TENSILE AND UNIAXIAL/MULTIAXIAL FATIGUE BEHAVIOR
OF CERAMIC MATRIX COMPOSITES AT AMBIENT AND
ELEVATED TEMPERATURES

by

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Increasing use of fiber reinforced ceramic matrix composites (CMC's) materials is needed, especially for hostile environments such as elevated temperatures. However, some fundamental issues regarding how these materials should be made for optimized performance are far from being settled. This study focuses on the modeling of the tensile behavior of unidirectional CMC using statistical methods and micro-mechanical analysis, based on laboratory observations. The model can be used to examine the effect of performance-influencing parameters on the strength of unidirectional CMC, thus shed light on how such material should be put together. The tensile strength model was then modified such that the behavior of unidirectional CMC under cyclic tensile load can be studied. Results from the tensile strength model suggest that the Weibull modulus, \( m \), of the strength of the reinforcing fibers and the fiber/matrix interfacial shear stress both have significant effect on the strength and toughness of the unidirectional composite: a higher \( m \) value and a lower interfacial shear stress result in a lower strength; a lower value of \( m \) and a higher interfacial shear stress results in a higher strength but lower toughness. Calculations from the tensile fatigue model suggest that a lower \( m \) value results in a longer fatigue life.
The performance of "real" components (tubes) made of CMC was also investigated. Experimental procedures and methods were developed and established, base-line data were obtained from testing CMC tubes under uniaxial and multiaxial loads, each for two types of loading history, quasi-static and cyclic; and, at both ambient and elevated temperatures. Damage and failure mechanisms in these components were examined in detail. It is shown that the cross-over regions of the fiber tows in the tube are very critical to the failure of the structure. Data interpretation schemes for structure failure are also suggested, in which a bundle strength approach and a fracture mechanics approach are used to predict the failure of the composite structure under torsion and tension, respectively, with fairly good agreement with experimental data.
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Introduction

For most engineering systems, the utilization of better structural materials is often a prerequisite for achieving better performance. The development of ceramic fiber reinforced ceramic matrix composites (CMCs) for applications in the aerospace and a number of other industries is one illustrative example. This class of material is intended to be used in hostile environments (e.g. elevated temperatures) in engineering systems such as heat engines, turbines, and heat exchangers, to name only a few applications. However, despite the enormous effort to develop CMCs, one fundamental issue is far from being settled. The problem can be summarized in a single sentence: How such material should be put together from its constituents (that is, fiber, matrix, and interphase region) such that optimum material properties can be obtained.

In many engineering applications, the ultimate tensile strength and the long-term fatigue behavior (life, or cycles to failure) of a material are two of the most important design parameters. A first step towards the understanding of these two fundamental material behavior for an engineering component made of CMC requires an understanding of the behavior of its simplest form, an unidirectional composite in which reinforcing fibers are
aligned in one direction, bound together by matrix material. In order to seek a solution, at least partially, for the problem of obtaining optimum material performance, a reasonable approach is to model its behavior analytically based on laboratory observations such that performance-influencing parameters can be identified, thus providing guidance for selecting material variables for optimized performance. However, unidirectional composites alone are seldom used in structural applications. In addition, real components, usually more complex in structure and used under complex loading conditions, may behave quite differently from laboratory coupons. To this end, experimental procedures and data interpretation methods must also be established for testing of representative real components.

**Damage Mechanisms and Their Modeling in Unidirectional CMCs Under Tensile Loading**

Modeling of the tensile behavior of CMCs usually begins with laboratory observations of damage events [1-18]. For strong fiber reinforced brittle matrix composites, the first noticeable damage event during tensile loading is the occurrence of matrix cracks perpendicular to the direction of the reinforcing fibers, which marks the deviation of the composite from linear stress-strain behavior [7]. This change from linear elastic behavior evoked many studies for predicting the stress at which the matrix crack occurs. In addition to the non-linear material behavior, environmental attack on the exposed fibers after matrix cracking (which is believed to cause strength degradation) is also a serious concern.
A prerequisite for analyzing the mechanical behavior of CMC is the understanding of load transfer mechanism between the fiber and the matrix. The load transfer problem, including the stress analysis around a broken fiber, and the effect of thermal residual stress, have been studied in detail by a number of researchers, although some of these studies are targeted at polymeric composites [19-25].

Aveston, et al [1] first explained the mechanism underlaying the observed phenomenon of multiple parallel matrix cracking in brittle matrix composite in their classic paper, know as the ACK model. In that model, the interfacial shear stress between the fiber and the matrix was assumed to be a constant. The regularly spaced matrix cracks are interpreted as results of stress transfer from the fibers in a matrix crack plane where its normal stress is maximum, back to the matrix (which has zero stress in the crack plane to matrix cracking stress over a distance approximately one half the crack spacing). In a later paper [2], they took into consideration the case where the fiber/matrix interface is partially debonded. Also in a classic paper, Marshall, et al [4] analyzed matrix fracture in brittle fiber composites prior to fiber failure, in which there is purely frictional bonding between the fibers and the matrix. The stress for matrix cracking is evaluated using a stress intensity approach. In another study, Budiansky, et al [5], presented an analytical model on the critical condition for onset of widespread matrix cracking based on fracture mechanics and energy balance approach. McCartney [6] established the relation between the matrix-cracking stress and the size of a pre-existing defect with accuracy in a refined treatment for the matrix cracking problem. Li et al [12] treated the same problem using an inclusion method and energy approach, where the stability and growth of matrix cracks in a semi-infinite fiber-reinforced composite are analyzed, and the applied stress at which such cracks cease to develop is predicted.
Accompanying the development of transverse matrix cracks during tensile loading, debonding of the fiber/matrix interface and slipping between the fiber and the matrix may occur. The debonding and slipping (and eventually pull-out) behavior depends on a variety of parameters and was the subject of a number of studies [26-39]. The study of debonding behavior of a continuous fiber embedded in matrix can be categorized as types, regarding the debonding criterion. One is the maximum shear strength approach in which debonding propagates when the interfacial shear stress exceeds that of the strength [26-29]. The other one is the fracture mechanics approach in which the strain energy change in the fiber and the matrix is equated to the fracture toughness of the interfacial region [30-32]. Kim, et al [30] compared the fracture mechanics approach and the shear strength approach for determining interfacial debonding and pull-out. They conclude that the debonding stress predicted by the two types of approaches agree exceptionally well with experiments for ceramic matrix composite systems. After analyzing the two types of approaches, Leung and Li [38] suggested that the strength-based approach is valid if the ratio of interfacial shear strength to that of the stress is low, for both small and large transition zone between the debonded and the bonded region. In addition, they indicate that fracture-based theory may be more suitable for cases with high shear strength to stress ratios.

Freund [39] studied the process of sliding a circular elastic fiber through a hole in an unbounded elastic matrix. He found that, for very long fibers, the force needed to slide the fiber against frictional resistance is “independent” of the length of the fiber and of the coefficient of friction. This is because the frictional resistance is localized to a small portion of the fiber near its free end. Sigl and Evans [40] considered the effect of residual stress and frictional sliding on cracking and pull-out behavior in CMCs. Chiang et al [41] adapted the approach of Marshall et al [4], but modified the approach by taking into
consideration the matrix shear deformation above the slipping zone. They conclude that the inclusion of matrix deformation significantly affects the crack tip stress intensity factor and the prediction of the critical matrix cracking stress. Curtin [42] approaches the matrix cracking problem using a fiber fragmentation analysis. Charalambides and Evans [43] studied the debonding and fiber pullout characteristics for interfaces subjected to residual tension. Hutchinson and Jensen [44], analyzed debonding behavior of fiber in matrix for the cases of constant friction and Coulomb friction. Thouless and Evans [45] discussed pull-out behavior using statistical analysis for fiber failure site in the presence of a matrix crack. Their results are strictly applied to weakly bonded fibers. Since the debonding mechanism is a direct consequence of interfacial properties, several studies are devoted solely to the effect of the interfacial region on the mechanical behavior [46-56].

**Tensile Strength Models for Unidirectional CMCs**

There are relatively fewer analytical models for the ultimate strength of unidirectional CMCs in the literature compared to analyses for damage mechanisms such as matrix cracking. Marshall and Cox [57] studied the mechanics of matrix cracking in unidirectional ceramic composites. By using a fracture mechanics approach, they established relationships between the composite stress intensity factor in front of a matrix crack (that characterizes the composite stress and strain field) bridged by fibers and the composite strength, characterized by a single-valued fiber strength. The analysis provided strength-crack-size relations for several different failure mechanisms using single-valued fiber strength, which is a major deviation from reality. Sutcu [60] developed a strength model by applying Weibull statistics to fiber fracture and fracture lo-
cation. Fiber pull-out length and work of pull-out can be determined for both single and multiple matrix cracking. The ultimate strength of the composite is determined by tracing the maximum stress supported by the intact and failed fibers in a matrix crack as the composite is loaded. The model overestimated the ultimate strength by about 300 MPa for Nicalon/LAS-Glass system. The strength model by Schwietert and Steif [61] begins with the assumption that multiple matrix cracks have already occurred. Statistical analysis is used to determine the probability of fiber fracture where each fiber is allowed to fail more than once. They found that the influence of second fiber fracture on the ultimate strength is insignificant. The model employed a simple stress distribution in the fiber. They concluded that the ultimate strength is very sensitive to interfacial shear stress. One major drawback of the model is that it does not generate direct predictions given the necessary material properties. Curtin [63,64] proposed a tensile strength model for brittle matrix composites where he utilized the result of single fiber fragmentation behavior in (elastic) matrix. He established pull-out lengths by intersecting the fiber fragmentation length and the matrix crack spacing. Based on those results, he formulated the tensile strength similar to the approach of Sutcu [60]. Weitsman and Zhu [62] presented a comprehensive model for multi-fracture process of unidirectional CMCs. They used an energy approach to account for progression of matrix cracks, and Weibull statistics to relate fiber fractures. Two special features of their model are that an iterative procedure is used to estimate matrix crack density, and a minimum potential energy principle is used for approximate solution. However, the model does not offer a direct numeric prediction for the tensile strength of the composite.

There are some common features in tensile strength models by Sutcu [60], Schwietert and Steif [61], Weitsman and Zhu [62], and Curtin [63]. First, Weibull statistics is used for
treatting fiber fracture. Such treatment was the subject of a number of reports [65-72]. Second, the phenomenon of multiple matrix cracking contributes to the ultimate strength only in the sense that it serves to define the geometry for the boundary-value problem. Third, the ultimate strength (or fracture criterion) depends on the stress supported by fibers (intact and broken) on the matrix crack plane. In the models of Sutcu, Curtin, Schweitzer and Steif, simplified stress distribution in the fiber and the matrix is used where some important material behavior (such as the effect of Poisson's contraction on residual radial stress between the fiber and the matrix) are not considered. Also, debonding of fiber/matrix interface is treated as a separate case.

**Long-Term Behavior of CMCs**

Experimental investigation on the long-term behavior of CMCs has been reported by a number of researchers [73-82]. In general, observable damage events in tensile-tensile fatigue of unidirectional CMCs are similar to those loaded monotonically, which include parallel matrix cracking and fiber/matrix debonding. They account partially for the usually observed stiffness degradation of the material as functions of applied load cycles. Lee [10] found that fiber surfaces of fatigue-loaded specimens were severely damaged, possibly as a result of frictional sliding against the debonded matrix. Holmes [73] also investigated the implication of frictional heating during cyclic loading of CMCs on the interfacial shear stress and life of the specimen. Despite these experimental investigations, to the author's knowledge, there is not any model in the literature for fatigue life prediction of unidirectional CMC at the micro-level.
Test Method for CMCs Tubes

The specimens used for experimental investigation are tubular in shape, which is one of the most common geometries found in real engineering components. In addition, multi-axial stress states, that represent more realistic service conditions, can be applied to tubular specimens. Existing literature on test methods for CMC components is not abundant. Petrovic published a number of papers on testing of monolithic $Al_2O_3$ tubes [83]. There are also a few reports on testing tubes made of polymer matrix composites [84]. In tube testing, perhaps the most important issue is specimen gripping. Complicated systems are usually required for gripping tubes. Liao, et al [84] developed a relatively simple gripping method for testing straight-sided CMC tubes in association with the available test frame. In addition, there is not a standardized method for testing tubes at elevated temperatures. As a result, such procedures need to be developed and established.

In the light of the previous paragraph, the objectives of this research effort are stated as follows. First, to model the ultimate tensile strength of unidirectional CMCs using micro-mechanical analysis such that performance-related parameters can be established. Second, to model the tensile fatigue behavior of unidirectional CMCs for the same purpose. Third, to develop experimental methodologies for testing representative real components at both ambient and elevated temperatures, and at "realistic" applied load (multi-axial loads), and to establish data interpretation schemes.

The content of this dissertation is arranged as follows. After the introduction and literature review in Chapter 1, an analytical model of the tensile behavior of unidirectional CMCs is presented in Chapter 2. In Chapter 3, the tensile model is modified to model
the fatigue behavior of unidirectional CMCs. In Chapter 4, experimental investigation of the mechanical behavior of CMC structures is presented. The structure (tubes) under investigation represents some of the actual components in "real life" applications. Detailed experimental procedures and results for characterizing the CMC structure under uniaxial and multi-axial loads, quasi-static and cyclic, and at both ambient and elevated temperatures, are presented. Methods are suggested for data interpretation. Finally, a summary, conclusions, as well as recommendations for future study are presented in Chapter 5.
Tensile Strength of Unidirectional Ceramic Matrix Composites

In this chapter, an analytical model for predicting the tensile strength of a unidirectional CMC is presented. A brief description of the phenomenological damage events in the CMC accompanying tensile loading is first provided, followed by a sketch of the concept of the model. Stress analyses for the fibers in a (matrix) cracked CMC within the debonded and bonded regions are then presented, followed by a general treatment of statistical analysis for brittle materials. The stress and the statistical analyses are then combined to describe probability of fiber fracture during tensile loading. Typical results are also presented and discussed in the last section of the chapter.

2.1 Damage and Failure Mechanisms

There have been numerous reports on the damage and failure mechanisms of unidirectional CMCs subjected to tensile loading [1-18]. In general, these mechanisms
can be depicted, in a simplified way, as shown schematically in Fig. 1. In many CMC systems, the failure strain of the matrix material is lower than that of the reinforcing fibers. As a result, the first noticeable damage event is transverse matrix cracking perpendicular to the fiber direction (Fig. 1b). The initial matrix cracking stress has been treated theoretically by a number of investigators [5,6]. As the applied load increases, more such matrix cracks will develop. Accompanying matrix crack growth, the fiber/matrix interface may debond, as shown in Fig. 1c. Fiber fractures may also occur at this point, some within the matrix crack plane, some inside the matrix. These matrix cracks become saturated at certain load level (Fig. 1d). Aveston et al. proposed an explanation for the phenomenon in their classic paper, also known as the ACK model [1,2]. At this stage, the debonding length between the fiber and the matrix also increased. Also, more fiber fractures may occur inside the matrix.

