PRECISE ENERGY DECAY RATES FOR
SOME VISCOELASTIC AND
THERMO-VISCOELASTIC RODS

by

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(ABSTRACT)

Energy dissipation in systems with linear viscoelastic damping is examined. It is shown that in such viscoelastically damped systems the use of additional dissipation mechanisms (such as boundary velocity feedback or thermal coupling) may not improve the rate of energy decay. The situation where the viscoelastic stress relaxation modulus decreases to its (positive) equilibrium modulus at a subexponential rate, e.g., like \((1 + t)^{-\alpha} + E\), where \(\alpha > 0, \ E > 0\) is examined. In this case, the nonoscillatory modes (the so-called creep modes) dominate the energy decay rate. The results are in two parts.

In the first part, a linear viscoelastic wave equation with infinite memory is examined. It is shown that under appropriate conditions on the kernel and initial history, the total energy is integrable against a particular weight if the kinetic energy component of the total energy is integrable against the same weight. The proof uses energy methods in an induction argument. Precise energy decay rates have recently been obtained using boundary velocity feedback. It is shown that the same decay rates hold for history value problems with conservative boundary conditions provided that an \textit{a priori} knowledge of the decay rate of the kinetic energy term is assumed.

In the second part, a simple linear thermo-viscoelastic system, namely, a viscoelastic wave equation coupled to a heat equation, is examined. Using Laplace transform methods, an integral representation formula for \(W(x, s)\), the transform of the displacement \(w(x, t)\), is obtained. After analyzing the location of the zeros of the appropriate characteristic equation, an asymptotic expansion for the displacement \(w(0, t)\) is obtained which is valid for large \(t\) and the specific kernel \(g(t) = g(\infty) + \delta t^{\eta-1} T(\eta), \ 0 < \eta < 1\). With this expansion it is shown that the coupled system tends to its equilibrium at a slower rate than that of the uncoupled system.
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Chapter 1
INTRODUCTION AND OVERVIEW

We wish to examine energy dissipation in systems with linear viscoelastic damping. Our goal is to show that in such viscoelastically damped systems, the use of additional dissipation mechanisms may not improve the rate of energy decay. The two mechanisms we will consider are boundary velocity feedback and thermal coupling. We examine the situation where the viscoelastic stress relaxation modulus decreases to its (positive) equilibrium modulus at a subexponential rate, e.g., like \((1 + t)^{-\alpha} + E\), where \(\alpha > 0\), \(E > 0\). In this case, the nonoscillatory modes, that is, the so-called creep modes, dominate the energy decay rate. (See [11] or [13]). Our results are in two parts.

In the first part, we examine a linear viscoelastic wave equation with infinite memory. After reviewing the semigroup formulation of the problem, we prove that, under appropriate conditions on the kernel and initial history, the total energy is integrable against a particular weight if the kinetic component of the total energy is integrable against the same weight. Our proof uses energy methods in an induction argument. Precise energy decay rates have recently been obtained using boundary velocity feedback [19, 20, 21]. Our results show that the same decay rates hold for history value problems with conservative boundary conditions provided that we assume an \textit{a priori} knowledge of the decay rate of the kinetic energy term.

In the second part, we examine a simple linear thermo-viscoelastic system, namely, a viscoelastic wave equation coupled to a heat equation. Using Laplace transform methods, we obtain an integral representation formula for \(W(x, s)\), the transform of the displacement \(w(x, t)\). We then analyze the location of the poles of \(W(x, s)\) by examining the zeros of the appropriate characteristic equation. Then for the specific kernel \(g(t) = g(\infty) + \delta^{m-1}_{1(\eta)}\), \(0 < \eta < 1\), we examine the asymptotic behavior of the displacement at the left end \(x = 0\).
To do this we obtain an asymptotic expansion for $W(0,s)$ valid near $s = 0$. Based on our analysis of the location of the poles of $W(x,s)$, we can invert the expansion for $W(0,s)$ to obtain an asymptotic expansion for the displacement $w(0,t)$ valid for large $t$. With this expansion, we show that for this kernel the displacement of the coupled system tends to its equilibrium at a slower rate than that of the uncoupled system.

In both parts, it is shown that the quantity of interest (the rate of energy decay or rate at which the displacement tends to its equilibrium) is dependent on the dissipation due to the viscoelastic stress-strain law inherent in the system and is not improved (and may in fact be made worse) by the use of additional dissipation mechanisms. This phenomenon occurs when the nonoscillatory creep modes dominate the asymptotic behavior of the viscoelastic system.

The thesis is organized as follows. In Chapter 2, we first describe the history space formulation for a linear viscoelastic rod with infinite memory. We then review recent results on energy decay in such viscoelastic systems both with conservative boundary conditions and with velocity feedback. In Chapter 3, we state and prove our results on precise energy decay rates for the history value problem with conservative boundary conditions described above. In Chapter 4, we review recent results on uniform energy decay in a linear thermo-elastic or thermo-viscoelastic rod with conservative boundary conditions. Finally, in Chapter 5, we give our example showing that the addition of thermal dissipation may actually slow the rate at which the displacement tends to equilibrium in a thermo-viscoelastic system.
Chapter 2

ENERGY DECAY FOR VISCOELASTIC SYSTEMS WITH INFINITE MEMORY

Throughout this Chapter, we consider the integro-partial differential equation

\[ u_{tt}(x,t) = g(0)u_{xx}(x,t) + \int_{-\infty}^{t} g'(t-s)u_{xx}(x,s)ds, \quad x \in \Omega := (0,a) \quad (2.1) \]

for \( t > 0 \), with initial conditions

\[ u(x, -t) = w^0(x,t) \text{ for } t > 0, \quad u(x, 0) = v^0(x), \quad u_t(x, 0) = v^0(x), \quad x \in \Omega, \quad (2.2) \]

where \( u^0 \in H_0^1(\Omega) \), \( v^0 \in H^0(\Omega) \), \( w^0 \in L^2(\mathbb{R}^+; |g'|; H_0^1(\Omega)) \). Here \( H^m(\Omega) \) denotes the Sobolev space

\[ H^m(\Omega) = \{ f : f, f', \ldots, f^{(m-1)} \text{ are absolutely continuous on } [0,a] \text{ with } f^{(m)} \in L^2(\Omega) \} \]

with norm

\[ \|f\|_{H^m}^2 := \sum_{j=0}^{m} \int_{0}^{a} |f^{(j)}(x)|^2 dx < \infty; \]

\[ H_0^1(\Omega) = \{ f \in H^1(\Omega) : f(0) = f(a) = 0 \}; \]

and \( H^0(\Omega) = L^2(\Omega) \). The “history space” \( L^2(\mathbb{R}^+; |g'|; H_0^1(\Omega)) \) consists of the space of \( H_0^1(\Omega) \)-valued functions \( h \) on \( \mathbb{R}^+ := [0, \infty) \) for which

\[ \|h\|_{L^2(\mathbb{R}^+; |g'|; H_0^1(\Omega))}^2 := \int_{0}^{\infty} \|h(s)\|_{H_0^1(\Omega)}^2 |g'(s)| ds < \infty. \]
We will be interested in homogeneous boundary conditions

\[ u(0,t) = u(a,t) = 0, \quad t \geq 0. \quad (2.3) \]

The energy of solutions of (2.1),(2.2),(2.3) is given by

\[ E(t) = \frac{1}{2} \int_0^a u_t^2(x,t)dx + \frac{1}{2} g(\infty) \int_0^a u_x^2(x,t)dx \]

\[ - \frac{1}{2} \int_{-\infty}^t g'(t-s) \int_0^a (u_x(x,t) - u_x(x,s))^2 dx ds. \quad (2.4) \]

We will assume throughout this Chapter that the kernel \( g(t) \) satisfies the following conditions:

\[ 0 < g(\infty) < g(0) < \infty, \quad (2.5) \]

\( g \) is positive, nonincreasing and convex, and \( g \in C^2[0, \infty) \).

We note that several of the results stated below require additional convexity assumptions on \( g \).

Problem (2.1), (2.2) is a model for torsional motion in a linear viscoelastic rod of constant mass density and of length \( a \). After scaling, \( u(z,t) \) represents the angular displacement from equilibrium at position \( z \) along the rod at time \( t \). The boundary condition (2.3) means that the ends of the rod are held fixed. For further discussion of viscoelastic constitutive laws and models, we refer the reader to the books [27] and [29].

The study of the decay of energy in viscoelastic systems of Boltzmann type began with the papers of C.M. Dafermos in the early 1970’s [3, 4]. [3] was motivated by the 1963 paper of J.J. Levin [22], in which he obtained conditions under which the solution \( u(t) \) (and its first two derivatives) of the nonlinear scalar Volterra integro-differential equation

\[ u'(t) = - \int_0^t g(t-\tau) h(u(\tau)) d\tau \]

tends to zero asymptotically. By extending Levin’s proof, Dafermos was able to prove similar results for a class of linear integro-partial differential equations with past histories. In [3], Dafermos assumed the kernel \( g \) as well as \(-g'\) were convex. In [4], he was able to get
a similar result without the assumption that \(-g'\) be convex, but with stronger assumptions on the function spaces. In a 1976 paper [5], Dafermos used dynamical systems methods to study the energy decay of

\[ u_{tt}(x, t) = g(0)u_{xx}(x, t) + \int_{-\infty}^{t} g'(t - \tau)u_{xx}(x, \tau)d\tau, \quad x \in [0, a], \quad t \in [0, \infty), \quad (2.6) \]

\[ u(0, t) = u(a, t) = 0. \quad (2.7) \]

Letting \(v = u_t\) be the angular velocity and \(w(x, \xi, t) = u(x, t - \xi)\) for \(\xi \in \mathbb{R}^+\) be the history of the angular displacement \(u(x, t)\), Dafermos writes (2.6) as a system as follows:

\[ u' = v \]
\[ v' = g(0)u_{xx} + \int_{0}^{\infty} g'(|\xi|)w_{xx}d\xi \]
\[ w' = -\frac{\partial w}{\partial \xi}. \quad (2.8) \]

This can be rewritten as:

\[
\begin{pmatrix}
\begin{bmatrix}
u\end{bmatrix}' \\
v \\
w
\end{bmatrix} =
\begin{bmatrix}
\begin{bmatrix}
u(\cdot, t) \\
(\cdot, t) \\
0
\end{bmatrix} \\
\begin{bmatrix}
g(0)u_{xx}(\cdot, t) + \int_{0}^{\infty} g'(|\xi|)w_{xx}(\cdot, \xi, t)d\xi \\
\frac{\partial w}{\partial \xi}(\cdot, \xi, t)
\end{bmatrix}
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
\]
\[
=: A \begin{bmatrix}
u \\
v \\
w
\end{bmatrix}, \quad (2.9)
\]

where the domain of \(A\) consists of \((u, v, w) \in H_0^1(\Omega) \times H^0(\Omega) \times L^2(\mathbb{R}^+; |g'|; H_0^1(\Omega))\) for which \(v \in H_0^1(\Omega), \frac{\partial w}{\partial \xi} \in L^2(\mathbb{R}^+; |g'|; H_0^1(\Omega)), w(\cdot, 0) = u(\cdot)\) and \(g(0)u_{xx} + \int_{0}^{\infty} g'(|\xi|)w_{xx}d\xi \in H^0(\Omega)\). Dafermos shows that \(A\) is a closed, densely defined maximally dissipative operator, and hence, by the Lumer-Phillips theorem, \(A\) generates a \(C_0\)-semigroup of contractions. Moreover, using the dynamical systems methods developed in [5], Dafermos shows that the

\[ 5 \]
energy \( E(t) \) defined in (2.4) tends to zero as \( t \to \infty \). However, none of the papers mentioned above gives rates of decay.

The first result giving an explicit rate at which the energy \( E(t) \) decays to zero was provided by Day [6] in 1980. Day considered the equation (2.6) with the boundary conditions

\[
 u(0,t) = 0, \quad \sigma(a,t) = \tau(t), \quad t > 0, \tag{2.10}
\]

where the stress \( \sigma(x,t) \) is given by

\[
 \sigma(x,t) = g(0)u_x(x,t) + \int_0^t g'(t-s)u_x(x,s)ds.
\]

In the case of a conservative boundary condition at \( x = a \), that is, when \( \tau(t) \equiv 0 \), Day’s result in [6] reduces to

**Theorem A.** Assume that \( g(t) \) satisfies (2.5) and that \( \log(g(t) - g(\infty)) \) is convex. In addition, assume that

\[
 \int_0^\infty (g(t) - g(\infty))dt < \infty. \tag{2.11}
\]

If \( u(x,t) \) is a solution of (2.6) satisfying (2.10) with \( \tau(t) \equiv 0 \), and if the history satisfies

\[
 |u_x(x,t)| + |u_{xt}(x,t)| \leq k \quad \text{for} \quad t \leq 0, \quad 0 \leq x \leq a, \tag{2.12}
\]

then there exists a constant \( c_1 \) such that for every \( T \geq 0 \),

\[
 \int_0^T E(t)dt \leq c_1.
\]

Furthermore, \( E(t) = o(t^{-1}) \) as \( t \to \infty \).

We remark that as formulated in [6], the solution \( u(x,t) \) of equation (2.6) is assumed to exist in a classical sense. Also, the history space formulation used by Dafermos, namely,
that the history $w^0$ belong to $L^2(\mathbb{R}^+; |g'|; H^1_0(\Omega))$, is not used in [6]. It is replaced by the requirement (2.12) that $u_x(x,t)$ and $u_{xt}(x,t)$ be uniformly bounded for $(x,t) \in \Omega \times (-\infty, 0]$.

During the late 80's, J. Lagnese [19] and G. Leugering [20, 21] obtained sharp estimates for the rate of energy decay in simple linear viscoelastic systems (membranes, beams, and thin plates) when the system is subjected to dissipative boundary feedback. These results were proved by semigroup theory and multiplier methods. In [20], Leugering investigates the behavior of a viscoelastic membrane modeled by

$$u_{tt}(x,t) = g(0)\Delta u(x,t) + \int_{-\infty}^{t} g'(t-s)\Delta u(x,s)ds \quad \text{for } x \in \Omega, \quad (2.13)$$

$$u(x,0) = u^0(x), \quad u_t(x,0) = v^0(x) \quad \text{and} \quad u(x,-s) = w^0(x,s) \quad \text{for } s \geq 0, \quad (2.14)$$

$$g(0)\frac{\partial}{\partial n}u(x,t) + \int_{-\infty}^{t} g'(t-s)\frac{\partial}{\partial n}u(x,s)ds = f(x,t) \quad \text{on } \Gamma_2, \quad (2.15)$$

$$u(x,t) = 0 \quad \text{on } \Gamma_1, \quad (2.16)$$

where $\Delta$ is the Laplacian, $\frac{\partial u}{\partial n}$ is the normal derivative of $u$, and $\Omega \subset \mathbb{R}^2$ is a bounded domain with piecewise smooth boundary $\Gamma = \Gamma_1 \cup \Gamma_2$. Here $f(x,t) = -ku_t(x,t)$, $k > 0$, is the boundary feedback. The energy of solutions is given by the two-dimensional analogue of (2.4). Concerning problem (2.13)--(2.16) Leugering proves:

**Theorem B.** Let $g(t)$ satisfy (2.5) and let $(u^0, v^0, w^0) \in H_0^1(\Omega) \times H_0^0(\Omega) \times L^2(\mathbb{R}^+; |g'|; H^1_0(\Omega))$.

Assume that the history $w^0$ has compact support in $[0, \infty)$. Then

$$\int_0^\infty E(t)dt < \infty. \quad (2.17)$$

If, in addition,
\[
\int_0^\infty (1 + t)^{n-1} (g(t) - g(\infty)) dt < \infty
\]
(2.18)

for some positive integer \(n\), then

\[
\int_0^\infty (1 + t)^n E(t) dt < \infty.
\]
(2.19)

We remark that conclusion (2.17) of Theorem B holds when the requirement that \(w^0\) has compact support is relaxed to

\[
\int_0^\infty (g(s) - g(\infty)) \|\partial_s w^0(\cdot, s)\|^2 ds < \infty.
\]
(2.20)

(See the proof of Theorem B in Leugering [20] or see Lagnese [19, p. 140].) Here \(\|\partial_s w^0(\cdot, s)\|^2 = \int_\Omega |\partial_s w(x, s)|^2 dx\). We note that in one space dimension, problem (2.13)-(2.16) reduces to

\[
\begin{align*}
&u_{tt}(x, t) = g(0)u_{xx}(x, t) + \int_{-\infty}^t g'(t - s)u_{xx}(x, s) ds \quad \text{for } x \in \Omega := (0, a), \\
&u(x, 0) = u^0(x), \quad u_t(x, 0) = v^0(x), \quad u(x, -s) = w^0(x, s) \quad \text{for } s \geq 0,
\end{align*}
\]
(2.21)

\[
\begin{align*}
&\sigma(a, t) := g(0)u_x(a, t) + \int_{-\infty}^t g'(t - s)u_x(a, s) ds = -ku_t(a, t), \\
&u(0, t) = 0.
\end{align*}
\]
(2.22)

(2.23)

(2.24)

In [21], Leugering studies the behavior of a viscoelastic beam

\[
\begin{align*}
u_{tt}(x, t) + g(0)u_{xxxx}(x, t) + \int_{-\infty}^t g'(t - s)u_{xxxx}(x, s) ds = 0, \quad x \in (0, a),
\end{align*}
\]
(2.25)
with left end clamped, that is,

$$u(0, t) = u_x(0, t) = 0 \quad \text{for} \quad t > 0,$$

(2.26)

with the same initial conditions (2.14) as before. At the right end he assumes that the beam is moment free and he imposes a dissipative shear boundary feedback, that is,

$$g(0)u_{xx}(a, t) + \int_{-\infty}^{t} g'(t - s)u_{xx}(a, s)ds = 0,$$

(2.27)

$$g(0)u_{xxx}(a, t) + \int_{-\infty}^{t} g'(t - s)u_{xxx}(a, s)ds = ku_t(a, t),$$

(2.28)

with $k > 0$. Under appropriate conditions on the kernel and the history, Leugering shows that the analogue of Theorem B holds for problem (2.25)-(2.28). (We remark that Leugering also allows nonconstant linear density and flexural rigidity in [21].) In [19], Lagnese considers thin plates with viscoelastic damping and obtains results similar to those of Leugering. We note that in [19],[20], and [21] it was assumed that the initial history had compact support.

