ONE-TO-ONE CORRESPONDENCE BETWEEN
MAXIMAL SETS OF ANTISYMMETRY AND
MAXIMAL PROJECTIONS OF ANTISYMMETRY

by

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(ABSTRACT)

Let \( X \) be a compact Hausdorff space and \( A \) a uniform algebra on \( X \). Let \( \pi \) be an isometric unital representation that maps \( A \) into bounded linear operators on a Hilbert space. This research investigated that there is a one-to-one correspondence between the collection of maximal sets of antisymmetry for \( A \) and that of maximal projections of antisymmetry for \( \pi(A) \) under the extension of \( \pi \) if \( \pi \) satisfies a certain regularity property.
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INTRODUCTION

1. Notation and Definitions

Let $X$ be a compact Hausdorff space and $C(X)$ the algebra of all complex-valued continuous functions on $X$. For $f \in C(X)$, we set $\|f\| \equiv \max \{|f(x)| : x \in X\}$ and call $\|f\|$ the uniform norm of $f$. A norm closed subset of $C(X)$ will be referred to as uniformly closed.

We shall call $A$ a uniform algebra on $X$ if $A$ is a uniformly closed subalgebra of $C(X)$ which contains the constants and separates the points of $X$.

Let $A$ be a nonempty subset of $C(X)$. A set $K \subseteq X$ is a set of antisymmetry for $A$ if $K \neq \emptyset$ and every $f \in A$ which is real-valued on $K$ is constant on $K$. Since the union of sets of antisymmetry for $A$ that contain a common point is a set of antisymmetry for $A$, every set of antisymmetry for $A$ is contained in a maximal set of antisymmetry for $A$. The collection $\mathcal{K}_A$ of maximal sets of antisymmetry for $A$ forms a pairwise disjoint, closed covering of $X$.

Let $\mathcal{H}$ be a Hilbert space over the complex numbers $\mathbb{C}$ with the inner product $\langle \; , \; \rangle$, and define

$$\|y\| \equiv \sqrt{\langle y, y \rangle}$$

for any $y \in \mathcal{H}$. Let $B(\mathcal{H})$ stand for the algebra of all bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, we set $\|T\| \equiv \sup \{|Ty| : y \in \mathcal{H}, \text{ and } \|y\| = 1\}$ and $T^*$ denotes the adjoint of $T$. If $T = T^*$ then $T$ is called self-adjoint. If $P \in B(\mathcal{H})$ satisfies $P = P^2 = P^*$, then $P$ is called a projection. Denote the identity map on $\mathcal{H}$ by I. Denote the zero map on $\mathcal{H}$ by O.
Let $\mathcal{U}$ be a nonempty subset of $\mathcal{B}(\mathcal{H})$. A projection $P$ in $\mathcal{B}(\mathcal{H})$ is called a projection of antisymmetry for $\mathcal{U}$ if

(a) $P \neq 0$,
(b) $P \cdot T = T \cdot P$ for all $T \in \mathcal{U}$, and
(c) if $T \cdot P$ is self-adjoint for some $T \in \mathcal{U}$ then $T \cdot P = \lambda \cdot P$ for some real number $\lambda$.

In a natural way we define a maximal projection of antisymmetry for $\mathcal{U}$:

A projection $P$ in $\mathcal{B}(\mathcal{H})$ is called a maximal projection of antisymmetry for $\mathcal{U}$ if

(a) $P$ is a projection of antisymmetry for $\mathcal{U}$, and
(b) if $Q$ is a projection of antisymmetry for $\mathcal{U}$ such that $Q \geq P$ then $P = Q$.

Applying Zorn’s lemma one can show the range of every projection of antisymmetry for $\mathcal{U}$ is contained in that of a maximal projection of antisymmetry for $\mathcal{U}$.

Let $A$ be a uniform algebra of $C(X)$. A mapping $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ is called a unital representation of $A$ into $\mathcal{B}(\mathcal{H})$ if

(a) $\pi(1) = I$,
(b) $\pi(f + g) = \pi(f) + \pi(g)$,
(c) $\pi(\lambda \cdot f) = \lambda \cdot \pi(f)$, and
(d) $\pi(f \cdot g) = \pi(f) \cdot \pi(g)$

for all $f, g \in A$, and $\lambda \in \mathbb{C}$

2. Historical Background

We begin with a history on sets of antisymmetry.

Let $A$ be a closed algebra on a compact Hausdorff space $X$. Denote the set of all regular complex measures on the Borel subsets of $X$ by $M(X)$. Let $b(A^\perp) \equiv \{ \mu \in M(X) \mid \int_X f \, d\mu = 0 \text{ for all } f \in A \text{ and the total variation of } \mu \text{ is at most } 1 \}$. Then
by Alaoglu's theorem and the fact that a closed subset of a compact set is compact,
b$(A^\perp)$ is compact in the weak topology induced by $C(X)$ under integration.

Suppose $A \neq C(X)$, then $b(A^\perp) \neq \{0\}$. Since $b(A^\perp)$ is also convex, by Krein-Milman theorem, $b(A^\perp)$ has a nonzero extreme point, say $\mu_0$. In [5], deBranges proved the Stone-Weierstrass theorem using the concept below:
If $f \in A$ and $f$ is real-valued on the support of $\mu_0$, then $f$ is constant on the support of $\mu_0$.

In [9], Glicksberg gave a formal definition of sets of antisymmetry and proved some important properties of maximal sets of antisymmetry.

Next we have some history on representations and projections of antisymmetry.

Let $T$ be a bounded linear operator on $\mathcal{H}$ with $X$ as a spectral set (in the sense von Neumann meant). Then there is a natural contractive unital representation, say $\pi$, from the uniform closure of the rational functions with poles off $X$ in $C(X)$, denoted by $R(X)$, into $B(\mathcal{H})$. Suppose that $\mathcal{T} \setminus X$ is connected. Sarason [19] saw how this representation determines a relationship between reducing subspaces for $T$ and the components of the interior of the spectrum of $T$. Lautzenheiser [12] extended the result by showing that nontrivial Gleason parts for $R(X)$ yielded reducing subspaces for $T$. The work of ([15], Melnikov) established that these subspaces are nontrivial.

Mlak [16] further replaced $\pi$ by any contractive unital representation, mapping a uniform algebra, say $A$, of $C(X)$ into $B(\mathcal{H})$. Mlak developed a process that associates to each generalized peak set for $A$ a projection that commutes with the range of $\pi$. The projection may be trivial.

Suppose $K$ is a maximal set of antisymmetry for $A$. In ([9], Glicksberg) it is shown that $K$ is also a generalized peak set for $A$. Hence it is natural to ask if the projection generated by the Mlak's process from $K$, denoted by $P_K$, is a projection of antisymmetry for $\pi(A)$, or even a maximal projection of antisymmetry for $\pi(A)$.
Furthermore, if the answer to the latter part is affirmative, we want to investigate whether there is a one-to-one correspondence between maximal sets of antisymmetry and maximal projections of antisymmetry.

In ([22], Szymanski) a maximal projection of antisymmetry was introduced. Szymanski also proved the following result:

Let $N$ be a normal operator on a Hilbert space $\mathcal{H}$ with spectrum $X$. Let $P(X)$ be the uniform closure of all polynomials in $C(X)$. Denote the collection of maximal sets of antisymmetry for $P(X)$ by $K_{P(X)}$. Define $\pi : P(X) \to B(\mathcal{H})$ by $\pi(p) = p(N)$ for any $p$ in $P(X)$. Suppose for every $K \in K_{P(X)}$ with $P_K \neq 0$, then $K = \text{the spectrum of } (N \cdot P_K)$. Then Szymanski showed that there is a one-to-one correspondence between the collection of maximal sets of antisymmetry $K$ for $P(X)$ that satisfy $P_K \neq 0$ and the collection of all maximal projections of antisymmetry for $\pi(P(X))$.

3. Objectives

We will prove that a contractive unital representation $\pi : A \to B(\mathcal{H})$ can be extended so that the characteristic function of a maximal set of antisymmetry for $A$ is in the domain of the extension. If the representation is an isometry and a certain regularity property is satisfied for each maximal set of antisymmetry for $A$, then the extension will map the characteristic function of a maximal set of antisymmetry for $A$ to a maximal projection of antisymmetry for $\pi(A)$. We show also that every maximal projection of antisymmetry for $\pi(A)$ is the image of the characteristic function of a unique maximal set of antisymmetry for $A$ under the extension of $\pi$. That is, there is a one-to-one correspondence between the collection of maximal sets of antisymmetry for $A$ and that of maximal projections of antisymmetry for $\pi(A)$.
CHAPTER 1

GENERALIZED PEAK SETS

In ([9], Glicksberg) it is shown that a maximal set of antisymmetry for a uniform algebra \( A \) is a generalized peak set for \( A \). In this chapter we will define generalized peak sets and prove some related properties of generalized peak sets we will use later.

We will continue our notation in Introduction. Let \( X \) denote a compact Hausdorff space, and \( A \) a uniform algebra on \( X \).

First we define peak sets.

**Definition (Peak Sets).** A set \( F \subseteq X \) is a peak set for \( A \) if \( F \neq \emptyset \) and there is a function \( f \in A \) such that \( f = 1 \) on \( F \) and \( |f| < 1 \) on \( X \setminus F \) (the complement of \( F \) in \( X \)). Any such function \( f \) is said to peak on \( F \).

**Notes:**
(a) Clearly \( F = \{ x \in X \mid f(x) = 1 \} \) is a closed set.
(b) Since \( F \) can also be expressed as the intersection of the countable open sets \( \{ x \in X : |f(x)| > 1 - \frac{1}{n} \}_{n=1}^{\infty} \), a peak set is a \( G_\delta \)-set.

Next we give some examples about peak sets.

1.1 Examples.
(a) For any uniform algebra \( A \) on \( X \), the constant function 1 peaks on \( X \). Hence \( X \) is always a peak set for any uniform algebra on \( X \).

(b) Denote the unit circle in the complex plane, \( \{ c \in \mathbb{C} : |c| = 1 \} \), by \( \Gamma \). Let \( A \) be the disk algebra, i.e., the uniform closure of all polynomials in \( C(\Gamma) \). For any
point $c$ in $\Gamma$, the polynomial $f(x) = \overline{c} \cdot (x + c)/2$ is a function peaking on \{c\}. Therefore, every point in the unit circle is a peak set for the disk algebra.

(c) Let $K_1, K_2$ be two disjoint compact subsets in the complex plane $\mathbb{C}$. Define $X \equiv K_1 \cup K_2$. Denote the uniform closure of the rational functions with poles off $X$ in $C(X)$ by $R(X)$. Then both $K_1$ and $K_2$ are peak sets for $R(X)$. This follows from the fact that $\mathbb{C}$ is a normal space and application of Runge's theorem.

By the following two lemmas, a finite or countable intersection of peak sets for a uniform algebra $A$ is still a peak set for $A$.

1.2 Lemma. Let $A$ be a uniform algebra on $X$. Then a finite intersection of peak sets for $A$ is a peak set for $A$.

proof: Let $F_1, F_2, F_3, \ldots, F_n$ be peak sets for $A$ with $f_j$ peaks on $F_j$. Set

$$f \equiv f_1 \cdot f_2 \cdot f_3 \cdots f_n.$$ 

Then $f \in A$ and $f$ peaks on $\bigcap \{F_j \mid j = 1, 2, 3, \ldots, n\}$. Therefore, $\bigcap \{F_j \mid j = 1, 2, 3, \ldots, n\}$ is a peak set for $A$.

