Extended Describing Function Method For Small-Signal Modeling of Resonant and Multi-Resonant Converters

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(ABSTRACT)

The extended describing function method is proposed as a systematic small-signal modeling approach to nonlinear switching circuits. This method offers significant simplification upon the previous work on using the multi-variable describing functions to treat the circuit nonlinearities. As an extension to the state-space averaging method, this modeling technique can incorporate any Fourier components for good model accuracy and provides continuous-time small-signal models for PWM topologies and various soft-switching resonant topologies.

The proposed method is demonstrated using four resonant topologies and two multi-resonant topologies. These circuits are strongly oscillatory, and thus they cannot be modeled by means of traditional averaging techniques. By employing the proposed modeling method, the dynamics of the resonant converters are analyzed with emphasis on the nonlinear interaction between the switching frequency and the circuit natural resonant frequency. Equivalent circuit models are provided for more convenience of practical designs. Small-signal analysis is also performed for two
challenging multi-resonant topologies with complex structure and operation. All of the theoretical models are verified experimentally and the predictions are well supported by the measurement data up to the Nyquist frequency.
To my parents
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1. Introduction

1.1. Background and Motivation

Switching converters are nonlinear forced oscillation systems with feedback control. Due to the complexity of the nonlinear oscillation, it is very difficult to optimize the dynamic performances related to the control design in the large-signal sense. However, small-signal analysis can treat the control issues very effectively by providing the small-signal model, which governs the dynamic behavior of the converter around a given operating point. Since the small-signal model describes the converter as a linear time invariant (LTI) plant, it allows us to apply the knowledge of linear system control to the switching converters. In addition, the control-loop design based on small-signal modeling can be verified experimentally when necessary.

There exist a variety of switching converter topologies, and new topologies have been invented at an astonishing rate in recent years. Among them, various soft
switching resonant topologies have been proposed. Considering major converter families, a list could include PWM converters, quasi-resonant converters (QRCs), resonant converters, multi-resonant converters (MRCs), and soft-switching PWM converters. The existing modeling techniques based on the averaging concept are essentially limited to model PWM converters and QRCs. The averaging concept methods breaks down for resonant and multi-resonant converters. There has been a strong desire to develop a general small-signal modeling approach to treat all of the switching converters in a unified way, and to implement such a method in the form of a computer program to provide a useful tool for practical designs.

The dynamic behaviors of the PWM converters and the QRCs are basically determined by the low pass filters; therefore, the average techniques [1-6, 68] can be employed to model these circuits. State-space averaging method [1-5] is a popular approach for PWM converters. The method can provide continuous-time small-signal models which are accurate up to one half of the output ripple frequency, or the Nyquist frequency. The analysis of the state-space averaging was simplified by using a circuit-averaging technique based on three-terminal PWM switch model [6]. This concept was extended successfully to quasi-resonant converters (QRCs) [68].

However, the averaging concept breaks down for resonant converters. For these circuits, some of the state variables do not have dc components but contain strong switching frequency harmonics, whereas for PWM and quasi-resonant converters, the dc components are the dominant parts of the state variables. Due to the strongly oscillatory nature, the switching frequency interacts with the natural resonant fre-
quency. This interaction is often referred to as the beat frequency dynamics [41], which cannot be investigated using the averaging concept because it eliminates the switching frequency information.

The MRCs are very similar in many aspects to the resonant converters: they all have a strongly oscillatory nature; switching frequency harmonics play important roles in energy delivery; and the interactions exist between the natural resonant frequencies and the switching frequency. The major differences between the MRCs and the resonant converters include:

1) some of the state variables in the resonant tank are dc biased and some are not; and

2) there exist two or three resonant frequencies corresponding to the on/off status of the active and the passives switches.

These features make the MRCs more complicated to model.

While the averaging techniques cannot be applied to any converter circuits, the sampled-data method is considered a general modeling approach [8-16]. The method provides discrete-time small-signal models which are LTI difference equations with the switching period as the sampling interval. Although the discrete-time approach is very general, continuous-time small-signal models are preferred for practical design due to the following reasons. First, the discrete-time models cannot be directly verified by commonly used measurement scheme using a network analyzer [37-40]. Second, if the voltage mode control is used and the compensator is a low pass filter, the control-signal fed into the modulator is usually switching-ripple free. In this

1. Introduction
situation, the modulator can be treated as a simple constant gain block [24], and the
sampling effect can be ignored. Since the most widely used sensing and compensation
circuitry is analog network, a continuous-time small-signal model of the power stage
will be consistent with the control-loop design.

Regarding the resonant converters and MRCs, little has been done in the past
to provide continuous-time small-signal models which can correctly predict the
small-signal responses when the modulation frequency is close to the Nyquist fre-
quency. The challenge involves capturing the high frequency dynamics caused by
the interaction of the resonant frequencies and the switching frequency. It is very
important to have the adequate small-signal models for the control-loop optimization,
because failure to account for the beat frequency dynamics (usually a double pole
for resonant converters) may result in instability. However, the continuous-time
small-signal models accurate up to the Nyquist frequency are not available, and
comprehensive study has not been done to date for the beat frequency dynamics.