It is not clear from the analysis in [19], [20], and [21] whether the rate of energy decay is due to the dissipative boundary feedback or to the internal viscoelastic damping. In fact, Lagnese asks [19, p. 144] whether the boundary feedback actually increases the margin of stability. Motivated by this, in 1990 Hannsgen and Wheeler [13, 14] showed that in the case of zero initial history one obtains the same rate of energy decay with conservative boundary conditions; hence, the boundary feedback does not enhance the margin of stability in this situation.

The results of Hannssen and Wheeler in [13, 14] are for problems with zero initial history, that is, they deal with initial value problems for abstract Volterra equations. The results of both papers are proved using resolvent theory. In [13], Hannssen and Wheeler consider the abstract Volterra equation

$$u_{tt}(t) = g(0)Lu(t) + \int_{0}^{t} g'(t - \tau)Lu(\tau)d\tau,$$

(2.29)
where $L$ is the generator of a strongly continuous cosine family in a Banach space $X$. We emphasize that this paper does not handle the case of past histories. In Section 4 of [13], it is further assumed that $X$ is a Hilbert space, $L$ is a negative definite self-adjoint operator, and $M$ is defined to be $(\sim L)^{\frac{1}{2}}$. For example, in our setting we could take $X = H^0(\Omega)$ and $Lu = \frac{\partial}{\partial x^2}u$ on $H^2(\Omega) \cap H^1_0(\Omega)$. It is shown that the energy

$$E(t) = \frac{1}{2} ||u_t(t)||^2 + \frac{1}{2} g(\infty) ||Mu(t)||^2 \quad (2.31)$$

is integrable against a weight $\rho(t)$ if the kernel satisfies appropriate assumptions. Here $\rho(t)$ is a weight on $\mathbb{R}^+$ means that

$$\rho$$

is positive, continuous and nondecreasing on $\mathbb{R}^+$, $\rho(0) = 1$, $\rho(t + s) \leq \rho(t)\rho(s)$ and $\rho_* := -\lim_{t \to \infty} t^{-1} \log \rho(t) = 0$. (2.32)

An example of a weight on $\mathbb{R}^+$ is given by $\rho(t) = (1 + t)^n$, $n \geq 0$.

More precisely, Hannsgen and Wheeler show in [13]

**Theorem C.** Let $g(t)$ satisfy (2.5) and assume, in addition, that

$$\int_0^\infty (|g'(t)| + t|g''(t)| + |t^2 g'''(t)|)|\rho(t)| dt < \infty \quad (2.33)$$

where $\rho(t)$ is a weight on $\mathbb{R}^+$. Then if $u_0 \in \text{Dom}(M)$ and $u_1 \in X$, the energy $E(t)$ for problem (2.29)–(2.30) defined by (2.31) satisfies

$$\int_0^\infty E(t)\rho(t) dt < \infty. \quad (2.34)$$
In particular, if \( \rho(t) = (1 + t)^n \), \( n \geq 0 \), Theorem C gives the same rate of energy decay as is obtained by Leugering in Theorem B. In [14], the same result is proved for a modification of equation (2.29) with nonzero forcing term, provided that the forcing term is in \( L^1(\mathbb{R}^+; \rho; X) \).

We wish to show that the precise energy decay rates obtained by Leugering and Lagnese for linear viscoelastic systems with nonzero initial history hold without the use of a dissipative boundary feedback, thus proving that the decay is attributable to the internal viscoelastic damping. Also, we would like to replace the restrictive assumption that the history must have compact support with a more general integrability condition. Ideally, our main theorem would be

**Conjecture.** Let \( g(t) \) satisfy (2.5) and assume that

\[
\int_0^\infty |g'(t)|(1 + t)^n dt < \infty \tag{2.35}
\]

for some nonnegative integer \( n \). Then there exists a positive constant \( C \) that depends only on \( n \) and the kernel \( g(t) \) such that

\[
\int_0^\infty (1 + t)^n E(t) dt \leq C \left( E(0) + \int_0^\infty \int_s^\infty |g'(t)|(1 + t - s)^n dt \|w^0_s(s)\|^2 ds \right) \tag{2.36}
\]

whenever \((u^0, v^0, w^0) \in H_0^1(\Omega) \times H^0(\Omega) \times L^2(\mathbb{R}^+; |g'|; H_0^1(\Omega)) \) with

\[
\int_0^\infty \int_s^\infty |g'(t)|(1 + t - s)^n dt \|w^0_s(s)\|^2 ds < \infty. \tag{2.37}
\]

We remark that when \( n = 0 \), (2.35) is a consequence of (2.5) and (2.37) is equivalent to (2.20) (in one dimension, so \( \Delta w^0(\cdot, s) = w^0_0(\cdot, s) \)). Also, if (2.35) holds for a positive integer
l, then (2.18) holds as well. (See Lemma 3.2, Chapter 3.) Notice that the conclusion of our conjecture is stronger than the conclusion of Day's theorem. In particular, to conclude that \( \int_0^\infty E(t)dt < \infty \), we conjecture that we only need \( 0 < g(\infty) < g(0) < \infty \). According to Day's result, we should need \( \int_0^\infty (g(t) - g(\infty))dt < \infty \) to conclude that \( \int_0^\infty E(t)dt < \infty \), but with this stronger assumption our conjecture gives \( \int_0^\infty (1+t)E(t)dt < \infty \). We note that the theorems of Lagnese, Leugering, and Hannsgen and Wheeler support our conjecture.

While we have been successful at replacing the compactness assumption on the history with a more general one, our result requires \textit{a priori} knowledge of the decay rate of the kinetic energy component of the total energy. Although the "ideal" result remains unproven, we have reduced the problem significantly. In Chapter 3 we state and prove our results.
Chapter 3

PRECISE DECAY RATES FOR A VISCOELASTIC SYSTEM WITH INFINITE MEMORY

We consider the integro-partial differential equation

\[ u_{tt}(x, t) = g(0)u_{xx}(x, t) + \int_{-\infty}^{t} g'(t-s)u_{xx}(x, s)ds, \quad x \in \Omega := (0, a), \tag{3.1} \]

for \( t > 0 \), with initial conditions

\[ u(x, -t) = w^0(x, t) \quad \text{for} \quad t > 0, \quad u(x, 0) = u^0(x), \quad u_t(x, 0) = v^0(x), \quad x \in \Omega, \tag{3.2} \]

where \( u^0 \in H^1_0(\Omega), \quad v^0 \in H^0(\Omega), \) and \( w^0 \in L^2(\mathbb{R}^+; |g'|; H^0_0(\Omega)) \), and with homogeneous boundary conditions

\[ u(0, t) = u(a, t) = 0. \tag{3.3} \]

The energy of solutions of (3.1)–(3.3) is given by

\[ E(t) = \frac{1}{2} \int_0^a u_t^2(x, t)dx + \frac{1}{2} g(\infty) \int_0^a u_x^2(x, t)dx \]

\[ - \frac{1}{2} \int_{-\infty}^{t} g'(t-s) \int_0^a (u_x(x, t) - u_x(x, s))^2 dx ds. \tag{3.4} \]

Throughout this Chapter, we will assume that the kernel \( g(t) \) satisfies the following conditions:

\[ 0 < g(\infty) < g(0) < \infty, \tag{3.5} \]

\( g \) is positive, nonincreasing and convex, and \( g \in C^2[0, \infty) \).

The following hypotheses on the initial history and on the kernel will appear frequently:
\[ \int_0^\infty \int_s^\infty |g'(t)|(1 + t - s)^n dt \langle w^0_x(s) \rangle^2 ds \leq K, \quad \text{and} \]
\[ \int_0^\infty |g'(t)|(1 + t)^n dt \leq L, \quad \text{(3.7)} \]

for nonnegative integers \( n \). Here
\[ \|w^0_x(s)\|^2 = \int_0^a |w^0_x(x, s)|^2 dx. \]

We observe that (3.6)(n) and (3.7)(n) imply (3.6)(n-1) and (3.7)(n-1), respectively, with the same \( K \) and \( L \). An example of the kind of kernel \( g(t) \) that we have in mind is provided by
\[ g(t) = \frac{1}{(1 + t)^\alpha} + g(\infty) \]
with \( \alpha > 0 \). If we let \( n_0 \) be the unique nonnegative integer such that \( n_0 < \alpha \leq n_0 + 1 \), we see that (3.7)(n) holds for \( n \leq n_0 \), but that it fails for \( n = n_0 + 1 \). Also, throughout this section, we will let \( k \) denote a positive constant that may change from line to line.

In our main theorem, we give conditions that yield precise rates of decay of the energy \( E(t) \). Recall from Chapter 2 that Dafermos [5] showed that \( E(t) \rightarrow 0 \) as \( t \rightarrow \infty \), but no rate was given. Before proceeding, we review the semigroup set up used by Dafermos.

The state space is
\[ H = H^1_0(\Omega) \times H^0(\Omega) \times L^2(\mathbb{R}^+; |g'|; H^0_0(\Omega)) \quad \text{(3.8)} \]
with inner product
\[ \langle (u, v, w), (\dot{u}, \dot{v}, \dot{w}) \rangle_H = g(\infty) \int_\Omega u_x \bar{u}_x dx + \int_\Omega v \bar{v} dx 
\[ - \int_\Omega \int_0^\infty g'(\xi)[u_x - w_x][\dot{u}_x - \dot{w}_x]d\xi dx. \quad \text{(3.9)} \]

Letting \( w(x, \xi, t) = u(x, t - \xi) \) for \( \xi \in \mathbb{R}^+ \), we can write the original problem (3.1)–(3.3) as

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a system as follows:

\[
\begin{pmatrix}
  u \\
  v \\
  w
\end{pmatrix}
\begin{pmatrix}
  \cdot \\
  \cdot \\
  \cdot \\
\end{pmatrix}
= \begin{pmatrix}
  v(\cdot, t) \\
  g(0)u_{xx}(\cdot, t) + \int_0^\infty g'(\xi)w_{xx}(\cdot, \xi, t)d\xi \\
  -\frac{\partial w}{\partial \xi}(\cdot, \xi, t)
\end{pmatrix}
\begin{pmatrix}
  \cdot \\
  \cdot \\
  \cdot \\
\end{pmatrix}
\]

\[=: A \begin{pmatrix}
  u \\
  \cdot \\
  w
\end{pmatrix}(\cdot, t) \tag{3.10}\]

with initial conditions

\[
\begin{pmatrix}
  u \\
  v \\
  w
\end{pmatrix}(0) = \begin{pmatrix}
  u^0 \\
  v^0 \\
  w^0
\end{pmatrix}. \tag{3.11}
\]

As is shown in Dafermos [5], \( A \) is the generator of a \( C_0 \)-semigroup of contractions \( T(t) \) on \( H \) with domain \( D(A) \) consisting of those \((u, v, w) \in H \) satisfying

\[v \in H^1_0(\Omega), \tag{3.12}\]

\[g(0)u_{xx} + \int_0^\infty g'(\xi)w_{xx}d\xi \in H^0(\Omega), \tag{3.13}\]

\[\frac{\partial w}{\partial \xi} \in L^2(\mathbb{R}^+; |g'|; H^1_0(\Omega)), \tag{3.14}\]

and the compatibility condition

\[w(\cdot, 0) = u(\cdot). \tag{3.15}\]

By Theorem 1.3 in Pazy’s book [26, p. 102], we know that (3.10) has a differentiable solution \((u, v, w)\) for every initial condition \((u^0, v^0, w^0) \in D(A)\). So, for initial data \((u^0, v^0, w^0) \in D(A)\) we have classical solutions of (3.1)–(3.3).

The weak solution of (3.10) is given by

\[(u, v, w)(t) = T(t)(u^0, v^0, w^0) \tag{3.16}\]

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for any initial condition \((u^0, v^0, w^0) \in H\). The energy of solutions is given by:

\[
E(t) = \frac{1}{2} \|T(t)(u^0, v^0, w^0)\|_H^2 = \frac{1}{2} \|(u, v, w)\|_H^2 = \frac{1}{2} \langle (u, v, w), (u, v, w) \rangle_H
\]

\[
= \frac{1}{2} \int_0^a v^2 dx + \frac{1}{2} g(\infty) \int_0^a u_x^2 dx - \frac{1}{2} \int_0^\infty g'(\xi) \int_0^a (u_x - w_x)^2 dx d\xi
\]

\[
= E_k(t) + E_p(t) + E_s(t).
\]

(3.17)

Here \(E_k(t), E_p(t)\) and \(E_s(t)\) are the kinetic, elastic potential, and viscoelastic stored components of the energy, respectively. Since \(T(t)\) is a semigroup of contractions on \(H\), the energy \(E(t)\) is a nonincreasing function of \(t\). We note that for smooth data \((u^0, v^0, w^0) \in D(A)\), we have classical solutions and that

\[
E_k(t) = \frac{1}{2} \int_0^a u_x^2(x, t) dx.
\]

(3.18)

Before stating the main theorem, we prove an important lemma.

**Lemma 3.1** Let \(n\) be a nonnegative integer. If the solution \(u\) of (3.1)-(3.3) satisfies

\[
\int_0^T \int_0^a (1 + t)^n u_x^2(x, t) dx dt \leq C_2,
\]

where \(C_2\) is a constant independent of \(T\), and (3.5), (3.6)(n) and (3.7)(n) hold, then

\[
- \frac{1}{2} \int_0^T (1 + t)^n \int_{-\infty}^t g'(t - s) \|u_x(t) - u_x(s)\|^2 ds dt \leq C_3,
\]

where \(C_3 = (g(0) - g(\infty))C_2 + LC_2 + K\).

**Proof:** By a change of variables,

\[
- \frac{1}{2} \int_0^T (1 + t)^n \int_{-\infty}^t g'(t - s) \|u_x(t) - u_x(s)\|^2 ds dt
\]

\[
\leq - \frac{1}{2} \int_0^\infty \int_0^\infty g'(\sigma) \|u_x(t) - u_x(t - \sigma)\|^2 d\sigma (1 + t)^n dt
\]

\[
\leq - \int_0^\infty \int_0^\infty g'(\sigma) \|u_x(t)\|^2 (1 + t)^n d\sigma dt
\]

\[
- \int_0^\infty \int_0^\infty g'(\sigma) \|u_x(t - \sigma)\|^2 (1 + t)^n d\sigma dt,
\]

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where the last inequality follows by the Parallelogram Law. An application of Fubini's Theorem and a change of variables then yields

\[-\frac{1}{2} \int_0^T (1 + t)^n \int_{-\infty}^t g'(t-s) ||u_x(t) - u_x(s)||^2 ds dt\]

\[\leq - \int_0^\infty g'(\sigma) d\sigma \cdot \int_0^\infty (1 + t)^n ||u_x(t)||^2 dt\]

\[= \int_0^\infty \int_{-\sigma}^\infty ||u_x(s)||^2 (1 + s + \sigma)^n \sigma s g'(\sigma) d\sigma d\sigma\]

\[\leq (g(0) - g(\infty)) \int_0^\infty (1 + t)^n ||u_x(t)||^2 dt\]

\[= \int_0^\infty \int_0^\infty ||u_x(s)||^2 (1 + s + \sigma)^n \sigma s g'(\sigma) d\sigma d\sigma\]

Here, the last inequality follows by noting that \((1 + s + \sigma)^n \leq (1 + s)^n \cdot (1 + \sigma)^n\). After a change of variables, use of \(u(x, -t) = w^0(x, t)\), and an application of Fubini's Theorem on the third term on the right hand side of the previous line, (3.6)(n) and (3.7)(n) give

\[-\frac{1}{2} \int_0^T (1 + t)^n \int_{-\infty}^t g'(t-s) ||u_x(t) - u_x(s)||^2 ds dt\]

\[\leq (g(0) - g(\infty)) C_2 + \int_0^\infty (1 + s)^n ||u_x(s)||^2 ds \cdot \int_0^\infty |g'(\sigma)|(1 + \sigma)^n d\sigma\]

\[+ \int_0^\infty \int_t^\infty |g'(\sigma)|(1 + \sigma - t)^n d\sigma ||w_x^0(t)||^2 d\tau\]

\[\leq (g(0) - g(\infty)) C_2 + C_2 \cdot L + K =: C_3.\]

Thus, the lemma is proved. □

As a consequence of Lemma 3.1, we see that the decay of the viscoelastic stored component of the energy can be determined in terms of the decay of the elastic potential component. We remark that Lemma 3.1 can be proved with weights other than \((1 + t)^n\).
In fact, for the abstract Volterra equation (2.29), Hannsgen and Wheeler [14, Section 4] proved the analogue of Lemma 3.1 for the stored energy term in equation (2.31) using a general weight \( \rho(t) \) as defined in equation (2.32).

An examination of the proof of Lemma 3.1 shows that inequalities occur only when we use the Parallelogram Law in the first step, and when we use the elementary inequality 
\[ (1 + s + \sigma)^n \leq (1 + s)^n(1 + \sigma)^n \]
in the second step. All other lines in the proof are equalities. Since the term
\[
- \int_0^\infty \int_{-\sigma}^0 \| u_x(s) \|^2 (1 + s + \sigma)^n ds g'(\sigma) d\sigma
\]
\[
= \int_0^\infty \int_t^\infty \| w_x^0(t) \|^2 (1 + \sigma - t)^n |g'(\sigma)| d\sigma dt
\]
is determined entirely by the past history, we conjecture that hypothesis (3.6)(n) is not only a sufficient condition, but that it is also a necessary condition to conclude that (3.19) follows from
\[
\int_0^T (1 + t)^n \| u_x(t) \|^2 dt \leq C_2.
\]

We now state our main theorem.

**Theorem 3.1** Let \( n \) be a nonnegative integer and let \( g(t) \) satisfy (3.5) and (3.7)(n). Suppose that there exists a constant \( C_1 \) depending only on \( n \) and \( g \) such that the kinetic energy component \( E_k(t) \) of the energy (3.17) satisfies

\[
\int_0^\infty (1 + t)^n E_k(t) dt 
\leq C_1 \left( E(0) + \int_0^\infty \int_s^\infty |g'(t)||1 + t - s|^n d\xi \| w_x^0(s) \|^2 ds \right)
\]

whenever \( (u^0, v^0, w^0) \in H \) with \( w^0 \) satisfying (3.6)(n). Then there exists a constant \( C \) depending only on \( n \) and \( g \) such that

\[
\int_0^\infty (1 + t)^n E(t) dt 
\leq C \left( E(0) + \int_0^\infty \int_s^\infty |g'(t)|(1 + t - s)^n dt \| w_x^0(s) \|^2 ds \right)
\]
whenever \((v^0, v^0, w^0) \in H\) with \(w^0\) satisfying (3.6)(n).