Since the strength of a fiber varies from point to point along its length, fiber fracture may occur at any point where the local fiber strength matches the applied stress. Fibers do not necessarily fail at the point of maximum stress (which is usually at the matrix crack plane). Further increase in applied load results in complete debonding of the fiber/matrix interface and more fiber fracture. At the final stage of the tensile loading, all the fibers have failed and are pulled out from the matrix (Fig. 1f). In the following sections, we attempt to describe, using statistical methods and elasticity theory, the scenario depicted in Fig. 1; we are particularly interested in the final stage, that is, the strength.

Since we hypothesizes that most of the fibers are still intact at the time matrix cracks are saturated, we assume that it does not make too much difference if we start modeling the failure process from Fig. 1a or Fig. 1d. In fact, it can be shown in the analysis that
follows that only a trivial amount of fibers have failed up to the matrix cracking stress. Therefore, for simplicity, and the aforementioned reason, we begin our modeling with Fig. 1d.

To facilitate our analysis, it is desirable to chose a "representative volume", with convenient geometry and known boundary conditions. This "representative volume" repeats itself throughout the composite, as shown in Fig. 2a. We chose the geometry inside the dotted rectangle, as shown in Fig. 2b, with the matrix crack plane in the middle. The thickness of the representative volume is $2s$, the distance between two adjacent parallel matrix cracks. For such a geometry, the fiber stress is known at the matrix crack plane, which is simply $\sigma/\nu$, where $\sigma$ is the remote applied stress in the fiber direction and $\nu$ is fiber volume fraction. To analyze stress distributions in the fiber and the matrix, we further chose a "unit cell" from our representative volume, indicated by the dotted rectangle in Fig. 2c. The unit cell contains a single fiber surrounded by a cylindrical matrix block, the diameter of it is determined by the matrix volume fraction of the bulk composite. A coordinate system for the unit cell is shown in Fig. 2c. The origin of the coordinate system is set at the middle of the matrix block, with matrix crack plane at $z = s$. A debonded region with length $l_d$ is shown, represented by the dashed lines.

In the present model, we assume that each fiber only breaks once. It has been shown by Schwietert and Steif [61] that the effect of fiber breaking twice on the mechanical behavior of the composite is negligible. The probabilities of fiber fractures and the fracture locations for each of these three regions can be determined using statistical analysis, once the stress distributions of the fibers in each of the three regions are known. These probabilities can be interpreted as the ratio of number of broken fibers (in each region) to that of the total number of fibers in the unit cell.
Figure 2. Representative volume and unit cell for stress analysis.
For fibers broken within the matrix crack plane, they will not carry any load upon further loading. For fibers broken within the debonded region, they still carry some applied load by the action of interfacial shear stress. It can be shown that the (normal) fiber stress in the matrix crack plane can be related to the interfacial shear stress between the fiber and the matrix and the location of fracture. For fibers broken within the bonded region, we assume that debonding will eventually propagate to their fracture location, such that the normal fiber stress in the matrix crack plane is supported by interfacial shear stress and depends on the fiber fracture locations, which is similar to the previous case where fiber fracture occurs within the debonded region.

Knowing the ratio (or probability) of fibers broken in the three regions, and the stress supported by broken fibers (for those broken in the debonded and bonded region), the applied stress on the intact fibers can be determined. With further increase of applied load, the previous analyses are carried out for the remaining intact fibers until the fibers (intact and broken) can no longer support the applied load, when fracture occurs.

2.2 Stress Analysis of the Unit Cell

To carry out the analysis for the modeling concept just described, a stress analysis for the fiber and the matrix in the unit cell must be established. In this unit cell, the portion of the fiber embedded in the matrix is assumed to be partially debonded at some stress level, as shown in Fig. 2. Stress analysis for the fiber and the matrix is carried out for the debonded and the bonded region, which is briefly described as follows. Here the analysis of Gao & Mai [58] is followed closely.
**Debonded Region**

Consider the cylindrical model of a single fiber embedded in a layer of matrix, as shown in Fig. 2. The origin of the coordinate system is at the middle between two matrix crack planes. Force equilibrium of the fiber gives

\[
\frac{\partial \sigma_f^z}{\partial z} = -\frac{2}{a} \tau_i \quad s - t_d < z < s
\]  

(2.2.1)

Force equilibrium of the matrix gives

\[
\frac{\partial \sigma_m^z}{\partial z} = \frac{2a}{b^2 - a^2} \tau_i
\]  

(2.2.2)

where

\[
\sigma_f^z = \text{fiber stress in the } z \text{ direction}
\]

\[
\sigma_m^z = \text{matrix stress in the } z \text{ direction}
\]

\[
a = \text{radius of the fiber}
\]

\[
b = \text{outer radius of the matrix cylinder}
\]

The interfacial shear stress, \(\tau_i\), is given by

\[
\tau_i = \mu_i (q_o - q_r)
\]  

(2.2.3)

where \(\mu_i\) is the coefficient of friction between the the debonded fiber and matrix, \(q_o\) and \(q_r\) denote the radial thermal residual stress and radial stress due to Poisson's effect, respectively. Expression for \(q_o\) has been suggested by Naik [25]. The expression for \(q_r\) is
\[ q_{\rho} = \frac{E_r \nu_f \sigma_f^2 - \nu_m \sigma_m^2}{E_r (1 - \nu_f) + 1 + \nu_m + 2V_r} \]  

(2.2.4)

where

\[ E_r = \frac{E_m}{E_f} \]

\( E_r \) = fiber modulus

\( E_m \) = matrix modulus

\( V_r = \frac{V_f}{V_m} \)

\( V_f \) = fiber volume fraction

\( V_m \) = matrix volume fraction

\( \nu_m \) = Poisson’s ratio of the matrix

\( \nu_f \) = Poisson’s ratio of the fiber

A detailed derivation for \( q_{\rho} \) can be found in Appendix A. Combining Eqs. (2.2.1) through (2.2.3), we can obtain the following differential equations for the normal stresses for the fiber and the matrix

\[ \frac{\partial^2 \sigma_f^2}{\partial z^2} + \Psi \frac{\partial \sigma_f^2}{\partial z} = 0 \]  

(2.2.5a)

\[ \frac{\partial^2 \sigma_m^2}{\partial z^2} + \Psi \frac{\partial \sigma_m^2}{\partial z} = 0 \]  

(2.2.5b)

where

\[ \Psi = -\frac{2\mu_i}{a} \left( q_o + \frac{V_r \nu_m}{C_o} \right) \]  

(2.2.6)

and

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At the matrix crack plane \((z = s)\), we have \(\sigma_f(s) = \sigma_a\), the applied stress, assuming that intact fibers share the same load; and \(\sigma_z(s) = 0\), since the matrix is stress free at the crack plane. With these boundary conditions, Eqs. (2.2.5) can be solved, and the fiber stress is given by

\[
\sigma_f = \sigma_a - \omega(\bar{\sigma} - \sigma_a)(e^{\lambda_0(z - s)} - 1)
\]  

(2.2.8)

where

\[
\omega = \frac{E_r v_f}{E_r v_f + V_r v_m}
\]

(2.2.9)

\[
\bar{\sigma} = \frac{q_o C_o}{E_r v_f}
\]

\[
\lambda_0 = \frac{2\mu_f(E_r v_f + V_r v_m)}{a C_o}
\]

where \(C_o\) is given by Eq.(2.2.7). Matrix and interfacial shear stress can also be found. However, those stresses will not be given here as they do not relate directly to fiber fracture. Details of the analysis can be found in Appendix B.

**Bonded Region**

Force equilibrium of the fiber and the matrix in the bonded region results in equations similar to Eqs. (2.2.1) and (2.2.2).
\[
\frac{\partial \sigma_f^z}{\partial z} = -\frac{2}{a} \tau_l \quad 0 < z < s - l_d \quad (2.2.11a)
\]

\[
\frac{\partial \sigma_m^z}{\partial z} = \frac{2a}{b^2 - a^2} \tau_l \quad (2.2.11b)
\]

Equations (2.2.11a,b) together with stress-strain-displacement equations for the fiber and the matrix can result in the following second order partial differential equation for fiber stress

\[
\frac{\partial^2 \sigma_f^z}{\partial z^2} - A_1 \sigma_f^z - A_2 \sigma_a = 0 \quad (2.2.12)
\]

where

\[
A_1 = \frac{2[E_r(1 - k_o \nu_f) + V_r(1 - 2k_o \nu_m)]}{(1 + \nu_m)[2V_r b^2 \ln(b/a) - a^2]}
\]

\[
A_2 = \frac{-2V_r(1 - 2k_o \nu_m)}{(1 + \nu_m)[2V_r b^2 \ln(b/a) - a^2]}
\]

\[
k_o = \frac{(E_r \nu_f + V_r \nu_m)}{[E_r(1 - \nu_f) + 1 + \nu_m + 2V_r]}
\]

The boundary conditions for the bonded region are

\[
\sigma_f^z = \sigma_f^d \quad z = s - l \quad (2.2.13)
\]

\[
\sigma_f^z = \sigma_f^d \quad z = - (s - l)
\]
where $\sigma_f$ is the fiber stress at the debonding front. A detailed analysis for stresses in the bonded region is found in Appendix C. Results from the analysis show that the fiber stress is

\[
\sigma_f^2 = 2 \left( \sigma_f^d + \frac{A_2}{A_1} \sigma_a \right) \frac{\sinh[\lambda_1(s - l_d)]}{\sinh[2\lambda_1(s - l_d)]} \cosh(\lambda_1 z) - \frac{A_2}{A_1} \sigma_a
\]  

(2.2.14)

where

\[
\lambda_1 = \sqrt{A_1}
\]

and the interfacial shear stress is

\[
\tau_i = -a\lambda_1 \left( 1 + \frac{A_2}{A_1} \right) \sinh^2[\lambda_1(s - l_d)]\sigma_a \sinh^{-1}(2\lambda_1 s)
\]  

(2.2.15)

**Debonding Criterion**

As the applied stress increases, the length of the debonding region also increases. There are two types of debonding criterion that can be found in the literature, the maximum shear strength approach [26-29] and the fracture mechanics approach [30-32]. The validity of these two types of approaches were discussed by Leung and Li [38]. They concluded that the strength-based approach is valid for material systems with low interfacial shear strength to shear stress ratio while the fracture mechanics approach is valid for cases with a large such ratio.

The debonding criterion adapted here is similar to that used by Hsueh [29], the maximum shear strength criterion. The fiber/matrix interface debonds when the interfacial
shear stress exceeds that of the maximum strength. From equation (2.2.15), at 
\(z = s - l_d\), the debonding front, \(\tau_s = \tau_i\), the interfacial strength.

\[
\tau_s = -a_1 \beta \left( 1 + \frac{A_2}{A_1} \right) \sinh^2[\lambda_1(s - l_d)] \sigma_f^d \sinh^{-1}(2\lambda_1 s) \tag{2.2.16}
\]

To relate the debonding stress at the debonding front, \(\sigma_f\), to the applied normal stress of the fiber on the matrix crack plane, we use Eq. (2.2.8). At \(z = s - l_d\), we have

\[
\sigma_f^d = \sigma_a - \omega(\bar{\sigma} - \sigma_a)(e^{\lambda_1 s} - 1) \tag{2.2.17}
\]

Equating Eqs. (2.2.16) and (2.2.17), we have

\[
\sigma_a = \frac{\tau_s \sinh[2\lambda_1(s - l_d)]}{a_1 \beta \sinh^2[\lambda_1(s - l_d)] + \omega(e^{\lambda_1 s} - 1)\bar{\sigma}} \tag{2.2.18}
\]

which relates the applied stress at the matrix crack plane to the debonding length. From this equation, the length of debonded fiber/matrix interface, \(l_d\), can be found at any given applied stress level. In the computer program, this is done using the secant method.

### 2.3 Statistical Analysis of Fiber Fracture

Since the strength of the composite is controlled for the most part by the strength of the fibers, and the strength of the fibers are statistically distributed, incorporation of the statistical analysis of fiber fracture in the strength theory is inevitable for an accurate description of the composite behavior. If such an analysis can provide us, during any given load increment, with the number and the location of fiber fracture, it is possible
for us to trace the process and accumulation of damage events and, from which to con-
struct a criterion that can predict the strength of the composite. Statistical treatments
on the fracture of brittle solids have been reported by several investigators [65,66]. Here
we follow closely the treatment by Oh and Finnie [65], applying to ceramic fibers.

Suppose that a single long ceramic fiber consists of N small elements, each of length (or
volume) $\xi$ units. Let the normal stress on an element located at $\xi$ be $\sigma$. Usually, local
stress can be expressed in terms of some reference stress, such that $\sigma = \sigma_0(\sigma, \xi)$, where
$\sigma_0$ is the applied stress. Let $G_{\Delta \sigma}(\sigma_0)$ denote the probability of failure of the element at a
stress $\leq \sigma$. Then the probability of survival of the element is

$$1 - G_{\Delta \sigma}(\sigma_0)$$  \hspace{1cm} (2.3.1)

and the probability of survival of all the elements is the product of each individual
probability of survival

$$\prod_{i=1}^{N} \left[1 - G_{\Delta \sigma}(\sigma_0)\right]$$  \hspace{1cm} (2.3.2)

The probability of failure of the element at $\xi$, between a stress increment from $\sigma_0$ to
$\sigma_0 + \Delta \sigma$, provided that it has survived up to $\sigma_0$, is

$$\left[\frac{\partial G_{\Delta \sigma}(\sigma_0, \xi)}{\partial \sigma_0}\right] \frac{d\sigma_0}{1 - G_{\Delta \sigma}(\sigma_0, \xi)}$$ \hspace{1cm} (2.3.3)

With Eqs. (2.3.2) and (2.3.3), the probability of failure of the fiber at $\xi$, (with all other
elements survived up to $\sigma_0$) is
\[
\left[ \frac{\partial G_{\Delta \xi_i}(\sigma, \xi_i)}{\partial \sigma} \right] \frac{d\sigma}{1 - G_{\Delta \xi_i}(\sigma, \xi_i)} \left[ \prod_{i=1}^{N} [1 - G_{\Delta \xi_i}(\sigma_i)] \right]
\]

(2.3.4)

If we define \( F(\sigma, \xi_i) \) as the probability of failure of a flaw on the fiber up to the stress \( \sigma \), (expressed in terms of \( \sigma_i \)), then the probability of survival of the element of length \( \Delta \xi_i \) is

\[
[1 - F(\sigma, \xi_i)]^{\gamma \Delta \xi_i}
\]

(2.3.5)

where \( \gamma \) is the flaw density such that \( \gamma \Delta \xi_i \) is the total number of flaws within the element \( \Delta \xi_i \). The probability of failure of the element up to a stress \( \sigma \), is

\[
G_{\Delta \xi_i}(\sigma, \xi_i) = 1 - [1 - F(\sigma, \xi_i)]^{\gamma \Delta \xi_i}
\]

(2.3.6)

Therefore, the probability of survival of the element at \( \xi_i \), subject to stress up to \( \sigma_i \), is

\[
1 - G_{\Delta \xi_i} = [1 - F(\sigma, \xi_i)]^{\gamma \Delta \xi_i}
\]

(2.3.7)

or

\[
1 - G_{\Delta \xi_i} = \exp \left\{ \ln[1 - F(\sigma, \xi_i)]^{\gamma \Delta \xi_i} \right\}
\]

Substituting Eq. (2.3.7) into Eq. (2.3.4), Eq. (2.3.4) becomes

\[
- \frac{\partial}{\partial \sigma} \left\{ \ln[1 - F(\sigma, \xi_i)]^{\gamma \Delta \xi_i} \right\} \exp \left\{ \sum_{i=1}^{N} \ln[1 - F(\sigma, \xi_i)]^{\gamma \Delta \xi_i} \right\} d\sigma
\]

(2.3.8)
The probability of failure of a flaw, or to be more exact, the probability of a characteristic length that contains exactly one flaw (i.e., $\gamma^{-1}$), is given by the well-known Weibull distribution

$$F(\sigma_c, \xi_l) = 1 - \exp \left[ -\frac{1}{\gamma} \left( \frac{\sigma_c}{\sigma_o} \right)^m \right]$$ (2.3.9)

Here we use a two-parameter Weibull distribution where $\sigma_c$ and $m$ are the characteristic strength and Weibull modulus, respectively. If we assume that the characteristic length within the element at $\xi_l$ (and, throughout the entire fiber) is the same, Eq. (2.3.9) can be used in Eq. (2.3.8). If we do so, Eq. (2.3.8) becomes

$$\frac{\partial}{\partial \sigma_o} \left( \frac{\sigma_c}{\sigma_o} \right)^m \exp \left\{ - \sum_{i=1}^{N} \left( \frac{\sigma_i}{\sigma_o} \right)^m \Delta \xi_i \right\}$$ (2.3.10)

As the number of elements in the fiber is very large, that is, $N \to \infty$, and the unit (length) of an element is very small, that is, $\Delta \xi_i \to d \xi$, Eq. (2.3.10) can be rewritten as

$$H(\sigma_o, \xi) d\sigma_o d\xi = \exp \left\{ - \int_{\xi} \left( \frac{\sigma}{\sigma_o} \right)^m d \xi \right\} \frac{\partial}{\partial \sigma_o} \left( \frac{\sigma}{\sigma_o} \right)^m d \xi d\sigma_o$$ (2.3.11)

where $H(\sigma_o, \xi)$ is the density function of fiber failure with respect to both applied load and location. Here $\sigma$ is changed to $\sigma_i$ to indicate stress distribution in the fibers.