1.3 Lemma. Let $A$ be a uniform algebra on $X$. Then a countable intersection of peak sets for $A$ is a peak set for $A$.

proof: Let $\{F_j \mid j = 1, 2, 3, \ldots\}$ be a sequence of peak sets for $A$ with $f_j$ peaks on $F_j$. Set

$$f \equiv \sum_{j=1}^{\infty} \frac{f_j}{2^j}.$$ 

Then $f \in A$ and $f$ peaks on $\bigcap \{F_j \mid j = 1, 2, 3, \ldots\}$. Therefore, $\bigcap \{F_j \mid j = 1, 2, 3, \ldots\}$ is a peak set for $A$. 

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Definition (Generalized Peak Sets). A generalized peak set is an intersection of peak sets.

Notes:
(a) A peak set for a uniform algebra $A$ is a generalized peak set for $A$.
(b) An intersection of closed sets is a closed set, so a generalized peak set is a closed set.
(c) For a compact metrizable space $X$, the collection of peak sets for a uniform algebra $A$ on $X$ coincides with that of generalized peak sets for $A$.

Proof: A compact metrizable space is second countable. For a second countable space, every open subspace is second countable, and hence Lindelöf. Considering the complement, then the intersection of a collection of closed sets can be expressed as the intersection of a finite or countable subcollection of closed sets. Therefore, a generalized peak set in a compact metrizable space can be written as the intersection of a finite or countable peak sets. Applying Lemma 1.2 or 1.3, the claim is proved.

The following example shows that a generalized peak set for a uniform algebra $A$ is not necessarily a peak set for $A$.

1.4 Example. Let $\Omega$ be an uncountable well-ordered set with a largest element $\omega_1$. The set $\Omega$ has the property that if $\alpha \in \Omega$ with $\alpha < \omega_1$, then $\{ \beta \in \Omega \mid \beta \leq \alpha \}$ is countable.

Take as a subbase of the topology on $\Omega$ all sets of the form $\{ \beta \in \Omega \mid \beta > \alpha \}$ and $\{ \beta \in \Omega \mid \beta < \alpha \}$, for all $\alpha \in \Omega$. By 17.2 (c) of ([24], Willard), $\Omega$ is a compact Hausdorff space.
Define $A \equiv \{ f \in C(\Omega) \mid \text{there exists a constant } c \in \mathfrak{C}, \text{ and } \alpha < \omega_1 \text{ such that } f(\beta) = c \text{ for all } \beta \geq \alpha \}$. Then one can check that $A$ is a uniform algebra on $\Omega$.

Let $\alpha$ be an element in $\Omega$ with $\alpha < \omega_1$. Since $\chi_{[\alpha, \omega_1]}$, the characteristic function of $[\alpha, \omega_1]$, belonging to $A$ peaks on $[\alpha, \omega_1]$, the closed interval $[\alpha, \omega_1]$ is a peak set for $A$. Hence $\{ \omega_1 \} = \bigcap [\alpha, \omega_1] \mid \alpha < \omega_1 \}$ is a generalized peak set for $A$. Clearly, $\{ \omega_1 \}$ is not a peak set for $A$ by the definition of $A$.

To prove the main theorems of this chapter we need the following technical lemma.

1.5 Lemma ([7], Gamelin). Let $B$ be a closed subspace of $C(X)$. Let $G \subseteq X$ be closed and $f \in C(G)$ satisfies

$$\|f\|_G \equiv \max \{ |f(x)| : x \in G \} \leq 1.$$

Suppose there exist two constants $M$ and $a$ with $0 < a < 1 < M$ such that for each open set $V \supseteq G$, there exists $g \in B$ such that (i) $g(x) = f(x)$ for all $x \in G$, (ii) $\|g\| \leq M$, and (iii) $|g| \leq a$ on $X \setminus V$. Then there exists $h \in B$ satisfying $\|h\| \leq 1$ and $h(x) = f(x)$ for all $x \in G$. Furthermore, $h$ can be chosen so that $|h| < 1$ on any prescribed $F_\sigma$-subset of $X \setminus G$.

Note: An $F_\sigma$-set is a countable union of closed sets.

Proof: Fix a number $s$ such that

$$1 > s > \frac{M - 1}{M - a}.$$

Let $\{ \delta_n \mid n = 1, 2, 3, \ldots \}$ be a decreasing sequence with $\delta_n$ converges to 0.

Let $\epsilon_1 = \delta_1$, and for $n \geq 2$, let

$$\epsilon_n = \min \{ \delta_n, \frac{s^k[s(M - a) - (M - 1)]}{1 - s^k} \mid k = 2, 3, 4, \ldots, n \}.$$
One can verify \( \{ \epsilon_n : n = 1, 2, 3, \ldots \} \) is a decreasing sequence with \( \epsilon_n \) converges to 0.

Now suppose \( E \) is an \( F_\sigma \)-subset of \( X \setminus G \), say \( E = \bigcup \{ E_n \mid n = 1, 2, 3, \ldots \} \), where each \( E_n \) is a closed set.

We construct a sequence \( \{ h_n \mid n = 1, 2, 3, \ldots \} \subseteq B \) as follows.

Since \( X \) is open and \( X \supseteq G \), by hypothesis, there exists \( h_1 \in B \) such that \( h_1(x) = f(x) \) for all \( x \in G \), and \( \|h_1\| \leq M \).

Suppose \( h_1, h_2, \ldots, h_n \) have been chosen such that for each \( j = 1, 2, \ldots, n \), \( h_j(x) = f(x) \) for all \( x \in G \) and \( \|h_j\| \leq M \). Let

\[
W_n \equiv \{ x \in X : \max_{1 \leq j \leq n} |h_j(x)| \geq 1 + \epsilon_n \}
= \{ x \in X : |h_j(x)| \geq 1 + \epsilon_n \text{ for some } j \text{ with } 1 \leq j \leq n \}.
\]

Clearly, \( W_n \) is a closed set, and \( W_n \subseteq W_{n+1} \) for all \( n = 1, 2, 3, \ldots \). Since \( \|f\|_G \leq 1 \), and for all \( x \in G \), \( h_j(x) = f(x) \) for each \( j = 1, 2, 3 \ldots \), we see that \( W_n \cap G = \emptyset \) for all \( n = 1, 2, 3, \ldots \).

From \( E_n \subseteq E \subseteq X \setminus G \) and \( W_n \cap G = \emptyset \), the open set \( X \setminus (W_n \cup E_n) \supseteq G \).

By hypothesis, there exists \( h_{n+1} \in B \) such that \( h_{n+1}(x) = f(x) \) for all \( x \in G \), \( \|h_{n+1}\| \leq M \), and \( |h_{n+1}| \leq a \) on \( W_n \cup E_n \).

Set

\[
h \equiv (1 - s) \sum_{j=1}^{\infty} s^{j-1} h_j.
\]

Since the \( h_j \)'s are uniformly bounded by \( M \), the series converges uniformly on \( X \). Hence \( h \in B \).
For any \( x \in G \),

\[
h(x) = (1 - s) \sum_{j=1}^{\infty} s^{j-1} h_j(x) \\
= (1 - s) \sum_{j=1}^{\infty} s^{j-1} f(x) \\
= (1 - s) \frac{f(x)}{1 - s} \\
= f(x).
\]

To complete the proof we need to show \(|h| < 1\) on \( E \) and \(|h| \leq 1\) on \( X \setminus G \).

It suffices to prove:

1. If \( x \in X \setminus (\bigcup \{ W_n \mid n = 1, 2, 3, \ldots \}) \), then \(|h(x)| \leq 1\);
2. If \( x \in E \cap \{ X \setminus (\bigcup \{ W_n \mid n = 1, 2, 3, \ldots \}) \} \), then \(|h(x)| < 1\); and
3. If \( x \in \bigcup \{ W_n \mid n = 1, 2, 3, \ldots \} \), then \(|h(x)| < 1\).

To prove (1), let \( x \in X \setminus (\bigcup \{ W_n \mid n = 1, 2, 3, \ldots \}) \). Then for any \( j = 1, 2, 3, \ldots \), the value \(|h_j(x)| < 1 + \epsilon_n\) for all \( n \geq j \). The condition \( \epsilon_n \) converges to 0 implies \(|h_j(x)| \leq 1\), for \( j = 1, 2, 3, \ldots \). Therefore,

\[
|h(x)| = |(1 - s) \sum_{j=1}^{\infty} s^{j-1} h_j(x)| \leq (1 - s) \sum_{j=1}^{\infty} s^{j-1} = 1.
\]

To prove (2), let \( x \in E \cap \{ X \setminus (\bigcup \{ W_n \mid n = 1, 2, 3, \ldots \}) \} \). Clearly, \( x \in X \setminus (\bigcup \{ W_n \mid n = 1, 2, 3, \ldots \}) \). From the result of (1), \(|h_j(x)| \leq 1\) for \( j = 1, 2, 3, \ldots \). Suppose \( x \in E_k\) for some positive integer \( k \). Then \(|h_{k+1}(x)| \leq a < 1\). Therefore, \(|h(x)| < 1\) by the definition of \( h \).

To prove (3), let \( x \in \bigcup \{ W_n \mid n = 1, 2, 3, \ldots \} \). Since \( W_1 \subseteq W_2 \subseteq \cdots \subseteq W_n \subseteq W_{n+1} \subseteq \cdots \), either \( x \in W_1 \) or (there exists an \( m \geq 2 \) such that \( x \in W_m \) and \( x \not\in W_j \) for all \( j < m \)).
If \( x \in W_1 \), then \( |h_j(x)| \leq a \) for all \( j > 1 \), and

\[
|h(x)| \leq (1 - s) \sum_{j=1}^{\infty} s^{j-1} |h_j(x)|
\]

\[
= (1 - s)|h_1(x)| + \sum_{j=2}^{\infty} s^{j-1} |h_j(x)|
\]

\[
\leq (1 - s)[M + \sum_{j=2}^{\infty} s^{j-1} a]
\]

\[
= (1 - s)[M + \frac{as}{1 - s}]
\]

\[
= (1 - s)M + as
\]

\[
= M - s(M - a)
\]

\[
< M - \frac{M - 1}{M - a}(M - a)
\]

\[
= 1.
\]

That is, \(|h(x)| < 1 \) if \( x \in W_1 \).

Suppose \( x \in W_m \) for some \( m \geq 2 \), and \( x \notin W_j \), for all \( j < m \). In particular, \( x \notin W_{m-1} \) and hence \(|h_j(x)| < 1 + \epsilon_{m-1} \) for all \( j = 1, 2, \ldots, m - 1 \). Therefore,

\[
|h(x)| \leq (1 - s) \sum_{j=1}^{\infty} s^{j-1} |h_j(x)|
\]

\[
= (1 - s)[\sum_{j=1}^{m-1} s^{j-1} |h_j(x)| + s^{m-1} |h_m(x)| + \sum_{j=m+1}^{\infty} s^{j-1} |h_j(x)|]
\]

\[
< (1 - s)[\sum_{j=1}^{m-1} s^{j-1}(1 + \epsilon_{m-1}) + s^{m-1} M + \sum_{j=m+1}^{\infty} s^{j-1} a]
\]

\[
= (1 - s)[\frac{1 - s^{m-1}}{1 - s} (1 + \epsilon_{m-1}) + s^{m-1} M + \frac{as^m}{1 - s}]
\]

\[
= (1 - s^{m-1})(1 + \epsilon_{m-1}) + (1 - s)s^{m-1} M + as^m
\]

\[
= 1 + \epsilon_{m-1}(1 - s^{m-1}) - s^{m-1}[1 - (1 - s)M - as]
\]

\[
\leq 1 + \frac{s^{m-1}[s(M - a) - (M - 1)]}{1 - s^{m-1}}(1 - s^{m-1}) - s^{m-1}[s(M - a) - (M - 1)]
\]

\[
= 1.
\]
Therefore, $|h(x)| < 1$ for all $x \in \bigcup\{ W_n \mid n = 1, 2, 3, \ldots \}$.