We remark that in the case \(n = 0\), (3.7) is automatically satisfied since \(g(0) < \infty\). Note that the extra condition on the history (3.6)(n) is expected. If the energy \(E(t)\) were integrable for every \((u^0, v^0, w^0) \in H\), then by Theorem 4.1 of Pazy’s book [26, p. 116] we would get exponential decay of \(E(t)\), which is unexpected since it is not assumed that the kernel decays exponentially. For example, we do not expect to get exponential energy decay for the kernel \(g(t) = (1 + t)^{-\alpha} + g(\infty), \alpha > 0\). (See Leugering [20, p. 78, Remark ii]).

Before proceeding with the proof of Theorem 3.1, we first state and prove some elementary facts that will be used in the proof.

**Lemma 3.2** Let \(g\) satisfy (3.5) and let \(n \geq 1\). If (3.7)(n) holds, then

\[
\int_0^\infty (g(t) - g(\infty))(1 + t)^{n-1} dt \leq L, \tag{3.22}
\]

and if (3.6)(n) holds, then

\[
\int_0^\infty \int_s^\infty (g(t) - g(\infty))(1 + t - s)^{n-1} dt \|w_2^0(s)\|^2 ds \leq K. \tag{3.23}
\]

**Proof:** To prove (3.22), first write

\[
\int_0^\infty (1 + t)^{n-1}(g(t) - g(\infty)) dt = -\int_0^\infty (1 + t)^{n-1} \int_t^\infty g'(s) ds dt.
\]

Now, applying Fubini’s Theorem and evaluating the integration in \(t\) gives

\[
\int_0^\infty (1 + t)^{n-1}(g(t) - g(\infty)) dt = \frac{1}{n} \int_0^\infty (1 + t)^n |g'(t)| dt + \frac{1}{n} (g(\infty) - g(0)).
\]

Since \(g(0) > g(\infty)\) and \(\frac{1}{n} \leq 1\) we have

\[
\int_0^\infty (1 + t)^{n-1}(g(t) - g(\infty)) dt \leq \int_0^\infty (1 + t)^n |g'(t)| dt \leq L.
\]

To prove (3.23), we begin as in the proof of (3.22) and write

\[
\int_0^\infty \int_s^\infty (1 + t - s)^{n-1}(g(t) - g(\infty)) dt \|w_2^0(s)\|^2 ds
\]

\[
= -\int_0^\infty \int_s^\infty (1 + t - s)^{n-1} \int_t^\infty g'(\tau) d\tau dt \|w_2^0(s)\|^2 ds.
\]
Interchanging the inner two integrations and evaluating the integration in $t$ gives

\[
\int_0^\infty \int_s^\infty (1 + t - s)^{n-1} (g(t) - g(\infty)) dt \|w_2^0(s)\|^2 ds \\
= \frac{1}{n} \int_0^\infty \int_s^\infty g'(\tau) d\tau \|w_2^0(s)\|^2 ds \\
+ \frac{1}{n} \int_0^\infty \int_s^\infty (1 + \tau - s)^n |g'(\tau)| d\tau \|w_2^0(s)\|^2 ds.
\]

Since $g'(t) \leq 0$ and $\frac{1}{n} \leq 1$ we have

\[
\int_0^\infty \int_s^\infty (1 + t - s)^{n-1} (g(t) - g(\infty)) dt \|w_2^0(s)\|^2 ds \\
\leq \int_0^\infty \int_s^\infty (1 + \tau - s)^n |g'(\tau)| d\tau \|w_2^0(s)\|^2 ds \leq K,
\]

and the proof of (3.23) is complete. \(\square\)

**Lemma 3.3** Let $n$ be a positive integer. If \(\int_0^\infty t^{n-1} E(t) dt < \infty\), then \(t^n E(t) \to 0\) as \(t \to \infty\).

**Proof:** For $0 \leq s \leq t$, we know that

\[E(t) \leq E(s)\]

since $E(t)$ is nonincreasing. Multiplying the above inequality by $s^{n-1}$ and integrating with respect to $s$ from $\frac{1}{2}$ to $t$ we have

\[
\int_{\frac{1}{2}}^t s^{n-1} E(t) ds \leq \int_{\frac{1}{2}}^t s^{n-1} E(s) ds.
\]

After evaluating the integral on the left hand side of this inequality, we see that

\[
\frac{2^n - 1}{n2^n} t^n E(t) \leq \int_{\frac{1}{2}}^t s^{n-1} E(s) ds.
\]

Now, as $t \to \infty$, the right hand side of the last inequality approaches 0 since $\int_0^\infty t^{n-1} E(t) dt < \infty$, and thus the lemma is proved. \(\square\)
We now turn to the

Proof of Theorem 3.1:

PART 1 (Smooth Data): We first deduce Theorem 3.1 for smooth initial data \((u^0, v^0, w^0) \in D(A)\). Our proof will be by induction on \(n\). Thus, we first assume that \((u^0, v^0, w^0) \in D(A)\) with \(w^0\) satisfying (3.6)\((n=0)\). We begin by showing that

\[
\int_0^\infty \int_0^a u_x^2(x,t) dx dt \\
\leq C_2 \left( E(0) + \int_0^\infty \int_0^\infty |g'(t)| dt \| w_x^0(s) \|^2 ds \right)
\]

(3.24)

where \(C_2\) depends only on \(g\). To do this we need some preliminary results. We note that Lemma 3.4 and Lemma 3.5 and their proofs are motivated by Day [6].

Lemma 3.4

\[
\left| \int_0^a u(x,t) u_t(x,t) dx \right| \leq k \left( \frac{1}{g(\infty)} \right)^{\frac{1}{2}} E(t).
\]

(3.25)

Proof of Lemma 3.4: Applying Schwarz's inequality followed by Wirtinger's inequality, that is,

\[
\int_0^a u^2(x,t) dx \leq \frac{4a^2}{\pi^2} \int_0^a u_x^2(x,t) dx
\]

(see [6, p. 269]), we obtain

\[
\left| \int_0^a u(x,t) u_t(x,t) dx \right| \leq \int_0^a |u(x,t)| |u_t(x,t)| dx
\]

\[
\leq k \left( \int_0^a u_x^2(x,t) dx \right)^{\frac{1}{2}} \cdot \left( \int_0^a u_t^2(x,t) dx \right)^{\frac{1}{2}}
\]

\[
= k \left( \frac{1}{g(\infty)} \right)^{\frac{1}{2}} \left[ g(\infty) \int_0^a u_x^2(x,t) dx \cdot \int_0^a u_t^2(x,t) dx \right]^{\frac{1}{2}}.
\]

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Use of the arithmetic-geometric mean property $\alpha \beta \leq \frac{1}{2}(\alpha + \beta)^2$ then yields

$$\left| \int_0^a u(x,t)u_t(x,t)dx \right| \leq k \left( \frac{1}{g(\infty)} \right)^{\frac{1}{2}} \left[ g(\infty) \int_0^a u_x^2(x,t)dx + \int_0^\infty u_t^2(x,t)dx \right]$$

$$\leq k \left( \frac{1}{g(\infty)} \right)^{\frac{1}{2}} E(t). \quad \square$$

**Lemma 3.5**

$$0 = \int_0^a u(x,t)u_{tt}(x,t)dx + g(0)\int_0^a u_x^2(x,t)dx$$

$$+ \int_0^a u_x(x,t)\int_{-\infty}^t g'(t-s)u_x(x,s)dsdx. \quad (3.26)$$

Proof of Lemma 3.5: A substitution from the original equation (3.1) shows that

$$\int_0^a u(x,t)u_{tt}(x,t)dx = g(0)\int_0^a u(x,t)u_{xx}(x,t)dx$$

$$+ \int_0^a u(x,t)\int_{-\infty}^t g'(t-s)u_{xx}(x,s)dsdx.$$ 

Since the boundary condition (3.3) holds, an integration by parts on each term on the right hand side produces the desired result. $\square$

We now complete the proof of Theorem 3.1 for smooth data. This will be done inductively.

**CASE n=0:** We first deduce the following lemma.

**Lemma 3.6** Under the hypotheses of Theorem 3.1 (n=0), we have

$$\int_0^\infty \int_0^a u_x^2(x,t)dxdt$$

$$\leq C_2 \left( E(0) + \int_0^\infty \int_s^\infty |g'(t)|dt\|u_x^0(s)\|^2ds \right) \quad (3.27)$$

where the constant $C_2$ depends only on $g$. 

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Proof of Lemma 3.6: Consider the quantity
\[
g(\infty)u_z^2(x, t) - \frac{1}{2} \int_{-\infty}^{t} g'(t - s)(u_z(x, t) - u_z(x, s))^2 \, ds
\]
\[+ \frac{i}{2} \frac{d}{dt} \int_{-\infty}^{t} (g(t - s) - g(\infty))u_z^2(x, s) \, ds.
\]
By squaring out the integrand in the second term and taking the derivative of the third term, we see that the above expression equals
\[
g(0)u_z^2(x, t) + u_z(x, t) \int_{-\infty}^{t} g'(t - s)u_z(x, s) \, ds.
\]
Thus, deleting the second term in the original expression, we get
\[
g(0)u_z^2(x, t) + u_z(x, t) \int_{-\infty}^{t} g'(t - s)u_z(x, s) \, ds
\]
\[\geq \frac{1}{2} g(\infty)u_z^2(x, t) + \frac{1}{2} \frac{d}{dt} \int_{-\infty}^{t} (g(t - s) - g(\infty))u_z^2(x, s) \, ds.
\] (3.28)
Now, fix \( T > 0 \). Integrating (3.28) with respect to \( x \) and \( t \), and then evaluating the second term on the right hand side yields
\[
g(0) \int_{0}^{T} \int_{0}^{a} u_z^2(x, t) \, dx \, dt + \int_{0}^{T} \int_{0}^{a} u_z(x, t) \int_{-\infty}^{t} g'(t - s)u_z(x, s) \, ds \, dx \, dt
\]
\[\geq \frac{1}{2} g(\infty) \int_{0}^{T} \int_{0}^{a} u_z^2(x, t) \, dx \, dt + \frac{1}{2} \int_{-\infty}^{T} \int_{0}^{a} (g(T - s) - g(\infty))u_z^2(x, s) \, dx \, ds
\]
\[\quad - \frac{1}{2} \int_{-\infty}^{0} \int_{0}^{a} (g(-s) - g(\infty))u_z^2(x, s) \, dx \, ds
\]
\[\geq \frac{1}{2} g(\infty) \int_{0}^{T} \int_{0}^{a} u_z^2(x, t) \, dx \, dt - \frac{1}{2} \int_{0}^{\infty} (g(t) - g(\infty)) \|w_0(t)\|^2 \, dt,
\]
where the last inequality follows by making a change of variables and discarding the second term. Using Lemma 3.5 on the second term on the left hand side of the last inequality, we obtain

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\[
\int_0^T \int_0^a u_2^2(x,t)dxdt \leq \frac{1}{g(\infty)} \int_0^\infty (g(t) - g(\infty)) ||u_2^0(t)||^2 dt \\
+ \frac{2}{g(\infty)} \int_0^T \int_0^a u(x,t)u_2(x,t)dxdt \\
=: I_1 + I_2.
\]

(3.29)

Now, by the fundamental theorem of calculus and (3.6)(n=0),

\[
I_1 \leq \frac{1}{g(\infty)} \int_0^\infty \int_s^\infty |g'(t)|dt ||u_2^0(s)||^2 ds \leq K.
\]

(3.30)

Next,

\[
I_2 = -\frac{2}{g(\infty)} \int_0^T \left[ \int_0^a (u(x,t)u_2(x,t) + u_2^2(x,t))dx \right] dt \\
+ \frac{2}{g(\infty)} \int_0^T \int_0^a u_2^2(x,t)dxdt \\
= -\frac{2}{g(\infty)} \int_0^T \frac{d}{dt} \int_0^a u(x,t)u_2(x,t)dxdt + \frac{2}{g(\infty)} \int_0^T \int_0^a u_2^2(x,t)dxdt.
\]

Evaluating the first term, we obtain

\[
I_2 \leq \frac{2}{g(\infty)} \left| \int_0^a u(x,T)u_2(x,T)dx \right| + \frac{2}{g(\infty)} \left| \int_0^a u(x,0)u_2(x,0)dx \right| \\
+ \frac{2}{g(\infty)} \int_0^T \int_0^a u_2^2(x,t)dxdt.
\]

Using Lemma 3.4, the fact that \( E(t) \leq E(0) \) for \( t \geq 0 \), and assumption (3.20) (n=0), we see that

\[
I_2 \leq \frac{k}{g(\infty)} \left( \frac{1}{g(\infty)} \right)^{\frac{1}{2}} E(0) \\
+ \frac{4}{g(\infty)} C_1 \left( E(0) + \int_0^\infty \int_s^\infty |g'(t)|dt ||u_2^0(s)||^2 ds \right).
\]

(3.31)
Since $T > 0$ is arbitrary, combining (3.29) with (3.30) and (3.31) gives (3.27) with

$$C_2 := \frac{1}{g(\infty)} + \frac{k}{g(\infty)} \left( \frac{1}{g(\infty)} \right)^\frac{1}{2} + \frac{4}{g(\infty)} C_1. \quad \Box$$

(3.32)

We can now easily prove Theorem 3.1 (n=0). Namely, since

$$\int_0^T E(t)dt = \frac{1}{2} \int_0^T \int_0^a u_t^2(x, t)dx dt + \frac{1}{2} g(\infty) \int_0^T \int_0^a u_x^2(x, t)dx dt$$

$$- \frac{1}{2} \int_0^T \int_{-\infty}^t g'(t-s) \|u_x(t) - u_x(s)\|^2 ds dt,$$

(3.33)

we immediately get the desired result (3.21)(n=0) with

$$C = C_1 + \frac{1}{2} g(\infty) C_2 + (g(0) - g(\infty)) C_2 + LC_2 + 1$$

by appealing to (3.20) (n=0), (3.27) and Lemma 3.1. Thus, the case n=0 is proved.

CASE n>0: We make the inductive assumption that the conclusion of Theorem 3.1 holds for $n-1$. Assume (3.6)(n), (3.7)(n) and (3.20) (n) hold. These imply the same hypotheses for $n-1$, so

$$\int_0^\infty \int_0^a (1+t)^{n-1} u_x^2(x, t)dx dt$$

$$\leq \tilde{C}_2 \left( E(0) + \int_0^\infty \int_0^\infty |g'(t)|(1+t-s)^{n-1} \|w_x^0(s)\|^2 ds \right)$$

(3.34)

and

$$\int_0^\infty (1+t)^{n-1} E(t)dt$$

$$\leq \tilde{C} \left( E(0) + \int_0^\infty \int_0^\infty |g'(t)|(1+t-s)^{n-1} dt \|w_x^0(s)\|^2 ds \right)$$

(3.35)

where $\tilde{C}_2$ and $\tilde{C}$ depend only on n and the kernel $g$.

As before, we first show:
Lemma 3.7 Under the hypotheses of Theorem 3.1 \((n>0)\), we have

\[
\int_0^\infty \int_0^a (1 + t)^n u_x^2(x, t)dxdt \\
\leq C_2 \left(E(0) + \int_0^\infty \int_s^{\infty} |g'(t)| (1 + t - s)^n dt \|w_x^0(s)\|^2 ds \right)
\]

(3.36)

where the constant \(C_2\) depends only on \(n\) and \(g\).

Proof of Lemma 3.7: Fix \(T > 0\). Multiplying (3.28) by \((1 + t)^n\) and integrating with respect to \(x\) and \(t\) we get

\[
g(0) \int_0^T \int_0^a (1 + t)^n u_x^2(x, t)dxdt \\
+ \int_0^T \int_0^a (1 + t)^n u_x(x, t) \int_{-\infty}^t g'(t - s)u_x(x, s)ds dx dt \\
\geq \frac{1}{2} g(\infty) \int_0^T \int_0^a (1 + t)^n u_x^2(x, t)dxdt \\
+ \frac{1}{2} \int_0^T \int_0^a (1 + t)^n \frac{d}{dt} \int_{-\infty}^t (g(t - s) - g(\infty)) u_x^2(x, s)ds dx dt \\
\geq \frac{1}{2} g(\infty) \int_0^T \int_0^a (1 + t)^n u_x^2(x, t)dxdt \\
- \frac{1}{2} \int_0^a \int_{-\infty}^0 (g(s) - g(\infty)) u_x^2(x, s) ds dx \\
- \frac{n}{2} \int_0^T \int_0^a (1 + t)^{n-1} \int_{-\infty}^t (g(t - s) - g(\infty)) u_x^2(x, s)ds dx dt,
\]

where the last inequality follows after an integration by parts on the second term. Using Lemma 3.5 on the second term on the left hand side and a change of variables on the second term on the right hand side of the last calculation, we obtain

\[
\int_0^T \int_0^a (1 + t)^n u_x^2(x, t)dxdt \\
\leq \frac{1}{g(\infty)} \int_0^\infty (g(t) - g(\infty)) \|w_x^0(t)\|^2 dt \\
- \frac{2}{g(\infty)} \int_0^T (1 + t)^n \int_0^a u(x, t) u_t(x, t) dx dt
\]
\[
+ \frac{n}{g(\infty)} \int_0^T \int_0^t (1 + t)^n \int_{-\infty}^t (g(t - s) - g(\infty)) u_x^2(x, s) ds dt \\
=: I_1 + I_2 + I_3.
\] (3.37)

Now, using the fact that (3.6) (n > 0) implies (3.6) (n = 0), and the fundamental theorem of calculus, we obtain

\[
I_1 \leq \frac{1}{g(\infty)} \int_0^\infty \int_s^\infty |g'(t)| dt \|w_x^0(s)\|^2 ds \\
\leq \frac{1}{g(\infty)} \int_0^\infty \int_s^\infty |g'(t)|(1 + t)^n dt \|w_x^0(s)\|^2 ds. 
\] (3.38)

Next, as in the proof of Lemma 3.6, we write

\[
I_2 = -\frac{2}{g(\infty)} \int_0^T (1 + t)^n \frac{d}{dt} \int_0^a u(x, t) u_t(x, t) dx dt \\
+ \frac{2}{g(\infty)} \int_0^T (1 + t)^n u_t^2(x, t) dx dt.
\]