To determine the probability of fracture of the entire fiber, $P_f$, between a load increment, Eq. (2.3.11) is integrated over the entire length of the fiber and then over the load increment.

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\[ P_f = \int_\sigma^{\sigma + \Delta \sigma} \int_\xi H(\sigma_a, \xi) d\xi d\sigma_a \]  

(2.3.12)

or

\[ P_f = \int_\sigma^{\sigma + \Delta \sigma} \exp \left\{ - \int_\xi \left( \frac{\sigma_f^2}{\sigma_o} \right)^m d\xi \right\} \int_\xi \frac{\partial}{\partial \sigma_a} \left( \frac{\sigma_f^2}{\sigma_o} \right)^m d\xi d\sigma_a \]

To determine the mean location of fracture, \( \bar{\xi} \), for a fiber subjected to probability of failure described by Eq. (2.3.12), we have

\[ \bar{\xi} = \int_\sigma^{\sigma + \Delta \sigma} \int_\xi \xi H(\sigma_a, \xi) d\xi d\sigma_a \]  

(2.3.13)

or

\[ \bar{\xi} = \int_\sigma^{\sigma + \Delta \sigma} \exp \left\{ - \int_\xi \left( \frac{\sigma_f^2}{\sigma_o} \right)^m d\xi \right\} \left[ \int_\xi \xi \frac{\partial}{\partial \sigma_a} \left( \frac{\sigma_f^2}{\sigma_o} \right)^m d\xi \right] d\sigma_a \]

In Eqs. (2.3.12) and (2.3.13), \( \sigma_f \) could be the fiber stress in any of the three regions (matrix crack plane, debonded, and bonded), and the probability of fiber failure (and fracture locations) can be found by using Eqs (2.3.12) and (2.3.13) with stress distributions presented in the previous section. Details are provided as follows.
2.4 Analysis of Probability of Fiber Fracture in the Composite

*Fiber Fracture due to Propagation of Matrix Cracks*

Fiber fracture may occur at any applied load level for which the composite strength is governed by the statistical distribution of the fiber strength, although the probability of fracture at low stress level is so small that it may be ignored.

As transverse matrix cracks start to propagate, even at relatively low stress level (200-300 MPa, Evans), fiber fracture in front of the propagating cracks may not be ignored. Thouless, et al [45] investigated the probability of fiber fracture in the crack wake. However, the effect of a propagating crack front on the probability of fiber fracture has not been analyzed. Evans, et al [4] have shown that the stress intensity factor of a transverse matrix crack bridged by unbroken fibers is

\[
K_I = 2 \left( \frac{c}{\pi} \right)^{1/2} \int_0^1 \frac{[\sigma_a - p(X)]x dx}{\sqrt{1 - X^2}}
\]

(2.4.1)

where \(X = x/c\) Subsequent analysis shows that for a long crack,

\[
K_I = \sigma_a^{3/2} \left[ \frac{\Omega}{\alpha} - \frac{2I}{\sqrt{\pi \alpha}} \right]^{1/2}
\]

(2.4.2)

where
\[ \Omega = \pi^{\frac{1}{2}} \]

\[ I = 1.2 \]

\[ \alpha = 8(1 - v_m^2) \tau_f V_f^2 E_f (1 + \eta)(1/E_0) a \pi^{1/2} \]

and

\[ \eta = \frac{E_f V_f}{E_m(1 - V_f)} \]

From elementary fracture mechanics, the stress field in front of a crack is

\[ \sigma_z = \frac{K_f}{\sqrt{2\pi r}} \cos(\theta/2)[1 + \sin(\theta/2) \sin(3\theta/2)] \]  \hspace{1cm} (2.4.3)

Transforming Eq. (2.4.3) from polar to Cartesian coordinates, we have,

\[ \sigma_z = K_f \sin^{1/2} \left[ \frac{\tan^{-1}(z/d)}{(2\pi z)^{1/2}} \cos \left( \frac{1}{2} \tan^{-1} \left( \frac{z}{d} \right) \right) \right] \times \]  \hspace{1cm} (2.4.4)

\[ \left\{ 1 + \sin \left[ \frac{1}{2} \tan^{-1} \left( \frac{z}{d} \right) \right] \sin \left[ \frac{3}{2} \tan^{-1} \left( \frac{z}{d} \right) \right] \right\} \]

where \( d \) is the distance between the crack tip and the fiber surface. To simplify our calculation, we assume that at the matrix cracking stress all the transverse matrix cracks propagate simultaneously such that the spacing between any two cracks is \( s \) (which is approximately 400 \( \mu \)m), the probability of fiber fracture in front of a matrix crack within a distance of \( 2s \) can be evaluated using Eqs. (2.3.12) and (2.4.4). An example is shown in Fig. 3 where the probability of fiber fracture is plotted against the normalized distance of propagating crack front from a fiber, for the following parameter values.
Figure 3. Probability of fiber fracture due to a propagations matrix crack.
\[ E_f = 200 \text{ GPa} \]
\[ E_m = 85 \text{ GPa} \]
\[ V_f = V_m = 0.5 \]
\[ a = 8 \mu m \text{ (fiber radius)} \]
\[ v_f = 0.12 \]
\[ \tau_f = 2 \text{ MPa} \]
\[ \sigma_a = 300 \text{ MPa} \]
\[ s = 400 \mu m \]
\[ \sigma_o = 1500 \text{ GPa} \]

Although the calculated probability of failure may not be significant for materials systems with weak interfacial bonding as shown in the figure, it may not be ignored in case of strong interfacial bonding as fiber/matrix debonding may not occur in front of the propagating crack. In such a case, probability of fiber failure may increase rapidly as the crack approaches a fiber.

**Probability of Fiber Fracture within the Matrix Crack Plane**

The length of the intact fibers within the matrix crack opening, \( U_o \), due to sliding between the debonded portion of fiber and the matrix is

\[
U_o = \int_{s-l_d}^{z} (e_f^z - e_m^z) \, dz \quad (2.4.5)
\]

where
\[ \varepsilon_f^* = \text{fiber strain in the z direction} \]
\[ \varepsilon_m^* = \text{matrix strain in the z direction} \]

which is also the relative displacement between the fiber and the matrix. In detail,

\[ U^o = U_f^z - U_m^z = (U_1 - U_2 - U_3 + U_4) l + (U_2 - U_4)(1 - e^{-\lambda d})\lambda_o^{-1} \quad (2.4.6) \]

where

\[ U_1 = \frac{\sigma_o}{E_f} \left( 1 - \frac{2 v_f^2 E_f}{C_o} \right) \]

\[ U_2 = \omega (\bar{\sigma} - \sigma_o) \left( \frac{2 v_f (v_f E_f + v_m V_r) - C_o}{E_f C_o} \right) \]

\[ U_3 = \frac{2 V_r E_f v_m \sigma_o}{E_m C_o} \]

\[ U_4 = \omega (\bar{\sigma} - \sigma_o) \left( \frac{C_o - 2 v_m (E_f v_f + V_r v_m)}{E_m C_o} \right) \]

Substituting Eq. (2.4.6) into Eq. (2.3.12), and integrating with respect to \( \zeta \), we get the probability density function for fiber fracture within the matrix crack plane, \( H'_{cp} \)

\[ H'_{cp} = \exp \left[ - \left( \frac{\sigma_o}{\sigma_o} \right)^m \left( U^o \right) m \sigma_o^{-m} \sigma_o^{-m} - 1 \right] \quad (2.4.7) \]

The probability of fiber fracture within this region can be evaluated by integrating Eq. (2.4.7) with respect to applied load.


Probability of Fiber Fracture in the Debonded Region

The density function, $H'_{debond}$, for fiber fracture in the debonded region with respect to applied load only, can be found by using Eq.(2.2.8) in Eq.(2.3.12) and integrate over the fiber length in the debonded region, that is, from $z = s - l_o$ to $z = s$. The resulting equation is

$$H'_{debond} = \exp \left\{ - \left( \frac{1}{\sigma_o} \right)^m \left[ \sum_{i=0}^{\infty} C_i^m \psi_1^m \psi_2^l \left( \frac{1}{\lambda_i^o} \right) \left( 1 - e^{-\lambda_i^o \ell} \right) \right] \right\} \times \left( \frac{m \sigma_o^{-m}}{\lambda_o} \left[ \sum_{i=0}^{\infty} C_i^{m-1} \psi_1^{m-1} \psi_2^l \left( \frac{1}{i} \right) \left( 1 - e^{-\lambda_i^o \ell} \right) \right] \right) + \left\{ \frac{\omega}{\lambda_o} \left[ \sum_{i=0}^{\infty} C_i^{m-1} \psi_1^{m-1} \psi_2^l \left[ \frac{1}{i+1} \right] \left( 1 - e^{-\lambda_i^o \ell} \right) \right] \right\}$$

where

$C_i^m =$ permutation symbol

$\psi_1 = \sigma_o (1 - \omega) + \omega \sigma$

$\psi_2 = - \omega (\sigma - \sigma_o)$

$\psi_3 = 1 - \omega$

The parameters $\omega$ and $\lambda_o$ are as previously defined. The density function for the mean fracture location, $L'_{debond}$, for the debonded region (with respect to the applied stress only) can be found by using Eqs. (2.2.8) and (2.3.13) and integrate over the debonded region.
\[ L_{\text{debond}}' = \exp \left\{ -\left( \frac{1}{\sigma_o} \right)^m \left[ \sum_{i=0}^{m} C_i^m \psi_1^m - i \psi_2^m \left( \frac{1}{i \lambda_o} \right) \left( 1 - e^{-i \lambda_d} \right) \right] \right\} \times (2.4.9) \]

\[ m \sigma_o \left\{ \frac{1}{2} \psi_3 \psi_1^m - i (2s - l_d) \frac{1}{\lambda_o^2} \left[ (\lambda_o s + 1) - e^{i \lambda_d} (\lambda_o (s - l_d) + 1) \right] + \right\} \]

\[ \frac{\psi_1}{-\lambda_o} \sum_{i=1}^{m-1} C_i^m - i \psi_2^m / (1 - i^2) \left[ (i \lambda_d s + 1) - e^{i \lambda_d} (i \lambda_o (s - l_d) + 1) \right] \]

\[ \frac{\omega}{-\lambda_o} \sum_{i=1}^{m-1} C_i^m - i \psi_2^m / (1 + i)^2 \times \]

\[ \left[ \left[ (i + 1) \lambda_d s + 1 \right] - e^{(i+1) \lambda_d} (i + 1) \lambda_o (s - l_d) + 1 \right] \]

The actual probabilities of fiber failure and fracture location can be evaluated by integrating Eqs (2.4.8) and (2.4.9) over applied load.

**Probability of Fiber Fracture in the Bonded Region**

Similar to that in the debonded region, the probability density functions of fiber fracture, \( H'_{\text{bond}} \), and its mean fracture location, \( L'_{\text{bond}} \), in the bonded region can be found by substituting Eq. (2.2.14) into Eqs. (2.3.12) and (2.3.13), respectively, and integrate over \( \zeta \) for the length of bonded region.

\[ H'_{\text{bond}} = \exp \left\{ -\sigma_o \sum_{i=0}^{m} C_i^m \phi_1^m - i \phi_2^m (1/2)^i \left[ \sum_{k=0}^{i} C_k^i \frac{1}{(i - 2k) \lambda_1} \left( e^{(i - 2k) \lambda (s - l_d) \lambda_1} - 1 \right) \right] \right\} \times (2.4.10) \]
\[ m\sigma_o^{-m} \left\{ \phi_5 \sum_{i=0}^{n-1} C_i^{m-1} \phi_4^{m-1} i \phi_3^{(1/2)^i} \left[ \sum_{k=0}^{i+1} C_k^{i+1} \frac{1}{(i+1-2k)\lambda_1} (e^{(i+1-2k)(s-l_d)\lambda_1} - 1) \right] \right\} + \left\{ \phi_6 \sum_{i=0}^{m-1} C_i^{m-1} \phi_4^{m-1} i \phi_3^{(1/2)^i} \left[ \sum_{k=0}^{i} C_k^{i} \frac{1}{(i-2k)\lambda_1} (e^{(i-2k)(s-l_d)\lambda_1} - 1) \right] \right\} \]

where

\[ \phi_1 = -\left( \frac{A_2}{A_1} \right) \sigma_a \]

\[ \phi_2 = 2 \left[ \sigma_a \left( \frac{A_2}{A_1} + e^{-\lambda_1d} \right) - \omega \left( e^{-\lambda_1d} - 1 \right) \bar{\sigma} \right] \frac{\sinh[\lambda_1(s-l_d)]}{\sinh[2\lambda_1(s-l_d)]} \]

\[ \phi_3 = 2 \left( \frac{\sigma_j}{A_1} + \frac{A_2}{A_1} \sigma_a \right) \frac{\sinh[\lambda_1(s-l_d)]}{\sinh[2\lambda_1(s-l_d)]} \]

\[ \phi_4 = -\frac{A_2}{A_1} \sigma_a \]

\[ \phi_5 = \frac{\sinh[\lambda_1(s-l_d)]}{\sinh[2\lambda_1(s-l_d)]} \left[ 1 + \omega \left( e^{\lambda_1d} - 1 \right) + \frac{A_2}{A_1} \right] \]

\[ \phi_6 = -\frac{A_2}{A_1} \]

And

\[ L'_{bond} = \exp \left\{ -\sigma_o^{-m} \sum_{i=0}^{m} C_i^{m} \phi_1^{m} i \phi_2^{(1/2)^i} \left[ \sum_{k=0}^{i} C_k^{i} \frac{1}{(i-2k)\lambda_1} (e^{(i-2k)(s-l_d)\lambda_1} - 1) \right] \right\} \times (2.4.11) \]
\[ m \sigma_o - m \left\{ \phi_5 \sum_{i=0}^{m-1} C_i^{m-1} \phi_{4}^{m-1-i} \phi_3^{i/2} + 1 \times \right\} \]

\[ \left[ \sum_{k=0}^{i+1} C_k^{i+1} \left( \frac{1}{i+1-2k} \right) \frac{s - \lambda_1}{(i+1-2k) \lambda_1} e^{(i+1-2k)(s - \lambda_1) + \frac{1}{(i+1-2k) \lambda_1}} \right] + \]

\[ \phi_6 \sum_{i=0}^{m-1} C_i^{m-1} \phi_{4}^{m-1-i} \phi_3^{i/2} \times \]

\[ \left\{ \sum_{k=0}^{l} C_k^{l} \left( \frac{1}{i-2k} \right) \frac{s - \lambda_1}{(i-2k) \lambda_1} e^{(i-2k)(s - \lambda_1) + \frac{1}{(i-2k) \lambda_1}} \right\} \]

The actual probabilities of fiber failure and fracture location can be evaluated by integrating Eqs (2.4.10) and (2.4.11) over applied load.