This completes the proof of the lemma. ■ ■

Applying the previous lemmas, we can prove the following theorem.

1.6 Theorem ([7], Gamelin). Let $A$ be a uniform algebra on $X$, and $G$ a generalized peak set for $A$. Let $f \in C(G)$ with $f(x) = q(x)$ for some $q \in A$ and all $x \in G$. Then there exists $h \in A$ such that $h(x) = f(x)$ for all $x \in G$ and $\|h\|_X = \|f\|_G$.

**Proof:** Without loss of generality, we assume $\|f\|_G = 1$.

Let $V$ be an open set that contains $G$. We claim there exists a peak set for $A$, say $F$, such that $G \subseteq F \subseteq W \equiv V \cap \{ x \in X : |q(x)| < 2 \}$.

**Proof of claim:** As an intersection of two open sets, $W$ is open. Since

$$G = \{ x \in G : |f(x)| \leq 1 \} = \{ x \in G : |q(x)| \leq 1 \} \subseteq \{ x \in X : |q(x)| \leq 1 \} \subseteq \{ x \in X : |q(x)| < 2 \},$$

we have $W \supseteq G$.

The set $G$ is a generalized peak set for $A$, by definition,

$$G = \bigcap \{ F_\alpha \mid \alpha \in \Lambda \},$$

where $\Lambda$ is an index set, and $F_\alpha$ is a peak set for $A$ for each $\alpha \in \Lambda$. 12
By taking the complement of \( W \supseteq G \), we have
\[
X \setminus W \subseteq X \setminus G
= X \setminus \left( \bigcap \{ F_\alpha | \alpha \in \Lambda \} \right)
= \bigcup \{ X \setminus F_\alpha | \alpha \in \Lambda \}.
\]
Since \( X \setminus W \) is compact, there exist \( \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \in \Lambda \) such that
\[
X \setminus W \subseteq \bigcup \{ X \setminus F_\alpha | \alpha = \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \}
= X \setminus \left( \bigcap \{ F_\alpha | \alpha = \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \} \right).
\]
So \( W \supseteq \bigcap \{ F_\alpha | \alpha = \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \} \). Let \( F \equiv \bigcap \{ F_\alpha | \alpha = \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \} \).
By Lemma 1.2, \( F \) is a peak set for \( A \) and \( G \subseteq F \subseteq W \). The claim is proved. ■

Let \( p \in A \) peak on \( F \). Then as \( n \) approaches \( \infty \), we see that \( p^n \) converges to 0 uniformly on
\[
X \setminus (V \cap \{ x \in X : |q(x)| < 2 \}) = (X \setminus V) \cup \{ x \in X : |q(x)| \geq 2 \}.
\]
For sufficiently large \( n \), the function \( g \equiv p^n \cdot q \in A \) satisfies (i) \( g(x) = f(x) \) for all \( x \in G \), (ii) \( \|g\|_X \leq 2 \), and (iii) \( |g| \leq \frac{1}{2} \) on \( X \setminus V \). By Lemma 1.5, with \( M = 2 \) and \( a = \frac{1}{2} \), the desired function \( h \in A \) exists, and this wraps up the proof of the theorem. ■■

The following theorem shows the restriction of a uniform algebra to a generalized peak set is still a uniform algebra on that set.

**1.7 Theorem.** Let \( A \) be a uniform algebra on \( X \) and \( G \) a generalized peak set for \( A \). Define \( i_G : G \to X \) by \( i_G(x) = x \) for all \( x \in G \). Then \( A_G \equiv \{ f \circ i_G | f \in A \} \), the restriction of \( A \) to \( G \), is a uniform algebra on \( G \).

**Proof:** The restriction of an algebra to a subset of \( X \) is an algebra on that set. The restriction of a constant function to a subset of \( X \) is a constant function on that set.
An algebra that separates points of a set separates points of a subset. Therefore, its restriction to that subset separates points. Hence we now proceed to prove that $A_G$ is uniformly closed to complete the claim.

To this end, define $\psi : A \to A_G$ by $\psi(f) = f \circ i_G$ for all $f \in A$. Then $\psi$ is linear, onto with kernel space, denoted by $\ker \psi$, the set of functions in $A$ that vanish on $G$.

Define $A/\ker \psi \equiv \{ f + \ker \psi \mid f \in A \}$ with the norm $\| f + \ker \psi \| \equiv \inf \{ \| f + g \| : g \in \ker \psi \}$ for any $f \in A$. Since $A$ is a complete normed space, by III.4.2 of ([2], Conway), $A/\ker \psi$ is a complete normed space.

The mapping $\Psi : A/\ker \psi \to A_G$ defined by $\Psi(f + \ker \psi) \equiv f \circ i_G$ is one-to-one, onto, and linear. Furthermore, by Theorem 1.6, $\Psi$ is an isometry. Therefore, $A_G$ is a complete normed space, and hence uniformly closed. ■■
CHAPTER 2
EXTENSION OF CONTRACTIVE UNITAL REPRESENTATIONS

Let $A$ be a uniform algebra on $X$. A unital representation $\pi : A \to B(H)$ is said to be contractive if $\|\pi(f)\| \leq \|f\|$ for all $f \in A$. Denote the collection of generalized peak sets for $A$ by $(GP)_A$. Then in this chapter we want to extend the domain of $\pi$ to include $\chi_G$, the characteristic function of $G$, for each $G \in (GP)_A$. We will see that the range of $\chi_G$ under the extension of $\pi$ is a projection that commutes with the range of $\pi$. Therefore, if the projection is not $O$ or $I$, then the range of the projection is a non-trivial reducing subspace for the range of $\pi$. The main result of this chapter, Theorem 2.7, is due to ([16], Mlak).

We first need some definitions for our following examples.

Definition (Spectral Sets). If $T$ is a bounded linear operator on a Hilbert space $H$, then a compact subset $X$ of the complex numbers $C$ is called a spectral set for $T$ if it contains the spectrum of $T$ and satisfies

$$\|r(T)\| \leq \sup \{ |r(x)| : x \in X \}$$

for all rational functions $r$ with poles off $X$.

(For definition of $r(T)$, see VII.4 of ([2], Conway).)

Definition (von Neumann Operators). A bounded linear operator $T$ on a Hilbert space $H$ is a von Neumann operator if the spectrum of $T$ is a spectral set for $T$.

Here we present some examples of contractive unital representations.
2.1 Examples

(a) Let $X$ be a compact Hausdorff space and let $\mu$ be a positive Borel measure on $X$. Denote the Hilbert space of square integrable functions on $(X, \mu)$ by $L^2(\mu)$. Define $\pi : C(X) \to B(L^2(\mu))$ by $\pi(f) = M_f$, where $M_f(y) = f \cdot y$ for any $y \in L^2(\mu)$. One can check that $\pi$ is a contractive unital representation.

(b) Let $N$ be a normal operator on a Hilbert space $\mathcal{H}$ and let $X$ be the spectrum of $N$. By the spectral theorem for normal operators, for every $f$ in $C(X)$ there exists an operator $f(N)$ on $\mathcal{H}$. Define $\pi : C(X) \to B(\mathcal{H})$ by $\pi(f) = f(N)$. Then one can use the spectral theorem to check that $\pi$ is a contractive unital representation. (Actually, $\pi$ is an isometry.)

(c) Let $T$ be a von Neumann operator on a Hilbert space $\mathcal{H}$. (For a brief discussion of von Neumann operators see ([4], Conway).) Let $X$ be the spectrum of $T$. Denote the uniform closure of all rational functions with poles off $X$ in $C(X)$ by $R(X)$. Define $\pi : R(X) \to B(\mathcal{H})$ by $\pi(f) = f(T)$. By invoking the definition of von Neumann operators one can verify that $\pi$ is a contractive unital representation.

(d) Let $X$ in $\mathcal{C}$ be a spectral set for an operator $T$ on a Hilbert space $\mathcal{H}$. (See ([4], Conway) for reference on spectral sets.) Let $R(X)$ and $\pi$ be defined as in (c) above. Then $\pi$ is a contractive unital representation.

Denote the collection of peak sets for $A$ by $\mathcal{P}_A$. Next we extend the domain of $\pi$ to include $\chi_F$, for any $F \in \mathcal{P}_A$. Before that we need several lemmas.

2.2 Lemma. Let $A$ be a uniform algebra on a compact Hausdorff space $X$, and $\pi$ a contractive unital representation that maps $A$ into $B(\mathcal{H})$. Then for any $y, z \in \mathcal{H}$
there exists a regular complex Borel measure $\mu_{y,z}$ on $X$ such that

$$\langle \pi(f) y, z \rangle = \int_X f \, d\mu_{y,z} \quad \text{for all } f \in A,$$

and

$$\|\mu_{y,z}\| \leq \|y\| \cdot \|z\|.$$  \hspace{1cm} (2)

(Note: $\|\mu_{y,z}\|$ denotes the total variation of $\mu_{y,z}$.)

**Proof:** Fix $y, z \in H$ and define the map $\phi : A \to \mathbb{C}$ by

$$\phi(f) = \langle \pi(f) y, z \rangle.$$

By linearity of inner product one can see that $\phi$ is a linear functional. Also

$$|\phi(f)| = |\langle \pi(f) y, z \rangle| \leq \|\pi(f) y\| \cdot \|z\| \leq \|\pi(f)\| \cdot \|y\| \cdot \|z\| \leq \|f\| \cdot \|y\| \cdot \|z\|$$

implies that $\|\phi\|$ is bounded by $\|y\| \cdot \|z\|$.

By the Hahn-Banach theorem there exists a bounded linear functional, say $\Phi$, defined on $C(X)$ such that $\Phi(f) = \phi(f)$ for all $f \in A$, and $\|\Phi\| = \|\phi\|$. From Riesz representation theorem there exists a complex regular Borel measure on $X$, say $\mu_{y,z}$, with $\|\mu_{y,z}\| = \|\Phi\| \leq \|y\| \cdot \|z\|$, and

$$\Phi(g) = \int_X g \, d\mu_{y,z} \quad \text{for all } g \in C(X).$$

In particular, for $f \in A \subseteq C(X)$

$$\langle \pi(f) y, z \rangle = \phi(f) = \Phi(f) = \int_X f \, d\mu_{y,z}.$$

The proof is complete.  \hfill \blacksquare

**Definition (Elementary Measures of $\pi$).** Any arbitrary regular complex Borel measure that satisfies (1) and (2) in Lemma 2.2 is called an elementary measure for $(y, z) \in H \times H$ of the contractive unital representation $\pi$. 