An integration by parts on the first term gives

\[
I_2 = -\frac{2}{g(\infty)} (1 + T)^n \int_0^a u(x, T) u_t(x, T) dx + \frac{2}{g(\infty)} \int_0^a u(x, 0) u_t(x, 0) dx \\
+ \frac{2n}{g(\infty)} \int_0^T (1 + t)^{n-1} \int_0^a u(x, t) u_t(x, t) dx dt \\
+ \frac{2}{g(\infty)} \int_0^T (1 + t)^n u_t^2(x, t) dx dt.
\]

By Lemma 3.4 and (3.20) (recall that (3.18) holds for data in D(A)),

\[
I_2 \leq \frac{k}{g(\infty)} \left( \frac{1}{g(\infty)} \right)^{\frac{1}{2}} (1 + T)^n E(T) + \frac{k}{g(\infty)} \left( \frac{1}{g(\infty)} \right)^{\frac{1}{2}} E(0) \\
+ \frac{4}{g(\infty) C_1} \left( E(0) + \int_0^\infty \int_s^\infty |g'(t)|(1 + t)^n dt \|w_x^0(s)\|^2 ds \right).
\]

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Since Theorem 3.1 (n-1) holds, we have (3.35). By Lemma 3.3, we also know that \( t^n E(t) \to 0 \) as \( t \to \infty \), and hence \((i + T)^n E(T) \leq E(0)\) provided that \( T \) is sufficiently large. Combining these facts with the last inequality, we get

\[
I_2 \leq \frac{2k}{g(\infty)} \left( \frac{1}{g(\infty)} \right)^{\frac{1}{2}} E(0) + \left( \frac{k}{g(\infty)^{\frac{3}{2}}} \tilde{C} + \frac{4}{g(\infty)} C_1 \right) \left( E(0) + \int_{s}^{\infty} \int_{s}^{\infty} |g'(t)|(1 + t - s)^n dt \|w_2^0(s)\|^2 ds \right).
\]  

(3.39)

Finally, replace \( T \) with infinity in the definition of \( I_3 \), and use Fubini's Theorem to get

\[
I_3 \leq \frac{n}{g(\infty)} \int_{-\infty}^{0} \int_{0}^{\infty} (1 + t)^{n-1}(g(t - s) - g(\infty)) dt \|u_x(s)\|^2 ds
+ \frac{n}{g(\infty)} \int_{s}^{\infty} \int_{s}^{\infty} (1 + t)^{n-1}(g(t - s) - g(\infty)) dt \|u_x(s)\|^2 ds.
\]

Using the fact that \((1 + \sigma + s)^{n-1} \leq (1 + \sigma)^{n-1} \cdot (1 + s)^{n-1}\) when \( s, \sigma \geq 0 \), and several changes of variables, we get

\[
I_3 \leq \frac{n}{g(\infty)} \int_{0}^{\infty} \int_{s}^{\infty} (1 + t - s)^{n-1}(g(t) - g(\infty)) dt \|w_2^0(s)\|^2 ds
+ \frac{n}{g(\infty)} \int_{0}^{\infty} (1 + \sigma)^{n-1}(g(\sigma) - g(\infty)) d\sigma \int_{0}^{\infty} (1 + s)^{n-1} \|u_x(s)\|^2 ds
\leq \frac{n}{g(\infty)} \int_{0}^{\infty} \int_{s}^{\infty} (1 + t - s)^{n-1}(g(t) - g(\infty)) dt \|w_2^0(s)\|^2 ds
+ \frac{2n}{g(\infty)^2} \int_{0}^{\infty} (1 + t)^{n-1}(g(t) - g(\infty)) dt \int_{0}^{\infty} (1 + s)^{n-1} E(s) ds.
\]

By assumptions (3.6) (n), (3.7) (n), Lemma 3.2 and (3.35),

\[
I_3 \leq \left( \frac{n}{g(\infty)} + \frac{2n}{g(\infty)^2} L \right) \left( E(0) + \int_{0}^{\infty} \int_{s}^{\infty} |g'(t)|(1 + t - s)^n dt \|w_2^0(s)\|^2 ds \right)
\]  

(3.40)

Combining (3.37) with (3.38), (3.39) and (3.40) yields (3.36) where \( C_2 \) depends only on \( n \) and \( g \), and the lemma is proved. □
To complete the proof of (3.21), simply write

\[ \int_0^T (1 + t)^n E(t)dt \]
\[ = \frac{1}{2} \int_0^T \int_0^a (1 + t)^n u_1^2(x, t)dxdt \]
\[ + \frac{1}{2} g(\infty) \int_0^T \int_0^a (1 + t)^n u_2^2(x, t)dxdt \]
\[ - \frac{1}{2} \int_0^T (1 + t)^n \int_{-\infty}^t g'(t - s)\|u_x(t) - u_x(s)\|^2 dsdt. \]  

(3.41)

We immediately get (3.21) with C depending only on n and g by appealing to (3.20), (3.36), and Lemma 3.1. This completes the proof of Theorem 3.1 (n>0) for smooth data \((u^0, v^0, w^0) \in D(A)\). \(\Box\)

PART 2 (General Data): We now wish to prove Theorem 3.1 for general data \((u^0, v^0, w^0) \in H\) with \(w^0\) satisfying (3.6)(n) for some nonnegative integer n. To do this we will use a density argument on the appropriate subspace of \(H\) endowed with a stronger norm that keeps track of the term

\[ \int_0^\infty \int_s^\infty |g'(t)|(1 + t - s)^n dt \|w_x^0(s)\|^2 ds \]

on the right hand side of conclusion (3.21).

In order to accomplish this, for each nonnegative integer n define

\[ h_n(s) = \int_s^\infty |g'(t)|(1 + t - s)^n dt, \]

and let \(H_n\) consist of those \((u, v, w) \in H\) with

\[ \int_0^\infty h_n(s)\|w_x^0(s)\|^2 ds < \infty. \]

Define the norm on \(H_n\) by

\[ \|(u, v, w)\|_{H_n}^2 = \|(u, v, w)\|_H^2 + \int_0^\infty h_n(s)\|w_x^0(s)\|^2 ds. \]

We have
Proposition 3.1 For each nonnegative integer \( n \), \( D(A) \cap H_n \) is dense in \( H_n \).

Proof of Proposition 3.1: Let \((u, v, w) \in H_n \) and \( \epsilon > 0 \) be given. We must find \((p, q, r) \in D(A) \cap H_n \) such that \( \|(u, v, w) - (p, q, r)\|_{H_n}^2 < \epsilon \). Note that for any \((p, q, r) \in H_n \),

\[
\|(u, v, w) - (p, q, r)\|_{H_n}^2 = \langle u_x - p_x \rangle^2 + \langle v - q \rangle^2
\]

\[
+ \int_0^\infty \|g'(\xi)\| \langle u_x - p_x \rangle - \langle w_x - r_x \rangle \| d\xi
\]

\[
+ \int_0^\infty h_n(\xi) \| w_x - r_x \|^2 d\xi
\]

\[
\leq \langle u_x - p_x \rangle^2 + \langle v - q \rangle^2
\]

\[
+ 2 \int_0^\infty \|g'(\xi)\| \langle u_x - p_x \rangle^2 d\xi
\]

\[
+ 2 \int_0^\infty \|g'(\xi)\| \langle w_x - r_x \rangle^2 d\xi
\]

\[
+ \int_0^\infty h_n(\xi) \| w_x - r_x \|^2 d\xi,
\]

where the last inequality follows by the triangle inequality and the elementary inequality \( 2ab \leq a^2 + b^2 \). We remark that since (3.7)(n) holds, \( h_n \in C[0, \infty) \) and \( h_n(\xi) \to 0 \) as \( \xi \to \infty \).

Since \( H^2(\Omega) \cap H^1_0(\Omega) =: H^2 \cap H^1_0 \) is dense in \( H^1_0 \) and since \( H^1_0 \) is dense in \( H^0 \), we can choose \( p \in H^2 \cap H^1_0 \) and \( q \in H^1_0 \) such that

\[
(2g(0) - g(\infty)) \langle u_x - p_x \rangle^2 < \frac{\epsilon}{4} \text{ and } \langle v - q \rangle^2 < \frac{\epsilon}{4}.
\]

By the definition of the Bochner integral (see [17, Section 3.7]), we can find a simple function \( w_\epsilon \) and numbers \( 0 < \delta < \Delta \) such that \( w_\epsilon \) is supported on \([\delta, \Delta] \) and

\[
\int_0^\infty (2|g'(\xi)| + h_n(\xi)) \langle (w_\epsilon)_x - w_x \rangle^2 d\xi < \frac{\epsilon}{8}.
\]

Moreover, since \( H^2 \cap H^1_0 \) is dense in \( H^1_0 \), we can assume that \( w_\epsilon \) takes values in \( H^2 \cap H^1_0 \). By modifying \( w_\epsilon \), we can find \( r \in C^1(\mathbb{R}^+; H^2 \cap H^1_0) \) such that \( r(0) = p \), \( r(t) \equiv 0 \) for \( t \geq \Delta + 1 \) and

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\[ \int_0^\infty (2|g'(\xi)| + h_n(\xi)) \|(w_n)_x - r_x\|^2 d\xi < \frac{\epsilon}{8}. \]  

(3.45)

Combining (3.42), (3.43), (3.44) and (3.45) we see that \( \|(u, v, w) - (p, q, r)\|_{H_n}^2 < \epsilon. \) In addition, \( g(0)p_{xx} + \int_0^\infty g'(\xi) r_{xx}(\cdot, \xi) d\xi \in H^0(\Omega) \) and \( \frac{\partial}{\partial \xi} \in L^2(\mathbb{R}^+; |g'|; H^1_0(\Omega)) \) since \( r \in C^1(\mathbb{R}^+; H^2 \cap H^1_0) \) and \( r \) has compact support. Thus, \( (p, q, r) \in D(A) \cap H_n \) and Proposition 3.1 is proved. \( \Box \)

Proof of Theorem 3.1 (General Data): Let \( n \) be a nonnegative integer, and let \( (u^0, v^0, w^0) \in H \) with \( w^0 \) satisfying (3.6)(n), that is, \( (u^0, v^0, w^0) \in H_n. \) By Proposition 3.1 we can find a sequence \( \{(u^0_m, v^0_m, w^0_m)\} \) in \( D(A) \cap H_n \) converging to \( (u^0, v^0, w^0) \) in \( H_n; \) in particular,

\[ \|(u^0_m, v^0_m, w^0_m) - (u^0, v^0, w^0)\|_H \to 0 \text{ as } m \to \infty \]  

(3.46)

and

\[ \int_0^\infty h_n(s)\|(w^0_m)_x(s)\|^2 ds \to \int_0^\infty h_n(s)\|w^0_x(s)\|^2 ds \text{ as } m \to \infty. \]  

(3.47)

By Theorem 3.1 for data in \( D(A) \cap H_n \) we have

\[ \int_0^\infty (1 + t)^n E_m(t) dt \leq C \left( E_m(0) + \int_0^\infty h_n(s)\|(w^0_m)_x(s)\|^2 ds \right), \]  

(3.48)

where \( C \) depends only on \( n \) and the kernel \( g. \) Here

\[ E_m(t) = \frac{1}{2} \|T(t)(u^0_m, v^0_m, w^0_m)\|_H^2. \]

Since \( \|(u^0_m, v^0_m, w^0_m) - (u^0, v^0, w^0)\|_H \to 0 \) and \( T(t) \) is a \( C_0 \)-semigroup of contractions, \( E_m(t) \to E(t) \) uniformly on \([0, \infty)\) as \( m \to \infty. \) Thus, letting \( m \to \infty \) in (3.48) we obtain,

\[ \int_0^\infty (1 + t)^n E(t) dt \leq C \left( E(0) + \int_0^\infty h_n(s)\|w^0_x(s)\|^2 ds \right), \]

and by the definition of \( h_n \) this is (3.21). \( \Box \)

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We remark that our proof of Theorem 3.1 does not appear to explicitly use the hypotheses that \( g \) be convex or that \( g \in C^2(0, \infty) \). However, these hypotheses are needed to conclude that \( E(t) \to 0 \), and we feel that they are necessary to prove the integrability of the kinetic energy term. For example, Hannsgen [10] constructs an example of a positive, decreasing, convex, piecewise linear kernel \( g(t) \) (so \( g \in C^2(0, \infty) \) does not hold) such that the Laplace transform of the resolvent equation corresponding to equation (2.29) of Chapter 2 has a pair of poles on the imaginary axis. For this \( g \) we cannot conclude that \( E(t) \to 0 \), and the integrability of \( E(t) \) must fail to hold.
Chapter 4

ENERGY DISSIPATION IN
THERMO-ELASTIC AND
THERMO-VISCOELASTIC SYSTEMS

Throughout this Chapter, we consider the thermo-viscoelastic system

\[ \rho w_{tt}(x, t) = g(\infty)w_{xx}(x, t) + \frac{d}{dt} \int_0^t [g(t - \tau) - g(\infty)]w_{xx}(x, \tau) d\tau - \gamma \theta_x(x, t), \]  \hspace{1cm} (4.1)  

\[ c\theta_t(x, t) = k\theta_{xx}(x, t) - \gamma \theta_a w_{xt}(x, t), \]  \hspace{1cm} (4.2)  

where \( 0 < x < 1 \) and \( t > 0 \), with initial conditions

\[ \theta(x, 0) = \theta_0(x), \ w(x, 0) = w_0(x) \ and \ w_t(x, 0) = w_1(x), \]  \hspace{1cm} (4.3)  

and boundary conditions

\[ \theta(i, t) = 0, \ i = 0, 1 \]  \hspace{1cm} (4.4)  

and

\[ g(\infty)w_x(i, t) + \frac{d}{dt} \int_0^t [g(t - \tau) - g(\infty)]w_x(i, \tau) d\tau = 0, \ i = 0, 1. \]  \hspace{1cm} (4.5)  

Here \( \rho > 0 \) is the mass density, \( \theta_a > 0 \) is the ambient reference temperature, \( \gamma \geq 0 \) is the coupling coefficient, \( k > 0 \) is the heat conductivity, and \( c > 0 \) is the specific heat. We will assume throughout this Chapter that the kernel \( g(t) \) satisfies the following conditions:
$g$ is completely monotonic on $(0, \infty)$, $g \in L^1(0, 1)$

and $0 < g(\infty) < g(0^+) \leq \infty$.

Problem (4.1)-(4.2) is a model for torsional motion in a linear viscoelastic rod which is assumed to have constant mass density and length 1. Mechanical energy is converted to heat, which is dissipated to affect damping. Here, $\theta(x, t)$ is the difference of the temperature at time $t$ and position $x$ from the reference temperature $\theta_0$, and $w(x, t)$ represents the angular displacement from equilibrium at position $x$ at time $t$. We assume that the displacement is identically zero for $t < 0$. We remark that boundary condition (4.5) means that the ends of the rod are stress-free. We note that equation (4.1) has been derived using a linear Boltzmann model for the viscoelastic stress-strain law. For a more detailed derivation of the form of the viscoelastic stress-strain law as it appears in (4.1), we refer the reader to [11, Introduction]. Note that when written in this specific form, equation (4.1) allows the case $g(0^+) = \infty$ (see (4.6)). For a thorough discussion and derivation of the equations of motion of thermo-viscoelasticity (in three dimensions) see Navarro [25].

We note that in the elastic case ($g(t) \equiv g(\infty)$), problem (4.1)-(4.2) reduces to the linear thermo-elastic system

$$\rho w_{tt}(x, t) = g(\infty)w_{xx}(x, t) - \gamma \theta_x(x, t), \quad (4.7)$$

$$c\theta_t(x, t) = k\theta_{xx}(x, t) - \gamma \theta_x w_t(x, t), \quad (4.8)$$

for $x \in \Omega := (0, 1)$, and $t > 0$. For a thorough discussion of thermo-elastic systems, and the physical meaning of the coupling coefficient $\gamma$, the reader is directed to Day’s book [7]. Recently, several authors [16], [18], [24] have shown that the system (4.7),(4.8) together with conservative boundary conditions is uniformly exponentially stable.

More precisely, Scott Hansen [16] examined the thermo-elastic system (4.7),(4.8) with boundary conditions
\[ w_x(i, t) = \theta(i, t) = 0, \quad i = 0, 1. \quad (4.9) \]

The energy of the system is given by

\[ E(t) = \frac{1}{2} \int_0^1 \left[ g(\infty)w_x^2 + \rho w_t^2 + \frac{c}{\theta_a} \theta^2 \right] dx. \quad (4.10) \]

Hansen writes the system in the form

\[ \frac{dy}{dt} = Ay \quad (4.11) \]

where \( y = (y_1, y_2, y_3)^T = (w_x, \rho w_t, \theta)^T \) and

\[
A = \begin{pmatrix}
0 & \frac{1}{\rho} D & 0 \\
g(\infty) D & 0 & -\gamma D \\
0 & -\frac{\gamma \theta_a}{\rho c} D & \frac{k}{c} D^2 \\
\end{pmatrix} \quad (4.12)
\]

with domain

\[ D(A) = H^1_0(\Omega) \times H^0(\Omega) \times (H^2(\Omega) \cap H^1_0(\Omega)). \quad (4.13) \]

Here \( D = \frac{d}{dx} \). Using the Lumer-Phillips theorem, he proves that \( A \) is the generator of a strongly continuous semigroup of contractions on the space \( H = H^0(\Omega) \times H^0(\Omega) \times H^0(\Omega) \) with energy norm

\[ \|y\|_H^2 = g(\infty)\|y_1\|_{H^0(\Omega)}^2 + \frac{1}{\rho}\|y_2\|_{H^0(\Omega)}^2 + \frac{c}{\theta_a}\|y_3\|_{H^0(\Omega)}^2. \quad (4.14) \]

Hansen then proves his main theorem and an important corollary using spectral estimates and Fourier methods:

**Theorem D.** Let \( A \) be the operator defined in (4.12), (4.13). Then the eigenfunctions of \( A \) form a Riesz basis for \( H \) and there exists \( \beta > 0 \) such that
\[
\sup_{\lambda \in \sigma(A) \setminus \{0\}} \Re \lambda \leq -\beta. \tag{4.15}
\]

A Riesz basis \( \{\phi_k\} \) for a separable Hilbert space \( X \) is the image of an orthonormal basis \( \{e_k\} \) through a bounded invertible operator \( T : X \to X \). For the boundary conditions given, \( A \) has nontrivial nullspace \( \mathcal{N}(A) = \text{span} \{(0,1,0)^T\} \) and Hansen defines \( \mathcal{H} = \mathcal{N}(A)^\perp \). Theorem D is now used to obtain a uniform exponential energy decay rate of solutions to (4.7), (4.8) and (4.9) as follows:

**Corollary D.** With \( A \) and \( \beta \) as in Theorem D, the restriction of \( A \) to \( \mathcal{H} \) is the generator of an exponentially stable semigroup \( T(t) \) on \( \mathcal{H} \) which satisfies

\[
\|T(t)\|_{\mathcal{L}(\mathcal{H})} \leq Me^{-\beta t} \tag{4.16}
\]

for some \( M \geq 1 \) and any \( t \geq 0 \).

