangasarian-Fromovitz qualification, but not conversely. The following example illustrates that point.

Example 1.

$$\text{minimize } x_2$$
$$\text{subject to } x_1^2 - x_2 \leqq 0$$
$$-x_1^2 - x_2 + \alpha \leqq 0$$

The minimization problem is shown in Fig. 6.1. The solutions to the Fritz John conditions for this problem are curves $(x_1, x_2, \mu_1, \mu_2, \mu_3)$: $(\frac{\sqrt{\alpha}}{\sqrt{2}}, \frac{\alpha}{2}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$, $(\frac{-\sqrt{\alpha}}{\sqrt{2}}, \frac{\alpha}{2}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}})$, and $(0, 0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. The first two curves are minima for $\alpha \geqq 0$, but they do not exist for $\alpha < 0$. The third curve consists of infeasible points for $\alpha > 0$ and of minima for $\alpha \leqq 0$. For $\alpha = 0$ there is also an additional curve of solutions obtained by fixing $x_1 = x_2 = 0$ and varying the Lagrange multipliers. In this example the gradients of the active constraints for $\alpha = 0$ at $x = (0,0)$ are $\nabla g_1 = \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \nabla g_2$, hence they are linearly dependent. The Mangasarian-Fromovitz criterion is satisfied, since the gradients of the active constraints form a pointed cone. Since the second order condition is also satisfied, the sufficient conditions for persistence of minima provided by Robinson hold, and indeed the path of minima is continuous in this example. Since the gradients of active constraints are not linearly independent, the sufficient conditions for persistence of minima obtained by Poore are not satisfied and we cannot determine if the path of minima will be continuous or not.

The Mangasarian-Fromovitz criterion can be checked computationally as follows. By Motzkin's Transposition Theorem [21], the Mangasarian-Fromovitz criterion is equivalent to the condition that the gradients of equality constraints are linearly independent and that the system of equations

$$(6.3) \qquad \sum_{i \in \bar{A}} \mu_i \nabla g_i(x_0) + \sum_{i=1}^{l} \lambda_i \nabla g_i(x_0) = 0, \qquad \bar{A} = \{l + 1 \leqq i \leqq l + k \mid g_i(x_0) = 0\},$$

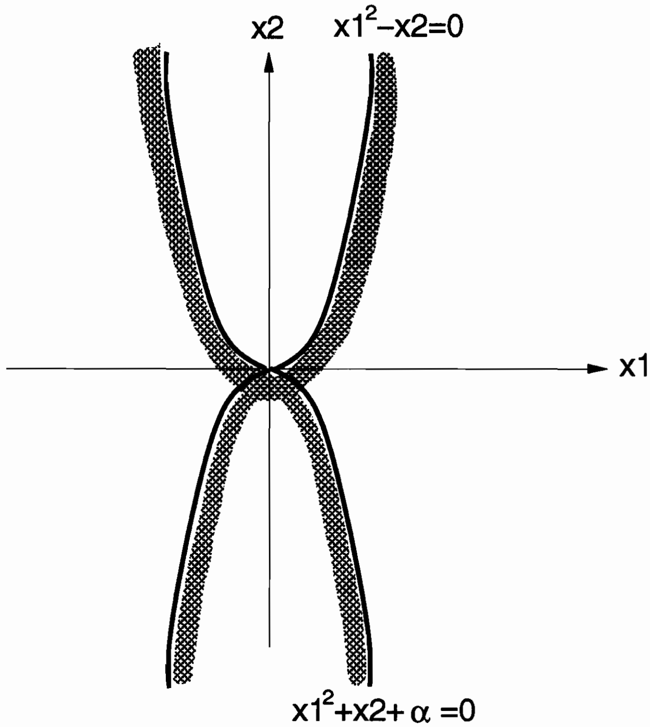

FIGURE 6.1. *Active constraints are linearly dependent, but satisfy Mangasarian-Fromovitz criterion.*

has no solution $\mu_i \geqq 0$ (not all μ_i equal to zero), $\lambda_i \in R$. This latter condition can be verified computationally as follows. We can write the system of equations (6.3) as

$$\begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow & \cdots & \uparrow & \uparrow & \uparrow & \cdots & \uparrow \\ \nabla g_1 & -\nabla g_1 & \nabla g_2 & -\nabla g_2 & \cdots & \nabla g_l & -\nabla g_l & \nabla g_{i_1} & \cdots & \nabla g_{i_r} \\ \downarrow & \downarrow & \downarrow & \downarrow & \cdots & \downarrow & \downarrow & \downarrow & \cdots & \downarrow \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \lambda_1^+ \\ \lambda_1^- \\ \lambda_2^+ \\ \lambda_2^- \\ \vdots \\ \lambda_l^+ \\ \lambda_l^- \\ \mu_{i_1} \\ \vdots \\ \mu_{i_r} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

where $\bar{A} = \{i_1, \ldots, i_r\}$, $\lambda_i = \lambda_i^+ - \lambda_i^-$, $\lambda_i^+ \geqq 0$, $\lambda_i^- \geqq 0$. The latter system can be written in matrix form as $Ay = b$. To verify condition (6.3) we need to find a feasible solution to the problem

$$(6.4) \qquad Ay = b, \qquad y \geqq 0.$$

This is just Phase I of the simplex method. We introduce additional variables $r = b - Ay$, and then we solve an auxiliary problem

$$(6.5) \qquad \begin{aligned} &\text{minimize} \sum_{j=1}^{n+1} r_j \\ &\text{subject to } Ay + r = b, \\ &\qquad y \geqq 0 \\ &\qquad r \geqq 0. \end{aligned}$$