The lack of good continuous-time small-signal models is due to the lack of an
appropriate modeling approach. An effort was made in [35] to extend the state-space
averaging technique to modeling the resonant converters. Instead of only considering
the dc components, as was done in the averaging methods, the amplitudes of the
switching frequency harmonics were also taken as the generalized average variables.
A resonant converter was used to demonstrate the modeling concept. A
continuous-time small-signal model was derived for a series resonant converter under
the assumption that the state variable in the resonant tank can be well approximated
by sinusoidal waves (at switching frequency). The derived model can correctly
predict the beat frequency dynamics. However, this model is questionable for the operating conditions where the resonant waveforms are highly distorted from sine waves. For the MRCs, the state variables in the resonant tank cannot be adequately approximated by their dc components or fundamental components. The approach shown in [35] cannot be carried out when several harmonic components are required to properly approximate the state variables. Although [35] presented a modeling concept, it did not propose a method to implement the concept in the general case.

In [28], J. Groves proposed a modeling approach using the harmonic balance technique. This method imitates the network analyzer to calculate the small-signal spectrum when a given converter is subject to modulation. There is no approximation (other than small-signal) made in Groves' method, therefore, the accurate model can match the measurement results even when the modulation frequency is beyond the Nyquist frequency. This method features very complex but detailed mathematic treatments for general modeling problems. The execution of the method depends upon knowing the steady-state responses of the converter over a long period which is commensurate with the switching period and the modulation period. Generally, obtaining such steady-state responses requires performing large-signal simulation over the commensurate period which usually contains hundreds of switching cycles. Therefore, the method is very time consuming.

This dissertation is an effort to develop a generalized modeling method which is named extended describing function method. This method is based on modification of Groves’ method [28] to provide simple continuous-time models for general switching converters. It offers considerable simplification to the modeling procedure.
in [28] on using multi-variable describing functions to treat the circuit nonlinearities, therefore, the extensive simulation effort is avoided. As a systematic yet relatively simple modeling approach, the proposed method allows us to incorporate any number of harmonic components for good model accuracy. This is a significant advantage comparing with the generalized averaging method [35]. The proposed method covers the state-space averaging and circuit-averaging as its special cases. Therefore, the PWM converters and the QRCs are the simplest applications. Using this method, continuous-time small-signal models are obtained for the resonant converters, MRCs, and possibly other more complicated topologies. The modeling technique is demonstrated using examples of four resonant topologies and two multi-resonant topologies. The experimental results confirm that the derived small-signal models are accurate up to the Nyquist frequency. The dynamic behavior of these converters will be analyzed with emphasis on the beat frequency dynamics and their impact on the control design. Equivalent circuit models are synthesized for more convenience in practical applications.

1.2. Dissertation Outline

Chapter 2 of this dissertation presents the extended describing function method. The existing modeling approaches proposed in [28] will be reviewed first.
Proposed by J. Groves, the method employs the harmonic balance technique over the commensurate period and provides accurate small-signal models which can predict the system dynamics beyond the Nyquist frequency. But the method depends on extensive simulation efforts because it needs to know the circuit steady-state responses over the commensurate period (with switching period and modulation period) under every modulation frequency. Therefore, Groves' methods is complicated and time consuming.

The extended describing function method is proposed as a modification to Groves' method [28]. It offers considerable simplification on using multi-variable describing functions to treat the circuit nonlinearities. The execution of the proposed approach only needs the steady-state information when the converter is not subject to the modulation. Therefore, the extensive simulation effort is avoid. The proposed method provides continuous-time small-signal models in state-space representation which can incorporate any number of harmonics for good model accuracy; whereas the generalized averaging method [35] only provided solutions for the cases when state variables can be well approximated by dc or fundamental components. All of the measurable small-signal transfer functions can be extracted from the extended describing function models. The method is implemented by a computer program to provide a useful modeling tool.

In Chapter 3, two useful series resonant converters (SRCs) are analyzed to demonstrate the proposed modeling technique. The modeling procedure and the capability of the modeling tool are presented. The effect of incorporating higher order harmonics will be investigated, especially for those operating conditions where
the resonant waveforms are highly distorted from pure sine waves. The dynamic behavior of the SRCs, in particular the beat frequency dynamics, is carefully studied for variant operating conditions. The small-signal models are verified by the experimental data; the results confirm that the models are accurate up to the Nyquist frequency. Based on the analysis results, a unified small-signal equivalent circuit is synthesized for both the SRC topologies. The model can be easily implemented in circuit simulation software for the convenience of practical design.

In Chapter 4, small-signal analysis is performed for two parallel resonant converters (PRCs). The issues of waveform distortion and the effect of higher order harmonics are investigated. The beat frequency dynamics are carefully studied especially when the converters are operated very close to the resonant frequency. Based on the analysis, a unified small-signal equivalent circuit model is derived to facilitate the computer simulation. The model predictions are confirmed experimentally for both low frequency and high frequency accuracy.