We note that Hansen considers several other sets of conservative boundary conditions, but he does not handle Dirichlet–Dirichlet (displacement and temperature, respectively). J.U. Kim felt that Hansen's methods could not be extended to handle the Dirichlet–Dirichlet boundary conditions, so he used energy methods and multiplier techniques in his analysis [18] of (4.7), (4.8) with boundary conditions

\[
w(i, t) = \theta(i, t) = 0, \quad i = 0, 1 \tag{4.17}
\]

and initial conditions

\[
w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \quad \theta(x, 0) = \theta_0(x). \tag{4.18}
\]

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After proving several regularity results, Kim proves his main theorem:

**Theorem E.** Let \((w, \theta)\) be a solution of (4.7), (4.8) with boundary and initial conditions (4.17), (4.18) with \(w_0 \in H_0^1(\Omega), w_1 \in H^0(\Omega)\) and \(\theta_0 \in H^0(\Omega)\). Then, it holds that the energy \(E(t)\) defined in (4.10) satisfies

\[
E(t) \leq Me^{-\alpha t}E(0)
\]

for all \(t \geq 0\), with some positive constants \(M\) and \(\alpha\) independent of \(w_0, w_1\) and \(\theta_0\).

We remark that Kim obtains analogous results for a linear thermo-elastic plate also.

Z. Liu and S. Zheng [24] showed that Dirichlet–Dirichlet boundary conditions could also be handled using semigroup methods, which they used in their analysis of (4.7), (4.8) with boundary conditions (4.17) and initial conditions (4.18). By introducing a new variable \(v = \rho w_t\), Liu and Zheng reduce the system (4.7), (4.8) and (4.17) to an abstract first order evolution equation

\[
\frac{dy}{dt} = Ay,
\]

where \(y = (y_1, y_2, y_3)^T = (w, v, \theta)^T\) and

\[
A = \begin{pmatrix}
0 & \frac{1}{\rho} I & 0 \\
g(\infty)D^2 & 0 & -\gamma D \\
0 & -\frac{\gamma}{\rho c} D & \frac{k}{c} D^2
\end{pmatrix},
\]

with domain \(D(A) = (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))\) (compare [5, Section 5]). The state space \(H\) is defined by \(H = H_0^1(\Omega) \times H^0(\Omega) \times H^0(\Omega)\) equipped with the energy norm

\[
\|y\|^2_H = g(\infty)\|Dy_1\|^2 + \frac{1}{\rho}\|y_2\|^2 + \frac{c}{\theta_a}\|y_3\|^2
\]

where \(\| \cdot \|\) is the usual \(L^2\) norm on \(H^0(\Omega)\).
The main result is:

**Theorem F.** The operator $A : \mathcal{D}(A) \subset H \to H$ defined in (4.21) generates a $C_0$-semigroup of contractions $T(t)$ on $H$. Moreover, $T(t)$ is exponentially stable, i.e. there exist constants $M > 0, \alpha > 0$ such that

$$\|T(t)\|_{\mathcal{L}(H,H)} \leq Me^{-\alpha t} \text{ for all } t > 0. \quad (4.22)$$

Recently, it has been shown that linear thermo-viscoelastic systems are asymptotically stable [23], [25]. More precisely, in his doctoral thesis Z. Liu [23] (see also [1]) examined the thermo-viscoelastic system described by the history value problem

$$\rho w_{tt}(x,t) = \frac{\partial}{\partial x} \left[ \alpha w_x(x,t) + \int_{-r}^{0} m(s)w_x(x,t+s)ds \right] - \gamma \theta_x(x,t) \quad (4.23)$$

$$c\theta_t(x,t) = k\theta_{xx}(x,t) - \gamma \theta_x w_x(x,t) \quad (4.24)$$

where $0 < x < 1$ and $t > 0$, with boundary conditions

$$w(i,t) = \theta(i,t) = 0, \quad i = 0, 1. \quad (4.25)$$

Here the delay $r > 0$ may be finite or infinite. It is assumed that the kernel $m(t) : [-r,0) \to \mathbb{R}$ satisfies

$$m \in L^1(-r,0), \quad m(t) \leq 0 \text{ on } [-r,0), \quad (4.26)$$

$$m \text{ is absolutely continuous on } [-r,-\delta] \text{ for all } r > \delta > 0 \quad (4.27)$$

and $m'(t) \leq 0$ for $-r \leq t < 0$,
and

there is a constant $\epsilon > 0$ such that $\epsilon = \alpha + \int_{-\tau}^{0} m(s)ds. \quad (4.28)$

We remark that in the notation of our thesis, $\alpha$ in equation (4.23) is $\alpha = g(0)$ and $m(s) = g'(-s)$ for $s \in [-\tau, 0)$ so

$$
\int_{-\tau}^{0} m(s)w_x(x, t + s)ds = \int_{0}^{\tau} g'(s)w_x(x, t - s)ds.
$$

Liu allows the kernel $m$ to have a singularity at $s = 0$ (that is, $g'(s)$ may have a singularity at $s = 0$ so the system (4.23), (4.24) is weakly singular). We remark that Navarro requires that $m(0^+)$ be finite in [25].

Liu shows that system (4.23), (4.24) with boundary conditions (4.25) may be recast as a first order abstract differential equation in a suitable state space. He then shows this system generates a $C_0$-semigroup and he proves

**Theorem G.** The thermo-viscoelastic system (4.23), (4.24) with boundary conditions (4.25) is asymptotically stable if in addition to (4.26)-(4.28), the kernel $m$ also satisfies:

$$
\lim_{t \to 0^-} t^2 m(t) = 0, \quad (4.29)
$$

and

$$
m'(t) \neq 0 \text{ in } -\sigma < t < 0 \text{ for some } \sigma > 0. \quad (4.30)
$$

Here we have used the terminology that a linear system

$$
\frac{d}{dt}z(t) = Lz(t), \quad z(0) = z_0 \in Z \quad (4.31)
$$

where $Z$ is a Hilbert space and $L$ is the infinitesimal generator of a $C_0$-semigroup $S(t)$, $t > 0$, is said to be asymptotically stable if for every initial condition $z_0 \in Z$, the corresponding
mild solution $z(t) \to 0$ as $t \to \infty$. We remark that Liu also considered system (4.23)-(4.25) with a control added.

In the next Chapter, we use Laplace transform methods to examine the rate at which the angular displacement $w(x, t)$ for problem (4.1)-(4.5) tends to its equilibrium. We give a general result concerning the complex solutions of the characteristic equation for this problem. These describe the oscillatory vibrations of $w(x, t)$. We also consider the specific kernel $g(t) = g(\infty) + \frac{s^{\eta-1}}{\Gamma(\eta)}$, $0 < \eta < 1$; for this kernel the creep modes dominate the rate at which $w(x, t)$ tends to equilibrium for the pure viscoelastic system with no thermal coupling ($\gamma = 0$), see [13]. We derive an asymptotic expansion (valid for large $t$) for the solution $w(0, t)$ monitored at the end $x = 0$. This example shows that the addition of thermal dissipation ($\gamma > 0$) can actually give a rate of decay to equilibrium for $w(0, t)$ that is slower than that for the system with no thermal coupling ($\gamma = 0$).
Chapter 5

ANALYSIS OF A THERMO-VISCOELASTIC SYSTEM

We consider the thermo-viscoelastic system

\[ \rho w_{tt}(x, t) = g(\infty)w_{xx}(x, t) + \frac{d}{dt} \int_0^t [g(t - \tau) - g(\infty)]w_{xx}(x, \tau)d\tau - \gamma \theta_x(x, t), \quad (5.1) \]

\[ c\theta_t(x, t) = k\theta_{xx}(x, t) - \gamma \theta_x w_{xt}(x, t), \quad (5.2) \]

where \(0 < x < 1\) and \(t > 0\), with initial conditions

\[ \theta(x, 0) = \theta_0(x), \quad w(x, 0) = w_0(x) \quad \text{and} \quad w_t(x, 0) = w_1(x), \quad (5.3) \]

and boundary conditions

\[ \theta(i, t) = 0, \quad i = 0, 1 \quad (5.4) \]

and

\[ g(\infty)w_x(i, t) + \frac{d}{dt} \int_0^t [g(t - \tau) - g(\infty)]w_x(i, \tau)d\tau = 0, \quad i = 0, 1. \quad (5.5) \]

We assume throughout this Chapter that

\[ g \text{ is completely monotonic on } (0, \infty), \quad g \in L^1(0, 1) \quad (5.6) \]

and \(0 < g(\infty) < g(0^+) \leq \infty.\)
We will analyze the behavior of the system (5.1)–(5.5) by using transform methods. We begin by taking the Laplace transform of (5.1) and (5.2) and examining the behavior of this new system with appropriate boundary conditions derived from (5.4) and (5.5). Recall that the Laplace transform of a function \( f(t) \) is defined by \( \hat{f}(s) = \int_0^\infty e^{-st}f(t)dt \). Our analysis will be broken down into five sections. In Section 5.1, we obtain an integral representation formula for \( W(x, s) := \hat{w}(x, s) \). In Section 5.2, we examine the location of the poles of \( W(x, s) \) by examining the location of the zeros of the appropriate characteristic equations \( \Delta_n(s) = 0, n = 1, 2, \ldots \). We prove two results about the zeros of \( \Delta_n(s) \). These results show that for each \( n, \Delta_n(s) \) has no zeros in \( \Re s \geq 0 \), and furthermore that for each \( n, \Delta_n(s) \) has at most one pair of complex conjugate zeros in \( \Re s < 0, 3s \neq 0 \). Also, for the specific kernel \( g(t) = g(\infty) + \frac{\delta t^{-1}}{1(\eta)}, 0 < \eta < 1 \), we show that there exists an angle \( \psi, \frac{\pi}{3} < \psi < \pi \), such that for all \( n, \Delta_n(s) \) has no zeros in the sector \( -\psi \leq \arg s \leq \psi \). This result will be used in Section 5.5. In Section 5.3, we monitor \( W(x, s) \) at the left end \( x = 0 \), and obtain an integral representation formula for \( W(0, s) \) in terms of a Green’s kernel when \( \theta_0(x) \equiv \emptyset \) and \( w_1(x) \equiv 0 \). In Section 5.4, we obtain an asymptotic series expansion for \( W(0, s) \) valid near \( s = 0 \) for the specific kernel \( g(t) = g(\infty) + \frac{\delta t^{-1}}{1(\eta)}, 0 < \eta < 1 \). In Section 5.5, we invert the asymptotic expansion for \( W(0, s) \) to obtain an asymptotic expansion for \( w(0, t) \) valid for large \( t \). We will find that for this kernel the rate of decay of \( w(0, t) \) to its equilibrium value is actually slower when there is thermal coupling (\( \gamma > 0 \)) than it is when there is no thermal coupling (\( \gamma = 0 \)).

5.1 An Integral Representation Formula for \( W(x, s) \)

We wish to simplify the coupled system of two equations in \( W \) and \( \Theta \) to one equation in \( W \). Taking transforms of (5.1) and (5.2) and letting \( W(x, s) \) and \( \Theta(x, s) \) be the Laplace transforms of \( w(x, t) \) and \( \theta(x, t) \), respectively, we get

\[
\rho s^2 W - \rho s w_0 - \rho w_1 = s\hat{g}(s)W'' - \gamma \Theta'
\]  

(5.7)
\[ c_s \Theta - c \theta_0 = k \Theta'' - \gamma \theta_a s W' + \gamma \theta_a w'_0. \] (5.8)

where \( ' = \frac{d}{dx} \) and the variables \( x \) and \( s \) are suppressed in \( W \) and \( \Theta \) for convenience. (Throughout this Chapter, we assume that \( s \) lies in the slit plane \( \mathcal{C}' \equiv \mathcal{C} \setminus (-\infty, 0] \).)

Dividing (5.7) by \( s \tilde{g}(s) \) and dividing (5.8) by \( k \) we get

\[ W'' - \frac{\rho s}{\tilde{g}} W - \frac{\gamma}{s \tilde{g}} \Theta' = -\frac{\rho (s w_0 + w_1)}{s \tilde{g}} =: f(x, s) \] (5.9)

\[ \Theta'' - \frac{c s}{k} \Theta - \frac{\gamma \theta_a s}{k} W' = -\frac{(c \theta_0 + \gamma \theta_a w'_0)}{k} =: h(x) \] (5.10)

after some simple rearranging. Differentiating (5.9) and substituting from (5.10) for \( \Theta'' \), then differentiating again and substituting from (5.9) for \( \Theta' \), we obtain

\[ W''' - \left( \frac{\rho s}{\tilde{g}} + \frac{\gamma^2 \theta_a}{k \tilde{g}} + \frac{c s}{k} \right) W'' + \frac{\rho c s^2}{k \tilde{g}} W = f'' + \frac{\gamma}{s \tilde{g}} h' - \frac{c s}{k} f =: F(x, s). \] (5.11)

Taking transforms of (5.4) and (5.5), we obtain boundary conditions

\[ \Theta(i, s) = 0, \quad i = 0, 1 \] (5.12)

and

\[ s \tilde{g}(s) W'(i, s) = 0, \quad i = 0, 1. \] (5.13)

Since \( s \tilde{g}(s) \) is not zero, (5.13) gives

\[ W'(i, s) = 0, \quad i = 0, 1. \] (5.14)

Since we have reduced the system to an equation in \( W \), we want the boundary conditions to also be in terms of \( W \). Using the differentiated version of (5.9) and substituting for \( \Theta'' \) from (5.10), equations (5.12) and (5.14) imply

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\[ W''(i, s) = 0, \quad i = 0, 1, \quad (5.15) \]

provided we assume \( \theta_0(i) = w'_0(i) = w'_1(i) = 0 \) for \( i = 0, 1 \). So, the new boundary conditions are (5.14) and (5.15).

The characteristic equation for the ordinary differential equation (5.11) is

\[ r^4 - \left( \frac{\rho s}{\bar{g}} + \frac{\gamma^2 \theta_a}{k \bar{g}} + \frac{cs}{k} \right) r^2 + \frac{\rho cs^2}{k \bar{g}} = 0. \]

Using the quadratic formula, we get

\[ r^2 = \frac{-\left( \frac{\rho s}{\bar{g}} + \frac{\gamma^2 \theta_a}{k \bar{g}} + \frac{cs}{k} \right) \pm \sqrt{\left( \frac{\rho s}{\bar{g}} + \frac{\gamma^2 \theta_a}{k \bar{g}} + \frac{cs}{k} \right)^2 - 4 \frac{\rho cs^2}{k \bar{g}}}}{2} \]

\[ = t_+, t_- \quad (5.16) \]

So, \( r = \pm \sqrt{t_+}, \pm \sqrt{t_-} \). We remark that both \( t_- \) and \( t_+ \) are functions of \( s \). We have suppressed the variable \( s \) for convenience. Assuming that \( t_- \neq t_+ \), we can now write a formula for \( W(x, s) \)

\[ W(x, s) = C_1 \cosh \sqrt{t_+} x + C_2 \sinh \sqrt{t_+} x \]

\[ + C_3 \cosh \sqrt{t_-} x + C_4 \sinh \sqrt{t_-} x + W_p(x, s) \quad (5.17) \]

where \( W_p(x, s) \) is a particular solution to (5.11). Differentiating \( W(x, s) \) and using the boundary conditions (5.14) and (5.15), we obtain the following set of equations

\[ W'(0, s) = C_2 t_+^{\frac{1}{2}} + C_4 t_-^{\frac{1}{2}} + W'_p(0, s) = 0 \]

\[ W'(1, s) = C_1 t_+^{\frac{1}{2}} \sinh \sqrt{t_+} + C_2 t_+^{\frac{1}{2}} \cosh \sqrt{t_+} \]

\[ + C_3 t_-^{\frac{1}{2}} \sinh \sqrt{t_-} + C_4 t_-^{\frac{1}{2}} \cosh \sqrt{t_-} + W'_p(1, s) = 0 \]
\[ W'''(0, s) = C_2 t_+^{\frac{3}{2}} + C_4 t_-^{\frac{3}{2}} + W_p'''(0, s) = 0 \]

\[ W'''(1, s) = C_1 t_+^{\frac{3}{2}} \sinh \sqrt{t_+} + C_2 t_+^{\frac{3}{2}} \cosh \sqrt{t_+} + C_3 t_-^{\frac{3}{2}} \sinh \sqrt{t_-} + C_4 t_-^{\frac{3}{2}} \cosh \sqrt{t_-} + W_p'''(1, s) = 0. \]

In matrix form:

\[
\begin{bmatrix}
0 & t_+^{\frac{1}{2}} & 0 & t_-^{\frac{1}{2}}
\end{bmatrix}
\begin{bmatrix}
0 & t_+^{\frac{1}{2}} & 0 & t_-^{\frac{1}{2}}
\end{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
C_4 
\end{bmatrix}
\]

\[
\begin{bmatrix}
-W_p'(0, s) \\
-W_p'(1, s) \\
-W_p''(0, s) \\
-W_p'''(1, s) 
\end{bmatrix}
\]

(5.18)

The Wronskian is

\[
W = \begin{vmatrix}
\cosh \sqrt{t_+} x & \sinh \sqrt{t_+} x & \cosh \sqrt{t_-} x & \sinh \sqrt{t_-} x \\
t_+^{\frac{1}{2}} \sinh \sqrt{t_+} x & t_+^{\frac{1}{2}} \cosh \sqrt{t_+} x & t_-^{\frac{1}{2}} \sinh \sqrt{t_-} x & t_-^{\frac{1}{2}} \cosh \sqrt{t_-} x \\
t_+ \cosh \sqrt{t_+} x & t_+ \sinh \sqrt{t_+} x & t_- \cosh \sqrt{t_-} x & t_- \sinh \sqrt{t_-} x \\
t_-^{\frac{3}{2}} \sinh \sqrt{t_+} x & t_-^{\frac{3}{2}} \cosh \sqrt{t_+} x & t_-^{\frac{3}{2}} \sinh \sqrt{t_-} x & t_-^{\frac{3}{2}} \cosh \sqrt{t_-} x
\end{vmatrix}
\]

