SUBNORMAL OPERATORS, HYPONORMAL OPERATORS AND MEAN POLYNOMIAL APPROXIMATION

by

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(ABSTRACT)

We prove quasisimilar subdecomposable operators without eigenvalues have equal essential spectra. Therefore, quasisimilar hyponormal operators have equal essential spectra. We obtain some results on the spectral pictures of cyclic hyponormal operators. An algebra homomorphism π from $H^\infty(G)$ to $\mathcal{L}(\mathcal{H})$ is a unital representation for T if $\pi(1) = I$ and $\pi(\chi) = T$. It is shown that if the boundary of G has zero area measure, then the unital norm continuous representation for a pure hyponormal operator T is unique and is weak star continuous. It follows that every pure hyponormal contraction is in $C_{.0}$

Let μ represent a positive, compactly supported Borel measure in the plane, $\mathcal{C}$. For each t in $[1, \infty)$, the space $P^t(\mu)$ consists of the functions in $L^t(\mu)$ that belong to the (norm) closure of the (analytic) polynomials. J.Thomson in [T] has shown that the set of bounded point evaluations, *bpe* μ, for $P^t(\mu)$ is a nonempty simply connected region G. We prove that the measure μ restricted to the boundary of G is absolutely continuous with respect to the harmonic measure on G and the space $P^2(\mu) \cap C(spt\mu) = A(G)$, where $C(spt\mu)$ denotes the continuous functions on $spt\mu$

and $A(G)$ denotes those functions continuous on $\overline{G}$ that are analytic on G.

We also show that if a function f in $P^2(\mu)$ is zero a.e. μ in a neighborhood of a point on the boundary, then f has to be the zero function. Using this result, we are able to prove that the essential spectrum of a cyclic, self-dual, subnormal operator is symmetric with respect to the real axis. We obtain a reduction into the structure of a cyclic, irreducible, self-dual, subnormal operator. One may assume, in this inquiry, that the corresponding $P^2(\mu)$ space has $bpe\mu = D$. Necessary and sufficient conditions for a cyclic, subnormal operator S_μ with $bpe\mu = D$ to have a self-dual are obtained under the additional assumption that the measure on the unit circle is log-integrable.

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Contents

CHAPTER I INTRODUCTION

We assume that the reader is familiar with the basic results of complex, real and functional analysis. In this introductory chapter, we will introduce some notation and concepts, summarize previous results, and discuss our theorems in this dissertation. We will keep everything as simple as possible, leaving the details for later chapters.

Let $\mathcal{H}$ be a separable Hilbert space over the complex field C and let $\mathcal{L}(\mathcal{H})$ be the collection of all linear bounded operators on $\mathcal{H}$. An operator S in $\mathcal{L}(\mathcal{H})$ is a subnormal operator if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and there is a normal operator N on $\mathcal{K}$ which leaves $\mathcal{H}$ invariant so that N restricted to $\mathcal{H}$ is S. An operator $T \in \mathcal{L}(\mathcal{H})$ is a hyponormal operator if

$$[T^*, T] = T^*T - TT^* \geq 0.$$

For $T \in \mathcal{L}(\mathcal{H})$, let $\sigma(T)$ denote the spectrum of T and let $\sigma_e(T)$ denote the essential spectrum of T. One says that an operator $T \in \mathcal{L}(\mathcal{H})$ is a M-hyponormal operator if there is a constant $M > 0$ so that

$$\|T^*x\| \leq M\|Tx\|$$

for every $x \in \mathcal{H}$. An operator $D \in \mathcal{L}(\mathcal{K})$ is decomposable if for every open cover $\{U_1 , U_2\}$ of $\sigma(T)$, there are two invariant subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ of D such that

(a) $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$;

(b) $\sigma(D|_{\mathcal{K}_1}) \subset U_1$ and $\sigma(D|_{\mathcal{K}_2}) \subset U_2$.

An operator S on $\mathcal{H}$ is called subdecomposable if there is a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ and there is a decomposable operator D on $\mathcal{K}$ with $D\mathcal{H} \subset \mathcal{H}$ such that S is the restriction of D to $\mathcal{H}$. In this case, we say that D is an extension of S.

Two operators $S \in \mathcal{L}(\mathcal{H}_1)$ and $T \in \mathcal{L}(\mathcal{H}_2)$ are quasisimilar if there are quasiaffinities X from $\mathcal{H}_1$ to $\mathcal{H}_2$ and Y from $\mathcal{H}_2$ to $\mathcal{H}_1$ (i.e. $kerX = kerY = \{0\}$, $\overline{RanX} = \mathcal{H}_2$ and $\overline{RanY} = \mathcal{H}_1$) so that $XS = TX$ and $SY = YT$. If this is the case, we write $T \sim S$. Let G be a bounded domain in $\mathcal{C}$ and let $H^\infty(G)$ denote the Banach algebra of all bounded analytic functions on G. Let χ denote the function whose value at λ is λ. An algebra homomorphism π from $H^\infty(G)$ to $\mathcal{L}(\mathcal{H})$ is a unital representation for T if $\pi(1) = I$ and $\pi(\chi) = T$.

In [27] M.Putinar shows that every hyponormal operator is similar to a subdecomposable operator. This is to say, a hyponormal operator turns out to be the restriction to a closed subspace of an decomposable operator which behaves in many ways like a normal operator. Using this result, S.Brown [4] proved that every hyponormal operator with rich spectrum has a nontrivial invariant subspace, which generalized his early work on subnormal operators. Using Putinar's model for hyponormal operators, we seek other properties that hyponormal operators have in common with subnormal operators. Chapter II enumerates some of these common characteristics.

In Section 2.1, we discuss the equivalence class of hyponormal (subdecomposable) operators under the equivalence relation of quasisimilarity.

Recently, J.Agler, E.Franks and D.A.Herrero [2] have given examples to show that there exists an operator T that is quasisimilar to the unilateral shift such that

$$\sigma(T) \neq \overline{U} \qquad \text{and} \qquad \sigma_e(T) \neq \partial U.$$

where U is the open unit disc. However, S.Clary [7] has shown that quasisimilar hyponormal operators do have equal spectra. Hence, a natural question occurs: What classes of operators preserve spectra and essential spectra under quasisimilarity? In particular, S.Clary [7] and J.Conway [8, p.225] (also [9, p.98]) asked the following question.

Question: Do quasisimilar hyponormal operators have equal essential spectra?

Many authors have obtained partial results on this question. L. Williams [33] showed that quasisimilar quasinormal operators have equal essential spectra.
M. Raphael [28] proved that quasisimilar cyclic subnormal operators have the same essential spectra. K. Yan [35] proved that if a subnormal operator is quasisimilar to a quasinormal operator, then they have equal essential spectra. K. Takahashi [30] has shown that if a contraction is quasisimilar to the unilateral shift, then they must have the same essential spectra. Recently, analyzing the structure of spectral pictures of subnormal operators, the author [36] proved if an operator T without eigenvalues is quasisimilar to a subnormal operator S, then

$$\sigma(T) \setminus \sigma_e(T) \subset \sigma(S) \setminus \sigma_e(S).$$

In particular, if T is hyponormal, then the essential spectrum of T contains the essential spectrum of S.

The main idea of the paper [36], was to use spectral decomposability of normal operators to analyze the spectral behavior of subnormal operators. It is natural then to study the spectral behavior of a subdecomposable operator using its decomposable extension. Doing that, we are able to prove that quasisimilar subdecomposable operators have equal spectra. We then establish one of our main results in this chapter: two quasisimilar subdecomposable operators without eigenvalues have equal essential spectra. In particular, we prove that quasisimilar M-hyponormal operators have equal essential spectra.

In Section 2.2, we study the set of bounded point evaluations for cyclic, pure, hyponormal operators. It is shown that a point λ_0 belongs to $\sigma(T) \setminus \sigma_e(T)$ of a pure cyclic hyponormal operator T if and only if there is a neighborhood $O(\lambda_0,\ \delta)$ of λ_0 and there is a coanalytic function k_λ on $O(\lambda_0,\ \delta)$ with valued in $\mathcal{H}$ so that $(T\ -\ \lambda)^* k_\lambda\ =\ 0$ for $\lambda \in O(\lambda_0,\ \delta)$.

Section 2.3 gives some results on representations for hyponormal operators if the boundary of G has area measure zero. Let $T \in \mathcal{L}(\mathcal{H})$ be a pure hyponormal operator and let $T_1\ \in\ \mathcal{L}(\mathcal{H}_1)$ be an arbitrary operator. Suppose π and π_1 are two norm continuous unital representations from $H^\infty(G)$ to $\mathcal{L}(\mathcal{H})$ and $\mathcal{L}(\mathcal{H}_1)$, respectively, such that $\pi(\chi)\ =\ T$ and $\pi_1(\chi)\ =\ T_1$. It is shown that if there is a linear bounded operator X from $\mathcal{H}_1$ to $\mathcal{H}$ such that $TX\ =\ XT_1$, then $\pi(f)X\ =\ X\pi_1(f)$, for all f in $H^\infty(G)$. As a result, we show that a unital norm continuous representation for T on G is unique. It follows that a unital norm continuous representation for T is weak-star continuous. Consequently, it is proved that every pure contraction hyponormal operator is in $C_{\cdot 0}$. The latter generalizes the results in Chapter 2 and 3 of [20] from subnormal operators to hyponormal operators provided the area measure of the boundary of G is zero.

In Chapter III, we study the Hilbert spaces obtained by taking the closure in mean of polynomials. For a positive regular Borel measure μ, having compact support in the complex plane $\mathcal{C}$, and any t in $[1\ ,\ \infty)$, let $P^t(\mu)$ be the closure of polynomials in $L^t(\mu)$ and let S_μ be the operator consisting of multiplication by z on $P^t(\mu)$. For $t\ =\ \infty$ we define $P^\infty(\mu)$ to be the weak-star closure of polynomials in $L^\infty(\mu)$. In this chapter, we always assume that S_μ is irreducible (i.e., $P^t(\mu)$ can not be decomposed into the direct sum of two nonzero closed subspaces which are invariant subspaces of S_μ).

D.Sarason [29] gave a complete description of $P^\infty(\mu)$. Using the terminology in

[11], we may paraphrase Sarason's results as follows:

(1) Each antisymmetric summand of $P^\infty(\mu)$ consists of the algebra of bounded analytic functions on G, where G is a very special region (consult [22]).

(2) The measure μ restricted to the boundary of G is absolutely continuous with respect to the harmonic measure on G.

Recently, J. Thomson [31] (also see [9, p.399]) obtained the following generalization to $P^t(\mu)$.

Thomson's Theorem: If μ is a finite positive measure on $\mathcal{C}$ with compact support, then there is a Borel partition $\{\Delta_0, \Delta_1, ...\}$ of the support of μ such that if $\mu_n = \mu|_{\Delta_n}$, then the following statements hold.

(a) $P^t(\mu) = L^t(\mu_0) \oplus P^t(\mu_1) \oplus ...$

(b) If $n \geq 1$, then S_{μ_n} is irreducible. Equivalently, $P^t(\mu_n)$ contains no nontrivial characteristic functions.

(c) If $n \geq 1$ and $G_n = abpe(\mu_n)$, then G_n is a simply connected region with $spt(\mu_n) \subset \overline{G}_n$ and $bpe(\mu_n) = G_n$.

(d) If S_μ is an irreducible operator and G is the set of analytic bounded point evaluations for μ, then the Banach algebras $P^t(\mu) \cap L^\infty(\mu)$ and $H^\infty(G)$ are algebraically and isometrically isomorphic and weak-star homeomorphic.

A natural question occurs by comparing Sarason's theorem with Thomson's theorem (Again we remind the reader we tacitly assume S_μ is irreducible.): Is the measure $\mu|_{\partial G}$ absolutely continuous with respect to the harmonic measure on G? We will give an affirmative answer to this question.

Before we state our results we need to introduce some notation which will be useful for this chapter. For a Banach space X and its dual space X^*, we define

$$(x \ , \ f) \ = \ f(x), \quad \text{for } x \in X \text{ and } f \in X^*.$$

For a fixed measure μ, a carrier of μ (denoted by $car(\mu)$) is a Borel subset of $spt\mu$ so that

$$\mu((car(\mu))^c) = 0.$$

A point λ in C is called a bounded point evaluation (bpe) for $P^t(\mu)$ if there is a positive constant M such that

$$|p(\lambda)| \leq M\|p\|_t, \text{ for every polynomial } p.$$

In this case, evaluation at λ extends uniquely to a bounded linear functional L_λ on $P^t(\mu)$. A point λ is an analytic bounded point evaluation (abpe) if there exists a neighborhood U of λ such that each point in U is a bpe and $\hat{f}(\omega) = L_\omega(f)$ is analytic in U for each f in $P^t(\mu)$. The books [8] and [9] contains the basic results on bpes. Thomson's theorem states that there exists an isometric isomorphism $\sim$ from $H^\infty(G)$ onto $P^t(\mu) \cap L^\infty(\mu)$, where G ($= bpe(\mu) = abpe(\mu)$) denotes the set of all bounded point evaluations for $P^t(\mu)$. For $\tilde{f}$ in $P^t(\mu) \cap L^\infty(\mu)$, the operator $T^\mu_{\tilde{f}}$ is the one obtained from multiplication by $\tilde{f}$ on $P^t(\mu)$; that is,

$$T^\mu_{\tilde{f}} g = \tilde{f} g, \quad \text{for } g \in P^t(\mu).$$

For λ in G, by using the Hahn-Banach and Riesz Representation Theorems, we see that there exists a vector $k^\mu_\lambda \in L^q(\mu)$ ($t^{-1} + q^{-1} = 1$) such that for each polynomial p,

$$(p \ , \ k^\mu_\lambda) = p(\lambda), \quad \lambda \in G$$

The function k^μ_λ is called the reproducing kernel for $P^t(\mu)$ (the kernel function is unique in the equivalence class $L^q(\mu)/P^t(\mu)^\perp$). Let $\phi : G \to D$ be a Riemann map; from the properties of our mapping $\sim$, we conclude that the function $\tilde{\phi}$ in $P^t(\mu) \cap L^\infty(\mu)$ has the property that

$$(\tilde{\phi} \ , \ k^\mu_\lambda) = \phi(\lambda), \qquad \text{for all } \lambda \in G.$$

Define a finite measure on $\overline{D}$ by setting $\nu = \mu \circ \tilde{\phi}^{-1}$, then ν is a finite measure with support in $\overline{D}$. Finally, for notational convenience we set $\psi = \phi^{-1}$.

In Section 3.1, we prove one of the major results of this chapter: we show that μ restricted to the boundary of G is absolutely continuous with respect to the harmonic measure ω on G (here ω is the harmonic measure with respect a fixed point in G). In Section 3.2, we study a subalgebra of $P^2(\mu) \cap L^\infty(\mu)$. We show that the subalgebra $P^2(\mu) \cap C(spt\mu)$ is precisely the disc algebra over G.

In Chapter IV, we investigate cyclic, irreducible, self-dual subnormal operators. Let S be a pure subnormal operator on $\mathcal{H}$ (that is, S is a subnormal operator with no normal direct summand). If N is the minimal normal extension of S, the dual T of S is the restriction of N^* to the space $\mathcal{K} \ominus \mathcal{H}$. A subnormal operator S is self-dual if S is unitarily equivalent to its dual T (this notion was introduced by J.Conway [10]). The reader can consult [8] and [9] for the basic results of subnormal operators.

The relations between subnormal operators and their dual have been studied by several people, [10] , [21] and [34]. J.Conway [10] provides some basic results about this. In this chapter, we study the structure of a cyclic irreducible self-dual subnormal operator by using Thomson's recent characterization of cyclic irreducible subnormal operators (see [31] or [9]). We give some necessary and sufficient conditions for cyclic irreducible subnormal operators to be self dual. Our approach also needs some results from Chapter III and a recent paper [25] by R.Olin and L.Yang in which the boundary behaviors of functions in the commutant of cyclic subnormal operators are studied.

In Section 4.1, we will introduce some notation and terminology. We also list some known facts which will be used in chapter IV.

In Section 4.2, it is shown that if a function f in $P^2(\mu)$ is zero in a neighborhood of a point on the boundary, then f has to be a zero function (see Section 4.1 for details). With this fact, we are able to prove the essential spectrum of a cyclic self-dual subnormal operator is symmetric with respect to the real axis.

Section 4.3 shows that the study of a cyclic irreducible self-dual subnormal operator can be reduced to the case of a cyclic self-dual subnormal operator with bounded point evaluations being the open disc.

Section 4.4 gives some necessary and sufficient conditions for cyclic subnormal operators with bounded point evaluations being the open disc to be self-dual under the condition that the scalar spectral measure restricted to the circle is log integrable.

CHAPTER II HYPONORMAL AND SUBDECOMPOSABLE OPERATORS

Section 2.1. Quasisimilarities

For a closed subspace $\mathcal{L}$ of $\mathcal{H}$, let $[\quad]_{\mathcal{L},\mathcal{H}}$ be the projection map from $\mathcal{H}$ to $\mathcal{H}/\mathcal{L}$. For an operator $T \in \mathcal{L}(\mathcal{H})$ with $T\mathcal{L} \subset \mathcal{L}$, let $[T]_{\mathcal{L},\mathcal{H}}$ be the induced operator on $\mathcal{H}/\mathcal{L}$. That is, $[T]_{\mathcal{L},\mathcal{H}}$ is defined as follows:

$$[T]_{\mathcal{L},\mathcal{H}}[f]_{\mathcal{L},\mathcal{H}} = [Tf]_{\mathcal{L},\mathcal{H}}, \quad f \in \mathcal{H}$$

Theorem 2.1.1. Let T be a linear bounded operator on $\mathcal{H}_0$ and let S be a subdecomposable operator on $\mathcal{H}$. Suppose X is a linear bounded operator from $\mathcal{H}_0$ to $\mathcal{H}$ with dense range such that $XT = SX$. Then $\sigma(S) \subset \sigma(T)$.

