BIPED ROBOT WITH A VESTIBULAR SYSTEM

by

Chuen-Chane Huang

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APPROVED:

J. W. Grant, Chairman

C. W. Smith

D. P. Tehonis

M. P. Singh

H. H. Robertshaw

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Abstract

The kinematics and dynamics of two legged or biped walking is considered. The resulting governing equations include actuator torques for a robot and muscle generated torques for a human. These torques are those necessary at each joint of a leg, including the foot, for a successful stride. The equations are developed from a consistent set variables with respect to a single inertial reference frame. This single reference frame approach has not been used by previous investigators.

Control of the joint torques makes biped walking an extraordinary complex problem from a dynamics and control viewpoint. The control scheme that is developed incorporates the use of the direction of gravity as an important element in the overall control. The inclusion of gravity in biped robot walking has not previously been properly considered in other works.

A way is described to separate gravity and acceleration which are measured by an accelerometer which is on the robot. This system incorporates the use of angular motion sensing of the robot segment that contains the linear accelerometers. This system was formulated based on human motion sensing and what probably is present in the human central nervous system for processing these signals.
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Vita
CHAPTER 1 INTRODUCTION

1.0 Introduction

The vestibular system is used by animals to measure linear and angular motion of their skulls. This information is then used by the central nervous system (CNS) to stabilize the visual image during periods of motion and coordinate muscular activity during static periods as well as periods of motion. In addition, our sensation of motion also comes from these organs. The vestibular system is located in the inner ear and forms a nonauditory section of this sensory organ. There is a set of vestibular organs with each ear.

In humans, which are bipedal (animals which walk on two legs), these organs perform a vital function of maintaining balance and equilibrium during static situations and are used to coordinate bipedal motion. In quadrupeds (animals which walk on four legs), this activity is not as important and does not occupy nearly as much brain volume as it does in bipedal animals which evolved much later.

The objective of this work is to first study how the otolith organs—the linear motion sensors of the vestibular system—function mechanically. Secondly, to study biped walking by examining a biped robot. And thirdly, to see how a vestibular system can be incorporated into the robot for motion control. It
should be noted that previous control attempts at bipedal walking have included only angular motion sensing, but have not included accelerometers which response to linear motion and gravity. It is known that in man, the direction of the gravity is extremely important in posture control. It is felt that gravity must play an equally important role in a biped robot.

This work is a two fold or two way undertaking. By studying both human motion sensing and bipedal robot walking, the two subjects can provide a synergism to further the understanding in each field. Understanding how vestibular signals are utilized in the central nervous system to control posture (static and dynamic) is a complex undertaking. However, determining what must be present to control a biped robot in its walking efforts can certainly identify the control element that must be present in human central nervous system.

In a similar fashion bipedal robot walking has been rather unsuccessful. In fact a prediction made 20 years ago, that we would have robots walking around and carrying out tasks has not come to pass. The artificial intelligence is available today, what is not available is a control system that will successfully control biped walking. This task was greatly underestimated. The portion of the human brain which carries out their control is larger than the frontal lobe where one's intellectual capacity resides.
In designing a control system for bipedal walking, complexities creep in very rapidly. Many unknowns and governing equations appear almost instantly. The fact that an accelerometer cannot distinguish between gravity and acceleration is a major problem. How the human can utilize its sensory inputs to distinguish between these two quantities—gravity and acceleration—can help the understanding of both systems.

Developing and linking these research areas and reducing the complexity of the systems to a manageable control effort is the main thrust of this work. It was started as an explanatory effort and has remained such. The attempt to synthesize and link sensory, motion and motor activity, with control is completed. Many more questions and topics were passed by than were answered. This was done deliberately to complete the entire control circuit.

A vestibular system contains two otolith organs as its linear motion sensors (accelerometers), and three semicircular canals as its angular motion sensors (integrating accelerometers). It is of interest to understand how the signals which are sensed by these motion transducers can be synthesized and turned into an inertial measurement unit (IMU). The IMU can determine the motion of the skull in six degrees of freedom.
Chapter 2 is designed to develop more understanding about the otolith organ. This chapter is a continued work furthering previous investigations [1,2,3,4]. Through all this work only the viewpoint of mechanics is concerned as well as an analytical approach where symbols and mathematics are used to describe their physical phenomenon. All other chapters comprise new contributions to the fields of robotics, bipedal control, and overall vestibular function.

This work contains five chapters; the content of each is explained below:

Chapter 1: Introduction—purpose, methodology and philosophy.

Chapter 2: Analytic Solution of an Otolith Organ Model

This chapter is a continued work to solve the governing equations analytically which were given by [4]. All assumptions and the nomenclature are the same.

Chapter 3: Skull Velocity and Gravity Sensing

There are four otocional layers and six semicircular canals. Each picks up a channel of signal. These signals are synthesized to determine the skull linear acceleration and the gravitational field, and angular velocity.
Chapter 4: Dynamic Model of the Biped Robot

In this chapter, a set of dynamic equations of the lower limbs are found. Uniqueness guarantees the controllability. Hence, it is possible to plan the angles, $\theta_i(t)$, and to compute the torques needed at the joints, $\Gamma_i(t)$.

Chapter 5: Biped Walking with the Vestibular Systems

In this chapter, the vestibular systems are combined with the dynamic model in a combined control system.

The robot which is studied in this work is shown in Fig.4-1 of Chapter 4. The reader should examine Fig.4-1 to understand the remainder of this introductory chapter. Important symbols and their definition are explained as needed. Details are given in Appendix 1. The philosophy behind stride planning and control is shown in Fig.1-1. This work forms a foundation to make a reasonable stride possible.

1.1 Hierarchical structure

Job designing--is to structure how a given motion can be accomplished. A track is decided next. A track is composed of a series of continued strides. Track planning--is to decide what strides are needed for a wanted track. Stride planning--is to decompose a stride into the $\theta$ at each joint. With the motion at each joint, a stride is being performed. Stride planning is not
Fig. 1-1  Hierarchical Functional Relationship of the Biped Robot. A biped walking design contains different levels of hierarchical functions.
included in this work. Only good stride planning can make successfully continued strides or a single stride. Stride planning involves decisions on limb angles as a function of time, $\theta_i(t)$. Accordingly, the actuators at each joint generate precisely the torques needed to perform a stride or sequential strides. Then, a piece of a stride is made in each time increment $\Delta t$. The sensors pick up the physical data to update or modify the remaining stride plan. Increment after time increment is completed and the stride is accomplished.

Fig.1-2 explains control designing; it is a combination of the hardware and the software used in the computers and the robot machinery. As a functional block, B3 in Fig.1-2 is the dynamic model used for providing the activating torque at the joints. For different control designing, the stride planning will vary.

The block diagram of Fig.1-2 contains a model referenced control about torques $\Gamma_i$, and a model referenced control about geometric variables $\theta_j$, $\dot{\theta}_j$, and $\ddot{\theta}_j$. Each servomechanism (for an example, B6, servo 1) accepts electrical signals and generates mechanical torque. The servomechanism gain varies depending on different engineering preference. The response should be quicker or meet the requirement of Stride Planning.

1.2 Adaptive Control
In Fig.1-1, the hierarchical concept is shown. Under Stride Planning, the mechanism concept of Control Designing is explained by Fig.1-2. However, it looks quite complicated in order to explain the concept. Fig.1-2 is reduced to a much simpler diagram. This simplified concept is presented in Fig.1-3. Fig.1-3 is obtained by simplifying the multiple channels to a single channel and directly re-drawing the functional relationship in the format of Fig.1-4. The fundamental adaptive feedback control is shown in Fig.1-4, where the output of Block G is measured by Block H. The measured variable is used as a reference signal and is used by Block G to adjust the gain. The use of the reference signal is illustrated by Fig.1-5. The command signal times a gain is the output of Block Ŕ. If the output, which shall be measured by Block H, is too small, then, the difference between the command and the readout will be positive. This difference will be used to adjust the gain. After the gain is increased, the difference will be zero. For an example, the command (voltage) wants the motor, Block Ŕ, to produce a torque as much as the command itself. Block Ŕ can change its powered voltage to have a correct output (torque). Block H, the torque sensor, converts torque into the voltage which shall be compared with the command. The difference will be used by Block Ŕ to adjust its powered voltage accordingly. Fig.1-3 shows that Control Designing is adaptive.
Fig. 1-2 Block Diagram of Control Designing for the Biped Robot. See next page for explanation.
Fig. 1-2  Explanation of Fig.1-2

B1: accepts variables $\theta_j$, $\dot{\theta}_j$, and $\ddot{\theta}_j$ at time $t = (n - 3) \Delta t$, $(n - 2) \Delta t$, and $(n - 1) \Delta t$ which measured by the appropriate sensors. And it sends out planned $\theta_j$, $\dot{\theta}_j$, and $\ddot{\theta}_j$ for $t = n \Delta t$.

B2: is not a functional block, just showing what have been sent out from B1.

B3: is the biped dynamic model, being composed of six dynamic equations.

B4: executes torque computation, a computer with a software.

B5: helps to reduce the burden of B4, a computing method.

B6 – B11: is a torque servomechanism.

B12: is the whole biped robot, the lower limbs and the upper limbs.

B13: is the vestibular systems, including the otolith organs and the semicircular canals as well as the other sensors.

B14: is a high gain function which is just mentioned without the details. Between B2 and B14, there is a switch. The switch is currently open.

This figure shows the functional relationship among Stride Planning, Control Designing, and Dynamic Model (Refer to Fig. 1-1). This figure contains a model referenced control about torques and a model referenced control about geometric variables $\theta_j$, $\dot{\theta}_j$, and $\ddot{\theta}_j$. Each servomechanism (for an example, B6, servo. 1) accepts electric
signals and generates mechanical torque. The servomechanism is various depending on different engineering preference. The response should be quicker or meet the requirement of Stride Planning. <End of Fig. 1-2>
Fig. 1-3 Reduced Diagram of Fig. 1-2. Only single servo loop is shown. To understand this diagram, refer to Fig. 1-4.

Fig. 1-4 Reduced Diagram of Fig. 1-3. In this diagram, a basic control loop is shown.

Fig. 1-5 Detailed Diagram of Block G in Fig. 1-4. Block G has two inputs. The command signal for G varies with time. The output will be measured by Block H in Fig. 1-4. The measured value is compared with the command. The difference between both is used to adjust Block $\bar{G}$. 
CHAPTER 2 ANALYTIC SOLUTION OF AN OTOLITH ORGAN MODEL

2.0 Introduction

The otolith organs and semicircular canals are contained in the nonauditory section of the inner ear and comprise the vestibular apparatus [1,2,3,4]. The vestibular organs are contained in hollowed-out section of the temporal bone of the skull. The semicircular canals respond to angular acceleration of the skull and the otolith organs respond to linear acceleration of the skull as well as to gravity. Their physiological function is that of motion and gravity transducers for the central nervous system. Their signals are used primarily for visual imaging fixation during the period of motion, muscle coordination and posture control during static as well as dynamic movement, and as one's perceptual sensation of motion.

The semicircular canals have been studied extensively. They have been shown to transduce skull motion into skull angular velocity and send this signal to the central nervous system. They form a second order system which is highly overdamped.

The otolith organs are linear sensors and have not been extensively studied. They form a second order system, which is
overdamped. The inertial mass or "proof" mass of an otolith organ is composed of a thin relative flat layer of dense calcium carbonate crystal, called otoconia, which are bound together by a weblike structure into a conglomerate rigid mass, called the otoconial membrane or layer. The otoconial membrane is in contract with a fluid called endolymph on one side and the opposite side is rigidly attached to thin, extremely deformable gelatinous layer of material, called the gel layer. The base of the gel layer is rigidly attached to the skull by a connective tissue layer. The cross-sectional thickness of the otoconial and gel layers have been measured and both are approximately 15\(\mu\)m in human's. The otoconial membrane has a surface area of approximately 2mm\(^2\). The calcium carbonate crystals in otoconial membrane have a specific gravity of 2.71, however the overall specific gravity of this layer is approximately 1.3.

Any linear acceleration of the skull results in accelerating the entire inner ear including the contained endolymph fluid. The component of this acceleration in the plane of the otoconial membrane accelerates the connective tissue base and endolymph; however, the otoconial membrane tends to remain at rest, and lags behind due to its inertia. The resulting relative motion of otoconial membrane w.r.t. its connective tissue base deform the gel layer in shear. In a like manner, any component of the gravity vector in the plane of the otolith will cause relative shear movement between the
otoconial membrane and its connective tissue, again, deforming the cupular membrane.

Sensory cells are located in the connective tissue base below the cupular membrane. These sensory cells have hairlike structures which penetrate the gel layer and respond to the deformation produced by the relative motion between the otoconial membrane and its connective tissue base. The sensory cells then transduce the gel layer shear deformation into trains of nerve action potentials which are carried by the vestibular nerve to the central nervous system. These components of otoconial membrane, gel layer, and sensory cell base, comprise the otolith organs.

The sensory cells of the otoliths are directionally sensitive, which allows for transduction of movement of the otoconial membrane in any direction of its plane. The planes of the two otoliths of an ear are orthogonal to one another and as a unit can respond to acceleration and gravity in three mutually perpendicular coordinated directions associated with the skull.

The relative displacement of the otoconial membrane with respect to its connective tissue base, or more simply with respect to the skull, is the chief measure of skull motion in the organ. A few previous attempts have been made to describe the system with a second-order lumped parameter model [1,2,3,4].
These models were relatively inconclusive since numerical values for all the lumped parameters were not ascertained. Experimental evidence indicates that the system is highly overdamped.

The objective here will be to develop a mathematical model which will describe the physical nature of the otoconial membrane displacement with respect to the skull. This model will show that the displacement is proportional to the sum of skull acceleration and gravity.

2.1 Problem Formulation

The otoconial membrane will be considered a rigid flat plate of thickness $b$ and density $\rho_0$, and having infinite extent. The edge effects of the actual organ, which has finite extent, can be shown to be negligible due to the small thickness $b$ (15\,\mu m) in relation to its flat plane area (2\,mm$^2$). For the present formulation, only a small element with surface area $dA$ of the plane will be considered. This plane is in contact with the endolymph fluid above and the gel layer below. A two-dimensional cross section of this elemental area is shown in Fig.2-1, where the gel layer is shown in a deformed state of simple shear. The displacement of the otoconial membrane w.r.t. the skull is $\delta c$ and $V_s$ is the velocity of the skull. A photomicrograph of histologic section of an otolith is shown in
Fig. 2–1 Diagram of Deformed Otolith Element. See Fig. 2–3 for a general view, and see Fig. 2–2 for more information.

Fig. 2–2 Free-Body Diagram of Otolith Layers [1, 2, 3, 4].
Fig. 2-3 Figure of the modeled 1-D Otolith Organ. This is a figure showing a modeled otolith organ. Please refer to Appendix 5 or [1,2].
reference [5]. What follows in this section is a compilation of reference [1,2,3,4], and is presented for the readers understanding of the problem.

The otolith organ structure shown in Fig.2-1 can be divided into three elemental parts: the endolymph fluid section, the otoconial membrane treated as a rigid plate, and the deformable gel layer. A free-body diagram of each elemental part is shown in Fig.2-2 with its associated forces. Note that the fluid shear stress $\tau_x$ couples the endolymph fluid to the top of the otoconial membrane plate and the elastic shear stress $\tau_x$ couples the cupular membrane to the plate. Governing equations of motion will now be developed for each of the three elemental parts.

Considering the otoconial membrane plate first and applying Newton's second law of motion of the elemental volume shown in Fig.2-2 yields

$$-\tau_{xx} \, dA - \tau_{ux} \, dA - \tau_{fx} \, dA + w_x + B_x = \rho_0 \, (bdA) \, \frac{dv_x}{dt}$$  \hspace{1cm} (2.1)

where $w_x$ is the weight component in the $x$ direction, $B_x$ is the buoyant force component in the $x$ direction, and $v_x$ is the velocity of the otoconial membrane with respect to an inertial reference frame. Expressions can now be developed to represent the force, shear stresses, and velocity in this equation.

(a) The fluid shear stress can be represented as

$$\tau_{fx} = -\mu \frac{\partial u_x}{\partial y}$$  \hspace{1cm} (2.2)
where the velocity gradient is evaluated at the plate wall \( y=0 \) as indicated by the subscript zero.

(b) The weight component can be expressed as

\[
w_x = \rho_0 g_x b \, dA \tag{2.3}
\]

where \( g_x \) is the component of gravity in the \( x \) direction.

(c) The buoyant force component in the \( x \) direction is presented during any period where there is a pressure gradient presents in the fluids contained in the inner ear. The pressure gradient is generated by gravity and acceleration of the skull. Fig.2-3 shows a closed container (skull) filled with fluid (endolymph) and an object submerged in the fluid. The buoyant force \( B \), on this submerged object is given by the surface integral

\[
B = - \int_S P \, n \, ds
\]

where \( n \) is an outward facing unit normal vector, \( P \) is pressure and \( ds \) is an elemental surface area. This expression can be converted into a volume integral and becomes

\[
B = - \int \nabla P \, n \, d\hat{V}
\]

where \( d\hat{V} \) is the elemental volume. The pressure gradient is constant over the volume of the submerged object, which allows the buoyant force to be written as

\[
B = - \nabla P \, \hat{V}
\]

and the \( x \) component of this force becomes
\[ B_x = -\frac{\partial p}{\partial x} b \, dA \]

The pressure gradient in the x direction is

\[ \frac{\partial p}{\partial y} = \rho_r \left( g_x - \frac{\partial v_x}{\partial t} \right) \quad (2.4) \]

and the buoyant force then becomes

\[ B_x = \rho_r \left( \frac{\partial v_x}{\partial t} - g_x \right) b \, dA \quad (2.5) \]

Note that a positive buoyant force is generated by a positive skull acceleration and negative gravity component.

(d) The elastic shear stress is developed from linear elasticity theory. The cupular membrane is composed of a gelatinous mucopolysaccharide material. Gelatinous materials in general are thought to be isotropic, linearly elastic, and deform without volume change. With these assumptions and simple shear deformation the elastic shear stress can be represented as the product of the shear modulus and the displacement gradient

\[ \tau_{xy} = G \frac{\partial \delta}{\partial y} \quad (2.6) \]

where \( G = \frac{E}{2(1 + \nu)} \)

Incompressibility requires \( \nu = 0.5 \). For a shear stress \( \tau_x \) at the otoconial membrane, the displacement gradient must be evaluated at \( y = c \), the otoconial membrane surface. In the simple shear deformation shown in this figure here, the displacement gradient is constant over the entire range of \( y (0 \leq y \leq c) \). This is not necessary the case in the general solution. Note also that the \( y \) coordinate is in the elastic system, is not to be confused
with the fluid system.

Converting the displacement to a velocity integral using

$$\delta(y, t) = \int_0^t w(y, \tau) \, d\tau$$

Eq. (2.6) becomes

$$\tau_{ex} = G \int_0^t \left. \frac{\partial w_x}{\partial y} \right|_0^c \, d\tau$$

(2.7)

where the subscript c indicates evaluations of the derivative at \( y = c \).

(e) The elastic viscous stress is developed from viscoelastic theory.

$$\tau_{ux} = \mu_s \left. \frac{\partial w_x}{\partial y} \right|_c$$

(2.8)

(f) The relative motion of the otoconial membrane with respect to the skull is of chief importance in this formulation. The inertial velocity is then written as the sum of the relative velocity of the otoconial membrane with respect to the skull and the velocity of the skull with respect to an inertial reference frame, \( \nu_s \), as

$$\nu_I = \nu + \nu_s$$

(2.9)

Utilizing Eq. (2.2), (2.3), (2.5), (2.7), (2.8), and (2.9),
Eq. (2.1) becomes

\[ \rho_0 b \frac{dv_x}{dt} + (\rho_0 - \rho_f) b \left( \frac{dv_{ux}}{dt} - g_x \right) = \mu_r \frac{\partial v_x}{\partial y} \bigg|_b - G \int_0^t \frac{\partial w_x}{\partial y} \bigg|_c \bigg|_c - \mu_s \frac{\partial w_x}{\partial y} \bigg|_c \]

(2.10)

The first item on the left represents the inertial motion of the otoconial membrane with respect to the skull. The second term contains the two stimuli to otoconial membrane movement, gravity and skull acceleration. These two stimulus terms are multiplied by \((\rho_0 - \rho_f)\) indicating that if the otoconial membrane density were not greater than the fluid density the system would not be perturbed. The first term on the right side of the equal sign represents endolymph fluid viscous effects on the system and the last term represents gel layer viscous effects on the system and the next to the last term represents the gel layer elastic effects. The fluid and the the elastic terms couple this equation to field equations which govern fluid and elastic motion.

In order to develop these fluid and elastic field equations, the endolymph fluid will be considered first. The fluid is shown as the upper element of the total system in Fig. 2-2. The linearized Navier-Stokes equation in the x direction describes the fluid motion

\[ \rho_f \frac{\partial u_t}{\partial t} = \rho_f g_x \frac{\partial p}{\partial x} + \mu_r \frac{\partial^2 u_t}{\partial y^2} \]  

(2.11)
where \( \mathbf{u}_i \) is the fluid velocity with respect to an inertial reference frame and

\[
\mathbf{u}_i = \mathbf{u} + \mathbf{v}_i
\]  

(2.12)

where \( \mathbf{u} \) is the velocity of the endolymph measured with respect to the skull.

The relative velocity sum is again introduced because we are interested in relative motion with respect to the skull. The pressure gradient is the same as before (Eq.(2.4)), and when this and the inertial velocity are substituted into Eq.(2.11), it becomes

\[
\rho_r \frac{\partial \mathbf{u}}{\partial t} = \mu_r \frac{\partial^2 \mathbf{u}}{\partial y_i^2}
\]  

(2.13)

Note that that the stimulus term \( \mathbf{v}_i \) and \( \mathbf{g}_i \) add out of the fluid field equation and fluid flow can be initiated only through movement of the otoconial membrane plate. The boundary conditions for the fluid Eq.(2.13) are

\[
u(\infty,t) = 0
\]

\[
u(0,t) = \mathbf{v}(t)
\]  

(2.14)

The fluid velocity at the plate surface is equal to the velocity of the plate. Thus the otoconial membrane plate is the source of fluid motion. The velocity at infinity is equal to zero. This boundary condition is justified on the basis of large fluid
depth above the plate in relation to its small displacement.

The field equation governing gel layer motion is developed from the momentum equation. The gel layer considered here is shown as the lower element in Fig.2-2. The momentum equation in x direction is

\[
\rho_\text{g} \frac{\partial w}{\partial t} = \rho_\text{g} g_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{yx}}{\partial y},
\]

(2.15)

where \( w \) is the velocity of the cupular membrane with respect to an inertial reference frame and

\[
w = w + v_x,
\]

(2.16)

where \( W \) is the gel velocity measured with respect to the skull, again the transformation to relative motion is introduced.

The density in Eq.(2.15) is that of the gel material. For the practical purposes of this formulation, the gel material density can be considered equal to that of the endolymph fluid. The pressure gradient in the cupular material is equal to that of fluid, Eq.(2.4), since each end of the gel layer is in contact with the endolymph fluid. The elastic shear stress is the same as Eq.(2.6). Utilizing this expression, and the integral of velocity with respect to time to represent displacement, Eq.(2.15) becomes

\[
\rho_\text{f} \frac{\partial \bar{w}}{\partial t} = \int_0^l \frac{\partial^2 w}{\partial y^2} \, dt + \mu_\text{s} \frac{\partial^2 w}{\partial y^2},
\]

(2.17)
This field equation governs gel layer displacement and again the stimulus terms add out of the equation. The boundary conditions for the field equations are

\[ w(1,t) = v(t) \]
\[ w(0,t) = 0 \]  

(2.18)

The velocity of the gel material is equal to that of the otocional membrane at their interface, and the gel material velocity is equal to zero at the skull.

The equation set Eq.(2.10),(2.13), and (2.17), with their associated boundary conditions, form a coupled set of equations which govern the motion of the otocional membrane in response to skull acceleration and gravity stimuli.