By Abel’s Theorem, the Wronskian is a constant and may be evaluated at any convenient
value of \( x \). We choose \( x = 0 \). Then

\[
W = \begin{vmatrix}
1 & 0 & 1 & 0 \\
0 & t_+^{\frac{1}{2}} & 0 & t_-^{\frac{1}{2}} \\
t_+ & 0 & t_- & 0 \\
0 & t_+^{\frac{3}{2}} & 0 & t_-^{\frac{3}{2}}
\end{vmatrix} = \begin{vmatrix}
t_+^{\frac{1}{2}} & 0 & t_-^{\frac{1}{2}} & 0 \\
0 & t_- & 0 & t_+ \\
t_+^{\frac{3}{2}} & 0 & t_-^{\frac{3}{2}} & 0
\end{vmatrix} + \begin{vmatrix}
1 & 0 & 1 & 0 \\
0 & t_+^{\frac{1}{2}} & 0 & t_-^{\frac{1}{2}} \\
t_+ & 0 & t_- & 0 \\
0 & t_+^{\frac{3}{2}} & 0 & t_-^{\frac{3}{2}}
\end{vmatrix} = (t_- - t_+)(t_+^{\frac{1}{2}}t_-^{\frac{1}{2}} - t_+^{\frac{3}{2}}t_-^{\frac{3}{2}})
\]

Thus,

\[
W = t_+^{\frac{1}{2}}t_-^{\frac{1}{2}}(t_- - t_+)^2. \quad (5.19)
\]

Now, \( W_p(x, s) \) is given by the formula

\[
W_p(x, s) = \cosh \sqrt{t_+} x \int_0^x \frac{W_1}{W} F(\tau, s) d\tau + \sinh \sqrt{t_+} x \int_0^x \frac{W_2}{W} F(\tau, s) d\tau \\
+ \cosh \sqrt{t_-} x \int_0^x \frac{W_3}{W} F(\tau, s) d\tau + \sinh \sqrt{t_-} x \int_0^x \frac{W_4}{W} F(\tau, s) d\tau \quad (5.20)
\]

where \( W \) is the Wronskian and the \( W_i \)'s are computed by replacing the \( i^{th} \) column of \( W \).
with \((0,0,0,1)^T\). We now compute the \(W_i\)'s:

\[
W_1 = \begin{vmatrix}
0 & \sinh \sqrt{t_+}x & \cosh \sqrt{t_-}x & \sinh \sqrt{t_-}x \\
0 & t_+^\frac{1}{2} \cosh \sqrt{t_+}x & t_-^\frac{1}{2} \sinh \sqrt{t_-}x & t_+^\frac{1}{2} \cosh \sqrt{t_-}x \\
0 & t_+ \sinh \sqrt{t_+}x & t_- \cosh \sqrt{t_-}x & t_- \sinh \sqrt{t_-}x \\
1 & t_+^\frac{3}{2} \cosh \sqrt{t_+}x & t_-^\frac{3}{2} \sinh \sqrt{t_-}x & t_-^\frac{3}{2} \cosh \sqrt{t_-}x
\end{vmatrix},
\]

\[
= -\sinh \sqrt{t_+}x \begin{vmatrix} t_-^\frac{1}{2} \sinh \sqrt{t_-}x & t_+^\frac{1}{2} \cosh \sqrt{t_-}x \\
t_- \cosh \sqrt{t_-}x & t_- \sinh \sqrt{t_-}x \end{vmatrix}
+ t_+^\frac{1}{2} \cosh \sqrt{t_+}x \begin{vmatrix} \cosh \sqrt{t_-}x & \sinh \sqrt{t_-}x \\
t_- \cosh \sqrt{t_-}x & t_- \sinh \sqrt{t_-}x \end{vmatrix}
- t_+ \sinh \sqrt{t_+}x \begin{vmatrix} \cosh \sqrt{t_-}x & \sinh \sqrt{t_-}x \\
t_-^\frac{1}{2} \sinh \sqrt{t_-}x & t_-^\frac{1}{2} \cosh \sqrt{t_-}x \end{vmatrix}
= -\sinh \sqrt{t_+}x(t_-^\frac{3}{2} \sinh^2 \sqrt{t_-}x - t_+^\frac{3}{2} \cosh^2 \sqrt{t_-}x)
+ t_+^\frac{1}{2} \cosh \sqrt{t_+}x(t_- \sinh \sqrt{t_-}x \cosh \sqrt{t_-}x - t_- \sinh \sqrt{t_-}x \cosh \sqrt{t_-}x)
- t_+ \sinh \sqrt{t_+}x(t_-^\frac{1}{2} \cosh^2 \sqrt{t_-}x - t_-^\frac{3}{2} \sinh^2 \sqrt{t_-}x)
= t_-^\frac{3}{2} \sinh \sqrt{t_+}x - t_-^\frac{1}{2} t_+ \sinh \sqrt{t_+}x.
\]
So,

\[ W_1 = \frac{1}{2} (t_+ - t_-) \sinh \sqrt{t_+} x. \]  

(5.21)

The other \(W_i\)'s are computed similarly and are given below.

\[ W_2 = \frac{1}{2} (t_+ - t_-) \cosh \sqrt{t_+} x, \]  

(5.22)

\[ W_3 = \frac{1}{2} (t_+ - t_-) \sinh \sqrt{t_-} x, \]  

(5.23)

and

\[ W_4 = \frac{1}{2} (t_+ - t_-) \cosh \sqrt{t_-} x. \]  

(5.24)

Substituting (5.21)–(5.24) into (5.20), and simplifying, we see that

\[
W_p(x, s) = \frac{1}{t_+^2 (t_+ - t_-)} \int_0^x \sinh \sqrt{t_+} \tau \cosh \sqrt{t_+} x F(\tau, s) d\tau
- \frac{1}{t_-^2 (t_+ - t_-)} \int_0^x \cosh \sqrt{t_-} \tau \sinh \sqrt{t_+} x F(\tau, s) d\tau
+ \frac{1}{t_-^2 (t_+ - t_-)} \int_0^x \sinh \sqrt{t_-} \tau \cosh \sqrt{t_-} x F(\tau, s) d\tau
- \frac{1}{t_-^2 (t_+ - t_-)} \int_0^x \cosh \sqrt{t_-} \tau \sinh \sqrt{t_-} x F(\tau, s) d\tau.
\]

Using the trigonometric identity for \(\sinh\) of a difference, we can write the above as

\[
W_p(x, s) = \frac{1}{t_+^2 (t_+ - t_-)} \int_0^x \sinh \sqrt{t_+} (\tau - x) F(\tau, s) d\tau
+ \frac{1}{t_-^2 (t_+ - t_-)} \int_0^x \sinh \sqrt{t_-} (\tau - x) F(\tau, s) d\tau.
\]  

(5.25)

We now solve for \(C_1, C_2, C_3,\) and \(C_4\) in (5.17) by using Cramer's Rule on (5.18). The \(C_i\)'s are given by

\[ C_i = \frac{D_i}{D} \]

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where $D$ is the determinant of the matrix of coefficients in (5.18), and the $D_i$'s are the determinants of the matrices obtained by replacing the $t^{th}$ column of $D$ by the column on the right hand side of (5.18). We now compute $D$,

$$D = -t_+^\frac{1}{2}$$

$$= -t_+^\frac{1}{2} \begin{vmatrix} t_+^\frac{1}{2} \sinh \sqrt{t_+} & t_+^\frac{1}{2} \sinh \sqrt{t_-} & t_+^\frac{1}{2} \cosh \sqrt{t_-} \\ 0 & 0 & t_+^\frac{3}{2} \\ t_+^\frac{3}{2} \sinh \sqrt{t_+} & t_+^\frac{3}{2} \sinh \sqrt{t_-} & t_+^\frac{3}{2} \cosh \sqrt{t_-} \end{vmatrix}$$

$$= -t_+^\frac{1}{2} \begin{vmatrix} t_+^\frac{1}{2} \sinh \sqrt{t_+} & t_+^\frac{1}{2} \sinh \sqrt{t_-} \\ 0 & t_+^\frac{3}{2} \sinh \sqrt{t_-} \end{vmatrix}$$

$$= t_+^\frac{1}{2} t_+^\frac{3}{2} \begin{vmatrix} t_+^\frac{1}{2} \sinh \sqrt{t_+} & t_+^\frac{1}{2} \sinh \sqrt{t_-} \\ t_+^\frac{3}{2} \sinh \sqrt{t_+} & t_+^\frac{3}{2} \sinh \sqrt{t_-} \end{vmatrix}$$

$$= t_+ t_+^2 (t_- - t_+) \sinh \sqrt{t_-} \sinh \sqrt{t_+}$$

$$= t_- t_+^2 (t_- - t_+) \sinh \sqrt{t_-} \sinh \sqrt{t_+}$$

So,

$$D = t_- t_+ (t_- - t_+)^2 \sinh \sqrt{t_-} \sinh \sqrt{t_+}. \quad (5.26)$$

We remark that $D$ is a function of $s$ since both $t_-$ and $t_+$ depend on $s$. The variable has been suppressed in equation (5.26) for convenience. The $D_i$'s are computed similarly, and are given below:

$$D_1 = t_- t_+^\frac{1}{2} (t_- - t_+) (W_p'''(1, s) - t_- W_p'(1, s)) \sinh \sqrt{t_-}$$

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\[ D_2 = 0 \]

\[ D_3 = t_+ t_+^{\frac{1}{2}} (t_+ - t_-) (W_p''(1, s) - t_+ W_p'(1, s)) \sinh \sqrt{I_+} \]

\[ D_4 = 0. \]

So,

\[ C_2 = C_4 = 0, \]

\[ C_1 = \frac{1}{t_+^\frac{1}{2} (t_+ - t_-) \sinh \sqrt{I_+}} \int_0^1 \cosh \sqrt{I_+} (\tau - 1) F(\tau, s) d\tau, \]

and

\[ C_3 = \frac{1}{t_-^\frac{1}{2} (t_+ - t_-) \sinh \sqrt{I_-}} \int_0^1 \cosh \sqrt{I_-} (\tau - 1) F(\tau, s) d\tau. \]

Thus,

\[ W(x, s) = \frac{1}{t_+^\frac{1}{2} (t_+ - t_-) \sinh \sqrt{I_+}} \int_0^1 \cosh \sqrt{I_+} (\tau - 1) F(\tau, s) d\tau \cosh \sqrt{I_+} x \]

\[ + \frac{1}{t_-^\frac{1}{2} (t_+ - t_-) \sinh \sqrt{I_-}} \int_0^1 \cosh \sqrt{I_-} (\tau - 1) F(\tau, s) d\tau \cosh \sqrt{I_-} x \]

\[ + \frac{1}{t_+^\frac{1}{2} (t_+ - t_-)} \int_0^x \sinh \sqrt{I_+} (\tau - x) F(\tau, s) d\tau \]

\[ + \frac{1}{t_-^\frac{1}{2} (t_+ - t_-)} \int_0^x \sinh \sqrt{I_-} (\tau - x) F(\tau, s) d\tau, \]

where \( F(x, s) = f'' + \frac{\gamma}{s^2} h' - \frac{\alpha^2}{k} f. \)

We remark that in general we have not been able to rule out the possibility that \( t_-(s) = t_+(s) \), in which case the form of the solution \( W(x, s) \) in (5.17) would be different. In the case where \( \gamma = 0 \), we can show that it is not possible to have \( t_- = t_+ \) and have a (nontrivial) solution to (5.11) satisfying the boundary conditions (5.14), (5.15). We suspect that this is true in general, however the proof of this for \( \gamma > 0 \) remains open at this time.
5.2 Zeros of the Characteristic Equation

We wish to examine the location of the zeros of \( D(s) \), which will give us the location of the poles of \( W(x, s) \). Recall equation (5.26)

\[
D(s) = t_- t_+ (t_- - t_+)^2 \sinh \sqrt{t_-} \sinh \sqrt{t_+}.
\]

Using the formula for the product of the roots of a polynomial, we get

\[
t_- t_+ = \frac{\rho c s^2}{k \tilde{g}(s)},
\]

and dividing by \( \frac{\rho c}{k} \) we see that \( D(s) = 0 \) when

\[
\frac{s^2}{\tilde{g}(s)} = 0
\]

Note that \( s = 0 \) is the only solution of (5.28), and this corresponds to a simple translation (see formula (5.60)).

Clearly \( D(s) = 0 \) if \( \sinh \sqrt{t_+} \sinh \sqrt{t_-} = 0 \), and this product is zero when either \( \sqrt{t_+} = i n \pi \) or \( \sqrt{t_-} = i n \pi \) for some \( n = \pm 1, \pm 2, \ldots \). Thus, if we let \( \mu_n = n \pi, n = 1, 2, \ldots \) we have \( D(s) = 0 \) if either \( t_+(s) = -\mu_n^2 \) or \( t_-(s) = -\mu_n^2 \) for some \( n = 1, 2, \ldots \). If we now use the expression in (5.16), rearrange to isolate the square root term on one side, then square both sides and multiply by \( k \tilde{g}(s) \), we see that if \( \sinh \sqrt{t_+} \sinh \sqrt{t_-} = 0 \), then

\[
\Delta_n(s) := (\rho s + \mu_n^2 \tilde{g}(s))(c s + k \mu_n^2) + \gamma^2 \theta \mu_n^2 = 0.
\]

for some \( n = 1, 2, \ldots \). Throughout this Chapter, we will refer to \( \Delta_n(s) = 0 \) for some \( n = 1, 2, \ldots \) as the characteristic equation of (5.1)–(5.5).

The rest of this Section deals with the location of the zeros of \( \Delta_n(s) \). Observe that \( \Delta_n(\overline{s}) = \overline{\Delta_n(s)} \), so it suffices to confine our analysis of \( \Delta_n(s) \) to the upper half-plane \( \Im s \geq 0 \). We first prove a general result about the location of the zeros of \( \Delta_n(s) \).

**Proposition 5.1** Let \( g(t) \) satisfy (5.6), let \( \rho, c, k \) and \( \theta \) be positive constants, and let \( \gamma \geq 0 \). Let \( \mu > 0 \) and define \( \Delta(s) \) by
\[ \Delta(s) := (\rho s + \mu^2 \hat{g}(s))(cs + k\mu^2) + \gamma^2 \theta_a \mu^2. \] (5.30)

Then \( \Delta(s) = 0 \) has no solutions in \( \Re s \geq 0 \).

Proof: Since \( g(t) \) is completely monotonic, we can use Bernstein’s Theorem [30, Theorem 12a, p. 160] to represent \( g(t) \) as

\[ g(t) = \int_0^\infty e^{-xt} d\nu(x), \]

where \( \nu \) is a positive measure; and hence \( \hat{g} \) is given by the Stieltjes transform

\[ \hat{g}(s) = \int_0^\infty \frac{d\nu(x)}{s + x} \text{ for } s \in \mathbb{C}' . \] (5.31)

It is easy to see that \( \Re \hat{g}(s) > 0 \). Therefore, \( \Re (\rho s + \mu^2 \hat{g}) > 0 \) and hence \(-\frac{\pi}{2} < \arg (\rho s + \mu^2 \hat{g}) < \frac{\pi}{2} \). Likewise, \( \Re (cs + k\mu^2) > 0 \) and hence \(-\frac{\pi}{2} < \arg (cs + k\mu^2) < \frac{\pi}{2} \). Combining these, we see that the argument of the product satisfies

\[-\pi < \arg (\rho s + \mu^2 \hat{g})(cs + k\mu^2) < \pi . \]

Clearly, the product \( (\rho s + \mu^2 \hat{g})(cs + k\mu^2) \) can never be a negative real number in \( \Re s \geq 0 \) and the proof for \( \gamma > 0 \) is complete. It is easy to see that the product \( (\rho s + \mu^2 \hat{g})(cs + k\mu^2) \) can never be zero in \( \Re s \geq 0 \), since \( \Re (\rho s + \mu^2 \hat{g}) > 0 \) and \( \Re (cs + k\mu^2) > 0 \), and hence the proof for \( \gamma = 0 \) is complete, and the proposition is proved. \( \square \)

We now prove a general result about the existence of complex conjugate zeros of \( \Delta(s) \).

**Theorem 5.1** Let \( g(t) \) be completely monotonic on \( (0, \infty) \) with \( g \in L^1(0,1) \). Let \( \rho, c, k, \theta_a, \) and \( \mu \) be positive constants, let \( \gamma \geq 0 \), and define \( \Delta(s) \) as in equation (5.30). Then \( \Delta(s) \) has at most one pair of complex conjugate zeros in \( \Re s < 0, \Im s \neq 0 \). Moreover, these zeros are simple.

We remark that when \( \gamma = 0 \), Theorem 5.1 was proved by Desch and Grimmer [8], who showed that \( \rho s + \mu^2 \hat{g}(s) \) has at most one pair of complex conjugate zeros in \( \Re s < 0, \Im s \neq 0 \).
Proof: Consider $s = \sigma + i\tau$ with $\sigma < 0$, $\tau > 0$. Clearly, for such $s$, $\Delta(s) = 0$ exactly when

$$\frac{\Delta(s)}{cs + k\mu^2} = \rho s + \mu^2 \hat{g}(s) + \frac{\gamma^2 \theta_a \mu^2}{cs + k\mu^2} = 0. \quad (5.32)$$

As in the proof of Proposition 5.1, since $g$ is completely monotonic we can express $\hat{g}(s)$ as in (5.31). Now, using (5.31) we can write

$$\frac{\Delta(s)}{cs + k\mu^2} = R(\sigma, \tau) + i\tau I(\sigma, \tau) \quad (5.33)$$

with

$$R(\sigma, \tau) = \rho \sigma + \mu^2 \left\{ \int_0^\infty \frac{(\sigma + x)}{(\sigma + x)^2 + \tau^2} d\nu(x) + \frac{\gamma^2 \theta_a (c\sigma + k\mu^2)}{(c\sigma + k\mu^2)^2 + c^2 \tau^2} \right\}, \quad (5.34)$$

and

$$I(\sigma, \tau) = \rho - \mu^2 \left\{ \int_0^\infty \frac{d\nu(x)}{(\sigma + x)^2 + \tau^2} + \frac{\gamma^2 \theta_a c}{(c\sigma + k\mu^2)^2 + c^2 \tau^2} \right\}. \quad (5.35)$$

Clearly, for each fixed $\sigma$, $I(\sigma, \tau)$ is a monotonically increasing function of $\tau$ for $\tau \in (0, \infty)$ and $I(\sigma, \tau) \to \rho$ as $\tau \to \infty$. Define $I(\sigma, 0) = \lim_{\tau \to 0^+} I(\sigma, \tau)$; so $I(\sigma, 0)$ exists in $[-\infty, \rho)$ for each $\sigma < 0$.