### 2.5 Effect of Ineffective Length on Strength

Details of the computational procedures will be described as follows. We begin with Fig. 4a. The drawing shows part of a composite with parallel matrix cracks, with representative fibers numbered 1 through 9. Let us first consider the "representative volume" inside the dotted rectangle, which repeats itself throughout the composite. Within that representative volume, our probability calculations are based on the "unit cell", represented by the shaded rectangle, which contains a single fiber in the cylindrical matrix. The length of the unit cell is s/2, half that of the thickness of the matrix block. Consider the damage event (fiber fracture) that can occur during a small load increment. If the
calculated probability of fiber failure during a load interval is \( P_n \) (which is \( 1/9 \) as shown in this case), each of the half representative volumes contains one such break, as represented by the back slash.

Here we make the assumption that all the fibers that break during a load increment do not occur simultaneously. If this is so, once a fiber breaks within the dotted representative volume has occurred, it immediately sets up an ineffective region on either side of the break, the length of which is the length required for the matrix to transfer stress to the fiber from zero (at the break) to applied fiber stress by interfacial shear stress. We further assume that no further fiber breaks can occur in the already broken fibers in adjacent representative volumes (encapsulating the broken fiber) within the ineffective length, due to a decrease in axial fiber stress. This is illustrated in Fig. 4a, where fiber breaks cannot occur in the shaded unit cells (in fibers 1 and 2) in the adjacent representative volume under the two breaks marked with back slash. Instead, they can break anywhere in fibers 3 through 9 in adjacent unit volumes.

Let one fiber break occur in fiber 4 and fiber 5 (represented by the “x” sign). Note that the probability of failure in the adjacent unit volume is still \( 1/9 \). At this point, ineffective regions are set up around the four breaks in fibers 1, 2, 4, and 5, as shown in Fig. 4b. However, on the “origin” crack-plane, fiber 4 and 5 are not supporting the load that an intact fiber is supposed to sustain, and therefore, should be considered as broken fibers within the dotted unit volume, although their fracture origin is outside that unit volume.

We are still within a particular load interval. Fiber breaks (1/9 in each half unit volume) keep repeating themselves in adjacent unit volumes, outside the “excluded” regions (ineffective lengths) set up by previous breaks. There can be a variety of combinations of fiber breaks when they do so. We illustrate two examples in Figs 4c and 4d. Fibers can
Figure 4. Ineffective length considerations on fiber fracture.
break in fiber 8 and 9 (represented by the "= ") sign, as shown in Fig. 4c. In this case, only fibers 3, 6, and 7 are considered intact on the "origin" crack plane, or, 2/3 of all fibers have failed within the dotted unit volume. Fiber breaks may also occur in fibers 4 and 5, just outside the ineffective length. In this case, overlapping of ineffective lengths occurs, and fibers 3, 6, 7, 8, 9 are still intact (or 4/9 of all fibers have failed). This situation may be rare compared to the previous case with a large amount of fibers in the composite. However, as the number of broken fibers increases, the probability of occurrence may increase. Nevertheless, a calculation based on this case may establish an upper bound for the ultimate strength whereas the previous case represents a lower bound.

In actual computation, an ineffective region of length $L_{ineff}$ is determined at each load interval

$$L_{ineff} = \frac{a \sigma_a}{2 \tau_i}$$  \hspace{1cm} (2.5.1)$$

The origin of this ineffective region is set at a matrix crack plane. The number of matrix blocks within the ineffective region, $N_m$, is $L_{ineff}/s$. The number of intact fibers left (considered within the ineffective region) after a load interval is

$$N_T (1 - P_f N_m)$$  \hspace{1cm} (2.5.2)$$

The force supported by broken fibers, $f_b$, during a load interval is

$$f_b = \sum_{i=1}^{N_T} \frac{2 \pi a l_i \tau_i P_f N_T}{l_i \tau_i P_f N_T}$$  \hspace{1cm} (2.5.3)$$
where \( l \) is the distance from the location of fracture to the matrix crack plane. The accumulative force supported by all the broken fibers, \( F_b \), up to a certain applied stress level, \( \sigma_a \), is

\[
F_b = \sum_{\Delta \sigma} \sum_{l=1}^{N_m} 2\pi l_\tau_l P_f N_l
\]  

(2.5.4)

where \( N_m \) is the number of current intact fibers instead of \( N_r \), the number of initial intact fibers. The force supported by intact fibers, \( F_i \), is

\[
F_i = N_i (1 - P_f N_m) \sigma_a
\]  

(2.5.5)

The composite can no longer support the applied load when

\[
\frac{\partial (F_b + F_i)}{\partial \sigma_a} = 0
\]  

(2.5.6)

### 2.6 Other Factors Affecting the Composite Strength

So far we have been focusing on the "intrinsic" strength of the fibers and their influence on the strength of the composite. There are some other factors which may have substantial influence on the composite strength.

When fiber/matrix interfacial debonding has occurred, interfacial sliding between the two constituents may provoke damage to the fiber surface, thus lowering its strength. The phenomenon of crack formation during scratching of brittle materials was investigated by Veldkamp et al [90]. Miyoshi et al [88,89] also investigated fracture of single-crystal
silicon carbide during sliding between each other. Lawn et al [86,87] looked into the problem of strength degradation of brittle materials subjected to surface damage by indentation. It is shown from their study that, depending on the indenter load, the strength of the specimen degraded significantly from those without surface damage. Lee [10] investigated fracture behavior of cross-ply Nicalon fiber reinforced LAS composites. From his fractography studies, there is evidence that mechanical interaction did happen between the fiber and the matrix (wear). The aforementioned studies suggest that the details of the sliding mechanism between the fiber and the matrix may not be ignored.

Another important factor needs to be considered is the effect of stress concentration around a broken fiber on the probability of failure of its adjacent fibers, especially in systems with high interfacial shear strength. This subject has been investigated by many researchers for polymeric composites. For CMC's, however, the stress concentration effect of a broken fiber which has already debonded from the matrix has not been well analyzed. Since the complexity of the problem renders an interesting dissertation topic by itself, we do not include analysis of the effect of stress concentration here.

2.7 Results

Typical results from the strength model will be illustrated in this section. The predictions by the strength model are in very good agreement with experimental data. A few examples are shown in Fig. 5, using a Nicalon reinforced LAS-II system. Experimental data are adapted from Ref. [64]. All measured data fall in between the lower and upper bound of the prediction. The measured data are closer to the upper bound of theoretical prediction, this is partly because the model only admits integer values of Weibull shape.
modulus, \( m \). For case 1 and 2, \( m \) value used in the model (4) is slightly higher than the data (3.8, 3.9), which results in slightly lower predicted values. In case 3, where the \( m \) value used in the model (3) is slightly lower than the data (3.1), the measured strength falls in the middle of the lower and upper bound.

**Effect of Weibull Modulus**

The influence of the Weibull modulus, \( m \), of the fibers on the strength of the composite is shown in Figs. 6 and 7. The input parameters used are the same as the third example in Fig. 5 but the \( m \) value. In Fig. 6, the normalized force supported by the broken and intact fibers on a matrix crack plane is plotted against the (artificial) remote applied stress. Failure of the composite is defined at the point when the total force supported by the fibers begins to drop. As shown in Fig. 6, the strength of the composite decreases with an increase of \( m \). As a lower \( m \) value implies that the strength of the fibers is scattered (or less uniform) with more high strength fibers (and also low strength fibers) than those with higher \( m \) values, it results in a higher composite strength. The same trend of predictions was also shown by Curtin [67]. A plot of applied stress versus cumulative fiber failure ratio is shown in Fig. 7. With a larger \( m \) value, fiber failure ratio increases more rapidly (as they are more uniform in strength, implying that a considerable amount of fibers may break within a relatively narrow load increment) than those with lower \( m \) values.
<table>
<thead>
<tr>
<th></th>
<th>case 1</th>
<th>case 2</th>
<th>case 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fiber Volume Fraction</td>
<td>0.46</td>
<td>0.44</td>
<td>0.44</td>
</tr>
<tr>
<td>Fiber Strength</td>
<td>1,580</td>
<td>1,470</td>
<td>1,450</td>
</tr>
<tr>
<td>m (data)</td>
<td>3.8</td>
<td>3.9</td>
<td>3.1</td>
</tr>
<tr>
<td>m (used in model)</td>
<td>4</td>
<td>4</td>
<td>3</td>
</tr>
<tr>
<td>Fiber Radius *</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>Interfacial Shear Stress</td>
<td>2.5</td>
<td>2.5</td>
<td>2.5</td>
</tr>
<tr>
<td>Experiment</td>
<td>753</td>
<td>670</td>
<td>680</td>
</tr>
<tr>
<td>Theory (lower bound)</td>
<td>709</td>
<td>633</td>
<td>658</td>
</tr>
<tr>
<td>Theory (upper bound)</td>
<td>754</td>
<td>671</td>
<td>717</td>
</tr>
</tbody>
</table>

All strength values are in MPa
* unit in micro-meter

Material: Nicalon/LAS-II
Fiber Modulus: 193 MPa
Matrix Modulus: 83 MPa
Fiber Gage Length: 25.4 mm

Figure 5. Results of strength predictions from the model.
Figure 6. Normalized fiber force versus applied stress for several m values.
Figure 7. Applied stress versus fiber failure ratio for several m values.
Effect of Interfacial Shear Stress

The interfacial shear stress between the fiber and the matrix plays a very important role in the strength and toughness of the composite. As the interfacial shear stress varies, the spacing of the parallel matrix cracks may also change, depending on the magnitude of the shear stress. Kimber and Keer [13] have shown that the average crack spacing, s, is 1.337x, where x is the length over which the additional load sustained by the fibers at the crack is transferred back to matrix.

\[ x = \frac{a \sigma_{mc}}{2 \tau_l} \]  

(2.7.1)

where \( \sigma_{mc} \) is the matrix cracking stress, determined experimentally. We adapt this relationship for estimating the matrix crack spacing with a given interfacial shear stress value. A plot of normalized fiber force at the crack plane versus applied stress is shown in Fig. 8, for three different interfacial stress values. The accompanied fiber failure ratio versus applied stress is shown in Fig. 9. It is shown in these figures that higher the interfacial stress, the higher the composite strength.

However, high interfacial shear stress may not be a favorable factor on the toughness of the composite. A higher interfacial shear stress results in a lower ineffective length. And, as a result, we have lower pull-out length which implies a less tough composite than those with lower interfacial shear stress. This is illustrated in Fig. 10, where the normalized pull-out work is plotted against applied stress for three different interfacial shear stress values. The pull-out work is monitored within each load increment and fracture length.
Figure 8. Normalized fiber force versus applied stress for different interfacial shear stress values.
Figure 9. Applied stress versus fiber failure ratio for different interfacial shear stress values.
\[ W = \sum_{\Delta \sigma} \sum_{i=1}^{N_m} \pi a \tau_i l_i^2 P_f \] (2.7.2)

A significant difference in pull out work is shown. With a low interfacial shear stress (1.0 MPa) the pull out work can be twice as much as for the case where \( \tau_i = 2.5 \) MPa and about four times as much for \( \tau_i = 5 \) MPa.
Figure 19. Normalized work of pullout versus applied stress for different interfacial shear stress values.
Modeling of Fatigue Behavior of Unidirectional Ceramic Matrix Composites

In this chapter, the fatigue behavior of unidirectional CMC under tensile loading is studied through micro-mechanical modeling. With the current model, the influence of material parameters (such as the Weibull shape parameter, \( m \), of the fibers, and the interfacial shear stress) on the long-term performance of unidirectional CMCs can be studied.

The fatigue model is developed based on the tensile strength model discussed in the last chapter. The geometry of the representative volume is the same as Fig. 2 in Chapter 2. Under uniaxial cyclic loading, the damage events that occur in the composite include transverse matrix cracking, fiber/matrix interfacial debonding, and fiber fracture. These damage events are very similar to those in quasi-static tensile loading [74-80]. It is assumed in the present model that multiple matrix cracking and saturation occurs after a few cycles. Fiber fractures that occur prior to the saturation of multiple matrix cracking is ignored. Fiber/matrix interfacial debonding occurs after matrix cracking has taken
place, and the debonding length increases with an increase of applied cycles. In a recent paper, Rousseau et al [79] studied the fatigue behavior of Nicalon fiber reinforced CAS glass ceramic composites using high-resolution micromechanical test methods. They found that interface debonding initiated adjacent to matrix cracks and increased with the number of loading cycles. For a load amplitude of 275 MPa, a debond length of 40 μm had been measured in the first cycle; by the tenth cycle the debond length had increased to 50 μm. In the present model, the debonding length is determined by an energy approach, in which the change in debonding length with applied cycles is related to the change in strain energy release rate in the vicinity of a fiber, a relationship analogous to the form of the well-known Paris law. The stress analysis part of the fatigue model is similar to that already described in the last chapter. The ratio of fiber fracture is determined at any arbitrary applied cycle for each of the three regions of the representative volume, namely, the matrix crack plane, the debonded, and the bonded region.

By quantifying the accumulated fiber fractures during the fatigue process, the life of the material can be estimated when a fracture criterion is satisfied. More will be said on this later. In addition, the remaining tensile strength of the material after cyclic loading for a certain number of cycles can also be estimated. The fraction of fiber fracture at the end of the cyclic loading is used as an input parameter to the tensile strength model to estimate the remaining strength can be estimated.