2.2 Nondimensionalization

The set of governing equations and boundary conditions can be nondimensionalized in order to determine the relative important nondimensional variable, denoted by circumflex "\(^\wedge\)" will be utilized

\[ \hat{\gamma} = \frac{y}{b} \text{ and } \hat{\tau} = \frac{\mu}{\rho_o b^2} \]

for space and time, and for velocities

\[ \hat{u} = \frac{u}{V} \quad \hat{v} = \frac{v}{V} \quad \hat{w} = \frac{w}{V} \quad \hat{v}_t = \frac{v_t}{V} \]  

(2.19)

where \( V \) is some characteristic velocity in the problem. The governing equations then become:
For the fluid,

$$R \frac{\partial \bar{u}}{\partial t} = \frac{\partial^2 \bar{u}}{\partial y_t^2}$$

(2.20)

with boundary conditions

B.C \quad \bar{u}(\infty, t) = 0

\bar{u}(0, t) = \mathcal{V}(t)

For the otoconial membrane plate,

$$\frac{d\tilde{\mathcal{W}}}{dt} + (1 - R) \left[ \frac{d\tilde{\mathcal{W}}}{dt} - \bar{g}_x \right] = \frac{\partial \bar{u}}{\partial y_t} \bigg|_0 - \varepsilon \int_0^t \frac{\partial \tilde{\mathcal{W}}}{\partial y_t} \bigg|_0 \ dt - m \frac{\partial \tilde{\mathcal{W}}}{\partial y_t} \bigg|_0$$

(2.21)

For the elastic cupular membrane,

$$R \frac{\partial \tilde{\mathcal{W}}}{\partial t} = \varepsilon \int_0^t \frac{\partial^2 \tilde{\mathcal{W}}}{\partial y_t^2} \ dt + m \frac{\partial^2 \tilde{\mathcal{W}}}{\partial y_t^2}$$

(2.22)

with boundary conditions

\tilde{\mathcal{W}}(t, 0) = \mathcal{V}(t)

\tilde{\mathcal{W}}(0, t) = 0

where the three naturally occurring nondimensional parameters

$$R = \frac{pr}{\rho_0} \quad m = \mu_A / \mu_t$$

(2.23)

and

$$\varepsilon = C \frac{b^2 \rho_0}{\mu^2 \mu_0}$$

(2.24)

were used, and

$$\bar{g}_x = b^2 \rho_0 g_x / \mu \nu$$

(2.25)

is the nondimensional gravity.
The coefficient $\varepsilon$ in Eq.(2.21) will determine the relative magnitude of the elastic term in relation to the inertial and viscous terms which both have coefficients of one. The sum of the stimulus terms is also multiplied by the factor $(1-R)$ which changes their relative magnitude with respect to the inertial and viscous terms. Again note that when $\rho_0=\rho_f$, there is no disturbance to the system.

The numerical value of $\varepsilon$, $m$ and $R$ can be determined using physical parameters for humans [4], and previous numerical solution. These parameters are

$$R = 0.75$$
$$m = 5 \text{ to } 20$$
$$\varepsilon = 0.01 \text{ to } 0.2$$

The resulting small value indicates that the elastic term is relatively unimportant in relation to the viscous and inertial terms in Eq.(2.21). Since Eq.(2.21) governs the displacement of the otoconial membrane, it can be concluded that the elastic terms are almost negligible, and otoconial membrane displacement is dominated by inertial and viscous effects.

This small value of $\varepsilon$ suggests that a meaningful solution to the problem can be developed for $\varepsilon=0$. A solution with $\varepsilon=0$ will completely eliminate the elastic term in Eq.(2.22) and (2.21). The resulting problem then simplifies to a plate in
contact with a viscous fluid which fills an infinite half
space above and a viscous gel layer below.

For clarity, the above governing equations are re-written
as below without the "^"(circumflex).

The equation of the endolymph field:

\[
R \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} \quad (2.26)
\]

I.C. \( u(y,0^+) = 0 \) \quad (2.27)

B.C. \( u(\infty,t) = 0 \) \quad (2.28)

\( u(0,t) = v(t) \) \quad (2.29)

The system of the gel layer:

\[
R \frac{\partial w}{\partial t} = \varepsilon \int_0^t \frac{\partial^2 w}{\partial y^2} dt + m \frac{\partial^2 w}{\partial y^2} \quad (2.30)
\]

I.C. \( w(y,0^+) = 0 \) \quad (2.31)

B.C. \( w(1,t) = v(t) \) \quad (2.32)

\( w(0,t) = 0 \) \quad (2.33)

The equation of the otoconial layer:

\[
\frac{\partial v}{\partial t} + (1 - R) \left( \frac{\partial v_x}{\partial t} - g_x \right) = \frac{\partial u}{\partial y} \int_0^t \frac{\partial w}{\partial y} dt - m \frac{\partial w}{\partial y} \quad (2.34)
\]

I.C. \( v(0^+) = 0 \) \quad (2.35)

Where subscript "s" indicates "gel" and subscript "f" indicates
"fluid"
In the next section, Section 2.3, an analytic solution will be determined.

2.3 Analytic Solution

The governing equation were given by [4]. The following is original work by the author. For the fluid (endolymph) field, the governing equation is:

\[
R \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y_f^2} \quad \text{(2.26)}
\]

I.C. \quad u(y_f,0^+) = 0 \quad \text{(2.27)}

B.C. \quad u(\infty,t) = 0 \quad \text{(2.28)}

\quad u(0,t) = v(t) \quad \text{(2.29)}

By applying Laplace transformation method Eq. (2.26) becomes:

\[
R[S \bar{u} - u(y_f,0^+)] = \frac{d^2 \bar{u}}{dy_f^2}
\]

where bar "-" represents Laplace Transformation of a variable, and S is the Laplace Transform variable.

Using Eq. (2.27), then,

\[
\frac{d^2 \bar{u}}{dy_f^2} - RS \bar{u} = 0
\]

Solution to the above ordinary differential equation is:
\[ \bar{u} = \alpha e^{-\sqrt{y_2}} + \beta e^{\sqrt{y_2}} \]

where \( \alpha \) and \( \beta \) are undetermined constants.

In order to satisfy the boundary condition at \( y_1 \to \infty, \beta = 0 \).

Hence,

\[ \bar{u} = \alpha e^{-\sqrt{y_2}} \]

(2.36)

Using the boundary condition at \( y_1 = 0 \), then gives:

\[ \bar{u}(y_1, S) = \bar{v}(S) e^{-\sqrt{y_2}} \]

where \( \bar{v}(S) \) is the otoconial layer velocity.

Taking the derivative and evaluating at \( y = 0 \), allows use of this result in the otoconial layer

\[ \frac{d \bar{u}}{dy_1} = \bar{v}(S) (-\sqrt{y_2}) e^{-\sqrt{y_2}} \]

(2.37)

\[ \frac{d \bar{u}}{dy_1} \bigg|_0 = -\sqrt{y_2} \bar{v}(S) \]

(2.38)

For the gel layer, the governing equation is:

\[ R \frac{\partial w}{\partial t} = \epsilon \int_0^t \frac{\partial^2 w}{\partial y_2^2} \, dt + m \frac{\partial^2 w}{\partial y_2^2} \]

(2.30)
I.C. \( w(y',0^+) = 0 \) \hspace{1cm} (2.31)

B.C. \( w(1,t) = v(t) \) \hspace{1cm} (2.32)

\( w(0,t) = 0 \) \hspace{1cm} (2.33)

By applying the Laplace Transformation method, Eq.(2.30) becomes:

\[
R[S \bar{w} - w(y',0^+)] = \varepsilon \frac{1}{S} \frac{d^2 \bar{w}}{dy_t^2} + m \frac{d \bar{w}}{dy_t}
\]

Using the initial condition (Eq.(2.31)), then,

\[
(\varepsilon \frac{1}{S} + m) \frac{d^2 \bar{w}}{dy_t^2} - R S \bar{w} = 0
\]

The solution to the above differential equation is:

\[
\bar{w} = C \sinh \sqrt{A} y_t + D \cosh \sqrt{A} y_t
\]

where

\[
\sqrt{A} = \sqrt{\frac{RS^2}{\varepsilon + Sm}}
\]

Using the boundary condition (Eq.(2.33)), \( \bar{w}(0,S) = 0 \), results in \( D = 0 \), and

\[
\bar{w} = C \sinh \sqrt{A} y_t
\]
And, using the other boundary condition (Eq. (2.32)), $\bar{w}(1, S) = \bar{v}(S)$. Gives:

$$\bar{w} = \bar{v}(S) \left( \frac{\sinh \sqrt{A} y_s}{\sinh \sqrt{A}} \right)$$  \hspace{1cm} (2.39)

Taking the derivative and evaluate at $y_s = 1$, for use in the otoconial layer equation.

$$\frac{\partial \bar{w}}{\partial y_s} = \bar{v}(S) \sqrt{A} \left( \frac{\coth \sqrt{A} y_s}{\sinh \sqrt{A}} \right)$$  \hspace{1cm} (2.40)

$$\left. \frac{\partial \bar{w}}{\partial y_s} \right|_0 = \bar{v}(S) \sqrt{A} \cosh \sqrt{A}$$  \hspace{1cm} (2.41)

For the otoconial layer, the governing equation is:

$$\frac{\partial \nu}{\partial t} + (1 - R) \left( \frac{\partial v_s}{\partial t} - g_s \right) = \frac{\partial u}{\partial y_s} \bigg|_0 - \varepsilon \int_0^t \frac{\partial \bar{w}}{\partial y_s} \bigg|_0 \, dt - m \frac{\partial \bar{w}}{\partial y_s}$$  \hspace{1cm} (2.34)

I.C. $\nu(0^+) = 0$  \hspace{1cm} (2.35)

By applying a Laplace Transformation and using the result from Eq. (2.38), Eq. (2.41) the otoconial equation becomes:

$$\bar{\nu}(S) + (1 - R) \left( \frac{\partial v_s}{\partial t} - g_s \right) = -\sqrt{RS} \bar{v}(S) - \frac{\varepsilon}{S} \bar{v}(S) \sqrt{A} \coth \sqrt{A} \\bar{v}(S) \sqrt{A} \coth \sqrt{A}$$

$$- m \bar{v}(S) \sqrt{A} \coth \sqrt{A}$$  \hspace{1cm} (2.42)

And upon rearrangement
\[ S \bar{v}(s) + (1 - R) K(s) = -\gamma RS \bar{v}(s) - \left( m + \frac{g_x}{s} \right) \bar{v}(s) \gamma^{A} \coth \gamma^{A} \]  

(2.43)

where \( K(s) \) is defined as

\[ K(s) \equiv L \left[ -\frac{\partial v_x}{\partial t} + g_x \right] = [-S \bar{v}_t(s) + v_r(0^*) + g_x(s)] \]

Eq. (2.43) is rearranged to become:

\[
\bar{v}(s) = \frac{- (1 - R)}{S + \gamma RS + \gamma R(mS + \varepsilon) \coth \gamma^{A}} K(s) = - (1 - R) K(s) \ E(s) 
\]

(2.44)

where

\[
E(s) = \frac{1}{S + \gamma RS + \gamma R(mS + \varepsilon) \coth \sqrt{\frac{RS^2}{\varepsilon + Sm}}}
\]

By taking the inverse Laplace Transformation and using convolution integration, the velocity \( v \) can be found as a function of time.

\[ v(t) = +(1 - R) \ K(t) \ast E(t) \]  

(2.45)
\[ K(t) = \mathcal{L}^{-1}[K(S)] = \mathcal{L}^{-1}[L \left( \frac{dv_x}{dt} - g_x \right)] = \frac{dv_x}{dt} - g_x \]

\[ E(t) = \mathcal{L}^{-1}[E(S)] \]

And * denotes convolution integration.

Therefore Eq. (2.45) can be rewritten as:

\[ v(t) = -(1-R) \left( \frac{dv_x}{dt} - g_x \right) * E(t) \]

\[ = (1-R) \left[- \frac{dv_x}{dt} + g_x \right] * E(t) \]

\[ = \left[ (1-R)E(t) \right] * \left[- \frac{dv(t)}{dt} + g_x(t) \right] \]

(2.46)

For Eq. (2.46), \( (1-R)E(t) \) * can be thought as an operator in the t-domain. Thus, for any \( v_x(t) \), Eq. (2.46) can provide the \( E(t) \) will be found in the next section. Therefore, substituting \( E(t) \) into Eq. (2.46) and inverse Laplace Transforming

\[ v(t) = \left[ (1-R) \frac{1}{(\pi t)^{1/2}} \int_0^\infty e^{-\sqrt{4t u}} \left( \frac{e^{-0.5R u}}{\sqrt{u - 0.25 R}} \sin(\sqrt{u - 0.25 R}) \right) du \right] * \left[- \frac{dv_x}{dt} + g_x \right] \]

(2.47)

The above expression, Eq. (2.47), is to show what \( v(t) \) looks like.
For the above case, \( m=5 \) and \( R=0.75 \) (\( \varepsilon \) is small), Eq.(2.47) becomes:

\[
\bar{v}(t) = 0.0363 \ t^{-1.5} \int_{0}^{\infty} u \ e^{-0.25 \ v^2 \ t^{1-0.433}} \ \sin(2.20 \ u) \ du \ast \left[-\frac{dv_z(t)}{dt} + g_z(t)\right]
\]

(2.48)

For the case \( v_z(t) = -U(t) \) and \( g_z(t) = 0, \ t > 0^+ \),

\[
\bar{v}(t) = 0.0363 \ t^{-1.5} \int_{0}^{\infty} u \ e^{-0.25 \ v^2 \ t^{1-0.433}} \ \sin(2.20 \ u) \ du
\]

(2.49)

2.3.1 Evaluation of \( E(t) \)

\( E(S) \) contains \( \coth \sqrt{R S^2 / m S + \varepsilon} \), a compound transcendental function of \( S \), as its factor. Therefore, this factor will be replaced by an approximate function of \( S \)

\[ \coth \sqrt{\frac{R S^2}{mS + \varepsilon}} \approx \coth \left( \sqrt{\frac{R}{m}} \sqrt{S} \right) \]

Since \( \varepsilon \) is small in regard to \( m S \), even for small \( S \) (see paragraph below). Then, using a Taylor's series expansion.

\[
\coth \left( \sqrt{\frac{R}{m}} \sqrt{S} \right) = \frac{1}{\sqrt{\frac{R}{m}} \sqrt{S}} + \frac{1}{3} \sqrt{\frac{R}{m}} \sqrt{S} - \frac{1}{45} \left( \frac{m^3}{R} \right) S^{3/2} + ...
\]

(2.50)
The reason for making \( \coth \sqrt{\frac{RS^2}{mS + \varepsilon}} \doteq \coth \left( \sqrt{\frac{R}{m} \sqrt{S}} \right) \) is because \( \varepsilon \) is small compared to \( m \) and \( R \). And, the solution for small \( t \) is the dominate part of initial solution where \( \varepsilon \) or the elastic effects are negligible. In Laplace Transformation, \( S \) is a parameter, and \( S \) is not arbitrary. \( S \) should be larger than another parameter \( S_0 \), such that \( |K(t)| \leq Me^{St} \), where \( K(t) \) is a function to be transformed and \( M \) is a positive number. In other words, for a function \( K(t) \) which is going to be Laplace transformed, \( S_0 \) should be larger enough to make it possible to find an \( M \) such that \( |K(t)| \leq Me^{St} \). For an example, \( \nu(t) = K(t) = V\sin\omega t \), where \( V \) is the magnitude and \( \omega \) is the angular velocity, \( S_0 = 0 \).

For \( S \) is small, then only the first term of the above series is required, and

\[
\coth \left( \sqrt{\frac{R}{m} \sqrt{S}} \right) \doteq \frac{1}{\sqrt{\frac{R}{m} \sqrt{S}}} \tag{2.51}
\]

Substitute Eq. (2.51) into \( E(S) \).

\[
E(S) = \frac{1}{S + \gamma RS + \gamma RM \sqrt{S} \coth \sqrt{\frac{RS^2}{mS + \varepsilon}}}
\]

\[
= \frac{1}{S + \gamma RS + \gamma RM \sqrt{S} \sqrt{\frac{R}{m} \sqrt{S}}}
\]

\[
= \frac{1}{S + \gamma RS + m \sqrt{\frac{R}{m} \sqrt{S}}}
\]

\[
= \frac{1}{(\sqrt{S} + 0.5\sqrt{R})^2 + (\sqrt{m} - 0.25)^2} \tag{2.52}
\]
Therefore, the inverse Laplace Transformation can be found in a set of tables [15], and is:

\[ E(t) = \frac{1}{(\pi t)^{1/2}} \int_0^\infty e^{-u/4t} \left( e^{-0.5\sqrt{m} u} \right) \sin \left( \sqrt{m} - 0.25R u \right) du \]  \hspace{1cm} (2.53)

Substitute \( m=5 \) and \( R=0.75 \) [4] into Eq. (2.53),

\[ E(t) = 0.142 \Gamma^{-1.5} \int_0^\infty u e^{-0.25 u^2 \Gamma^{-1.5}} \sin(2.20 u) du \]  \hspace{1cm} (2.54)

\( E(t) \) is decreasing very rapidly. This function is shown in Fig.2-4.

For the case of step change in skull velocity, \( v_4(t) = -U(t) \), then \( \frac{dv_4}{dt} = -\delta(t), s\to\infty \) at very small time. Hence, \( \coth \sqrt{\frac{RS^2}{mS + \epsilon}} \to 1 \). For convenience, denote \( \coth \sqrt{\frac{RS^2}{mS + \epsilon}} = \gamma \). Thus,

\[ E(S) = \frac{1}{S + \gamma RS + \gamma Rm \sqrt{S} \gamma} \]  \hspace{1cm} (2.55)

And, \( E(t) \) can be found by using a table of inverse Laplace Transformation as:

\[ E(t) = e^{(\sqrt{R} + \sqrt{Rm} \gamma)^2 t} \text{erfc}[\sqrt{\gamma R + \sqrt{Rm} \gamma} / \gamma t] \]  \hspace{1cm} (2.56)
Fig. 2-4 Diagram for Eq. (2.54). This figure shows the result of Eq. (2.54).
Fig. 2-5 Diagram for Eq. (2.56). This figure shows the result of Eq. (2.56).
E(t) is decreasing very rapidly. This function is shown in Fig. 2-5.

E(t) is a function which is composed of Eq. (2.53) and (2.56). At t very close to 0⁺, E(t) is expressed by Eq. (2.56). For t small but sufficiently away from t = 0⁺, E(t) is expressed by Eq. (2.53). For an example, if \( \nu_s(t) = -U(t) \), the E(t) is composed of \( \sqrt{\gamma} = 1 \) for Eq. (2.56) when \( t \) is very close to zero plus, and Eq. (2.53) when \( t \) is small but sufficiently away from zero plus. So-called "sufficient" depends on how good the evaluation of Eq. (2.53) is.

2.3.2 A Short Time Solution for \( \nu_s(t) \) and \( m = 0 \) When \( g_x = 0 \)

For a short time solution of \( m = 0 \) and \( g_x = 0 \), \( \varepsilon \) can be set to be zero because \( t \) is very small. And, the restoring force is negligible. The restoring force is provided by \( \varepsilon \). Besides, \( m > R > \varepsilon \). Therefore,

\[
\nu(S) = \frac{1 - R}{S + \sqrt{RS}} \tag{2.57}
\]

\[
\nu(t) = (1 - R) e^{Rt} \text{erfc}(\sqrt{Rt}) \tag{2.58}
\]

Eq. (2.58) is the solution shown in [1].
2.3.3 Evaluation of $v(t)$ at $t = 0^+$ and $t \to \infty$ for $v_s(t) = -U(t)$
and $g_x = 0$

It is helpful and can aid in understanding of the problem by applying the initial value and the final value theorem to examine $v(t)$ without solving $E(t)$ from $E(S)$ when gravity has vanished.

Applying the initial value theorem to Eq.(2.44),

$$\lim_{t \to 0^+} v(t) = \lim_{t \to 0} S\bar{v}(S)$$

$$= \lim_{t \to 0} \frac{1 - R}{1 + \sqrt{\frac{R}{S}} + \sqrt{\frac{R\epsilon + Rm}{S^2}} \coth\left(\frac{RS^2}{mS + \epsilon}\right)^{1/2}}$$

$$= 1 - R \equiv v(0^+) \quad (2.59)$$

Applying the final value theorem to Eq.(2.44),

$$\lim_{t \to \infty} v(t) = \lim_{t \to \infty} S\bar{v}(S)$$

$$= \lim_{t \to \infty} \frac{1 - R}{1 + \sqrt{\frac{R}{S}} + \sqrt{\frac{R\epsilon + Rm}{S^2}} \coth\left(\frac{RS^2}{mS + \epsilon}\right)^{1/2}}$$

$$= 0 \equiv v(\infty) \quad (2.60)$$

Note, observing Eq.(2.59), one can see that $\epsilon$ has no chance of showing much influence because the $\frac{R\epsilon}{S^2}$ factor contains $S^{-2}$ and shall vanish quicker, and the $\frac{RS^2}{mS + \epsilon}$ factor contains $S^2$ and
shall make \( \coth \sqrt{\frac{RS^2}{mS + \epsilon}} \) grow smaller much more rapidly. In Eq. (2.60), the \( \frac{Re}{S^2} \) factor and the \( \frac{RS^2}{mS + \epsilon} \) factor have the strongest influence on \( v(t) \) to force it go to zero. In other words, the elastic restoring force is not important at the beginning, but it is the strongest force to restore the system to its equilibrium position later in the problem. This factor supports that \( \epsilon \) is negligible for short time solution.

2.3.4 Evaluation of Otoconial Displacement at \( t = 0^+ \) and \( t \to \infty \)

For notational convenience, denote \( D(t) = \text{displacement} = \int_0^t v(t) \, dt \), the otoconial layer displacement with respect to the skull. Then by applying the initial value theorem and the final value theorem give:

\[
\lim_{t \to 0^+} D(t) = \lim_{S \to \infty} \frac{Sv(S)}{S} = \lim_{S \to \infty} \frac{1 - R}{S + \gamma RS + \gamma Re + RmS} \coth\left(\frac{RS^2}{mS + \epsilon}\right)^{1/2} = 0 = D(0^+) \quad (2.61)
\]

\[
\lim_{t \to \infty} D(t) = \lim_{S \to 0} v(S) = 0 = D(\infty) \quad (2.62)
\]

Eq. (2.59), (2.60), (2.61), and (2.62) reveal some physical properties of \( v(t) \) when \( g_x = 0 \).
2.3.5 Evaluation of Otoconial Displacement at $t = 0^+$ and $t \to \infty$

for No Skull Velocity Change

For the case $g_x = \text{constant} \neq 0$, denote $(1-R)g_x\left(\frac{1}{S}\right)E \equiv G_x$ in Eq.(2.44). There are:

$$\lim_{t \to 0^+} G_x(t) = \lim_{s \to \infty} SG_x(S)$$

$$= \lim_{s \to \infty} \frac{(1-R)g_x}{S + \gamma RS + \gamma Re + RmS \coth(RS^2/mS + \varepsilon)^{1/2}} = 0$$

(2.63)

$$\lim_{t \to \infty} G_x(t) = \lim_{s \to 0} SG_x(S)$$

$$= \lim_{s \to 0} \frac{(1-R)g_x}{S + \gamma RS + \gamma Re + RmS \coth(RS^2/mS + \varepsilon)^{1/2}} = 0$$

(2.64)

Besides, denote $D_x(S) \equiv (1-R)g_x\left(\frac{1}{S^2}\right)E$, the displacement due to $g_x \neq 0$. There are:

$$\lim_{s \to \infty} SD_x(S) = \lim_{t \to 0^+} D_x(t)$$

$$= \lim_{s \to \infty} \frac{(1-R)g_x}{S \left[ S + \gamma RS + \gamma Re + RmS \coth(RS^2/mS + \varepsilon)^{1/2} \right]} = 0$$

(2.65)

$$\lim_{s \to 0} SD_x(S) = \lim_{t \to \infty} D_x(t) \neq 0$$

(2.66)

The value of Eq.(2.66) is $(1-R)g_x/\varepsilon$ by using Eq.(2.6).
2.4 Discussion and Conclusion

When the governing equation, Eq. (2.34), is solved with the initial condition Eq. (2.35), this leads to the solution shown as Eq. (2.34). An I.C. like Eq. (2.35) is proper because Eq. (2.31) is thus assumed due to the thickness of the gel layer being very thin which was introduced at the beginning of this chapter. If time lag or wave propagation is considered, this will lead to quite complicated additional terms in Eq. (2.44). But these terms vanish quickly without making any contribution to the solution. Thus these effects can be ignored.