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2.3 Lemma. Let $A, X,$ and $\pi$ be as in Lemma 2.2. If $\mu_{y,z},$ and $\mu'_{y,z}$ are elementary measures for $(y,z) \in H \times H$ of $\pi,$ then for any $F \in P_A$

$$\int_X \chi_F d\mu_{y,z} = \int_X \chi_F d\mu'_{y,z},$$

where $\chi_F$ denotes the characteristic function of $F.$

Proof: Let $F \in P_A,$ and let $p \in A$ peak on $F.$

Since $p^n$ converges to $\chi_F$ pointwise, by Lebesgue's dominated convergence theorem, we have

$$\int_X \chi_F d\mu_{y,z} = \lim_{n \to \infty} \int_X p^n d\mu_{y,z}$$

$$= \lim_{n \to \infty} \langle \pi(p^n) y, z \rangle$$

$$= \lim_{n \to \infty} \int_X p^n d\mu'_{y,z}$$

$$= \int_X \chi_F d\mu'_{y,z}.$$ 

Thus the lemma is proved. $\blacksquare$

With Lemma 2.3 we can now extend $\pi$ to include $\{ \chi_F \mid F \in P_A \}$ in its domain. That is, for any $F \in P_A$ we define $\pi(\chi_F)$ to be $P_F$ in the following theorem.

2.4 Theorem ([16], Mlak). Let $A, X,$ and $\pi$ be as in Lemma 2.2. For any $F \in P_A$ there exists a unique projection $P_F \in B(H)$ such that

$$\langle P_F y, z \rangle = \int_X \chi_F d\mu_{y,z},$$

and $P_F \cdot \pi(f) = \pi(f) \cdot P_F$ for all $f \in A.$

Proof: For any fixed $F \in P_A,$ define $S_F : H \times H \to \mathcal{H}$ by:

$$S_F(y, z) = \int_X \chi_F d\mu_{y,z}.$$
where \( y, z \in \mathcal{H} \), and \( \mu_{y,z} \) is an elementary measure for \((y, z)\) of \( \pi \).

By Lemma 2.3, \( S_{F} \) is well-defined.

Let \( p \in A \) peak on \( F \), we have

\[
S_{F}(y, z) = \int_{X} \chi_{F} d\mu_{y,z}
= \lim_{n \to \infty} \int_{X} p^{n} d\mu_{y,z}
= \lim_{n \to \infty} \langle \pi(p^{n}) y, z \rangle. \tag{3}
\]

From the above relation one can verify \( S_{F} \) is a sesquilinear functional.

Also

\[
|S_{F}(y, z)| = \left| \int_{X} \chi_{F} d\mu_{y,z} \right|
\leq \langle \mu_{y,z} \rangle
\leq \|y\| \cdot \|z\|.
\]

Hence \( S_{F} \) is a bounded sesquilinear functional on \( \mathcal{H} \times \mathcal{H} \). By Theorem 1 of § 22 in ([10], Halmos) there exists a unique operator, say \( P_{F} \), on \( \mathcal{H} \) such that

\[
S_{F}(y, z) = \langle P_{F} y, z \rangle \quad \text{for all } y, z \in \mathcal{H}.
\]

For any \( y \in \mathcal{H} \),

\[
\|y\|^{2} = \langle y, y \rangle = \langle \pi(1) y, y \rangle
= \int_{X} 1 d\mu_{y,y}
= \mu_{y,y}(X)
\leq \|\mu_{y,y}\|
\leq \|y\| \cdot \|y\|
= \|y\|^{2}.
\]
This implies $\mu_{y,y}$ is a positive measure.

For any $y \in \mathcal{H}$,

$$
\langle P_F y, y \rangle = S_F(y, y) = \int_X \chi_F \, d\mu_{y,y} = \mu_{y,y}(F) \geq 0.
$$

Therefore, $P_F$ is a positive operator on $\mathcal{H}$.

Next we show $P_F$ commutes with $\pi(f)$ for any $f \in A$.

For any $y, z \in \mathcal{H}$, and any $f \in A$,

$$
\langle P_F \cdot \pi(f) y, z \rangle = \lim_{n \to \infty} \langle \pi(p^n) \cdot \pi(f) y, z \rangle \\
= \lim_{n \to \infty} \langle \pi(p^n \cdot f) y, z \rangle \\
= \lim_{n \to \infty} \langle \pi(f \cdot p^n) y, z \rangle \\
= \lim_{n \to \infty} \langle \pi(f) \cdot \pi(p^n) y, z \rangle \\
= \lim_{n \to \infty} \langle \pi(p^n) y, \pi(f)^* z \rangle \\
= \langle P_F y, \pi(f)^* z \rangle \\
= \langle \pi(f) \cdot P_F y, z \rangle.
$$

Therefore, $P_F \cdot \pi(f) = \pi(f) \cdot P_F$ for all $f \in A$.

To complete the proof we need to show $P_F^2 = P_F$.

For any fixed positive integer $k$, and any $y, z \in \mathcal{H}$,

$$
\langle P_F y, z \rangle = \lim_{n \to \infty} \langle \pi(p^n) y, z \rangle \\
= \lim_{n \to \infty} \langle \pi(p^{n+k}) y, z \rangle \\
= \lim_{n \to \infty} \langle \pi(p^n) \cdot \pi(p^k) y, z \rangle \\
= \langle P_F \cdot \pi(p^k) y, z \rangle \\
= \langle \pi(p^k) y, P_F z \rangle.
$$
Since this is true for any \( k \),

\[
\langle P_F y, z \rangle = \lim_{k \to \infty} \langle \pi(p^k)y, P_F z \rangle
= \langle P_F y, P_F z \rangle
= (P_F^2 y, z) .
\]

(Use (3).)

This proves \( P_F = P_F^2 \), and the theorem is proved completely. 

To further extend the domain of \( \pi \) to include \{ \chi_G \mid G \in (\mathcal{G}\mathcal{P})_A \} \ we need some more preparations.

2.5 Lemma. Let \( G \in (\mathcal{G}\mathcal{P})_A \) and \( \mu \) a regular complex Borel measure on \( X \). Then there is a sequence \( \{ F_n \mid n = 1, 2, 3, \ldots \} \subseteq \mathcal{P}_A \) such that \( G \subseteq F_n \) for each \( n = 1, 2, 3, \ldots \), and

\[
\lim_{n \to \infty} |\mu|(F_n \setminus G) = 0.
\]

Note: \( |\mu|(F_n \setminus G) \) denotes the total variation of \( \mu(F_n \setminus G) \).

Proof: Clearly \( G = \bigcap \{ F_\alpha \mid G \subseteq F_\alpha \ \text{and} \ F_\alpha \in \mathcal{P}_A \} \).

The set \( X \setminus G \) is open, and \( |\mu| \) is regular, so there is a sequence \( \{ K_n \mid n = 1, 2, 3, \ldots \} \) of compact sets such that \( K_n \subseteq X \setminus G \) and

\[
\lim_{n \to \infty} |\mu|(K_n) = |\mu|(X \setminus G).
\]

We claim for any \( K_n \) there is some \( F_n \) with \( G \subseteq F_n \in \mathcal{P}_A \) such that \( K_n \cap F_n = \emptyset \).

Proof of claim: Suppose not. Then for some fixed \( n \), \( \{ K_n \cap F_\alpha \mid G \subseteq F_\alpha \ \text{and} \ F_\alpha \in \mathcal{P}_A \} \) is a collection of nonempty closed sets with finite intersection property.
By compactness of $X$, the set $\bigcap\{K_n \cap F_\alpha \mid G \subseteq F_\alpha, \text{ and } F_\alpha \in \mathcal{P}_A\} \neq \emptyset$. But

$$\emptyset = K_n \cap G$$
$$= K_n \cap (\bigcap\{F_\alpha \mid G \subseteq F_\alpha \text{ and } F_\alpha \in \mathcal{P}_A\})$$
$$= \bigcap\{K_n \cap F_\alpha \mid G \subseteq F_\alpha \text{ and } F_\alpha \in \mathcal{P}_A\}$$
$$\neq \emptyset.$$  

A contradiction arrives. Thus the claim is proved. ■

Continuing the proof of the lemma, we have

$$0 \leq |\mu|(F_n \setminus G) = |\mu|(F_n) - |\mu|(G)$$
$$\leq |\mu|(X \setminus K_n) - |\mu|(G)$$
$$= |\mu|((X \setminus K_n) \setminus G)$$
$$= |\mu|((X \setminus G) \setminus K_n)$$
$$= |\mu|(X \setminus G) - |\mu|(K_n).$$

Since $|\mu|(K_n)$ converges to $|\mu|(X \setminus G)$ as $n$ approaches $\infty$, $|\mu|(F_n \setminus G)$ converges to 0 as $n$ approaches $\infty$. Therefore, the lemma is proved. ■■

The key to the proof of the next theorem is the following:

2.6 Lemma. Let $G \in (G\mathcal{P})_A$ and $\mu$ a regular complex Borel measure on $X$ such that

$$\int_X f \, d\mu = 0 \quad \text{for all } f \in A.$$  

Then

$$\int_X f \cdot \chi_G \, d\mu = 0 \quad \text{for all } f \in A.$$  

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Proof: By Lemma 2.5, there exists a sequence \( \{ F_n \mid n = 1, 2, 3, \ldots \} \subseteq \mathcal{P}_A \) such that \( G \subseteq F_n \) for each \( n = 1, 2, 3, \ldots \), and

\[
\lim_{n \to \infty} |\mu|(F_n \setminus G) = 0.
\]

Let \( p_n \in A \) peak on \( F_n \) for each \( n = 1, 2, 3, \ldots \). For any positive integer \( n \), and any \( f \in A \),

\[
\int_X f \cdot \chi_{F_n} \, d\mu = \lim_{m \to \infty} \int_X f \cdot p_n^m \, d\mu = 0. \tag{4}
\]

For any \( f \in A \),

\[
| \int_X f \cdot \chi_G \, d\mu | = | \int_X f \cdot (\chi_{F_n} - \chi_{F_n \setminus G}) \, d\mu | \\
\leq | \int_X f \cdot \chi_{F_n} \, d\mu | + | \int_X f \cdot \chi_{(F_n \setminus G)} \, d\mu | \\
= | \int_X f \cdot \chi_{(F_n \setminus G)} \, d\mu | \tag{Use (4).} \\
\leq \| f \| \cdot |\mu|(F_n \setminus G).
\]

As \( n \) approaches \( \infty \), \( |\mu|(F_n \setminus G) \) converges to \( 0 \). This proves \( \int_X f \chi_G \, d\mu = 0 \) for all \( f \in A \). \( \blacksquare \)

Now we are ready to extend the domain of \( \pi \) to include \( \{ \chi_G \mid G \in (\mathcal{G} \mathcal{P})_A \} \). We define, for any \( G \) in \( (\mathcal{G} \mathcal{P})_A \), the image \( \pi(\chi_G) \) to be \( P_G \) in the following theorem.

2.7 Theorem ([16], Mlak). Let \( A, X, \) and \( \pi \) be as in Lemma 2.2. For any \( G \in (\mathcal{G} \mathcal{P})_A \) there exists a unique projection \( P_G \in \mathcal{B}(\mathcal{H}) \) such that

\[
\langle P_G y, z \rangle = \int_X \chi_G \, d\mu_{y,z},
\]

and \( P_G \cdot \pi(f) = \pi(f) \cdot P_G \) for all \( f \in A \).