Proof: Let $D \in \mathcal{L}(\mathcal{K})$ be a decomposable extension of S. Suppose $\lambda_0 \in \sigma(T)^c$, then there is $\delta > 0$ such that

$$O(\lambda_0\ ,\ \delta) \subset \sigma(T)^c$$

where $O(\lambda_0\ ,\ \delta) = \{z : |z - \lambda_0| < \delta\}$. Hence, for every $f \in \mathcal{H}_0$ there is an analytic vector-valued function g_λ^f on $O(\lambda_0\ ,\ \delta)$ such that

$$f = (T - \lambda)g_\lambda^f$$

Therefore,

$$\begin{aligned} Xf &= X(T - \lambda)g_\lambda^f \\ &= (S - \lambda)Xg_\lambda^f \\ &= (D - \lambda)Xg_\lambda^f \end{aligned} \tag{$*$}$$

Let $U_1 = O(\lambda_0 , \frac{\delta}{2})$ and $U_2 = \overline{O(\lambda_0 , \frac{\delta}{4})}^c$, then $\{U_1 , U_2\}$ is an open cover of $\sigma(D)$. So there are two invariant subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ of D such that

(a) $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$;

(b) $\sigma(D|_{\mathcal{K}_1}) \subset U_1$ and $\sigma(D|_{\mathcal{K}_2}) \subset U_2$.

Using (*), we have

$$[Xf]_{\mathcal{K}_2,\mathcal{K}} = ([D]_{\mathcal{K}_2,\mathcal{K}} - \lambda)[Xg_\lambda^f]_{\mathcal{K}_2,\mathcal{K}}.$$

We know that $[Xg_\lambda^f]_{\mathcal{K}_2,\mathcal{K}}$ is an analytic vector-valued function on $O(\lambda_0 , \delta)$. Also it is easy to show that $[D]_{\mathcal{K}_1\cap\mathcal{K}_2,\mathcal{K}_1}$ and $[D]_{\mathcal{K}_2,\mathcal{K}}$ are similar because $\mathcal{K}/\mathcal{K}_2 \cong \mathcal{K}_1/\mathcal{K}_1\cap\mathcal{K}_2$. We also have

$$\sigma([D]_{\mathcal{K}_1\cap\mathcal{K}_2,\mathcal{K}_1}) \subset \sigma(D|_{\mathcal{K}_1}) \cup \sigma(D|_{\mathcal{K}_1\cap\mathcal{K}_2}) \subset U_1.$$

Hence,

$$\sigma([D]_{\mathcal{K}_2,\mathcal{K}}) \subset U_1$$

and

$$\begin{aligned}
&[Xf]_{\mathcal{K}_2,\mathcal{K}} \\
=&\frac{1}{2\pi i}\int_{\partial O(\lambda , \delta)} ([D]_{\mathcal{K}_2,\mathcal{K}} - \lambda)^{-1}[Xf]_{\mathcal{K}_2,\mathcal{K}} d\lambda \\
=&\frac{1}{2\pi i}\int_{\partial O(\lambda , \delta)} [Xg_\lambda^f]_{\mathcal{K}_2,\mathcal{K}} d\lambda \\
=&0.
\end{aligned}$$

Thus, $[Xf]_{\mathcal{K}_2,\mathcal{K}} = 0$. This means $Xf \in \mathcal{K}_2$. Since the spectrum of $D|_{\mathcal{K}_2}$ is contained in U_2, there is a constant $M > 0$, so that for each $g \in \mathcal{K}_2$ the following inequality holds

$$\|g\| \leq M\|(D - \lambda_0)g\|.$$

Therefore, for every $f \in \mathcal{H}_0$ we have

$$\|Xf\| \leq M\|(S - \lambda_0)Xf\|.$$

Since X has dense range, we conclude that S is bounded below at λ_0. Using $X^*S^* = T^*X^*$ and $ker(T - \lambda_0)^* = 0$, we see that $ker(S - \lambda_0)^*$ has to be zero. Hence, $\lambda_0 \in \sigma(S)^c$ because S has no eigenvalues due to S being bounded below at point λ_0. The proof is completed.

Corollary 2.1.2. Let $S_1 \in \mathcal{L}(\mathcal{H}_1)$ and $S_2 \in \mathcal{L}(\mathcal{H}_2)$ be two subdecomposable operators. Suppose $S_1 \sim S_2$, then $\sigma(S_1) = \sigma(S_2)$.

As a result, we have the following conclusion.

Corollary 2.1.3. Let $S_1 \in \mathcal{L}(\mathcal{H}_1)$ and $S_2 \in \mathcal{L}(\mathcal{H}_2)$ be two M-hyponormal operators. Suppose $S_1 \sim S_2$, then $\sigma(S_1) = \sigma(S_2)$.

Proof: This is direct conclusion of [27] and Theorem 2.1.1.

Remark 2.1.4: Corollary 2.1.3 is a generalization of S. Clary's result. The proofs are totally different.

With the help of a few more lemmas we can prove our main results.

Lemma 2.1.5. Suppose $T \in \mathcal{L}(\mathcal{H}_0)$ has no eigenvalues and $\lambda_0 \in \sigma(T) \setminus \sigma_e(T)$. Let $n = -ind(T - \lambda_0)$. Then there exists a constant $\delta > 0$ such that

(a) $O(\lambda_0,\ \delta) \subset \sigma(T) \setminus \sigma_e(T)$;

(b) For each $j = 1, 2, ..., n$, there exists a conjugate analytic vector-valued function k_λ^j on the open disc $O(\lambda_0,\ \delta)$ (i.e., $(f,\ k_\lambda^j)$ is an analytic function in λ for each f in $\mathcal{H}_0$) ;

(c) $\{k_\lambda^1, ..., k_\lambda^n\}$ is linearly independent set for each $\lambda \in O(\lambda_0, \delta)$;

(d) $(T - \lambda)^* k_\lambda^j = 0$, for each $j = 1, 2, ..., n$ and for all $\lambda \in O(\lambda_0,\ \delta)$.

(See Cowen and Douglas [12], [8] or [9].)

Lemma 2.1.6. Suppose $T \in \mathcal{L}(\mathcal{H}_0)$ has no eigenvalues, $\lambda_0 \in \sigma(T) \setminus \sigma_e(T)$. We use the notation and results of Lemma 2.1.5 and we set $f_1 = k_{\lambda_0}^1$,...,$f_n = k_{\lambda_0}^n$. There exists a positive constant δ_0 with $\delta_0 < \delta$ such that for every $f \in \mathcal{H}_0$, there exist bounded analytic functions $p_i^f(\lambda)$ on $O(\lambda_0, \delta_0)$ for $i = 1, 2, ...n$, and an analytic vector-valued function g_λ^f on $O(\lambda_0, \delta_0)$ satisfying

$$f \ - \ \sum_{i=1}^{n} p_i^f(\lambda) f_i \ = \ (T - \lambda) g_\lambda^f.$$

(see [36, p.434-436].)

Lemma 2.1.7. Let T be a linear bounded operator without eigenvalues acting on a Hilbert space $\mathcal{H}_0$, let S be a subdecomposable operator on $\mathcal{H}$ and let D be a decomposable extension of S acting on $\mathcal{K}$. Suppose X is a linear bounded operator from $\mathcal{H}_0$ to $\mathcal{H}$ such that $XT = SX$. Let $\lambda_0 \in \sigma(T) \setminus \sigma_e(T)$ and choose δ_0 as specified in Lemma 2.1.6. Set $U_1 = O(\lambda_0, \frac{\delta_0}{2})$ and $U_2 = \overline{O(\lambda_0, \frac{\delta_0}{4})}^c$. There are two invariant subspaces $\mathcal{K}_1$ and $\mathcal{K}_2$ of D such that

(a) $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$;

(b) $\sigma(D|_{\mathcal{K}_1}) \subset U_1$ and $\sigma(D|_{\mathcal{K}_2}) \subset U_2$.

Then there is a positive constant $M > 0$ such that for each $f \in \mathcal{H}_0$,

$$\|[Xf]_{\mathcal{K}_2,\mathcal{K}}\| \ \leq \ M \sup_{\substack{1 \leq i \leq n \\ \lambda \in O(\lambda_0, \delta_0)}} |p_i^f(\lambda)|$$

Proof: Looking at the proof of Theorem 2.1.1, we have $\sigma([D]_{\mathcal{K}_2,\mathcal{K}}) \subset U_1$. Using Lemma 2.1.6, we get

$$Xf \ - \ \sum_{i=1}^{n} p_i^f(\lambda) X f_i \ = \ X(T - \lambda) g_\lambda^f \ = \ (S - \lambda) X g_\lambda^f.$$

So

$$[Xf]_{\mathcal{K}_2,\mathcal{K}} \ - \ \sum_{i=1}^{n} p_i^f(\lambda) [X f_i]_{\mathcal{K}_2,\mathcal{K}} \ = \ ([D]_{\mathcal{K}_2,\mathcal{K}} - \lambda) [X g_\lambda^f]_{\mathcal{K}_2,\mathcal{K}}.$$

Since $[Xg_\lambda^f]_{\mathcal{K}_2,\mathcal{K}}$ is an analytic vector-valued function on $O(\lambda_0, \delta_0)$, it follows that

$$\frac{1}{2\pi i} \int_{\partial O(\lambda_0 \ , \ \delta_0)} ([D]_{\mathcal{K}_2,\mathcal{K}} \ - \ \lambda)^{-1} ([Xf]_{\mathcal{K}_2,\mathcal{K}} \ - \ \sum_{i=1}^{n} p_i^f(\lambda) [X f_i]_{\mathcal{K}_2,\mathcal{K}}) d\lambda$$
$$= \frac{1}{2\pi i} \int_{\partial O(\lambda_0 \ , \ \delta_0)} [X g_\lambda^f]_{\mathcal{K}_2,\mathcal{K}} d\lambda$$
$$= 0.$$

Thus,

$$[Xf]_{\mathcal{K}_2,\mathcal{K}} = \frac{1}{2\pi i}\int_{\partial O(\lambda_0\ ,\ \delta_0)}([D]_{\mathcal{K}_2,\mathcal{K}} - \lambda)^{-1}\sum_{i=1}^{n}(p_i^f(\lambda)[Xf_i]_{\mathcal{K}_2,\mathcal{K}})d\lambda.$$

Therefore,

$$\begin{aligned}\|[Xf]_{\mathcal{K}_2,\mathcal{K}}\| &\leq \frac{1}{2\pi}\int_{\partial O(\lambda_0\ ,\ \delta_0)}\|([D]_{\mathcal{K}_2,\mathcal{K}} - \lambda)^{-1}\sum_{i=1}^{n}p_i^f(\lambda)[Xf_i]_{\mathcal{K}_2,\mathcal{K}}\||d\lambda| \\ &\leq (\frac{1}{2\pi}\max\{\|([D]_{\mathcal{K}_2,\mathcal{K}} - \lambda)^{-1}\|, \quad \lambda\in\partial O(\lambda_0,\delta_0)\} \\ &\quad \sum_{i=1}^{n}\|[Xf_i]_{\mathcal{K}_2,\mathcal{K}}\|) \sup_{\substack{1\leq i\leq n \\ \lambda\in O(\lambda_0,\delta_0)}} |p_i^f(\lambda)|.\end{aligned}$$

Let

$$M = \frac{1}{2\pi}max\{\|([D]_{\mathcal{K}_2,\mathcal{K}} - \lambda)^{-1}\|, \quad \lambda\in\partial O(\lambda_0,\delta_0)\}\sum_{i=1}^{n}\|[Xf_i]_{\mathcal{K}_2,\mathcal{K}}\|,$$

it follows that

$$\|[Xf]_{\mathcal{K}_2,\mathcal{K}}\| \leq M \sup_{\substack{1\leq i\leq n \\ \lambda\in O(\lambda_0,\delta_0)}} |p_i^f(\lambda)|.$$

The proof is completed.

Theorem 2.1.8. Let $T\in\mathcal{L}(\mathcal{H}_0)$ have no eigenvalues and $S\in\mathcal{L}(\mathcal{H})$ be a subdecomposable operator. Suppose that X and Y are two linear bounded operators from $\mathcal{H}_0$ to $\mathcal{H}$ and $\mathcal{H}$ to $\mathcal{H}_0$, respectively, with both operators having dense ranges. If

$$XT = SX, \quad YS = TY,$$

then

$$\sigma(T)\setminus\sigma_e(T)\subset\sigma(S)\setminus\sigma_e(S)$$

Proof: Let $\lambda_0\in\sigma(T)\setminus\sigma_e(T)$. Using the same proof of Theorem 1 of [36, p.438-439], we see that there are $\delta_0>0$ and $M_1>0$ so that

$$O(\lambda_0\ ,\ \delta_0)\subset\sigma(T)\setminus\sigma_e(T);$$

for each $f \in \mathcal{H}_0$

$$f - \sum_{i=1}^{n} p_i^f(\lambda) f_i = (T - \lambda) g_\lambda^f;$$

and

$$\sup_{\substack{1 \le i \le n \\ \lambda \in O(\lambda_0, \delta_0)}} |p_i^f(\lambda)| \le M \|(S - \lambda_0) X f\|.$$

Suppose $\mathcal{K}_1$, $\mathcal{K}_2$ and U_1, U_2 are as in Lemma 2.1.7. From the inclusion $\sigma(D|_{\mathcal{K}_2}) \subset U_2$, we can find a constant $M_2 > 0$ such that for each $g \in \mathcal{K}_2$

$$\|g\| \le M_2 \|(D - \lambda_0) g\|.$$

So for each g in $\mathcal{K}_2$ and each $f \in \mathcal{H}_0$, we have

$$\begin{aligned} \|Xf\| &\le \|Xf + g\| + \|g\| \\ &\le \|Xf + g\| + M_2 \|(D - \lambda_0) g\| \\ &\le \|Xf + g\| + M_2 (\|(D - \lambda_0) X f\| + \|D - \lambda_0\| \|Xf + g\|) \\ &= (1 + M_2 \|D - \lambda_0\|) \|Xf + g\| + M_2 \|(D - \lambda_0) X f\|. \end{aligned}$$

Hence,

$$\|Xf\| \le (1 + M_2 \|D - \lambda_0\|) \|[Xf]_{\mathcal{K}_2, \mathcal{K}}\| + M_2 \|(S - \lambda_0) X f\|$$

where $DXf = SXf$. Using Lemma 2.1.7, we have

$$\begin{aligned} \|Xf\| &\le (1 + M_2 \|D - \lambda_0\|) M \sup_{\substack{1 \le i \le n \\ \lambda \in O(\lambda_0, \delta_0)}} |p_i^f(\lambda)| + M_2 \|(S - \lambda_0) X f\| \\ &\le (1 + M_2 \|D - \lambda_0\|) M M_1 \|(S - \lambda_0) X f\| + M_2 \|(S - \lambda_0) X f\|. \end{aligned}$$

Thus, there is $M > 0$ so that

$$\|Xf\| \le M \|(S - \lambda_0) X f\|.$$

This implies for each g in $\mathcal{H}$, we have

$$\|g\| \leq M\|(S - \lambda_0)g\|$$

because the range of X is dense in $\mathcal{H}$. So $\lambda_0 \in \sigma(S)\setminus\sigma_e(S)$. The theorem is proved.

Corollary 2.1.9. Suppose $T \in \mathcal{L}(\mathcal{H}_0)$ is a subdecomposable operator without eigenvalues and $S \in \mathcal{L}(\mathcal{H})$ is a subdecomposable operator. Suppose X and Y are as in Theorem 2.1.8 such that

$$XT = SX \quad \text{and} \quad YS = TY$$

we have

$$\sigma_e(T) = \sigma_e(S).$$

Proof: Combine the results of Theorem 2.1.1 and Theorem 2.1.8.

Theorem 2.1.10. Suppose $S_1 \in \mathcal{L}(\mathcal{H}_1)$ and $S_2 \in \mathcal{L}(\mathcal{H}_2)$ are two M-hyponormal operators and $S_1 \sim S_2$, then $\sigma_e(S_1) = \sigma_e(S_2)$.

Proof: By virtue of the inclusion $\sigma_p(S_1) \subset \overline{\sigma_p(S_1^*)}$ (complex conjugate), we can assume

$$S_1 = N_1 \oplus S_1', \qquad \text{and} \qquad S_2 = N_2 \oplus S_2'$$

where

$$N_1 = \lambda_1 I_1 \oplus ... \oplus \lambda_n I_n \oplus ...$$

$$N_2 = \lambda_1' I_1' \oplus ... \oplus \lambda_n' I_n' \oplus ...$$

and $\{\lambda_i\}$, $\{\lambda_i'\}$ are the eigenvalues of S_1 and S_2, respectively. We now write

$$X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}$$

with respect to the decompositions of S_1 and S_2. It is easy to show that $X_3 = 0$, $Y_3 = 0$ and N_1 is unitary equivalent to N_2. Therefore, X_4 and Y_4 have dense ranges, $X_4 S'_1 = S'_2 X_4$, and $S'_1 Y_4 = Y_4 S'_2$. Using Corollary 2.1.9, we conclude $\sigma_e(S'_1) = \sigma_e(S'_2)$. Thus,

$$\sigma_e(S_1) = \sigma_e(S_2).$$

Remark 2.1.11: As an application, we are able to answer S.Clary and J.Conway's question affirmatively. That is, two quasisimilar hyponormal operators have equal essential spectra.

Corollary 2.1.12. Suppose S_1 and S_2 are essential normal, M-hyponormal operators which are quasisimilar, then S_1 and S_2 are essentially unitary equivalent.

Proof: See [6, p.63].

Section 2.2. Bounded point evaluations

Let T be a linear bounded operator on Hilbert space $\mathcal{H}$. Recall that T is cyclic if there is a vector x in $\mathcal{H}$ so that span$\{p(T)x\}$ is dense in $\mathcal{H}$ where p stands for a polynomial. Suppose T is cyclic, a point λ is called a bounded point evaluation (bpe) for T if there is a constant $M > 0$ so that

$$|p(\lambda)| \leq M\|p(T)x\|$$

for every polynomial p. A point λ is called an analytic bounded point evaluation (abpe) if there exists a open subset U containing λ so that the above inequality holds for all points in U.

T.Trent [32] has shown that the set $\sigma(S) \setminus \sigma_e(S)$ for a pure cyclic subnormal operator S is precisely the set of abpes. In the section, we will generalize Trent's result to subdecomposable cyclic operators.