Let y' and r' solve problem (6.5). If $r' = 0$ then y' is feasible in (6.4), whereas if $r' \neq 0$, then there is no feasible point for (6.4). The first part of the latter statement is obvious. The second part is true, since if y' is feasible in (6.4) then y', 0 is feasible in (6.5) with $\sum_{j=1}^{n+1} r_j = 0$, which contradicts the optimality of $r' \neq 0$. Since (6.5) is a standard linear program, it can be easily solved by standard methods.

Definitions of the constraint regularity for linear and nonlinear problems given in Chapters 5.1 and 5.2, respectively, look different, but in fact the definition for linear problems is a special case of the definition for nonlinear problems, which we show next.

We say that the constraint

$$(6.6) \qquad g(x) \in Q^0, \qquad x \in C$$

is regular at x_0 if

$$0 \in \operatorname{int}\{g(x_0) + g'(x_0)(C - x_0) - Q^0\},$$

where Q^0 and C are a closed convex cone and a closed convex set, respectively.

We define the constraints for linear programming as

$$Ax - b \in Q^*, \qquad x \in P \tag{6.7}$$

where Q^* and P are closed convex cones. The constraints of (6.7) are regular if

$$b \in \operatorname{int}\{A(P) - Q^*\}.$$

Since Q^0 and C above are a closed convex cone and a closed convex set, then we can take $Q^0 = Q^*$ and $C = P$. Writing (6.7) as a nonlinear problem, we say that the constraints for a linear problem (6.7) are regular at x_0 if

$$0 \in \operatorname{int}\{Ax_0 - b + A(P - x_0) - Q^*\} = \operatorname{int}\{-b + A(P) - Q^*\},$$

which can be written equivalently as

$$b \in \operatorname{int}\{A(P) - Q^*\}.$$

Therefore the regularity of the linear constraints can be checked in the same way as the regularity of the nonlinear constraints.

Since we can check computationally the regularity of the nonlinear constraints when the set C in (6.6) is R^n, we want to write the linear programming problem (6.7) so that $P = R^n$. If we show that this transformation preserves the regularity of the constraints, then we will be able to use the Mangasarian-Fromovitz criterion also for standard linear programming problems. For the standard linear problem the constraints $Ax - b \in Q^*$, $x \in R^n_+$ can be written as

$$A_1 x - b_1 \in R^m_-, \qquad A_2 x - b_2 \in \{0\}^l, \qquad x \in R^n_+. \tag{6.8}$$

We can also state this problem as

$$\tilde{A}_1 x - \tilde{b}_1 \in \begin{pmatrix} R^m_- \\ R^n_+ \end{pmatrix}, \qquad A_2 x - b_2 \in \{0\}^l, \qquad x \in R^n, \tag{6.9}$$

where $\tilde{A}_1 = \begin{pmatrix} A_1 \\ I \end{pmatrix}$, ($I$ an identity matrix) and $\tilde{b}_1 = \begin{pmatrix} b_1 \\ 0 \end{pmatrix}$. We will show that the constraints (6.8) are regular if and only if the constraint (6.9) are regular.

Suppose that (6.9) are regular. Since int $\begin{pmatrix} R_-^m \\ R_+^n \end{pmatrix} \neq \emptyset$, then, by Theorem 5.1.3, the constraints $A_2x - b_2 \in \{0\}^l$, $x \in R^n$ are regular and there is $x_0 \in R^n$ such that $A_1x_0 - b_1 \in$ int $\{R_-^m\}$, $x_0 \in$ int $\{R_+^n\}$ and $A_2x_0 - b_2 \in \{0\}^l$. Therefore there is x_0 strictly positive such that $A_1x_0 - b_1 \in$ int $\{R_-^m\}$ and $A_2x_0 - b_2 \in \{0\}^l$. Moreover, since $A_2x - b_2 \in \{0\}^l$ are regular equality constraints, then, by Theorem 5.1.2, the matrix $(\, A_2 \quad b_2 \,)$ has full rank. Hence, by Theorem 5.1.2, using also the fact that there is x_0 strictly positive such that $A_2x_0 - b_2 \in \{0\}^l$, the constraints $A_2x - b_2 \in \{0\}^l$, $x \in R_+^n$ are regular. Therefore, by Theorem 5.1.3, the constraints (6.8) are regular.