Proof: For any fixed \( G \in (\mathcal{G} \mathcal{P})_A \), define \( S_G : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \) by

\[
S_G(y, z) = \int_X \chi_G \, d\mu_{y,z},
\]

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where \( y, z \in H \), and \( \mu_{y,z} \) is an elementary measure for \( (y, z) \) of \( \pi \).

First we check that \( S_G \) is well-defined.

Let \( \mu_{y,z} \) and \( \mu'_{y,z} \) be two elementary measures for \( (y, z) \) of \( \pi \). Then \( \mu_{y,z} - \mu'_{y,z} \) is a regular complex Borel measure on \( X \) such that

\[
\int_X f \, d(\mu_{y,z} - \mu'_{y,z}) = 0 \quad \text{for all } f \in A.
\]

From Lemma 2.6, it follows that

\[
\int_X f \cdot \chi_G \, d(\mu_{y,z} - \mu'_{y,z}) = 0 \quad \text{for all } f \in A.
\]

Take \( f = 1 \), then

\[
\int_X \chi_G \, d\mu_{y,z} = \int_X \chi_G \, d\mu'_{y,z}.
\]

Consequently, \( S_G \) is well-defined.

For \( y_1, y_2, z \in H \), and any \( f \in A \),

\[
\int_X f \, d\mu_{y_1+y_2,z} = \langle \pi(f)(y_1 + y_2), z \rangle
\]

\[
= \langle \pi(f) y_1, z \rangle + \langle \pi(f) y_2, z \rangle
\]

\[
= \int_X f \, d\mu_{y_1,z} + \int_X f \, d\mu_{y_2,z}.
\]

This proves that for any \( f \in A \),

\[
\int_X f \, d(\mu_{y_1+y_2,z} - \mu_{y_1,z} - \mu_{y_2,z}) = 0.
\]

Using Lemma 2.6 again, we have

\[
\int_X f \cdot \chi_G \, d(\mu_{y_1+y_2,z} - \mu_{y_1,z} - \mu_{y_2,z}) = 0,
\]

or, equivalently, we have for all \( f \) in \( A \) that

\[
\int_X f \cdot \chi_G \, d\mu_{y_1+y_2,z} = \int_X f \cdot \chi_G \, d\mu_{y_1,z} + \int_X f \cdot \chi_G \, d\mu_{y_2,z}.
\]
If $f = 1$, we see that

$$S_G((y_1 + y_2), z) = S_G(y_1, z) + S_G(y_2, z).$$

In a similar way we can prove $S_G$ is a sesquilinear functional. Also

$$|S_G(y, z)| = |\int_X \chi_G \, d\mu_{y,z}| \leq \|\mu_{y,z}\| \leq \|y\| \cdot \|z\|.$$

Hence, $S_G$ is a bounded sesquilinear functional. There exists a unique operator, say $P_G$, on $\mathcal{H}$ such that $S_G(y, z) = (P_G y, z)$ for all $y, z \in \mathcal{H}$.

Also for any $y \in \mathcal{H},$

$$(P_G y, y) = \int_X \chi_G \, d\mu_{y,y} = \mu_{y,y}(G) \geq 0,$$

since $\mu_{y,y}$ is a positive measure as proved in Theorem 2.4. Therefore, $P_G$ is a positive operator on $\mathcal{H}$.

Next we show $P_G$ commutes with $\pi(f)$ for any $f \in A$.

For any $y, z \in \mathcal{H}$, and any $f \in A$, by Lemma 2.5, there exists a sequence $\{F_n \mid G \subseteq F_n, \text{ where } n = 1, 2, 3, \ldots \} \subseteq \mathcal{P}_A$ such that $|\mu_{\pi(f)y,z}|(F_n \setminus G)$ approaches 0, and $|\mu_{y,\pi(f)x}|(F_n \setminus G)$ approaches 0 as $n$ tends to $\infty$.  

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Then for any $f \in A$, and any $y, z \in \mathcal{H}$,

$$
\langle P_G \cdot \pi(f) \cdot y, z \rangle = \int_X \chi_G \, d\mu_{\pi(f)} \cdot y, z
$$

$$
= \lim_{n \to \infty} \int_X \chi_{F_n} \, d\mu_{\pi(f)} \cdot y, z
$$

$$
= \lim_{n \to \infty} \langle P_{F_n} \cdot \pi(f) \cdot y, z \rangle
$$

$$
= \lim_{n \to \infty} \langle \pi(f) \cdot P_{F_n} \cdot y, z \rangle
$$

$$
= \lim_{n \to \infty} \langle P_{F_n} \cdot y, \pi(f)^* \cdot z \rangle
$$

(Use Theorem 2.4.)

$$
= \lim_{n \to \infty} \int_X \chi_{F_n} \, d\mu_{\pi(f)^*} \cdot z
$$

(Use Theorem 2.4)

$$
= \int_X \chi_G \, d\mu_{\pi(f)^*} \cdot z
$$

$$
= \langle \pi(f) \cdot P_G \cdot y, z \rangle.
$$

Thus $P_G \cdot \pi(f) = \pi(f) \cdot P_G$ for any $f \in A$.

It remains to show $P_G^2 = P_G$.

To this end, fix $y \in \mathcal{H}$. We can find, by Lemma 2.5, a sequence $\{ F_n \mid G \subseteq F_n, \text{ where } n = 1, 2, 3, \ldots \} \subseteq \mathcal{P}_A$ such that $|\mu_{y,y}|(F_n \setminus G)$ converges to 0, as $n$ approaches $\infty$. Then

$$
0 \leq \langle (P_{F_n} - P_G) \cdot y, (P_{F_n} - P_G) \cdot y \rangle
$$

$$
\leq \langle (P_{F_n} - P_G) \cdot y, y \rangle^{1/2} \langle (P_{F_n} - P_G)(P_{F_n} - P_G) \cdot y, (P_{F_n} - P_G) \cdot y \rangle^{1/2}
$$

$$
\leq |\mu_{y,y}|(F_n \setminus G) \cdot \|P_{F_n} - P_G\|^2 \cdot \|y\|^2
$$

where the second inequality follows from Schwartz inequality. This proves that as $n$ approaches $\infty$,

$$
P_{F_n} \cdot y \to P_G \cdot y.
$$

(5)
Therefore,

\[
\langle P_G y, y \rangle = \int_{\mathbf{x}} \chi_G \, d\mu_{y,y}
\]

\[
= \lim_{n \to \infty} \int_{\mathbf{x}} \chi_{F_n} \, d\mu_{y,y}
\]

\[
= \lim_{n \to \infty} \langle P_{F_n} y, y \rangle
\]

\[
= \lim_{n \to \infty} \langle P_{F_n}^2 y, y \rangle \quad \text{(Use Theorem 2.4.)}
\]

\[
= \lim_{n \to \infty} \langle P_{F_n} y, P_{F_n} y \rangle
\]

\[
= \langle P_G y, P_G y \rangle \quad \text{(Use (5).)}
\]

\[
= \langle P_G^2 y, y \rangle.
\]

This implies \( P_G = P_G^2 \), and the proof is complete. ■■
CHAPTER 3

ONE-TO-ONE CORRESPONDENCE BETWEEN
MAXIMAL SETS OF ANTISYMMETRY AND
MAXIMAL PROJECTIONS OF ANTISYMMETRY

Let $A$ be a uniform algebra on $X$, and $\pi$ a contractive unital representation mapping $A$ into $B(H)$. Later in this chapter we assume further that (i) $\pi$ is an isometry and (ii) for each maximal set of antisymmetry for $A$, say $K$, and any $\varepsilon > 0$, there is an open set, say $V$, with $V \supset K$ such that

$$|\mu_{y,z}((V \setminus K)) < \varepsilon \text{ for all } y, z \in H \text{ with } \|y\| = \|z\| = 1,$$

where $\mu_{y,z}$ is an elementary measure for $(y, z)$.

Under these conditions, in Corollary 3.9 we show that there is a one-to-one correspondence between the collection of maximal sets of antisymmetry for $A$ and that of maximal projections of antisymmetry for $\pi(A)$.

In Chapter 2 we proved for each $G \in (GP)_A$ there exists a projection $P_G \in B(H)$ such that $P_G \cdot \pi(f) = \pi(f) \cdot P_G$ for any $f \in A$, and

$$\langle P_G y, z \rangle = \int_X \chi_G \, d\mu_{y,z},$$

where $\mu_{y,z}$ is any elementary measure for $(y, z)$ of $\pi$.

In actuality, we also have an integral form for the expression $\langle \pi(f) \cdot P_G y, z \rangle$, for any $f \in A$, $y, z \in H$, and $G \in (GP)_A$. First we obtain the desired form for those $F \in \mathcal{P}_A$.

3.1 Lemma. Let $F$ be a peak set for $A$. Then for any $f \in A$, and $y, z \in H$, we have

$$\langle \pi(f) \cdot P_F y, z \rangle = \int_X f \cdot \chi_F \, d\mu_{y,z},$$
where \( \mu_{y,z} \) is any elementary measure for \((y,z)\) of \(\pi\).

**Proof:** Let \(p \in A\) peak on \(F\), we have

\[
\langle \pi(f) \cdot P_F y, z \rangle = \langle P_F \cdot \pi(f) y, z \rangle
\]

(Use Theorem 2.4.)

\[
= \lim_{n \to \infty} \int_X p^n \cdot d\mu_{\pi(t)y,z}
\]

\[
= \lim_{n \to \infty} \langle \pi(p^n) \cdot \pi(f) y, z \rangle
\]

\[
= \lim_{n \to \infty} \langle \pi(p^n \cdot f) y, z \rangle
\]

\[
= \lim_{n \to \infty} \int_X p^n \cdot f \cdot d\mu_{y,z}
\]

\[
= \int_X f \cdot \chi_F \cdot d\mu_{y,z}. \quad \blacksquare
\]

Next we prove the equality in Lemma 3.1 holds true for any \(G \in (\mathcal{G}P)_A\).

3.2 Lemma. Let \(G\) be a generalized peak set for \(A\). For any \(y, z \in H\), and \(f \in A\), we have

\[
\langle \pi(f) \cdot P_G y, z \rangle = \int_X f \cdot \chi_G \cdot d\mu_{y,z},
\]

where \(\mu_{y,z}\) is any elementary measure for \((y,z)\) of \(\pi\).

**Proof:** Fix \(y, z \in H\), any elementary measure \(\mu_{y,z}\) for \((y,z)\) of \(\pi\), and \(f \in A\).

By Lemma 2.5, there exists a sequence \(\{F_n \mid n = 1, 2, 3, \ldots\} \subseteq \mathcal{P}_A\) such that

\[
\lim_{n \to \infty} |\mu_{y,z}|(F_n \setminus G) = 0,
\]

and

\[
\lim_{n \to \infty} |\mu_{\pi(t)y,z}|(F_n \setminus G) = 0.
\]
Then

\[
\langle \pi(f) \cdot P_G y, z \rangle = \langle P_G \cdot \pi(f) y, z \rangle
\]

(Use Theorem 2.7.)

\[
= \int_X \chi_G \ d\mu_{\pi(f) y, z}
\]

\[
= \lim_{n \to \infty} \int_X \chi_{F_n} \ d\mu_{\pi(f) y, z}
\]

(Use Theorem 2.4.)

\[
= \lim_{n \to \infty} \langle \pi(f) \cdot P_{F_n} y, z \rangle
\]

(Use Theorem 2.4.)

\[
= \lim_{n \to \infty} \int_X f \cdot \chi_{F_n} \ d\mu_{y, z}
\]

(Use Lemma 3.1.)

\[
= \int_X f \cdot \chi_G \ d\mu_{y, z}.
\]

3.3 Corollary. Let $G_1, G_2$ be two generalized peak sets for $A$. For any $y, z \in H$, we have

\[
\langle P_{G_1} \cdot P_{G_2} y, z \rangle = \int_X \chi_{G_1 \cap G_2} \ d\mu_{y, z},
\]

where $\mu_{y, z}$ is any elementary measure for $(y, z)$ of $\pi$.

proof: Fix $y, z \in H$.