Theorem 2.2.1. Let T be a subdecomposable cyclic operator on $\mathcal{H}$ with no eigenvalues, then the set $\sigma(T) \setminus \sigma_e(T)$ is precisely the set of abpes for T.

Proof: Suppose $\lambda_0 \in \sigma(T) \setminus \sigma_e(T)$. There is a neighborhood U of λ_0 contained in $\sigma(T) \setminus \sigma_e(T)$ there is a reproducing kernel k_λ on U so that $T^* k_\lambda = \overline{\lambda} k_\lambda$ (Lemma 2.1.5). Suppose that x is a cyclic vector for T. Clearly $(x\ ,\ k_\lambda) \neq 0$, for all $\lambda \in U$. There is a constant $M > 0$ and a neighborhood U_0 of λ_0 so that

$$\left\| \frac{k_\lambda}{(x\ ,\ k_\lambda)} \right\| \leq M, \qquad \text{for all } \lambda \in U_0.$$

Therefore, for each polynomial p, we have that

$$|p(\lambda)| \leq M\|p(T)x\|$$

for each point λ in U_0 because of the fact

$$p(\lambda) = \frac{(p(T)x,\ k_\lambda)}{(x,\ k_\lambda)}.$$

This means that λ_0 is an analytic bounded point evaluation.

Suppose that λ_0 is an abpe, then there is a constant $M > 0$, so that

$$|p(\lambda)| \leq M\|p(T)x\|$$

for each λ in a neighborhood U of λ_0. On the other hand, we know that

$$p(T) - p(\lambda) = (T - \lambda)g_\lambda$$

where $g_\lambda = \left[\dfrac{p(z) - p(\lambda)}{(z - \lambda)}\right](T)$. Using the same proof of as in Theorem 2.1.8, we have

$$\begin{aligned}\|p(T)x\| &\leq M(\|(T - \lambda_0)p(T)x\| + \sup_{\lambda \in U}|p(\lambda)|) \\ &\leq (M + \frac{M}{\delta})\|(T - \lambda_0)P(T)x\|.\end{aligned}$$

Hence, λ_0 is in $\sigma(T) \setminus \sigma_e(T)$.

Corollary 2.2.2. Suppose T is a pure cyclic hyponormal operator with cyclic vector x on $\mathcal{H}$. Then the following conditions are equivalent:

(1) λ_0 is an abpe;

(2) $\lambda_0 \in \sigma(T) \setminus \sigma_e(T)$;

(3) There is a neighborhood U of λ_0 such that for each $\lambda \in U$ there exists a nonzero vector k_λ in $\mathcal{H}$ having the properties that

$$(T - \lambda)^* k_\lambda = 0$$

and

$$(p(T)x \ , \ k_\lambda) = p(\lambda)$$

for every polynomial p. In the case, for each $f \in \mathcal{H}$, $(f, \ k_\lambda)$ is an analytic function on U.

Suppose $\pi : H^{\infty}(D) \to \mathcal{L}(\mathcal{H})$ is an isometric isomorphism so that $\pi(1) = I$ and $\pi(\chi) = T$ where T is a contraction operator. S.Brown and B. Chevreau [5] showed that T has a lot of full analytic subspaces (first introduced by Olin and Thomson [25], and recently J.Thomson [31] showed that every invariant cyclic subspace of a subnormal operator is a kind of full analytic subspace). As an application of the last result, we have the following conclusion.

Corollary 2.2.3. Let π and T be as above. If T is pure hyponormal and M is a cyclic full analytic subspace of T, then

$$\sigma(T|_M) = \overline{D}$$

and

$$\sigma_e(T|_M) = \partial D.$$

Olin and Thomson [25] used Corollary 2.2.3 together with the construction of full analytic subspaces to characterize the class of cyclic, cellular-indecomposable, subnormal operators. So naturally we have the following problem:

Open Problem: Can the cyclic, cellular-indecomposable, hyponormal operators satisfying the hypotheses of Corollary 2.2.3 be characterized?

Section 2.3. Representations of hyponormal operators

From now on, we fix a pure hyponormal operator T on $\mathcal{H}$, the subdecomposable operator S on $\mathcal{H}_0$, and the invertible operator M from $\mathcal{H}$ to $\mathcal{H}_0$ such that

$$T = M^{-1}SM.$$

Let D be a decomposable extension of S on $\mathcal{K}$ and let $\{U_1 , U_2\}$ be an open cover of $\sigma(D)$ such that $\overline{U}_1 \subset G$. Let $\mathcal{K}_1$ and $\mathcal{K}_2$ denote two invariant subspaces of D such that

(1) $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2$;

(2) $\sigma(D|_{\mathcal{K}_1}) \subset U_1$ and $\sigma(D|_{\mathcal{K}_2}) \subset U_2$

Lemma 2.3.1. Let G be an bounded domain on $\mathcal{C}$ and let π be a unital norm continuous representation from $H^\infty(G)$ to $\mathcal{L}(\mathcal{H})$ such that $\pi(\chi) = T$. Suppose T is a pure hyponormal operator having the properties stated immediately before this lemma, then

$$[M\pi(f)M^{-1}x]_{\mathcal{K}_2,\mathcal{K}} = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - [D]_{\mathcal{K}_2,\mathcal{K}})^{-1}[x]_{\mathcal{K}_2,\mathcal{K}}d\lambda$$

for every $x \in \mathcal{H}_0$, where Γ is a Jordan curve in G surrounding $\overline{U_1}$.

Proof: Let $\pi_1 = M\pi M^{-1}$, then $\pi_1(\chi) = S$ and π_1 is a unital continuous representation from $H^\infty(G)$ to $\mathcal{L}(\mathcal{H}_\prime)$ such that

$$\begin{aligned}\pi_1(f) - f(\lambda)I &= \pi_1((z - \lambda)f_\lambda)\\ &= (S - \lambda)\pi_1(f_\lambda)\\ &= (D - \lambda)\pi_1(f_\lambda),\end{aligned}$$

where

$$f_\lambda(z) = \begin{cases}\dfrac{f(z) - f(\lambda)}{z - \lambda}, & z \neq \lambda\\ f'(\lambda) & z = \lambda.\end{cases}$$

It is easy to show that $f_\lambda(z)$ is a vector-valued analytic function with respect to λ. In fact, if $\lambda_0 \in G$, define

$$f_0(z) = \begin{cases} \dfrac{f(z) - f(\lambda_0)}{z - \lambda_0}, & z \neq \lambda_0 \\ f'(\lambda_0), & z = \lambda_0 \end{cases}$$

$$f_1(z) = \begin{cases} \dfrac{f(z) - f(\lambda_0) - f'(\lambda_0)(z - \lambda_0)}{(z - \lambda_0)^2}, & z \neq \lambda_0 \\ \dfrac{1}{2!} f''(\lambda_0), & z = \lambda_0 \end{cases}$$

$$f_2(z) = \begin{cases} \dfrac{f(z) - f(\lambda_0) - f'(\lambda_0)(z - \lambda_0) - \frac{1}{2!}f''(\lambda_0)(z - \lambda_0)^2}{(z - \lambda_0)^3}, & z \neq \lambda_0 \\ \dfrac{1}{3!} f'''(\lambda_0), & z = \lambda_0 \end{cases}$$

$$\cdots$$

In a neighborhood of λ_0, f_λ can be written as

$$f_\lambda = f_0 + (\lambda - \lambda_0) f_1 + (\lambda - \lambda_0)^2 f_2 + \ldots$$

For every $x \in \mathcal{H}_0$, we have

$$[\pi_1(f)x]_{\mathcal{K}_2,\mathcal{K}} - f(\lambda)[x]_{\mathcal{K}_2,\mathcal{K}} = ([D]_{\mathcal{K}_2,\mathcal{K}} - \lambda I)[\pi_1(f_\lambda)x]_{\mathcal{K}_2,\mathcal{K}}$$

where $[\pi_1(f_\lambda)x]_{\mathcal{K}_2,\mathcal{K}}$ is a vector-valued analytic function with respect to λ. It follows from

$$\sigma([D]_{\mathcal{K}_2,\mathcal{K}}) \subset U_1$$

that

$$\frac{1}{2\pi i}\int_\Gamma (\lambda - [D]_{\mathcal{K}_2,\mathcal{K}})^{-1}([M\pi(f)M^{-1}x]_{\mathcal{K}_2,\mathcal{K}} - f(\lambda)[x]_{\mathcal{K}_2,\mathcal{K}})d\lambda = 0.$$

This implies

$$[M\pi(f)M^{-1}x]_{\mathcal{K}_2,\mathcal{K}} = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - [D]_{\mathcal{K}_2,\mathcal{K}})^{-1}[x]_{\mathcal{K}_\epsilon,\mathcal{K}} d\lambda.$$

The lemma is proved.

Corollary 2.3.2. Let T, π and G be as in Lemma 2.3.1. Let T_1 be a linear bounded operator on $\mathcal{H}_1$ and X be a linear bounded operator from $\mathcal{H}_1$ to $\mathcal{H}$, such that $TX = XT_1$. Suppose π_1 is a unital continuous representation from $H^\infty(G)$ to $\mathcal{L}(\mathcal{H}_1)$ such that $\pi_1(\chi) = T_1$. We then have

$$[M\pi(f)Xx]_{\mathcal{K}_2,\mathcal{K}} = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - [D]_{\mathcal{K}_2,\mathcal{K}})^{-1}[MXx]_{\mathcal{K}_2,\mathcal{K}}d\lambda,$$

and

$$[MX\pi_1(f)x]_{\mathcal{K}_2,\mathcal{K}} = \frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - [D]_{\mathcal{K}_2,\mathcal{K}})^{-1}[MXx]_{\mathcal{K}_2,\mathcal{K}}d\lambda,$$

for every $x \in \mathcal{H}_1$.

Proof: The first equality is a direct conclusion of Lemma 2.3.1. For the second one, we observe that

$$\begin{aligned} MX\pi_1(f) - f(\lambda)MX &= MX(T_1 - \lambda)\pi_1(f_\lambda) \\ &= M(T - \lambda)X\pi_1(f_\lambda) \\ &= (S - \lambda)MX\pi_1(f_\lambda) \\ &= (D - \lambda)MX\pi_1(f_\lambda). \end{aligned}$$

With this last fact, the same proof of Lemma 2.3.1 shows the second equality .

Theorem 2.3.3. Let G be a bounded domain with boundary having area measure zero. Let T, π, T_1, π_1, and X be as above such that $TX = XT_1$, then $\pi(f)X = X\pi_1(f)$ for every $f \in H^\infty(G)$.

Proof: For $x \in \mathcal{H}_1$, we construct the subspace

$$\mathcal{H}_x = \overline{span\{M\pi(f)Xx - MX\pi_1(f)M^{-1}x,\ f \in H^\infty(G)\}}.$$

Clearly $\mathcal{H}_x$ is an invariant subspace of S.

Let $\lambda_0 \in G$ and choose $\delta > 0$ such that $O(\lambda_0 \ , \ \delta) \subset G$. If U_1 and U_2 are as in Theorem 2.1.1, then $\{U_1 \ , \ U_2\}$ is an open cover of $\sigma(D)$. By Corollary 2.3.2, for every $x \in \mathcal{H}_1$, we get

$$
\begin{aligned}
&[M\pi(f)Xx]_{\mathcal{K}_2,\mathcal{K}} - [MX\pi_1(f)x]_{\mathcal{K}_2,\mathcal{K}} \\
=&\frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - [D]_{\mathcal{K}_\in,\mathcal{K}})^{-1}[MXx]_{\mathcal{K}_\in,\mathcal{K}} d\lambda - \\
&\frac{1}{2\pi i}\int_\Gamma f(\lambda)(\lambda - [D]_{\mathcal{K}_\in,\mathcal{K}})^{-1}[MXx]_{\mathcal{K}_\in,\mathcal{K}} d\lambda \\
=&0.
\end{aligned}
$$

This means

$$M\pi(f)Xx - MX\pi_1(f)x \in \mathcal{K}_2.$$

By the definition of $\mathcal{K}_2$, we see that there is a positive number N such that for every $y \in \mathcal{K}_2$ we have

$$\|y\| \ \leq \ N\|(D - \lambda_0)y\|.$$

Hence,

$$\|M\pi(f)Xx - MX\pi_1(f)x\| \leq N\|(S - \lambda_0)(M\pi(f)Xx - MX\pi_1(f)x)\|.$$

Therefore, there exists $N_0 > 0$ such that

$$\|\pi(f)Xx - X\pi_1(f)x\| \leq N_0\|(T - \lambda_0)(\pi(f)Xx - X\pi_1(f)x)\|.$$

Let

$$\mathcal{H}_{Xx} = span\{\pi(f)Xx - X\pi_1(f)x, \ f \in H^\infty(G)\overline{\}};$$

we know $T\mathcal{H}_{Xx} \subset \mathcal{H}_{Xx}$ and $(T - \lambda_0)|_{\mathcal{H}_{Xx}}$ is bounded below. We also have

$$
\begin{aligned}
&(\pi(f)X - X\pi_1(f))x \\
=&(\pi(f)X - f(\lambda_0)X - (X\pi_1(f) - f(\lambda_0)X))x \\
=&(T - \lambda_0)((\pi(f_{\lambda_0})X - X\pi_1(f_{\lambda_0}))x.
\end{aligned}
$$

Thus, the range of $(T - \lambda_0)|_{\mathcal{H}_{Xx}}$ is dense in $\mathcal{H}_{Xx}$. Hence,

$$\lambda_0 \notin \sigma(T|_{\mathcal{H}_{Xx}}).$$

It follows that

$$\sigma(T|_{\mathcal{H}_{Xx}}) \subset \partial G.$$

By hypotheses, the boundary of G has area measure zero and T is a pure hyponormal operator, so the subspace $\mathcal{H}_{Xx}$ has to be zero. (Otherwise, $T|_{\mathcal{H}_{Xx}}$ is a normal operator according to Putnam's inequality and it turns out that this operator becomes the normal part of T.) This is a contradiction to the purity of T. Hence, $\pi(f)X = X\pi_1(f)$. The theorem is now proved.

Corollary 2.3.4. Let T be a pure hyponormal operator on $\mathcal{H}$ and π_i be two unital continuous representations from $H^\infty(G)$ to $\mathcal{L}(\mathcal{H})$ such that $\pi_1(\chi) = \pi_2(\chi) = T$, then $\pi_1 = \pi_2$

Proof: Apply Theorem 2.3.3 to the case $X = I$.

Next, we discuss the weak star continuity of the representations. We assume G is a bounded open set whose boundary has area measure zero. Let $\pi^*(f) = \pi(f)^*$, then π^* is also a unital continuous representation with $\pi^*(\chi) = T^*$.

Theorem 2.3.5. The map π^* is weak-star, s.o.t, sequentially continuous. That is, if $\{f_n\}$ is a sequence in $H^\infty(G)$ that converges in weak star topology, then$\{\pi^*(f_n)\}$ converges in the strong topology, s.o.t.

Proof: Suppose not, we may assume that there is a sequence $\{f_n\}$ in $H^\infty(G)$ with $\|f_n\| \leq 1$ and there is $x \in \mathcal{H}$ with $\|x\| \leq 1$ such that

$$\|\pi^*(f_n)x\| \to a \neq 0.$$

Let $x_n = \pi^*(f_n)x$, by passing to a subsequence we can assume that $\{\pi(f_n)x_n\}$ converges to y weakly. We also have

$$(y,\ x) = \lim_{n\to\infty}(\pi(f_n)x_n,\ x) = a^2 \neq 0.$$

Hence, $y \neq 0$. Using [20] and [1], we can construct a linear bounded map Γ from $C(\partial G)$ to $\mathcal{H}$ such that

$$q(T)\Gamma(\frac{p}{q}) = p(T)y$$

where p and q are polynomials. It is easy to show

$$(T - \lambda)\Gamma(f) = \Gamma(f(\chi - \lambda))$$

for every $f \in C(\partial G)$ and $\lambda \in G$. Let $\Gamma_1(f) = M\Gamma(f)M^{-1}$, then

$$\begin{aligned}\Gamma_1(f) &= (S - \lambda)\Gamma_1(f/(\chi - \lambda)) \\ &= (D - \lambda)\Gamma_1(f/(\chi - \lambda)).\end{aligned}$$

Obviously $\Gamma_1(f/(\chi - \lambda))$ is an analytic, vector-valued function on G. Using the same notation and the same proof of Theorem 2.3.3, we have

$$[\Gamma_1(f)]_{\mathcal{K}_2,\mathcal{K}} = 0.$$

Therefore, for every $\lambda \in G$, there is a constant $M_\lambda > 0$ such that

$$\|\Gamma(f)\| \leq M_\lambda\|(T - \lambda)\Gamma(f)\|$$

for $\lambda \in G$. Also

$$\overline{Ran(T - \lambda)|_{\mathcal{H}_\Gamma}} = \mathcal{H}_\Gamma = \overline{\{\Gamma(f) : f \in C(\partial G)\}}.$$

Thus,

$$\sigma(T|_{\mathcal{H}_\Gamma}) \subset \partial G.$$

By the hypotheses, we know ∂G has area measure zero. So $T|_{\mathcal{H}_\Gamma}$ is a normal operator which contradicts the purity of T because $y \neq 0$ and $y \in \mathcal{H}_\Gamma$. The theorem is proved.

Corollary 2.3.6. Suppose T is a pure contraction hyponormal operator on $\mathcal{H}$, then $(T^*)^n \to 0$ s.o.t. That is, $T \in C_{.0}$.

Corollary 2.3.7. The map π is weak star continuous.

Proof: The same proof in [20] carries over.

CHAPTER III THE COMMUTANT OF MULTIPLICATION BY Z ON THE CLOSURE OF POLYNOMIALS IN $L^t(\mu)$

Section 3.1. The measure restricted to the boundary

In the chapter, the measures μ, ν and the Riemann maps ϕ, ψ will be as in the introduction. We begin with some preliminary lemmas.

Lemma 3.1.1. The operator S_ν is irreducible on $P^t(\nu)$ and $bpe(\nu) = D$.