To model the fatigue process of a unidirectional CMC, two important parameters are subjected to variation with applied cycles. The residual stress between the fiber/matrix interface in the debonded region (which arises as a result of thermal mismatch between the two) is assumed to decrease with applied cycles only, due to frictional wear. A direct consequence of this is a decrease in interfacial shear stress. The effect of interfacial shear
stress on strength has been discussed in the last Chapter. Here we assume that the change of interfacial shear stress with applied cycles is

\[
\frac{\tau_i^\prime}{\tau_i^0} = 1 - p_1 (\log n)^{p_2}
\]  

(3.1)

where

\(n\) = applied cycles

\(\tau_i^0\) = initial interfacial shear stress

\(\tau_i^\prime\) = interfacial shear stress at \(n\)

\(p_i (i = 1, 2)\) = constants

The effect of decreasing \(\tau_i\) is that less stress is transferred back to the matrix material of the debonded region from the fibers in the crack plane. As a result, the entire (intact) fiber is under higher stress as applied cycles increases. Moreover, it causes a longer ineffective length on both sides of a broken fiber.

The characteristic strength of the fibers is also assumed to decrease with an increase of applied cycles, as a result of possible flaw development or surface abrasion in the fibers during sliding with the matrix. In addition, it may also be a function of applied load. We propose the following relationship for the characteristic fiber strength and applied cycles.

\[
\frac{\sigma_i^\prime}{\sigma_i^0} = A \exp \left( - \frac{\sigma_i}{\sigma_i^0} \right) \left[ 1 - p_3 (\log N)^{p_4} \right]
\]  

(3.2)

where
\( \sigma_a = \) applied stress  
\( \sigma_0 = \) initial characteristic strength  
\( \sigma^n_0 = \) characteristic fiber strength at \( N \) cycles  
\( A, p_3, p_4 = \) constants

The exponential term takes care of the effect of stress level: the higher the applied stress, the lower the characteristic strength after a certain applied cycles. The logarithmic variation with applied cycles, \( N \), is an inheritance of the conventional form of S-N curve for unidirectional composites. For some CMC systems, a fatigue limit (below which no noticeable damage occurs to the material) is observed. This suggests that Eq. (3.2) may only apply to cases in which the applied cyclic load exceeds the fatigue limit. The consequence of the decreasing characteristic fiber strength is obvious, a fiber is more likely to fail as it is being cyclically loaded. Evidence of fiber damage during cyclic load is reported by Lee [10]. It is realized that Eqs. (3.1) and (3.2) are in empirical forms, The constants appeared in the two equations are unknowns that need to be characterized for each material system under various operating conditions.

**Debond Length**

Debonding between the fiber/matrix interface occurs during/after multiple matrix cracking. The stress state of the fiber in the debonded and the bonded region will influence the calculation of the probability of fiber fractures. Therefore, the debonding length needs to be known, as in the case of the tensile strength model. The problem of crack propagation between two dissimilar materials bonded together has been studied by a number of investigators [93-95]. However, the solution methods are usually com-
plicated, not practical for use in the present model. To circumvent the problem, it is proposed that the change in debonding length, $l_d$, with that of the applied cycle is related to change of strain energy in the vicinity of a fiber, in a form analogous to the well-known Paris law of fatigue crack propagation in isotropic materials.

$$\frac{\partial l_d}{\partial n} = \beta \left( \frac{1}{2\pi a} \frac{\partial U}{\partial l_d} \right)^\alpha$$  \hspace{1cm} (3.3)

or

$$n = \frac{(2\pi a)^\alpha}{\beta} \int_0^L \left( \frac{\partial U}{\partial l_d} \right)^{-\alpha} dl_d$$  \hspace{1cm} (3.4)

where

$\beta, \alpha = \text{constants}$

$a = \text{fiber radius}$

$U = \text{total strain energy around a fiber}$

$n = \text{number of applied cycles}$

The debond length at any arbitrary applied cycle can be evaluated by numerically integrating Eq. (3.4).

The total strain energy, $U$, around a fiber is the sum of strain energy terms from the normal fiber stress, normal matrix stress and matrix shear stress

$$U = U_1 + U_2 + U_3$$  \hspace{1cm} (3.5)

where
\[ U_1 = \int_{s-l}^{s} \int_{r_0}^{a} \frac{(\sigma_f^2)^2}{E_f} \pi r dr \] (3.6a)

\[ U_2 = \int_{s-l}^{s} \int_{a}^{b} \frac{(\sigma_m^2)^2}{E_m} \pi r dr \] (3.6b)

\[ U_3 = \int_{s-l}^{s} \int_{a}^{b} \frac{2(1 + \nu_m)(\tau_m^2)}{E_m} \pi r dr \] (3.6c)

where

\[ \sigma_m^2 = V_r \omega (\overline{\sigma} - \sigma_a)(e^{A_0(z-s)} - 1) \]

\[ \tau_m^2 = \frac{V_r \omega A_0 (b^2 - r^2)}{2r} (\overline{\sigma} - \sigma_a)e^{A_0(z-s)} \]

\( \sigma_f, \sigma_m, \) and \( \tau_m \) are normal fiber stress, normal matrix stress, and matrix shear stress, respectively. Functional form of \( \sigma_f \) is given by Eq. (2.2.8). Here only terms in the debonded regions are included in calculations since contributions from similar terms in the bonded regions to the value of \( l_d \) is insignificant (several orders of magnitude smaller).

**Fracture Criterion**

The fracture criterion adapted in the case of quasi-static loading may not be applicable in case of cyclic loading. Numerically, it is not feasible to use the same procedure as it is very time consuming to calculate fiber fracture ratio for each small load increment.
within each increment of applied cycles. Also, in contrast to the case of monotonic loading, the total force supported by the composite is constant until some very last cycles during cyclic loading under constant load amplitude. In the present case, we determine the probability of fiber fracture for each increment of applied cycles, treating the applied load amplitude as one load increment. We propose that fatigue failure is defined when the force supported by broken fibers is greater than the force supported by intact fibers on a matrix crack plane. When this condition is satisfied, fibers are gradually pulled out to cause separation of the specimen. This criterion is rigorous as the probability of fiber fracture at composite failure is always higher than those calculated in the static case.

The computational procedures are shown in Fig. 11. During each increment of applied cycles, the ratio of fiber fracture, change in interfacial shear stress and characteristic fiber strength is determined. Calculation of fiber fracture has been described in detail in the last chapter. Change of interfacial shear stress and characteristic fiber strength are described by Eqs. (3.1) and (3.2). With fiber fractures occur at each interval of applied cycle, stress in the intact fibers changes. Here we still assume that all the intact fibers share the same stress. Broken fibers support a "constant" amount of load once they are broken, the magnitude depends on the interfacial shear stress and the distance of fracture location from a reference matrix crack plane. As the interfacial shear stress is decreasing with applied cycles, the force supported by a broken fiber also decreases. But, as the total number of broken fibers increases, the total force supported by the broken fibers increases. The composite life is defined when the fracture criterion is satisfied.
Figure 11. Schematic of the computational procedures for the fatigue model.
Results

Several examples will be illustrated in this section. Because of lack of supporting data, we only consider cases with relatively weak interfacial strength in which complete debonding occurs within a few applied cycles. To this end, we use $\alpha = 0.1$ and $\beta = 100$, which causes complete debonding of the fiber/matrix interface within a few cycles.

An example from the fatigue model is shown in Fig. 12, which is a S-N curve for unidirectional CMC. Experimental data are adapted from Ref. [76]. The material is unidirectional Nicalon/CAS. The two constants for Eq. (3.1) are:

\begin{align*}
  p_1 &= 0.1 \\
  p_2 &= 1.0
\end{align*}

which is a linear degradation curve for $\tau_c$ with log of applied cycles. At one million cycles, the interfacial shear stress decreased to 40% of its initial value. The three constants used for Eq. (3.2) are:

\begin{align*}
  A &= 2.0 \\
  p_3 &= 1.0 \\
  p_4 &= 0.2
\end{align*}

The constant $A$ is to offset the exponential form of the effect of applied stress level on characteristic strength. For instance, if $\sigma_d/\sigma_c = 0.8$, the fiber characteristic strength is 90% of its initial value before fatigue.
Figure 12. Prediction of S-N curve for unidirectional Nicalon/CAS.
A second example, shown in Fig. 13, is for unidirectional Nicalon/1723, data is adapted from Ref. [74]. In this case, we assume that there is no change in interfacial shear stress with applied cycles. The three constants used for Eq. (3.2) are:

\[ A = 2.0 \]
\[ p_3 = 0.02 \]
\[ p_4 = 1.0 \]

Some details of the computational results for the second example are shown in Fig. 14. Here normalized intact fiber ratio, broken fiber force, and intact fiber force are plotted against applied cycles. As mentioned earlier, failure is defined when force supported by broken fibers exceeds that of intact fibers.

The effect of Weibull shape modulus, \( m \), on the long-term behavior is illustrated in Fig. 15. All material properties used in generating the curves are the same as Fig. 13. The applied load amplitude is 60% uts. As can be see from the plot, the result of increasing \( m \) is a decrease in life. More importantly, the ratio of intact fibers drops more abruptly in case of higher \( m \) values. The reason behind this result is similar to the quasi-static case. For fibers with higher \( m \) values, they are more uniform in strength. Fewer fibers fail early in the load history than fibers with lower \( m \) values. As the intact fiber stress increases as a result of stress redistribution and increasing applied cycles, a considerable amount of fibers fail within a relative short period (because of their uniform strength), which causes a sudden drop in intact fiber ratio (Fig. 15).
Figure 13. Prediction of S-N curve for unidirectional Nicalon/1723.
Figure 14. Normalized fiber forces and fracture ratio versus cycles for Nicalon/1723.
Figure 15. Ratio of intact fibers versus cycles for different m values during fatigue.
Experimental: Performance of CMC Components

In this chapter, experimental work on the mechanical behavior of CMC tubes is presented. To support the goal of developing advanced CMC for structural applications (such as heat exchangers, regenerators, turbines, and heat engines, to name only a few applications), careful characterization of material properties as well as accessing the general performance of the actual structure are inevitably important. Testing of tubular specimens seems to satisfy these two demands: first, all the ply-level elastic properties can be determined from a single specimen by applying to it axial, torsional, as well as mutiaxial loads and measuring the corresponding structural response, which is more efficient than coupon testing where several specimens are needed to determined those properties [85]; second, some engineering components (for example, heat exchangers) are in fact, tubular in shape. As a result, tube testing provide a direct way of accessing the structural performance. However, standard test methods for characterizing CMC tubes have not yet been well established at present, especially at elevated temperatures, and similar data simply does not exist in the literature. Therefore, test method development and base-line data generation for guiding future testing directions are of primary concern, which is the subject we will discuss in this chapter.
We will first discuss test method development for testing CMC tubular specimens at room and elevated temperatures. Base-line data are obtained from uniaxial and torsional quasi-static tests as well as uniaxial and multiaxial fatigue tests, at both ambient and elevated temperatures. Results will be presented and discussed. Some data interpretation scheme concerning the torsional and axial strength of the tube will also be discussed.

Test Facilities

A biaxial testing frame, developed under a cooperative effort between the Materials Response Group at Virginia Tech and the Instron Corporation, was used to perform tests on the specimens. The test frame is a biaxial, two post system rated for 4.448E5 N (100 kips) axial and 5.65E3 Nm (50 kip-in) torsional load, as shown in Fig. 16. A high-stiffness frame to reduce the amount of twist in the crosshead under rated loads is a basic requirement for testing CMC materials. The frame stiffness is rated at 8.76E9 N/m (5.0E6 lb/in) axial and 4.29E5 Nm/degree (3.8E6 in-lb/degree) torsional with a 101.6 cm (40 in) separation between the crosshead and base. The load cell and actuators are rated for use to 2.224E5 N (50 kips) axial and 2.824E3 Nm (25 kip-in) torsional. An Instron 8500 controller was used to operate the test frame.

Worth mentioning here is the simple gripping system of the test frame, shown schematically in Fig. 17. Each grip contains a collet assembly designed to provide a cylindrical gripping surface 3.81 cm (1.5 in) in diameter and 6.35 cm (2.5 in) in length. When activated, the piston moves upward, forcing the collet into the head assembly and thereby causing the collet to grip the specimen. The grips are equipped with precision alignment
Figure 16. Instron hydraulic multiaxial test frame.
Figure 17. Schematic of the gripping system.
capabilities to prevent bending of the specimen. Water cooling system was built into the grips for operation at elevated temperatures. One of the advantages of the gripping system is that its mechanism is relatively simple compared to other gripping systems for tubes. As a result, it can be used to test straight-sided tubes, which are less expensive and much easier to fabricate than tapered tubes [84].

An Instron Two-Zone Short Furnace along with an Eurotherm 818 controller were used for heating. The furnace heating chamber is about 10.2 cm (4 in) long, housing the section of the specimen between two grips. An Instron high temperature capacitive extensometer was used for measuring axial displacement at elevated temperatures. An Hewlett Packard 3852 control unit along with a data acquisition program developed at the Materials Response Group was used for data acquisition.

Test Specimens

The tubular specimens, supplied by Babcock & Wilcox Company, were fabricated by winding of continuous ceramic fibers into preform, in a ± 45 degree arrangement. The matrix material is introduced into the preform by a sol-gel infiltration technique. The preform is then dried and fired after each infiltration. Typically fifteen such infiltration cycles are needed to reduce the matrix porosity to about 26%. The dimensions for the resulting tubes are approximately 3.8 cm for outside diameter, 0.25 cm in wall thickness, and 20 cm in length. These dimensions vary slightly within the length of the specimen and from one specimen to another.

Three batches (batch I, II and III) of specimens consisting a total of 23 tubes were provided for testing. The purpose of batch I specimens (5 tubes) was for test method
development; results from trail tests on these specimens will not be discussed. Batch II specimens (14 tubes) were fabricated using Almax fiber (an alumina based fiber) and alumina matrix while batch III specimens (4 tubes) were fabricated with a relatively larger winding pattern, using DuPont PRD-166 fibers (an alumina-zirconia fiber) and a Yttria stabilized zirconia matrix. The winding patterns for these two types of specimens (II, III) are illustrated in Fig. 18. The porosity for these specimens was about 26%.

Specimen Preparation

Specimen preparation is perhaps the most challenging task in the entire process of characterization. As a result of the fabrication process, the as-received tubes are usually not exactly circular in shape. Moreover, in many cases, the size of the two ends of a single tube were not the same. These geometrical irregularities imposed considerable difficulty on gripping of the specimens, given the small radial tolerance of the collets (0.01 cm). The grip region of a specimen either had to be trimmed down or an extra layer had to be built up depending on the dimensions of the outside diameter.

In addition to the modification of specimen surfaces in the grip region, extra radial support was needed in that region to prevent the tube from collapsing in that region. This can be accomplished by introducing end-plugs into both ends of the specimen and bond them to the inside wall of the specimen. For room temperature tests, epoxy was used for bonding. The procedures for bonding end plugs were illustrated in Fig. 19. A detailed description for the bonding procedures can be found in Ref. 84. For specimens that were trimmed down in the grip region, longer end-plug were used and bonded to the specimen to guarantee failure outside the grip region. This method worked well for
Figure 18. Schematic of the specimen architecture.
room temperature stiffness and strength tests. However, caution must be exercised such that epoxy in fill covering the end-plug must not in excessive amount as there is a possibility of crack development between the interface of the two which may act as stress raiser to cause "immature" failure around the transition region. Two such cases occurred during tensile testing.