Using Lemma 2.5, there exists a sequence $\{F_n \mid n = 1, 2, 3, \ldots\} \subseteq \mathcal{P}_A$ with $G_1 \subseteq F_n$ for each $n = 1, 2, 3, \ldots$, such that

\[
\lim_{n \to \infty} |\mu_{y, z}|(F_n \setminus G_1) = 0,
\]

and

\[
\lim_{n \to \infty} |\mu_{P_{G_2} y, z}|(F_n \setminus G_1) = 0.
\]
Also let $p_n$ peak on $F_n$ for each $n = 1, 2, 3, \ldots$. Then

$$
\langle P_{G_1} \cdot P_{G_2} y, z \rangle = \int_X \chi_{G_1} d\mu_{P_{G_2} y, z} \quad (\text{Use Theorem 2.7.})
$$

$$
= \lim_{n \to \infty} \int_X \chi_{F_n} d\mu_{P_{G_2} y, z}
$$

$$
= \lim_{n \to \infty} \lim_{m \to \infty} \int_X P_n^m d\mu_{P_{G_2} y, z}
$$

$$
= \lim_{n \to \infty} \lim_{m \to \infty} (\pi(P_n^m) \cdot P_{G_2} y, z)
$$

$$
= \lim_{n \to \infty} \lim_{m \to \infty} \int_X P_n^m \cdot \chi_{G_2} d\mu_{y, z} \quad (\text{Use Lemma 3.2.})
$$

$$
= \lim_{n \to \infty} \int_X \chi_{F_n} \cdot \chi_{G_2} d\mu_{y, z}
$$

$$
= \lim_{n \to \infty} \int_X \chi_{F_n \cap G_2} d\mu_{y, z}
$$

$$
= \int_X \chi_{G_1 \cap G_2} d\mu_{y, z} \quad \blacksquare
$$

Remarks:

(a) $P_{G_1 \cap G_2} = P_{G_1} \cdot P_{G_2}$.

(b) If $G_1 \cap G_2 = \emptyset$, then $P_{G_1} \cdot P_{G_2} = 0$.

In the following we further assume that $\pi$ is an isometry, i.e., $\|\pi(f)\| = \|f\|$ for all $f \in A$.

3.4 Lemma. Let $\pi$ be a unital representation that maps $A$ into $B(H)$. Also assume $\|\pi(f)\| = \|f\|$ for all $f \in A$. If $f_0 \in A$, and $\pi(f_0)$ is self-adjoint, then $f_0$ is real-valued on $X$.

Proof: Suppose on the contrary $f_0$ is not real-valued on $X$. Then there exists $x_0 \in X$ such that $f_0(x_0) = a + ib$ with $a, b$ real numbers and $b \neq 0$.  

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Define $\phi : \pi(A) \to \mathbb{C}$ by

$$
\phi(\pi(f)) = f(x_0) \quad \text{for} \; f \in A.
$$

We check $\phi$ is well-defined.

Suppose $\pi(f) = \pi(g)$ for some $f, g \in A$. Then

$$
0 = \|\pi(f) - \pi(g)\| = \|\pi(f - g)\| = \|f - g\|.
$$

This implies $(f - g)(x_0) = 0$, or $f(x_0) = g(x_0)$.

One can check $\phi$ is a nonzero multiplicative linear functional defined on $\pi(A)$. By Theorem VII.8.6 of ([2], Conway), the spectrum of $\pi(f_0)$ is:

$$
\{ \psi(\pi(f_0)) \mid \text{where} \; \psi \; \text{is any nonzero multiplicative linear functional on} \; \pi(A) \}.
$$

Since $\pi(f_0)$ is self-adjoint, the spectrum of $\pi(f_0)$ is a subset of real numbers. But $\phi(\pi(f_0)) = f_0(x_0)$ is not a real number. A contradiction arrives. Thus $f_0$ must be real-valued on $X$. ■■

Recall that the collection of maximal sets of antisymmetry for $A$ is denoted by $\mathcal{K}_A$. In ([9], Glicksberg), it is shown $\mathcal{K}_A \subseteq (\mathcal{GP})_A$.

**3.5 Corollary.** Let $\pi$ be a unital representation that maps $A$ into $B(H)$. Also assume that $\|\pi(f)\| = \|f\|$ for all $f \in A$, and $\mathcal{K}_A = \{ X \}$. Then the only self-adjoint operator in $\pi(A)$ is $\lambda \cdot I$ for some real number $\lambda$.

**Proof:** Suppose $\pi(f)$ is self-adjoint for some $f \in A$. We see from Lemma 3.4 that $f$ is real-valued on $X$. Since $X$ is the maximal set of antisymmetry for $A$, we have $f = \lambda \cdot 1$ for some real number $\lambda$. Thus, $\pi(f) = \pi(\lambda \cdot 1) = \lambda \cdot \pi(1) = \lambda \cdot I$. ■■

Remark: The corollary says $I$ is a projection of antisymmetry, and hence a maximal projection of antisymmetry, for $\pi(A)$ in the case $X$ is the only maximal set of
antisymmetry for $A$.

3.6 Theorem. Let $\pi$ be a unital representation mapping a uniform algebra $A \subseteq C(X)$ into $B(\mathcal{H})$ with $\|\pi(f)\| = \|f\|$ for all $f \in A$. Let $K$ be a maximal set of antisymmetry for $A$ such that for any $\epsilon > 0$ there exists an open set $V$ with $K \subseteq V \subseteq X$ such that

$$|\mu_{\gamma,z}(V \setminus K) < \epsilon$$

for all $\gamma, z \in \mathcal{H}$ with $\|\gamma\| = \|z\| = 1$,

where $\mu_{\gamma,z}$ is an elementary measure for $(\gamma, z)$. Then $P_K$ is a projection of antisymmetry for $\pi(A)$.

Proof: Let $i_K$ denote the mapping from $K$ to $X$ such that $i_K(x) = x$ for all $x \in K$.

Define $A_K = \{f \circ i_K | f \in A \}$. Then, by Theorem 1.7, $A_K$ is a uniform algebra on $K$ with $K$ as the only maximal set of antisymmetry for $A_K$.

Define $\pi_K : A_K \to B(P_K \mathcal{H})$ by $\pi_K(f \circ i_K) = \pi(f) \cdot P_K$.

We need to check $\pi_K$ is well-defined.

Suppose $f \circ i_K = g \circ i_K$ for some $f, g \in A$. We want to show $\pi(f) \cdot P_K = \pi(g) \cdot P_K$.

To this end, for any $\gamma, z \in \mathcal{H}$,

$$\langle \pi(f) \cdot P_K \gamma, z \rangle = \int_X f \cdot \chi_K \, d\mu_{\gamma,z}$$

(Use Lemma 3.2.)

$$= \int_X g \cdot \chi_K \, d\mu_{\gamma,z}$$

$$= \langle \pi(g) \cdot P_K \gamma, z \rangle.$$

Thus $\pi(f) \cdot P_K = \pi(g) \cdot P_K$.

Using the property that $P_K$ commutes with $\pi(f)$ for all $f \in A$, one can verify $\pi_K$ is a unital representation that maps $A_K$ into $B(P_K \mathcal{H})$.

To apply Corollary 3.5 we need to show $\pi_K$ is an isometry, i.e., $\|\pi_K(f \circ i_K)\| = \|f \circ i_K\|$, or $\|\pi(f) \cdot P_K\| = \|f\|_K$. 

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Suppose this is so, we claim $P_K$ is a projection of antisymmetry for $\pi(A)$. We see that

$$\|P_K\| = \|\pi(1) \cdot P_K\| = \|1\|_K = 1,$$

hence $P_K \neq 0$. By Theorem 2.7, $\pi(f) \cdot P_K = P_K \cdot \pi(f)$ for any $f \in A$. Suppose $\pi(f) \cdot P_K$ is self-adjoint. By Corollary 3.5 we have $\pi(f) \cdot P_K = \lambda \cdot P_K$ for some real number $\lambda$.

To show $\|\pi(f) \cdot P_K\| = \|f\|_K$, we first claim that $\|\pi(f) \cdot P_K\| \leq \|f\|_K$.

Let $y, z \in \mathcal{H}$ with $\|y\| = \|z\| = 1$, and $f \in A$. Then

$$\langle \pi(f) \cdot P_K y, z \rangle = \int_X f \cdot \chi_K \, d\mu_{y,z} \quad \text{(Use Lemma 3.2.)}$$

$$\leq \|f\|_K \cdot \|\mu_{y,z}\|$$

$$\leq \|f\|_K \cdot \|y\| \cdot \|z\|$$

$$= \|f\|_K.$$

Thus $\|\pi(f) \cdot P_K\| \leq \|f\|_K$.

To prove the other direction of inequality, fix any $f \in A$.

Since $K$ is a maximal set of antisymmetry for $A$, by ([9], Glicksberg), $K$ is a generalized peak set for $A$. By Theorem 1.6, there exists $h \in A$ such that $h(x) = f(x)$ for all $x \in K$ and $\|h\|_K = \|f\|_K$.

Given any $\epsilon > 0$, by hypothesis, there exists an open set $V$ with $V \supseteq K$ such that $|\mu_{y,z}(V \setminus K)| < \epsilon$ for any $y, z \in \mathcal{H}$ with $\|y\| = \|z\| = 1$.

Write $K = \bigcap \{ F_\alpha | K \subseteq F_\alpha \in \mathcal{P}_A \} \subseteq V$. By a standard argument about compactness and the property that a finite intersection of peak sets for $A$ is a peak set for $A$, one can show there exists a peak set, say $F$, for $A$ such that $K \subseteq F \subseteq V$.

Let $g$ peak on $F$. By raising $g$ to a sufficiently large power we may assume $|g(x)| < \epsilon$ for any $x \in X \setminus V$. 