Decompose $(-\infty, 0] = S_1 \cup S_2 \cup S_3$ where $\sigma \in S_1$ if $I(\sigma, 0) < 0$, $\sigma \in S_2$ if $I(\sigma, 0) > 0$ and $\sigma \in S_3$ if $I(\sigma, 0) = 0$. For each $\sigma \in S_1$, there exists a unique $\tau(\sigma) > 0$ so that $I(\sigma, \tau(\sigma)) = 0$. For $\sigma \in S_2 \cup S_3$, define $\tau(\sigma) := 0$, and define the curve $C = C_1 \cup C_2 \cup C_3$ by $(\sigma, \tau(\sigma))$ where $(\sigma, \tau(\sigma)) \in C_j$ if $\sigma \in S_j$ ($j = 1, 2, 3$).

To complete the proof, we show that

$$\frac{dR(\sigma, \tau(\sigma))}{d\sigma} \geq 0 \text{ for } \sigma < 0 \quad (5.36)$$

with strict inequality on $S_1 \cup S_2$.

To prove (5.36), note that for $\sigma \in S_2 \cup S_3$,

$$R(\sigma, 0) = \rho \sigma + \mu^2 \left( \int_0^\infty \frac{d\nu(x)}{(\sigma + x)} + \frac{\gamma^2 \theta_a}{c(\sigma + k\mu^2)} \right),$$

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so

\[ \frac{dR(\sigma, 0)}{d\sigma} = I(\sigma, 0) \geq 0, \]

with strict inequality when \( \sigma \in S_2 \).

Since everything is analytic when \((\sigma, \tau(\sigma)) \in C_1\), we have for \( \sigma \in S_1 \),

\[ \frac{dR(\sigma, \tau(\sigma))}{d\sigma} = \frac{\partial R}{\partial \sigma} + \frac{\partial R}{\partial \tau} \frac{d\tau}{d\sigma} \]

with

\[ \frac{d\tau}{d\sigma} = -\frac{\partial I}{\partial \sigma}. \]

Routine calculations show that

\[ \frac{\partial R}{\partial \sigma} = \rho + \mu^2 \left\{ \int_0^{\infty} \frac{\tau^2 - (\sigma + x)^2}{[(\sigma + x)^2 + \tau^2]^2} d\nu(x) \right. \]

\[ + \frac{\gamma^2 \theta_a e^2(\tau^2 - (\sigma + k\mu^2)^2)}{[(\sigma + k\mu^2)^2 + \tau^2]^2} \right\}, \]

(5.40)

\[ \frac{\partial I}{\partial \sigma} = 2\mu^2 \left\{ \int_0^{\infty} \frac{(\sigma + x)}{[(\sigma + x)^2 + \tau^2]^2} d\nu(x) \right. \]

\[ + \frac{\gamma^2 \theta_a e^2(\sigma + k\mu^2)}{[(\sigma + k\mu^2)^2 + \tau^2]^2} \right\}, \]

(5.41)

and

\[ \frac{\partial I}{\partial \tau} = 2\mu^2 \tau \left\{ \int_0^{\infty} \frac{d\nu(x)}{[(\sigma + x)^2 + \tau^2]^2} \right. \]

\[ + \frac{\gamma^2 \theta_a e^3}{[(\sigma + k\mu^2)^2 + \tau^2]^2} \right\}. \]

(5.43)
Combining (5.38), (5.39) and (5.42), we can write
\[ \frac{dR(\sigma, \tau(\sigma))}{d\sigma} \]}
\[ = \frac{\partial R}{\partial \sigma} + \tau \frac{\partial R}{\partial \tau}. \]

Since \( \frac{\partial R}{\partial \tau} > 0 \), it suffices to show that \( \frac{\partial R}{\partial \sigma} \geq 0 \), which we prove with the following estimates:

\[ \mu^2 \left\{ \int_0^\infty \frac{(\sigma + x)^2}{(\sigma + x^2 + \tau^2)^2} \, \nu(x) \, dx \right\} + \gamma^2 \theta c(\sigma + k\mu^2)^2 \]
\[ \leq \mu^2 \left\{ \int_0^\infty \frac{d\nu(x)}{(\sigma + x)^2 + \tau^2} + \frac{\gamma^2 \theta}{(\sigma + k\mu^2)^2 + c^2 \tau^2} \right\} \]
\[ = \rho, \]

where the last equality follows by (5.35) and the fact that \( I(\sigma, \tau(\sigma)) = 0 \) on \( C_1 \). So, it follows that
\[ \frac{dR(\sigma, \tau(\sigma))}{d\sigma} > 0 \text{ for } \sigma \in S_1, \]
and (5.36) is proved. This completes the proof of Theorem 5.1. □

Notice that Proposition 5.1 and Theorem 5.1 require no knowledge of the specific form of \( g(t) \), only that \( g(t) \) be completely monotonic. The next result, however, depends on the specific form of \( g(t) \). We remind the reader that \( \Delta_n(\bar{s}) = \overline{\Delta_n(s)} \), so it suffices to confine our analysis to the upper half plane \( \Re s \geq 0 \).

**Proposition 5.2** Let \( g(t) = g(\infty) + \frac{\delta s^{n-1}}{1(0)} \) with \( 0 < \eta < 1, \ g(\infty) > 0, \) and \( \delta > 0 \). Let \( \rho, c, k \) and \( \theta \) be positive constants and \( \gamma \geq 0 \). Set \( \mu_n = n\pi \) for \( n = 1, 2, \ldots \). Then

(i) For each sufficiently large \( n \), \( \Delta_n(s) = 0 \) has a solution \( s_n \) in \( \Re s > 0 \). The sequence \( \{s_n\} \) satisfies
\[ |s_n| \sim \left( \frac{\delta \mu_n^2}{\rho} \right)^{\frac{1}{1+\gamma}}, \ \arg s_n \rightarrow \frac{\pi}{1+\eta} \text{ as } n \rightarrow \infty. \] (5.44)
(ii) There exists an angle $\psi$, $\frac{\pi}{2} < \psi < \pi$, such that $\Delta_n(s)$ has no zero in $-\psi \leq \arg s \leq \psi$ for all $n$.

Proof: To prove Part (i), we will use Rouché's Theorem. Define $F_n$ and $G_n$ by

$$F_n(s) = \frac{\Delta_n(s)}{cs + k\mu_n^2}$$

$$= \rho s + \mu_n^2 \left( \frac{g(\infty)}{s} + \frac{\delta}{s^\nu} \right) + \frac{\gamma^2 \theta_\alpha \mu_n^2}{cs + k\mu_n^2},$$

$$G_n(s) = \rho s + \mu_n^2 \frac{\delta}{s^{\nu'}}.$$  

Clearly, the nonreal zeros of $F_n$ and $\Delta_n$ are the same, and for each $n$, $G_n(s) = 0$ has a solution

$$w_n = \left( \frac{\mu_n^2 \delta}{\rho} \right)^{\frac{1}{1+\nu'}} e^{i\frac{\pi}{1+\nu'}}. \quad (5.45)$$

Fix $\nu, \nu' \in \left( \frac{2\eta}{1+\eta}, \frac{2}{1+\eta} \right)$. For each $n$, let $C_n$ denote the circle with center $w_n$ and radius $\mu_n$. Clearly, for $s$ inside or on $C_n$, $s \sim w_n$ as $n \to \infty$.

For $s$ on $C_n$,

$$|F_n(s) - G_n(s)| \leq \left| \frac{\mu_n^2 g(\infty)}{s} \right| + \left| \frac{\gamma^2 \theta_\alpha \mu_n^2}{cs + k\mu_n^2} \right|$$

$$= \mathcal{O} \left( \frac{\mu_n^{2\eta}}{\mu_n^{1+\eta}} \right) + \mathcal{O}(1) \quad (5.46)$$

$$= \mathcal{O} \left( \frac{2\eta}{\mu_n^{1+\eta}} \right) \quad \text{as } n \to \infty.$$

Next, $G'_n(s) = \rho - \eta \mu_n^2 \delta s^{-1-\eta}$, and since $s \sim w_n$ for $s$ inside or on $C_n$, we have that

$$G'_n(s) \to (1+\eta)\rho \quad \text{as } n \to \infty.$$  

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when \(s\) is inside or on \(C_n\). Since \(G_n(w_n) = 0\),

\[
|G_n(s)| \sim |(1 + \eta)\rho \mu_n^s| \quad \text{when } s \in C_n \quad \text{as } n \to \infty.
\] (5.47)

It follows by (5.46) and (5.47) and our choice of \(\nu\) that for all large \(n\)

\[
|F_n(s) - G_n(s)| < |G_n(s)| \quad \text{when } s \in C_n.
\] (5.48)

By Rouché's Theorem, when \(n\) is sufficiently large \(F_n(s)\) has exactly one zero \(s_n\) inside \(C_n\). Since \(s_n \sim w_n\) the proof of Part (i) is complete.

Part (ii) is an easy consequence of Part (i) and Proposition 5.1 and Theorem 5.1. Namely, by Theorem 5.1, for all large \(n\), the zero \(s_n\) constructed in Part (i) is the only solution of \(\Delta_n(s) = 0\) in \(\Re s > 0\). Also, by Proposition 5.1, \(\Delta_n(s) = 0\) never has a solution in \(\Re s \geq 0\). Since each \(\Delta_n(s)\) has at most one pair of complex conjugate zeros, and since \(\Delta_n(\Re) = \Delta_n(\Re)\), it follows that there exists \(\psi, \frac{\pi}{2} < \psi < \pi\), so that \(\Delta_n(s) = 0\) has no solution in \(-\psi \leq \arg s \leq \psi\) for all \(n\) and the proof of Part (ii) is complete. □

As we saw in Proposition 5.2, Theorem 5.1 is very useful in studying the complex solutions of \(\Delta_n(s) = 0\). Observe that for the kernel \(g(t)\) examined in Proposition 5.2 the asymptotic behavior of the complex zeros \(s_n\) as described in (5.44) is independent of the coupling coefficient \(\gamma\). We plan to study the question of the \(\gamma\) dependence of the asymptotic behavior of the complex zeros \(s_n\) of \(\Delta_n(s)\) for general completely monotonic kernels \(g(t)\). We also plan to numerically investigate the \(\gamma\) dependence of the complex zeros of \(\Delta_n(s)\) for small values of \(n\) and various model kernels \(g(t)\).

We now turn to the study of the asymptotic behavior of \(W(0, s)\) near \(s = 0\) when \(g(t) = g(\infty) + \frac{\delta t^{\eta-1}}{1(\eta)}, 0 < \eta < 1\). Inversion of the asymptotic series we obtain for \(W(0, s)\) will give us the asymptotic behavior of the solution \(w(0, t)\) as \(t \to \infty\).
5.3 An Integral Representation Formula for $W(0, s)$

For the specific kernel $g(t) = g(\infty) + \delta(t_0)$, we will study the dependence on the thermal coupling coefficient $\gamma$ of the displacement $w(x, t)$ at a fixed point $x \in [0, 1]$. For convenience, we choose $x = 0$. In this Section we obtain an integral representation formula for $W(0, s)$.

Using (5.9), (5.10) and (5.11) to substitute for $F(\tau, s)$ in (5.27) we see that

$$W(0, s) = \frac{1}{t_+^2 (t_+ - t_-) \sinh \sqrt{t_+}} \int_0^1 \cosh \sqrt{t_+} (\tau - 1) F(\tau, s) d\tau$$

$$+ \frac{1}{t_-^2 (t_+ - t_-) \sinh \sqrt{t_-}} \int_0^1 \cosh \sqrt{t_-} (\tau - 1) F(\tau, s) d\tau$$

$$= \frac{1}{t_+^2 (t_+ - t_-) \sinh \sqrt{t_+}} \int_0^1 \cosh \sqrt{t_+} (\tau - 1)$$

$$\cdot \left[ - \left( \frac{\rho}{\bar{g}} + \frac{\gamma^2 \theta_0}{k s g} \right) w_0'' + \frac{c p s}{k \bar{g}} w_0 - \frac{\rho \theta_0}{k s g} w_1'' + \frac{c p s}{k \bar{g}} w_1 - \frac{\gamma c}{k s g} \theta_0' \right] d\tau$$

$$+ \frac{1}{t_-^2 (t_+ - t_-) \sinh \sqrt{t_-}} \int_0^1 \cosh \sqrt{t_-} (\tau - 1)$$

$$\cdot \left[ - \left( \frac{\rho}{\bar{g}} + \frac{\gamma^2 \theta_0}{k s g} \right) w_0'' + \frac{c p s}{k \bar{g}} w_0 - \frac{\rho \theta_0}{k s g} w_1'' + \frac{c p s}{k \bar{g}} w_1 - \frac{\gamma c}{k s g} \theta_0' \right] d\tau.$$

After several integrations by parts and the fact that $\theta_0(i) = w_0'(i) = w_1'(i) = 0, i = 0, 1$, we have

$$W(0, s) = \frac{1}{s \bar{g} t_+^2 (t_+ - t_-) \sinh \sqrt{t_+}} \int_0^1 \cosh \sqrt{t_+} (\tau - 1)$$

$$\cdot \left[ w_0 \left( \frac{c p s}{k} \frac{1}{t_+} - \theta_0 \left( \frac{\rho s}{k} + \frac{\gamma^2 \theta_0}{k} \right) \right) + w_1 \left( \frac{c p s}{k} - \theta_0 \right) \right] d\tau$$

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\[ + \frac{1}{\sinh t_+ - t_+} \int_0^1 \cosh \sqrt{t_+} \left( \tau - 1 \right) \]
\[ \cdot \left[ w_0 \left( \frac{c \rho s^2 - t_+}{k} \right) \left( \rho s + \frac{\gamma^2 \theta_+}{k} \right) + w_1 \left( \frac{\rho s}{k} - t_- \right) \right] d\tau \]
\[ + \frac{\gamma_c}{k \sinh t_+} \int_0^1 \cosh \sqrt{t_+} \left( \tau - 1 \right) \theta_0(\tau) d\tau \]
\[ + \frac{\gamma_c}{k \sinh t_+} \int_0^1 \cosh \sqrt{t_-} \left( \tau - 1 \right) \theta_0(\tau) d\tau. \]

For simplicity we will assume that \( \theta_0(x) \equiv 0 \) and \( w_1(x) \equiv 0 \). In this case the expression for \( W(0, s) \) becomes

\[ W(0, s) = \frac{c \rho s^2 - t_+ (\rho k s + \gamma^2 \theta_+)}{k \sinh t_+} \int_0^1 \cosh \sqrt{t_+} \left( \tau - 1 \right) w_0(\tau) d\tau \]
\[ + \frac{c \rho s^2 - t_- (\rho k s + \gamma^2 \theta_+)}{k \sinh t_-} \int_0^1 \cosh \sqrt{t_-} \left( \tau - 1 \right) w_0(\tau) d\tau. \]
(5.50)

5.4 Series Expansion for \( W(0, s) \)

We wish to obtain an asymptotic series expansion for \( W(0, s) \) in formula (5.50) which is valid near \( s = 0 \). We restrict our attention to the kernel \( g(t) = g(\infty) + \frac{\eta}{t_+} \), \( \eta < 1 \). Let \( w(n) := \int_0^1 (1 - y)^n w_0(y) dy \). If we expand \( \cosh \) in a series and carry out the integration in (5.50) we get

\[ W(0, s) = \frac{c \rho s^2 - t_+ (\rho k s + \gamma^2 \theta_+)}{k \sinh t_+} \left[ w_0^{(0)} + \frac{t_+ w_0^{(2)}}{2!} + \frac{t_+^2 w_0^{(4)}}{4!} + \ldots \right] \]
\[ + \frac{c \rho s^2 - t_- (\rho k s + \gamma^2 \theta_+)}{k \sinh t_-} \left[ w_0^{(0)} + \frac{t_- w_0^{(2)}}{2!} + \frac{t_-^2 w_0^{(4)}}{4!} + \ldots \right]. \]

Next, we expand \( \frac{1}{t_{\pm}^2 \sinh \sqrt{t_{\pm}}} \) in a series to get

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\[ \frac{1}{t_{\pm}^2 \sinh \sqrt{t_{\pm}}} = \frac{1}{t_{\pm}} - \frac{1}{3!} + t_{\pm} \left[ \frac{1}{(3!)^2} - \frac{1}{5!} \right] + \ldots \]

Substituting this into our formula for \( W(0, s) \) yields

\[ W(0, s) = \frac{c^2 p s^2 - t_+ (p k s + \gamma^2 \theta_\alpha)}{k s \tilde{g}(t_+ - t_-)} \left[ \frac{w^{(0)}}{t_+} + \tilde{w}^{(2)} t_+ + \tilde{w}^{(4)} t_+^3 + \ldots \right] \]

\[ + \frac{c^2 p s^2 - t_- (p k s + \gamma^2 \theta_\alpha)}{k s \tilde{g}(t_- - t_+)} \left[ \frac{w^{(0)}}{t_-} + \tilde{w}^{(2)} t_- + \tilde{w}^{(4)} t_-^3 + \ldots \right] \]

where

\[ \tilde{w}^{(2)} = \frac{w^{(2)}}{2!} - \frac{w^{(6)}}{3!} \]

\[ \tilde{w}^{(4)} = \frac{w^{(4)}}{4!} - \frac{w^{(2)}}{2!3!} + w^{(0)} \left[ \frac{1}{(3!)^2} - \frac{1}{5!} \right] \]

(5.51)

etc. We note that by the formulae for the sum and product of the roots of a polynomial, we have

\[ t_+ t_- = \frac{p s^2}{k \tilde{g}} \]

(5.52)

and

\[ t_+ + t_- = \frac{p s}{\tilde{g}} + \frac{\gamma^2 \theta_\alpha}{k \tilde{g}} + \frac{c s}{k} \]

(5.53)

Thus,

\[ \frac{c^2 p s^2 - t_+ (p k s + \gamma^2 \theta_\alpha)}{k s \tilde{g} t_+ (t_- - t_+)} - \frac{c^2 p s^2 - t_- (p k s + \gamma^2 \theta_\alpha)}{k s \tilde{g} t_- (t_- - t_+)} = \frac{c p s^2 (t_- - t_+)}{k s \tilde{g} t_+ (t_- - t_+)} = \frac{1}{s}, \]
\[
\frac{c p s^2 - t_+ (p k s + \gamma^2 \theta_a)}{k s \tilde{g}(t_- - t_+)} - \frac{c p s^2 - t_- (p k s + \gamma^2 \theta_a)}{k s \tilde{g}(t_- - t_+)} = \frac{(p k s + \gamma^2 \theta_a)(t_- - t_+)}{k s \tilde{g}(t_- - t_+)}
= \frac{p k s + \gamma^2 \theta_a}{k s \tilde{g}} ,
\]

and
\[
t_+ \frac{c p s^2 - t_+ (p k s + \gamma^2 \theta_a)}{k s \tilde{g}(t_- - t_+)} - t_- \frac{c p s^2 - t_- (p k s + \gamma^2 \theta_a)}{k s \tilde{g}(t_- - t_+)}
= \frac{c p s^2}{k s \tilde{g}} + \frac{(t_+ + t_-) p k s + \gamma^2 \theta_a}{k s \tilde{g}}
= \frac{(p k s + \gamma^2 \theta_a)^2 + c \gamma^2 \theta_a \tilde{g}}{k^2 s \tilde{g}^2} .
\]

Hence,
\[
W(0, s) = \frac{1}{s} w^{(0)} + \frac{p k s + \gamma^2 \theta_a}{k s \tilde{g}} \tilde{w}^{(2)} + \frac{(p k s + \gamma^2 \theta_a)^2 + c \gamma^2 \theta_a \tilde{g}}{k^2 s \tilde{g}^2} \tilde{w}^{(4)} + \ldots \tag{5.54}
\]

We remark that no square roots of \( \tilde{g} \) or other quantities appear in this series expression for \( W(0, s) \).