Two multi-resonant converters (MRCs) are modeled in Chapter 5 using the extended describing function method. These circuits are very complex because of their structure and the operation, but they can be treated straightforwardly by our systematic modeling technique. The details are discussed on how to take different Fourier components for each state variables. The effects of higher order harmonics are also studied. The control-to-output transfer functions of these circuits are characterized by Bode plots and the pole-zero distribution. The small-signal analysis is verified experimentally. The measurement results support the model predictions up to the Nyquist frequency.
Conclusions are presented in Chapter 6. Those readers who want to use the proposed modeling method to investigate other circuits can find the computer program and related documentation in Appendix A. Also provided in Appendix B are the SPICE node lists of the equivalent small-signal models of the SRCs and the PRCs.
2. Extended Describing Function Method

2.1. Introduction

In this chapter, we will first provide basic definitions and relations with respect to the Fourier analysis of the switching converters. Then, we will review the existing modeling approach proposed by J. Groves in [28]. This method imitates the network analyzer to calculate the small-signal spectrum of a given converter. The method makes no simplified assumption, and it can predict the measurement results beyond the Nyquist frequency. Groves’ method is rather complex, but it provides very detailed mathematical treatment of the general modeling problem. The execution of the method depends upon knowing the steady-state responses of the converter over a long commensurate period

\[ T_c = NT_s = MT_m, \]  

(2.1)
for every given modulation frequency, where \( M \) and \( N \) are integers, and \( T_s, T_m \) stand for switching period and modulation period, respectively. Usually, \( T_c \) includes hundreds of switching cycles. Generally speaking, getting the steady-state responses requires performing large-signal simulation over the commensurate period. Therefore, the method is time consuming and difficult to execute.

The extended describing function method is proposed as a generalized modeling approach based on modification of Groves' method. It offers considerable simplification to Groves' method on using the multi-variable describing functions to treat the system nonlinearities. The method only needs the steady-state information for a given operation-point (without being subject to a disturbance), and gives a continuous-time small-signal model in state-space representation which can incorporate any number of harmonics to improve the model accuracy. All of the measurable small-signal transfer functions can be extracted from the model.

A comparison is made between the proposed method and the generalized averaging method [35]. It will be shown that the proposed method provides systematic implementation for the general modeling problem, whereas the analytical derivation suggested in [35] cannot be carried out when several harmonic components are required to properly approximate the state variables. The last section gives a brief introduction to the computer program implementing the extended describing function method. More details about the program can be found in Appendix A.
2.2. Fourier Analysis of Switching Converters

A switching converter can be generally described by a nonlinear state equation

\[ \dot{x} = f(x, u, t) \]  \hspace{1cm} (2.2)

where the state vector and the input vector are denoted by \( x \) and \( u \), respectively. Usually, the state variables are the inductor currents and the capacitor voltages in the circuits, and the input variables are the line voltage and the control reference.

Under steady-state, the input \( u \) is a constant vector and the state trajectory is periodic over the switching period, \( T_s \). Therefore, the state vector can be expanded into Fourier series:

\[ x^{ss}(t) = \sum_{k=-\infty}^{\infty} X_k^{ss} e^{j k \omega_s t} \] \hspace{1cm} (2.3)

where

\[ X_k^{ss} = \frac{1}{T_s} \int_0^{T_s} x^{ss}(t)e^{-j k \omega_s t} \, dt \] \hspace{1cm} (2.4a)

\[ \omega_s = \frac{2\pi}{T_s} \] \hspace{1cm} (2.4b)

The superscript, \( ss \), represents the steady-state operation.

Similarly, the nonlinear function defined in (2.2) can also be expanded into Fourier series under steady-state operation:

2. Extended Describing Function Method

12
\[ f(x^{ss}, U_0, t) = \sum_{k = -\infty}^{\infty} F_k^{ss}(X^{ss}, U_0)e^{jk\omega_0 t}, \quad (2.5) \]

where

\[ F_k^{ss}(X^{ss}, U_0) = \frac{1}{T_s} \int_0^{T_s} f(x^{ss}(t), U_0, t)e^{-jk\omega_0 t} dt, \quad (2.6) \]

\[ X^{ss} = (\ldots, X_{-k}^{ss}, \ldots, X_0^{ss}, \ldots, X_k^{ss}, \ldots) \quad (2.7) \]

The Fourier coefficient, \( F_k^{ss}(\cdot) \), is a function of the steady-state input, \( U_0 \), and all of the Fourier coefficients of \( x^{ss}(t) \). Therefore, \( F_k^{ss}(\cdot) \) can be called a multi-variable describing function.

The steady-state spectrum of \( x(t) \), \( u \), and \( f(x, u, t) \) is illustrated in Fig. 2.1. Notice that

\[ F_0^{ss}(\cdot) = \frac{1}{T_s} \int_0^{T_s} f(x^{ss}, u, t)dt = 0 \quad (2.8) \]

is required by the flux balance and the charge balance of the energy-storage components in the circuit.