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Since \( \pi \) is an isometry, we can find \( y_0, z_0 \in \mathcal{H} \) with \( \|y_0\| = \|z_0\| = 1 \) such that
\[
\left| \int \mathbf{g} \cdot \mathbf{h} \, d\mu_{y_0, z_0} \right| = |(\pi(\mathbf{g} \cdot \mathbf{h}) y_0, z_0)|
\geq \|\mathbf{g} \cdot \mathbf{h}\| \cdot \| \mathbf{x} - \epsilon \|
= \|\mathbf{h}\| \cdot \| \mathbf{x} - \epsilon \|
= \|\mathbf{f}\| \mathcal{K} - \epsilon.
\]

Thus
\[
|\langle \pi(\mathbf{f}) \cdot \mathbf{P}_{\mathcal{K}} y_0, z_0 \rangle| = \left| \int \mathbf{f} \cdot \chi_{\mathcal{K}} \, d\mu_{y_0, z_0} \right| \quad \text{(Use Lemma 3.2.)}
\geq \left| \int \mathbf{g} \cdot \mathbf{h} \, d\mu_{y_0, z_0} \right| - \left| \int (\mathbf{g} \cdot \mathbf{h} - \mathbf{f} \cdot \chi_{\mathcal{K}}) \, d\mu_{y_0, z_0} \right|
= \left| \int \mathbf{g} \cdot \mathbf{h} \, d\mu_{y_0, z_0} \right| - \left| \int (\mathbf{g} \cdot \mathbf{h} - \mathbf{f} \cdot \chi_{\mathcal{K}}) \, d\mu_{y_0, z_0} \right|
\geq \left| \int \mathbf{g} \cdot \mathbf{h} \, d\mu_{y_0, z_0} \right| - \left| \int \mathbf{g} \cdot \mathbf{h} \, d\mu_{y_0, z_0} \right| - \left| \int \mathbf{g} \cdot \mathbf{h} \, d\mu_{y_0, z_0} \right|
= \left| \int \mathbf{g} \cdot \mathbf{h} \, d\mu_{y_0, z_0} \right| - \left| \int \mathbf{g} \cdot \mathbf{h} \, d\mu_{y_0, z_0} \right| - \left| \int \mathbf{g} \cdot \mathbf{h} \, d\mu_{y_0, z_0} \right|
\geq (\|\mathbf{f}\| \mathcal{K} - \epsilon) - \|\mathbf{g} \cdot \mathbf{h}\| \cdot \| \mathbf{y}_0, z_0 \| (\mathcal{V} \setminus \mathcal{K}) - \|\mathbf{g} \cdot \mathbf{h}\| \cdot \| \mathbf{y}_0, z_0 \| (\mathcal{X} \setminus \mathcal{V}) \quad \text{(Use (1).)}
\geq (\|\mathbf{f}\| \mathcal{K} - \epsilon) - \|\mathbf{f}\| \mathcal{K} \cdot \epsilon - \|\mathbf{h}\| \cdot \| \mathbf{y}_0, z_0 \|
\geq (\|\mathbf{f}\| \mathcal{K} - \epsilon) - \|\mathbf{f}\| \mathcal{K} \cdot \epsilon - \|\mathbf{f}\| \mathcal{K}
\geq \|\mathbf{f}\| \mathcal{K} (1 - 2\epsilon) - \epsilon.
\]

Since \( \epsilon \) was arbitrary, \( \|\pi(\mathbf{f}) \cdot \mathbf{P}_{\mathcal{K}} \| \geq \|\mathbf{f}\| \mathcal{K} \) for all \( \mathbf{f} \in \mathcal{A} \). Hence \( \pi_{\mathcal{K}} \) is an isometry.

\[
\square
\]

Remark: Suppose \( \mathbf{G} \in (\mathcal{GP})_{\mathcal{A}} \) satisfies the regularity property in Theorem 3.6, i.e., for any \( \epsilon > 0 \) there exists an open set \( \mathcal{V} \) with \( \mathbf{G} \subseteq \mathcal{V} \subseteq \mathcal{X} \) such that
\[
|\mu_{y,z}(\mathcal{V} \setminus \mathbf{G})| < \epsilon \text{ for all } y, z \in \mathcal{H} \text{ with } \|y\| = \|z\| = 1,
\]

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where \( \mu_{y,z} \) is an elementary measure for \((y,z)\).

In the proof of Theorem 3.6 we actually showed that \( \|\pi(f) \cdot P_G\| = \|f\|_G \) for all \( f \in A \). In particular, \( P_G \neq O \).

The following example taken from ([22], Szymanski) shows the regularity hypothesis in Theorem 3.6 is indispensable.

3.7 Example. Let
(a) \( \Gamma \equiv \{ c \in \mathbb{C} : |c| = 1 \} \), i.e., \( \Gamma \) denotes the unit circle,
(b) \( \{ c_n \mid n = 1, 2, 3, \ldots \} \) be a bounded sequence of complex numbers with \( |c_n| > 1 \) for each \( n \) and such that \( \Gamma \) is the cluster points of \( \{ c_n \mid n = 1, 2, 3, \ldots \} \),
(c) \( K \) denote the real number interval \([0, \frac{1}{2}]\), and
(d) \( X \equiv \Gamma \cup \{ c_n \mid n = 1, 2, 3, \ldots \} \cup K \).

Let \( P(X) \equiv \) the uniform closure of all polynomials in \( C(X) \). Denote the point mass at \( c_n \) and the linear Lebesgue measure restricted to \( K \) by \( \delta_n \) and \( m_K \), respectively. Define \( \mu \equiv m_K + \sum_{n=1}^{\infty} 2^{-n}\delta_n \). It is clear \( \mu \) is a positive, finite Borel measure on \( X \), and \( \mu(\Gamma) = 0 \).

Denote the collection of square integrable functions on the measure \( \mu \) by \( L^2(\mu) \). Define \( \pi : P(X) \to B(L^2(\mu)) \) by \( \pi(f) = M_f \) for any \( f \in P(X) \), where \( M_f(y) = f \cdot y \) for any \( y \in L^2(\mu) \).

One can check:
(i) The mapping \( \pi \) is a unital representation with \( \|\pi(f)\| = \|f\| \) for all \( f \in P(X) \);
(ii) For any \( y, z \in L^2(\mu) \), the measure \( y \cdot \overline{z} \, d\mu \) is an elementary measure for \((y,z)\);
(iii) The set \( \Gamma \cup K \) is a maximal set of antisymmetry for \( P(X) \); and
(iv) For any open set \( V \) with \( X \supset V \supset (\Gamma \cup K) \), there exists a \( c_n \in V \setminus (\Gamma \cup K) \). Let
\( y_n \equiv \sqrt{2^n} \cdot \chi_{\{c_n\}} \). Then \( y_n \in L^2(\mu) \), and \( \|y_n\| = 1 \), and yet \( |\mu_{y_n,Y_n}|(V \backslash (\Gamma \cup K)) = 1 \). Hence the regularity hypothesis in Theorem 3.6 is not satisfied.

In the following we show \( P_{\Gamma \cup K} \) is not a projection of antisymmetry for \( \pi(P(X)) \).

**Proof:** The function \( i : X \to X \) defined by \( i(x) = x \) is in \( P(X) \). For any \( y \in L^2(\mu) \),

\[
\langle \pi(i) \cdot P_{\Gamma \cup K} y, y \rangle = \int_X i \cdot \chi_{\Gamma \cup K} \cdot d\mu_y_y = \int_X i \cdot \chi_{\Gamma \cup K} \cdot y \cdot \bar{y} \cdot d\mu = \int_{\Gamma \cup K} i \cdot y \cdot \bar{y} \cdot d\mu.
\]

Since \( i \) is real-valued on \( K \) and \( y \cdot \bar{y} \geq 0 \), the operator \( \pi(i) \cdot P_{\Gamma \cup K} \) is self-adjoint.

If \( P_{\Gamma \cup K} \) is a projection of antisymmetry for \( \pi(P(X)) \), then \( \pi(i) \cdot P_{\Gamma \cup K} = \lambda \cdot P_{\Gamma \cup K} \) for some real number \( \lambda \). This is absurd, because for any real number \( \lambda \), we can find \( y \in P_{\Gamma \cup K}(L^2(\mu)) \) with \( \|y\| = 1 \) such that \( \int_k i \cdot y \cdot \bar{y} \cdot d\mu \neq \lambda \).

### 3.8 Theorem

Let \( A \) be a uniform algebra on \( X \) and \( \pi \) a unital representation mapping \( A \) into \( B(H) \) with \( \|\pi(f)\| = \|f\| \) for all \( f \in A \). Suppose also for any \( K \in K_A \), and \( \epsilon > 0 \) there exists an open set \( V \) with \( K \subseteq V \subseteq X \) such that

\[
|\mu_{y,z}|(V \backslash K) < \epsilon \quad \text{for all} \quad y, z \in H \quad \text{with} \quad \|y\| = \|z\| = 1,
\]

where \( \mu_{y,z} \) is an elementary measure for \( (y,z) \).

If \( P \) is a maximal projection of antisymmetry for \( \pi(A) \), then \( P = P_K \) for some \( K \in K_A \).

**Proof:** Let \( Z \equiv \bigcup \{ \text{support of } \mu_{P_y,Py} | y \in H \text{ and } \mu_{P_y,Py} \text{ is any elementary measure for } (P_y,Py) \} \).
We first claim $Z$ is a set of antisymmetry for $A$.

**Proof of Claim:** Let $f \in A$ be real-valued on $Z$. Then for any $y \in \mathcal{H},$

$$\langle \pi(f) \cdot P y, y \rangle = \langle \pi(f) \cdot P y, P y \rangle = \int_X f \, d\mu_{P y, P y}.$$

Since $f$ is real-valued on support of $\mu_{P y, P y}$, and $\mu_{P y, P y} \geq 0$, we conclude that $\langle \pi(f) \cdot P y, y \rangle$ is real-valued for all $y \in \mathcal{H}$. Therefore, $\pi(f) \cdot P$ is self-adjoint.

By the definition of projection of antisymmetry, we have $\pi(f) \cdot P = \lambda \cdot P$ for some real number $\lambda$. Thus $O = [\pi(f) - \lambda] \cdot P = \pi(f - \lambda) \cdot P$. Since $\pi(f - \lambda) \cdot P = P \cdot \pi(f - \lambda)$, we see

$$P \cdot \pi(f - \lambda) \cdot P = \pi(f - \lambda) \cdot P^2 = \pi(f - \lambda) \cdot P. \quad (2)$$

Thus

$$O = [\pi(f - \lambda) \cdot P] \cdot [\pi(f - \lambda) \cdot P]$$

$$= \pi(f - \lambda) \cdot [P \cdot \pi(f - \lambda) \cdot P]$$

$$= \pi(f - \lambda) \cdot [\pi(f - \lambda) \cdot P] \quad (\text{Use } (2)).$$

$$= \pi((f - \lambda)^2) \cdot P.$$

Therefore, for any $y \in \mathcal{H},$

$$0 = \langle \pi((f - \lambda)^2) \cdot P y, y \rangle = \langle \pi((f - \lambda)^2) \cdot P y, P y \rangle = \int_X (f - \lambda)^2 \, d\mu_{P y, P y}.$$

Since $(f - \lambda)^2 \geq 0$, and $\mu_{P y, P y} \geq 0$, this implies $f - \lambda = 0$ on support of $\mu_{P y, P y}$. Hence $f = \lambda$ on support of $\mu_{P y, P y}$, for any $y \in \mathcal{H}$. Thus $Z$ is a set of antisymmetry for $A$. ■

We return to the proof of the theorem. Let $K$ be the maximal set of antisymmetry
for \( A \) that contains \( Z \). Then for any \( y \in \mathcal{P} \), we have

\[
\langle P_K y, y \rangle = \int_X \chi_K d\mu_{y,y} \\
= \int_X \chi_{(\text{support of } \mu_{y,y})} d\mu_{y,y} \\
= \|\mu_{y,y}\| \\
= \langle y, y \rangle \\
= \langle P y, y \rangle.
\]

This proves \( P_K \geq P \).

We also know \( P_K \) is a projection of antisymmetry for \( \pi(A) \) from Theorem 3.6. Thus \( P_K = P \), by the fact \( P \) is a maximal projection of antisymmetry for \( \pi(A) \).