Review Eq. (2.59), \( \lim_{t \to 0^+} v(t) = 1 - R \neq v(0^+) \). It is proper that there is a jump at \( t = 0^+ \), from \( v(t) = 0 \) to \( v(t) = 1 - R \). The physical reason is at \( t = 0 \), the skull undergoes a \( v_s(t) = -U(t) \). At \( t = 0^+ \), the perturbation effects into the interior of the otolith organ with a very short elastic propagation time. As \( v_s(t) = -U(t) \) is a step function at \( t = 0 \), there is a jump of \( v(t) \) at \( t = 0^+ \) naturally making the \( v(t) \) jump from zero to \( 1 - R \) as the skull can be thus stimulated as \( -U(t) \).
CHAPTER 3 SKULL VELOCITY AND GRAVITY SENSING

3.0 Introduction

The goal of this chapter is to provide the logic for how the vestibular system can measure the skull acceleration as well as the direction of the gravitational field.

In Chapter 2, \( v(t) \), the otocional layer velocity, is a function of \( \frac{dv_s}{dt} - g_x \). Sec.3.1 provides a method to find \( v_s \) as a function of \( v(t) \) with an assumption that \( g_x \) is known. Because the skull may rotate, \( v_s \) is denoted as \( v_{so} \), the otolith velocity, to specify the position in the skull where a velocity is sensed.

Sec.3.2 provides a method to analyze and synthesize \( \bar{V}_R \) and \( \bar{V}_L \) (\( \bar{V}_R \): the right otolith velocity; \( \bar{V}_L \): the left otolith velocity) into \( \bar{V}_{s} \), the skull velocity, which is referred to a specified point. The point is midway between the two otolith organs. The angular velocity of the skull, \( \bar{\omega} \), is measured by the semicircular canals.

In Sec.3.3, a logic, which is expressed in mathematic form, is given to calculate the direction of the gravitational field. A human does not merely use the vestibular system but also the other kinds of sensors to determine the direction of gravitation. Only the vestibular systems are considered in this section for the academic interest and the closeness of this
work.

3.1 Acquisition of Otolith Velocity

For the purpose of Section 3.2, the velocity of each otolith organ or the velocity of each vestibule must be known. Recall Eq. (2.17),

$$\bar{V}(S) = (1 - R) [-S \bar{V}_i + V_s(0^*) + \bar{g}_x(S)] E(S)$$

Because the skull may rotate, the position within the skull where the otolith organ is located must be specified. For this purpose, a subscript "o" is used to specify "otolith". Then, Eq.(3.1) is rewritten as Eq.(3.2).

$$\bar{V}(S) = (1 - R) [-S \bar{V}_{i0} + V_{s0}(0^*)] E(S) + (1 - R) \bar{g}_x(S) E(S)$$

Eq.(3.2) can then be solved for $V_{s0}$ in term of $v(s)$.

$$S \bar{V}_{i0}(S) = \frac{-\bar{V}(S)}{(1 - R) S E(S)} + \frac{1}{S} V_s(0^*) + \frac{1}{S} \bar{g}_x(S)$$

Taking the inverse Laplace Transformation, Eq.(3.3) becomes

$$V_{s0}(t) = \frac{-1}{(1 - R) E(t)} \int_0^t v(t) dt + V_{s0}(0^*) + \int_0^t g_x(t) dt$$

where * is convolution integration. The velocity $v(t)$ is $v(t) = \int_0^t \frac{dv(t)}{dt} dt$, and $\frac{dv(t)}{dt}$ is measured by the cilia of the hair cells in
the otolith organs. $E(t)$ is solved in Chapter 2, $g_x(t)$ will be found in Sec. 3.3, and $V_{s0}(0^\circ)$ is just a constant (the inertial velocity at the otolith organ).

Thus when terms on the right side of Eq. (3.4) are known, $V_{s0}(t)$ can be determined. Extending $V_{s0}$ into a three dimensional case (3-D), the velocities of the right and left otolith become

$$
\omega_{R} V_{Rx} \quad \omega_{R} V_{ Ry} \quad \omega_{R} V_{Rz} \quad \mathbf{J}^T = \omega^R \tilde{V}_R
$$

(3.5)

$$
\omega_{L} V_{Lx} \quad \omega_{L} V_{ Ly} \quad \omega_{L} V_{Lz} \quad \mathbf{J}^T = \omega^L \tilde{V}_L
$$

(3.6)

where $\omega_{R} V_{Ri} = \omega_{R} V_{Ri}(t)$ is the component of $\omega^R \tilde{V}_R$ in $i$ direction.

Eq. (3.4) is for the one dimensional (1-D) case without specifying any direction. Therefore, each component in Eq. (3.5) and (3.6) can be found by using Eq. (3.4) individually. That is, for instance, $\omega_{R} V_{Rx}$ is obtained from $V_{s0}(0^\circ)$ for the $x$ direction. Therefore, the measurements of the otolith organ can be written as the above equations (refer to Fig. 3-1).

In the above equations, the left superscript "oR" or "oL" indicates the frame which the vector refers to. "I" is a scale factor which is defined in 4x4 matrix expression method [12]. Where the 4x4 matrix method or quaternion method is used for convenience (refer to Appendix 6&7 for an explanation). "T" means "transpose" of the matrix. When a point $i$ is used in "$i\tilde{V}_i$", this "i" is just a point to identify the point where a vector $\tilde{V}$ comes from as an arrow. "j" is the frame to which $\tilde{V}$ is with
Fig. 3-1 Figure of the Frame System Used in Chapter 3. Inside the skull, there are two vestibules. "R" and "L" denote "right" and "left" respectively. "o" denotes "otolith", for an example, the left side subscript "o" in oR or oRx specifies "otolith". But, in R or Rx, there is no left side subscript. This specifies "semicircular canals", for an example, R is the origin of R-frame, the right side semicircular canal frame.
respect. When a point $i$ is used in "$\vec{V}_i$", this $i$ also identifies the point where a vector $\vec{V}$ comes from as an arrow. But, this point "$i$" is not just a point, it is a frame having its directional sense. For details, please see [12,13] and Appendix 6&7. Refer to Fig.3-1, there two suitable points inside the two vestibules. Each point is used as the origin of the frame, i.e., $R$-frame and $oR$-frame, or $L$-frame and $oL$-frame respectively. $\vec{oR} \parallel \vec{sx}$, where point $s$ is at the midpoint of the segment $\vec{RL}$, and $\vec{sy}$ is a projected line which is parallel to $\vec{Ty}$. And, $\vec{R} \rightarrow R_x$ has a $-45^\circ$ rotation about $\vec{oR} \rightarrow R_z$, and has a $-30^\circ$ rotation about $\vec{oRs}$ from $\vec{oR} \rightarrow R_z$. Plane $xz$ is on the walking plane.

Quantities: $oV_{Rx}$, $oV_{Ry}$, $oV_{Rz}$, $oV_{Lx}$, $oV_{Ly}$, and $oV_{Lz}$ are the measured values by using Eq.(3.4), the equation for $V_{s0}$.

3.2 Acquisition of Skull Velocity and Acceleration

The semicircular canals measure the angular velocity of the skull

$$R[\omega_{Rx} \omega_{Ry} \omega_{Rz} 0]^T = R\vec{\omega} = S\vec{\omega} = L\vec{\omega} = L[\omega_{Lx} \omega_{Ly} \omega_{Lz} 0]^T \quad (3.7)$$

The relation of coordinate transformation are (refer to Appendix 6):
\[
\mathbf{\tilde{\omega}}^S = [I \ S][S \ R]^R\mathbf{\tilde{\omega}}^S = [I \ R]^R\mathbf{\tilde{\omega}}^S = [I \ S][S \ L]^L\mathbf{\tilde{\omega}}^S = [I \ L]^L\mathbf{\tilde{\omega}}^S
\]

\[
= [I \ S] \ S \mathbf{\tilde{\omega}}^S
\]

(3.8)

where matrix \([I \ J]_{4 \times 4}\) denotes a transformation, \(I\) and \(J\) are points, and \(I\) or \(J\) may also be the origins of frames. Refer to Fig.3-1, some transformation matrices can be found by inspecting geometric relation. They are:

\[
[L \ R] = \begin{bmatrix}
0 & 1 & 0 & -\frac{RL}{\sqrt{T}} \\
-1 & 0 & 0 & -\frac{RL}{\sqrt{T}} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(3.9)

\[
[R \ L] = \begin{bmatrix}
0 & -1 & 0 & -\frac{RL}{\sqrt{T}} \\
1 & 0 & 0 & \frac{RL}{\sqrt{T}} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(3.10)

\[
[S \ L] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & SL \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos -30^\circ & 0 & \sin -30^\circ & 0 \\
0 & 1 & 0 & 0 \\
-\sin -30^\circ & 0 & \cos -30^\circ & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos 45^\circ & -\sin 45^\circ & 0 & 0 \\
\sin 45^\circ & \cos 45^\circ & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

(3.11)
\[
[S \ R] = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & -SR \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos -30^\circ & 0 & \sin -30^\circ & 0 \\
0 & 1 & 0 & 0 \\
-\sin -30^\circ & 0 & \cos -30^\circ & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
\begin{bmatrix}
\cos -45^\circ & -\sin -45^\circ & 0 & 0 \\
\sin -45^\circ & \cos -45^\circ & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos 30^\circ \cos 45^\circ & \cos 30^\circ \sin 45^\circ & -\sin 30^\circ & 0 \\
-\sin 45^\circ & \cos 45^\circ & 0 & -SR \\
\sin 30^\circ \cos 45^\circ & \sin 30^\circ \sin 45^\circ & \cos 30^\circ & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
(3.12)
\]

Generally, inside [S L] and [S R], the following matrices, "-30°" and "±45°" are used. These values vary slightly from a person to a person.

From the knowledge of kinematics,

\[
1\vec{V}_R = I\vec{V}_S + I\vec{\omega}_S \times R\vec{p}_SR = [I \ S]^S\vec{V}_R
(3.13)
\]

\[
1\vec{V}_L = I\vec{V}_S + I\vec{\omega}_S \times R\vec{p}_SL = [I \ S]^S\vec{V}_L = 1\vec{V}_S - I\vec{\omega}_S \times R\vec{p}_SR
(3.14)
\]

Adding Eq. (3.13) and Eq. (3.14), yields:

\[
1\vec{V}_S = \frac{1}{2} (1\vec{V}_R + 1\vec{V}_L) = \frac{1}{2} ([I_o R] d\vec{V}_R + [I_o L] d\vec{V}_L)
(3.15)
\]

Recall Eq. (3.5) and Eq. (3.6), and that [I oR]=[I R]=[I S][S R], [I_o L]=[I L]=[I S][S L], it is known that
Eq. (3.15) provides $\mathbf{iV}_s$ a known quantity. $\mathbf{iV}_s$ is the velocity of point S (or skull) with respect to the initial frame. Examine Eq. (3.8) and Eq. (3.7), $\mathbf{i\omega}_s$ is known. Thus, both $\mathbf{iV}_s$ and $\mathbf{i\omega}_s$ are known.

Since $\mathbf{iV}_s$ and $\mathbf{i\omega}_s$ are known, for any point j inside the skull, there is:

$$\mathbf{iV}_j = \mathbf{iV}_s + \mathbf{i\omega}_s \times \mathbf{i\rho}_{sj} \tag{3.16}$$

For the special case of a skull which can rotate on a neck and the whole body is contacting on the ground, the line of rotation is determined with the sensors which installed at each joint of the limbs of the body. Suppose this rotation line is j. Then, for any point i inside the skull

$$\mathbf{iV}_i = \mathbf{iV}_j + \mathbf{i\omega}_j \times \mathbf{i\rho}_{ji} = \mathbf{iV}_j + \mathbf{i\omega}_s \times \mathbf{i\rho}_{ji}$$

$$= \mathbf{iV}_s + \mathbf{i\omega}_s \times \mathbf{i\rho}_{si} \tag{3.17}$$

Eq. (3.17) describes the general motion of the body, or biped robot, while it is on the ground.

### 3.3 Abstraction of Gravitational Vector and Vestibular Vector

Recall Eq. (3.4) for the velocity of an individual otolith, and remember that $g_x(t)$ is undetermined. In this section a
possible method to extract $g_x(t)$, the gravitational vector, from the given information is presented. This is probably how the brain carries out this determination and would be required for a biped robot to use this system.

$g_x(t)$ is the x direction component of $\vec{g}(t)$ with respect to an otolith frame. However, an otolith frame is not a fixed frame or inertial frame (I-frame). The following discussion is based on the right vestibule, the similar discussion can totally be made on the left side. oR-frame has a permanent relation with the R-frame. The semicircular canals can determine the posture or angular velocity of the R-frame w.r.t. I-frame. Therefore, the motion of oR-frame w.r.t. I-frame can be found.

The $\vec{g}$ is a fixed vector which is an invariant for a robot or human on the earth. For oR-frame, $\vec{g}$ shall have a set of projected components with respect to this frame. Eq. (3.18) is an expression for the above description.

$$\vec{g} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$$

$$= [I \ R(0)] \ R(0) \ \vec{g}(0)$$

$$= [I \ L(0)] \ L(0) \ \vec{g}(0)$$

where, the left side superscript "I" on $\vec{g}$ is used to emphasize with respect to I-frame; for I-frame, $\vec{g}$ has a set of projected components, namely $\begin{bmatrix} 0 & 0 & 1/g \end{bmatrix}^T$. $\ R(0) \ \vec{g}(0)$ represents a set of projected components with respect to R(0)-frame, the right side otolith frame at time 0. $[I \ R(0)]$ is a coordinate type transformation matrix to transform $\ R(0) \ \vec{g}(0)$ w.r.t. the I-frame at time 0.
Because the \( oR \)-frame is not a fixed frame, it may move in space. Therefore, Eq. (3.18) can be extended as Eq. (3.19) for \( t = n \Delta t \) (\( n \) is a positive integer and \( \Delta t \) is a convenient time step).

\[
\tilde{\mathbf{g}} = [I \ oR(0)] [oR(0) \ oR(\Delta t)] [oR(\Delta t) \ oR(2\Delta t)] \ldots
\]

\[
[oR((n-1)\Delta t) \ oR(n\Delta t)] [oR(n\Delta t)] [oR(n\Delta t) \ oR(n\Delta t) \ oR(n\Delta t)] \ldots
\]

(3.19)

Eq. (3.19) is written with discretized time intervals because any computation takes time of length \( \Delta t \). Eq. (3.19) is just a continued coordinate transformation (see Appendix 6). However, any coordinate transformation matrix (for an example, \([oR(\Delta t) \ oR(2\Delta t)]\) is a coordinate transformation matrix) transforms a vector or a matrix from one frame to another. For an example, \([oR(\Delta t) \ oR(2\Delta t)]\) transforms a vector from w.r.t. \( oR(\Delta t) \)-frame w.r.t. \( oR(2\Delta t) \)-frame.

For any \([oR((m-1)\Delta t) \ oR(m\Delta t)]\), there is:

\[
[oR((m-1)\Delta t) \ oR(m\Delta t)] = [oR((m-1)\Delta t) \ I] [I \ oR(m\Delta t)]
\]

\[
= [oR((m-1)\Delta t) \ R((m-1)\Delta t)] [R((m-1)\Delta t) \ I] [I \ R(m\Delta t)]
\]

\[
[R(m\Delta t) \ oR(m\Delta t)]
\]

(3.20)

where \( m \) is a positive integer less than \( n \).

Either \([oR((m-1)\Delta t) \ R((m-1)\Delta t)]\) or \([R(m\Delta t) \ oR(m\Delta t)]\) in the above equation is a constant matrix which is given by Eq. (3.12)
with \( \mathbf{-SR}=0 \) as an element of the matrix.

With

\[
\theta_i(t + \Delta t) - \theta_i(t) = \int_t^{t+\Delta t} \omega_i(t) \, dt
\]  

(3.21)

at each axis of \( \mathbf{R} \)-frame, for a given \( j \), where \( \omega_i \) is the angular velocity signal from the semicircular canals, and \( \theta_i \) is the angular displacement, then

\[
[\begin{bmatrix} I & \mathbf{R}(j \Delta t) \end{bmatrix}] = [\begin{bmatrix} I & \mathbf{R}(0) \end{bmatrix}] [\begin{bmatrix} \mathbf{R}(0) & \mathbf{R}(\Delta t) \end{bmatrix}] [\begin{bmatrix} \mathbf{R}(\Delta t) & \mathbf{R}(2\Delta t) \end{bmatrix}] \ldots [\begin{bmatrix} \mathbf{R}((j-1)\Delta t) & \mathbf{R}(j \Delta t) \end{bmatrix}]
\]  

(3.22)

Eq. (3.22) shows that \( [\begin{bmatrix} I & \mathbf{R}(j \Delta t) \end{bmatrix}] \) can be formed by a series of matrices, which are current frame type body rotation matrices (see Appendix 6).

Each matrix on the right side of Eq. (3.22) can be determined from Eq. (3.21), except \( [\begin{bmatrix} I & \mathbf{dR}(0) \end{bmatrix}] \), which shall be given below by Eq. (3.23). Therefore, all the matrices on the right of Eq. (3.19) are known. The following is the expression of

\[
[\begin{bmatrix} I & \mathbf{dR}(0) \end{bmatrix}] = [\begin{bmatrix} I & \mathbf{s}(0) \end{bmatrix}] [\begin{bmatrix} \mathbf{s}(0) & \mathbf{dR}(0) \end{bmatrix}]
\]  

(3.23)

where \( [\begin{bmatrix} I & \mathbf{S}(0) \end{bmatrix}] \) is assigned, and \( [\begin{bmatrix} \mathbf{S}(0) & \mathbf{oR}(0) \end{bmatrix}] \) is given by Eq. (3.12). Thus, all the matrices on the right side of Eq. (3.19) are known. \( \mathbf{\vec{g}} \) is known, therefore, \( \mathbf{\omega}^{(n\Delta t)} \mathbf{\vec{g}}(n \Delta t) \) is determined. That is, the set of projected components of \( \mathbf{\vec{g}} \) at time \( t \) with respect
to $\omega R(t)$-frame can be determined.

Recall Eq. (3.4) and the paragraph next to it.

$$V_{s0}(t) = \frac{-1}{(1-R)E(t)} \int_0^t v(t)dt + V_{s0}(0^+) + \int_0^t g(t)dt$$  \hspace{1cm} (3.4)$$

$v(t)$ is measured by the otolith organ, $E(t)$ is solved in Chapter 2, and $V_{s0}(0^+)$ is the initial value of the skull at the otolith organ which is zero for the case that the skull is originally at rest or assigned a value. For an example, though the earth is moving w.r.t. the inertial frame of the universe, a human or a robot on the earth can not feel the motion of the earth. When $g(t)$ is determined, all the unknowns on the right side of Eq. (3.4) are known. Then, $V_{s0}(t)$ can be determined. For more details, rewrite Eq. (3.4) with frame notations as:

$$\omega_{R(t)} V_{s0}(t) = \frac{-1}{(1-R)E(t)} \int_0^t \omega_{R(t)} v(t)dt + \omega_{R(0)} V_{s0}(0^+) + \int_0^t \omega_{R(t)} g(t)dt$$  \hspace{1cm} (3.24)$$

or

$$\omega_{L(t)} V_{s0}(t) = \frac{-1}{(1-R)E(t)} \int_0^t \omega_{L(t)} v(t)dt + \omega_{L(0)} V_{s0}(0^+) + \int_0^t \omega_{L(t)} g(t)dt$$  \hspace{1cm} (3.25)$$

Then, the utilization of Eq. (3.4) can be implemented.
The above discussion doesn't use the information about left side otolith, and the information for Eq.(3.21) doesn't specify right or left side of the semicircular canals because $\varphi_1$ is the same for the entire skull.

3.4 Discussion and Conclusion

Discussion

Section 3.1 shows how the acceleration or velocity at an otolith organ is measured. $R^{\Delta\varphi}(n\Delta t) = \mathbf{g}(t)$ in Eq.(3.19) is required by Eq.(3.4). With the information of Eq.(3.4) and that of the semicircular canals, Section 3.2 provides a method to determine skull velocity at point S (see Fig.3.1).

In this chapter, there are many coordinate frames used. Therefore, when dealing with vector or quantity, which frame that vector is referring to should be specified. For this purpose, left side superscript is used. And, for pointing out where the vector or quantity is located, a left side or right side subscript is used. Among so many coordinate frames, the matrices are all expressed by two variables to be named. Note, a frame is named as its origin, and the two variables for naming a matrix are the origins of two frames because the transformation is transforming from one point or frame to another. There are different types of the transformation, for more details, see Appendix 6.

Inside a robot, there are many parts which move relatively to one another. For convenience, the so-called quaternion
method is used to simplify the mathematic transforms. For more details, see Appendix 7 and (12). $\ddot{\omega}^s$ is assumed to be provided by the semicircular canals without any discussion since such a description is beyond the content of this work [19].

Conclusion

1. Chapter 2 provides a method to determine $v(t)$, the otoconial velocity, in terms of $\frac{dv_x}{dt}$. Section 3.1 provides $V_{s0}$, the skull velocity at the otoolith organ, in terms of $v(t)$ and $g_x$.

2. In Section 3.2, $\dot{R}(t)\vec{V}_{s0}(t)$, $\dot{\lambda}(t)\vec{V}_{s0}(t)$ and $\dot{\omega}^S$ can be synthesized to find $\dot{\vec{V}}_S$.

3. Section 3.3 develops a technique to determine $\dot{\lambda}(t)\vec{g}(t)$ or $\dot{R}(t)\vec{g}(t)$ which is needed in Section 3.1.

4. Because $\dot{R}(t)\vec{g}(t)$ can be found at any time, this sense of gravity can be used in Section 5.1 as well as it can be used in flying object (for an example, ballistic missiles) posture control.

5. $\dot{\omega}(t)\vec{g}(t)$ can be found with at least three mutually perpendicular semicircular canals and at least two mutually perpendicular otoconial layers.
CHAPTER 4 DYNAMIC MODEL OF THE BIPED ROBOT

4.0 Introduction

In this chapter the dynamic equations involved in controlling biped robot are developed. These equations give the moments required at joints based on the kinematics and kinetics of movements of the body segments. These are the control elements shown in Block B3 in Fig.1-2.

There are many different types of biped or two legged robots, however, only two different types of biped vertebrate animals. Birds are one type, and men and apes are the other. The foot is very important or key element necessary for two-legged robot or animals to achieve walking. This will be clear after carefully examining the dynamic equations of biped walking.

For each limbs locomotion of a biped robot, the mass distribution remains the same or almost the same when it is moving. For a man, the distribution is variable because the limbs are activated by the contraction of skeletal muscles, and the distribution differs position to position.

The upper limbs and torso of a human are flexible because there are many joints. Fortunately, the torso can be treated as a whole piece that connects to the lower limbs at the hip joints. The biped robot has the similar configuration.

Torsional type dampers and springs exist at each joint in a
human or a biped robot, but for simplicity, they will be ignored. And also, only two dimensional (2-D) walking is considered for simplicity. The complexities with these simplifying assumptions are tremendous, however, the analysis offers insight into the walking mechanism.

Please examine the figures of this chapter to understand the sense (positive or negative) of torques and details of the assumptions which are made. All figures in this chapter as well as the whole work have unified symbols and nomenclature.

4.1 Principles of the Analysis

Consider Fig.4-1, Fig.4-2 and Fig.4-3. "Soma" (defined as the whole body except legs and hands, and hands are not considered in this chapter) connects itself to the legs at joint J7 (please see Fig.4-1). O-frame is fixed to the space with oz parallel to the gravitation vector and pointing upwards. The robot walks in plane xz. The limbs of the legs are denoted by L1, L2, etc., where L1 is the segment between nodes 1 and 2, and having length $l_1$. Only limb L1, L2, L3 and L4 are assumed to have mass. Denote $\vec{H}$ as the angular momentum about J7; and $\vec{P}$ as the linear momentum about J7. Then, there are:

\[-\vec{M}_7 = \frac{d\vec{H}}{dt} - \vec{M}_4 \sim \vec{r}_i \times \vec{E}_i - \sum_{i=1}^{N} \vec{r}_i \times \vec{g} \Delta m_i + \sum_{i=1}^{N} \vec{r}_i \times \vec{J}_i \Delta m_i\]
Fig. 4-1 Figure of the Biped Robot System. A biped robot contains two parts, the soma (upper body) and the lower limbs. The soma is represented by a random circle because it is flexible. The lower limbs are represented by segments. All the flexure points are designated as nodes as are numbered. "CG" means center of gravity or center of mass for each segment. The right hand system is used for rotational sense. For an example, $\theta_2$ is negative value. "$\Delta$" doesn't mean "difference". $\Delta \theta$ means a completely new variable other than $\theta$. This is a double-letter symbol. Frame oxyz (or named as o-frame) is not arbitrary. It has $Oz$ always parallel with the gravitational direction, or upwards. "0" (zero) always goes with the supporting foot. "l" means length. Point 3, 7, and 8 are different points, see Fig.4-2. $d_i$ is the distance between joint $i$ and CGi ($i$-th limb center of gravity).
Fig. 4-2  Figure of the Linkage Between the Upper Half and the Lower Half of the Biped Robot. This figure shows how the soma links to the lower limbs.