Proof: For each $\lambda \in D$, we know $\phi^{-1}(\lambda) \in G$, hence, there exists a positive constant $M > 0$ so that

$$\begin{aligned} |p(\lambda)|^t &= |p \circ \phi(\phi^{-1}(\lambda))|^t \\ &\le M \int |p \circ \tilde{\phi}(z)|^t d\mu \\ &= M \int |p|^t d\mu \circ \tilde{\phi}^{-1} \\ &= M \int |p|^t d\nu \end{aligned} \tag{0}$$

Therefore, we have the inclusion $D \subset bpe(\nu)$.

By the Hahn-Banach and Riesz Representation Theorems for each λ in D there exists a function k_λ^ν in $L^q(\nu)$ such that

$$p(\lambda) = (p \ , \ k_\lambda^\nu)$$

Claim: If $f \in P^t(\nu)$ and $(f \ , \ k_\lambda^\nu) = 0$ for each λ in an open subset U of D, then $f = 0$.

To see the claim, suppose that U is an open subset of D and $f \in P^t(\nu)$ with $(f\ ,\ k_\lambda^\nu) = 0$ for each $\lambda \in U$. There exist polynomials p_n so that $p_n \to f$ in $P^t(\nu)$. So for $\lambda \in U$,

$$p_n(\lambda) \ = \ (p_n\ ,\ k_\lambda^\nu) \to (f\ ,\ k_\lambda^\nu) = 0.$$

Hence, p_n converges uniformly to zero in any compact subset of U (because for each compact subset of U, there exists a constant M such that

$$|p_n(\lambda)| \leq M$$

for λ in a neighborhood of the compact set). This implies that $p_n \circ \tilde{\phi}$ converges uniformly to zero in any compact subset of $\psi(U)$. By [31] Lemma 5.4, we know that $p_n \circ \tilde{\phi}$ converges to zero in $P^t(\mu)$. Using the change of variable formula

$$\int |p_n|^t d\nu \ = \ \int |p_n \circ \tilde{\phi}|^t d\mu,$$

we now conclude

$$\int |p_n|^t d\nu \to 0, \ n \to \infty.$$

Hence, $f \ = \ 0$.

Now suppose that S_ν is not pure, then there exist a nontrivial Borel partition $\{\Delta_0\ ,\ \Delta\}$ of the support of ν such that

$$P^t(\nu) \ = \ L^t(\nu|\Delta_0) \oplus P^t(\nu|\Delta)$$

and $P^t(\nu|\Delta)$ contains no L^t summand (see [31], Theorem 1.3). From duality theory we know

$$P^t(\nu)^* \ = \ L^q(\nu|\Delta_0) \oplus P^t(\nu|\Delta)^*$$

From the previous claim we know for any open subset U of D that $\text{span}\{k_\lambda^\nu\ ,\ \lambda \in U\}$ is dense in $P^t(\nu)^*$. Hence if P is the projection of $P^t(\nu)^*$ onto $L^q(\nu|\Delta_0)$, then there is some point $\lambda_0 \in U$ such that $Pk^{\nu}_{\lambda_0} \neq 0$. For any $f \in L^t(\nu|\Delta_0)$, we see

$$((z-\lambda_0)f\ ,\ k^{\nu}_{\lambda_0}) = ((z-\lambda_0)f\ ,\ Pk^{\nu}_{\lambda_0}) = 0$$

thus, it follows

$$\overline{(z-\lambda_0)}Pk^{\nu}_{\lambda_0} = 0.$$

Hence, $\nu\{\lambda_0\} \neq 0$ and $\lambda_0 \in \Delta_0$. Let f be the characteristic function of the point λ_0, then $f \in P^t(\nu)$ and $(z - \lambda_0)f = 0$. But for $\lambda \in U$ we have

$$((z-\lambda_0)f\ ,\ k^{\nu}_{\lambda}) = (\lambda - \lambda_0)(f\ ,\ k^{\nu}_{\lambda})$$

This means for $\lambda \neq \lambda_0$

$$(f, k^{\nu}_{\lambda}) = 0$$

Therefore, by the claim f has to be zero. This is a contradiction. So S_ν is pure. The irreducibliy of S_ν easily follows from Thomson's theorem. Using Thomson's theorem again, we see $bpe(\nu) = D$.

By our map $\sim$, we know that $\tilde{\psi}$ is in $P^t(\nu) \cap L^\infty(\nu)$. Let $\mu_0 = \nu \circ \tilde{\psi}^{-1}$, we want to show $\mu_0 = \mu$. Before we prove that, we need the following lemma.

Lemma 3.1.2. Let U be a simply connected region containing G and f be a Riemann map of U onto D, then $|\tilde{f}(z)| = 1$ a.e. with respect to $\mu|_{\partial U}$.

Proof: Fix a point $\lambda \in U$. Define a function g as follows:

$$g(z) = \begin{cases} \dfrac{f(z) - f(\lambda)}{z - \lambda}, & z \neq \lambda \\ f'(\lambda), & z = \lambda \end{cases}$$

clearly g is in $H^\infty(G)$ and it is easy to check $\tilde{g}(z)(z - \lambda) = \tilde{f}(z) - f(\lambda)$. Since f is a Riemann mapping, g is bounded below in G. Hence, g is invertible in $H^\infty(G)$ and by our isomorphism $\sim$ we know that $\tilde{g}$ is invertible in $P^t(\mu) \cap L^\infty(\mu)$. Choose a positive constant c_λ such that $|\tilde{g}(z)| \geq c_\lambda$ a.e. with respect to μ. Therefore,

$$|\tilde{f}(z) - f(\lambda)| \geq c_\lambda|z - \lambda| \qquad a.e.\ \mu$$

Now suppose there is a Borel set $E \subset \partial G \cap \partial U$ such that $\mu(E) > 0$ and

$$|\tilde{f}(z)| \;<\; 1 \quad \text{on} \;\; E.$$

We can assume E is a compact subset and $\tilde{f}|_E$ is a continuous function. From Thomson's theorem, we see $\mu|_{\partial G}$ has no atoms. Therefore, we can assume there is a point $z_0 \in E$ and

$$\mu(B(z_0, \frac{1}{n}) \cap E \setminus \{z_0\}) \;>\; 0$$

Choose $\lambda_0 \in U$ such that $f(\lambda_0) \;=\; \tilde{f}(z_0)$. There exists a constant $c > 0$ such that

$$|\tilde{f}(z) \;-\; f(\lambda_0)| \;>\; c|z \;-\; \lambda_0| \qquad a.e.\ \mu$$

We can find points $z_n \in B(z_0, \frac{1}{n}) \cap E$ such that

$$|\tilde{f}(z_n) \;-\; f(\lambda_0)| \;>\; c|z_n \;-\; \lambda_0| \qquad a.e.\ \mu$$

Taking the limit of both sides of last inequality, we see that z_0 has to be λ_0. This is a blatant contradiction that $U \cap E = \emptyset$. Hence,

$$|\tilde{f}(z)| \;=\; 1 \qquad a.e.\ \mu|_{\partial G \cap \partial U}$$

Notice that $\mu(\partial U \setminus \partial G) \;=\; 0$. The lemma is proved.

Proposition 3.1.3. Let μ_0 and μ as above, then $\mu \;=\; \mu_0$.

Proof : It is obvious that $\mu|_G$ equals $\mu_0|_G$. Therefore, we only need to show $\mu|_{\partial G} \;=\; \mu_0|_{\partial G}$. For every polynomial p, clearly $p \circ \tilde{\psi}$ is in $P^t(\nu)$. There exists a sequence of polynomials $\{p_n\}$ such that

$$\int |p_n \;-\; p \circ \tilde{\psi}|^t d\nu \to 0, \qquad \text{as } n \to \infty.$$

On the other hand,

$$\int |p_n \;-\; p_m|^t d\nu \;=\; \int |p_n \circ \tilde{\phi} \;-\; p_m \circ \tilde{\phi}|^t d\mu$$

So $\{p_n \circ \tilde{\phi}\}$ is a Cauchy sequence of $P^t(\mu)$. This implies that $p_n \circ \tilde{\phi}$ converges to an element $\tilde{f}$ in $P^t(\mu)$. For $\lambda \in G$,

$$\begin{aligned}
(\tilde{f} \ , \ k_\lambda^\mu) &= \lim_{n \to \infty} (p_n \circ \tilde{\phi} \ , \ k_\lambda^\mu) \\
&= \lim_{n \to \infty} p_n(\phi(\lambda)) \\
&= \lim_{n \to \infty} (p_n \ , \ k_{\phi(\lambda)}^\nu) \\
&= p(\psi \circ \phi(\lambda)) \\
&= p(\lambda) \\
&= (p \ , \ k_\lambda^\mu).
\end{aligned}$$

Hence, $\tilde{f} = p$ because span $\{k_\lambda^\mu \ , \ \lambda \in G\}$ is dense in $P^t(\mu)^*$. Therefore,

$$\begin{aligned}
\int |p|^t d\mu_0 &= \int |p \circ \tilde{\psi}|^t d\nu \\
&= \lim_{n \to \infty} \int |p_n|^t d\nu \\
&= \lim_{n \to \infty} \int |p_n \circ \tilde{\phi}|^t d\mu \\
&= \int |p|^t d\mu. \qquad (1)
\end{aligned}$$

Therefore, the operator V defined by $Vp = p$ from $P^t(\mu)$ to $P^t(\mu_0)$ is an isometric invertible operator. Hence, S_{μ_0} is pure (also, from the proof of Lemma 3.1.1, we can see that) and $bpe(\mu_0) = G$. Let U be a simply connected region containing G and f be the Riemann map from U to D, then the properties of the mappings $\sim_1$, $\sim_2$, we know that $\tilde{f}^1$ is in $P^t(\mu) \cap L^\infty(\mu)$ and $\tilde{f}^2$ is in $P^t(\mu_0) \cap L^\infty(\mu_0)$. Moreover, by (1) we have for each polynomial p the following equality:

$$\int |p(\tilde{f}^2)^n|^t d\mu_0 = \int |p(\tilde{f}^1)^n|^t d\mu \qquad (2)$$

Now letting n tend to infinity on both sides of (2), and applying Lemma 3.1.2, we get

$$\int_{\partial U} |p|^t d\mu_0 = \int_{\partial U} |p|^t d\mu \tag{3}$$

Now let W be the union of the interior of the polynomial convex hull of $\overline{G}$ and a small disc $B(\lambda\ ,\ c)$ with λ on the boundary of the polynomial convex hull of $\overline{G}$. Clearly $\mu_0(\partial W \setminus \partial\overline{G}) = \mu(\partial W \setminus \partial\overline{G}) = 0$; so we have

$$\int_{\partial W \cap \partial\overline{G}} |p|^t d\mu_0 = \int_{\partial W \cap \partial\overline{G}} |p|^t d\mu \tag{4}$$

Since

$$P(\partial W \cap \partial\overline{G}) = C(\partial W \cap \partial\overline{G}),$$

(see [25, p.219] or [9, p.223]) we have (using (4)) that for every g in $C(\partial W \cap \partial\overline{G})$,

$$\int_{\partial W \cap \partial\overline{G}} |g|^t d\mu_0 = \int_{\partial W \cap \partial\overline{G}} |g|^t d\mu$$

Hence, $\mu_0|_{\partial W \cap \partial\overline{G}} = \mu|_{\partial W \cap \partial\overline{G}}$. Since (3) holds for all simply connected regions U containing G, we have

$$\int_{\partial\hat{\overline{G}}} |p|^t d\mu_0 = \int_{\partial\hat{\overline{G}}} |p|^t d\mu$$

where $\hat{A}$ is the polynomial convex hull of A. Therefore, from (4) and our last equality,, we get that

$$\int_{\partial\hat{\overline{G}} \setminus \partial W} |p|^t d\mu_0 = \int_{\partial\hat{\overline{G}} \setminus \partial W} |p|^t d\mu$$

Using the same reasoning as above, where

$$P(cl(\partial\hat{\overline{G}} \setminus \partial W)) = C(cl(\partial\hat{\overline{G}} \setminus \partial W))$$

This implies

$$\int\limits_{\partial\hat{\overline{G}}\setminus\partial W} |g|^t d\mu_0 = \int\limits_{\partial\hat{\overline{G}}\setminus\partial W} |g|^t d\mu$$

for every continuous function g on the set over which we last integrated. Therefore,

$$\mu_0|_{\partial\hat{\overline{G}}\setminus\partial W} = \mu|_{\partial\hat{\overline{G}}\setminus\partial W}$$

Hence, μ_0 restricted to the outer boundary of G equals μ restricted to the outer boundary of G. Let Γ be another component of the boundary of $\mathcal{C} \setminus \overline{G}$ and let $U = \mathcal{C} \cup \{\infty\} \setminus \hat{\Gamma}$. For λ in Γ, choose $B(\lambda\ ,\ c)$ a sufficiently small disc so that for every polynomial p we have

$$\int\limits_{\partial(U\cup B(\lambda,c))} |p|^t d\mu_0 = \int\limits_{\partial(U\cup B(\lambda,c))} |p|^t d\mu$$

Therefore, for all polynomials p

$$\int\limits_{\Gamma\setminus B(\lambda,c)} |p|^t d\mu_0 = \int\limits_{\Gamma\setminus B(\lambda,c)} |p|^t d\mu$$

Since

$$P(\Gamma \setminus B(\lambda, c)) = C(\Gamma \setminus B(\lambda, c)),$$

we can conclude that μ_0 restricted to $\Gamma \setminus B(\lambda, c)$ agrees to μ restricted to $\Gamma \setminus B(\lambda, c)$. Using the same argument as above , we can prove that μ_0 restricted to Γ equals μ restricted to Γ. Therefore,

$$\mu_0|_{\partial\overline{G}} = \mu|_{\partial\overline{G}}$$

Let K be the closure of a component of $\partial G \setminus \partial\overline{G}$ and let $U = \mathcal{C} \cup \{\infty\} \setminus K$. Clearly U is a simply connected region which contains G. So from (3), we have

$$\int\limits_K |p|^t d\mu_0 = \int\limits_K |p|^t d\mu$$

This implies $\mu_0|_K = \mu|_K$ because again we have that $P(K) = C(K)$. Therefore, $\mu_0 = \mu$.

Remark: In the case $t = 2$, the proof of this proposition is very easy by using the fact that if two subnormal operators are unitarily equivalent, then their minimal normal extensions are unitarily equivalent (see [9, p.38]).

The next theorem shows how S_μ on $P^t(\mu)$ can be pulled back to the disc as the operator $T^\nu_{\tilde{\psi}}$, multiplication by $\tilde{\psi}$, on $P^t(\nu)$.

Theorem 3.1.4. There is an isometry U from $P^t(\mu)$ onto $P^t(\nu)$ such that

$$US_\mu U^{-1} = T^\nu_{\tilde{\psi}}$$

Proof: Define U: $P^t(\mu_0) = P^t(\mu) \to P^t(\nu)$ by

$$Up = p \circ \tilde{\psi}$$

for all polynomials p. Note that this is an isometric map and then extend U to an isometry on all of $P^t(\nu)$. We only need to show the range of U is $P^t(\nu)$. For each polynomial p, the function $p \circ \phi \in H^\infty(G)$, so $p \circ \tilde{\phi}$ is in $P^t(\mu) \cap L^\infty(\mu)$. Choose a sequence of polynomials $\{p_n\}$ such that p_n converges to $p \circ \tilde{\phi}$ in $P^t(\mu)$. Since

$$\begin{aligned} \int |p_n - p_m|^t d\mu &= \int |p_n - p_m|^t d\mu_0 \\ &= \int |p_n \circ \tilde{\psi} - p_m \circ \tilde{\psi}|^t d\nu, \end{aligned}$$

we see that $\{p_n \circ \tilde{\psi}\}$ is a Cauchy sequence in $P^t(\nu)$. There exists $\tilde{g} \in P^t(\nu)$ so that $p_n \circ \tilde{\psi}$ converges to $\tilde{g}$ in $P^t(\nu)$. It is easy to show $\tilde{g} = p$ (by using the same method in the proof of last proposition). Hence, $p_n \circ \tilde{\psi}$ converges to p in $P^t(\nu)$. We already know that $RanU$ is closed, so the range of U is $P^t(\nu)$. Thus U maps $P^t(\mu)$ isometrically onto $P^t(\nu)$. It is easy to check that $US_\mu U^{-1} = T^\nu_{\tilde{\psi}}$.

For f in $H^\infty(D)$, we can define the nontangential limit of f in the customary way:

$$f^*(e^{i\theta}) = \lim_{r\to 1-} f(re^{i\theta}) \quad a.e.\ m$$

Where m is normalized Lebesgue measure on ∂D (see [13], [17] or [18]). By virtue of Lemma 3.1.1 and [9, p.301], we know that ν restricted to ∂D is absolutely continuous respect to m.

Lemma 3.1.5. For f in $H^\infty(D)$ and $\tilde{f}$ in $P^t(\nu)\cap L^\infty(\nu)$, we have

$$f^*(e^{i\theta}) = \tilde{f}(e^{i\theta}) \qquad a.e.\ \nu|_{\partial D}.$$

Proof: Let $\tilde{\nu}$ be the sweep of the measure of ν. Since S_ν is irreducible - hence, $\nu|_{\partial D}$ is absolutely continuous with respect to m - it follows that $\tilde{\nu}$ is absolutely continuous respect to m. For p a polynomial, $|p|^t$ is a subharmonic function for t ≥ 1; thus

$$\int |p|^t d\nu \leq \int |p|^t d\tilde{\nu}.$$

For each $\lambda \in D$, there exists a constant M such that

$$|p(\lambda)|^t \leq M\int |p|^t d\nu \leq M\int |p|^t d\tilde{\nu}. \tag{6}$$

So there exists $k_\lambda^{\tilde{\nu}}$ in $P^t(\tilde{\nu})^*$ ($\subset L^q(\tilde{\nu})$) such that

$$p(\lambda) = (p\ ,\ k_\lambda^{\tilde{\nu}}).$$

We now claim that

$$\int log\frac{d\tilde{\nu}}{dm} > -\infty$$

In fact, if not, by Szego's theorem (see [15, p.136]), there is a sequence of polynomials $\{p_n\}$ such that $p_n(0) = 0$ and

$$\int |1 - p_n|^t d\tilde{\nu} \to 0, \qquad \text{as } n\to\infty \tag{7}$$

Now, on one hand we know that

$$(1 - p_n , k_0^{\tilde{\nu}}) = 1 - 0 = 1.$$

On the other hand, using (7) and (6) we have

$$(1 - p_n , k_0^{\tilde{\nu}}) \to 0 \qquad \text{as } n \to \infty.$$

This is a contradiction. Hence, we can find an outer function $g \in H^t(\partial D)$ so that $\dfrac{d\tilde{\nu}}{dm} = |g|^t$ and

$$\int |p|^t d\nu \leq \int |p|^t d\tilde{\nu} = \int |p|^t |g|^t dm \tag{8}$$

Let $f \in H^\infty(D)$, Choose a sequence of polynomials $\{p_n\}$ so that $p_n g \to f^* g$ in $H^t(\partial D)$.