Now suppose that the constraints (6.8) are regular. Then the constraints $A_2x - b_2 \in \{0\}^l$, $x \in R_+^n$ are regular. Therefore, by Theorem 5.1.2, the constraints $A_2x - b_2 \in \{0\}^l$, $x \in R^n$ are regular, and there exists $x_1 > 0$ such that $A_2x_1 - b_2 \in \{0\}^l$. Since (6.8) are regular, by Theorem 5.1.3, there exists $x_0 \in R_+^n$ such that $A_1x_0 - b_1 \in$ int $\{R_-^m\}$, and $A_2x_0 - b_2 \in \{0\}^l$. Then $x_2 = (1-\lambda)x_0 + \lambda x_1)$, for small $\lambda > 0$, satisfies $x_2 \in$ int R_+^n, $A_1x_2 - b_1 \in$ int $\{R_-^m\}$, and $A_2x_2 - b_2 \in \{0\}^l$. Hence, by Theorem 5.1.3, the constraints (6.9) are regular.

Therefore the regularity of the primal linear constraints can be verified computationally by checking the Mangasarian-Fromovitz criterion. Similarly, we can verify the regularity of the dual constraints. Therefore for linear programming problems it is possible to check computationally the necessary and sufficient conditions for solvability of the perturbed problems. Since the second order sufficient optimality condition is not satisfied for linear problems, the path of parametrized optima can be discontinuous in the sense that the solution can jump for arbitrary small perturbation. The following example illustrates this point.

Example 2.

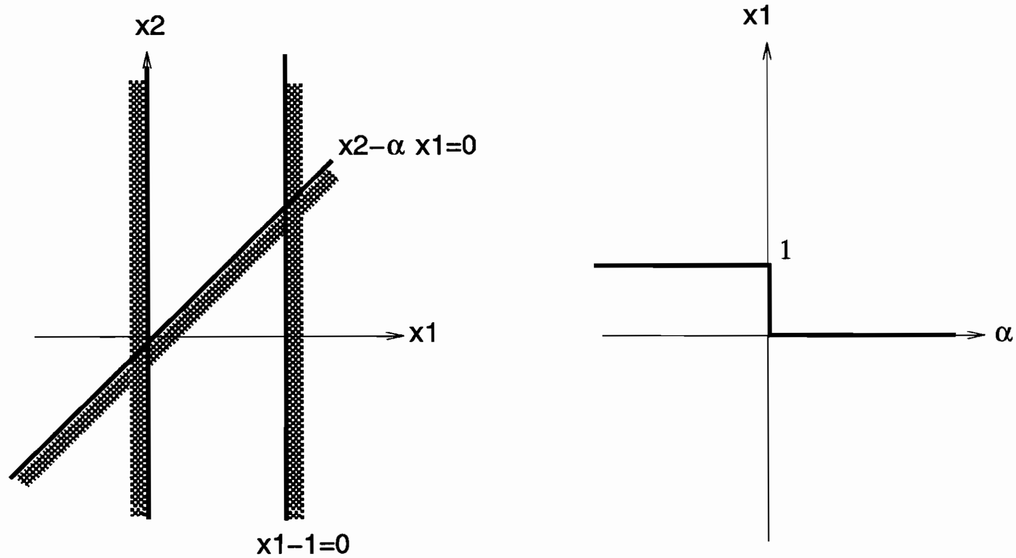

FIGURE 6.2. *Path of optima for the linear problem can be discontinuous even when the constraints are regular.*

$$\text{minimize } x_2$$
$$\text{subject to } \alpha x_1 - x_2 \leqq 0$$
$$-x_1 \leqq 0$$
$$x_1 - 1 \leqq 0$$

The minimization problem is shown in Fig. 6.2. The solution (x_1, x_2) to this problem is (0,0) for $\alpha > 0$, any point $(x_1, 0)$, $0 \leqq x_1 \leqq 1$ for $\alpha = 0$, and $(1, \alpha)$ for $\alpha < 0$. The plot of $x_1(\alpha)$ is given in Fig. 6.2. The solution is discontinuous for $\alpha = 0$, since we can find a neighborhood of (0,0) that would not contain a minimizer for a negative value of α, no matter how small α we take. This cannot happen if the second order sufficient optimality condition is satisfied, since then, by Thm. 5.2.4., for each neighborhood N of the minimizer $\bar{x}$ corresponding to the perturbation $\bar{\alpha}$ there is a neighborhood M of $\bar{\alpha}$ such that for any perturbation α in M we can find a minimizer in N.

The Mangasarian-Fromovitz qualification and the second order sufficient optimality condition are sufficient for solvability of the perturbed system, but neither of these conditions is necessary for minima to persist.

Example 3.

$$\text{minimize } -x_1$$
$$\text{subject to } x_1^3 + x_2 \leqq 0$$
$$x_1^3 - x_2 \leqq 0$$

This minimization problem is shown in Fig. 6.3. The solution to this problem is $(0, 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, and it corresponds to the point of intersection of the two constraints. Consider an arbitrary perturbation of the problem above, i.e., a small perturbation of any of the coefficients defining the constraints, or the objective function. Since for any neighborhood of the point (x_1, x_2)=(0,0) the perturbation may be made small enough so that the constraints intersect in that neighborhood, the minima will persist. And yet the Mangasarian-Fromovitz qualification is not satisfied at this point, since the gradients of the active constraints are $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, so they do not form a pointed cone.

The second order sufficient condition also is not necessary for persistence of minima – it is not satisfied, for example, for linear programming problems.