Fig. 4-3  Figure of a Modeled Foot. This figure shows a modeled foot. Only points 9, 0, and 11 can touch the ground. And, points 0 and 1 are joints.
\[
\lim_{\Delta t \to 0} \frac{\ddot{H}(t + \Delta t) - \ddot{H}(t)}{\Delta t} = -\ddot{\bar{M}}_7 - \bar{M}_6 - \sum_{i=1}^{N} \vec{r}_i \times g \Delta m_i - \sum_{i=1}^{N} \vec{r}_i \Delta m_i \quad (4.1)
\]

\[
\ddot{H} = \int \vec{r} \times \dot{\vec{r}} \, dm = \sum_{i=1}^{N} \vec{r}_i \times \dot{\vec{r}}_i \Delta m_i \quad (4.2)
\]

\[
-\ddot{E}_7 = \frac{d\vec{E}}{dt} - \ddot{E}_4 - \sum_{i=1}^{N} \vec{g} \Delta m_i + \vec{1} \bar{7} m_7
\]

\[
= \lim_{\Delta t \to 0} \frac{\ddot{P}(t + \Delta t) - \ddot{H}(t)}{\Delta t} - \ddot{E}_4 - \sum_{i=1}^{N} \vec{g} \Delta m_i + \sum_{i=1}^{N} \vec{1} \bar{7} \Delta m_i \quad (4.3)
\]

\[
\ddot{P} = \int \vec{r} \, dm = \sum_{i=1}^{N} \vec{r}_i \Delta m_i \quad (4.4)
\]

Then \(\ddot{E}_7\) and \(\ddot{M}_7\) are the external loads to the legs, they will cause torques at each joints. These torques can be found by using Newtonian Mechanics. The motion of the legs cause torque at each joint, which can be derived by using Lagrange's equation. Combine both at each joint, which means the effect of motion and the effect of loadings are added up. In fact, proper sensors can be built at J3, J7 and J8 to measure \(\ddot{E}_7\), \(\ddot{M}_7\), \(\ddot{E}_3\), \(\ddot{M}_3\), \(\ddot{E}_8\) and \(\ddot{M}_8\). Then, without applying Eq.(4.1) and (4.3) to compute, they can be known by measurement.

### 4.2 Dynamic Equations of Biped Walking

By applying Lagrange's equation, and referring to Fig.4-1 and Fig.4-4, the coordinate of the gravity centers of the limbs respectively are:
Fig. 4-4 Free-Body Diagram for the Biped Robot System. This free-body diagram shows the force $E_i$ and moments $M_i$ acting on the lower limbs and soma. When the biped robot is treated as a two-element body.
\[ x_1 = -l_0 + d_1 \sin(\theta_0 + \theta_1 + \Delta \theta_1) \]
\[ z_1 = d_1 \cos(\theta_0 + \theta_1 + \Delta \theta_1) \]
\[ x_2 = -l_0 + l_1 \sin(\theta_0 + \theta_1) + d_2 \sin(\theta_0 + \theta_1 + \theta_2 + \Delta \theta_2) \]
\[ z_2 = l_1 \cos(\theta_0 + \theta_1) + d_2 \cos(\theta_0 + \theta_1 + \theta_2 + \Delta \theta_2) \]
\[ x_3 = -l_0 + l_1 \sin(\theta_0 + \theta_1) + l_2 \sin(\theta_0 + \theta_1 + \theta_2) + d_3 \sin(\theta_0 + \theta_1 + \theta_2 + \theta_3 + \Delta \theta_3) \]
\[ z_3 = l_1 \cos(\theta_0 + \theta_1) + l_2 \cos(\theta_0 + \theta_1 + \theta_2) + d_3 \cos(\theta_0 + \theta_1 + \theta_2 + \theta_3 + \Delta \theta_3) \]
\[ x_4 = -l_0 + l_1 \sin \varphi_1 + l_2 \sin \varphi_2 + l_3 \sin \varphi_3 + d_4 \sin (\varphi_4 + \Delta \theta_4) \]
\[ z_4 = l_1 \cos \varphi_1 + l_2 \cos \varphi_2 + l_3 \cos \varphi_3 + d_4 \cos (\varphi_4 + \Delta \theta_4) \]  

where
\[ \varphi_i = \sum_{j=0}^{i} \theta_j \]

Denote kinetic energy as KE and potential energy as PE.

\[ PE_1 = m_1 g d_1 \cos(\varphi_1 + \Delta \theta_1) \]  
\[ PE_2 = m_2 g \left[ l_1 \cos \varphi_1 + d_2 \cos(\varphi_2 + \Delta \theta_2) \right] \]  
\[ PE_3 = m_3 g \left[ l_1 \cos \varphi_1 + l_2 \cos \varphi_2 + d_3 \cos(\varphi_3 + \Delta \theta_3) \right] \]  
\[ PE_4 = m_4 g \left[ l_1 \cos \varphi_1 + l_2 \cos \varphi_2 + l_3 \cos \varphi_3 + d_4 \cos(\varphi_4 + \Delta \theta_4) \right] \]  
\[ KE_i = \frac{1}{2} m_i (\dot{x}_i^2 + \dot{z}_i^2) \]

where, \( i = 1, 2, 3, \) and 4.

The torque which is required to provide the motion is:

\[ T_i = \frac{d}{dt} \left( \frac{\partial KE}{\partial \dot{\theta}_i} - \frac{\partial KE}{\partial \theta_i} + \frac{\partial PE}{\partial \theta_i} \right) \]  

(4.11)
where \( i = 1, 2, 3 \) and 4. \( KE = \sum_{j=1}^{4} KE_j \), and \( PE = \sum_{j=1}^{4} PE_j \).

\( Ti \) are the torque required by leg motion.

Because they are very lengthy, explicit forms of \( Ti \) are given in Appendix 2.

Denote \( \bar{M}_j \) as the torque caused by \( \bar{E}_i \) at joint \( J_j \) (\( j \)-th joint). The value of \( \bar{M}_j \) are the moment at each joint produced by the joint loads.

\[
\bar{E}_3 \bar{M}_2 = 23 \times \bar{E}_3 = (E_{3x} l_2 \cos \varphi_2 - E_{3z} l_2 \sin \varphi_2) \bar{e}_y
\]

(4.12)

where \( E_{3x} \) is the \( \bar{E}_3 \) component in \( x \) direction, where \( 23 \) is the position vector from node 2 and node 3. Other position vectors are likewise designated. The joint moments are

\[
\bar{E}_3 \bar{M}_1 = 13 \times \bar{E}_3
\]

\[
\bar{E}_3 \bar{M}_4 = 43 \times \bar{E}_4 = -34 \times \bar{E}_8
\]

\[
\bar{E}_3 \bar{M}_5 = -35 \times \bar{E}_8
\]

\[
\bar{E}_3 \bar{M}_6 = [E_{3x} (l_5 \sin \varphi_3 + l_4 \sin \varphi_4) - E_{3x} (l_3 \cos \varphi_3 + l_4 \cos \varphi_4)] \bar{e}_y
\]

(4.13)

\[
\bar{E}_3 \bar{M}_5 = -35 \times \bar{E}_8
\]

\[
\bar{E}_3 \bar{M}_6 = [E_{3x} (l_5 \sin \varphi_3 + l_4 \sin \varphi_4 + l_5 \sin \varphi_5) - E_{3x} (l_3 \cos \varphi_3 + l_4 \cos \varphi_4 + l_5 \cos \varphi_5)] \bar{e}_y
\]

(4.15)
where $\vec{E}_1 = \vec{E}_5 + \vec{E}_8$. Besides, $\vec{M}_7 = \vec{M}_3 + \vec{M}_8$.

The individual torques required at each joints can now be calculated. Whether J8 (as shown in Fig. 4.2) is locked or unlocked makes the function different because different joint connection have different mechanical status. When J8 is locked, J8 connected to limb L2 equivalently. Therefore, the two cases should be shown in the following.

When J8 is locked, $\Gamma_i$, the net torque required at each joint are:

$$\Gamma_1 = T_1 - M_3 - E_3 M_{11}$$  
(4.17)

$$\Gamma_2 = T_2 + M_3 - E_3 M_2$$  
(4.18)

$$\Gamma_3 = T_3 - M_7 - E_4 M_6 + E_3 M_{13}$$  
(4.19)

$$\Gamma_4 = T_4 + M_8 + E_4 M_4$$  
(4.20)

$$\Gamma_5 = - M_8 + E_4 M_5$$  
(4.21)

where

$E_3 M_1 = E_3 M_{11} + E_3 M_{13}$

And,

$T_1 =$ motion torque

$\vec{M}_1 =$ reaction torque from external loads, and

$E_3 \vec{M}_j =$ reaction torque to external forces.

When J8 is unlocked, there are:

$$\Gamma_1 = T_1 - M_3 - E_3 M_{11}$$  
(4.17)

$$\Gamma_2 = T_2 + M_3 - E_3 M_2$$  
(4.18)

$$\Gamma_3 = T_3 - M_3 + E_4 M_6 + E_3 M_{13}$$  
(4.22)

$$\Gamma_4 = - T_{38} + M_8 - E_4 M_6$$  
(4.23)
\[ \Gamma_4 = T_4 + M_8 + E_4 M_4 \]  
\[ \Gamma_5 = -M_8 + E_8 M_5 \]  

where \( T_3 = T_{33} + T_{38} \). \( T_{33} \) or \( T_{38} \) is a portion of \( T_3 \) related to \( J_3 \) or \( J_8 \) respectively. The direction and sense of each torque is shown in Fig. 4-5.

The following is to discuss the forces at two peds related to the ground. Refer to Fig. 4-6 and Fig. 4-3. The frictional forces are very important to biped walking, therefore, the forces exerted by the ground are studied first. The forces written as magnitude and angular direction are

\[ \vec{E}_6 = -\frac{\Gamma_6}{L_5}, \varphi_4 - 90^\circ \]  
\[ \vec{E}_{11} = \frac{\Gamma_1}{L_{11}}, \varphi_0 + 90^\circ \]  
\[ \vec{E}_1 = \frac{\Gamma_2}{L_1}, \varphi_1 - 90^\circ \]  
\[ \vec{E}_5 = -\frac{\Gamma_4}{L_{46}}, \varphi_4 - q + 90^\circ \]  

where

\[ q = \tan^{-1} \left( \frac{L_5 \sin \theta_5}{L_5 \sin \theta_3 + L_4} \right) \]  
\[ L_{46} = \frac{L_5 \sin \theta_5}{\sin q} \]

Breaking the external exerted forces into components,

\[ \vec{E}_3 = E_{3x} \vec{e}_x + E_{3z} \vec{e}_z \]  
\[ \vec{E}_8 = E_{8x} \vec{e}_x + E_{8z} \vec{e}_z \]  
\[ \vec{e}_1 = \sin \varphi_1 \vec{e}_x + \cos \varphi_1 \vec{e}_z \]  
\[ \vec{e}_2 = \sin \varphi_2 \vec{e}_x + \cos \varphi_2 \vec{e}_z \]
Fig. 4-5 Figure For Showing the Sense of the Torque of Each Joint. This figure shows the details about the sense of the motors. The sense is for the rotor w.r.t. the stator (the shell or house of a motor).
Fig. 4-6 Figure for Showing the Forces Exerted by the Ground to the Peds. This figure shows $\vec{E}_1$, $\vec{E}_{11}$, $\vec{E}_5$, and $\vec{E}_6$. $\vec{E}_{j1}$ and $\vec{E}_{j5}$ (Eq. (4.28) and (4.29)) are the forces caused by the soma, which are not shown in this figure. These two forces are not delivered by the torques. The similar forces created by $m_1$, $m_2$, $m_3$, and $m_4$ are neglected compared to $\vec{E}_{j1}$ and $\vec{E}_{j5}$.
\[ \vec{e}_3 = \sin \varphi_3 \vec{e}_x + \cos \varphi_3 \vec{e}_z \]
\[ \vec{e}_4 = \sin \varphi_4 \vec{e}_x + \cos \varphi_4 \vec{e}_z \]
\[ \vec{e}_5 = \sin \varphi_5 \vec{e}_x + \cos \varphi_5 \vec{e}_z \]

The responses due to \( \vec{E}_3 \) or \( \vec{E}_8 \) at point 1 and point 5 respectively are:

\[ \vec{E}_{11} = -((\vec{E}_3 \cdot \vec{e}_2) \vec{e}_2 \cdot \vec{e}_1) \vec{e}_1 \]
\[ = -((E_{3x} \sin \varphi_2 + E_{3z} \cos \varphi_2)(\sin \varphi_2 \vec{e}_x + \cos \varphi_2 \vec{e}_z) \cdot (\sin \varphi_1 \vec{e}_x + \cos \varphi_1 \vec{e}_z)) \vec{e}_1 \]
\[ = -((E_{3x} \sin \varphi_2 + E_{3z} \cos \varphi_2)(\sin \varphi_2 \sin \varphi_1 + \cos \varphi_2 \cos \varphi_1)) \vec{e}_1 \]
\[ = -((E_{3x} \sin \varphi_2 + E_{3z} \cos \varphi_2)(\sin \varphi_2 \sin \varphi_1 + \cos \varphi_2 \cos \varphi_1))(\sin \varphi_1 \vec{e}_x + \cos \varphi_1 \vec{e}_z) \]
\[ \equiv E_{11x} \vec{e}_x + E_{11z} \vec{e}_z \quad (4.28) \]

\[ \vec{E}_{15} = -((\vec{E}_8 \cdot \vec{e}_3) \vec{e}_3 \cdot \vec{e}_5) \vec{e}_5 \]
\[ = -((E_{8x} \sin \varphi_3 + E_{8z} \cos \varphi_3)(\sin \varphi_3 \sin \varphi_5 + \cos \varphi_3 \cos \varphi_5)) \vec{e}_5 \]
\[ \equiv E_{15x} \vec{e}_x + E_{15z} \vec{e}_z \quad (4.29) \]

where

\[ E_{11x} = -\alpha \sin \varphi_1 \]
\[ E_{11z} = -\alpha \cos \varphi_1 \]
\[ E_{15x} = -\beta \sin \varphi_5 \]
\[ E_{15z} = -\beta \cos \varphi_5 \]
\[ \alpha = (E_{3x} \sin \varphi_2 + E_{3z} \cos \varphi_2)(\sin \varphi_2 \sin \varphi_1 + \cos \varphi_2 \cos \varphi_1) \]
\[ \beta = (E_{8x} \sin \varphi_3 + E_{8z} \cos \varphi_3)(\sin \varphi_3 \sin \varphi_5 + \cos \varphi_3 \cos \varphi_5) \]
Fig. 4-7 Comparison of a Robotic Arm and a Ped. The above figure shows a robotic arm is hinged to the ground, the lower figure shows a foot which is put on the ground. The ped which is discussed in this chapter is assumed to hinged to the ground first. Then, some conditions are adopted to recover the ped to put on the ground.
Because a foot is an object which is put on the ground, and it is not an object hinged to the ground, it is necessary to consider the condition of 1, non-skidding of the whole body both feet must not slide, and 2, non-skidding of any of the two feet (each individual foot must not slide). Biped walking will be restricted to non-skidding.

Eq.(4.30), below equation, is an approximately accurate equation showing that the sum of the lateral forces should be smaller or equal to the sum of the vertical forces times \( \mu \), the static coefficient of friction. The lateral force must be less than or equal to the maximum friction force. For no slip thus:

\[
\| E_1 \sin(\phi_1 - 90^\circ) + E_{11} \sin(\phi_0 + 90^\circ) + E_5 \sin(\phi_4 - q + 90^\circ) + E_6 \sin(\phi_5 - 90^\circ) \\
+ E_{J1x} + E_{J5x} \| \leq \mu [E_1 \cos(\phi_1 - 90^\circ) + E_{11} \cos(\phi_0 + 90^\circ) + E_5 \cos(\phi_4 - q + 90^\circ) \\
+ E_6 \cos(\phi_5 - 90^\circ) + E_{J1z} + E_{J5z}] 
\]

(4.30)

where \( \mu \) is the frictional coefficient.

Eq.(4.30) is the condition for non-skidding of the whole body. \( \bar{E}_1, \bar{E}_{11}, \bar{E}_5, \bar{E}_6, \bar{E}_{J1}, \) and \( \bar{E}_{J5} \) are the forces which are provided by the ground. For the supporting foot:

\[
\| E_1 \sin(\phi_1 - 90^\circ) + E_{11} \sin(\phi_0 + 90^\circ) + E_{J1x} \| \leq \mu [E_1 \cos(\phi_1 - 90^\circ) \\
+ E_{11} \cos(\phi_0 + 90^\circ) + E_{J1z}] 
\]

(4.31)

For the condition for non-skidding of the rear foot:
\[ ||E_5 \sin(\phi_4 + 90^\circ) + E_6 \sin(\phi_5 - 90^\circ) + E_{J5z}|| \leq \mu \left[ E_5 \cos(\phi_4 - q + 90^\circ) + E_6 \cos(\phi_5 - 90^\circ) + E_{J5z} \right] \] (4.32)

Besides Eq. (4.30), (4.31) and (4.32), the system should satisfy the following three conditions for making the peds contact the ground. Eq. (4.33) effects up to the swing ped leaving the ground.

\[
E_5 \cos(\phi_4 - q + 90^\circ) + E_6 \cos(\phi_5 - 90^\circ) + E_{J5z} \geq 0 \] (4.33)

\[
E_{11} \cos(\phi_0 + 90^\circ) + E_{J1z} \frac{I_0}{I_0 + I_{11}} \geq 0 \] (4.34)

\[
E_1 \cos(\phi_1 - 90^\circ) + E_{J1z} \geq 0 \] (4.35)

Unlike a biped robot, a human body has additional restrictions about joint rotation range. All the above non-skidding conditions are expected to be fulfilled automatically if a biped robot has similar dimensions and dynamic properties as the human body. For a biped robot which is not like a human body, a trial and error method will help find what a stride or strides should be used for no skid. Besides, Stride Planning (refer to Fig. 1-1 and Fig. 1-2) determines \( \Theta_j(t) \) for biped walking. The above development provides a function to control the actuator at each joint knowing how much torque is needed, no matter what \( \Theta_j(t) \) combination is ordered by upper controlling center.
4.3 Uniqueness and Controllability

Uniqueness

In Section 4.2, \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_8, \Gamma_4 \) and \( \Gamma_5 \) are determined, which are going to be added at each joint respectively. They are:

\[
\begin{align*}
\Gamma_1 &= T_1 - M_3 - \bar{E}_3 M_{11} & (4.17) \\
\Gamma_2 &= T_2 + M_5 - \bar{E}_3 M_2 & (4.18) \\
\Gamma_3 &= T_{33} - M_3 + \bar{E}_4 M_6 + \bar{E}_3 M_{13} & (4.22) \\
\Gamma_8 &= -T_{38} + M_8 - \bar{E}_4 M_6 & (4.23) \\
\Gamma_4 &= T_4 + M_8 + \bar{E}_4 M_4 & (4.20) \\
\Gamma_5 &= -M_8 + \bar{E}_4 M_5 & (4.21)
\end{align*}
\]

where

\( \Gamma_i \)-the net torque needed at \( i \)-th joint  \\
\( T_i \)-the torque caused by motion at \( i \)-th joint  \\
\( M_i \)-the external moment at \( i \)-th joint  \\
\( \bar{E}_3 M_j \)-the external moment caused by \( E_j \) at \( i \)-th joint  \\
\( \bar{E}_i \)-the external force at \( i \)-th joint

Besides, there are some moment relations which are introduced in Section 4.2, they are:

\[
\begin{align*}
T_3 &= T_{33} + T_{38} & (4.36) \\
\bar{E}_7 &= \bar{E}_3 + \bar{E}_8 & (4.37)
\end{align*}
\]
\[ \vec{M}_7 = \vec{M}_3 + \vec{M}_8 \]  
\[ (4.38) \]

\[ \varepsilon_3 M_1 = \varepsilon_3 M_{11} + \varepsilon_3 M_{13} \]  
\[ (4.39) \]

Where \( \vec{E}_j \equiv E_{jx} \hat{e}_x + E_{jz} \hat{e}_z \), for 2-D case, \( \hat{e}_x \) and \( \hat{e}_z \) are unit vectors. For Eq. (4.36), \( T_3 \) is obtained from applying Lagrange's equation. If \( T_{33} \) is known, \( T_{38} \) will be determined, and vice versa. For Eq. (4.36), there is one unknown ratio between \( T_{33} \) and \( T_{38} \) \( \left( \frac{T_{33}}{T_{38}} \right) \). For Eq. (4.37), \( \vec{E}_7 \) is given by the soma (robot or human). For the same reason as in Eq. (4.36), there are two unknown ratios between \( E_{3x} \) and \( E_{8x} \), and \( E_{3z} \) and \( E_{8z} \). For Eq. (4.39), \( \varepsilon_3 M_1 \) is obtained since \( \vec{E}_3 \) is known. For the same reason as in Eq. (4.36), there is one unknown ratio between \( \varepsilon_3 M_{11} \) and \( \varepsilon_3 M_{13} \). For Eq. (4.38), \( M_7 \) is given by the soma.

There are totally five unknown ratios. Here is the uniqueness question:

If \( \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_8, \Gamma_4 \), and \( \Gamma_5 \) are added into the robot system, can the above five unknown ratios be determined and uniquely determined?

For answering the above uniqueness question, some analysis is given as the following.

Recall Eq. (4.17), (4.18), (4.22), (4.23), (4.20) and (4.21), they are the equations of the net torques needed at each joint. They are totally six equations, but there are only five
unknowns. Examine these six equations, one can find that Eq. (4.21) and Eq. (4.20) are dependent on each other because both have terms which can be derived from the same source. That is, for both equations, they have the same $M_8$ and either $M_4$ or $M_1 M_8$ which are derived from $E_8$. $T_4$ is already known from Lagrange's equation. Therefore, Eq. (4.21) can be ruled out in analyzing uniqueness. Thus there are five equations, $T_1, T_2, T_3, T_5$ and $T_4$, for five unknowns.

Recall Eq. (4.36), $T_3 = T_{33} + T_{38}$. If either the unknown ratio is known ( $T_{33} / T_{38}$ ) or any one of T33 and T38 is known, both $T_{33}$ and $T_{38}$ will be determined. Eq. (4.37), which is equivalent to two equations $E_{7x} = E_{3x} + E_{8x}$ and $E_{7z} = E_{3z} + E_{8z}$, Eq. (4.38) $M_7 = M_3 + M_8$, and Eq. (4.39) $E_3 M_1 = E_3 M_{11} + E_3 M_{13}$ have the same analogy. There are totally five unknowns (or unknown ratios) and five simultaneous equations, $T_1, T_2, T_3, T_5$, and $T_4$. However, it is not easy to directly solve the five unknowns in terms of the other knowns to prove the uniqueness. Therefore, a technique will be used as follows: First, assume $E_{8x}$ and $E_{8z}$ are two knowns though they are actually two of the five unknowns. Then, prove the other three of the five unknowns can be expressed in terms of the knowns. Finally, the five equations will be deduced into two equations in terms of $E_{8x}$ and $E_{8z}$. Then, the two remained equations will be solved for $E_{8x}$ and $E_{8z}$.

Inspect Eq. (4.20). $M_8$ is determined since $T_4$ is what to be added into $J4$, $T_4$ is obtained from Lagrange's equation, and $E_3 M_6$
is obtained from assuming $\bar{E}_g$ is known. And, $\bar{M}_3$ is determined since $\bar{M}_8$ is determined.

Inspect Eq. (4.23), $T_{38}$ is determined since $M_8$, $\bar{E}_s M_6$ and $\Gamma_8$ are knowns. Then, $T_{33}$ is determined.

Inspect Eq. (4.17), $E_3 M_{11}$ is determined.

Inspect Eq. (4.22), $E_3 M_{13}$ is determined.