(Note: the closure of $\{p_n g\}$ in $H^t(\partial D)$ contains $H^\infty(\partial D)g$). From (8), we see that $\{p_n\}$ is a Cauchy sequence in $P^t(\nu)$. Consequently, for each $\lambda \in D$

$$\begin{aligned}(\tilde{f} , k_\lambda^\nu) &= f(\lambda) \\ &= \lim_{n\to\infty} p_n(\lambda) \\ &= \lim_{n\to\infty} (p_n , k_\lambda^\nu) \\ &= (\lim_{n\to\infty} p_n , k_\lambda^\nu).\end{aligned}$$

Thus $\{p_n\}$ converges to $\tilde{f}$ in $P^2(\nu)$. By passing to a subsequence if necessary, we have

$$p_n(e^{i\theta}) \to f^*(e^{i\theta}) \qquad a.e.\ m;$$

thus,

$$p_n(e^{i\theta}) \to f^*(e^{i\theta}) \qquad a.e.\ \nu|_{\partial D}.$$

Thus, $\tilde{f}(e^{i\theta}) = f^*(e^{i\theta})$ $a.e.$ $\nu|_{\partial D}$.

The following theorem is one of our main theorems in this chapter.

Theorem 3.1.6. Let μ , ν be as above, then $\mu|_{\partial G}$ is absolutely continuous to the harmonic measure ω on G and $\tilde{\psi}$ is one-to-one a.e. ν from a carrier of $\nu|_{\partial D}$ to a carrier of $\mu|_{\partial G}$. That is, there exists a Borel set $E_1 \subset \partial D$ such that

(1) $\nu(\partial D \setminus E_1) = 0$;

(2) $\tilde{\psi}|_{E_1}$ is one-to-one;

(3) $\tilde{\psi}(E_1) \subset \partial G$ and

$$\mu(\partial G \setminus \tilde{\psi}(E_1)) = 0.$$

Proof: Using previous lemmas, we have

$$\mu = \mu_0 = \nu \circ \tilde{\psi}^{-1},$$

thus,

$$\begin{aligned} \mu|_{\partial G} &= \nu \circ \tilde{\psi}^{-1}|_{\partial G} \\ &= \nu \circ \psi^{*-1}|_{\partial G} \\ &<< m \circ \psi^{*-1} \\ &= \omega \end{aligned}$$

For notational convenience let $\mu_1 = \mu|_{\partial G}$ and $\nu_1 = \nu|_{\partial D}$. Let $\{p_n\}$ be a sequence of polynomials such that $p_n \to \tilde{\phi}$ in $P^t(\mu)$. Observing the change of variable formula

$$\int |p_n - p_m|^t d\mu = \int |p_n \circ \tilde{\psi} - p_m \circ \tilde{\psi}|^t d\nu,$$

we conclude that $\{p_n \circ \tilde{\psi}\}$ is a Cauchy sequence in $P^t(\nu)$. An elementary argument yields that $p_n \circ \tilde{\psi}$ goes to z in $P^t(\nu)$.

Choose a subset $E_0 \subset \partial D$, a ν_1 null set, with the following properties: If

$$E = car\nu \setminus E_0$$

then

(a) E is σ-compact, say $E = \cup_{n=1}^{\infty} E_n$, where each E_n is closed;

(b) $\tilde{\psi}|_{E_n}$ is a continuous function.

Let $F = \cup_{n=1}^{\infty} \tilde{\psi}(E_n)$. Its complement is a μ_1 null set because

$$\begin{aligned}
\mu_1(F^c) &= \nu_1 \circ \tilde{\psi}^{-1}(F^c) \\
&= \nu_1 \left(\cup_{n=1}^{\infty} \tilde{\psi}^{-1}(\tilde{\psi}(E_n)) \right)^c \\
&\leq \nu_1(\cup_{n=1}^{\infty} E_n)^c \\
&= 0.
\end{aligned}$$

Choose a μ_1 null set $F_0 \subset F$ such that the sequence p_n (pass to a subsequence if necessary) converges pointwise to $\tilde{\phi}$ on $F \setminus F_0$. Letting $E_1 = \tilde{\psi}^{-1}(F \setminus F_0) \cap E$, we have for each $e^{i\theta} \in E_1$

$$\begin{aligned}
\tilde{\phi} \circ \tilde{\psi}(e^{i\theta}) &= \lim p_n \circ \tilde{\psi}(e^{i\theta}) \\
&= e^{i\theta},
\end{aligned}$$

hence, $\tilde{\psi}|_{E_1}$ is one-to-one. Furthermore, for $x \in F \setminus F_0$, there is a $y \in E_1$ so that $\tilde{\psi}(y) = x$; hence, $\tilde{\phi}(x) = \tilde{\phi} \circ \tilde{\psi}(y) = y$. Thus,

$$\tilde{\phi}(F \setminus F_0) = E_1$$

and

$$\begin{aligned}
\nu_1(E_1^c) &= \nu_1(\tilde{\phi}(F \setminus F_0))^c \\
&= \mu \circ \tilde{\phi}^{-1}(\tilde{\phi}(F \setminus F_0))^c \\
&= \mu \left(\tilde{\phi}^{-1}(\tilde{\phi}(F \setminus F_0)) \right)^c \\
&\leq \mu_1(F \setminus F_0)^c \\
&= 0
\end{aligned}$$

So $\tilde{\psi}$ is a one-to-one function from E_1 onto $F \setminus F_0$ and E_1 is the set we want.

Section 3.2. On the subalgebra $P^2(\mu) \cap C(spt\mu)$

Let $A(G)$ be the subalgebra of $C(\overline{G})$ consisting of those functions that are analytic on G. As before, we assume S_μ is an irreducible operator and $G = abpe(\mu)$.

Lemma 3.2.1. Let $\lambda_0 \in \partial G$ and k_λ be the reproducing kernel of $P^2(\mu)$, then $\frac{k_\lambda}{\|k_\lambda\|}$ weakly goes to zero as λ approaches λ_0.

Proof: Suppose there is a sequence of $\{\lambda_n\}$ such that λ_n converges to λ_0 and

$$\|k_{\lambda_n}\| \leq M < \infty$$

Without loss of generality, we can assume that k_{λ_n} weakly converges to k in $L^2(\mu)$; therefore,

$$p(\lambda_0) = (p\ ,\ k)$$

for every polynomial p. It follows that λ_0 is a bpe of $P^2(\mu)$. This is a contradiction to Thomson's theorem. Hence,

$$\|k_\lambda\| \to \infty \quad \text{as } \lambda \to \lambda_0$$

For every polynomial p, we have

$$(p\ ,\ \frac{k_\lambda}{\|k_\lambda\|}) = \frac{p(\lambda)}{\|k_\lambda\|} \to 0 \quad \text{as } \lambda \to \lambda_0$$

Because $\|\frac{k_\lambda}{\|k_\lambda\|}\| = 1$, it follows that $\frac{k_\lambda}{\|k_\lambda\|}$ converges weakly to zero as λ approaches λ_0.

Theorem 3.2.2. Let μ satisfy above conditions, then

$$P^2(\mu) \cap C(spt\mu) = A(G)$$

Proof: The difficult part is to show $P^2(\mu) \cap C(spt\mu) \subset A(G)$. First we prove $A(G) \subset P^2(\mu) \cap C(spt\mu)$.

Let $f \in A(G)$, then we extend f as a continuous function on $\mathcal{C}_\infty$ with compact support. Using the same proof of Lemma 5.5 of [31] and noticing that the Vitushkin scheme gives uniform approximation for continuous functions, we can find a sufficiently small constant C (as in the proof of Lemma 5.5.of [31]) so that h (in Lemma 5.5 of [31]) uniformly approximates f in $\overline{G}$. Passing to a limit, we conclude that f is in $P^2(\mu) \cap C(spt\mu)$.

Now we prove the reverse inclusion: $P^2(\mu) \cap C(spt\mu) \subset A(G)$. Fix $\tilde{f} \in P^2(\mu) \cap C(spt\mu)$; observe $f = \tilde{f}$ is analytic in G. Since $R(\partial G) = C(\partial G)$ (see [25, p.219] and [9, p.223]), for $\epsilon > 0$, there exists a rational function $r(z) = \dfrac{q}{p}$ with poles off ∂G satisfying

$$|r(z) - \tilde{f}(z)| < \frac{\epsilon}{2} \quad \text{for all} \quad z \in \partial G$$

(It is obvious that the support of μ contains ∂G). There is $\delta > 0$ so that

$$|r(z) - \tilde{f}(z)| < \epsilon \quad for \ z \in G_\delta \cap spt\mu$$

where $G_\delta = \{z : \ dist(z, \partial G) < \delta\}$. Since p has no zeros on ∂G, we see that multiplication by p (denoted by M_p) is an invertible operator from $P^2(\mu)$ to $pP^2(\mu)$. Since $dim(P^2(\mu) \ominus pP^2(\mu))$ is finite, we conclude that there is a bounded linear operator A on $P^2(\mu)$ such that

$$AM_p = I, \text{ and } M_pA = Q$$

where Q is the orthogonal projection from $P^2(\mu)$ to $pP^2(\mu)$ and $I - Q$ is a finite rank operator.

We make the following computation:

$$
\begin{aligned}
&((M_q A)^* k_\lambda, k_\lambda)\\
=&\overline{q(\lambda)}(A^* k_\lambda, k_\lambda)\\
=&\overline{q(\lambda)}(A^* k_\lambda, Q k_\lambda) + \overline{q(\lambda)}(A^* k_\lambda, (I-Q)k_\lambda)\\
=&\overline{q(\lambda)}(k_\lambda, \frac{1}{p} Q k_\lambda) + \overline{q(\lambda)}(A^* k_\lambda, (I-Q)k_\lambda)\\
=&\overline{r(\lambda)}(k_\lambda, Q k_\lambda) + \overline{q(\lambda)}(A^* k_\lambda, (I-Q)k_\lambda)\\
=&\overline{r(\lambda)}(k_\lambda, k_\lambda) + \overline{r(\lambda)}(k_\lambda, (Q-I)k_\lambda) + \overline{q(\lambda)}(A^* k_\lambda, (I-Q)k_\lambda)
\end{aligned}
$$

where λ is near a boundary point λ_0. Hence,

$$
(((M_q A)^* - \overline{r(\lambda)})k_\lambda, k_\lambda) = \overline{r(\lambda)}(k_\lambda, (Q-I)k_\lambda) + \overline{q(\lambda)}(A^* k_\lambda, (I-Q)k_\lambda)
$$

Therefore,

$$
\begin{aligned}
&|(((M_q A)^* - \overline{r(\lambda)})\frac{k_\lambda}{\|k_\lambda\|}, \frac{k_\lambda}{\|k_\lambda\|})|\\
\leq&|\overline{r(\lambda)}|\|(I-Q)\frac{k_\lambda}{\|k_\lambda\|}\| + |\overline{q(\lambda)}|\|A(I-Q)\frac{k_\lambda}{\|k_\lambda\|}\|
\end{aligned}
$$

Using the fact that $(I-Q)$ is a compact operator and using Lemma 3.2.1, we see that

$$
\begin{aligned}
&\lim_{\lambda\to\lambda_0\in\partial G} |(((M_q A)^* - \overline{r(\lambda)})\frac{k_\lambda}{\|k_\lambda\|}, \frac{k_\lambda}{\|k_\lambda\|})|\\
\leq& \lim_{\lambda\to\lambda_0} |\overline{r(\lambda)}|\|(I-Q)\frac{k_\lambda}{\|k_\lambda\|}\| + \lim_{\lambda\to\lambda_0} |\overline{q(\lambda)}|\|A(I-Q)\frac{k_\lambda}{\|k_\lambda\|}\|\\
=&0
\end{aligned}
$$

So if λ is sufficiently close to λ_0, then we have

$$
|(((M_q A)^* - \overline{r(\lambda)})\frac{k_\lambda}{\|k_\lambda\|}, \frac{k_\lambda}{\|k_\lambda\|})| < \epsilon
$$

We also have the computation:

$$
\begin{aligned}
&\Big(((M_qA)^* - M_{\tilde{f}}^*)k_\lambda,\ k_\lambda\Big)\\
=&(k_\lambda,\ (M_qA - M_{\tilde{f}})k_\lambda)\\
=&(k_\lambda,\ (M_qA - M_{\tilde{f}})Qk_\lambda) + (k_\lambda,\ (M_qA - M_{\tilde{f}})(I-Q)k_\lambda)\\
=&(k_\lambda,\ PM_{\chi_{G_\delta}}(M_qA - M_{\tilde{f}})Qk_\lambda) + (k_\lambda,\ PM_{\chi_{G_\delta^c}}(M_qA - M_{\tilde{f}})Qk_\lambda)\\
&+ (k_\lambda,\ (M_qA - M_{\tilde{f}})(I-Q)k_\lambda)\\
=&(k_\lambda,\ PM_{\chi_{G_\delta}}(r - \tilde{f})k_\lambda) + (k_\lambda,\ PM_{\chi_{G_\delta^c}}(M_qA - M_{\tilde{f}})Qk_\lambda)\\
&+ (k_\lambda,\ (M_qA - M_{\tilde{f}})(I-Q)k_\lambda) + (k_\lambda,\ PM_{\chi_{G_\delta}}(r - \tilde{f})(Q-I)k_\lambda)
\end{aligned}
$$

where P is the orthogonal projection from $L^2(\mu)$ to $P^2(\mu)$. We can estimate the first term on the right of the last equality as follows:

$$
\begin{aligned}
&|(k_\lambda,\ PM_{\chi_{G_\delta}}(r - \tilde{f})k_\lambda)|\\
\leq&\int_{G_\delta} |k_\lambda|^2|r - \tilde{f}|d\mu\\
\leq&\epsilon\int_{G_\delta} |k_\lambda|^2 d\mu\\
\leq&\epsilon\|k_\lambda\|^2.
\end{aligned}
$$

So we have

$$
\begin{aligned}
&|\left(((M_qA)^* - M_{\tilde{f}}^*)\frac{k_\lambda}{\|k_\lambda\|},\ \frac{k_\lambda}{\|k_\lambda\|}\right)|\\
\leq&\epsilon + \|(PM_{\chi_{G_\delta^c}}(M_qA - M_{\tilde{f}})Q)\frac{k_\lambda}{\|k_\lambda\|}\|\\
&+ \|((M_qA - M_{\tilde{f}})(I-Q))\frac{k_\lambda}{\|k_\lambda\|}\| + \|(PM_{\chi_{G_\delta}}(r - \tilde{f})(I-Q))\frac{k_\lambda}{\|k_\lambda\|}\|.
\end{aligned}
$$

Recalling $PM_{\chi_{G_\delta^c}}$ and $I - Q$ are compact operators, we see that the these terms on the right side of the last inequality can be made small because

$$\begin{aligned}
&\lim_{\lambda\to\lambda_0} \|(PM_{\chi_{G_\delta^c}}(M_qA \;-\; M_{\tilde{f}})Q)\frac{k_\lambda}{\|k_\lambda\|}\| \\
=&\lim_{\lambda\to\lambda_0} \|((M_qA \;-\; M_{\tilde{f}})(I-Q))\frac{k_\lambda}{\|k_\lambda\|}\| \\
=&\lim_{\lambda\to\lambda_0} \|(PM_{\chi_{G_\delta}}(r \;-\; \tilde{f})(I-Q))\frac{k_\lambda}{\|k_\lambda\|}\| \\
=&0.
\end{aligned}$$

Hence, for λ near λ_0, we have

$$|\left(((M_qA)^* \;-\; M_{\tilde{f}}^*)\frac{k_\lambda}{\|k_\lambda\|},\; \frac{k_\lambda}{\|k_\lambda\|}\right)| \le 2\epsilon.$$

Therefore,

$$\begin{aligned}
&|\left((\overline{r(\lambda)} \;-\; M_{\tilde{f}}^*)\frac{k_\lambda}{\|k_\lambda\|},\; \frac{k_\lambda}{\|k_\lambda\|}\right)| \\
=&|\left((\overline{r(\lambda)} \;-\; \overline{f(\lambda)})\frac{k_\lambda}{\|k_\lambda\|},\; \frac{k_\lambda}{\|k_\lambda\|}\right)| \\
=&|f(\lambda) \;-\; r(\lambda)| \\
=&|(((M_qA)^* - \overline{r(\lambda)})\frac{k_\lambda}{\|k_\lambda\|}, \frac{k_\lambda}{\|k_\lambda\|})| \;+\; |\left(((M_qA)^* \;-\; M_{\tilde{f}}^*)\frac{k_\lambda}{\|k_\lambda\|},\; \frac{k_\lambda}{\|k_\lambda\|}\right)| \\
<&3\epsilon.
\end{aligned}$$

For λ close λ_0,

$$|r(\lambda) \;-\; r(\lambda_0)| \;<\; \epsilon.$$

Hence, for λ sufficiently near to λ_0, we have

$$\begin{aligned}
&|f(\lambda) \;-\; f(\lambda_0)| \\
\le&|f(\lambda) \;-\; r(\lambda)| \;+\; |r(\lambda) \;-\; r(\lambda_0)| \;+\; |\tilde{f}(\lambda_0) \;-\; r(\lambda_0)| \\
<&3\epsilon \;+\; \epsilon \;+\; \epsilon \\
=&5\epsilon
\end{aligned}$$

This means $\tilde{f} \in A(G)$. The theorem is established.