Although the hypotheses of sufficient conditions for persistence of minima in Poore's bifurcation theory are stronger than those assumed by Robinson (hence some well behaved problems will not be recognized), the results stated by Poore are useful for other reasons, for example: (1) they give necessary conditions for discontinuity of the path of minima, (2) they characterize the singularities of the path of stationary points, (3) they are easily

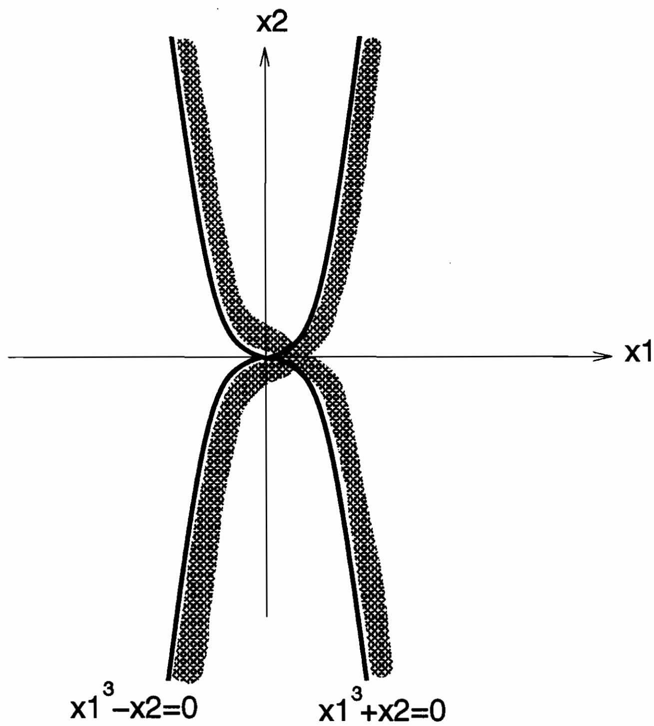

FIGURE 6.3. *Constraints do not satisfy Mangasarian-Fromvitz criterion, but minima persist.*

computable and for many continuation algorithms checking these conditions does not require much additional work.

It is useful to know when a discontinuity can happen, since this allows verifying such points and excluding the cases when minima cannot be found due to the inadequacy of the tracking algorithm.

The results stated by Poore are also valuable because they characterize the possible singularities of the path of stationary points. For example, if minima disappear at some point, but the path of stationary points is continuous and the active set does not change, then a bifurcation point could have been crossed. Hence it may be worthwhile to locate such points and check other branches of the solutions to the Fritz John conditions (4.6). The analysis of the singularities of the path of solutions to the Kuhn-Tucker conditions also has some practical consequences for the algorithm described in Chapter 2. For example, if at the transition point (i.e., the point where the active set changes) the constraints are not linearly dependent and the Hessian of the Lagrangian is positive definite on the least constrained branch of the solutions, then, by Theorem 4.2, the path of minima through this point exists and is unique. Hence (if we are interested only in minima) we do not need to check all possible active sets for the next segment – we can stop at the first found set that satisfies the second order sufficient condition.

Numerical methods for checking the conditions for discontinuity of the minima path for several continuation algorithms were presented by Lundberg and Poore in [20]. In the case of an active set homotopy algorithm described in Chapter 2 the conditions for discontinuity can be checked with no additional effort. Since this algorithm is an active set algorithm, the violation of the strict complementarity condition results in terminating the current segment and starting a new one with a different set of active constraints. All possible active sets for a new segment are checked and the case of multiple solutions is signaled by a flag. The algorithm also has an option for checking the second order optimality conditions. These conditions are verified as follows. Let N be a matrix whose columns are the gradients of active constraints g_i, $i \in \bar{\mathcal{A}}=\{i : g_i(x_0) = 0\}$. Then a QR factorization of N with column pivoting (represented by the permutation P)

$$NP = QR = [Q_1 : Q_2]R,$$

gives a basis (columns of Q_2) for ker N^T=(im $N)^\perp$, i.e., a basis of all vectors $h \perp \nabla g_i$, $i \in \bar{\mathcal{A}}$. Next the inertia of the matrix $Q_2^T[\nabla_x^2 \mathcal{L}]Q_2$ is computed to verify that the matrix is positive semidefinite. Since $Q_2^T[\nabla_x^2 \mathcal{L}]Q_2$ is the projection of the Hessian of the Lagrangian $\nabla_x^2 \mathcal{L}$ on the space tangent to active constraints, this procedure can be used to detect the singularity of this matrix, which is one of the necessary conditions for path discontinuity given by Poore. This procedure can also be used to check the violation of the constraint independence qualification, since a QR factorization with column pivoting reveals (in principle) the rank of N.

In summary, the sufficient conditions for persistence of minima derived by Robinson in his stability theory can be used to analyze the continuity of the path of parametrized optima. They can be checked computationally, since for standard nonlinear programming problems they are equivalent to the Mangasarian-Fromovitz qualification and the standard second order sufficient condition. In the case of linear programming problems the necessary and sufficient conditions for solvability of the perturbed problems can be verified by checking that the constraints for the primal and dual problems satisfy the Mangasarian-Fromovitz criterion. The results of the bifurcation theory are also useful in the analysis of the discontinuity of the path of minima. They establish the necessary conditions for discontinuity of the path of minima and thus allow excluding the possibility that the minima are not found due to, for instance, the inadequacy of the path tracking algorithm. The results of the bifurcation theory can also be used to characterize the possible singularities of the path of stationary points.