It is known that when the switching converter is under modulation, the side-band spectrum will appear to each of the Fourier components located at multiplies of the switching frequency, provided the modulation signal is a small-amplitude sinusoid:

2. Extended Describing Function Method
Fig. 2.1. The steady-state spectrum of a switching converter. The flux balance and the charge balance of the energy storage components require that

\[ F_{0}^{ss} = 0 \]
u(t) = U_0 + U_M e^{j\omega_m t} + U_M e^{-j\omega_m t} . 

(2.9)

If the modulation frequency, \( \omega_m \), and the switching frequency, \( \omega_s \), are commensurate, or if the modulation frequency is restricted to the values

\[
\omega_m = M \left( \frac{\omega_s}{N} \right) \quad M = 1, 2, \ldots N, N+1, \ldots ,
\]

(2.10)

then a commensurate period, \( T_c \), can be defined, which contains integer numbers of switching period, \( T_s \), and modulation period, \( T_m \):

\[ T_c = N T_s = M T_m . \]

(2.11)

The state trajectory is periodic over this commensurate period when system reaches the steady-state under the modulation.

In this situation, the state variables can also be expanded into Fourier series over \( T_c \):

\[
x(t) = \sum_{l=-\infty}^{\infty} X_l e^{j l \omega_c t},
\]

(2.12)

where

\[
X_l = \frac{1}{T_c} \int_{0}^{T_c} x(t) e^{-j l \omega_c t} dt ,
\]

(2.13a)

\[
\omega_c = 2\pi/T_c .
\]

(2.13b)

2. Extended Describing Function Method
The spectrum of the converter under modulation is illustrated in Fig. 2.2. Notice that the nonzero spectrum components are those located at $kN\omega_c$ or $k\omega$, which correspond to the steady-state spectrum

$$X_{kN} = X_k^{ss},$$

and those located at $(kN \pm M)\omega_c$ or $k\omega_s \pm \omega_m$, which correspond to the side-bands of the kth harmonics.

By expanding the nonlinear function in (2.2) into Fourier series, and assuming the $lth$ harmonic component is a function of the Fourier components of $x(t)$ and $u$, we have:

$$f(x, u, t) = \sum_{l=-\infty}^{\infty} F_l(X, U) e^{jl\omega_c t},$$

where

$$F_l(\cdot) = \frac{1}{T_c} \int_0^{T_c} f(x, u, t) e^{-jl\omega_c t} dt.$$  

$F_l(X, U)$ is also a multi-variable describing function term, and $X$ and $U$ are the collections of all the Fourier coefficients of $x(t)$ and $u$, respectively:

$$X = (\ldots X_{-l}, \ldots, X_0, \ldots, X_l, \ldots),$$

$$U = (U_{-M}, U_0, U_M).$$
Fig. 2.2. The spectrum of a switching converter under small-signal modulation.
2.3. Review of Groves’ Method

J. Groves proposed a modeling method based on the Fourier analysis over the commensurate period [28]. Harmonic balance was employed to relate the state vector spectrum and the nonlinear function spectrum (or the describing function terms). Differentiating $x(t)$ using (2.12), and letting it equal the nonlinear function (2.15), we have:

$$
\sum_{l=-\infty}^{\infty} j l \omega_i X_i e^{j l \omega_i t} = \sum_{l=-\infty}^{\infty} F_i(X, U) e^{j l \omega_i t} .
$$

(2.19)

The Fourier coefficients of the $l$th harmonics should be equal, i.e.

$$
j l \omega_i X_i = F_i(X, U) .
$$

(2.20)

This procedure is called harmonic balance. As shown in [28], linearizing (2.20) for the spectrum component located at $k \omega_i + \omega_m$, we can get

$$
j (kN + M) \omega_i X_{kN + M} = \sum_{m=-\infty}^{\infty} \frac{\partial F_{kN + M}}{\partial X_{mN + M}} X_{mN + M} + \frac{\partial F_{kN + M}}{\partial U_M} U_M ,
$$

(2.21)

$$
k = 0, \pm 1, \pm 2, \ldots .
$$

The objective is to solve the small-signal spectrum

$$\{ X_{kN + M}, \quad k = 0, \pm 1, \pm 2, \ldots \}$$

(2.22)
using (2.21) for every modulation frequency. By truncating the (2.21) into finite terms, it becomes solvable if the partial derivative terms are known.

In order to find the partial derivatives, piecewise linear description of the switching converter is employed for further derivation. It is assumed that the converter is formed by $Q$ pieces of linear network in one switching cycle, the linear circuits are described by the following state equations:

$$\dot{x} = A_i x + B_i u \quad ,$$

$$i = 1, 2, \ldots, Q \quad ,$$

where the $ith$ network is valid for the time interval

$$t_{i-1}^{(p)} < t < t_i^{(p)} \quad ,$$

$$p = 1, 2, \ldots, N \quad .$$

The superscript, $(p)$, denotes the interval belonging to $p$th switching cycle in the total period $T_c$, as illustrated in Fig. 2.3.