Then, re-inspect Eq. (4.22), re-arrange the terms

$$E_3 M_6 = \Gamma_3 - T_{33} + M_3 - E_3 M_{13} \quad (4.40)$$

Inspect Eq. (4.18), re-arrange the terms

$$E_3 M_2 = \Gamma_2 - T_2 - M_3 \quad (4.41)$$

Rewrite Eq. (4.41) as

$$\bar{E}_2 - \bar{E}_4 M_2 = \Gamma_2 - T_2 - M_3 \quad (4.42)$$

Rewrite Eq. (4.40) as

$$E_3 M_6 = \Gamma_3 - T_{33} + M_3 - \bar{E}_2 - \bar{E}_4 M_{13} \quad (4.43)$$

Clearly, Eq. (4.42) and (4.43) are two equations contain two unknown variables, $E_{8x}$ and $E_{8z}$. There are three cases, no solution, many sets of solutions, and unique solution set. Obviously, Eq. (4.42) and (4.43) can't be contradictory or compatible, that is, two lines parallel to each other or overlapped in $E_{8x}$-$E_{8z}$ Plane respectively. Therefore, both the cases of no solution and many sets of solutions are not true. $E_{8x}$
and $E_{3x}$ can only determined uniquely.

Thus, five unknowns (terms or ratios) can be uniquely determined.

Controllability

There is a question: How can it be true, if $\Gamma_1$, $\Gamma_2$, .... are added into the dynamic system at each joint, will the dynamic system act exactly as $\theta_i$ were designed? In other words, the above question is equivalent to assume:

1. $e_3M_{11}/E_3M_{13}$, $T_{33}/T_{38}$, $E_{3x}$, $E_{3z}$ and $M_8$ are set to be parameters (automatically, $E_{3x}$, $E_{3z}$ and $M_1$ also),

2. $T_1$, $T_2$, $T_3$, and $T_4$ are functions as:

   $$T_i = T_i(\theta_1, \theta_2, \theta_3, \dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3, \ddot{\theta}_1, \ddot{\theta}_2, \ddot{\theta}_3, \dddot{\theta}_1, \dddot{\theta}_2, \dddot{\theta}_3, \dddot{\theta}_4)$$

3. $\Delta \theta_1$, $\Delta \theta_2$, $\Delta \theta_3$ and $\Delta \theta_4$ are assumed to be differentiable and:

   $$\Delta \theta_i = \Delta \theta_i(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_8, \Gamma_4)$$

Is it possible to have unique $\theta_i$ as:

$$\theta_i = \theta_i(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_8, \Gamma_4)$$

From an engineering standpoint, $\theta_i$ and $\Delta \theta_i$ are differentiable because these variables are natural quantities. Then, $T_i$ can be rewritten as:
\[ T_i = T_i(\theta_1, \theta_2, \theta_3, \theta_4, \Delta \theta_1, \Delta \theta_2, \Delta \theta_3, \Delta \theta_4) \text{; and then,} \]
\[ T_i = T_i(\theta_1, \theta_2, \theta_3, \theta_4; \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4) \]  
(4.44)

by applying assumption (3). According to assumption (1), \( \Gamma_8 \) is redundant because the uniqueness of the last paragraph. By assigning five quantities, the other five quantities can be found with the remaining quantities as parameters. Therefore, there is:

\[ T_i = T_i(\theta_1, \theta_2, \theta_3, \theta_4; \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4) \]  
(4.45)

Re-using assumption (1) and the uniqueness assesment, Eq.(4.45) becomes:

\[ T_i = T_i(\theta_1, \theta_2, \theta_3, \theta_4; T_1, T_2, T_3, T_4) \]  
(4.46)

\[ = T_i(\theta_1, \theta_2, \theta_3, \theta_4) \]  
(4.47)

The reason that Eq.(4.46) is true is that \( \Gamma_1, \Gamma_2, \Gamma_3 \) and \( \Gamma_4 \) can be determined from \( T_1, T_2, T_3 \) and \( T_4 \).

If Eq.(4.47) is nonsingular, then

\[ \theta_i = \theta_i(T_1, T_2, T_3, T_4) \]  
(4.48)

By merging Eq.(4.45) into (4.48), it becomes:

\[ \theta_i = \theta_i(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4) = \theta_i(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_8, \Gamma_4) \]  
(4.49)
Is Eq. (4.47) nonsingular? The following is a discussion of this question. Because the system is a mechanical system, there is \( \theta_i = \theta_i(T_1, T_2, T_3, T_4) \) always. Suppose there are \( \theta_{i1} = \theta_{i1}(T_1, T_2, T_3, T_4) \) and \( \theta_{i2} = \theta_{i2}(T_1, T_2, T_3, T_4) \), two different sets of solutions; the uniqueness is not true. But, from the beginning time, there is only one initial value set: \( \theta_1(0), \theta_2(0), \theta_3(0), \) and \( \theta_4(0) \). \( \theta_{i1}, \theta_{i2}, \theta_{i3}, \theta_{i4} \), and \( \theta_{i1}, \theta_{i2}, \theta_{i3}, \theta_{i4} \) cannot be far away from \( \theta_1(0), \theta_2(0), \theta_3(0), \) and \( \theta_4(0) \). If the time is very small, difference between \( \theta_{i1} \) and \( \theta_{i2} \) is zero. Hence, \( \theta_{i1} = \theta_{i2} = \theta_i \). This means only one value set is expected. From an engineering standpoint, the time usually can be small enough to renew one computation cycle. Therefore, Eq. (4.48) is expected without solving the inverse function. (refer to Fig. 1-2, B14).

Eq. (4.49) means giving initial values \( \theta_1(0), \theta_2(0), \theta_3(0), \) and \( \theta_4(0) \), and giving \( \Gamma_1, \Gamma_2, \Gamma_3 \), and \( \Gamma_4 \), then \( \theta_1, \theta_2, \theta_3 \), and \( \theta_4 \) are uniquely determined. Therefore, assign \( \theta_1(t), \theta_2(t), \theta_3(t), \) and \( \theta_4(t) \). Next assign \( \bar{E}_3M_{11}/\bar{E}_5M_{13}, T_{33}/T_{33}, \bar{E}_{33}, \bar{E}_{32} \) and \( M_8 \) (also \( M_7 \) and \( \bar{E}_7 \) are known objectively), a set of net torques \( \Gamma_1 \) can be obtained by using the dynamic equations. And then, when applying \( \Gamma_1 \) at each point, the system shall have a response \( \theta_1(t), \theta_2(t), \theta_3(t), \) and \( \theta_4(t) \). This is the foundation of controllability. It is obviously possible from an engineering viewpoint because all the \( \theta_1, \theta_2, \theta_3, \Delta \theta_1, \Delta \theta_2, \Delta \theta_3, \bar{E}_7, M_7, \bar{E}_8, M_8, \bar{E}_5M_7, \) etc., are measurable.
4.4 Discussion and Conclusion

Discussion

In this chapter only one fully ground touched foot case is completely studied. However, a two-foot ground touched case and a non-foot fully ground touched case (ballet) can be studied further in the same way without innovation. To walk afterwards can be treated in the same way, too. Although only one fully ground touched foot case is studied completely, this case contains both the fully ground touched foot and the one point touched foot to cover all the possible cases. For all cases, Eq.(4.17),(4.18),(4.23),(4.20) and (4.21) are still valid. Eq.(4.24)—Eq.(4.35) will be slightly different.

In treating this chapter, no damper or spring is assumed. However, it is not difficult to assume:

\[
F_{DJ} = C_{Dj}(\theta_j, \dot{\theta}_j) \tag{4.50}
\]

\[
F_{ij} = k_{ij}(\theta_j) \tag{4.51}
\]

where

- \(F_{Dj}\) = damping torque at \(Jj\)
- \(C_{Dj}\) = damper at \(Jj\), a function
- \(F_{ij}\) = springy torque at \(Jj\)
- \(k_{ij}\) = spring at \(Jj\), a function.
Include Eq.(4.50) and Eq.(4.51) in Eq.(4.11) [14] on the right side. Matrix type computation scheme can be used in treating more complicated cases and leading to structure eigenproblems.

Biped walking has several phases in a stride, such as single leg supporting phase and two leg supporting phase. Jump can also be described up to the moment of two-foot-lifted above the ground. Above this moment, the body obeys linear momentum conservation law and angular momentum conservation law. The dynamic equation will be somewhat different, however, the difference is limited.

Conclusion

1. A dynamic model of a biped robot is given to have a set of dynamic equations.

2. Dynamic equation of the soma can be added which created $\vec{E}_7$ and $M_7$ (refer to Appendix 3).

3. Uniqueness and controllability are examined. Depending on past experience of the control system, a biped robot or a human body is controllable.

4. The principle of the treatment in this chapter forms a wide
base for many related topics in biped walking, especially, when the $\Gamma_i$, the net torque at each joint, are found.

5. Previously published work [9,10,11] used many reference frames for their analysis. Difficulty between two adjacent stride phases is not a problem in this work because the dynamic model developed, uses only one frame for the supporting ped (0-frame) at all times of a stride.

6. Previously published works used centers of gravity which lie on the axes of the limbs. This dynamic model is valid for any human because the $\Delta \theta_i$ accounts for the location of any center of gravity (see Fig. 4-1). $\Delta \theta_i$ and $d_i$ are good for muscle motion because muscles may change their center of mass distribution while in motion.
CHAPTER 5  BIPED WALKING WITH THE VESTIBULAR SYSTEMS

5.0  Introduction

In Chapter 3, it was shown that the vestibular system can provide the direction of gravitational field. The gravitational sense and the function of the vestibular systems are based on inertial effect of the otoconial layer, and the fluid inside the semicircular canals, not to the ground upon which a biped robot or a human body walks.

Section 5.1 will use this sense and develop a method to determine the positions and directions of all the limbs and feet or peds. This section utilizes a method where all the angles of the limbs and peds are determined with respect to the gravitational vector. In addition, with the knowledge of Section 4.2, which provides a method to calculate all the torques needed at each joint no matter what the angular status is, there is no restriction that a biped robot should walk on smooth surface. That is, a biped robot with a vestibular systems can walk on nonlevel ground.

Because the robot body has larger mass than its limbs, a robot can be thought of as an inverse pendulum, a simplified model for dynamic analysis can be worked out. Section 5.2 is an attempt to develop some status variables to describe the path depended motion of a robot.
5.1 To Function as an Angular Sensor

The angular positions of all joints in a human are measured by proprioceptive sensors and can be measured in a robot with angular sensors. Fig.5-1 is schematic arrangement for an angular positioning system. At the skull, there is a s-frame. With respect to the s-frame, there is a gravity vector \( \mathbf{g} \) which is known via the principle of Chapter 3. Besides, s-frame has its relation clear with the proprioceptive sensors with respect to 3-frame, which is located at the lower part of the soma of the robot. Fig.5-1 is a figure which has the same structure and nomenclature as Fig.4-1. For the convenience sake, \( \overline{3z_3} \) axis of 3-frame is chosen to be parallel with the negative \( \mathbf{g} \). In this way, with the help of angular sensors, \( \gamma \), the angle between \( \overline{3z_3} \) and \( l_2 \) (the limb between J2 and J3), can be found. And, \( \theta_2 \), the angle between \( l_1 \) and \( l_2 \), can be measured by angular sensors which are installed at J2. Thus, the attitude of \( l_1 \) can be found. In the same way, all the attitudes of the limbs and peds can be sensed. Therefore, \( \theta_0 \) can be found using Eq.(5.1) shown in Fig.5-1, and the relation between \( \gamma \) and \( \delta \) (the angle between \( l_3 \) and \( \overline{3z_3} \)) can also be found as Eq.(5.1) with very straight forward geometry.

\[
\theta_0 = \gamma - \theta_1 - \theta_2 - 180^\circ \\
(5.1)
\]

\[
\gamma = \varphi_2 + 180^\circ = \delta - \theta_3 + 180^\circ \\
(5.2)
\]
Fig. 5-1 Figure of the Angular Positioning System. This figure is used to show the definition of $\gamma$ and $\delta$. Frame I is an inertial frame. Frame I and Frame $o$ parallel with each other. Frame $s$ is the skull frame. Refer to Fig. 3-1, please.
where \( \varphi_i = \sum_{j=0}^{i} \theta_j \) was given in Chapter 4, and 0-frame is set to have \( \vec{\delta z} \) (an axis of 0-frame) parallel with the negative gravity vector.

\( l_0 \), the supporting ped, is put on the ground, \( \theta_0 \) is given by Eq.(5.1), which describes the attitude of the ped. Therefore, the 0-frame can be set with its origin at point 0 (or Jo). Then, all the attitude can be known uniquely with respect to 0-frame (review Chapter 4). This section (Section 5.1) provides a way to measure all the angles which were required by Chapter 4.

Because with the gravitational sense, which provided by Section 3.3, all the angles are known, therefore, the peds (\( l_0 \) and \( l_f \)) can be put upon any kind of surface without regarding its smoothness or evenness. The robot dynamic system accounts for what \( \theta_0 \) is, not the geometric shape of the ground.

5.2 Inverse Pendulum Analysis

The goal of this section is to develop a status variable \( c(t) \). The development and definition of \( c(t) \) will be given later. The utilization of \( c(t) \) is not the purpose of this work. A status variable is needed because the robotic dynamic system is very complicated and path dependent. An inverse pendulum model is a convenient and easy way to reduce the complexity.

Chapter 4 provides a method, with finite status concept, to compute all the torque needed at each joint. However, Chapter 4
doesn't provide a tool which can be used in Stride Planning (refer to Chapter 1 to understand the hierarchical concept in designing a whole robotic system). No matter what kind of motion the robot has, Chapter 4 can take care of all of them. This provides a foundation to allow the robot to walk on any attitude, but the computation is terribly complicated already. Therefore, there should be some way, from the top to the bottom, to design the Stride Planning (see Fig.1-2). \( c(t) \) is the variable which can be used in Stride Planning. The design is to keep the value of \( c(t) \) of the robot between two energy levels, which will make biped walking successful. Name these two energy levels \( L_u \) (upper) and \( L_d \) (lower). The following is a way to develop \( c(t) \).

\[ L_u \text{ and } L_d \text{ are decided by experiments because:} \]

1. Biped walking doesn't have to have many patterns, just a few good patterns.

2. The dynamic analysis (Section 4.2) is very complicated and many simplifications are assumed. It is not reasonable to use Section 4.2 for Stride Planning because it is necessary to compute over several time intervals. Too many computation steps cause too much inaccuracy, and it is difficult for real time computation.

3. The robotic system is attracted by the earth and resets on the ground due to its weight. The normal and frictional forces
acting on the ground and on the robot are limited. The maximum velocity the robot can develop during a walk depends on how large the magnitude of the maximum friction force is.

For these reasons, the robot has an upper limit of its kinetic energy. The robot should keep at least one ped touching the ground at all times. Therefore, there is obviously an upper bound for its potential energy. Sum the upper limits of kinetic and potential energy, and define this upper mechanical energy bound $\text{Lu}$. For analogous reasons, $\text{Ld}$ exists.

(4) Because kinetic energy and potential energy of a human body or a robot can be transformed from one to the other and vice versa. It is easier if describing the motion of a human body or a robot with the sum of the kinetic energy and potential energy. For a certain value of the sum, a human body or a robot may have many different combinations of its state of static and kinetic posture. Therefore, the sum will be used to handle the ensemble. $\text{Lu}$ and $\text{Ld}$ are the extremities of the sum.

In brief, the meaning of the above four points are summarized as the following:

(1) An experiment is possible because the patterns for successful walking are limited.

(2) An experiment is better than pure computation.

(3) $\text{Lu}$ and $\text{Ld}$ exist.

(4) $\text{Lu}$ and $\text{Ld}$ are good status limits because kinetic energy can be transformed to be potential energy and vice
Fig. 5-2 Figure of the Inverse Pendulum. The biped robot is idealized to be an inverse pendulum. The rear leg is represented as a force actuator.
versa.

For an inverse pendulum model (see Fig.5.2; note, all figures in this work are unified and have the same symbols and nomenclature), the soma is supported by a leg $\overline{13}$. This is a reduced model of Fig.4-1. Leg $\overline{13}$ can be found directly from Fig.5.2,

$$
\overline{13}(t) = -(l_1 + l_2) \cos(\gamma - \theta_2 - 90^\circ) \, \vec{e}_x + (l_1 + l_2) \sin(\gamma - \theta_2 - 90^\circ) \, \vec{e}_z
$$

(5.3)

Point 3 (or $J3$) has a track in space and its velocity is $\vec{v}_3(t)$, which can be obtained from Eq.(5.3) by taking derivative.

$$
\vec{v}_3(t) = (l_1 + l_2) (\dot{\gamma} - \dot{\theta}_2) \left[ \sin(\gamma - \theta_2 - 90^\circ) \, \vec{e}_x + \cos(\gamma - \theta_2 - 90^\circ) \, \vec{e}_z \right]
$$

(5.4)

Because $\vec{E}_7$ and $\vec{M}_7$ are given by the soma and are measured by some sensors at $J7$ or computed by Section 3.1, the differential motion of $J3$ must be found from Eq.(5.3). This is then used to compute the translational energy, $\vec{w}_b$. $\gamma$ is obtained in Section 5.1, and will be used in computing $\vec{w}_b$, the rotational energy. Note, in Fig.5-2, $J3$ and $J7$ are overlapped.

The translational and rotational energies are computed as follows

$$
d \overline{13} = (l_1 + l_2) \sin(\gamma - \theta_2 - 90^\circ) \, (\dot{\gamma} - \dot{\theta}_2) \, dt \, \vec{e}_x
$$

+ $(l_1 + l_2) \cos(\gamma - \theta_2 - 90^\circ) \, (\dot{\gamma} - \dot{\theta}_2) \, dt \, \vec{e}_z
$$

(5.5)
\[ sW_b = \int_a^b \overline{E}_7 \cdot d\overline{13} \]

\[ = \int_a^b \left[ +E_{7y} (l_1 + l_2) \sin(\gamma - \theta_2 - 90^\circ) (\dot{\gamma} - \dot{\theta}_2) \\
+ E_{7z} (l_1 + l_2) \cos(\gamma - \theta_2 - 90^\circ) (\ddot{\gamma} - \ddot{\theta}_2) \right] dt \]

(5.6)

\[ s\overline{W}_b = \int_a^b \overline{M}_7 \cdot d\overline{\gamma} = \int_a^b M_7 \dot{\gamma} dt \]

(5.7)

where

\[ \overline{13}(t) \] is a vector starting from J1 and ending at J3, "(t)" means this vector is a function of time.

\[ sW_b \] is translational work done by soma to outside through J3. A control surface is built around the natural surface of the soma.

\[ s\overline{W}_b \] is rotational work done by soma to outside through J3.

a and b are times.

All the other variables or symbols were appeared in other chapters. Each chapter uses the same definitions and symbols.

Eq.(5.6) and Eq.(5.7) will be needed later in defining c(t).
Using the first law of thermodynamics,

\[ Q + U_1 = U_b + \dot{a}w_b + \ddot{a}\overline{w}_b \]  \hspace{1cm} (5.8)

where \( Q \) = heat added into soma=0,

\( U \) = internal energy, \( a \) and \( b \) are time, subscripts.

Rewriting Eq.(5.8) as Eq.(5.9)

\[ PE_1 + EL_1 = PE_b + EL_b + \dot{a}w_b + \ddot{a}\overline{w}_b \]  \hspace{1cm} (5.9)

where

\( PE \) = potential energy,

\( EL \) = elastic energy of tension of muscles.

Set \( EL_1 = EL_b \), (this is negligible) then,

\[ PE_1 = PE_b + \dot{a}w_b + \ddot{a}\overline{w}_b \]  \hspace{1cm} (5.10)

Because the soma swings clockwise and counterclockwise (swings periodically, not rotates), it is possible to choose a proper pair of times, \( a \) and \( b \), to have \( \ddot{a}\overline{w}_b = 0 \) (for an example, choose two zero rotation status points).

Then, Eq.(5.10) becomes:

\[ PE_1 = PE_b + \dot{a}w_b = C(t) \]  \hspace{1cm} (5.11)
where \( C(t) \) is defined as a total energy level. \( C(t) \) is a relatively defined value like potential energy is w.r.t. height. Therefore, just define \( C(t) = PE_s \) in the value, and \( C(t) = PE_b + \Delta \omega_b \) in the meaning.

Then using the equations which are derived above in this section, we can directly have:

\[
C(t) = \sum_{i=1}^{N} g Z_i \Delta m_i \\
+ (l_1 + l_2) \int_{s}^{b} (\dot{\gamma} - \dot{\theta}_2) \left[ E_{7x} \sin(\gamma - \theta_2 - 90^\circ) + E_{7z} \cos(\gamma - \theta_2 - 90^\circ) \right] dt \\
= m_7 g \bar{Z} + \frac{1}{2} m_7 \bar{V}^2 \tag{5.12}
\]

where

\[
\bar{Z} = \sum_{i=1}^{N} \frac{Z_i \Delta m_i}{m_7}
\]

\[
\bar{V} = \left( \frac{2}{m_7} (l_1 + l_2) \int_{s}^{b} (\dot{\gamma} - \dot{\theta}_2) \left[ E_{7x} \sin(\gamma - \theta_2 - 90^\circ) + E_{7z} \cos(\gamma - \theta_2 - 90^\circ) \right] dt \right)^{1/2}
\]

\( Z_i \) = height of \( \Delta m_i \)

\( \Delta m_i \) = a piece of mass of the soma which is marked as i-piece, the soma is divided into \( N \) pieces.

\( \bar{Z} \) can be thought of to be an equivalent height, and \( \bar{V} \) an equivalent velocity.

Rewrite Eq.(5.12) as Eq.(5.13) by dividing with \( m_7 \), the mass of the soma.
Fig. 5-3  Figure of Energy Levels for Navigation. Ck (k=1,2,3,...) are the elliptic curves. They are equi-energy lines. This figure provides a concept to treat biped walking with energy method together with the concept of the processes and cycles of thermodynamics.
\[ c(t) = gz + \frac{1}{2} v^2 = pc + kc \]  \hspace{1cm} (5.13)

where \( c(t) = \frac{C(t)}{m_\gamma} \), and remove "-", bar.
\[ gz = pc, \hspace{0.5cm} \frac{1}{2} v^2 = kc. \]

Curves for Eq. (5.13) are shown in Fig. 5-3. Curve Ck are the equi-energy level curves for different \( C(t) \) value. Because "v" (velocity) contributes a squared effect to \( c \), and height contribute linearly to \( c \), the function is elliptic. An ellipse is also shown in Fig. 5-3, which shows a vestibular reading \( (v_s, Z_s) \) at the center of this ellipse, where \( v_s \) = velocity of the skull, and \( Z_s \) = the height of the skull s-frame. The size of the ellipse stands for the size of measurement error. Any continuous type measurement has its error. Any digital type measurement has its value stairs.

\( c(t) \) is the energy which is concerned to measure the walking potential, especially if a walking speed is assigned. \( c(t) \) is the energy which is related to walking and composed of two types of energy, position (potential) type and kinetic (motion) type. The potential energy exchanges with kinetic energy mutually. Along a Ck curve the two types of energy are exchanging. While walking, \( c(t) \) is moving from a Ck curve to another by controlling its actuators.

For many reasons, which are given at the beginning of this section, Lu and Ld lines can be found. The vestibular ellipse should be kept inside or between Lu and Ld. The smaller the vestibular ellipse is, the finer the vestibules function, and
the wider safety margin for possible walking. Because "ke" and "pe" are values, c(t) is a variable of status as well as a point function. The abstract meaning of the development of c(t) is to take out "velocity" and "height" from path variable "aWb" (refer to Eq.(5.6) and the other equations of this section). Velocity and height are point functions of time, therefore, it is possible to develop c(t). And, "aWb" is path variable, however, it appears in calculating the dynamic energy for walking. Therefore, the developing of c(t) is needed and possible. Besides, ke and pe can be exchanged mutually, a variable c(t) to cover the sum is needed for more advanced treatment of biped walking. Thus, c(t) is a tool for more complicated dynamic analysis. The researcher can avoid the trouble in taking care of the energy exchange problem by giving it a curve Ck. Along a single Ck curve, infinite status of walking and posture is possible. This is an ensemble, it should be categorized.

5.3 Discussion and Conclusion

Section 5.1:

1. The gravitational sense which is provided by Section 3.3 does a great job to make \( \theta_0 \) (the foot angle) known without regarding the geometric condition of the ground that the robot walks on.
2. Since $\theta_0$ is known, all the angles are known with respect to 0-frame. All the data needed in Section 4.2 are ready.

Section 5.2:

1. In the biped robot system, only potential energy and kinetic energy are concerned with walking. This leads to the possibility to derive $c(t)$ as a function of velocity and height varying with time. A $C_k$ represents an ensemble and makes $c(t)$ more meaningful and convenient.