7. Analysis of discontinuities of the efficient curve for multi-objective control-structure optimization

In this chapter we analyze the discontinuities of the efficient curve for the bi-objective control-structure optimization problem described in Chapter 3.3.

Singularity of the Jacobian matrix D_zF due to the loss of strict complementarity occurred at all transition points between segments (these points are marked by dots in Figure 7.1, 7.2, and 7.3). The minimizers did not persist at the transition points between Segments 5 and 6 of the path for medium nonstructural weight (see Figure 7.1), and Segments 2 and 3, 5 and 6, 22 and 23, 24 and 25 of the path for high nonstructural weight (Figure 7.3). At these points the Hessian of the Lagrangian on the tangent space to active inequality constraints with strictly positive Lagrange multipliers was indefinite, hence the assumptions of Theorem 4.2 were not satisfied, and minima were not guaranteed to persist. The sufficient conditions for persistence of minima by Robinson (Theorem 5.2.4) were not satisfied at these points, since although the constraints were regular at these points (their gradients were linearly independent), the second order sufficient optimality condition was not satisfied.

At all the other transition points the Hessian of the Lagrangian on the space tangent to the active inequality constraints corresponding to the least constrained branch was positive definite, hence, by Theorem 4.2, the path of minima through this point was continuous and unique. At these points the sufficient conditions for persistence of minima (Theorem 5.2.4) were satisfied, since the constraints were regular and the second order sufficient condition was satisfied.

The singularity due to violation of the second order condition (i.e., the Hessian of the Lagrangian projected on the tangent space to active constraints became singular) occurred within Segment 8 (see Figure 7.2) and in the beginning of Segment 9 for medium nonstructural weight. In both cases the singularity of the Jacobian matrix D_zF had codimension zero. The path of minima turned smoothly at these points. In both cases one eigenvalue of the projected Hessian of the Lagrangian changed sign at the singular points, hence the path of minima became a path of saddle points. Turning points caused by an eigenvalue of the projected Hessian of the Lagrangian becoming zero occurred also in Segments 11, 12, and 16 for high nonstructural weight (Figure 7.3), but these segments consisted of the saddle points anyway (Hessian of the Lagrangian projected on the tangent space to active constraints had at least one negative eigenvalue in Segments 6-16). The sufficient conditions for persistence of minima by Robinson were not satisfied at these points, since although the constraints were regular the second order sufficient optimality condition was not satisfied.

For high nonstructural weight the path of solutions consists of three disconnected parts (see Fig. 7.3). Since different optima were obtained for the same value of the parameter α, we can conclude that some solutions were only local minima for this problem. Two parts of the path could not be traced beyond a certain value of the parameter α: (1) Part I beyond α=0.07236 in Segment 16, and (2) Part II beyond α=0.19586 in Segment 27. In both cases the direct reason for the singularity was that the gradient of one of the constraints became equal to zero, which caused the linear dependence of the active constraints and the singularity of the Hessian of the Lagrangian projected on the tangent space to active constraints. The real reason, however, was the extreme sensitivity of the problem at these two points, which were the turning points of the path. The ill - conditioning of the Jacobian matrix D_zF caused excessive increase of one of the design variables during the correction iterations, which was the reason for the zero gradient of one of the active constraints.

For bi-objective control-structure optimization the discontinuities of the path of parametrized optima were caused by the loss of strict complementarity or the Hessian of the Lagrangian becoming singular. To end this chapter we want to mention the third possible reason for the discontinuity, this is, the violation of linear independence of the

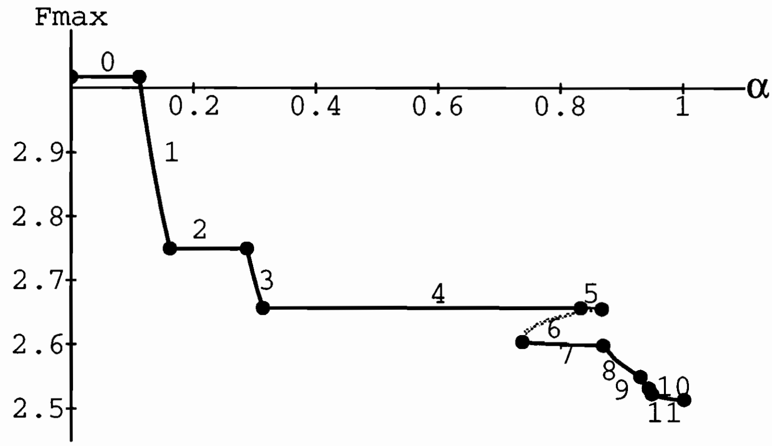

FIGURE 7.1. *$F_{\max}$ (pounds) for medium nonstructural weight (gray line denotes stationary nonoptimal points, black line denotes optimal points).*

gradients of active constraints. The gradients of the active constraints become linearly dependent at transition points whenever the number of the active constraints is the same as the number of active design variables, and one more constraint becomes active. This happened, for example, in the spring problem (Table 3.1.7, Segments 2 – 3). In this case the constraints were still regular in the sense of Robinson, since the derivatives of the active constraints with respect to the active design variables were $\frac{\partial g_8}{\partial k_5} = -3.109$ and $\frac{\partial g_{10}}{\partial k_5} = -96.0126$, so they formed a pointed cone. No discontinuities of the path of optima due to linear dependence of the gradients of the active constraints were found in any of the applications described in Chapter 3.