3.9 Corollary. Let \( A \) be a uniform algebra on \( X \) and \( \pi \) a unital representation mapping \( A \) into \( B(\mathcal{H}) \) with \( \|\pi(f)\| = \|f\| \) for all \( f \in A \). Suppose also for any \( K \in \mathcal{K}_A \), and \( \epsilon > 0 \) there exists an open set \( V \) with \( K \subseteq V \subseteq X \) such that

\[
|\mu_{y,z}|(V \setminus K) < \epsilon \quad \text{for all } y, z \in \mathcal{H} \quad \text{with } \|y\| = \|z\| = 1,
\]

where \( \mu_{y,z} \) is an elementary measure for \( (y,z) \). Then there is a one-to-one correspondence between \( \mathcal{K}_A \) and the collection of maximal projections of antisymmetry for \( \pi(A) \).

Proof: Define \( \phi : \mathcal{K}_A \rightarrow B(\mathcal{H}) \) by \( \phi(K) = P_K \) for all \( K \in \mathcal{K}_A \).

Let \( K \in \mathcal{K}_A \). Using Theorem 3.6, we see that \( P_K \) is a projection of antisymmetry for \( \pi(A) \). Hence \( P_K \leq P \), where \( P \) is a maximal projection of antisymmetry for \( \pi(A) \). By Theorem 3.8, we have \( P = P_{K_0} \) for some \( K_0 \in \mathcal{K}_A \).

Since \( \mathcal{K}_A \) forms a pairwise disjoint partition of \( X \), either \( K \cap K_0 = \emptyset \), or \( K = K_0 \).

If \( K \cap K_0 = \emptyset \), then, by Remark (b) after the proof of Corollary 3.3, \( P_K = P_K \cdot P_{K_0} = \)
O. A contradiction to the fact $P_K \neq O$. Thus $K = K_0$. Therefore, $\phi(K) = P_K$ is a maximal projection of antisymmetry for $\pi(A)$ for all $K \in \mathcal{K}_A$. Hence $\phi$ maps $\mathcal{K}_A$ into the collection of maximal projections of antisymmetry for $\pi(A)$.

By Theorem 3.8, every maximal projection of antisymmetry for $\pi(A)$ is $P_K$ for some $K \in \mathcal{K}_A$. Thus, $\phi$ maps $\mathcal{K}_A$ onto the collection of maximal projections of antisymmetry for $\pi(A)$.

To complete the proof it remains to show $\phi$ is one-to-one. If $K_1, K_2 \in \mathcal{K}_A$ and $K_1 \neq K_2$, then $K_1 \cap K_2 = \emptyset$. Hence, $P_{K_1} \cdot P_{K_2} = O$. Since $P_{K_1} \neq O$ and $P_{K_2} \neq O$, $P_{K_1} \neq P_{K_2}$. This proves $\phi$ is one-to-one. ■ ■ ■

The following corollary strengthens the result of Theorem 3.6. Note in the following we do not assume that every maximal set of antisymmetry for $A$ satisfies the regularity property.

3.10 Corollary. Let $\pi$ be a unital representation mapping a uniform algebra $A \subseteq C(X)$ into $B(H)$ with $\|\pi(f)\| = \|f\|$ for all $f \in A$. Let $K$ be a maximal set of antisymmetry for $A$ such that for any $\varepsilon > 0$ there exists an open set $V$ with $K \subseteq V \subseteq X$ such that

$$|\mu_{y,z}(V \setminus K) < \varepsilon$$

for all $y, z \in H$ with $\|y\| = \|z\| = 1$.

where $\mu_{y,z}$ is an elementary measure for $(y, z)$. Then $P_K$ is a maximal projection of antisymmetry for $\pi(A)$.

Proof: By Theorem 3.6, the projection $P_K$ is a projection of antisymmetry for $\pi(A)$. Hence $P_K \leq P$, where $P$ is a maximal projection of antisymmetry for $\pi(A)$.

By following the proof of Theorem 3.8, we see that there is $K_0 \in \mathcal{K}_A$ such that $P \leq P_{K_0}$. Thus $P_K \leq P_{K_0}$. Since $\mathcal{K}_A$ forms a pairwise disjoint partition of $X$,
$K \cap K_0 = \emptyset$, or $K = K_0$.

If $K \cap K_0 = \emptyset$, then, by Remark (b) after the proof of Corollary 3.3, $O = P_K \cdot P_{K_0} = P_K$. A contradiction to the fact that $P_K$ is a projection of antisymmetry for $\pi(A)$.

Thus $K = K_0$, and therefore, $P_K = P_{K_0} = P$. This completes the proof that $P_K$ is a maximal projection of antisymmetry for $\pi(A)$. ■ ■

In the next example we show that the regularity hypothesis is not necessary in Corollary 3.9.

3.11 Example. Let $\Gamma, \{c_n \mid n = 1, 2, 3, \ldots\}$, and $\delta_n$ be as defined in Example 3.7. Let $X = \Gamma \cup \{c_n \mid n = 1, 2, 3, \ldots\}$.

Denote the uniform closure of all polynomials in $C(X)$ by $P(X)$. For each $n = 1, 2, 3, \ldots$, the singleton $\{c_n\} \in K_{P(X)}$ by Runge's theorem. To claim that $K_{P(X)} = \{\{c_n\} \mid n = 1, 2, 3, \ldots\} \cup \{\Gamma\}$, it remains to show that $\Gamma$ is a set of antisymmetry for $P(X)$.

To this end, let $p \in P(X)$ with $p$ real-valued on $\Gamma$. The function $p$ restricted to $\Gamma$ is in the uniform closure of all polynomials in $C(\Gamma)$, or the disk algebra. That is, $p$ restricted to $\Gamma$ is in the disk algebra and real-valued. This implies that for all $x \in \Gamma$, $p(x) = \lambda$ for some real number $\lambda$, by the fact that $\Gamma$ is the maximal set of antisymmetry for the disk algebra.

Denote the normalized Lebesgue measure on $\Gamma$ by $m$. Let $\mu \equiv m + \sum_{n=1}^{\infty} 2^{-n} \delta_n$. Then $\mu$ is a positive, finite Borel measure on $X$. As in Example 3.7, regularity hypothesis is not satisfied by $\Gamma$. Define $\pi : C(X) \to B(L^2(\mu))$ as in Example 3.7.

For each $n = 1, 2, 3, \ldots$, the singleton $\{c_n\}$ satisfies the regularity hypothesis. By Corollary 3.10, the projection $P_{\{c_n\}}$ is a maximal projection of antisymmetry for
\[ \pi(P(X)). \]

Next we show \( P_\Gamma \) is also a maximal projection of antisymmetry for \( \pi(P(X)) \). For any \( f \in P(X) \),

\[
\|\pi(f) \cdot P_\Gamma\| = \sup \{ |\langle \pi(f) y, z \rangle| : y, z \in L^2(m), \text{ and } \|y\| = \|z\| = 1. \}
= \sup \{ |\int f \cdot y \cdot z \, dm| : y, z \in L^2(m), \text{ and } \|y\| = \|z\| = 1. \}
= \|f\|_{\Gamma}.
\]

This implies the mapping \( \pi_\Gamma : P(X)_\Gamma \rightarrow B(L^2(m)) \) defined by \( \pi_\Gamma(f \circ i_\Gamma) = \pi(f) \cdot P_\Gamma \) is an isometry. We conclude that \( P_\Gamma \) is a projection of antisymmetry for \( \pi(P(X)) \) by following the proof of Theorem 3.6.

There exists some maximal projection of antisymmetry for \( \pi(P(X)) \), say \( P \), such that \( P_\Gamma \leq P \). Following the proof of Theorem 3.8 we can find \( K \in \mathcal{K}_{P(X)} \) such that \( P \leq P_K \). But \( P_\Gamma \leq P \leq P_K \), thus \( \Gamma = K \). For otherwise \( P_\Gamma = P_\Gamma \cdot P_K = 0 \), a contradiction to the fact \( P_\Gamma \neq 0 \).

Hence \( \pi \) is a unital representation with \( \|\pi(f)\| = \|f\| \) for all \( f \in P(X) \). There exists a one-to-one correspondence between \( \mathcal{K}_{P(X)} \) and the collection of all maximal projections of antisymmetry for \( \pi(P(X)) \). But \( \Gamma \in \mathcal{K}_{P(X)} \) does not satisfy the regularity hypothesis.

Suppose \( A \) is a uniform algebra on \( X \) and \( f \in C(X) \) with \( f \circ i_K \in A_K \), where \( i_K \) and \( A_K \) as defined in the proof of Theorem 3.6, for each \( K \in \mathcal{K}_A \). Then, in ([9], Glicksberg), it is shown that \( f \in A \). We have a similar result on \( \pi(A) \).

3.12 Theorem. Let \( A \) be a uniform algebra on \( X \) and \( \pi \) a unital representation mapping \( A \) into \( B(H) \) with \( \|\pi(f)\| = \|f\| \) for all \( f \in A \). Suppose also \( K \) is both open and closed for each \( K \in \mathcal{K}_A \). If \( T \in B(H) \) with \( T \cdot P_K \in \pi(A) \cdot P_K \) for each
$K \in \mathcal{K}_A$, then $T = \pi(f)$ for some $f \in A$.

Notes:

(a) Since each $K \in \mathcal{K}_A$ is open, the hypotheses of Theorem 3.6 are satisfied.

(b) Since $X$ is compact, each $K \in \mathcal{K}_A$ is open is equivalent to $\mathcal{K}_A$ is a finite collection.

**Proof:** Suppose $T \cdot P_K = \pi(f_K) \cdot P_K$ for some $f_K \in A$. Define

$$f \equiv \sum_{K \in \mathcal{K}_A} f_K \cdot \chi_K.$$

The function $f$ is well-defined since the sets in $\mathcal{K}_A$ are pairwise disjoint.

Furthermore, $f$ is continuous since each $K \in \mathcal{K}_A$ is an open set, and for any open set $V \subseteq \mathfrak{F}$,

$$f^{-1}(V) = \bigcup \{K \cap f_K^{-1}(V) \mid K \in \mathcal{K}_A\}.$$

By the fact mentioned before the statement of the theorem, $f \in A$ since $f \circ i_K = f_K \circ i_K$ for all $K \in \mathcal{K}_A$.

Also for each $K \in \mathcal{K}_A$, the representation $\pi_K : A_K \to B(P_K \mathfrak{H})$ defined by $\pi_K(g \circ i_K) = \pi(g) \cdot P_K$ for each $g \in A$ is well-defined as shown in the proof of Theorem 3.6.

For any $K \in \mathcal{K}_A$, $f \circ i_K = f_K \circ i_K$, so we have $\pi_K(f \circ i_K) = \pi_K(f_K \circ i_K)$, or $\pi(f) \cdot P_K = \pi(f_K) \cdot P_K$.

Hence,

$$T = \sum_{K \in \mathcal{K}_A} T \cdot P_K = \sum_{K \in \mathcal{K}_A} \pi(f_K) \cdot P_K$$

$$= \sum_{K \in \mathcal{K}_A} \pi(f) \cdot P_K$$

$$= \pi(f) \cdot \sum_{K \in \mathcal{K}_A} P_K$$

$$= \pi(f),$$

and the proof is complete. \[\square\]
REFERENCES


VITAE

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