2. Within $L_u$ and $L_d$, the stride pattern is possible. This provides a necessary condition for successful Stride Planning.
CHAPTER 6 CONCLUSION

A close loop system is developed to control biped walking. This loop includes a set of equations which describe the dynamics of human or robot leg motion (see Chapter 4). The equation set is shown to be unique and the motion can be controlled. A consistent reference frame system is used in these equations which eliminates many difficulties with previous works. The variable location of the center of gravity of each limb is also included in the analysis.

As part of the control system for biped walking a motion sensing unit, similar to that of the human vestibular system, is included. The overall control is introduced in Chapter 1 and Chapter 5 shows the importance of using the direction of the gravity vector in this control. Previous works do not properly include gravity direction in biped walking control. Also in Chapter 5 a status variable based on energy principles is developed. It is shown how this variable, which can be thought of as the sum of potential and kinetic energy, can be used in robot walking control.

An accelerometer, which is used as a linear motion sensor in both robots and humans, cannot distinguish between gravity and acceleration. Chapter 3 develops a scheme to separate these two signals. This scheme includes angular motion sensors which are
also present in the human motion sensing system. The system that is developed utilizes the fact that the gravity vector has constant magnitude and direction on the earth's surface. Rotation of the accelerometer can be accounted for by the angular motion sensing system and the measurement of the accelerometer can be processed to obtain pure acceleration and the direction of gravity.

Chapter 2 is a detailed analysis of the human linear accelerometer, the otolith. This is a continuation of previous work where numeric solution were found. This chapter finds an analytic solution to the same set of governing equations.

The significant problems which are solved in this work are:

Chapter 1: A hierarchical biped robot architecture, which is model reference torque controlled, is proposed.

Chapter 2: Otoconial response is solved analytically.

Chapter 3: Skull's motion is determined and the gravity sense is obtained.

Chapter 4: A dynamic model governing equation set of a biped robot is derived. This set of equations makes the control system proposed in Chapter 1 possible.

Chapter 5: The ability of a biped robot with gravity sense to walk on uneven terrain is investigated. In
addition, an energy variable is suggested for stride planning.

In this work, several significant problems are solved analytically, however, there is something much more significant, i.e. a series of methodologies to solve this kind of problem is developed. The methods can applied to biped and human walking situations as well as for complicated coordinate system problems, such as complicated gyro systems.

More questions like biped robot core problems can be further studied based on this work and the methods developed. Since the methodology in developing the dynamic model is available, aircraft and missiles can be controlled better by applying the method of de-gravitation and others which are proposed in Chapter 3 and the remainder of this work.
REFERENCES


APPENDIX 1

NOMENCLATURE

\( \vec{ab} \)  
position vector from point a to point b

A  
Area

A/D  
analog to digital

b  
thickness of the otocional layer

B_i  
the i-th block in Fig. 1-1, a functional box

B  
buoyancy in Chapter 2; a solid body

B.C.  
boundary condition

c  
thickness of the gel layer

[cd]  
a transformation matrix, transforms a thing from originally w.r.t. d to c

CG_i  
center of gravity of limb i

C(t), c(t)  
mechanical energy level constant at time t

d_i  
the distance between Ji and CGi

D/A  
digital to analog

D(t)  
displacement, function of t or function of S

Dg_x(S)  
\( (1 - R) g_x \left( \frac{1}{S^2} \right) E(S) \), in x-direction

\( \vec{e}_i \)  
unit vector in i-direction

E  
Young's modulus of cupular membrane material

\( \vec{E} \)  
force exerted by external causes

\( \vec{E}_x \)  
external force

\( E(S) \)  
\[ \frac{1}{S + \gamma R S + \gamma R (m S + \varepsilon) \coth \sqrt{\frac{R S^2}{\varepsilon + S m}}} \]

\( E(t) \)  
\[ \mathcal{L}^{-1}[E(S)] \sqrt{\varepsilon + S m} \]

\( \vec{F} \)  
force exerted by internal causes

g_x  
gravity component in x-direction
$\bar{g}_x \quad b^2 \rho_0 g_x / \mu V, \quad V \text{ a characteristic velocity}\ [1], \ b \text{ may be replaced by } c$

$G \quad \text{shear modulus}$

$G_x(S) \quad (1 - R) g_x (\frac{1}{S}) E(S), \text{ in } x\text{-direction}$

$\vec{H} \quad \text{angular momentum}$

$i, j \quad \text{integers}$

$I \quad \text{inertial}$

I.C. \quad \text{initial condition}$

IMU \quad \text{inertial measurement unit}$

INU \quad \text{inertial navigation unit}$

$J_i \quad \text{joint } i$

$l_i \quad \text{length of bone } i \text{ or limb } i$

$L \quad \text{a point, left vestibule; left semicircular canal frame}$

$L \quad \text{operator of Laplace Transformation}$

$L_d \quad \text{lower limitation line, above which stable stride is impossible}$

$L_i \quad \text{limb } i$

$Lu \quad \text{upper limitation line, below which stable stride is impossible}$

$L_0 \quad \text{left otolith frame}$

$m \quad \mu_0 / \mu_r$

$m_i \quad \text{mass of } i\text{-th limb}$

$\Delta m_i \quad \text{mass of } i\text{-th small element}$

$\vec{M} \quad \text{torque exerted by external causes}$

$\vec{M}_i \quad \text{the moment at } i\text{ point due to } \vec{E}_j$

$n \quad \text{integer}$

$N \quad \text{integer, number of the pieces of soma}$

$o \quad \text{otolith, a subscript}$

$\vec{p} \quad \text{linear momentum}$
\[ q = \tan^{-1}\left(\frac{l_s \sin \theta_s}{l_s \sin \theta_s + l_4}\right), \text{ in Chapter 4} \]

\[ \vec{r} \]
position vector

\[ \dot{\vec{r}} \]
first time derivative of \( \vec{r} \)

\[ R \]
\( \frac{\rho_t}{\rho_0} \); a point, right vestibule; right semicircular canal frame

\[ oR \]
right otolith frame

\[ s \]
skull, a point, a subscript

\[ S \]
parameter of Laplace Transformation

\[ t \]
time, natural time or \( \hat{t} \) after circumflex \( ^\sim \) being neglected

\[ \hat{t} \]
\( \mu_t/\rho_0 b^2 \), nondimensional or \( \mu_t/\rho_0 c^2 \)

\[ T \]
transpose

\[ T_i \]
inner torque, torque required which exerted by the motion of lower limbs

\[ u, u(y,t) \]
velocity of endolymph fluid in x-direction w.r.t. skull, if no subscript such as x appears, then \( y = y_f \)

\[ U_i \]
internal energy at time \( i \)

\[ U(t) \]
unit step function

\[ v(t) \]
velocity of otoconial membrane (layer) in x-direction w.r.t. skull if no subscript such as x appears; a velocity in Chapter 5

\[ v_s \]
velocity of skull in x-direction w.r.t. inertial reference frames, if no subscript such as x appears

\[ \vec{v}_{s_0}(t) \]
velocity of the otolith organ or the vestibule

\[ V \]
a characteristic velocity used in nondimensionalization [1]

\[ \bar{V} \]
characteristic velocity, see Eq. (5.12) of Chapter 5

\[ \tilde{V} \]
measurement of the otolith organ

\[ w, w(y,t) \]
velocity of cupular membrane (gel layer) in x-direction w.r.t. skull, if no subscript such as x appears then, \( y = y_s \), a gel coordinate, see Fig. 2-1
\( a_{Wb} \) the translational type work done by the soma to outside
\( a_{Wb} \) the rotational type work done by the soma to outside
\( w \) gravitational force
\( w.r.t. \) with respect to
\( x, X \) coordinate direction on the plan of otoconial membrane, or some other definition, see Fig. 4-1
\( y, Y \) coordinate direction perpendicular to otoconial membrane. or some other definition, see Fig. 4-1
\( y_f \) see Fig. 2-1, the origin is on the top of otoconial membrane
\( y_s \) see Fig. 2-1, the origin is on the bottom of gel membrane
\( z, Z \) perpendicular to \( x \) and \( y \), right hand coordination
\( \bar{Z} \) characteristic velocity, see Eq. (12) of Chapter 5

**Greek Symbols and the Miscellaneous**

\( \alpha, \beta \) constant, or undetermined constant in solving differential equations
\( \gamma \) \( \varphi_2 + 180^\circ \); a parameter in Chapter 4 and Chapter 5
\( \delta \) \( \varphi_3 \), see Fig. 5-1; \( \delta(t) \) a singular function, \( \delta(t) = \frac{d}{dt} u(t) \)
\( \delta_c \) the displacement of otoconial layer or the displacement of gel layer at \( c \)
\( \varepsilon \) \( \frac{1}{2} \left( \frac{E}{1 + v} \right) \frac{b^2}{\mu^2} \rho_0 \), \( b \) may be replaced by \( c \)
\( \theta_i \) the angle between limb \( i \) and limb \( i+1 \), detail see Fig. 4-1. \( \theta_i \) used as generalized coordinates
\( \theta \) \( \theta_{ii}, \theta_i \) first and second derivative of \( \theta_i \)
\( \Delta \theta_i \) see Fig. 4-1, the angle between bone and the line between \( J_i \) and \( CG_i \). It is not a finite difference of \( \theta_i \)
\( \varphi_i \) \( \sum_{j=0}^{i} \theta_j \)
\[ \rho_f \] density of the endolymph
\[ \rho_g \] density of the gel layer
\[ \rho_0 \] density of the otoconial membrane
\[ \vec{r}_{SR} \] position vector from \( S \) to \( R \) inside skull
\[ \mu \] frictional coefficient
\[ \mu_t \] dynamic viscous coefficient of endolymph
\[ \mu_g \] dynamic coefficient of gel membrane
\[ \tau_e \] elastic force of gel layer
\[ \tau_t \] viscous force of endolymph
\[ \tau_u \] viscous force of gel layer
\[ \tau_{xy} \] stress on \( x \) plane in \( y \)-direction
\[ \Gamma_i \] total torque required at joint \( i \) w.r.t. \( L_{q_i} \), detail see Fig. 4-1
\[ \nu \] Poisson's ratio of cupular membrane material
\[ \dot{\omega}_b \] angular velocity of \( b \)-frame or \( b \) point w.r.t. \( a \)-frame
\[ \times \] vector cross product
\[ \cdot \] vector dot product
\[ \| \| \] absolute value
\[ "-" \] bar, Laplace transformed
\[ \sqrt{\frac{R S^2}{(\varepsilon + S m)}} \] convolution integral \[ \int_0^t f(\lambda) g(t - \lambda) \, d\lambda \]
APPENDIX 2

DERIVATION OF T1, T2, T3, AND T4.

\[ x_1 = -l_0 + d_1 \sin(\phi_1 + \Delta \theta_1) \]

\[ z_1 = d_1 \cos(\phi_1 + \Delta \theta_1) \]

\[ x_2 = -l_0 + l_1 \sin \phi_1 + d_2 \sin(\phi_2 + \Delta \theta_2) \]

\[ z_2 = l_1 \cos \phi_1 + d_2 \cos(\phi_2 + \Delta \theta_2) \]

\[ x_3 = -l_0 + l_1 \sin \phi_1 + l_2 \sin \phi_2 + d_3 \sin(\phi_3 + \Delta \theta_3) \]

\[ z_3 = l_1 \cos \phi_1 + l_2 \cos \phi_2 + d_3 \cos(\phi_3 + \Delta \theta_3) \]

\[ x_4 = -l_0 + l_1 \sin \phi_1 + l_2 \sin \phi_2 + l_3 \sin \phi_3 + d_4 \sin(\phi_4 + \Delta \theta_4) \]

\[ z_4 = l_1 \cos \phi_1 + l_2 \cos \phi_2 + l_3 \cos \phi_3 + d_4 \cos(\phi_4 + \Delta \theta_4) \]

\[ \dot{x}_1 = \dot{d}_1 \sin(\phi_1 + \Delta \theta_1) + \dot{d}_1 \cos(\phi_1 + \Delta \theta_1) (\phi_1 + \Delta \theta_1) \]

\[ \dot{z}_1 = \dot{d}_1 \cos(\phi_1 + \Delta \theta_1) - \dot{d}_1 \sin(\phi_1 + \Delta \theta_1) (\phi_1 + \Delta \theta_1) \]

\[ \dot{x}_2 = l_1 \cos \phi_1 \dot{\phi}_1 + \dot{d}_2 \sin(\phi_2 + \Delta \theta_2) + d_2 \cos(\phi_2 + \Delta \theta_2) (\dot{\phi}_2 + \Delta \theta_2) \]

\[ \dot{z}_2 = -l_1 \sin \phi_1 \dot{\phi}_1 + \dot{d}_2 \cos(\phi_2 + \Delta \theta_2) - d_2 \sin(\phi_2 + \Delta \theta_2) (\dot{\phi}_2 + \Delta \theta_2) \]

\[ \dot{x}_3 = l_1 \cos \phi_1 \dot{\phi}_1 + l_2 \cos \phi_2 \dot{\phi}_2 + \dot{d}_3 \sin(\phi_3 + \Delta \theta_3) + d_3 \cos(\phi_3 + \Delta \theta_3) (\dot{\phi}_3 + \Delta \theta_3) \]

\[ \dot{z}_3 = -l_1 \sin \phi_1 \dot{\phi}_1 - l_2 \sin \phi_2 \dot{\phi}_2 + \dot{d}_3 \cos(\phi_3 + \Delta \theta_3) - d_3 \sin(\phi_3 + \Delta \theta_3) (\dot{\phi}_3 + \Delta \theta_3) \]

\[ \dot{x}_4 = l_1 \cos \phi_1 \dot{\phi}_1 + l_2 \cos \phi_2 \dot{\phi}_2 + l_3 \cos \phi_3 \dot{\phi}_3 + \dot{d}_4 \sin(\phi_4 + \Delta \theta_4) + d_4 \cos(\phi_4 + \Delta \theta_4) (\dot{\phi}_4 + \Delta \theta_4) \]

\[ \dot{z}_4 = -l_1 \sin \phi_1 \dot{\phi}_1 - l_2 \sin \phi_2 \dot{\phi}_2 - l_3 \sin \phi_3 \dot{\phi}_3 + \dot{d}_4 \cos(\phi_4 + \Delta \theta_4) - d_4 \sin(\phi_4 + \Delta \theta_4) (\dot{\phi}_4 + \Delta \theta_4) \]

\[ kE_1 = \frac{1}{2} m_1 \left[ \dot{d}_1^2 + \dot{\phi}_1^2 + \left( \phi_1 + \Delta \theta_1 \right)^2 \right] = \frac{1}{2} m_1 \left( \dot{x}_1^2 + \dot{z}_1^2 \right) \]

\[ kE_2 = \frac{1}{2} m_2 \left[ l_1^2 \dot{\phi}_1^2 + d_2^2 + d_2^2 \left( \phi_2 + \Delta \theta_2 \right)^2 + 2l_1 d_2 \sin \left( \theta_2 + \Delta \theta_2 \right) \dot{\phi}_1 \right] \]
\[ kE_3 = \frac{1}{2} m_3 \left[ l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + d_3^2 + d_4^2 \left( \ddot{\varphi}_3 + \Delta \dot{\theta}_3 \right)^2 \right. \\
+ 2l_1 l_2 \cos \theta_2 \dot{\varphi}_1 \dot{\varphi}_2 + 2l_1 d_3 \sin \left( \theta_2 + \theta_3 \right) \dot{\varphi}_1 \\
+ 2l_1 d_3 \cos \left( \theta_2 + \theta_3 + \Delta \theta_3 \right) \dot{\varphi}_1 \left( \dot{\varphi}_3 + \Delta \dot{\theta}_3 \right) \]

\[ kE_4 = \frac{1}{2} m_4 \left[ l_1^2 \dot{\varphi}_1^2 + l_2^2 \dot{\varphi}_2^2 + l_3^2 \dot{\varphi}_3^2 + d_4^2 + d_4 \left( \dot{\varphi}_4 + \Delta \dot{\theta}_4 \right)^2 \right. \\
+ 2l_1 l_2 \cos \theta_2 \dot{\varphi}_1 \dot{\varphi}_2 + 2l_1 l_3 \cos \left( \theta_2 + \theta_3 \right) \dot{\varphi}_1 \dot{\varphi}_3 \\
+ 2l_1 d_4 \sin \left( \theta_2 + \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\varphi}_1 \\
+ 2l_1 d_4 \cos \left( \theta_2 + \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\varphi}_1 \left( \dot{\varphi}_4 + \Delta \dot{\theta}_4 \right) \]

\[ pE_1 = m_1 g d_1 \cos (\varphi_1 + \Delta \theta_1) \]

\[ pE_2 = m_2 g \left[ l_1 \cos \varphi_1 + d_2 \cos (\varphi_2 + \Delta \theta_2) \right] \]

\[ pE_3 = m_3 g \left[ l_1 \cos \varphi_1 + l_2 \cos \varphi_2 + d_3 \cos (\varphi_3 + \Delta \theta_3) \right] \]

\[ pE_4 = m_4 g \left[ l_1 \cos \varphi_1 + l_2 \cos \varphi_2 + l_3 \cos \varphi_3 + d_4 \cos (\varphi_4 + \Delta \theta_4) \right] \]

\[ T_i = \frac{d}{dt} \left( \frac{\partial KE}{\partial \dot{\theta}_i} \right) - \frac{\partial KE}{\partial \theta_i} + \frac{\partial PE}{\partial \theta_i} \]

where

\[ KE = KE_1 + KE_2 + KE_3 + KE_4 \]

\[ PE = PE_1 + PE_2 + PE_3 + PE_4 \]

And, \( T_1, T_2, T_3, \) and \( T_4 \) are:

\[ T_1 = \text{Eq. (1)} - \text{Eq. (5)} + \text{Eq. (9)} \]
\( T_2 = \text{Eq. (2)} - \text{Eq. (6)} + \text{Eq. (10)} \)

\( T_3 = \text{Eq. (3)} - \text{Eq. (7)} + \text{Eq. (11)} \)

\( T_4 = \text{Eq. (4)} - \text{Eq. (8)} + \text{Eq. (12)} \)

where, Eq. (1) - Eq. (12) see the followings.

\[
\frac{\partial KE}{\partial \theta_1} = \frac{1}{2} m_1 \left[ 2 d_1^2 \left( \dot{\phi}_1 + \Delta \dot{\phi}_1 \right) \right]
+ \frac{1}{2} m_2 \left[ 2 l_2^2 \phi_1 + 2 d_2^2 \left( \phi_2 + \Delta \dot{\phi}_2 \right) + 2 l_1 d_2 \sin(\theta_2 + \Delta \theta_2) \right.
+ 2 l_1 d_2 \cos(\theta_2 + \Delta \theta_2) \left( \phi_1 + \phi_2 + \Delta \dot{\phi}_2 \right) \]
+ \frac{1}{2} m_3 \left[ 2 l_2^2 \phi_1 + 2 l_2^2 \phi_2 + 2 d_3^2 \left( \phi_3 + \Delta \dot{\phi}_3 \right) \right.
+ 2 l_1 l_2 \cos \theta_2 \left( \phi_1 + \phi_2 \right) + 2 l_1 d_3 \sin \left( \theta_2 + \theta_3 \right) \]
+ \frac{1}{2} m_4 \left[ 2 l_3^2 \phi_1 + 2 l_2^2 \phi_2 + 2 l_3^2 \phi_3 + 2 d_4^2 \left( \phi_4 + \Delta \dot{\phi}_4 \right) \right.
+ 2 l_1 l_2 \cos \theta_2 \left( \phi_1 + \phi_3 \right) + 2 l_1 d_4 \cos \left( \theta_2 + \theta_3 + \theta_4 + \Delta \theta_4 \right) \left( \phi_1 + \phi_3 + \Delta \dot{\phi}_4 \right) \]
+ 2 l_1 d_4 \cos \left( \theta_2 + \theta_3 + \theta_4 + \Delta \theta_4 \right) \left( \phi_1 + \phi_3 + \Delta \dot{\phi}_4 \right) \]