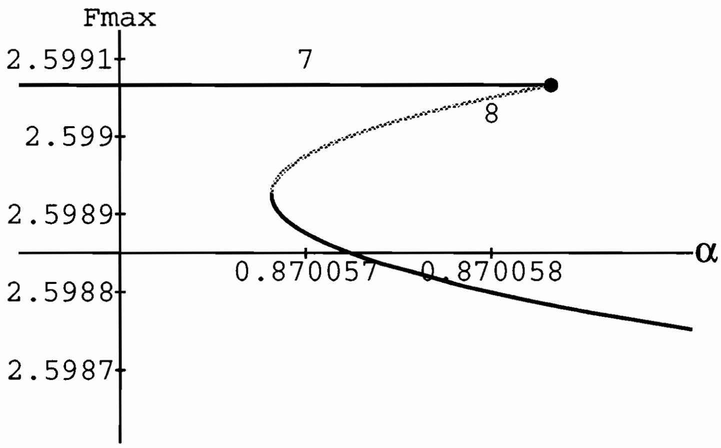

FIGURE 7.2. *$F_{\max}$ (medium nonstructural weight) at the beginning of Segment 8 (black line denotes optimal solutions, gray line denotes stationary nonoptimal points).*

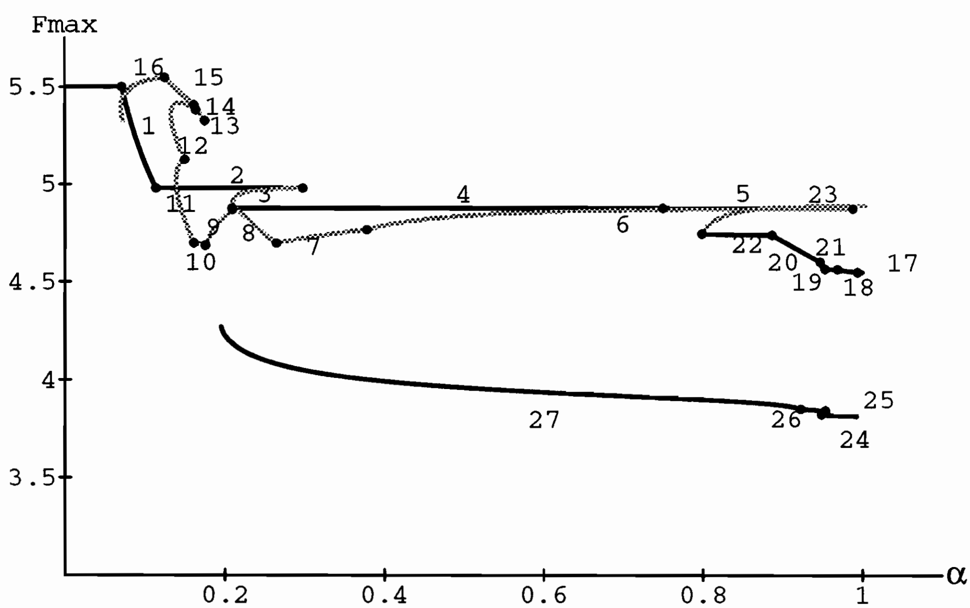

FIGURE 7.3. *$F_{\max}$ (pounds) for high nonstructural weight (gray line denotes stationary nonoptimal points, black line denotes optimal points).*

Concluding Remarks

In this work we developed a general algorithm for tracing the path of optima of inequality constrained optimization problems as a function of a parameter. The algorithm is an active set algorithm using a homotopy method to trace the path. A new feature of the algorithm is a capability of handling the transition points between segments in a routine way. The algorithm locates the transition points, and finds an active set for the next segment by considering all possible sets of active constraints. The nonoptimal sets are eliminated on the basis of the Lagrange multipliers and the derivatives of the optimal solutions with respect to the parameter.

The algorithm was implemented for three different problems. The first application, a spring-mass problem, was used to illustrate various kinds of transition events between segments. The second application, a well known ten-bar truss structural optimization problem, was used to validate the algorithm, since the numerical results for this problem have been obtained by other methods. The third application, bi-objective control-structure optimization, had an important engineering application. The numerical results obtained in this application could be used in the design process – they allowed selection of the best designs and provided some insight into behavior of the structure.

The algorithm can be used only for tracing the path of the stationary points as long as the path is continuous. Therefore our objective was also to determine the conditions for continuity of the path of stationary points and optima for parametrized problems. There exist theoretical results that address this problem, but many of them are too abstract and general to be of immediate use. There are also theories which are motivated by different problems, but whose results can be applied for our purpose. This work provides a synthesis of existing theories useful in determination of the continuity of the path.