The switching boundary conditions [31] are used to determine the switching instant from the $ith$ interval to $(i + 1)th$ interval:

$$0 = a_i x(t_i^{(p)}) + b_i u(t_i^{(p)}) + c_i t_i^{(p)} + d_i \quad .$$

After a lengthy derivation, the partial derivative terms are found in [28]:

2. Extended Describing Function Method
Fig. 2.3. Piecewise linear description of the converter and the switching instants.
\[
\frac{\partial F_i}{\partial X_v} = \sum_{p=1}^{N} \sum_{i=1}^{Q} \left\{ \delta[x(t_i^{(p)}), u(t_i^{(p)})] e^{-j\omega_c t_i^{(p)}} \frac{\partial \bar{f}_i}{\partial X_v} + A_i \gamma_{i,v}[t_i^{(p)}, t_{i-1}^{(p)}] \right\}, \quad (2.26a)
\]
\[
\frac{\partial F_i}{\partial U_M} = \sum_{p=1}^{N} \sum_{i=1}^{Q} \left\{ \delta[x(t_i^{(p)}), u(t_i^{(p)})] e^{-j\omega_c t_i^{(p)}} \frac{\partial \bar{f}_i}{\partial U_M} + B_i \gamma_{i,v}[t_i^{(p)}, t_{i-1}^{(p)}] \right\}, \quad (2.26b)
\]

where

\[
\delta[x(t_i^{(p)}), u(t_i^{(p)})] = \begin{cases} 
\frac{1}{T_c} [A_i x(t_i^{(p)}) + B_i u(t_i^{(p)})] & i = Q \\
\frac{1}{T_c} [(A_i - A_{i+1}) x(t_i^{(p)}) + (B_i - B_{i+1}) u(t_i^{(p)})] & i = 1, \ldots Q - 1
\end{cases}
\]

(2.27)

\[
\gamma_{i,v}[t_i^{(p)}, t_{i-1}^{(p)}] = \frac{1}{T_c} \int_{t_{i-1}^{(p)}}^{t_i^{(p)}} e^{-j\omega_c (l-v)\tau} d\tau, \quad (2.28)
\]

\[
l = kN + M, \quad v = mN + M
\]

The partial derivatives of the switching instant, \(t_i^{(p)}\), with respect to the Fourier coefficient \(X_v\) and \(U_M\) was found by perturbing the switching boundary condition (2.25) to give:

\[
\frac{\partial t_i^{(p)}}{\partial X_v} = \left( \frac{-a_i}{a_i x(t_i^{(p)}) + b_i u(t_i^{(p)}) + c_i} \right) e^{j\omega_c t_i^{(p)}}, \quad (2.29a)
\]
\[
\frac{\partial t_i^{(p)}}{\partial U_M} = \left( \frac{-b_i}{a_i x(t_i^{(p)}) + b_i u(t_i^{(p)}) + c_i} \right) e^{jM\omega_c t_i^{(p)}}, \quad (2.29b)
\]

This concludes the derivation.

2. Extended Describing Function Method
To execute the method, one needs to know all of the switching instants and the values of the state variables at those instants

\[ \{ t_i^{(p)}, x(t_i^{(p)}), i = 1, \ldots, Q, p = 1, \ldots, N \} \quad (2.30) \]

In other words, the partial derivatives become calculatable only if the steady-state response over the commensurate period has been known for every modulation frequency.

To summarize, the advantages of Groves’ method are as follows:

1) It makes no other assumptions but the small-signal assumption. By performing the Fourier analysis over the commensurate period, \( T_c \), this method can precisely calculate small-signal spectrum components, even when modulation frequency is higher than the Nyquist frequency.

2) The method provides a detailed mathematical derivation using the piecewise linear description of the switching network. It can incorporate any number of harmonics into the solution to improve the model accuracy.

The disadvantages of this method is that it depends on knowing the information (2.30) for every modulation frequency. This means the steady-state over the commensurate period has to be reached before the solving procedure can be carried on. The simulation is the only method in this situation for finding the steady-state response. For example, if \( N = 100 \) is used (\( N \) determines the resolution of the Bode plot, and \( N = 100 \) is reasonable for most applications), and if the modulation frequency sweeps through the values
\[ \omega_m = M \left( \frac{\omega_s}{N} \right), \quad M = 1, 2, 3, \ldots, N, N + 1, \ldots, M_{\text{max}} \] (2.31)

then the steady-state response has to be found over the commensurate period (in this case, \( T_c = 100T_s \)) for every \( M \). Assuming that the algorithm built in the simulator iterates three times to fine the steady-state, then three hundred switching cycles have to be simulated to get just one point on the Bode plot. This procedure is very complicated and computationally inefficient.

### 2.4. Extended Describing Function Method

#### 2.4.1. Basic Assumption and Formulation

In this section, the extended describing function method is proposed based on modification of Groves' method. The objectives are to reduce the model complexity and to simplify the implementation procedure.