To measure structural response during loading, strain gages were mounted on selected specimens. Prior to installation, a thin layer of epoxy was evenly spread over the gage section of the tube, providing a smooth surface. Both rosettes and single element gages were installed on the specimens. Only single element gages were installed on flat coupons. In some case, four rosettes, and four single element gages (i.e., a total of 16 single gages) were installed on a single tube. A typical gage arrangement is shown in Fig. 20.

Test Procedures at Ambient Temperature

Quasi-static Test. For each tubular specimen intended for the purpose of determining material properties, tension, compression, and torsion were applied separately while the response of the strain gages was recorded. In some cases, a multi-axial load was also applied (i.e., tension-torsion, compression-torsion). Usually, three measurements were made for each type of loading. The maximum applied tensile load was about 1780 N (400 lb), or approximately 6.895 MPa (1.0 ksi), which was under 20% of the tensile strength, and was well within the linear response region. The loading rate was 111.2 N/sec (25 lb/sec).

Cyclic Test. For cyclic tests (tension-tension, torsion-torsion, and multi-axial), the axial stiffness of the tubes was measured at selected intervals using the Instron high-
Figure 19. Schematic of the procedures for bonding end-plugs.
Figure 20. Strain gage arrangement in a tubular specimen.
temperature extensometer. More will be said about this later. For specimens that survived at one million cycles, monotonic tensile or torsional tests (none of the tubes tested under multi-axial fatigue survived to a million cycles) were performed to determine the residual strengths.

**Test Procedures at Elevated Temperatures**

Prior to any testing at elevated temperatures, the performance of the furnace must be known. For the Instron Two-Zone Short Furnace used, it usually takes more than an hour for the temperature to stabilize between any temperature increment over 200°C. A typical temperature-time curve of a specimen in the furnace is shown in Fig. 21. The temperature and temperature gradient along the gage section of the specimen also need to be known, since the feedback temperature shown on the furnace controller does not usually represent the temperature of the specimen. This information was obtained by attaching several thermocouples to the specimen. In our case, five high-temperature thermocouples 2.54 cm (1.0 in) apart from each other were attached to the specimen over a span of 10.2 cm (4 in), using a holder located outside the furnace. Good contact between the thermocouples and the specimen was assured by filling the small gap (if any) between them with ceramic cement. A typical temperature distribution along the specimen length is shown in Fig. 22. The result indicates that specimen temperature is fairly uniform in the mid section within a length of 5.1 cm (2.0 in).

**Quasi-Static Test.** Prior to any mechanical testing at elevated temperature after the specimen was installed in the test frame, two thermal couples were inserted through the two small holes 2.54 cm (1.0 in) apart from each other located in the front ceramic in-
Figure 21. A typical specimen temperature in the furnace versus time.
Figure 22. Temperature gradient along the length of the specimen.
sulation block of the furnace (which is used to accommodate the two arms of the high-temperature extensometer) contacting the mid section of the specimen. The thermocouples were withdraw from the furnace after the specimen temperature was stabilized as monitored in real time, and the temperature of the specimen become uniform within the 2.54 cm (1.0 in.) long section. The latter can be done by adjusting the temperature setting of the two heating zones. The two arms of the high temperature extensometer were then inserted into the furnace. Again, a waiting period was needed for extensometer output signal to become stabilized. In addition, maintaining a constant temperature environment around the extensometer assembly outside the furnace was very important in minimizing signal drifting. As the extensometer works on capacity change of the capacitor, temperature fluctuation around the assembly will cause such a change. Covering the entire extensometer assembly with thermal shielding using cardboard wrapped with aluminum foil is a simple and effective way to prevent such drifting.

**Cyclic Test.** For cyclic tests at elevated temperatures, the axial stiffness of the specimen was measured at both room and the testing temperatures at 0 cycles, and at other selected cyclic intervals (usually at 10E2, 10E3 cycles, and so on) at the testing temperature. Procedures for stiffness measurement were the same as for quasi-static testing.

A total of 12 batch II and 2 batch III specimens were tested. The test matrix is shown in Fig. 23.

**Results**

The results from the mechanical testing are presented and discussed in the following sections.
<table>
<thead>
<tr>
<th>specimen ID</th>
<th>test type</th>
<th>test temperature</th>
<th>axial load (% uts)</th>
<th>torsional load (% uts)</th>
<th>strength (ksi)</th>
<th>cycles to fail</th>
<th>failure location</th>
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<tr>
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<td>-</td>
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<td>-</td>
<td>gage</td>
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<td>-</td>
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<td>gage</td>
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<td>run out</td>
<td>gage</td>
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<td>grip</td>
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<td>95</td>
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* based on residual strength of 72A.
△ survived at one million cycles.

Figure 23. Test matrix for tube specimens.
Mechanical Response at Ambient and Elevated Temperatures

Typical structural (global) response of the CMC tubes under uniaxial (tension and compression), torsional, as well as multiaxial applied loads are presented as follows.

Uniaxial Response at Ambient Temperature. A typical response of a batch II specimen (68A) under quasi-static tensile load, measured by a rosette, is shown in Fig. 24. The axial stiffness for this specimen is about 54 GPa (6.62 Msi). The response of specimen 66A is shown in Fig. 25, where its axial stiffness is about 63.9 GPa (9.27 Msi). The reason for the 15% stiffness difference within the same batch is unclear at this point.

The structural response of batch III specimens were also measured. A typical stress-strain curve of a specimen (28A) is shown in Fig. 26. For this specimen, the measured axial stiffness (27.7 GPa) is only about 50% of the batch II specimens (based on the stiffness of 68A). Although unclear at this point, the reason for this difference could be attributed to both material and tube architecture (a larger winding pattern for the type III specimens).

Uniaxial Response at Elevated Temperatures. Selected specimens from batches II and III were tested under tensile load to obtain their axial stiffness at elevated temperatures using an Instron High-Temperature extensometer. The stress-strain behavior of a batch II specimen (65A) is shown in Fig. 27. No significant stiffness change was observed up to 650°. At 1100°C, there was a 33% reduction in stiffness compare to the room temperature measurement. The stress-strain behavior of a batch III specimen (25B) is shown in Fig. 28. This specimen behaved quite differently from specimen 65A. At 430° C, there was already a 37% reduction in stiffness. At 540°C and 650°C, the re-
Figure 24. Response of specimen 68A under tensile load.
Figure 25. Response of specimen 66A under tensile load.
Figure 26. Response of specimen 28A under tensile load.
duction in stiffness are 67 and 69%, respectively. It is also interesting to notice that the specimen displayed a nonlinear behavior at elevated temperatures. The axial stiffness increases with an increase of applied load at 540°C and 650°C, which is not observed in specimen 65A (batch II). One plausible explanation may be that at higher temperatures, the fibers are aligning themselves with the axial load (as a result of the larger winding pattern), thus increasing the axial stiffness.

**Torsional Response at Ambient Temperature.** The shear modulus of the tubes is determined by applying torsional load and measuring the response of the strain gages installed at 45 degrees to the longitudinal axis of the tube. Typical responses of a batch II and III specimen are shown in Figs. 29 and 30, respectively. The shear modulus of a batch II specimen (68A) is about 47.1 GPa (6.83 Msi), while that of a batch III specimen (28A) is about 34% less, at 30.7 GPa (4.45 Msi).

**Multiaxial Response at Ambient Temperature.** Typical multiaxial response of specimen 66A is shown in Figs. 31-32. Strain gage response with axial (tensile) and torsional load were shown in Figs. 31a-b, respectively. Strain gage response with axial (compressive) and torsional load were shown in Figs. 32a-b, respectively. As mentioned earlier, these responses can be used to determine ply-level material properties, using an "inverted" elasticity solution [85].

**Strength at Ambient and Elevated Temperatures**

**Tensile Failure Mode at Ambient Temperatures.** A stress-strain curve for specimen 68A loaded to failure in tension is shown in Fig. 33. The tensile strength of the tube in the axial direction is about 44.5 MPa (6.45 ksi), not taking into account the 25% porosity.
Figure 27. Response of specimen 65A at elevated temperatures.
Figure 28. Response of specimen 28B at elevated temperatures.
Figure 29. Response of specimen 68A under torsional load.
Figure 30. Response of specimen 28A under torsional load.
Figure 31. Response of specimen 66A under multiaxial load.
Figure 32. Response of specimen 66A under multiaxial load.
The failure strain is about 0.13%. The curve is linear up to about 27.6 MPa (4.0 ksi). Acoustic emission (AE) data taken along with the tensile test indicates that no significant damage event occurs until about 31 MPa (4.5 ksi), whereas the stress-strain curve already shows some nonlinearity. The discrepancy may arise as a result of poor surface contact between the AE sensor and the specimen (as the specimen surface is quite bumpy at the level of fiber tow, in addition to a curved surface).

In general, the tensile failure surfaces are perpendicular to the loading axis, and localized near the cross-over regions between the +45 and the -45 fiber tows. It is believed that bending of the fibers at those regions (as a result of crossing each other), stress concentrations arises from the geometrical arrangement of fiber tows, as well as porosity and voids all contribute to the localized failure.

Within the failure surface, two distinct features were observed. Usually, the surface ply exhibits some "clean" major failure surfaces, right on the cross-overs. On these clean surfaces, fibers within a tow failed uniformly on one plane; the normal of the plane is about 45 degrees from the loading axis, an strong indication of shear-type failure, very possibly as a result of rapid crack propagation. This is shown in Figs 34a-b. This particular feature is also seen in specimen 71B, loaded cyclically in tension-tension at 50% uts (the residual strength of which is about the same as 68A and 66A). As the strength of the tube is weaken by the propagating cracks along the cross-overs of the surface ply(ies), the global failure surface is localized. The separation of the specimen at the final stage results in some pullout-like feature in inner plies, distinct from the cleaner fracture surface of the surface ply(ies) described earlier. An SEM image is shown in Fig. 35. A combination of these two types of features in the inner plies were also seen. Other damage modes include splitting between the fiber tows and delamination. The two dis-
Figure 33. Stress-strain curve of specimen 68A loaded to failure under tension.
Distinct fracture modes may be a result of nonuniform penetration of matrix material during processing, with the matrix of the outer plies being denser which enhances crack propagation, while the relatively poor matrix adherence to the fibers of inner plies causes energy to dissipate through splitting of fibers and enhances pull-out type failure.

A stress-strain curve for a batch III specimen (28A) loaded to failure is shown in Fig. 36. The general pattern of the curve is similar to those of the batch II specimens. However, the tensile strength in the axial direction (15.2 MPa) is only about 40% of the batch II measurements. Since the material and architecture for these two types of tubes are both different, it is not possible to interpret which of the two factors contributes the reduction in tensile strength. The failure surface of these specimen 28A only exhibit the first feature that previously described, that is, clean failure surface, resulting in a more brittle-like failure surface. This may be a result of the larger winding pattern which allows a better matrix penetration and adherence.

**Tensile Failure Mode at Elevated Temperature.** Specimen 65A was loaded to failure at 1100°C under tension. The failed specimen is shown in Fig. 37, along with specimen 66B, failed at room temperature. The failure load was 3514 N (790 lb) or approximately 13.37 MPa. This is only about 31% of the tensile strength at ambient temperature. Difference in the failure surfaces of the two specimens is readily seen. The high temperature fracture shown a more brittle-like surface than the room temperature fracture. The fracture surface only exhibits the clean, shear-type failure similar to specimen 28B. An SEM image of the failure surface is shown in Fig. 38. One possible explanation for the brittle-like surface compared to the "less brittle-like" room temperature failure is that the adherence of the matrix to the fibers may be improved at high temperature, thus promoting rapid crack propagation.
Figure 34. SEM of tensile failure surface showing "clean" fiber bundle failure.
Figure 35. SEM of tensile failure surface showing fiber pullout.
Figure 36. Stress-strain curve of specimen 28A loaded to failure under tension.
Torsional Strength at Room Temperature. Specimen 68B was loaded to failure under pure torsion. The failed specimen is shown in Fig. 39. The torsional strength, calculated from a strength of materials approach, is about 56.54 MPa (8.2 ksi). A well defined 45-degree failure surface can be seen from the figure. Also, some fiber "pull-out" was seen in plies failed under tension. Specimen 72B was cyclically tested in torsion at 50% (quasi-static) torsional strength; its residual strength showed no significant reduction after surviving one million cycles. The hydraulic test frame was stopped just in time during residual strength testing preventing specimen separation. Two large and sharp surface cracks, running at +45 and -45 to the loading axis, were "frozen" on the specimen, as shown schematically in Fig. 40. The failure surface of 72B and 68B together provided valuable information regarding the quasi-static torsional failure mechanism of the tube.

Knowing the direction of the applied torque, crack 1 traverses two triangular patterns where fibers are under tension. There is no fiber pull-out type feature observed along the crack, a feature similar to the clean surface ply failure of tensile specimens, which is an indication of rapid crack propagation. This long surface crack may be originated from the cross-over point (indicated by the dotted circle in Fig. 40). Similar to those of the tensile specimens, as the strength of the tube is reduced by the propagating cracks, final separation results in some pull out type feature, as observed in specimen 68B. Although the torsional strength of the tube is thought to be controlled by the +45 plies, cracks did develop and propagate along the cross-over points, as seen from Fig. 39. But they do not contribute primarily to failure.
Figure 37. A comparison of tensile failure surfaces.
Figure 38. SEM of fracture surface at failed at 1100 C under tension.
Figure 39. Specimen 68B after failed under torsion.
Figure 40. Surface damage features of specimen 72B failed under torsion.
Fatigue Behavior at Ambient Temperature

Fatigue tests were performed only on batch II specimens. A total of 5 specimens were cyclically tested at room temperature: two under uniaxial, one under torsional, and two under multiaxial loading.

Specimen 71B was cyclically tested (at 5 Hz) under uniaxial tension at a maximum load of 50% of the average ultimate quasi-static tensile strength of the tubes 66A and 68A (42.75 MPa (6.2 ksi)), at $R = 0.1$, where $R$ is the ratio of minimum load to the maximum load. It survived after one million cycles. During cyclic testing, the axial stiffness of the tube was measured at $0$, $10^E2$, $10^E3$, $10^E4$, $10^E5$, $10^E5.45$, $10^E5.66$, and $10^E6$ cycles. Usually, three stiffness measurements were made at each interval. A plot of the axial stiffness for 71B versus log (cycles) is shown in Fig. 41. The initial stiffness (at 0 cycles) is rather peculiar in that the value is lower than measurements taken afterwards. At this point, there has not been an explanation for this peculiarity. Error in instrumentation is a possible cause. In general, the reduction in axial stiffness was not significant until the end of one million cycles, which suggest that no serious fatigue damage was introduced to the tube. The residual strength of the specimen was 6.05 ksi, which is slightly lower than the average of 66A and 68A, but higher than that of specimen 66A. The failure mode of this specimen is very similar to those of 66A and 68A.