CHAPTER IV A SUBNORMAL OPERATOR AND ITS DUAL

Section 4.1. Preliminaries

For an operator T in $\mathcal{L}(\mathcal{H})$, the sets $\sigma(T)$ and $\sigma_e(T)$ consist of the spectrum and essential spectrum of T, respectively. If S is a subnormal operator, then the minimal normal extension N can be written in a matrix format as follows:

$$N = \begin{bmatrix} S & X \\ 0 & T^* \end{bmatrix}$$

where T is the dual of S. The minimal normal extension of S_μ is N_μ, multiplication by z on $L^2(\mu)$. Let μ^* denote the measure obtained from μ as follows:

$$\mu^*(\Delta) = \mu(\Delta^*)$$

where Δ is any Borel subset of $\mathcal{C}$ and $\Delta^* = \{z : \bar{z} \in \Delta\}$. It is well-known that every cyclic, subnormal operator is unitarily equivalent to S_μ for a suitable choice of μ.

In this chapter, we assume that S_μ is an irreducible, subnormal operator. Let φ be a Riemann map of G onto the unit disc D. From the properties of the isomorphism $\sim$, the function $\tilde{\varphi}$ is in $P^2(\mu) \cap L^\infty(\mu)$ and

$$(\tilde{\varphi}\ ,\ K_\lambda^\mu) = \varphi(\lambda), \qquad \text{for all } \lambda \in G$$

where K_λ^μ is the kernel function for $P^2(\mu)$. For $f \in P^2(\mu)$, let

$$\hat{f}(\lambda) = (f,\ K_\lambda^\mu).$$

The pull back of μ to the closed unit disc is denoted by ν; that is, $\nu = \mu \circ \tilde{\varphi}^{-1}$.

For $f \in H^{\infty}(G)$ and $\tilde{f} \in P^2(\mu) \cap L^{\infty}(\mu)$, we will need nontangential limits of $\tilde{f}$ on a carrier of μ restricted to ∂G. Looking at Theorem 3.1.6, we may assume that $\tilde{\psi}$ is a one-to-one map from a Borel set $E \subset \partial D$ to a Borel set $F \subset \partial G$ with $\nu(E^c) = 0$ and $\mu(F^c) = 0$. Observing $f \circ \psi$ is in $H^{\infty}(D)$, we can choose a Borel set $E_1 \subset E$ with $m(E_1) = 0$ and for every point $e^{i\theta}$ in $E \setminus E_1$, we have the radial limit of $f \circ \psi$

$$\lim_{r \to 1-} f \circ \psi(re^{i\theta}) = (f \circ \psi)^*(e^{i\theta}).$$

(Actually we may compute $(f \circ \psi)^*(e^{i\theta})$ as a nontangential limit m a.e.)

Define

$$f^*(w) = (f \circ \psi)^*(e^{i\theta}) \tag{11}$$

for each $w \in \tilde{\psi}(E \setminus E_1)$, where we find a unique $e^{i\theta}$ in $E \setminus E_1$ so that $w = \psi(e^{i\theta})$. This radial limit $f^*(w)$ is well-defined on a carrier of $\mu|_{\partial G}$. The following theorem was also proved in [26].

Theorem 4.1.1. If $f \in H^{\infty}(G)$ and $\tilde{f} \in P^2(\mu) \cap L^{\infty}(\mu)$-, then $\tilde{f}(w) = f^*(w)$ almost everywhere with respect to $\mu|_{\partial G}$.

Section 4.2. Essential spectra of self-dual subnormal operators

A fundamental inclusion in this area of operator theory is that

$$\sigma(N) \subset \sigma(S),$$

a fact originally proved by Paul Halmos. We now present another central inclusion between the essential spectrum of N and the essential spectra of S and its dual T. (Recalling the fact that

$$\sigma(S) = \sigma(T^*)$$

(see [10]), we can derive the Halmos result from our Proposition 4.2.1.

Proposition 4.2.1. Let S be a pure subnormal operator on $\mathcal{H}$ with minimal normal extension N on $\mathcal{K}$ and let T be the dual of S, then

$$\sigma_e(N) \subset \sigma_e(S) \cup \sigma_e(T^*).$$

Proof: Suppose to the contrary that there is a point $\lambda_0 \in \sigma_e(N) \backslash (\sigma_e(S) \cup \sigma_e(T^*))$. Choose an infinite sequence of unit vectors $\{f_n\}$ in $\mathcal{K}$ which converges to zero weakly and

$$\|(N - \lambda_0)f_n\| \to 0.$$

For each n, let $f_n = g_n + h_n$ be the decomposition of f_n with respect to the orthogonal decomposition of $\mathcal{K} = \mathcal{H} \oplus \mathcal{H}^\perp$. It follows that

$$\|(S - \lambda_0)^* g_n\| \to 0,$$

and that

$$\|(T - \overline{\lambda}_0)^* h_n\| \to 0.$$

For each n let $g_n = g_n^1 + g_n^2$ be the decomposition of g_n with respect to the orthogonal decomposition

$$\mathcal{H} = Ker(S - \lambda_0)^* \oplus Ran(S - \lambda_0).$$

We now see g_n^2 converges to zero in norm since $\lambda_0 \in \sigma_e(S)^c$. Therefore, there is a subsequence $\{g_{n_k}\}$ converging in norm to a vector g since g_n^1 is in $Ker(S - \lambda_0)^*$, a finite dimensional space. Using the same argument, we can show that there is a subsequence $\{h_{n_{kl}}\}$ converging in norm to a vector h. Hence, $f_{n_{kl}}$ converges in norm to a unit vector f. This is a contradiction to the fact that f_n goes to zero weakly.

Remark: If S has a compact self-commutator, then an easy matricial argument shows

$$\sigma_e(N) = \sigma_e(S) \cup \sigma_e(T^*).$$

When S is a self-dual, subnormal operator, J.Conway [C1] shows that the spectrum of S and the spectrum of N are symmetric with respect to the real axis. It is natural to ask the following question.

Question: Is the essential spectrum of a self-dual subnormal operator symmetric with respect to the real axis?

We have not resolved this issue. The goal for this section is to supply an affirmative answer when S has a cyclic vector. First, we establish a result which is similar to the Riesz theorem for the classical Hardy spaces.

Theorem 4.2.2. Let S_μ be an irreducible subnormal operator on $P^2(\mu)$ with $bpe\mu = G$. Let $\lambda_0 \in \partial G$ and let $\Delta = O(\lambda_0, \delta_0)$ be the open disc with center at λ_0 and radius δ_0. Suppose $f \in P^2(\mu)$ is zero almost everywhere with respect to $\mu|_\Delta$. Then $f \equiv 0$ almost everywhere with respect to μ.

The proof of Theorem 4.2.2 requires several lemmas.

Lemma 4.2.3. Asumme the same hypotheses as Theorem 4.2.2. If there does exist a function f in the theorem which is not identically zero, then

$$\Delta \cap G \cap spt\mu = \{\lambda_n\}$$

where $\{\lambda_n\}$ is a sequence of isolated points (possibly empty) and

$$span\{K_{\lambda_n},\ n = 1, 2, 3, ...\overline{\}} \neq P^2(\mu).$$

Proof: It is well-known that

$$f(z) = (f,\ K_z) \qquad a.e.\ \mu|_G.$$

Suppose the first conclusion were false; then we can choose a sequence of numbers $\{\eta_n\} \subset \Delta \cap G \cap spt\mu$ such that

$$\eta_n \to \eta \in \Delta \cap G \cap spt\mu$$

and

$$f(\eta_n) = (f,\ K_{\eta_n}) = 0.$$

Since $(f,\ K_z)$ is an analytic function on G, it follows that $(f,\ K_z)$ is identically equal to zero on G. Hence f is the zero function in $P^2(\mu)$ since $span\{K_z,\ z \in G\}$ is dense in $P^2(\mu)$. This contradicts our assumption regarding f.

If the last conclusion of the lemma were false, that is, if $span\{K_{\lambda_n}\} = P^2(\mu)$, then

$$f(\lambda_n) = (f,\ K_{\lambda_n}) = 0$$

which implies that f is the zero function, a blatant contradiction again.

Lemma 4.2.4. Again we assume the same hypotheses as Theorem 4.2.2. and we assume there exists a function f that is zero $\mu|_\Delta$ a.e. but not identically zero. Write

$$\Delta \cap G \cap spt\mu = \{\lambda_n\}$$

and suppose $\lambda \in \partial G \cap \Delta$ is a limit point of the sequence $\{\lambda_n\}$. There exists an invariant subspace $\mathcal{H}$ for S_μ that contains f and

$$\sigma(S_\mu|_{\mathcal{H}})^c \cap \Delta \cap \partial G \neq \emptyset.$$

Proof: It is easy to describe the desired subspace. Let

$$\mathcal{H} = \{g \in P^2(\mu) : \; g = 0 \qquad a.e. \; \mu|_\Delta\}.$$

Clearly $\mathcal{H}$ is an invariant subspace of S_μ containing f. We want to show $\lambda \in \sigma(S_\mu|_{\mathcal{H}})^c$.

Let $\{\lambda_{n_l}\}$ be a subsequence of $\{\lambda_n\}$ which converges to λ. From the definition of $\mathcal{H}$, we see for each g in $\mathcal{H}$ that

$$\frac{g \; - \; \hat{g}(\lambda_{n_l})}{z \; - \; \lambda_{n_l}} \in P^2(\mu), \qquad \text{and} \qquad \hat{g}(\lambda_n) \; = \; 0.$$

Hence, for all n_l we have

$$\frac{g}{z \; - \; \lambda_{n_l}} \in P^2(\mu).$$

On the other hand, if $g \in \mathcal{H}$, then

$$\int |\frac{g}{z \; - \; \lambda_{n_l}} \; - \; \frac{g}{z \; - \; \lambda}|^2 d\mu \to 0$$

because $g = 0$ almost everywhere $\mu|_\Delta$. Therefore,

$$\frac{g}{z \; - \; \lambda} \in P^2(\mu)$$

for all g in $\mathcal{H}$. Using Theorem 3.1.6, we know that μ restricted to the boundary is absolutely continuous with respect to the harmonic measure. Hence $\mu\{\lambda\} = 0$. Consequently

$$\frac{g}{z \; - \; \lambda} \; = \; 0 \qquad a.e. \; \mu|_\Delta.$$

This implies $\frac{g}{z \; - \; \lambda} \in \mathcal{H}$. Hence, $\lambda \notin \sigma(S_\mu|_{\mathcal{H}})$.

Lemma 4.2.5. We still assume the hypotheses of Theorem 4.2.2. We now consider the case that (shrinking Δ if need be)

$$spt\mu \cap \Delta \cap G \; = \; \emptyset.$$

In this case, we still find an invariant subspace $\mathcal{H}$ which is not zero such that

$$\sigma(S_\mu|_{\mathcal{H}})^c \cap \Delta \cap \partial G \neq \emptyset.$$

Proof: From Theorem 3.1.6, we know that $\mu|_{\partial G}$ is absolutely continuous with respect to the harmonic measure ω. So

$$\omega(\Delta \cap \partial G) \neq 0.$$

But $\omega = m \circ (\psi^*)^{-1}$, where ψ is a Riemann map from D to G and ψ^* is the boundary function of ψ. Hence,

$$m \circ (\psi^*)^{-1}(\Delta \cap \partial G) \neq 0.$$

Choose a point $\eta \in (\psi^*)^{-1}(\Delta \cap \partial G)$ such that

$$\lim_{r \uparrow 1-0} \psi(r\eta) = \psi^*(\eta) \in \Delta \cap \partial G.$$

We now choose a sequence $\{\eta_n\} \subset D$ such that $\eta_n = r_n \eta$ and

$$\sum(1 - r_n) < \infty.$$

With the last inequality, we can construct a nonzero function $\varphi \in H^\infty(D)$ with $\varphi(\eta_n) = 0$. Let $\psi_0 = \varphi \circ \psi^{-1}$ and define $\beta_n = \psi(\eta_n)$. Clearly β_n converges to $\psi^*(\eta) = \beta_0$. We may assume $\beta_n \in \Delta$, for all n. From our construction, we see $\psi_0 \in H^\infty(G)$ and $\psi_0(\beta_n) = 0$. Let

$$\mathcal{H} = \{g \in P^2(\mu) : g = 0 \; a.e. \; \mu|_\Delta \text{ and } \hat{g}(\beta_n) = 0\}.$$

Clearly $\mathcal{H}$ is a closed invariant subspace for S_μ and is nonzero since $\tilde{\psi_0} f \in \mathcal{H}$. If $g \in \mathcal{H}$, then

$$\frac{g - \hat{g}(\beta_n)}{z - \beta_n} \in P^2(\mu) \qquad \text{and} \qquad \hat{g}(\beta_n) = 0.$$

Using the same argument as in the proof of Lemma 4.2.4, we see

$$\frac{g}{z - \beta_0} \in P^2(\mu), \qquad \text{and} \qquad \frac{g}{z - \beta_0} = 0 \qquad a.e.\mu|_\Delta.$$

We now want to show that $\frac{g}{z - \beta_0} \in \mathcal{H}$ for all $g \in \mathcal{H}$. The only thing left to show is

$$\widehat{\left(\frac{g}{z - \beta_0}\right)}(\beta_n) = 0$$

for all β_n. To this end, let $\{p_m\}$ be a sequence of polynomials such that $p_m \rightarrow \frac{g}{z - \beta_0}$ in $L^2(\mu)$ norm. Then for each n

$$p_m(\beta_n) \rightarrow \widehat{\left(\frac{g}{z - \beta_0}\right)}(\beta_n).$$

On the other hand, as m goes to infinity,

$$(\beta_n - \beta_0)p_m(\beta_n) = ((z - \beta_0)p_m, K_{\lambda_n}) \rightarrow (g, K_{\beta_n}) = 0.$$

Hence, $p_m(\beta_n)$ converges to zero. Therefore,

$$\widehat{\left(\frac{g}{z - \beta_0}\right)}(\beta_n) = 0.$$

Therefore, $\frac{g}{z - \beta_0} \in \mathcal{H}$, Thus

$$\sigma(S_\mu|_\mathcal{H})^c \cap \Delta \cap \partial G \neq \emptyset.$$

Proof of Theorem 4.2.2: Suppose to the contrary that f is not the zero function; using Lemmas 4.2.3, 4.2.4, and 4.2.5, there is a nonzero invariant subspace $\mathcal{H}$ for S_μ and a point $\lambda_0 \in \sigma(S_\mu|_\mathcal{H})^c \cap \partial G$. Choose $\delta_1 > 0$ so that

$$O(\lambda_0, \delta_1) \subset \sigma(S_\mu|_\mathcal{H})^c.$$

Claim:

$$G \subset \sigma(S_\mu|_{\mathcal{H}})^c.$$

If the claim were false, then there is a point $\lambda_1 \in G \cap \sigma(S_\mu|_{\mathcal{H}})$. Let

$$\lambda_2 \in O(\lambda_0,\ \delta_1) \cap G.$$

Because G is a simply connected, we can find a path in G from λ_1 to λ_2. Therefore,

$$\partial\sigma(S_\mu|_{\mathcal{H}}) \cap G \ \neq\ \emptyset.$$

Hence,

$$\sigma_a(S_\mu|_{\mathcal{H}}) \cap G \ \neq\ \emptyset.$$

This implies

$$\sigma_a(S_\mu) \cap G \ \neq\ \emptyset,$$

a contradiction that $G \ = \ abpe(\mu)$. This establishes the claim. Hence,

$$\sigma(S_\mu|_{\mathcal{H}}) \subset \partial G.$$

It is well-known $R(\partial G) \ = \ C(\partial G)$ (Every point in ∂G is a peak point for $R(\partial G)$). Consequently, $S_\mu|_{\mathcal{H}}$ is a normal operator. This contradicts the fact that S_μ is a pure subnormal operator. The proof of the theorem is completed.

The following lemma is an old chestnut. For the sake of completeness we include its proof.

Lemma 4.2.6. Let S_μ be an irreducible subnormal operator on $P^2(\mu)$ with $bpe\mu \ = \ G$ and $\lambda_0 \in G$. Then there is a small positive constant $\delta_0 \ > \ 0$ such that S_{μ_0} is similar to S_μ where $\mu_0 \ = \ \mu|_{\Delta^c}$ and $\Delta_0 \ = \ O(\lambda_0,\ \delta_0)$.

Proof: There is a constant $M > 0$ so that for all polynomial p

$$\begin{aligned}
&\int |p|^2 d\mu \\
&\leq M \int |z - \lambda_0|^2 |p|^2 d\mu \\
&= M \int_{\Delta_0} |z - \lambda_0|^2 |p|^2 d\mu + M \int_{\Delta_0^c} |z - \lambda_0|^2 |p|^2 d\mu \\
&= M\delta_0^2 \int_{\Delta_0} |p|^2 d\mu + M \|z - \lambda_0\|_\infty^2 \int_{\Delta_0^c} |p|^2 d\mu.
\end{aligned}$$

The first inequality follows since $bpe\mu = abpe\mu$. Choose δ_0 to be small enough such that

$$1 - M\delta_0 > 0.$$

We then have

$$(1 - M\delta_0) \int |p|^2 d\mu \leq M \|z - \lambda_0\|_\infty^2 \int |p|^2 d\mu_0.$$

Obviously, we have

$$\int |p|^2 d\mu_0 \leq \int |p|^2 d\mu.$$

The last two results yield the desired result: S_{μ_0} is similar to S_μ.