As discussed in last section, the model complexity comes from the Fourier analysis over the commensurate period. Meanwhile, we know that the analysis also makes Groves’ method capable of predicting the small-signal responses accurately. Now, we want to simplify the analysis over \( T_c \) under certain condition to obtain a useful approximate model.
To achieve this, we need to recall the small-signal model (2.21) derived by Groves:

\[ j(kN + M)\omega \omega X_{kN + M} = \sum_{m=-\infty}^{\infty} \frac{\partial F_{kN + M}}{\partial X_{mN + M}} X_{mN + M} + \frac{\partial F_{kN + M}}{\partial U_M} U_M, \]

\[ k = 0, \pm 1, \pm 2, \ldots, \pm K. \]

To better explain the modeling idea and avoid too complicated mathematic expressions, let us assume the circuit under analysis is a PWM converter. Since the average components of the state variables dominant the dynamic behavior, we only need to consider the side-bands of the average components (refer Fig. 2.2). Then the model (2.32) reduces to the following form for \( K = 0 \):

\[ jM \omega \omega X_M = \frac{\partial F_M}{\partial X_M} X_M + \frac{\partial F_M}{\partial U_M} U_M. \]

Multiplying both side of (2.33) with \( e^{j\omega t} \), we can define the slowly varying small-signal stimulus and the perturbed state vector:

\[ \hat{u} = U_M e^{j\omega t}, \]

\[ \hat{x}_0 = X_M e^{j\omega t}. \]

Then (2.33) can be deduced to

\[ \frac{d\hat{x}_0}{dt} = \frac{\partial F_M}{\partial X_M} \hat{x}_0 + \frac{\partial F_M}{\partial U_M} \hat{u}. \]
Notice that the following relations are used to derive (2.36):

\[ M \omega_c = \omega_m \]  \hspace{1cm} (2.37)

\[ jM \omega_c X_M e^{j\omega_m t} = \frac{d\hat{x}_0}{dt} \]  \hspace{1cm} (2.38)

It should be pointed out that although (2.36) has a linear form, it is not a linear time invariant (LTI) model because the model parameters (partial derivatives) are functions of the stimulus frequency or the modulation frequency. In [28], equation (2.36) is solved for every modulation frequency by evaluating the partial derivatives over the commensurate period.

While Groves’ method provides accurate but complex solutions, we know that state space averaging method can provide an LTI model which is based on the average over one switching cycle and the model is quite accurate when the modulation frequency is lower than one half of the switching frequency (the Nyquist frequency).

More specifically, if the partial derivatives defined over \( T_c \) (under modulation) can be approximated by their counterparts defined over \( T_s \) (under steady-state), \( i.e. \)

\[ \frac{\partial F_M}{\partial X_M} \approx \frac{\partial F_0^{ss}}{\partial X_0^{ss}} \]  \hspace{1cm} (2.39)

\[ \frac{\partial F_M}{\partial U_M} \approx \frac{\partial F_0^{ss}}{\partial U_0^{ss}} \]  \hspace{1cm} (2.40)

then an approximate LTI small-signal model can be derived from (2.36):

2. Extended Describing Function Method
\[
\frac{d\hat{x}_0}{dt} = \frac{\partial F_{00}^{ss}}{\partial X_0^{ss}} \hat{x}_0 + \frac{\partial F_{00}^{ss}}{\partial U_0} \hat{u}.
\] (2.41)

Now, we are going to prove that the approximations in (2.39) and (2.40) are valid under the following conditions:

1) the perturbation is at small-signal level;
2) the modulation frequency is much lower than the switching frequency.

In the first step, we will derive an approximate expression of the side-band \(X_M\), which is the spectrum component of \(x(t)\) located at \(\omega_m\) (refer Fig. 2.2). According to (2.13), \(X_M\) is defined by

\[
X_M = \frac{1}{T_c} \int_0^{T_c} x(t) e^{-j\omega_m t} dt.
\] (2.42)

By chopping the time interval \([0, T_c]\) into \(N\) pieces with equal length of \(T_s\), (2.42) is deduced to

\[
X_M = \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{T_s} \int_{kT_s}^{(k+1)T_s} x(t) e^{-j\omega_m t} dt.
\] (2.43)

Since we have assume that the modulation frequency is much lower than the switching frequency, the exponential term, \(e^{-j\omega_m t}\), changes very little over the interval \([kT_s, (k + 1)T_s]\). Therefore, we have the following approximation:

\[
X_M = \frac{1}{N} \sum_{k=0}^{N-1} e^{-j\omega_m kT_s} \left[ \frac{1}{T_s} \int_{kT_s}^{(k+1)T_s} x(t) dt \right],
\] (2.44)
The term in the bracket is the average of the state vector over the \(k\)th switching period. This average can be expressed by two terms:

\[
\frac{1}{T_s} \int_{kT_s}^{(k+1)T_s} x(t) dt = X_{0}^{ss} + \hat{x}_0(kT_s),
\]  

(2.45)

where the first term represents the steady-state average, and the second term is the perturbed average over the \(k\)th switching period. Substituting (2.45) into (2.44), we have

\[
X_M = \frac{1}{N} \sum_{k=0}^{N-1} X_0^{ss} e^{-j\omega_m kT_s} + \frac{1}{N} \sum_{k=0}^{N-1} \hat{x}(kT_s) e^{-j\omega_m kT_s}
\]

\[
= \left(\frac{X_0^{ss}}{N}\right) \left(1 - e^{-j\omega_m T_s N}\right) + \frac{1}{N} \sum_{k=0}^{N-1} \hat{x}(kT_s) e^{-j\omega_m kT_s}.
\]