Specimen 73A was cyclically tested under tension-tension at a maximum load of 80% of the average quasi-static ultimate axial strength. A plot of axial stiffness versus log (cycles) is shown in Fig. 42. In this case, there is about 20% stiffness reduction at the end of $10^E3$ cycles. The specimen failed at 1763 cycles. A picture of the failed specimen is shown in Fig. 43. A variety of damage/failure modes were seen. They include: cracks
Figure 41. Stiffness versus applied cycles for specimen 71B.
Figure 42. Stiffness versus applied cycles for specimen 73A.
Figure 43. Failure surface of specimen 73A.
Figure 44. Stiffness versus applied cycles for specimen 72B.
along all four of the cross-over lines within the gage section, long splitting (about 2.54 cm) between tows of fibers, and delamination between plies. The damage zone span over a region of about 13 cm (5 in), compared to the highly localized damage zone in specimen 71B. Moreover, the two distinct features of the fracture surfaces of the tensile specimens were also observed.

Specimen 72B was cyclically loaded in torsion-torsion at a maximum load of 50% the torsional strength of 68B. The specimen survived after one million cycles. There is only about 6% axial stiffness reduction at the end of the cyclic test, as shown in Fig. 44. The residual strength (torsional) of the specimen is about 65.5 MPa (9.5 ksi), 14% higher than specimen 68B, which was loaded to failure under pure quasi-static torsional load. The result suggested that the damage (if any) introduced to the tube at 50% torsional load is insignificant. A plot of the angular displacement versus torsion for specimen 72B is shown in Fig. 45. Failure mechanism of this specimen had been discussed together with specimen 68B.

**Multiaxial Fatigue**

Specimen 73B was cyclically tested under a maximum tensile load of 50% quasi-static ultimate tensile strength and a maximum of 50% quasi-static torsional strength (R = 0.1 for tensile and torsional loads) at 5 Hz. It failed at 10E5.48 (302814) cycles. A plot of axial stiffness versus cycles is shown in Fig. 46. There was a 18% reduction in axial stiffness at 10E5 cycles. This tube failed just outside the grip region, along a cross-over line. The failure surface was similar to those failed around the grip region. Therefore, the result (failure mode, life, etc) may not be representative under the applied load level.
Figure 45. Torque versus angular displacement curve for specimen 72B.
Figure 46. Stiffness versus applied cycles for specimen 73B.
Figure 47. Stiffness versus applied cycles for specimen 65B.
Specimen 65B was cyclically tested under a maximum tensile load of 50% quasi-static ultimate strength and a maximum of 80% quasi-static torsional strength (R = 0.1 for both types of loads). The specimen failed at 137381 cycles (10E5.14). A plot of axial stiffness versus cycles is shown in Fig. 47. An almost 50% reduction in axial stiffness is seen just prior to failure. The failure surface of this specimen is very interesting. Fracture along three cross-over regions, which is the characteristic of tensile failure, was seen. In addition, the global failure surface also shown an 45 degree across the tube, similar to those of specimen 72B, an indication of the effect of torsional load. Delamination and splitting between fiber tows were also observed. In general, the failure mode has the characteristics of both pure tensile failure and pure torsional failure.

Fatigue at Elevated Temperature

Two tubes from batch II were cyclically tested at 1000°C. Specimen 72A were cyclically tested at a maximum tensile load of 2891 N (650 lb), or, approximately 12 GPa (1.74 ksi), R = 0.1. This load level was about 90% of the quasi-static tensile strength of specimen 65A (tested at 1100°C). A plot of axial stiffness versus applied cycles is shown in Fig. 48. Room temperature stiffness was first measured. The axial stiffness at 1000°C was about 88% of the room temperature stiffness, as shown in Fig. 48. The specimen survived after one million cycles. From Fig. 48, reduction in axial stiffness during the test was not significant, as the average measured axial stiffness at the end of one million cycles was about 95% of the initial stiffness. Room temperature stiffness at the end of one million cycles was also measured. However, the average stiffness was about 87% of the initial room temperature axial stiffness. The reason for the unproportioned stiffness ratios is unclear at this point. The specimen was loaded to failure at
1000°C monotonically at a loading rate of 111.2 N/sec (25 lb/sec). The specimen failed near the lower gripping region, as a result of the uneven pressure (19.3 MPa average, or 2.8 ksi) exerted on that region by the broken collet (the collet was broken into four separate piece, one small piece was removed for the convenient of gripping). The residual strength was 25.88 MPa (3.754 ksi), which is almost twice the quasi-static strength of 65A at 1100°C. Since the specimen failed near the grip region, it is reasonable to assume that its residual strength was higher than 25.88 MPa. As a result, specimen 72A was actually only cyclically loaded under 46% of its ultimate strength, assuming that there was not much reduction in strength during fatigue.

Based on the result of specimen 72A, specimen 66B was cyclically loaded at a maximum load of 90% of the residual strength of 72A (21.87 MPa (3.173 ksi)), at R = 0.1, at 1000°C. Initial stiffness at that temperature was 80% of the room temperature stiffness, as shown in Fig. 49. The stiffness of the specimen was also measured after 100 cycles. From Fig. 49, only a small reduction was seen. The specimen failed at 880 cycles near the grip region, similar to 72A, despite the lowered applied grip pressure (6.895 MPa, or 1.0 ksi).

A summary of the results of quasi-static and uniaxial cyclic tests (from 6 tubes) at both room and 1000°C is shown in Fig. 50. Here we take the residual strength of specimen 72A as the quasi-static strength at 1000°C. Arrows in the figure indicate run out.

**Analysis of Experimental Data**

We have developed tensile strength and tensile fatigue theories for unidirectional CMCs based on micro-mechanical analysis in the last two chapters. These models describe the
Figure 48. Stiffness versus applied cycles for specimen 72A.
Figure 49. Stiffness versus applied cycles for specimen 66B.
Figure 50. A summary of fatigue test data at 1000°C.
behavior of the material in a simplistic and idealistic way: they are used as guidance for making the most basic form of CMCs. For real components such as tubes, a direct application of the theories for data interpretation is not viable, as the strength of the composite structure is influenced factors such as the processing technique and processing-related micro-structural details. We have discussed that several distinct failure modes are observed even under one single type of loading. As a result, failure process causing these failure modes and the subsequent structural failure are very difficult to describe in a comprehensive way. However, with some reasonable assumptions and analysis, the strength of the tubes can roughly be estimated. Two such examples are illustrated as follows.

**Torsional Strength**

The torsional strength of the tube seems to be an relatively easy case to analyze, as the fibers are arranged in a ± 45-degree pattern, and, with pure torsional load, fibers are mostly under tensile or compressive load in either the longitudinal (fiber) or transverse direction. We have suggested from observation of the failure surface that the final failure event is the failure of fiber bundles in tension while the matrix may not carry substantial load. Hence, a bundle strength theory (where the composite strength is estimated based on the strength of fiber bundles) may be applied here to estimate tube strength. We proceed with the analysis as follows. We treat the wall of the tube as a \([45^\circ]_4\) laminate where ply stresses are found by using classical laminate analysis. At the failure load of the tube, the tensile stress on the +45 lamina is estimated and compared to the prediction from bundle theory.
From laminate analysis (8 plies) under a unit shear load \((N_s)\) of 17.5 kN/m (100 lb/in), tensile stress in the +45 plies is 11.5 MPa (1.67 ksi), whereas stress in the transverse direction is about 2.27 MPa (0.33 ksi) in compression. From tube geometry and strength of materials formulation, we found that the maximum shear stress on the tube surface is about 4T, where T is the torsional failure load. As the thickness of the tube is approximately 0.254 cm (0.1 in), we have \(N_s = 147\) kN/m. Accounting for the 25% porosity, the failure tensile stress on the fibers is 128.9 MPa (18.70 ksi). As we have noticed that some +45 plies are failed by propagation of sharp cracks prior to final failure, these plies will not be counted in the laminate analysis. It is observed from the fracture surface that about half of the plies failed under tension (which exhibit pull-out type feature). If this is so, we have a ply stress in the fiber direction = 257.87 MPa (37.4 ksi).

The strength of a bundle of fibers, \(X_b\), is [92]

\[
X_b = E_f \varepsilon_o (mLe)^{-1/m}
\]

where

\(E_f\) = fiber modulus, (176 GPa)  
\(\varepsilon_o\) = characteristic strain  
\(m\) = Weibull modulus (10)  
\(L\) = bundle length (18 cm)

Here \(L\) is the fiber length within the gage section of the tube, taking into account the helical geometry around the tube. The strength of unidirectional coupons gives failure strains between 0.16 - 0.18\% [91]. With this strain value, the bundle strength ranged from 237.2 MPa (34.4 ksi) \((m = 8)\) to 270.3 MPa (39.2 ksi) \((m = 10)\). If our estimation
of failure ply stress using laminate analysis is close to reality, then bundle strength theory gives a fairly good estimation of the torsional strength.

*Axial Strength*

Analysis for the axial strength of the tube may not be as easy as for the previous case. In this case, fibers are under tension (in the fiber direction) as well as transverse shear. It can be shown from laminate analysis that under an applied load of 175.1 N/m (1000 lb/in), stress in the fiber direction is about 53.8 MPa (7.8 ksi) while shear stress is about 34.5 MPa (5.0 ksi). As the shear to axial stress ratio is as high as 64%, application of the bundle strength approach may not be appropriate.

As we have mentioned that sharp crack propagation was suggested by the features of the fracture surface, a fracture mechanics type approach to the tensile strength may be worth trying. Examination of a polished section revealed voids between fiber tows, with size comparable to the size of the tow (Fig.5). Here, we treat the composite as a bulk material with many initial collinear cracks (voids) as a result of the processing and tube architecture. The stress intensity factor, $K_n$, for an array of $n$ periodically spaced cracks when $n \to \infty$ is [95]:

$$K_n = \sigma \sqrt{2d_c \tan \left( \frac{\pi r_c}{2} \right)}$$

where

$\sigma =$ applied stress

$2d_c =$ crack spacing

$r_c =$ ratio of crack size to crack spacing
Figure 51. A polished surface of a specimen showing voids between fiber tows.
Figure 52. Strength versus crack spacing for material with many collinear cracks.
A plot of strength versus crack spacing, $d_c$, for several values of crack size to spacing ratio, $r_c$, is shown in Fig. 52. From SEM images, crack spacing may vary from 1000 to 2000 $\mu$m while crack (void) size is similar. As $K_e$ for a $\pm 45$ alumina/alumina composite laminate is not available, we use $K_e$ of bulk alumina, which is 3.12 MPa m$^{1/2}$. With this $K_e$ value, the axial strength of the tube (44 MPa) falls between $0.5 < r_c < 0.9$, and 1000 $< d_c < 2500$ $\mu$m. The result suggests that the fracture mechanics approach may be a reasonable way for interpreting the tensile strength of the tube.
Summary and Conclusions

Tensile Strength

A model for estimating the tensile strength of unidirectional CMCs is presented. The construction of the model is based on the observed behavior of a number of CMC systems that share some common damage features under tensile load. The present model is constructed in a comprehensive way with several distinct features which are not seen in other models in the literature. First, the model incorporates detailed (approximate) stress analysis for fibers embedded in the matrix after the occurrence of saturated parallel matrix cracking with probability analysis such that the probability as well as the location of fiber fracture can be estimated during a load increment. Second, the debonding process between the fiber/matrix interface during tensile loading is also described by using a shear strength debonding criterion. Third, the present model takes into consideration the effect of fiber fracture within the distance over which the fiber stress regains its remote value from zero at the broken end by the fiber/matrix interfacial shear stress, an important concept that has been over-looked in strength models in the current liter-
ature. It has been shown that the resulting model provides very accurate strength predictions for a number of CMC systems. The influence of a number of parameters (such as Weibull modulus and interfacial shear stress) on the composite strength were studied. Keeping the same Weibull characteristic strength of the fibers, an increase of Weibull modulus results in a decrease of tensile strength. The model can also quantify the influence of fiber/matrix interfacial shear stress on strength and toughness. Higher interfacial shear stress results in higher tensile strength as the broken fibers can support more applied load on a crack plane. However, high interfacial shear stress shortens the ineffective length, which reduces the work of pull-out, thus decreasing the composite toughness. The results of our study indicate that an optimum composite system with balanced strength and toughness can be made, theoretically, by proper selection of fiber strength distributions (represented by Weibull modulus) and the fiber/matrix interfacial shear stress.

**Long-Term Behavior**

A analytical model for describing the long-term behavior of unidirectional CMCs under cyclic tensile loading is developed based on the tensile strength model. Similar to the tensile strength model, the debonding process is described. However, an energy approach is used in the fatigue model instead of shear strength approach. The variation of the Weibull characteristic strength, \( \sigma_c \), and the fiber/matrix interfacial shear stress, \( \tau_f \), with applied cycles is considered in the model. Although it is recognized that such variations are still empirical in form because of lack of supporting data, which leave much room for future work, analytically and experimentally. Using coefficients determined by curve fitting for \( \sigma_c(n) \) and \( \tau_f(n) \) (Eqs (3.1) and (3.2)), the fatigue model gives satisfac-
tory predictions of the S-N curve for two materials systems chosen for study, Nicalon/CAS and Nicalon/1723. The influence of the Weibull modulus of the reinforcing fibers on life of the composite was studied. It has been shown that the lower the value of $m$, the longer the fatigue life.

**Experimental**

Experimental methods for tube testing at both ambient and elevated temperatures were developed and were proven to be successful. Base-line data for quasi-static strength (tensile and torsional), and fatigue behavior (uniaxial and multi-axial) were obtained. A few such tests were conducted at elevated temperatures. These data serve as guidelines for future testing and evaluation. It has been shown that the tensile and torsional (quasi-static) strength of the tubes are critically influenced by geometry at the cross-over region, which, as a result, also influences the long-term behavior. The damage/failure modes of the tubes under cyclic loads resemble that of the quasi-static cases except that they are scattered throughout the specimens with splitting of fiber tows. The damage/failure modes under multiaxial loading (tensile and torsional) have the characteristics of the uniaxial cases. In view of this, modifications in tube architecture at the cross-over region may have a positive result on both the strength and life of the tube. Moreover, an improved processing technique may also change failure modes. However, the effect of processing on the strength and toughness cannot be extrapolated at this point. Based on the observation of the specific failure modes, a bundle strength approach is used to interpret the result of torsional loading while a fracture mechanics approach is used for the case of tensile loading. Both schemes seem to offer some logical explanation of the failure mechanisms.
References


The radial stress in the fiber due to Poisson's effect is derived here. It has been shown that the radial and circumferential stress of a hollow cylinder (in this case, the matrix surrounding the fiber) under uniform pressure on the inner and outer surfaces are [96]

\[ \sigma_r = \frac{B}{r^2} + C \]  
(A1)

\[ \sigma_\theta = -\frac{B}{r^2} + C \]  
(A2)

where

\( r \) = radial coordinate

\( \sigma_r \) = radial stress in the matrix

\( \sigma_\theta \) = circumferential stress in the matrix

\( B, C \) = constants

Stress-strain-displacement equation for the circumferential strain of the elastic, isotropic matrix material is
\[ \varepsilon_m = \frac{U'_m}{r} = \frac{1}{E_m} \left[ \sigma_m - \nu_m (\sigma'_r + \sigma^2_m) \right] \quad (A3) \]

Substituting Eqs. (A1) and (A2) into Eq. (A3), we have

\[ U'_m = -\frac{1}{E_m} \left[ \frac{B}{r} (1 + \nu_m) + (1 - \nu_m) r C + \nu_m r \sigma^2_m \right] \quad (A4) \]

For cylindrical rod (i.e., the fiber), the corresponding expressions for the radial and circumferential stresses, \( \sigma'_r \) and \( \sigma'_\theta \), are

\[ \sigma'_r = \sigma'_\theta = A \quad (A5) \]

where \( A \) is a constant. The boundary conditions for the two concentric cylinders are

\[ \sigma'_m = 0 \quad \text{at} \quad r = b \quad (A6) \]

\[ \sigma'_m = \sigma'_f \quad \text{at} \quad r = a \quad (A7) \]

\[ U'_m = U'_f \quad \text{at} \quad r = a \quad (A8) \]

The constants \( B \) and \( C \) can be found by using Eqs. (A1), (A2), (A4), (A6) and (A8). The constant \( A \) can then be found by using Eq. (A7).

\[ A = a_p = \frac{E_f \nu_f \sigma^2_f - \nu_m \sigma^2_m}{E_r (1 - \nu_f) + 1 + \nu_m + 2V_r} \quad (A9) \]
Appendix B.