We gave the sufficient conditions for persistence of the minima using the results of the stability theory, and we showed the connection between these results and classical optimization theory. For the standard nonlinear programming problem these conditions are equivalent to the Mangasarian-Fromovitz criterion and the standard second order sufficient optimality condition.

We also considered the possibility of computational verification of the conditions for persistence of the minima. By using Motzkin's Transposition Theorem, the Mangasarian-Fromovitz criterion can be reduced to the problem of finding a feasible point for a linear optimization problem. In this form it can be easily solved by Phase I of the simplex method. Therefore **the sufficient conditions for persistence of minima for general nonlinear problems can be checked computationally**. We showed that the linear programming problem can be transformed to the form of general nonlinear problem in such a way that the regularity of the constraints is preserved. Since for linear problems the regularity of the primal and dual constraints is a necessary and sufficient condition for solvability of the perturbed system, we can verify computationally if the path of parametrized optima can be continued.

The results of bifurcation theory were used to characterize the possible points of discontinuity of the path. It is useful to know when the discontinuity of the path can happen because it allows excluding the cases when minima cannot be found due to the inadequacy of the tracking algorithm. The results of bifurcation theory were also used to describe the possible singularities of the path of optima and the behavior of the path near singular points.

Further research could be done in the future to weaken the hypotheses of the sufficient conditions for persistence of minima for nonlinear optimization problems. In Chapter 6 an example was given to show that the Mangasarian-Fromovitz criterion is a sufficient, but not necessary condition for continuity of the path. This suggests that some other constraint qualification might be a more appropriate condition for persistence of minima.

It might also be reasonable to improve the algorithm described in Chapter 2 by providing an option for checking the sufficient conditions for persistence of the minima. Another amendment to the algorithm could be to include a mechanism for detecting and handling singularities other than the loss of strict complementarity.

REFERENCES

[1] E. L. ALLGOWER AND K. GEORG, *Introduction to numerical continuation methods*, Springer Verlag, Berlin, Heidelberg, New York, 1990.

[2] H. ESCHENANER, J. KOSKI AND A. OSYCZKA, *Multicriteria design optimization*, Springer-Verlag, New York, 1990.

[3] A. V. FIACCO, *Sensitivity analysis for nonlinear programming using penalty methods*, Math. Programming 10 (3) (1976) pp. 287–311.

[4] A. V. FIACCO AND G. P. MCCORMICK, *Nonlinear programming: sequential unconstrained minimization technique*, Willey, New York, 1968.

[5] A. V. FIACCO, *Introduction to sensitivity and stability analysis in nonlinear programming*, Academic Press, New York, 1983.

[6] A. V. FIACCO, ED., *Sensitivity and parametric analysis*, Math. Programming Stud. 21 (1984), North Holland, Amsterdam.

[7] G. H. GOLUB AND C. F. VAN LOAN, *Matrix computations*, The Johns Hopkins University Press, Baltimore, 1989.

[8] R. T. HAFTKA, Z. GÜRDAL AND M. P. KAMAT, *Elements of structural optimization*, 2nd ed., Kluwer, Dordrecht, The Netherlands, 1990.

[9] R. T. HAFTKA, *Integrated structures-controls optimization of space structures*, Proc. AIAA Dynamics Specialist Conference, Long Beach, CA, April 5-6, 1990, pp. 1–9.

[10] S. P. HAN AND O. L. MANGASARIAN, *Exact penalty functions in nonlinear programming*, Math. Programming 17 (1979) pp. 251–269.

[11] K. JITTORNTRUM, *Solution point differentiability without strict complementarity in nonlinear programming*, Math. Programming 21 (1984) pp. 127–138.

[12] H. TH. JONGER, P. JONKER AND F. TWILT, *Critical sets in parametric optimization*, Math. Programming 34 (1984) pp. 333–353.

[13] H. B. Keller, *The bordering algorithm and path following near singular points of higher nullity*, SIAM J. Sci. and Statist. Comp. 4 (1983) pp. 573–582.

[14] U. Kirsch, *Optimum structural design*, McGraw-Hill, New York, 1981.

[15] M. Kojima, *Strongly stable stationary solutions in nonlinear programs*, Analysis and computation of fixed points, Academic Press, New York, pp. 93–138.

[16] M. Kojima and R. Hirabashi, *Continuous deformation of nonlinear programs*, Math. Programming Stud. 21 (1984) pp. 150–198.

[17] K. Lim and J. Junkins, *Robust optimization of structural and controller parameters*, J. Guidance 12 (1989) pp. 89–96.

[18] B. N. Lundberg and A. B. Poore, *Bifurcations and sensitivity in parametric programming*, Proc. Third Air Force/NASA Symposium on Recent Advances in Multidisciplinary Analysis and Optimization, September 24–26, 1990, San Francisco, CA, pp. 50–55.

[19] B. N. Lundberg and A. B. Poore, *Variable order Adams-Bashforth predictors with an error-stepsize control for continuation methods*, SIAM J. Sci. Stat.Comput. 12 (3) (1991) pp. 695–723.

[20] B. N. Lundberg and A. B. Poore, *Numerical continuation and singularity detection methods for parametric nonlinear programming*, submitted to SIOPT.