(2.45)

Since

\[
1 - e^{-j\omega_m T_s N} = 1 - e^{-j2\pi M} = 0,
\]

(2.47) becomes

\[
X_M = \frac{1}{N} \sum_{k=0}^{N-1} \hat{x}(kT_s) e^{-j\omega_m kT_s}.
\]

(2.48)

This equation represents the relation of the side-band component \(X_M\) and the perturbed average of \(x(t)\) over the \(k\)th switching period. In fact, (2.48) is the discrete Fourier transform (DFT) of \(\{\hat{x}(kT_s), k = 0, 1, \ldots, N-1\}\).
In the following step, we use the same idea to derive a similar expression for the side-band $F_m$, which is the spectrum component of the nonlinear function, $f(x, u, t)$, located at $\omega_m$ (refer Fig. 2.2). According to (2.16), we have:

$$F_m = \frac{1}{T_c} \int_0^{T_c} f(x, u, t)e^{-j\omega_m t} dt$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \frac{1}{T_s} \int_{kT_s}^{(k+1)T_s} f(x, u, t)e^{-j\omega_m t} dt$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} e^{-j\omega_m kT_s} \left[ \frac{1}{T_s} \int_{kT_s}^{(k+1)T_s} f(x, u, t) dt \right].$$  \hspace{1cm} (2.49)

Notice that under steady-state the average of the nonlinear function over the $k$th switching period can be represented by the describing function:

$$\frac{1}{T_s} \int_{kT_s}^{(k+1)T_s} f(x^{ss}, U_0, t) dt = F_0^{ss}(x_0^{ss}, U_0) = 0.$$  \hspace{1cm} (2.50)

If the system is subject to small-signal modulation and $\omega_m$ is much lower than $\omega_s$, we can assume the perturbed input and the perturbed average of the $x(t)$ are constants over the $k$th switching period, $[kT_s, (k+1)T_s]$, then we have

$$\frac{1}{T_s} \int_{kT_s}^{(k+1)T_s} f(x, u, t) dt = F_0^{ss}[X_0^{ss} + \hat{x}_0(kT_s), U_0 + \hat{u}(kT_s)].$$  \hspace{1cm} (2.51)

Expanding (2.51) into Taylor series and dropping the higher order small-signal terms, we get
\[ F_0^{ss}[X_0^{ss} + \hat{x}_0(kT_s), U_0 + \hat{u}(kT_s)] = F_0^{ss}(X_0^{ss}, U_0) + \frac{\partial F_0^{ss}}{\partial X_0^{ss}} \hat{x}(kT_s) + \frac{\partial F_0^{ss}}{\partial U_0} \hat{u}(kT_s) . \]

(2.52)

According to (2.50), the first term of the Taylor expansion is zero. Substituting (2.52) into (2.49), we have

\[ F_M = \left( \frac{\partial F_0^{ss}}{\partial X_0^{ss}} \right) \frac{1}{N} \sum_{k=0}^{N-1} \hat{x}(kT_s) e^{-j\omega_m kT_s} + \left( \frac{\partial F_0^{ss}}{\partial U_0} \right) \frac{1}{N} \sum_{k=0}^{N-1} \hat{u}(kT_s) e^{-j\omega_m kT_s} . \]

(2.53)

From (2.48), we know that the first summation is \( X_M \) or the DFT of the averaged state vector. The second summation of (2.53) represents the DFT of the small-signal stimulus, which is \( U_M \) or the side-band of the input located at \( \omega_m \) (refer Fig. 2.2). Substituting \( X_M \) and \( U_M \) into (2.53), it becomes

\[ F_M = \frac{\partial F_0^{ss}}{\partial X_0^{ss}} X_M + \frac{\partial F_0^{ss}}{\partial U_0} U_M . \]

(2.54)

From (2.54), it is straightforward to see that

\[ \frac{\partial F_M}{\partial X_M} = \frac{\partial F_0^{ss}}{\partial X_0^{ss}} , \]

(2.55)

\[ \frac{\partial F_M}{\partial U_M} = \frac{\partial F_0^{ss}}{\partial U_0} . \]

(2.56)

2. Extended Describing Function Method
At this point, we have proved that if the modulation is at small-signal level and the modulation frequency is low, the partial derivatives defined over $T_c$ can be approximated by the partial derivatives defined over $T_s$.