Theorem 4.2.7. Let S_μ be an irreducible, self-dual, subnormal operator on $P^2(\mu)$ and $bpe\mu = G$. Then ∂G is symmetric with respect to real axis

Proof: There is a unitary operator U from $P^2(\mu)$ to $P^2(\mu)^\perp$ so that

$$US_\mu U^* = T_\mu.$$

Suppose that there exists $\lambda_0 \in \partial G$ and $\overline{\lambda}_0 \notin \partial G$. Since $\sigma(S_\mu)$ is symmetric with respect to the real axis and

$$\sigma_e(S_\mu) = \partial G,$$

it follows that $\overline{\lambda_0} \in G$. We choose $\delta_0 > 0$ small enough such that $O(\overline{\lambda}_0,\ \delta_0) \subset G$ and S_{μ_0} is similar to S_μ where

$$\mu_0 \;=\; \mu|_{O(\overline{\lambda}_0,\ \delta_0)^c}.$$

It follows then that S_{μ_0} is an irreducible subnormal operator. Using [8] or [9], we choose a function $g \in L^2(\mu_0)$ which is orthogonal to $P^2(\mu_1)$ and $|g| \;>\; 0$ almost everywhere with respect to μ_1. Define

$$h \;=\; \begin{cases} g, & O(\overline{\lambda}_0,\ \delta_0)^c \\ 0, & O(\overline{\lambda}_0,\ \delta_0). \end{cases}$$

Clearly h is orthogonal to $P^2(\mu)$. Let $f \;=\; U1$; plainly $|f| \;>\; 0$ almost everywhere with respect to μ. Since h is orthogonal to $P^2(\mu)$ and f is a cyclic vector, we may choose a sequence of polynomials $\{p_n\}$ so that

$$p_n(\overline{z})f \to h$$

in $L^2(\mu)$ norm. Since U is a unitary, we note that for any polynomial p

$$\int |p|^2 d\mu \;=\; \int |p(\overline{z})|^2 |f|^2 d\mu.$$

Hence $\{p_n\}$ must converge to a function t in $P^2(\mu)$. It is easy to show that

$$h(z) \;=\; f(z)t(\overline{z}) \qquad a.e.\ \mu.$$

Therefore, $t(z) \;=\; 0$ almost everywhere with respect to $\mu|_{O(\lambda_0,\ \delta_0)}$. However, according to Theorem 4.2.2, the function t has to be zero. This is a contradiction since h is not the zero function. The proof is completed.

In [9, p.408] Conway uses our last theorem in the proof of proposition 6.5. Conway dose not prove the theorem; he asserts its validity follows from the fact that the spectrum is symmetric with respect to the real axis. To see that more justification is needed one should ponder why the following subnormal operator is not self-dual.

It is easy to construct a measure μ enjoying the following properties:

(1) $\sigma(S_\mu) = \overline{D}$.

(2) The support of μ is symmetric with respect to the real axis.

(3) $abpe(\mu) = D \setminus L$, where $L = \{|z| < 1, Rez \leq 0, Imz = \frac{1}{2}\}$. Note: It then follows from Thomson's theorem that

$$\sigma_e(S_\mu) = \partial D \cup L.$$

Corollary 4.2.8. Suppose S_μ is an irreducible, self-dual, subnormal operator on $P^2(\mu)$ with $bpe(\mu) = G$. We then have

$$\sigma(N_\mu) = \partial G \cup \{\lambda_n\}$$

where $\{\lambda_n\} \subset G$ is a sequence of isolated points.

Proof: It is well-known that for any normal operator N

$$\sigma(N) \setminus \sigma_e(N) = \{\lambda_n\}$$

where $\{\lambda_n\}$ is a sequence of isolated points. Using the remark after Proposition 4.2.1 and the result of Theorem 4.2.7, we have

$$\sigma_e(N_\mu) = \sigma_e(S_\mu) = \partial G.$$

Section 4.3. Reformulation of the problem; a reduction to the unit disc.

Suppose S_μ is an irreducible, cyclic subnormal operator on $P^2(\mu)$ with $bpe(\mu) = G$. Suppose S_μ is self-dual. Theorem 4.2.7 implies that G is equal to G^*. Let ψ be a Riemann map from D to G where $\psi(0) = a \in G$ is a real number and $\psi'(0) > 0$. If we define the analytic function ψ_0 on D by setting $\psi_0(z) = \overline{\psi(\overline{z})}$, then ψ_0 is also a Riemann map with the properties $\psi_0(0) = a$ and $\psi_0'(0) > 0$. From the uniqueness of the Riemann map, one sees $\psi_0(z) = \psi(z)$. We define the measure ν on $\overline{D}$ as done in Theorem 3.1.6.

Theorem 4.3.1. We use the notation and results of preceding paragraph. Let S_μ be a cyclic irreducible operator on $P^2(\mu)$ with bounded point evaluations $bpe\mu = G$. The transformation S_μ is self-dual if and only if the operator S_ν has the following two properties:

(1) The operator S_ν is a self-dual subnormal operator on $P^2(\nu)$.

(2) The operator S_μ is unitarily equivalent to $M^\nu_{\tilde{\psi}}$ (multiplication by $\tilde{\psi}$ on the space $P^2(\nu)$) and $\psi(z) = \overline{\psi(\overline{z})}$.

Proof: Suppose conditions (1) and (2) are satisfied. Let U be a unitary operator operator from $P^2(\nu)$ to $P^2(\nu)^\perp$ such that

$$U^* S_\nu U = T_\nu.$$

where T_ν is the dual of S_ν. If $\tilde{\psi}(N_\nu)$ denote operator multiplication by $\tilde{\psi}$ on $L^2(\nu)$, then it can be written matrically as

$$\tilde{\psi}(N_\nu) = \begin{bmatrix} T^\nu_{\tilde{\psi}} & * \\ 0 & T_1^* \end{bmatrix}$$

on $L^2(\nu) = P^2(\nu) \oplus P^2(\nu)^\perp$. If we establish that

$$U^* T^\nu_{\tilde{\psi}} U = T_1,$$

then we are done with one implication of the theorem if we recall condition (2) and use the fact that $\widetilde{\psi}(N_\nu)$ is the minimal normal extension of $T^\nu_{\widetilde{\psi}}$, see [16]. Using the lifting theorem [8, p128], there exists a unitary operator V on $L^2(\mu)$ so that

$$V^* N_\nu V \;=\; N^*_\nu.$$

Thus, for every function $f \in L^\infty(\mu)$, we have

$$V^* f(N_\nu) V \;=\; f(N^*_\nu)$$

where $f(N^*_\nu)$ is the operator obtained by multiplication by $f(\overline{z})$ on $L^2(\mu)$. Using Theorem 4.1.1, we know that $\widetilde{\psi}|_{\partial D}$ is equal to the nontangential limit of ψ almost everywhere with respect to $\nu|_{\partial D}$. This implies

$$\widetilde{\psi}(\overline{z}) \;=\; \overline{\widetilde{\psi}(z)} \; a.e. \; \nu$$

Hence,

$$V^* \widetilde{\psi}(N_\nu) V \;=\; \widetilde{\psi}(N_\nu)^*$$

However, V can be expressed matrically as

$$V \;=\; \begin{bmatrix} 0 & U \\ U^* & 0 \end{bmatrix}$$

with respect to the decomposition $L^2(\nu) \;=\; P^2(\nu) \oplus P^2(\nu)^\perp$. A trivial matrix computation shows

$$U^* T^\nu_{\widetilde{\psi}} U \;=\; T_1$$

Therefore, S_μ is self-dual.

Now suppose that S_μ is a self-dual subnormal operator. Let $\varphi \;=\; \psi^{-1}$ be the Riemann map from G to D, then $\widetilde{\varphi} \in P^2(\mu)$. Set $\nu \;=\; \mu \circ \widetilde{\varphi}^{-1}$, then according to

Theorem 3.1.4, we know that S_μ is unitary equivalent to $T^{\nu}_{\underset{\psi}{\sim}}$ on $P^2(\nu)$. Also by the argument before the theorem, we can show

$$\psi(z) = \overline{\psi(\overline{z})}$$

So (2) is proved.

Looking at Theorem 3.1.4 again, we know that S_ν is unitary equivalent to $M^{\mu}_{\underset{\varphi}{\sim}}$ on $P^2(\mu)$. It also follows from Theorem 4.1.1 that every function in the algebra $P^2(\mu) \cap L^\infty(\mu)$ has "nontangential limit" almost everywhere with respect to $\mu|_{\partial G}$ which guarantees

$$\overline{\widetilde{\varphi}(z)} = \widetilde{\varphi}(\overline{z}) \ a.e.\ \mu.$$

Using the same argument as above, we can show that $T^{\mu}_{\underset{\varphi}{\sim}}$ is self-dual. That is, S_ν is self-dual. The assertion in (1) is verified.

Theorem 4.3.1 says that the study of a cyclic self-dual subnormal operator can be done under the additional assumption that $bpe\mu = D$.

Section 4.4. Self-dual, cyclic subnormal operators having the unit disc as their set of bounded point evaluations

In this section, we study the class of self-dual operators mentioned at the end of last section. That is, a cyclic, self-dual subnormal operator S_μ with $bpe\mu = D$. We always assume that $\frac{d\mu|_{\partial D}}{dm}$ is log-integrable. That is,

$$\int log \frac{d\mu|_{\partial D}}{dm} > -\infty.$$

where m is the normalized Lebesgue measure ($dm = \frac{1}{2\pi} d\theta$). The later assumption implies (in fact it's equivalent to) that the operator, multiplication by z on $P^2(\mu|_{\partial D})$, is pure. By Sezgo's Theorem (see [15, p136]), there is an outer function $r \in H^2$ such that

$$\mu|_{\partial D} = |r|^2 m.$$

From the results in [10] and Corollary 2.8, we can assume

$$\mu = |r|^2 m + \sum_{i=1}^{\infty} \beta_i \delta_{a_i} + \sum_{j=1}^{\infty} (\gamma_j \delta_{b_j} + \gamma_j' \delta_{\overline{b_j}}) \qquad (*)$$

where the notation δ_a denotes point mass measure at a; the constants β_i, γ_j, γ_j' are strictly positive; the constants a_i are real and the constants b_j have a nonzero imaginary part. For $a \in D$, we define

$$\varphi_a(z) = \frac{z - a}{1 - \overline{a}z}.$$

Lemma 4.4.1. Let φ be an infinite Blaschke product whose zeros are exactly $a_1, a_2, ...$ and each has multiplicity one and if

$$\sum_{n=1}^{\infty} \frac{1 - |a_n|^2}{|\frac{\varphi}{\varphi_{a_n}}(a_n)|} < \infty,$$

then

$$\int pz\overline{\varphi}dm \;=\; \sum_{n=1}^{\infty} \frac{1 \;-\; |a_n|^2}{\frac{\varphi}{\varphi_{a_n}}(a_n)} p(a_n)$$

for every polynomial p.

Proof: If $\varphi_n \;=\; \varphi_{a_1}\varphi_{a_2}...\varphi_{a_n}$, then the sequence $\{\varphi_n\}$ converges to φ with respect to weak-star topology. Hence, using Cauchy integral formula too, we have

$$\begin{aligned} \int pz\overline{\varphi}dm \;&=\; \lim \int pz\overline{\varphi}_n dm \\ &=\; \lim \frac{1}{2\pi i} \int \frac{p(z)}{\varphi_n(z)} dz \\ &=\; \lim \sum_{i=1}^{n} \frac{1 \;-\; |a_i|^2}{\frac{\varphi_n}{\varphi_{a_i}}(a_i)} p(a_i). \end{aligned}$$

Now note that

$$\sum_{i=1}^{n} \frac{1 \;-\; |a_i|^2}{|\frac{\varphi_n}{\varphi_{a_i}}(a_i)|} |p(a_i)| \;\leq\; \sum_{i=1}^{n} \frac{1 \;-\; |a_i|^2}{|\frac{\varphi}{\varphi_{a_i}}(a_i)|} |p(a_i)| \;\leq\; \|p\|_\infty \sum_{n=1}^{\infty} \frac{1 \;-\; |a_n|^2}{|\frac{\varphi}{\varphi_{a_n}}(a_n)|} \;<\; \infty.$$

An easy application of the Lebesgue dominated theorem yields the desired result.

Theorem 4.4.2. Let μ be as in (*) where $|r(z)| \;=\; 1$. The operator S_μ is self-dual if and only if

(a) The set $\{a_i,\; b_j,\; \overline{b}_j\}$ is the zero set of a nonzero function in H^∞. Note this is the case, our notation for the Blaschke factor of this function is

$$\varphi \;=\; \Pi\varphi_{a_i}\Pi\varphi_{b_j}\varphi_{\overline{b}_j};$$

(b) The zeros of φ and the weights of μ are related as follows:

$$\beta_i \;=\; \frac{1 \;-\; a_i^2}{|\frac{\varphi}{\varphi_{a_i}}(a_i)|},$$

$$\sqrt{\gamma_j\gamma_j'} = \frac{1 - |b_j|^2}{|\frac{\varphi}{\varphi_{b_j}}(b_j)|},$$

and

$$\sum \frac{1 - a_i^2}{|\frac{\varphi}{\varphi_{a_i}}(a_i)|} + \sum(\gamma_j + \gamma_j') < \infty.$$

Proof: Suppose both (a) and (b) hold. Let

$$f(z) = \begin{cases} \overline{z}\varphi(z) & \text{on } \partial D, \\ -|\frac{\varphi}{\varphi_{a_i}}(a_i)| \,/\, \overline{(\frac{\varphi}{\varphi_{a_i}}(a_i))} & z = a_i, \\ -\sqrt{\frac{\gamma_j'}{\gamma_j}}|\frac{\varphi}{\varphi_{b_j}}(b_j)| \,/\, \overline{(\frac{\varphi}{\varphi_{b_j}}(b_j))} & z = b_j, \\ -\sqrt{\frac{\gamma_j}{\gamma_j'}}|\frac{\varphi}{\varphi_{\overline{b_j}}}(\overline{b_j})| \,/\, \overline{(\frac{\varphi}{\varphi_{\overline{b_j}}}(\overline{b_j}))} & z = \overline{b_j}. \end{cases}$$

Claim 1. $f \perp P^2(\mu)$.

The validity of the claim is a simple computation:

$$\begin{aligned} < p, f > &= \int pz\overline{\varphi}dm - \sum \beta_i p(a_i)|\frac{\varphi}{\varphi_{a_i}}(a_i)| \,/\, (\frac{\varphi}{\varphi_{a_i}}(a_i)) \\ &\quad - \sum \gamma_j \sqrt{\frac{\gamma_j'}{\gamma_j}} p(b_j)|\frac{\varphi}{\varphi_{b_j}}(b_j)| \,/\, (\frac{\varphi}{\varphi_{b_j}}(b_j)) \\ &\quad + \sum \gamma_j' \sqrt{\frac{\gamma_j}{\gamma_j'}} p(\overline{b}_j)|\frac{\varphi}{\varphi_{\overline{b_j}}}(\overline{b_j})| \,/\, (\frac{\varphi}{\varphi_{\overline{b_j}}}(\overline{b_j})) \\ &= \int pz\overline{\varphi}dm - \sum(1 - a_i^2)p(a_i) \,/\, (\frac{\varphi}{\varphi_{a_i}}(a_i)) \\ &\quad - \sum \left((1 - |b_j|^2)p(b_j)/(\frac{\varphi}{\varphi_{b_j}}(b_j)) + (1 - |\overline{b}_j|^2)p(\overline{b}_j)/(\frac{\varphi}{\varphi_{\overline{b}_j}}(\overline{b}_j))\right) \\ &= 0. \end{aligned}$$

Note that the last equality follows from Lemma 4.4.1.

Claim 2. $\{\overline{p(z)}f : p \text{ a polynomial}\}$ is dense in $P^2(\mu)^\perp$.

It is sufficient to show that if $g \in L^2(\mu)$ is orthogonal to both $P^2(\mu)$ and $span\{\overline{p(z)}f\}$, then g is the zero function. If g is the function with these property, then for every polynomial p

$$< \overline{\varphi p} f,\ g > = 0$$

because

$$\overline{P^{\infty}(\mu)} f \subset L^2(\mu) \text{ closure of } \{\overline{p(z)f}\}.$$

It follows that for every polynomial p

$$\int pzgdm = 0.$$

Thus, there is a function $g_0 \in H^2(\partial D)$ such that $g|_{\partial D} = g_0$. On the other hand, for every polynomial p we see that

$$\begin{aligned}
&< \overline{p}(\overline{\frac{\varphi}{\varphi_{a_i}}}) f,\ g > \\
&= \int \overline{p}(\overline{\frac{\varphi}{\varphi_{a_i}}}) \overline{z} \varphi \overline{g} dm - \beta_i \overline{p(a_i)} (\overline{\frac{\varphi}{\varphi_{a_i}}(a_i)}) | \frac{\varphi}{\varphi_{a_i}}(a_i) | \overline{g(a_i)} \ / \ (\overline{\frac{\varphi}{\varphi_{a_i}}(a_i)}) \\
&= 0;
\end{aligned}$$

the last equality following as a corollary of Lemma 4.1 and the fact that $g_0 \in H^2(\partial D)$. This implies that for any polynomial p

$$\int pz\overline{\varphi}_{a_i} gdm = (1 - a_i^2)p(a_i)g(a_i) = \int pg_0(1 - a_i z)\overline{k}_{a_i} dm$$

where $k_{\lambda}(z) = \dfrac{1}{1 - \overline{\lambda} z}$. Therefore,

$$(1 - a_i^2)p(a_i)\tilde{g}_0(a_i) = (1 - a_i^2)p(a_i)g(a_i)$$

where $\tilde{g}_0$ is the analytic extension to the disc. It follows that for all i

$$g(a_i) = \tilde{g}_0(a_i).$$

Using the same method, we can show that for all j

$$g(b_j) = \tilde{g}_0(b_j), \qquad g(\overline{b}_j) = \tilde{g}_0(\overline{b}_j).$$

Now let K_λ be the reproducing kernel for $P^2(\mu)$ and let

$$\varphi_n = \Pi_{i=1}^n \varphi_{a_i} \Pi_{j=1}^n \varphi_{b_j} \varphi_{\overline{b}_j}.$$