Before proceeding, we must find an expression for \( \frac{p k s + \gamma^2 \theta_a}{k s \tilde{g}} \) when \( \tilde{g} = \frac{g(\infty)}{s} + \frac{s}{g(\infty)}, 0 < \eta < 1 \). First we note that
\[
k s \tilde{g} = k [g(\infty) + \delta s^{1-\eta}] .
\]

Thus,
\[
\frac{p k s + \gamma^2 \theta_a}{k s \tilde{g}} = \frac{p k s + \gamma^2 \theta_a}{k g(\infty) + \delta s^{1-\eta}} = \frac{p k s + \gamma^2 \theta_a}{k g(\infty)} \left[ 1 + \frac{\delta s^{1-\eta}}{g(\infty)} \right]^{-1}
= \frac{p k s + \gamma^2 \theta_a}{k g(\infty)} \left[ 1 - \frac{\delta s^{1-\eta}}{g(\infty)} - \frac{\delta^2 s^{2(1-\eta)}}{g(\infty)^2} - \frac{\delta^3 s^{3(1-\eta)}}{g(\infty)^3} + \ldots \right]
\]

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\[
= \gamma^2 \left( \frac{\theta_a}{kg(\infty)} - \frac{\delta \theta_a s^{1-\eta}}{kg(\infty)^2} + \frac{\delta^2 \theta_a s^{2-2\eta}}{kg(\infty)^3} - \ldots \right) \\
+ \left( \frac{\rho s}{g(\infty)} - \frac{\rho \delta s^{2-\eta}}{g(\infty)^2} + \frac{\rho \delta^2 s^{3-2\eta}}{g(\infty)^3} - \ldots \right).
\]

In a similar manner we can show that the coefficient of \(w^{(4)}\) can be expanded as

\[
\frac{(\rho ks + \gamma^2 \theta_a)^2 + \dot{c} \gamma^2 \theta_a s \dot{g}}{k^2 s \dot{g}^2} \\
= \gamma^2 \left( \frac{c \theta_a g(\infty)}{k^2 g(\infty)^2} + \frac{\gamma^2 \theta_a^2}{k^2 g(\infty)^3} s^{-\eta} + \ldots \right) \\
+ \left( \frac{\rho^2 s^3}{g(\infty)^2} - \frac{2 \rho^2 \delta s^{4-\eta}}{g(\infty)^3} + \frac{3 \rho^2 \delta^2 s^{5-2\eta}}{g(\infty)^4} - \ldots \right).
\]

Using these expressions, we see that

\[
W(0,s) - \frac{1}{s} w^{(0)} \\
\approx w^{(2)} \left[ \gamma^2 \left( \frac{\theta_a}{kg(\infty)} - \frac{\delta \theta_a s^{1-\eta}}{kg(\infty)^2} + \frac{\delta^2 \theta_a s^{2-2\eta}}{kg(\infty)^3} - \ldots \right) \\
+ \left( \frac{\rho s}{g(\infty)} - \frac{\rho \delta s^{2-\eta}}{g(\infty)^2} + \frac{\rho \delta^2 s^{3-2\eta}}{g(\infty)^3} - \ldots \right) \right] \\
+ w^{(4)} \left[ \gamma^2 \left( \frac{c \theta_a g(\infty)}{k^2 g(\infty)^2} + \frac{\gamma^2 \theta_a^2}{k^2 g(\infty)^3} s^{-\eta} + \ldots \right) \\
+ \left( \frac{\rho^2 s^3}{g(\infty)^2} - \frac{2 \rho^2 \delta s^{4-\eta}}{g(\infty)^3} + \frac{3 \rho^2 \delta^2 s^{5-2\eta}}{g(\infty)^4} - \ldots \right) \right] + \ldots, \quad s \to 0. \tag{5.55}
\]

5.5 Asymptotic Behavior of \(w(0,t)\) for Large \(t\)

Now that we have an asymptotic expansion for \(W(0,s)\) valid near \(s = 0\), we will use it to obtain a formal asymptotic expansion for \(w(0,t)\) valid for \(t\) near infinity. To do this we
use Theorem 37.1 in Doetsch's book [9, p.254] which we now state

**Inversion Theorem** Suppose that the function \( f(t) \) can be presented as the \( \mathcal{W} \)-transform of \( F(s) \) for \( t > T \), employing a contour \( \mathcal{W} \) centered at 0 with the half-angle of opening \( \psi \), \( \frac{\pi}{2} < \psi \leq \pi \). This, in particular, is true when, in fact, we have initially

\[
f(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{ts} F(s) ds \quad (a > 0),
\]

\( F(s) \) being analytic in the region between the contour \( \mathcal{W} \) and the line \( \Re s = a \), and tending towards zero when \( s \) tends two-dimensionally in this region towards \( \infty \). Suppose, furthermore, that \( F(s) \) has in \( |\arg s| \leq \psi \) the asymptotic expansion

\[
F(s) \approx \sum_{\nu=0}^{\infty} c_{\nu} s^{\lambda_{\nu}} \quad (\Re \lambda_{0} < \Re \lambda_{1} < \ldots) \quad \text{as} \quad s \to 0.
\]

Then we conclude that \( f(t) \) has the asymptotic expansion

\[
f(t) \approx \sum_{\nu=0}^{\infty} c_{\nu} \frac{1}{\Gamma(-\lambda_{\nu})} \frac{1}{t^{\lambda_{\nu}+1}} \quad \left( \frac{1}{\Gamma(-\lambda_{\nu})} = 0 \text{ for } \lambda_{\nu} = 0, 1, 2, \ldots \right),
\]

as \( t \) tends two-dimensionally in the angular region \( |\arg t| \leq \psi - \frac{\pi}{2} - \delta \) towards \( \infty \). The function \( f(t) \) being a \( \mathcal{W} \)-transform is analytic in the angular region \( |\arg (t - i_0)| < \psi - \frac{\pi}{2} \).

Here, a \( \mathcal{W} \)-transform is the inverse Laplace transform of a function along the contour \( \mathcal{W} \), which consists of the lines \( \arg s = \pm \psi, \frac{\pi}{2} < \psi < \pi \), with a small circular indentation at \( s = 0 \).

By Proposition 5.2, we have shown that there is an angle \( \psi \), \( \frac{\pi}{2} < \psi < \pi \), such that \( W(0, s) \) has no nonzero poles, that is, for \( n = 1, 2, \ldots \), \( \Delta_n(s) \) as defined in (5.29) has no zeros, in the sector \( -\psi \leq \arg s \leq \psi \). In order to show that \( W(0, s) \to 0 \) as \( s \to \infty \), we will assume that \( w_0(x) \) is bounded on \([0, 1]\). In this case we have

**Lemma 5.1** Let \( w_0(x) \) be bounded on \([0, 1]\), \( g(t) = g(\infty) + \frac{\delta \pi - 1}{\Gamma(\eta)} \), and let \( \phi \) be any angle satisfying \( \frac{\pi}{2} < \phi < \frac{\pi}{1+\eta} \). Then \( W(0, s) \) defined by (5.50) satisfies

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uniformly in $|\text{arg} s| \leq \phi$.

Proof of Lemma 5.1: Let $S_{\phi}$ denote the sector $S_{\phi} = \{ s : |\text{arg} s| \leq \phi \}$. In this proof, all estimates will hold as $s \to \infty$ uniformly in $S_{\phi}$. Since $\dot{g}(s) = \delta s^{-\eta}(1 + o(1))$, we have

$$\frac{\rho s}{\dot{g}(s)} + \frac{\gamma^2 \theta_{a}}{k \dot{g}(s)} + \frac{cs}{k} = \frac{\rho s^{n+1}}{\delta}(1 + o(1)).$$

Thus, using the binomial expansion we see that

$$t_+(s) = \frac{1}{2} \left( \frac{\rho s}{\dot{g}} + \frac{\gamma^2 \theta_{a}}{k \dot{g}} + \frac{cs}{k} \right)$$

$$\cdot \left\{ 1 + \sqrt{1 - \frac{4 \rho \rho s^2}{k \dot{g}} \left( \frac{\rho s}{\dot{g}} + \frac{\gamma^2 \theta_{a}}{k \dot{g}} + \frac{cs}{k} \right)^{-2}} \right\}$$

$$= \frac{\rho s^{1+\eta}}{\delta}(1 + o(1)).$$

A similar calculation gives (or use (5.52))

$$t_-(s) = \frac{cs}{k}(1 + o(1)),$$

and taking square roots gives

$$\sqrt{t_+(s)} = \sqrt{\frac{\rho}{\delta} s^{n+1}}(1 + o(1)), \quad \sqrt{t_-(s)} = \sqrt{\frac{c}{k}} s^{\frac{1}{2}}(1 + o(1)).$$

Routine calculations now yield

$$\left| \frac{c\rho s^2 - t_+(\rho s + \gamma^2 \theta_{a})}{k \dot{g} t_+(t_- - t_+)} \right| = \mathcal{O}(|s|^{\frac{n+1}{2}}),$$

and

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\[
\left| \frac{c p s^3 - t_-(p k s + \gamma^2 \theta_a)}{k s^\frac{1}{2} (t_+ - t_-)} \right| = O \left( |s|^{-\frac{1}{2}} \right). \\
(5.58)
\]

Next note that \( \Re \sqrt{t_+(s)} \to \infty \) and \( \Re \sqrt{t_-(s)} \to \infty \) as \( s \to \infty \) uniformly in \( S_\phi \). It follows that uniformly for \( 0 \leq \tau \leq 1 \),

\[
\left| \frac{\cosh \sqrt{t_\pm(s)}(1 - \tau)}{\sinh \sqrt{t_\pm(s)}} \right|
\]

\[
= \left| \frac{\exp(-\sqrt{t_\pm(s)}\tau) + \exp(-\sqrt{t_\pm(s)}(2 - \tau))}{1 - \exp(-2\sqrt{t_\pm(s)})} \right| \\
(5.59)
\]

\[
= 1 + o(1) \quad \text{as} \quad s \to \infty
\]

uniformly in \( S_\phi \). Thus, if \( w_0(\tau) \) is bounded on \( [0, 1] \), (5.57), (5.58) and (5.59) can be combined to yield

\[
|W(0,s)| = O \left( |s|^{\frac{\alpha - 1}{2}} \right) = o(1) \quad \text{as} \quad s \to \infty
\]

uniformly in \( S_\phi \). \( \Box \)

If we use the Inversion Theorem to invert the asymptotic expansion for \( W(0,s) \) we obtain
\[ w(0, t) - w^{(0)} \]
\[ \approx \tilde{w}^{(2)} \left[ \gamma^2 \left( -\frac{\delta \theta_a}{k g(\infty)^2 \Gamma(\eta - 1)} t^{\eta - 2} + \frac{\delta^2 \theta_a}{k g(\infty)^3 \Gamma(2\eta - 2)} t^{2\eta - 3} - \ldots \right) \right. \]
\[ + \left. \left( -\frac{\rho \delta}{g(\infty)^2 \Gamma(\eta - 2)} t^{\eta - 3} + \frac{\rho \delta^2}{g(\infty)^3 \Gamma(2\eta - 3)} t^{2\eta - 4} - \ldots \right) \right] \]
\[ + \tilde{w}^{(4)} \left[ \gamma^2 \left( -\frac{(c\theta_g g(\infty) + 2\gamma^2 \theta_g^2) \delta}{k^2 g(\infty)^3} \frac{1}{\Gamma(\eta - 2)} t^{\eta - 3} + \ldots \right) \right. \]
\[ + \left. \left( -\frac{2\rho \delta^2}{g(\infty)^3 \Gamma(\eta - 4)} t^{\eta - 5} + \ldots \right) \right] + \ldots, \quad t \to \infty. \] (5.60)

We note that \( w^{(0)} = \int_0^1 w_0(x) \, dx \) is a constant equilibrium displacement which is expected because of the stress-free boundary condition (5.5). In the special case where \( w_0(x) \equiv \tilde{w}_0 \), we have \( w^{(0)} = w_0 \) and it is easy to show that \( \tilde{w}^{(2n)} = 0 \) for \( n = 1, 2, \ldots \), which we expect.

We remark that our use of the Inversion Theorem has not been completely justified since we have not dealt rigorously with questions of existence of \( w(0, t) \). We have merely assumed that \( w(0, t) \) is representable as the inverse of its transform \( W(0, s) \). We believe that this is the case for sufficiently smooth initial data \( w_0(x) \). However, for this reason expansion (5.60) must be regarded as a formal expansion at this time.

For the case of a purely viscoelastic rod with no thermal coupling \( (\gamma = 0) \), it is much easier to obtain an integral representation formula for the transform \( W(x, s) \). Namely, the Laplace transform of the solution of equation (5.1) with \( \gamma = 0 \), satisfying the boundary condition (5.5) and with initial condition \( w(x, 0) = w_0(x) \), \( w_t(x, 0) = 0 \) is given by

\[ W(x, s) = \frac{\rho}{\tilde{g} \beta \sinh \beta} \int_0^1 \cosh \beta (1 - \tau) w_0(\tau) \, d\tau \]
\[ - \frac{\rho}{\tilde{g} \beta} \int_0^x \sinh \beta (1 - \tau) w_0(\tau) \, d\tau \] (5.61)

where \( \beta = \beta(s) = \frac{s}{\tilde{g} \beta(s)} \) with \( \alpha(s) = \sqrt{\rho s \tilde{g}(s)} \) (compare [11]). (Here we use the branch of the
square root which is positive on the positive real axis.) Detailed analysis of the mapping properties of \( \beta(s) \) has been carried out (see [11],[12]). Moreover, rigorous existence and asymptotic decay results have been obtained for the integro-partial differential equation modelling pure linear viscoelasticity, both with conservative boundary conditions and with dissipative boundary conditions. (See, e.g. [2],[11],[13],[15],[28].) We note that if we let
\[
g(t) = g(\infty) + \frac{\delta^t - n}{t^{(n)}}, \quad \text{set } x = 0 \text{ in } (5.61),
\]
use (5.61) to develop an asymptotic expansion for \( W(0, s) \), and then use the theorem of Doetsch to develop an asymptotic expansion for \( w(0, t) \) valid for large \( t \) (a step which is completely justified in this case), we obtain the expression that is found by setting \( \gamma = 0 \) in equation (5.60).

Assuming that \( \bar{w}(2) \neq 0 \), we see that the rate at which the displacement tends to the equilibrium \( w^{(0)} \) when \( \gamma > 0 \) is one power of \( t \) slower than when \( \gamma = 0 \). Thus, the uncoupled system tends to equilibrium faster than the coupled system. This result differs considerably from a previous analysis of the viscoelastic rod with boundary feedback as the dissipation mechanism (see [13, p. 526, Note 2]). In that case, we have the boundary condition

\[
g(\infty)w_x(1, t) + \frac{d}{dt} \int_0^t [g(t - \tau) - g(\infty)] w_x(1, \tau) d\tau = -kw_t(1, t), \quad k > 0 \tag{5.62}
\]

instead of the stress-free condition at \( x = 1 \) and we have the Dirichlet condition \( w(0, t) = 0 \) at the left end. The boundary condition (5.62) is velocity feedback. If we carry out the asymptotic expansion for \( w(1, t) \) in this situation, we see that \( k \) does not appear in the leading term of the expansion, which shows that the displacement tends to zero at essentially the same rate whether \( k > 0 \) or \( k = 0 \).

We feel that our example indicates an interesting phenomenon which warrants further investigation. As we noted at the end of Section 5.2, in future work we plan to examine the asymptotic dependence of the zeros of \( \Delta_n(s) \) on \( \gamma \) for large \( n \). Also, for several specific kernels, we intend to numerically “track” the zeros of \( \Delta_n(s) \) for small \( n \) as we vary the parameter \( \gamma \). We will also investigate the behavior of zeros of \( \Delta_n(s) \) for exponentially decaying kernels. If \( g(t) \) decays to \( g(\infty) \) like \( e^{-\delta t} \) we can show that each \( \Delta_n(s) \) will have a
real zero on \((-\delta, 0]\); these correspond to creep modes. Preliminary investigations indicate that these zeros move to the right as \(\gamma\) increases from 0.

Lastly, we would like to study the energy decay of the system (5.1)–(5.5) instead of the rate at which the displacement \(w(0, t)\) tends to equilibrium. To do this, we will need a rigorous existence result in an appropriate energy space setting. We hope to obtain precise rates of decay of this energy when \(g(t) - g(\infty)\) decays to zero at a subexponential rate in the spirit of those obtained by Hannsgen and Wheeler in [13] for the viscoelastic rod with no thermal coupling.
REFERENCES


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