\[
\frac{\partial KE}{\partial \theta_2} = \frac{1}{2} m_2 \left[ 2 d_2 \left( \phi_2 + \Delta \dot{\phi}_2 \right) + 2 l_1 d_2 \cos \left( \theta_2 + \Delta \theta_2 \right) \phi_1 \right]
+ \frac{1}{2} m_3 \left[ 2 l_2^2 \phi_2 + 2 d_3^2 \left( \phi_3 + \Delta \dot{\phi}_3 \right) \right.
+ 2 l_1 l_2 \cos \theta_3 \phi_2 + 2 l_1 d_3 \cos \left( \theta_2 + \theta_3 + \Delta \theta_3 \right) \phi_1 \]
+ \frac{1}{2} m_4 \left[ 2 l_2^2 \phi_2 + 2 l_2^2 \phi_3 + 2 d_4^2 \left( \phi_4 + \Delta \dot{\phi}_4 \right) \right.
+ 2 l_1 d_4 \cos \left( \theta_2 + \theta_3 + \theta_4 + \Delta \theta_4 \right) \left( \phi_2 + \phi_3 + \Delta \dot{\phi}_4 \right) \]
+ 2 l_1 d_3 \sin \left( \theta_3 + \Delta \theta_3 \right) + 2 l_2 d_3 \cos \left( \theta_3 + \Delta \theta_3 \right) \left( \phi_2 + \phi_3 + \Delta \dot{\phi}_3 \right) \]
\[ + \frac{1}{2} m_4 \left[ 2 l_2 \dot{\phi}_2 + 2 l_3 \dot{\phi}_3 + 2 d_3 \left( \dot{\phi}_4 + \Delta \dot{\theta}_4 \right) \right. \\
+ 2 l_2 l_3 \cos \theta_2 \phi_1 + 2 l_2 l_5 \cos \left( \theta_2 + \theta_3 \right) \phi_1 + 2 l_3 d_4 \cos \left( \theta_2 + \theta_3 + \theta_4 + \Delta \theta_4 \right) \phi_1 \\
+ 2 l_2 d_3 \cos \left( \phi_2 + \phi_3 \right) + 2 l_2 d_4 \sin \left( \theta_3 + \theta_4 + \Delta \theta_4 \right) \\
+ 2 l_3 d_4 \cos \left( \theta_3 + \theta_4 + \Delta \theta_4 \right) \left( \dot{\phi}_2 + \dot{\phi}_4 + \Delta \dot{\theta}_4 \right) + 2 l_3 d_4 \sin \left( \theta_4 + \Delta \theta_4 \right) \\
+ 2 l_3 d_4 \cos \left( \theta_4 + \Delta \theta_4 \right) \left( \dot{\phi}_3 + \dot{\phi}_4 + \Delta \dot{\theta}_4 \right) \left] \right. \\
\frac{\partial \mathbf{KE}}{\partial \theta_3} = \frac{1}{2} m_3 \left[ 2 d_3 \left( \dot{\phi}_3 + \Delta \dot{\theta}_3 \right) + 2 l_1 d_3 \cos \left( \theta_2 + \theta_3 + \Delta \theta_3 \right) \dot{\phi}_1 \\
+ 2 l_2 d_3 \cos \left( \theta_3 + \Delta \theta_3 \right) \dot{\phi}_2 \right] \\
+ \frac{1}{2} m_4 \left[ 2 l_2 \dot{\phi}_2 + 2 d_3 \left( \dot{\phi}_4 + \Delta \dot{\theta}_4 \right) + 2 l_2 l_3 \cos \left( \theta_2 + \theta_3 \right) \dot{\phi}_1 \\
+ 2 l_3 d_4 \cos \left( \theta_2 + \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\phi}_1 \\
+ 2 l_2 d_3 \dot{\phi}_2 + 2 l_2 d_4 \cos \left( \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\phi}_2 \\
+ 2 l_3 d_4 \sin \left( \theta_4 + \Delta \theta_4 \right) + 2 l_3 d_4 \cos \left( \theta_4 + \Delta \theta_4 \right) \left( \dot{\phi}_3 + \dot{\phi}_4 + \Delta \dot{\theta}_4 \right) \right] \\
\frac{\partial \mathbf{KE}}{\partial \theta_4} = \frac{1}{2} m_4 \left[ 2 d_3 \left( \dot{\phi}_4 + \Delta \dot{\theta}_4 \right) + 2 l_1 d_4 \cos \left( \theta_2 + \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\phi}_1 \\
+ 2 l_2 d_4 \cos \left( \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\phi}_2 + 2 l_3 d_4 \cos \left( \theta_4 + \Delta \theta_4 \right) \dot{\phi}_3 \right] \\
\frac{d}{dt} \left( \frac{\partial \mathbf{KE}}{\partial \dot{\theta}_1} \right) = m_1 \left[ 2 d_3 \dot{d}_1 \left( \dot{\phi}_1 + \Delta \dot{\theta}_1 \right) + 2 d_3 \left( \ddot{\phi}_1 + \ddot{\theta}_1 \right) \right] \\
+ m_2 \left[ 2 l_1 \ddot{\phi}_1 + l_1 \ddot{\phi}_1 + 2 l_2 d_2 \left( \ddot{\phi}_2 + \Delta \ddot{\theta}_2 \right) + d_2 \left( \ddot{\phi}_2 + \ddot{\theta}_2 \right) \right. \\
\left. + \left( l_1 d_2 + l_1 d_2 \right) \sin \left( \theta_2 + \Delta \theta_2 \right) + l_1 d_3 \cos \left( \theta_2 + \Delta \theta_2 \right) \left( \dot{\theta}_2 + \Delta \dot{\theta}_2 \right) \right] \\
+ \left( l_1 d_2 + l_1 d_2 \right) \cos \left( \theta_2 + \Delta \theta_2 \right) \left( \dot{\phi}_1 + \dot{\phi}_2 + \Delta \dot{\theta}_2 \right) \\
- l_1 d_3 \sin \left( \theta_2 + \Delta \theta_2 \right) \left( \dot{\theta}_2 + \Delta \dot{\theta}_2 \right) \left( \dot{\phi}_1 + \dot{\phi}_2 + \Delta \dot{\theta}_2 \right) \\
+ l_1 d_3 \cos \left( \theta_2 + \Delta \theta_2 \right) \left( \dot{\phi}_1 + \dot{\phi}_2 + \Delta \dot{\theta}_2 \right) \right] \\
+ m_3 \left[ 2 l_1 \ddot{\phi}_1 + l_1 \ddot{\phi}_1 + 2 l_2 d_2 \ddot{\phi}_2 + l_2 \ddot{\phi}_2 + 2 l_3 d_4 \left( \ddot{\phi}_3 + \Delta \ddot{\theta}_3 \right) + d_3 \left( \ddot{\phi}_3 + \ddot{\theta}_3 \right) \right] \\
+ m_4 \left[ 2 l_2 \ddot{\phi}_2 + 2 l_3 \ddot{\phi}_3 + 2 d_4 \left( \ddot{\phi}_4 + \Delta \ddot{\theta}_4 \right) + d_4 \left( \ddot{\phi}_4 + \ddot{\theta}_4 \right) \right] \]
\[ \begin{align*}
&+ (l_1 l_2 + l_2 l_3) \cos \theta_2 (\dot{\phi}_1 + \dot{\phi}_2) - l_1 l_2 \sin \theta_2 \dot{\theta}_2 (\dot{\phi}_1 + \dot{\phi}_2) + l_5 l_3 \cos \theta_2 (\ddot{\phi}_1 + \ddot{\phi}_2) \\
&+ (l_1 d_3 + l_1 d_4) \sin (\theta_2 + \theta_3) + l_1 d_3 \cos (\theta_2 + \theta_3) (\dot{\theta}_2 + \dot{\theta}_3) \\
&+ (l_1 d_3 + l_1 d_4) \cos (\theta_2 + \theta_3 + \Delta \theta_3) (\dot{\phi}_1 + \dot{\phi}_3 + \Delta \dot{\theta}_3) \\
&- l_1 d_3 \sin (\theta_2 + \theta_3 + \Delta \theta_3) (\dot{\theta}_2 + \dot{\theta}_3 + \Delta \dot{\theta}_3) (\dot{\phi}_1 + \dot{\phi}_3 + \Delta \dot{\theta}_3) \\
&+ l_1 d_3 \cos (\theta_2 + \theta_3 + \Delta \theta_3) (\ddot{\phi}_1 + \ddot{\phi}_3 + \Delta \ddot{\theta}_3) \\
&+ (l_2 d_3 + l_2 d_4) \sin (\theta_3 + \Delta \theta_3) + l_2 d_3 \cos (\theta_3 + \Delta \theta_3) (\dot{\theta}_3 + \dot{\theta}_3) \\
&+ (l_2 d_3 + l_2 d_4) \cos (\theta_3 + \Delta \theta_3) (\dot{\phi}_2 + \dot{\phi}_3 + \Delta \dot{\theta}_3) \\
&- l_2 d_3 \sin (\theta_3 + \Delta \theta_3) (\dot{\theta}_3 + \Delta \dot{\theta}_3) (\dot{\phi}_2 + \dot{\phi}_3 + \Delta \dot{\theta}_3) \\
&+ l_2 d_3 \cos (\theta_3 + \Delta \theta_3) (\ddot{\phi}_2 + \ddot{\phi}_3 + \Delta \ddot{\theta}_3) \\
&+ m_4 \left[ l_1 l_2 \dot{\phi}_1 + l_1 l_3 \dot{\phi}_1 + 2l_1 l_3 \dot{\phi}_2 + l_2 l_3 \ddot{\phi}_2 + 2l_2 l_3 \ddot{\phi}_3 + l_3 \ddot{\phi}_3 \right] \\
&+ 2l_1 l_3 (\dot{\phi}_4 + \Delta \dot{\theta}_4) + l_3 (\dot{\phi}_4 + \Delta \dot{\theta}_4) \\
&+ (l_1 l_2 + l_2 l_4) \cos \theta_2 (\dot{\phi}_1 + \dot{\phi}_2) - l_1 l_2 \sin \theta_2 \dot{\theta}_2 (\dot{\phi}_1 + \dot{\phi}_2) + l_4 l_3 \cos \theta_2 (\ddot{\phi}_1 + \ddot{\phi}_2) \\
&+ (l_1 l_3 + l_1 l_4) \cos (\theta_2 + \theta_3) (\dot{\phi}_1 + \dot{\phi}_3) - l_1 l_3 \sin (\theta_2 + \theta_3) (\dot{\theta}_2 + \dot{\theta}_3) (\dot{\phi}_1 + \dot{\phi}_3) \\
&+ l_1 l_3 \cos (\theta_2 + \theta_3) (\dot{\phi}_1 + \dot{\phi}_3) \\
&+ (l_1 l_4 + l_1 l_4) \sin (\theta_2 + \theta_3 + \theta_4 + \Delta \theta_4) \\
&+ l_1 l_4 \cos (\theta_2 + \theta_3 + \theta_4 + \Delta \theta_4) (\dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4 + \Delta \dot{\theta}_4) \\
&+ (l_1 l_3 + l_1 l_4) \cos (\theta_2 + \theta_3 + \theta_4 + \Delta \theta_4) (\dot{\phi}_1 + \dot{\phi}_4 + \Delta \dot{\theta}_4) \\
&- l_1 l_4 \sin (\theta_2 + \theta_3 + \theta_4 + \Delta \theta_4) (\dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4 + \Delta \dot{\theta}_4) (\dot{\phi}_1 + \dot{\phi}_4 + \Delta \dot{\theta}_4) \\
&+ l_1 l_4 \cos (\theta_2 + \theta_3 + \theta_4 + \Delta \theta_4) (\ddot{\phi}_1 + \ddot{\phi}_4 + \Delta \ddot{\theta}_4) \\
&+ (l_2 l_3 + l_2 l_3) \cos \theta_3 (\dot{\phi}_2 + \dot{\phi}_3) - l_2 l_3 \sin \theta_3 \dot{\theta}_3 (\dot{\phi}_2 + \dot{\phi}_3) + l_3 l_3 \cos \theta_3 (\ddot{\phi}_2 + \ddot{\phi}_3) \\
&+ (l_2 l_4 + l_2 l_4) \sin (\theta_3 + \theta_4 + \Delta \theta_4) + l_3 l_4 \cos (\theta_3 + \theta_4 + \Delta \theta_4) (\dot{\theta}_3 + \dot{\theta}_4 + \Delta \dot{\theta}_4) \\
&+ (l_3 l_4 + l_3 l_4) \cos (\theta_3 + \theta_4 + \Delta \theta_4) (\dot{\phi}_2 + \dot{\phi}_4 + \Delta \dot{\theta}_4)
\end{align*}\]
\[-l_{d4} \sin \left( \theta_3 + \theta_4 + \Delta \theta_4 \right) \left( \dot{\theta}_3 + \dot{\theta}_4 + \Delta \dot{\theta}_4 \right) \left( \ddot{\theta}_2 + \ddot{\theta}_4 + \Delta \ddot{\theta}_4 \right) \]
\[+ l_{d4} \cos \left( \theta_3 + \theta_4 + \Delta \theta_4 \right) \left( \ddot{\phi}_2 + \ddot{\phi}_4 + \Delta \ddot{\phi}_4 \right) \]
\[+ \left( l_{d4} + l_{d4} \right) \sin \left( \theta_4 + \Delta \theta_4 \right) + l_{d4} \cos \left( \theta_4 + \Delta \theta_4 \right) \left( \dot{\theta}_4 + \Delta \dot{\theta}_4 \right) \]
\[+ \left( l_{d4} + l_{d4} \right) \cos \left( \theta_4 + \Delta \theta_4 \right) \left( \ddot{\phi}_3 + \ddot{\phi}_4 + \Delta \ddot{\phi}_4 \right) \]
\[-l_{d4} \sin \left( \theta_4 + \Delta \theta_4 \right) \left( \dot{\theta}_4 + \Delta \dot{\theta}_4 \right) \left( \ddot{\phi}_3 + \ddot{\phi}_4 + \Delta \ddot{\phi}_4 \right) \]
\[+ l_{d4} \cos \left( \theta_4 + \Delta \theta_4 \right) \left( \ddot{\phi}_3 + \ddot{\phi}_4 + \Delta \ddot{\phi}_4 \right) \]

\[\frac{d}{dt} \left( \frac{\partial K_E}{\partial \dot{\theta}_2} \right) = m_2 \left[ 2l_{d2} \dddot{\phi}_2 + \dddot{\phi}_2 \right] + d_2 \left( \dddot{\phi}_2 + \Delta \dddot{\phi}_2 \right) \]
\[+ \left( l_{d2} + l_{d2} \right) \cos \left( \theta_2 + \Delta \theta_2 \right) \left( \dot{\theta}_2 + \Delta \dot{\theta}_2 \right) \left( \ddot{\phi}_2 + \Delta \ddot{\phi}_2 \right) \]
\[+ l_{d2} \cos \left( \theta_2 + \Delta \theta_2 \right) \left( \ddot{\phi}_2 + \Delta \ddot{\phi}_2 \right) \]
\[+ m_3 \left[ 2l_{d3} \dddot{\phi}_3 + l_{d3} \dddot{\phi}_3 + 2d_3 \dddot{\phi}_3 + \dddot{\phi}_3 \right] + d_3 \left( \dddot{\phi}_3 + \Delta \dddot{\phi}_3 \right) \]
\[+ \left( l_{d3} + l_{d3} \right) \cos \left( \theta_3 + \Delta \theta_3 \right) \left( \dot{\theta}_3 + \Delta \dot{\theta}_3 \right) \left( \ddot{\phi}_3 + \Delta \ddot{\phi}_3 \right) \]
\[+ l_{d3} \cos \left( \theta_3 + \Delta \theta_3 \right) \left( \ddot{\phi}_3 + \Delta \ddot{\phi}_3 \right) \]
\[+ \left( l_{d3} + l_{d3} \right) \sin \left( \theta_3 + \Delta \theta_3 \right) + l_{d3} \cos \left( \theta_3 + \Delta \theta_3 \right) \left( \dot{\theta}_3 + \Delta \dot{\theta}_3 \right) \]
\[+ \left( l_{d3} + l_{d3} \right) \cos \left( \theta_3 + \Delta \theta_3 \right) \left( \ddot{\phi}_3 + \dddot{\phi}_3 + \Delta \dddot{\phi}_3 \right) \]
\[-l_{d3} \sin \left( \theta_3 + \Delta \theta_3 \right) \left( \dot{\theta}_3 + \Delta \dot{\theta}_3 \right) \left( \ddot{\phi}_3 + \dddot{\phi}_3 + \Delta \dddot{\phi}_3 \right) \]
\[+ l_{d3} \cos \left( \theta_3 + \Delta \theta_3 \right) \left( \ddot{\phi}_3 + \dddot{\phi}_3 + \Delta \dddot{\phi}_3 \right) \]
\[+ m_4 \left[ 2l_{d4} \dddot{\phi}_4 + l_{d4} \dddot{\phi}_4 + 2d_4 \dddot{\phi}_4 + \dddot{\phi}_4 \right] + d_4 \left( \dddot{\phi}_4 + \Delta \dddot{\phi}_4 \right) \]
\[+ \left( l_{d4} + l_{d4} \right) \cos \left( \theta_4 + \Delta \theta_4 \right) \left( \dot{\theta}_4 + \Delta \dot{\theta}_4 \right) \left( \ddot{\phi}_4 + \Delta \ddot{\phi}_4 \right) \]
\[
+ (l_d^3 + l_d^4) \cos (\theta_2 + \theta_3) \hat{\phi}_1 - l_d^3 \sin (\theta_2 + \theta_3) (\dot{\theta}_2 + \dot{\theta}_3) \hat{\phi}_1 + l_d^3 \cos (\theta_2 + \theta_3) \dot{\phi}_1 \\
+ (l_d^4 + l_d^3) \cos (\theta_2 + \theta_3 + \theta_4 + \Delta \theta_4) \dot{\phi}_1 \\
- l_d^4 \sin (\theta_2 + \theta_3 + \theta_4 + \Delta \theta_4) (\dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4 + \Delta \dot{\theta}_4) \hat{\phi}_1 \\
+ l_d^4 \cos (\theta_2 + \theta_3 + \theta_4 + \Delta \theta_4) \dot{\phi}_1 \\
+ (l_d^3 + l_d^2) \cos \theta_3 (\dot{\phi}_2 + \dot{\phi}_3) - l_d^2 \sin \theta_3 \dot{\theta}_3 (\dot{\phi}_2 + \dot{\phi}_3) + l_d^2 \cos \theta_3 (\ddot{\phi}_2 + \ddot{\phi}_3) \\
+ (l_d^4 + l_d^2) \sin (\theta_3 + \theta_4 + \Delta \theta_4) + l_d^2 \cos (\theta_3 + \theta_4 + \Delta \theta_4) (\dot{\theta}_3 + \dot{\theta}_4 + \Delta \dot{\theta}_4) \\
+ (l_d^4 + l_d^3) \cos (\theta_3 + \theta_4 + \Delta \theta_4) (\dot{\phi}_2 + \dot{\phi}_4 + \Delta \dot{\theta}_4) \\
- l_d^4 \sin (\theta_3 + \theta_4 + \Delta \theta_4) (\dot{\theta}_3 + \dot{\theta}_4 + \Delta \dot{\theta}_4) (\dot{\phi}_2 + \dot{\phi}_4 + \Delta \dot{\theta}_4) \\
+ l_d^4 \cos (\theta_3 + \theta_4 + \Delta \theta_4) (\ddot{\phi}_2 + \ddot{\phi}_4 + \Delta \ddot{\theta}_4) \\
+ (l_d^4 + l_d^3) \sin (\theta_4 + \Delta \theta_4) - l_d^4 \cos (\theta_4 + \Delta \theta_4) (\dot{\theta}_4 + \Delta \dot{\theta}_4) \\
+ (l_d^4 + l_d^3) \cos (\theta_4 + \Delta \theta_4) (\dot{\phi}_3 + \dot{\phi}_4 + \Delta \dot{\theta}_4) \\
- l_d^4 \sin (\theta_4 + \Delta \theta_4) (\dot{\theta}_4 + \Delta \dot{\theta}_4) (\dot{\phi}_3 + \dot{\phi}_4 + \Delta \dot{\theta}_4) \\
+ l_d^4 \cos (\theta_4 + \Delta \theta_4) (\ddot{\phi}_3 + \ddot{\phi}_4 + \Delta \ddot{\theta}_4) \\
\] (2)

\[
\frac{d}{dt} \left( \frac{\partial KE}{\partial \theta_3} \right) = m_3 \left[ 2l_d^3 \ddot{\theta}_3 (\dot{\phi}_3 + \Delta \dot{\theta}_3) + d_3^3 (\ddot{\phi}_3 + \Delta \ddot{\theta}_3) \right] \\
+ (l_d^3 + l_d^3) \cos (\theta_2 + \theta_3 + \Delta \theta_3) \dot{\phi}_1 - l_d^3 \sin (\theta_2 + \theta_3 + \Delta \theta_3) (\dot{\theta}_2 + \dot{\theta}_3 + \Delta \dot{\theta}_3) \dot{\phi}_1 \\
+ l_d^3 \cos (\theta_2 + \theta_3 + \Delta \theta_3) \dot{\phi}_1 + (l_d^3 + l_d^3) \cos (\theta_3 + \Delta \theta_3) \dot{\phi}_2 \\
- l_d^3 \sin (\theta_3 + \Delta \theta_3) (\dot{\theta}_3 + \Delta \dot{\theta}_3) \dot{\phi}_2 + l_d^3 \cos (\theta_3 + \Delta \theta_3) \dot{\phi}_2 \\
+ m_4 \left[ 2l_d^3 \ddot{\phi}_3 + \ddot{\theta}_3 \dot{\phi}_3 + 2l_d^3 \ddot{\phi}_4 (\Phi_4 + \Delta \Phi_4) + d_3^3 (\ddot{\phi}_4 + \Delta \ddot{\theta}_4) \right] \\
+ (l_d^3 + l_d^3) \cos (\theta_2 + \theta_3) \dot{\phi}_1 - l_d^3 \sin (\theta_2 + \theta_3) (\dot{\theta}_2 + \dot{\theta}_3) \dot{\phi}_1 + l_d^3 \cos (\theta_2 + \theta_3) \dot{\phi}_1 \\
+ (l_d^3 + l_d^3) \cos (\theta_2 + \theta_3 + \theta_4 + \Delta \theta_4) \dot{\phi}_1 \\
- l_d^3 \sin (\theta_2 + \theta_3 + \theta_4 + \Delta \theta_4) (\dot{\theta}_2 + \dot{\theta}_3 + \dot{\theta}_4 + \Delta \dot{\theta}_4) \dot{\phi}_1 \\
+ l_d^3 \cos (\theta_2 + \theta_3 + \theta_4 + \Delta \theta_4) \dot{\phi}_1 
\]
\[ + \left( l_2 \dot{\phi}_2 + l_3 \dot{\phi}_3 \right) \cos \theta_2 \dot{\phi}_2 - l_2 \sin \theta_3 \theta_3 \dot{\phi}_2 + l_2 \cos \theta_3 \dot{\phi}_2 \\
+ \left( l_2 \dot{\phi}_4 + l_3 \dot{\phi}_4 \right) \cos \left( \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\phi}_2 \\
- l_2 \dot{\phi}_4 \sin \left( \theta_3 + \theta_4 + \Delta \theta_4 \right) \left( \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\phi}_2 + l_2 \dot{\phi}_4 \cos \left( \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\phi}_2 \\
+ \left( l_3 \dot{\phi}_4 + l_4 \dot{\phi}_4 \right) \sin \left( \theta_4 + \Delta \theta_4 \right) + l_3 \dot{\phi}_4 \cos \left( \theta_4 + \Delta \theta_4 \right) \left( \theta_4 + \Delta \theta_4 \right) \\
+ \left( l_3 \dot{\phi}_4 + l_4 \dot{\phi}_4 \right) \cos \left( \theta_4 + \Delta \theta_4 \right) \left( \dot{\theta}_3 + \dot{\phi}_4 + \Delta \theta_4 \right) \\
- l_3 \dot{\phi}_4 \sin \left( \theta_4 + \Delta \theta_4 \right) \left( \theta_4 + \Delta \theta_4 \right) \left( \dot{\theta}_3 + \dot{\phi}_4 + \Delta \theta_4 \right) \\
+ l_3 \dot{\phi}_4 \cos \left( \theta_4 + \Delta \theta_4 \right) \left( \dot{\theta}_3 + \dot{\phi}_4 + \Delta \theta_4 \right) \right] \] (3)

\[
\frac{d}{dt} \left( \frac{\partial KE}{\partial \theta_4} \right) = m_4 \left[ 2l_4 d_4 \dot{\phi}_4 + \Delta \theta_4 \right] + d_4 \left( \dot{\phi}_4 + \Delta \dot{\theta}_4 \right) \\
+ \left( l_4 \dot{\phi}_4 + l_4 \dot{\phi}_4 \right) \cos \left( \theta_2 + \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\phi}_1 \\
- l_4 \dot{\phi}_4 \sin \left( \theta_2 + \theta_3 + \theta_4 + \Delta \theta_4 \right) \left( \theta_2 + \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\phi}_1 \\
+ l_4 \dot{\phi}_4 \cos \left( \theta_2 + \theta_3 + \theta_4 + \Delta \theta_4 \right) \left( \theta_2 + \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\phi}_1 \\
- l_4 \dot{\phi}_4 \sin \left( \theta_3 + \theta_4 + \Delta \theta_4 \right) \left( \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\phi}_2 + l_4 \dot{\phi}_4 \cos \left( \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\phi}_2 \\
+ \left( l_4 \dot{\phi}_4 + l_4 \dot{\phi}_4 \right) \cos \left( \theta_4 + \Delta \theta_4 \right) \dot{\phi}_3 \\
- l_4 \dot{\phi}_4 \sin \left( \theta_4 + \Delta \theta_4 \right) \left( \theta_4 + \Delta \theta_4 \right) \dot{\phi}_3 + l_4 \dot{\phi}_4 \cos \left( \theta_4 + \Delta \theta_4 \right) \dot{\phi}_3 \right] \] (4)

\[
\frac{\partial KE}{\partial \theta_1} = 0 
\frac{\partial KE}{\partial \theta_2} = m_2 \left[ l_2 \dot{\phi}_2 \cos \left( \theta_2 + \Delta \theta_2 \right) \dot{\phi}_1 - l_2 \dot{\phi}_2 \sin \left( \theta_2 + \Delta \theta_2 \right) \dot{\phi}_1 \right] \\
+ m_3 \left[ - l_2 \dot{\phi}_2 \sin \theta_2 \dot{\phi}_2 + l_3 \dot{\phi}_3 \cos \left( \theta_2 + \theta_3 \right) \dot{\phi}_1 - l_3 \dot{\phi}_3 \sin \left( \theta_2 + \theta_3 + \Delta \theta_3 \right) \dot{\phi}_1 \dot{\phi}_3 + \Delta \dot{\phi}_3 \right] \\
+ m_4 \left[ - l_4 \dot{\phi}_2 \sin \theta_2 \dot{\phi}_2 - l_4 \dot{\phi}_3 \sin \left( \theta_2 + \theta_3 \right) \dot{\phi}_1 \dot{\phi}_3 + l_4 \dot{\phi}_4 \cos \left( \theta_2 + \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\phi}_1 \\
- l_4 \dot{\phi}_4 \sin \left( \theta_2 + \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\phi}_4 + \Delta \dot{\phi}_4 \right] \] (5)

\[
\frac{\partial KE}{\partial \theta_3} = m_3 \left[ l_3 \dot{\phi}_3 \cos \left( \theta_2 + \theta_3 \right) \dot{\phi}_1 - l_3 \dot{\phi}_3 \sin \left( \theta_2 + \theta_3 + \Delta \theta_3 \right) \dot{\phi}_1 \dot{\phi}_3 + \Delta \dot{\phi}_3 \right] \\
- l_4 \dot{\phi}_3 \sin \left( \theta_2 + \theta_3 + \theta_4 + \Delta \theta_4 \right) \dot{\phi}_4 + \Delta \dot{\phi}_4 \right] \] (6)
\[ + l_2 \dot{\phi}_3 \cos(\theta_3 + \Delta \theta_3) \dot{\phi}_2 - l_2 \dot{\phi}_3 \sin(\theta_3 + \Delta \theta_3) \dot{\phi}_3 (\dot{\phi}_3 + \Delta \dot{\phi}_3) \]
\[ + m_4 [ - l_1 l_3 \sin(\theta_2 + \theta_3) \dot{\phi}_1 \dot{\phi}_3 + l_1 \dot{d}_4 \cos(\theta_2 + \theta_3 + \theta_4 + \Delta \theta_4) \dot{\phi}_1 - l_2 \dot{d}_4 \sin(\theta_2 + \theta_3 + \theta_4 + \Delta \theta_4) \dot{\phi}_2 (\dot{\phi}_4 + \Delta \dot{\phi}_4) - l_3 \dot{d}_4 \sin(\theta_3 + \theta_4 + \Delta \theta_4) \dot{\phi}_3 (\dot{\phi}_4 + \Delta \dot{\phi}_4) ] \] (7)

\[ \frac{\partial KE}{\partial \theta_4} = m_4 [ l_2 \dot{d}_4 \cos(\theta_2 + \theta_3 + \theta_4 + \Delta \theta_4) \dot{\phi}_1 - l_2 \dot{d}_4 \sin(\theta_2 + \theta_3 + \theta_4 + \Delta \theta_4) \dot{\phi}_2 (\dot{\phi}_4 + \Delta \dot{\phi}_4) \]
\[ + l_3 \dot{d}_4 \cos(\theta_3 + \theta_4 + \Delta \theta_4) \dot{\phi}_3 - l_3 \dot{d}_4 \sin(\theta_3 + \theta_4 + \Delta \theta_4) \dot{\phi}_3 (\dot{\phi}_4 + \Delta \dot{\phi}_4) \] (8)

\[ \frac{\partial PE}{\partial \theta_1} = -m_1 g \, d_1 \sin(\phi_1 + \Delta \theta_1) \]
\[ - m_2 g \left [ l_1 \sin \phi_1 + d_2 \sin(\phi_2 + \Delta \theta_2) \right ] \]
\[ - m_3 g \left [ l_1 \sin \phi_1 + l_2 \sin \phi_2 + d_3 \sin(\phi_3 + \Delta \theta_3) \right ] \]
\[ - m_4 g \left [ l_1 \sin \phi_1 + l_2 \sin \phi_2 + l_3 \sin \phi_3 + d_4 \sin(\phi_4 + \Delta \theta_4) \right ] \] (9)

\[ \frac{\partial PE}{\partial \theta_2} = -m_2 g \, d_2 \sin(\phi_2 + \Delta \theta_2) \]
\[ - m_3 g \left [ l_2 \sin \phi_2 + d_3 \sin(\phi_3 + \Delta \theta_3) \right ] \]
\[ - m_4 g \left [ l_2 \sin \phi_2 + l_3 \sin \phi_3 + d_4 \sin(\phi_4 + \Delta \theta_4) \right ] \] (10)

\[ \frac{\partial PE}{\partial \theta_3} = -m_3 g \, d_3 \sin(\phi_3 + \Delta \theta_3) \]
\[ - m_4 g \left [ l_3 \sin \phi_3 + d_4 \sin(\phi_4 + \Delta \theta_4) \right ] \] (11)

\[ \frac{\partial PE}{\partial \theta_4} = -m_4 g \, d_4 \sin(\phi_4 + \Delta \theta_4) \] (12)
APPENDIX 3

SOME EXTENDED DISCUSSION TO A HUMAN BODY

A biped robot does not necessarily have to be like a human. But, if it is, there are many advantages. If a biped robot is human like, Stride Planning (refer to Chapter 1) will be very simple. The reason is that human being's stride planning can be used as the referenced stride plan. In other words, at the lower level control of a biped robot walking, the Dynamic Model is the lowest reference model; and the Stride Planning reference model is the higher one. Tracking Planning reference model and the Behavior Planning reference model are the more higher one and the even more higher one respectively.