The detailed stress analysis for the fiber and the matrix in the debonded region is provided as follows. Substituting Eqs. (2.2.3) and (2.2.4) into Eq. (2.2.1), we have

$$\frac{\partial \sigma_f}{\partial z} = \frac{2 \mu_1 q_0}{a} \left[ q_0 - \frac{E_r v_f \sigma_f - \nu_m \sigma_m}{E_r (1 - v_f) + 1 + \nu_m + 2 V_r} \right]$$

Rearranging, we have

$$\frac{\partial \sigma_f}{\partial z} + F_1 \sigma_f + F_2 \sigma_m = F_o$$

(B1)

where

$$F_o = \frac{-2 \mu_1 q_0}{a}$$

$$F_1 = \frac{-2 \mu_1 E_r v_f}{a C_o}$$
\[ F_2 = \frac{2 \mu_i v_m}{a C_o} \]

The expression for \( C_o \) is given in Eq. (2.2.7). Similarly, by substituting Eqs. (2.2.3) and (2.2.4) into Eq. (2.2.2), we get

\[
\frac{\partial \sigma_m^2}{\partial z} = \frac{2 \mu_i}{b^2 - a^2} \left[ q_o - \frac{E_r v_f \sigma_f^2 - v_m \sigma_m^2}{E_r (1 - v_f) + 1 + v_m + 2V_r} \right]
\]

Rearranging, we have

\[
\frac{\partial \sigma_m^2}{\partial z} + G_1 \sigma_f^2 + G_2 \sigma_m^2 = G_o \quad \text{(B2)}
\]

where

\[
G_o = \frac{2 V_r \mu_i q_o}{a} \]

\[
G_1 = \frac{2 \mu_i E_r V_r v_f}{a C_o} \]

\[
G_2 = -\frac{2 V_r \mu_i v_m}{a C_o} \]

We get, after differentiating Eq. (B1) once with respect to \( z \),

\[
\frac{\partial^2 \sigma_f^2}{\partial z^2} + F_1 \frac{\partial \sigma_f^2}{\partial z} + F_2 \frac{\partial \sigma_m^2}{\partial z} = 0 \quad \text{(B3)}
\]

Rearranging Eq. (B2), we have
\[ \frac{\partial \sigma_m^z}{\partial z} = G_o - G_1 \sigma_f^z - G_2 \sigma_m^z \]  \hspace{1cm} (B4)

From Eq. (B1), we have
\[
\sigma_m^z = \frac{1}{F_2} \left[ F_o - \frac{\partial \sigma_f^z}{\partial z} - F_1 \sigma_f^z \right] \hspace{1cm} (B5)
\]

Substituting Eqs. (B4) and (B5) into Eq. (B3), we get
\[
\frac{\partial^2 \sigma_f^z}{\partial z^2} + (F_1 + G_2) \frac{\partial \sigma_f^z}{\partial z} = 0 \hspace{1cm} (B6)
\]

Similarly, by differentiating Eq. (B2) once with respect to \( z \), we have
\[
\frac{\partial^2 \sigma_m^z}{\partial z^2} + G_1 \frac{\partial \sigma_f^z}{\partial z} + G_2 \frac{\partial \sigma_m^z}{\partial z} = 0 \hspace{1cm} (B7)
\]

Rearranging Eq. (B1), we get
\[
\frac{\partial \sigma_f^z}{\partial z} = F_o - F_1 \sigma_f^z - F_2 \sigma_m^z \hspace{1cm} (B8)
\]

and from Eq. (B2)
\[
\sigma_f^z = \frac{1}{G_1} \left[ G_o - \frac{\partial \sigma_m^z}{\partial z} - G_2 \sigma_m^z \right] \hspace{1cm} (B9)
\]

Substituting Eqs. (B8) and (B9) into (B7), we get
\[
\frac{\partial^2 \sigma_m^z}{\partial z^2} + (F_1 + G_2) \frac{\partial \sigma_m^z}{\partial z} = 0 \hspace{1cm} (B10)
\]
To solve Eq. (B10), let

\[ \sigma_m^z = Y_1 + Y_2 e^{k_0 z} \]  \hspace{1cm} (B11)

The matrix is stress free at the matrix crack plane \((\sigma_m^z(s) = 0)\). Therefore, we get

\[ Y_1 = -Y_2 e^{k_0 s} \]  \hspace{1cm} (B12)

and

\[ \sigma_m^z = Y_2 \left( e^{k_0 z} - e^{k_0 s} \right) \]  \hspace{1cm} (B13)

\[ \frac{\partial \sigma_m^z}{\partial z} = Y_2 k_0 e^{k_0 z} \]  \hspace{1cm} (B14)

Substituting Eqs. (B13) and (B14) into Eq. (B2), we get

\[ \sigma_f^z = \frac{1}{G_1} \left[ G_0 - Y_2 k_0 e^{k_0 z} - G_2 \left( e^{k_0 z} - e^{k_0 s} \right) \right] \]  \hspace{1cm} (B15)

At the matrix crack plane \((z = s)\), the fiber stress is simply the applied stress, \(\sigma_a\). Using this boundary condition in Eq. (B15), we get

\[ Y_2 = \frac{1}{k_0} \left( G_0 - G_1 \sigma_a \right) \left[ e^{k_0 (z-s)} - 1 \right] \]  \hspace{1cm} (B16)

and Eq. (B13) becomes

\[ \sigma_m^z = \frac{1}{k_0} \left( G_0 - G_1 \sigma_a \right) \left[ e^{k_0 (z-s)} - 1 \right] \]  \hspace{1cm} (B17)

Differentiate Eq. (B17) once with respect to \(z\) and use Eqs. (B9) and (B17), we get Eq. (2.2.8).
Appendix C.

Detailed stress analysis for the fiber and the matrix in the bonded region is presented here. Stress-strain-displacement relationships of an elastic, isotropic fiber

\[ \varepsilon_f^z = \frac{\partial U_f^z}{\partial z} = \frac{1}{E_f} \left[ \sigma_f^z - v_f (\sigma_f^r + \sigma_f^\theta) \right] \]  \hspace{1cm} (C1)

\[ \varepsilon_f^\theta = \frac{U_f^r}{r} = \frac{1}{E_f} \left[ \sigma_f^\theta - v_f (\sigma_f^r + \sigma_f^\theta) \right] \]  \hspace{1cm} (C2)

where

\[ \varepsilon_f^\theta, \varepsilon_f^z \] = fiber strain in the \( \theta \) and \( z \) directions, respectively.

\[ U_f^r, U_f^z \] = fiber displacement in the \( r \) and \( z \) directions, respectively.

\[ \sigma_f^r, \sigma_f^\theta, \sigma_f^z \] = fiber stresses in the \( r \), \( \theta \), and \( z \) directions, respectively.

Stress-strain-displacement relationships of the elastic, isotropic matrix material:
\[ \varepsilon_m^z = \frac{\partial U_m^z}{\partial z} = \frac{1}{E_m} \left[ \sigma_m^z - v_m (\sigma_m^r + \sigma_m^\theta) \right] \]  \hspace{1cm} (C3)

\[ \varepsilon_m^\theta = \frac{U_m^\theta}{r} = \frac{1}{E_m} \left[ \sigma_m^\theta - v_m (\sigma_m^r + \sigma_m^z) \right] \]  \hspace{1cm} (C4)

\[ \gamma_m^{rz} = \frac{\partial U_m^z}{\partial r} = \frac{2 (1 + v_m)}{E_m} \tau_m^{rz} \]  \hspace{1cm} (C5)

where

\[ \varepsilon_m^\theta, \varepsilon_m^z, \gamma_m^{rz} \] = matrix strains in the \( \theta, z, \) and \( rz \) directions, respectively.

\[ U_m^r, U_m^z \] = displacement in the matrix in the \( r \) and \( z \) direction, respectively.

\[ \sigma_m^r, \sigma_m^\theta, \sigma_m^z \] = matrix stresses in the \( r, \theta, \) and \( z \) directions, respectively.

In our analysis, we assume that the shear stress in the matrix, \( \tau_m^{rz} \), is linearly related to the fiber/matrix interfacial shear stress, \( \tau_i \), in the radial direction through the relationship

\[ \tau_m^{rz} = \frac{V_r (b^2 - r^2)}{a r} \tau_i \]  \hspace{1cm} (C6)

Substituting Eq. (C6) into Eq. (C5), we get

\[ \gamma_m^{rz} = \frac{\partial U_m^z}{\partial r} = \frac{2 (1 + v_m) V_r (b^2 - r^2)}{a r E_m} \tau_i \]  \hspace{1cm} (C7)

Integrating Eq. (C7) with respect to \( r \), we get the displacement function of the matrix in the \( z \) direction.
\[ U_m^z = \frac{2 (1 + v_m) V_r}{E_m a} \left( b^2 \ln(r) - \frac{r^2}{2} \right) \tau_i \]  

(C8)

Differentiating Eq. (C8) with respect to \( z \), we get

\[ \varepsilon_m^z = \frac{\partial}{\partial z} \left( \frac{U_m^z}{E_m} \right) = \frac{2(1 + v_m) V_r}{a E_m} \left( b^2 \ln(r) - \frac{r^2}{2} \right) \frac{\partial \tau_i}{\partial z} \]  

(C9)

From Eq. (C9), we get

\[ \varepsilon_m^z(b,z) = \frac{2(1 + v_m) V_r}{a E_m} \left( b^2 \ln b - \frac{b^2}{2} \right) \frac{\partial \tau_i}{\partial z} \]  

(C10)

\[ \varepsilon_m^z(a,z) = \varepsilon_f^z(a,z) = \frac{2(1 + v_m) V_r}{a E_m} \left( b^2 \ln a - \frac{a^2}{2} \right) \frac{\partial \tau_i}{\partial z} \]  

(C11)

From Eqs. (C10) and (C11), we formulate

\[ \varepsilon_m^z(b,z) - \varepsilon_f^z(a,z) = \frac{2(1 + v_m) V_r}{a E_m} \left( b^2 \ln(b/a) - \frac{b^2}{2} + \frac{a^2}{2} \right) \frac{\partial \tau_i}{\partial z} \]  

(C12)

Rearranging, we get

\[ \frac{\partial \tau_i}{\partial z} = \frac{a E_m \left[ \varepsilon_m^z(b,z) - \varepsilon_f^z(a,z) \right]}{(1 + v_m)[2V_r b^2 \ln(b/a) - a^2]} \]  

(C13)

Now we need to know \( \varepsilon_m(b,z) \) and \( \varepsilon_f(a,z) \) in order to solve Eqs. (2.2.11a) and (2.2.11b).

From Eqs. (C1) and (A10), we get

\[ \varepsilon_f^z(a,z) = \frac{1}{E_f} \left[ \sigma_f^z - 2 v_f q_{\rho z} \right] \]
or

\[
\varepsilon^z_f(a,z) = \frac{1}{E_f} \left[ \left( 1 - \frac{2 v_f^2 E_r}{k_1} \right) \sigma^z_f + \frac{2 v_f v_m}{k_1} \sigma^z_m \right]
\]  \hspace{1cm} (C14)

where

\[k_1 = E_r (1 - v_f) + 1 + v_m + 2 V_r \]

From Eqs. (C3) and (A10), we have

\[
\varepsilon^z_m(b,z) = \frac{1}{E_m} (\sigma^z_m + 2 V_r v_m q_p)
\]

or

\[
\varepsilon^z_m(b,z) = \frac{1}{E_m} \left[ \left( 1 - \frac{2 V_r v_m^2}{k_1} \right) \sigma^z_m + \frac{2 V_r E_r v_m v_f}{k_1} \sigma^z_f \right]
\]  \hspace{1cm} (C15)

From force equilibrium of the composite, we have

\[
\sigma_a = \sigma^z_f + \frac{1}{V_r} \sigma^z_m
\]  \hspace{1cm} (C16)

Substituting Eqs. (C14)-(C16) into Eq. (C13), we get

\[
\frac{\partial \tau_f}{\partial z} = g_o [V_r g_1 \sigma_a + (g_2 - V_r g_1) \sigma^z_f]
\]  \hspace{1cm} (C17)

where

\[
g_o = \frac{a E_m}{(1 + v_m)[2 V_r \sigma^2 \ln(b/a) - a^2]}
\]
\[ g_1 = \frac{1}{E_m} \left( 1 - \frac{2 V_r v_m^2}{k_1} \right) - \frac{2 v_f v_m}{E_f k_1} \]

\[ g_2 = \frac{2 V_r E_r v_m v_f}{k_1 E_m} - \frac{1}{E_f} \left( 1 - \frac{2 v_f^2 E_r}{k_1} \right) \]

Differentiating Eq. (2.2.11a) once with respect to \( z \) and substituting Eq (C17) into it, we get

\[ \frac{\partial^2 \sigma_f}{\partial z^2} + \frac{2 g_o (g_2 - V_r g_1)}{a} \sigma_f = -2 g_o g_1 V_r \sigma_a \]

which is essentially the same as Eq. (2.2.12). Using the boundary conditions in Eq. (2.2.13), the solution for Eq. (C17) is Eq. (2.2.14).
Vita

Kin Liao was born on December 19, 1961, to Lu-Ping and Chi-Ren Liao in Beijing, China. He immigrated to Hong Kong with his family in 1974. In the fall of 1982, after graduated from Ho Fung College in Hong Kong, he was awarded the C. W. Chu Scholarship and came to the United States where he enrolled in Vincennes University in Vincennes, Indiana. He transferred to Virginia Tech a year later and graduated with a B.S. in Engineering Science and Mechanics in 1987. After receiving his M.S. in Engineering Mechanics in 1991, he joined the Materials Engineering Science program to pursue his Ph.D. degree.

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