[21] O. L. Mangasarian and S. Fromovitz, *The Fritz John necessary optimality conditions in the presence of equality and inequality constraints*, J. Math. Anal. Appl. 17 (1967) pp. 37-47.

[22] O. L. Mangasarian, *Nonlinear programming*, McGraw-Hill Book Company 1969.

[23] Z. N. Martinovic, G. Schamel, R. T. Haftka and W. L. Hallauer, *An analytical and experimental investigation of output feedback vs. linear quadratic regulator*, J. Guidance 13 (1990) pp. 160–167.

[24] M. MILMAN, R. E. SCHEID, M. SALAMA AND R. BRUNO, *Methods for combined control-structure optimization*, Proc. Conference on Dynamics and Control of Large Structures, Blacksburg, VA, May 8–10, 1989, pp. 191–206.

[25] A. B. POORE AND C. A. TIAHRT, *Bifurcation problems in nonlinear parametric programming*, Math. Programming 39 (1987) pp. 189–205.

[26] J. RAKOWSKA, R. T. HAFTKA AND L. T. WATSON, *An active set algorithm for tracing parametrized optima*, Structural Optimization 3 (1991) pp. 29–44.

[27] J. R. J. RAO AND P. Y. PAPALAMBROS, *Extremal behavior of one parameter families of optimal design models*, Proc. ASME Design Automation Conference, Montreal, Quebec, Canada, Sept. 17–20, 1989, pp. 91–100.

[28] J. R. J. RAO AND P. Y. PAPALAMBROS, *A nonlinear programming continuation strategy for one parameter design optimization problems*, Proc. ASME Design Automation Conference, Montreal, Quebec, Canada, Sept. 17–20, 1989, pp. 77–89.

[29] W. C. RHEINBOLDT, *Numerical analysis of continuation methods for nonlinear structural problems*, Computers & Structures 13 (1981) pp. 103–113.

[30] S. M. ROBINSON, *Perturbed Kuhn-Tucker points and rates of convergence for a class of nonlinear programming algorithms*, Math. Programming 7 (1) (1974) pp. 1–16.

[31] S. M. ROBINSON, *Stability theory for systems of inequalities Part I: Linear systems*, SIAM J. Numer. Anal. 12 (1975) pp. 754–769.

[32] S. M. ROBINSON, *Stability for systems of inequalities, Part II: Differentiable nonlinear systems* , SIAM J. Numer. Anal. 13 (1976) pp. 497-513.

[33] S. M. ROBINSON, *First order conditions for general nonlinear optimization*, SIAM J. Appl. Math. 30 (1976) pp. 597-607.

[34] S. M. ROBINSON, *A characterization of stability in linear programming*, Opns. Res. 3 (1977) pp. 435–447.

[35] S. M. Robinson, *Strongly regular generalized equations*, Math. Opns. Res. 5 (1) (1980) pp. 43–62.

[36] S. M. Robinson, *Generalized equations and their solutions Part II: Applications to nonlinear programming*, Math. Stud. 19 (1982) pp. 200–221.

[37] R. T. Rockafellar, *Convex analysis*, Princeton University Press, 1970.

[38] M. Salama and J. Garba, *Simultaneous optimization of controlled structures*, Comput. Mech. 3 (1988) pp. 275–282.

[39] Y. S. Shin, R. T. Haftka and L. T. Watson, *Tracing structural optima as a function of available resources by a homotopy method*, Comp. Meth. Appl. Mech. Eng. 70 (2) (1988) pp. 151–164.

[40] C. A. Tiahrt and A. B. Poore, *A bifurcation analysis of the nonlinear parametric programming problem*, Math. Programming 47 (1990) pp. 117–141.

[41] H. Wacker, *Continuation methods*, Academic Press, New York, 1978.

[42] L. T. Watson, *Numerical linear algebra aspects of globally convergent homotopy methods*, Tech. Report TR-85-14, Dept. of Computer Sci., VPI&SU, Blacksburg, VA, 1985, and SIAM Rev. 28 (1986) pp. 529–545.

[43] L. T. Watson, *A globally convergent algorithm for computing fixed points of C^2 maps*, Appl. Math. Comput. 5 (1979) pp. 297–311.

[44] L. T. Watson, S. C. Billups and A. P. Morgan, *HOMPACK: A suite of codes for globally convergent homotopy algorithms*, Tech. Report 85-34, Dept. of Industrial and Operations Eng., Univ. of Michigan, Ann Arbor, MI, 1985, and ACM Trans. Math. Software 13 (1987) pp. 281–310.

Vita

Joanna Rakowska was born in Zabrze, Poland. In 1973 she joined the program at the Warsaw Technical University, Warszawa. She graduated with the degree of Master of Science in Electrical Engineering.

Since 1978 to 1988 she worked at the Institute of Electron Technology CEMI in Warszawa, Poland. She developed testing programs for VLSI circuits working closely with designers of VLSI circuits.

In 1988 she enrolled in the graduate program at V. P. I. & S. U. From 1988 to 1992 she was a Graduate Research Assistant in Department of Aerospace and Ocean Engineering, V. P. I. & S. U. During her graduate work she published three papers.

J Rakowska

Joanna Rakowska