It can be seen that the control of biped walking is hierarchical model referenced. From this work, the status of biped walking can be understood and the difficulties of many other investigators. Indeed, the philosophy of this work is quite different from previous works.

The soma of a human body is complicated with many degrees of freedom. It is nearly impossible or not worthwhile trying to describe it in a certain way for simplification. Dividing a soma into N portions (each portion may have different size or mass) with 6N positions (3N linear positions and 3N angular positions) is required. By applying the principle of linear momentum conservation and the principle of angular momentum conservation, the \( \vec{E}_7 \) and \( \vec{M}_7 \) can be found. For a human body, two legs directly attach to the pelvic girdle, so the axis between J3 and J8 is an imaginary one.

Human legs are so large compared to the whole body. Massless assumption can
never be true (refer to Sec. 5.2). Muscles may change their mass centers and shapes. Therefore, "Δθ_i" and "d_i" considerations are necessary (refer to Fig. 4-1). Even the flex axis of two bones is not a fixed joint, which means the axis of a joint changes position with movement. The contact point between the two bones may vary while the two bones are in motion. To account for this small variation "θ_i" is still okay. All the variation should be accounted for by "Δθ_i"; and their is a new consideration, that is, "l_i" is no longer a constant. Ji is the contact point for one bone rolling over the other.

The writer of this work believes that the cerebellum doesn't actually process the equivalent computation upon the principle of conservation for the soma though the computation is much easier than computing the differential equations of the lower limbs. It guesses that at some certain points, the pressure or tension is picked up; and the cerebellum estimates the dynamic status of the soma as well as \( \vec{E}_7 \) and \( \vec{M}_7 \). And it expects that \( \vec{E}_7 \) and \( \vec{M}_7 \) are very periodically regular.

Because the body is very flexible, any sharp force or momentum change is retarded by local deformation (think of cat). Basically, the whole body keeps its response to a smoother environment. Besides, there is a better chance for those interferences to cancel each other to a certain level.

The cerebellum probably does not compute Lagrange's equations for the motion of the legs. It is not necessary to compute \( T_i \) (the torques for legs motion) so precisely if Stride Planning is skilled. All that is needed is to keep the available sum of the kinetic energy and the potential energy above a certain level which is shown in Chapter 5.

The writer also believes that all biped walking is 3-D alking. The knowledge of the 2-D case is referenced so well for 3-D walking. Even the lateral motion doesn't need to be controlled, the body dynamic system regulates the lateral motion
Fig. A-3 Figure for Showing the Analogy Between the Arms and the Lower Limbs.
automatically.

Refer to Fig. 4-3, there are \( \vec{M}_4 \) and \( \vec{E}_4 \). Therefore, by applying in a similar way, like the treatment for the legs, the upper limb system (arms and hands) can be modeled and give "\( \vec{M}_4 \)" and "\( \vec{E}_4 \)" to the soma. \( \vec{M}_4 \) and \( \vec{E}_4 \) are just analogy to \( \overline{T}_0 \) or \( \overline{E}_i, \overline{E}_{H1} \) and \( \overline{E}_{H1} \) of the supporting ped. Thus, a whole body can be modeled by the dynamic equations with the same methods.

For the 3-D case, the writer believes that the philosophy for mechanical analysis is the same as the 2-D case, except for a tremendous quantity of geometric treatment. This can be seen in the papers or books on robotic arms. It is not in the range of this work.
APPENDIX 4

DISCUSSION ABOUT STRIDE PLANNING

Stride Planning is defined as giving all $\theta_1$ during one stride. Besides, a $\theta_1$ is not necessarily an exact value, it can be within an interval of $\theta_1$. Moreover, such an interval can be coordinated to the Track Planning. Refer to Fig.1-1.

Because there is a large quantity of computation in practical engineering for biped walking, the capacity of a computer, precision of Stride Planning and Control Designing should be considered and decided. After these, a computer for Stride Designing should make traded-offs with the other factors. Also one cycle time for one routine of computation is known. Within one cycle, data acquisition and data release takes time other than the time for computation. Computation is expected to be very complicated and the time really for one computation cycle is very short. Therefore, except for using a powerful computer, there should be some way to shorten or reduce the burden on the computer. A good complier or directly using machine language to make the computation processes short and effective may be possible. Good technique for programming can shorten the the process or move away redundant processes. Computational accuracy should be reviewed for reducing ineffective precision. Many ways can reduce the burden of computation.

Refer to B6 to B11 in Fig.1-2. Control Designing of a local joint servomechanism should be quick enough, but no oscillation is allowed. The writer of this work wonders if there is a way which could be conceived by using variable mechanical dampers. One of the other ways is to give more than one object value for sequential times for local joint control design. This method would increase the burden of the
computer for dynamic model computation and the burden of the computer for Stride Planning. Inside each local control design, a simple chip and device for varying gain and the other characters for control quality is expected to be strict. A typical characteristic diagram of a servo-motor is shown below.

![Characteristics of a DC Servomotor](image)

**Fig. A-4-1** Characteristics of a DC Servomotor [adopted from Inland Motor]

The true character of mechanical property of a local joint servomechanism should involve the other characteristics of the other part of the system. The model of the servomechanism can be found and used in Stride Planning for simulation or expectation. A preceding simulation in Stride Planning center is expect to have, upon the basis of Tracking Planning instruction and physical real time data fed back from sensors. As for the computer of the dynamic model, a FIFO (First IN and First Out) is expected. Because the dynamic model is a fixed thing, pure hardware computation is possible. This means using ROM for cosine and sine, circuits for differentiation, multiplication, and summation. Parallel devices are used for MIMO (Multi-In Multi-Out).

When a stride begins from rest status, $\theta_i = \theta_i(0)$. According to Job Designing and Track Planning, $\ddot{\theta}_i = \ddot{\theta}_i(0)$ is chosen. $\dot{\theta}_i(0) = 0$. Thus, B1 sends $\theta_i$, $\dot{\theta}_i$ and $\ddot{\theta}_i$ to B3 at
$t = 0$. At $t = \Delta t$, $\ddot{\theta}_i = \ddot{\theta}_i(\Delta t)$, $\dot{\theta}_i = \dot{\theta}_i(\Delta t) = \ddot{\theta}(0)\Delta t$, and $\theta_i(\Delta t) = \theta_i(0) + \frac{1}{2} \ddot{\theta}(0) \Delta t^2$; these are sent to B3 from B1. When B3 received $\theta_i(0), \dot{\theta}_i(0),$ and $\ddot{\theta}_i(0)$ at $t=0$, $T_i$ is given at $t = \Delta t$. At $t = 2\Delta t$, receives physical measurements from B12 and B13. At $t = 4\Delta t$, B1 can have useful measurements and use them.

The following table shows how the signal flows. The signals of B1 were sent out at $t = 0$, and the physical measurements return at $t = 4\Delta t$. The feed back measurements are used inside the signals sent out from B1 at $t = 5\Delta t$. Therefore the time lag is $5\Delta t$. $\Delta t$ may not be a fixed interval. At the beginning of walking, $\Delta t$ should be small enough.

For easy reading, the details are not shown in Fig. 1-1. Actually, the joint sensors are angular type. They read position codes. Digital position sensors are reliable in the opinion of the writer.

B6~B11 are expected to be analogy type.
Fig. A-4-2 Data Flow Diagram for Fig. 1-2
APPENDIX 5

PHYSIOLOGY

The following brief knowledge about the vestibular system is written from the viewpoint of an engineer, not from a viewpoint of medical science. There are many books that tell the details from the viewpoint of medical science.

A human head has two ears. Inside each ear there is a part of the ear called the vestibule. A vestibule looks like Fig.A-5-1; and the position of the two vestibules is shown in Fig.A-5-2. Two different cuts show how to take them out from the skull. Fig.A-5-3 is a cross view of one semicircular canal. Three semicircular canals are almost mutually perpendicular to each other. A vestibule contains three semicircular canals, one sacculus and one utriculus. The last two are called otolith organs. There is a macula sacculus inside a sacculus. And, there is a macula utriculus inside a utriculus. The sacculus and utriculus are like what is shown in Fig.2-3. Fig.A-5-4 shows where they are. Fig.A-5-5 shows the details of a macular sacculus or a macular utriculus (All figures are adopted originally from [5,6,7], and re-edited by the writer).

Fig.A-5-6, A-5-7 and A-5-8 show more details. In Fig.A-5-7 or A-5-9, the direction of the arrows show the directional sensitivity or directional measurement capabilities.

Fig.A-5-10 is a drawing which shows the related position, especially for the otolith organs. Fig.A-5-11 shows the assumption which is used in this work, i.e. oR-frame or oL-frame. Though two otolith organs are not so close to each other, they are assumed to be adjacent to each other. Either oL-frame or oR-frame is thought to
be parallel to S-frame (see Fig. 3-1).

Assume the semicircular canals measure three directions of skull rotation, namely x, y and z directions. Thus it forms R-frame and L-frame. R-frame can be obtained from oR-frame by (1) rotating about vertical axis 45° out, (2) rotating about y axis of S-frame for 30° up. R-frame and L-frame are symmetric to x-z plane of S-frame.
Fig. A-5-1 Outer Bony Portion of an Inner Ear

Fig. A-5-2 Position of the Vestibules in the Skull

Fig. A-5-3 Cross Section of a Cupula of a Semicircular Canal
Fig. A-5-4 Figure for Showing the Position of a Sacculus and a Utriculus

Fig. A-5-5 Cross Section of an Otoconial Layer
Kinocilium lined up in opposite direction, between them is striola

Fig. A-5-7 Drawing of a Macula Sacculus (Top View)

Fig. A-5-6 Side View of the Cell of a Macula Sacculus

Fig. A-5-9 Drawing of a Macula Utriculus (Top View)

Fig. A-5-8 Side View of the Cells of Macula Utriculus
Fig. A-5-10 Drawing of a Vestibule System

Fig. A-5-11 Figure of a L-Frame
Appendix 6

The Spatial Transformation

For a two dimensional example, suppose a set of coordinate axes have rotation $\theta$, see Fig. A6-1. Then, the axes rotation matrix is:

$$R = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

Therefore, there is a transformation law as:

$$\vec{P}' = R \vec{P}$$

where, $\vec{P}=(a,b)$ is a position vector with respect to the original frame (unprimed), $\vec{P}'$ is the position vector with respect to the new frame (primed).

$a$ and $b$ are scalars which are the components of $\vec{P}$.

This type of transformation is called a coordinate rotation. The matrix $R$ is called a coordinate transformation matrix.

Suppose the coordinate axes remain unchanged, and $\vec{P}$ has a rotation $\theta$, see Fig. A6-2. Then, there is a transformation law as:

$$\vec{S} = R^T \vec{P}$$

where, $\vec{S}$ is the position vector of the object after the rotation.

"$T$" means transpose.

This type of rotation is called body rotation. The matrix $R^T$ is called an object transformation matrix.
Fig.A6–1 Coordinate Transformation. θ is positive when counterclockwise sense is taken. For 3-D, z or z' axis is perpendicular to x–y plane.

Fig.A6–2 Object Transformation. θ is positive when counterclockwise sense is taken.

In fact, $R^T = R^{-1}$ (R inverse) because $\mathbf{x}$ with respect to the orginal frame is equivalent to $\mathbf{P}$ fixed relatively in the space and the frame has $-\theta$ coordinate transformation.
Whenever a rotation is mentioned, what the frame rotation is with respect to should be specified for either a coordinate axes rotation or an object rotation. Therefore, there are four combinations for the rotation shown as the following table. This table is valid for 2-D and 3-D rotation for 2x2, 3x3 or 4x4 matrix (see Appendix 7) for Cartesian coordinate systm. (For more general consideration, replace "T" by "-1" for inverse)

<table>
<thead>
<tr>
<th>fixed</th>
<th>current</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>R_1 R_2 ...</td>
<td>... R_2 R_1</td>
<td>axis rotation</td>
</tr>
<tr>
<td>... R_2^T R_1^T</td>
<td>R_1^T R_2^T ...</td>
<td>body (object) rotation</td>
</tr>
</tbody>
</table>

For the above case, \( \vec{p}' = R \vec{p} \) is a simplified case of \( R_1 R_2 \ldots \). For the above case, \( S = R^T \vec{p} \) is a simplified case of \( ... R_2^T R_1^T \). There are four types of spatial rotation in the above table.

1. \( R_1 R_2 \ldots \) means an object fixed in the space and the coordinate axes rotates first as \( R_1 \), then as \( R_2 \) and subsequence as required. \( R_1, R_2 \) and \( \ldots \) are written with respect to the fixed original frame. For showing the form, a two dimensional example is given:
\[ \vec{p}' = \begin{bmatrix} \cos \theta_1 & \sin \theta_1 \\ -\sin \theta_1 & \cos \theta_1 \end{bmatrix} \begin{bmatrix} \cos \theta_2 & \sin \theta_2 \\ -\sin \theta_2 & \cos \theta_2 \end{bmatrix} \begin{bmatrix} \cos \theta_3 & \sin \theta_3 \\ -\sin \theta_3 & \cos \theta_3 \end{bmatrix} \cdots \vec{p} \]

where \[ R_i = \begin{bmatrix} \cos \theta_i & \sin \theta_i \\ -\sin \theta_i & \cos \theta_i \end{bmatrix} \quad i = 1, 2, 3, \text{ and } 4. \]

2. \( \cdots R_2 R_i \) is the same as \( 1 \cdot (R_1 R_2 \cdots) \), except \( R_2 \) is written with respect to the new coordinate axes which rotated by \( R_1^T \). Thus each successive rotation is with respect to the previous axes.

For a two dimensional example:

\[ \vec{p}' = \begin{bmatrix} \cos \theta_3 & \sin \theta_3 \\ -\sin \theta_3 & \cos \theta_3 \end{bmatrix} \begin{bmatrix} \cos \theta_2 & \sin \theta_2 \\ -\sin \theta_2 & \cos \theta_2 \end{bmatrix} \begin{bmatrix} \cos \theta_1 & \sin \theta_1 \\ -\sin \theta_1 & \cos \theta_1 \end{bmatrix} \cdots \vec{p} \]

3. \( \cdots R_2^T R_1^T \) means an object has a rotation \( R_1^T \) with respect to the original coordinate system. And, the object has another rotation \( R_2^T \) with respect to the original coordinate system again, and so forth.

For a two dimensional example,

\[ \vec{p}' = \cdots \begin{bmatrix} \cos \theta_3 & -\sin \theta_3 \\ \sin \theta_3 & \cos \theta_3 \end{bmatrix} \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 \\ \sin \theta_2 & \cos \theta_2 \end{bmatrix} \begin{bmatrix} \cos \theta_1 & -\sin \theta_1 \\ \sin \theta_1 & \cos \theta_1 \end{bmatrix} \vec{p} \]
4. $R_1^T R_2^T \ldots$ is like $3 \ldots$ $R_2^T R_1^T$ except it is not with respect to the original frame (only $R_1^T$ is with respect to the original frame). The original frame is fixed on the object and also rotates $R_1$. $R_2$ is with respect to the rotated frame. This type rotation is very useful in robotics. The object is thought to be a frame. The frame has an origin. This origin stands for the translational position of the object. And, the coordinate is the attitude of the point (the object).

For a two dimensional example:

$$
\vec{p}' = \begin{bmatrix}
\cos \theta_1 & -\sin \theta_1 \\
\sin \theta_1 & \cos \theta_1 
\end{bmatrix}
\begin{bmatrix}
\cos \theta_2 & -\sin \theta_2 \\
\sin \theta_2 & \cos \theta_2 
\end{bmatrix}
\begin{bmatrix}
\cos \theta_3 & -\sin \theta_3 \\
\sin \theta_3 & \cos \theta_3 
\end{bmatrix}
\ldots \vec{p}
$$

Eq.(3.11) is an example of $\ldots R_2^T R_1^T$ type. Eq.(3.22) is an example of $R_1^T R_2^T \ldots$ type. Eq.(3.8) is an example of $\ldots R_2 R_1 \ldots$ type.

For a matrix in Eq.(3.8), $[I \ s] = [s \ I]^T$. $[s \ I]^T$ is an object type transformation matrix. $[s \ I]$ is a coordinate type transformation matrix. Between a coordinate transformation matrix and an object transformation matrix is tranpose.

About symbolic system used in Chapter 3, the writer uses a two-variable notation to identify a matrix, for an example $[I \ s]$. This symbolic system is very valuable in making complicated transformation easily written down and treated without making errors. Though there are four types of basic transformation, the reader doesn't have to worry about which type is being mentioned in an equation. For an example, $[I \ s]$ is a coordinate transformation, which transforms a vector with respect to the skull's frame (s-frame) to the inertial frame (I-frame). The reader only has to worry about if
the symbolic variables can be cancelled in pairs (for an example, see Eq.(3.8)). Moreover, review the above table, \[ [ R_1^T R_2^T \ldots ]^T = \ldots R_2 R_1, \] and \[ \ldots R_2^T R_1^T ]^T = R_1 R_2 \ldots . \] These relationships explain why the reader doesn't have to worry about this. In fact, \( R_1^T R_2^T \ldots \) type can be treated by \( \ldots R_2 R_1 \) type (coordinate transformation type with respect to current frame). Finally, reverse the result \( (\ldots R_2 R_1) \) by a transpose because \[ [ \ldots R_2^T R_1^T ]^T = R_1 R_2 \ldots . \] This has an advantage for computation because \( \ldots \text{CBA type matrix multiplication is more convenient than ABC... } \) type matrix multiplication, where \( A, B, C, \ldots \) are matrices. The reason for convenience is: \( \ldots \text{CBA}\bar{P} \) is operated first \( A\bar{P} \), then \( B(A\bar{P}) \), and then \( C[B(A\bar{P})] \) and so forth. But, \( \text{ABC...}\bar{P} \) is operated first \( A\bar{P} \), then \( (AB)\bar{P} \), and then \( [(AB)C]\bar{P} \) and so forth. For \( B(A\bar{P}) \), there is only one operation between \( B \) and \( (A\bar{P}) \). For \( (AB)\bar{P} \), there are two operations between \( A \) and \( B \), and \( (AB) \) and \( \bar{P} \).
Appendix 7

Introduction to the 4x4 matrix, Translation and rotation

The so-called 4x4 matrix or quaternion method can be a coordinate axes rotation and translation matrix or an object rotation and translation transformation matrix. The usual 3x3 matrix or an object rotation transformation matrix. The difference between a 4x4 matrix and a 3x3 matrix is the 4x4 matrix is an expanded matrix involving the 3x3 rotation matrix and additionally the information of the coordinate origin or of the object. The properties of a 4x4 matrix is easily understood by carefully inspecting the followings:

(1) A normal coordinate transformation or object rotation in a 3x3 matrix

\[
\begin{bmatrix}
    a & b & c \\
    d & e & f \\
    g & h & i \\
\end{bmatrix} = \begin{bmatrix}
    aA+bb+cC & P \\
    dA+eb+fC & Q \\
    gA+hb+iC & R \\
\end{bmatrix}
\]

(2) A combined rotation and translation

\[
\begin{bmatrix}
    a & b & c & j \\
    d & e & f & k \\
    g & h & i & l \\
    0 & 0 & 0 & 1 \\
\end{bmatrix} = \begin{bmatrix}
    aA+bb+cC+j & P+J \\
    dA+eb+fC+k & Q+K \\
    gA+hb+iC+l & R+L \\
    \cdots & \cdots \\
\end{bmatrix}
\]

(A7.1)

(A7.2)
where the squared matrix in Eq.(A7.2) is the 4x4 matrix which is expanded from the 3x3 matrix of Eq.(A7.1). Comparing Eq.(A7.1) and (A7.2), J, K and L are the translational elements which perform the translational operation together with the operation for rotation. For coordinate axes transformation J, K and L have the negative values to quantities which the axes moved. For object transformation, J, K and L have the same values as the object moved. The right bottom element, "1", in Eq.(A7.2), is the scale factor. The lowest row in the matrix is always [0 0 0 1] for the convenience just like that the scale factor in [A B C 1]^T is always set to be "1". For more details, see [12]. For what is concerned in Chapter 3, A=\( v_{Rx} \), B=\( v_{Ry} \) and C=\( v_{Rz} \) in Eq.(3.5).

In Eq.(3.5), the "quaternion" method first appears [12]. "1" is the scale factor. \( v_{Rx} \), \( v_{Ry} \) or \( v_{Rz} \) is a component of the quaternion vector, \( R\vec{v}_R \), the right side otolith measurement. \( v_{Rx} \) is obtained from Eq.(3.4), which is the measurement of the otolith velocity at x direction. The same Eq.(3.4) is applied for \( v_{Ry} \) and \( v_{Rz} \) in y direction and z direction respectively. The superscript "oR" is used to specify the right otolith coordinate which \( R\vec{v}_R \) is referred to. And, "oL" in \( L\vec{v}_L \) (see Eq.(3.6)) shows that the coordinate which \( L\vec{v}_L \) is referred to is the left side otolith coordinate. Both \( R\vec{v}_R \) and \( L\vec{v}_L \) indicate the "skull velocity" at the right side otolith and at the left otolith respectively.
Vita

Chuen-Chane Huang was born in 1954, Chang-Hwa City, Taiwan, Republic of China. He received Bachelor of Science in Engineering Science (BSES) from National Chen Kung University, Tainan, Taiwan, in 1977. 1983, he received Master of Science in Mechanical Engineering (MSME) from Tamkang University, Taipei, Taiwan. His major was Air Bearing with analytic approach.

Before his attending the Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, he was an Assistant Researcher of Chung San Institute of Science and Technology, Taiwan, Republic of China. He worked in the field of Guidance and Control as well as a Machine Tool engineer.