For each $\lambda \in D$ and for each polynomial p we have

$$\begin{aligned}
&\int \frac{\varphi(z)}{\varphi_n(z)} p(z) \overline{k}_\lambda dm \\
&= p(\lambda) \frac{\varphi(\lambda)}{\varphi_n(\lambda)} \\
&= \int \frac{\varphi(z)}{\varphi_n(z)} p(z) \overline{K}_\lambda d\mu \\
&= \int \frac{\varphi(z)}{\varphi_n(z)} p(z) \overline{K}_\lambda dm + \sum_{i=1}^n \beta_i \frac{\varphi}{\varphi_n}(a_i) p(a_i) \overline{K_\lambda(a_i)} \\
&\quad + \sum_{i=1}^n \left(\gamma_i \frac{\varphi}{\varphi_n}(b_i) p(b_i) \overline{K_\lambda(b_i)} + \gamma_i' \frac{\varphi}{\varphi_n}(\overline{b}_i) p(\overline{b}_i) \overline{K_\lambda(\overline{b}_i)} \right) \\
&= \int \frac{\varphi(z)}{\varphi_n(z)} p(z) \overline{\left(K_\lambda + \sum_{i=1}^n (\beta_i K_\lambda(a_i) k_{a_i} + \gamma_i K_\lambda(b_i) k_{b_i} + \gamma_i' K_\lambda(\overline{b}_i) k_{\overline{b}_i} \right)} dm.
\end{aligned}$$

Since there are polynomials p_n that converges to g_0 in H^2, we now have that for all n

$$\begin{aligned}
&\int \frac{\varphi(z)}{\varphi_n(z)} g_0(z) \overline{k}_\lambda dm \\
&= \int \frac{\varphi(z)}{\varphi_n(z)} g_0(z) \overline{\left(K_\lambda + \sum_{i=1}^n (\beta_i K_\lambda(a_i) k_{a_i} + \gamma_i K_\lambda(b_i) k_{b_i} + \gamma_i' K_\lambda(\overline{b}_i) k_{\overline{b}_i} \right)} dm.
\end{aligned}$$

Meanwhile, since g is orthogonal to $P^2(\mu)$,

$$\begin{aligned}
&\int \frac{\varphi(z)}{\varphi_n(z)} g_0(z) \overline{k}_\lambda dm \\
&= \int \frac{\varphi(z)}{\varphi_n(z)} g_0(z) \overline{\left(K_\lambda + \sum_{i=1}^n (\beta_i K_\lambda(a_i) k_{a_i} + \gamma_i K_\lambda(b_i) k_{b_i} + \gamma_i' K_\lambda(\overline{b}_i) k_{\overline{b}_i} \right)} dm \\
&= \int \frac{\varphi(z)}{\varphi_n(z)} g(z) \overline{\left(K_\lambda + \sum_{i=1}^n (\beta_i K_\lambda(a_i) k_{a_i} + \gamma_i K_\lambda(b_i) k_{b_i} + \gamma_i' K_\lambda(\overline{b}_i) k_{\overline{b}_i} \right)} dm \\
&= \int \frac{\varphi(z)}{\varphi_n(z)} g(z) \overline{K}_\lambda d\mu.
\end{aligned}$$

The sequence of functions $\frac{\varphi}{\varphi_n}$ converges to 1 in both $P^\infty(\mu)$ and $H^\infty(\partial D)$ in weak-star topology. Hence, for all $\lambda \in D$ we have

$$\int g\overline{K}_\lambda d\mu \;=\; \int g_0\overline{k}_\lambda dm \;=\; 0.$$

Therefore, $g_0 \;=\; 0$. It now follows that g is the zero function. This establishes Claim 2.

Let U be the operator form $P^2(\mu)$ to $P^2(\mu)^\perp$ defined by

$$(Up)(z) \;=\; p(\overline{z})f(z)$$

for every polynomial p. By the assumption, it is easy to check $\mu \;=\; (|f(z)|^2\mu)^*$. Thus,

$$\|Up\|^2 \;=\; \int |p(\overline{z})|^2|f(z)|^2 d\mu \;=\; \int |p(z)|^2(|f(z)|^2 d\mu)^* \;=\; \int |p(z)|^2 d\mu.$$

This means U is a unitary operator. Also we have

$$US_\mu p \;=\; \overline{z}p(\overline{z})f(z) \;=\; \overline{z}Up \;=\; T_\mu Up$$

for every polynomial p. Therefore, S_μ is a self-dual subnormal operator.

Now suppose S_μ is a self-dual subnormal operator, that is, there is a unitary operator U such that

$$US_\mu U^* \;=\; T_\mu.$$

If

$$h \;=\; \begin{cases} \overline{z}, & \partial D \\ 0, & D, \end{cases}$$

then h is orthogonal to $P^2(\mu)$. Let $s \;=\; U^*h$ and let K_λ be the kernel function for S_μ, then UK_λ is the kernel function for $P^2(\mu)^\perp$ and

$$< s,\; K_\lambda > = < U^*h,\; K_\lambda > = < h,\; UK_\lambda > .$$

From the definition of h and the fact that the defining values of h agree with the analytic extension of h to D almost everywhere μ, we see

$$< h,\ UK_{a_i} > = < h,\ UK_{b_j} > = < h,\ UK_{\overline{b}_j} > =\ 0.$$

Hence,

$$< s,\ K_{a_i} > = < s,\ K_{b_j} > = < s,\ K_{\overline{b}_j} > =\ 0.$$

It now follows that $\{a_i,\ b_j,\ \overline{b}_j\}$ is a zero set of Blaschke product φ. As before, we write $\varphi\ =\ \Pi\varphi_{a_i}\Pi\varphi_{b_j}\varphi_{\overline{b}_j}$ and let $U1\ =\ f$. It follows then that for all polynomials p that $(Up)(z)\ =\ p(\overline{z})f(z)$. Since U is a unitary we now see that for all polynomials p_1 and p_2 that

$$\begin{aligned}\int p_1(z)\overline{p}_2(z)d\mu(z)\ &=\ \int p_1(\overline{z})\overline{p}_2(\overline{z})|f(z)|^2 d\mu(z)\\ &=\ \int p_1(z)\overline{p}_2(z)|f(\overline{z})|^2 d\mu^*(z).\end{aligned}$$

It now follows from Stone-Weierstrass theorem that

$$\mu\ =\ |f(\overline{z})|^2\mu^*.$$

It follows then that

$$|f(z)|\ =\ 1\ a.e.\ m\ \text{ on }\ \partial D,$$

$$|f(a_i)|\ =\ 1,$$

and

$$|f(b_j)|\ =\ \sqrt{\frac{\gamma_j'}{\gamma_j}},\qquad |f(\overline{b}_j)|\ =\ \sqrt{\frac{\gamma_j}{\gamma_j'}}.$$

For every polynomial p,

$$\int p\varphi\overline{f}d\mu\ =\ 0;$$

hence, on ∂D the function $\varphi \overline{f} \in H_0^2$. Recalling $|f| = 1$ on ∂D, we can choose an inner function ϕ such that

$$f(z) = \overline{z}\varphi\overline{\phi}(z)$$

for z in ∂D. Let

$$q = \begin{cases} \overline{\phi}\varphi, & \text{on } \partial D \\ 0, & \text{on } D, \end{cases}$$

we notice that

$$\int p(\overline{z}) f \overline{q} d\mu = \int p(\overline{z}) \overline{z} dm = 0.$$

Therefore, $q \in P^2(\mu) \cap L^\infty(\mu)$. It follows now from Theorem 1.2 that $\overline{\phi}\varphi$ is an inner function, I, that has zeros at all atoms of μ. However, since

$$\phi I = \varphi$$

and φ has single zeros at precisely those atoms, it follows that ϕ is a constant. Without loss of generality, we may assume on ∂D, the function f is equal to $\overline{z}\varphi$. For every polynomial p,

$$\int \frac{\varphi}{\varphi_{a_i}} p \overline{f} d\mu = 0.$$

Hence,

$$\int \frac{\varphi}{\varphi_{a_i}} p \overline{f} dm + \frac{\varphi}{\varphi_{a_i}}(a_i) p(a_i) \beta_i \overline{f(a_i)} = 0.$$

Therefore,

$$\int p z \overline{\varphi_{a_i}} dm + \frac{\varphi}{\varphi_{a_i}}(a_i) p(a_i) \beta_i \overline{f(a_i)} = 0.$$

A simple computation shows that

$$|\frac{\varphi}{\varphi_{a_i}}(a_i) p(a_i) \beta_i f(a_i)| = (1 - a_i^2)|p(a_i)|.$$

This implies

$$\beta_i = (1 - a_i^2)/|\frac{\varphi}{\varphi_{a_i}}(a_i)|.$$

Using the same method, we can prove for all j that

$$\sqrt{\gamma_j \gamma_j'} \;=\; (1 \;-\; |b_j|^2)/|\frac{\varphi}{\varphi_{b_j}}(b_j)| \;=\; (1 \;-\; |\overline{b}_j|^2)/|\frac{\varphi}{\varphi_{\overline{b}_j}}(\overline{b}_j)|.$$

Also we have

$$\sum(1 \;-\; a_i^2)/|\frac{\varphi}{\varphi_{a_i}}(a_i)| \;+\; \sum(\gamma_j \;+\; \gamma_j') < \infty$$

since μ is a finite measure. The proof of the Theorem is now completed.

Let $SD = \{\mu : \; \mu$ satisfies the conditions stated in (a) and (b) in Theorem 4.2 $\}$.

Theorem 4.4.3. Suppose S_μ is a pure subnormal operator on $P^2(\mu)$ with $bpe\mu \;=\; D$ and $\frac{\mu|_{\partial D}}{m}$ is log-integrable. The operator S_μ is self-dual if and only if there is $\mu_0 \in SD$ such that S_μ is unitary equivalent to S_{μ_0}

Proof: The sufficiency is obvious, so we need only show the necessity. Let S_μ be a self-dual subnormal operator, then by (*), the measure μ has the following form:

$$\mu \;=\; |r|^2 m \;+\; \sum_{i=1}^{\infty} \beta_i \delta_{a_i} \;+\; \sum_{j=1}^{\infty}(\gamma_j \delta_{b_j} \;+\; \gamma_j' \delta_{\overline{b_j}}).$$

Using the same proof of the last theorem, we can show that $\{a_i, \; b_j, \; \overline{b}_j\}$ is a zero set of a Blaschke product. As before, we write $\varphi \;=\; \Pi\varphi_{a_i}\Pi\varphi_{b_j}\varphi_{\overline{b}_j}$ and $f \;=\; U1$. Using the same argument as in the proof of Theorem 4.4.2, we have

$$|f(z)| \;=\; \frac{|r(\overline{z})|}{|r(z)|} \text{ on } \partial D,$$

$$|f(a_i)| \;=\; 1$$

and

$$|f(b_j)| \;=\; \sqrt{\frac{\gamma_j'}{\gamma_j}}, \; |f(\overline{b}_j)| \;=\; \sqrt{\frac{\gamma_j}{\gamma_j'}}.$$

On the other hand, for all polynomials p we have

$$\int \varphi p \overline{f} |r|^2 dm \;=\; 0.$$

Because r is an outer function, one easily sees that

$$\int \varphi p \overline{f} \overline{r} dm = 0,$$

that is, $\varphi \overline{f} \overline{r} \in H_0^2$. Hence, there are an inner function ϕ and an outer function h so that

$$\varphi \overline{f} \overline{r} = z \phi h.$$

Thus, on ∂D, we have

$$|h(z)| = |\overline{r(\overline{z})}|.$$

So $h(z) = a\overline{r(\overline{z})}$ where a is a constant of modulus one. This means on ∂D, we may assume that

$$f = \frac{r(\overline{z})}{r(z)} \overline{z} \varphi \overline{\phi}.$$

It follows by using a similar argument given in the proof of last theorem that ϕ is a constant function, so we can assume

$$f = \frac{r(\overline{z})}{r(z)} \overline{z} \varphi.$$

Borrowing again from the proof of the last theorem, one can show

$$\beta_i = |r(a_i)|^2 \frac{1 - a_i^2}{|\frac{\varphi}{\varphi_{a_i}}(a_i)|},$$

and

$$\sqrt{\gamma_j \gamma_j'} = |r(b_j)||r(\overline{b}_j)| \frac{1 - |b_j|^2}{|\frac{\varphi}{\varphi_{b_j}}(b_j)|}.$$

Let

$$\beta_i^0 = \beta_i / |r(a_i)|^2,$$

$$\gamma_j^0 = \gamma_j / |r(b_j)|^2, \ \gamma_j^{0'} = \gamma_j' / |r(b_j)|^2,$$

and

$$\mu_0 = m + \sum_{i=1}^{\infty} \beta_i^0 \delta_{a_i} + \sum_{j=1}^{\infty} (\gamma_j^0 \delta_{b_j} + \gamma_j^{0'} \delta_{\overline{b_j}}).$$

one sees that $\mu_0 \in SD$ and $r \in L^2(\mu_0)$. Define the function

$$f_0(z) = \begin{cases} \overline{z}\varphi(z), & \text{on } \partial D \\ -|\frac{\varphi}{\varphi_{a_i}}(a_i)|/\overline{(\frac{\varphi}{\varphi_{a_i}}(a_i))}, & z = a_i \\ -\sqrt{\frac{\gamma_j^{0'}}{\gamma_j^0}}|\frac{\varphi}{\varphi_{b_j}}(b_j)|/\overline{(\frac{\varphi}{\varphi_{b_j}}(b_j))}, & z = b_j \\ -\sqrt{\frac{\gamma_j^0}{\gamma_j^{0'}}}|\frac{\varphi}{\varphi_{\overline{b_j}}}(\overline{b_j})|/\overline{(\frac{\varphi}{\varphi_{\overline{b_j}}}(\overline{b_j}))}, & z = \overline{b_j}. \end{cases}$$

From Lemma 4.4.1, it is easy to check

$$\int pr\overline{f}_0 dm = 0.$$

Therefore, $r \in P^2(\mu_0)$ since f_0 is a cyclic vector of T_{μ_0}; a fact that follows by claim 2 in the proof of Theorem 4.4.2.

Claim: r is a cyclic vector of $P^2(\mu_0)$.

Looking at Clary's Theorems (see [8] or [9]), one sees that the operator S_{μ_0} is quasisimilar to S_m. So there is a cyclic vector ϕ_0 in $P^2(\mu_0)$ such that

$$\int |p\phi_0|^2 d\mu_0 \le M \int |p|^2 dm. \qquad (**)$$

Hence,

$$\int |p\phi_0|^2 dm \le M \int |p|^2 dm.$$

A routine argument yields the fact that ϕ_0 in H^∞. So $\phi_0 r \in P^2(\mu_0)$. Choose a sequence of polynomials p_n that converges to ϕ_0 in the weak-star topology of $L^\infty(\mu_0)$. Therefore, $p_n r$ converges to $r\phi_0$ in weak topology of $L^\infty(\mu_0)$. Thus,

$$\phi_0 r \in \overline{span\{pr\}},$$

the closure being in the norm topology of $L^\infty(\mu_0)$. Because r is an outer function, fix a polynomial p, we can choose another sequence of polynomials q_n so that $q_n r$ converges to p in $H^2(m)$. Hence, it follows from (**) that

$$p\phi_0 \in \overline{span\{pr\}}.$$

Recalling ϕ_0 is a cyclic vector for $P^2(\mu_0)$, we now have,

$$\overline{span\{pr\}} = P^2(\mu_0).$$

Now let U be the operator from $P^2(\mu)$ to $P^2(\mu_0)$ defined by

$$Up = rp$$

for every polynomial p. It turns out that U is a unitary operator and

$$US_\mu U^* = S_{\mu_0}.$$

The proof of the Theorem is now finished.

Section 4.5. Approaches to the general cases

The results of Theorem 4.4.2 and Theorem 4.4.3 show that the atoms of the scalar spectral measure play an important role in the study of self-dual cyclic subnormal operators. The next theorem shows if the set of atoms is not too large, then the structure of this cyclic operator is understood.

Theorem 4.5.1. Let S_μ be a cyclic irreducible subnormal operator on $P^2(\mu)$ with $bpe\mu = G$. Suppose the set of atoms of the scalar spectral measure μ (equivalently, the set $\sigma(N) \setminus \sigma_e(N)$) is a zero set of a nonzero function in $H^\infty(G)$. Then S_μ is a self-dual subnormal operator if and only if the following two properties hold:

(1) G is symmetric with respect to the real axis. In this case, we let ψ be the Riemann map from D to G so that $Im\psi(0) = 0$, $\psi'(0) > 0$ and $\psi(z) = \overline{\psi(\overline{z})}$. Note that the analytic Toeplitz operator T_ψ is cyclic on $H^2(\partial D)$; in fact 1 is a cyclic vector.

(2) There is $\mu_0 \in SD$ such that S_μ is unitarily equivalent to the the operator of multiplication by $\tilde{\psi}$ $(=M^{\mu_0}_{\tilde{\psi}})$ on $P^2(\mu_0)$.

Proof: The sufficiency is obvious. We assume that S_μ is a self-dual subnormal operator. Using Theorem 4.2.7, we know that G is symmetric with respect to the real axis; so we choose a Riemann map as in (1). Let $\nu = \mu \circ \tilde{\varphi}^{-1}$, where $\varphi = \psi^{-1}$. According to Theorem 4.3.1, the operator S_μ is unitarily equivalent to multiplication by $\tilde{\psi}$ on $P^2(\nu)$ and S_ν is a self-dual subnormal operator with $bpe\nu = D$. Using the hypotheses, we conclude that there is a nonzero function in $H^\infty(D)$ whose zero set is the set of all atoms of ν. Let $f \in P^2(\nu)^\perp$ be a cyclic vector and let ϕ be an inner function in H^∞ whose zero set is precisely the set of all atoms of ν. We have then that

$$\int p\phi\overline{f}d\nu = 0.$$

Thus, $P^2(\nu|_{\partial D})$ is pure. Therefore,

$$\int log(\frac{\nu|_{\partial D}}{dm})dm \ > \ -\infty.$$

According to Theorem 4.4.3, there is a measure μ_0 so that S_ν is unitary equivalent to S_{μ_0}. Hence, S_μ is unitarily equivalent to the multiplication by $\widetilde{\psi}$ on $P^2(\mu_0)$.

Now we need only show T_ψ is cyclic. In fact, using Clary's Theorem, we know that S_{μ_0} is quasisimilar to S_m; therefore, $T^{\mu_0}_{\underset{\psi}{\sim}}$ is quasisimilar to T_ψ. This means T_ψ has a cyclic vector. The theorem is proved.

Remark: We believe that the conditions (1) and (2) are the necessary and sufficient conditions for an irreducible cyclic subnormal operator to be self-dual. We believe that our hypothesis on the atoms is a by product of the hypothesis of self-duality.

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