To further demonstrate the procedure of evaluating the partial derivative terms and to reveal the relation between the approximate model (2.41) and the state space averaging model [2], we need to employ the piecewise linear description of the PWM converter:

\[ \dot{x} = A_1 x + (b_1, 0) \begin{pmatrix} v_g \\ d \end{pmatrix}, \quad 0 \leq t < t_1, \quad (2.57a) \]

\[ \dot{x} = A_2 x + (b_2, 0) \begin{pmatrix} v_g \\ d \end{pmatrix}, \quad t_1 \leq t < T_s, \quad (2.57b) \]

where

\[ t_1 = d T_s. \quad (2.57c) \]

According to (2.6), the steady-state describing function term is defined as

\[ F_{0}^{ss}(X_{0}^{ss}, U_{0}) = \frac{1}{T_s} \int_{0}^{T_s} f(x^{ss}, U_{0}, t) dt \quad (2.58) \]

\[ = \frac{1}{T_s} \int_{0}^{T_1} \left[ A_1 x^{ss}(t) + (b_1, 0) \begin{pmatrix} V_g \\ D \end{pmatrix} \right] dt + \frac{1}{T_s} \int_{T_1}^{T_s} \left[ A_2 x^{ss}(t) + (b_2, 0) \begin{pmatrix} V_g \\ D \end{pmatrix} \right] dt, \]

where

\[ x^{ss}(t) = \sum_{k=-\infty}^{\infty} X_{k}^{ss} e^{j k \omega_0 t}, \quad (2.59) \]
\[ T_1 = DT_s \]  
\[ (2.60) \]

Then it is quite easy to find the partial derivatives of the steady-state describing function to give:

\[
\frac{\partial F_0^{ss}}{\partial X_0^{ss}} = \frac{1}{T_s} \int_0^{T_s} A_1 \frac{\partial x^{ss}(t)}{\partial X_0^{ss}} \, dt + \frac{1}{T_s} \int_{T_1}^{T_s} A_2 \frac{\partial x^{ss}(t)}{\partial X_0^{ss}} \, dt
\]
\[ = A_1 D + A_2 (1 - D) \]
\[ (2.61) \]

\[
\frac{\partial F_0^{ss}}{\partial V_g} = \frac{1}{T_s} \int_0^{T_1} b_1 \, dt + \frac{1}{T_s} \int_{T_1}^{T_s} b_2 \, dt
\]
\[ = b_1 D + b_2 (1 - D) \]
\[ (2.62) \]

\[
\frac{\partial F_0^{ss}}{\partial D} = \frac{1}{T_s} [A_1 x^{ss}(T_1) + b_1 V_g] \frac{\partial T_1}{\partial D} - \frac{1}{T_s} [A_2 x^{ss}(T_1) + b_2 V_g] \frac{\partial T_1}{\partial D}
\]
\[ = (A_1 - A_2) x^{ss}(T_1) + (b_1 - b_2) V_g \]
\[ (2.63) \]

Then the approximate model (2.41) becomes:

\[
\frac{d \hat{x}_0}{dt} = [A_1 D + A_2 D'] \hat{x}_0 + [b_1 D + b_2 D'] \hat{V}_g + [(A_1 - A_2) x^{ss}(T_1) + (b_1 - b_2) V_g] D
\]
\[ (2.64) \]

where

\[ D' = 1 - D \]

Now we have shown that for PWM converters, the approximate model (2.64) has a simple form and it is equivalent to the state space averaging model. The Groves’ method has been significantly simplified by replacing the partial derivatives under

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modulation with the partial derivatives under steady-state. However, this simplification paid price for loosing the capability to predict the high frequency dynamics, when the modulation frequency is close to or high than the Nyquist frequency.

In order to generalize the idea discussed above for modeling resonant type of converters, we need to take into account not only the average components but also the amplitudes of the switching frequency harmonics. Rewriting (2.32) which is the accurate small-signal model

\[ j(kN + M)\omega_c X_{kN + M} = \sum_{m = -\infty}^{\infty} \frac{\partial F_{kN + M}}{\partial X_{mN + M}} X_{mN + M} + \frac{\partial F_{kN + M}}{\partial U_M} U_M, \quad (2.65) \]

\[ k = 0, \pm 1, \pm 2, \ldots \pm K, \]

we can define the small-signal stimulus and the new state variables of the model which are the slowly varying amplitudes of the \( k \)th harmonics:

\[ \dot{u} = U_M e^{j\omega_c t}, \quad (2.66) \]

\[ \dot{x}_k = X_{kN + M} e^{j\omega_m t}. \quad (2.67) \]

By multiplying both side of (2.65) with \( e^{j\omega_m t} \) and using the following relations:

\[ N\omega_c = \omega_s, \quad (2.68) \]

\[ M\omega_c = \omega_m, \quad (2.69) \]

\[ jM\omega_c X_{kN + M} e^{j\omega_m t} = j\omega_m \dot{x}_k = \frac{d\dot{x}_k}{dt}, \quad (2.70) \]
model (2.65) is deduced to

\[
\frac{d\hat{x}_k}{dt} + jk \omega \hat{x}_k = \sum_{m=-k}^{k} \frac{\partial F_{kN+M}}{\partial X_{mN+M}} \hat{x}_m + \frac{\partial F_{kN+M}}{\partial U_M} \hat{u}, \tag{2.71}
\]

\[k = 0, \pm 1, \pm 2, \ldots, \pm K.\]

In order to avoid the difficulties of evaluating the partial derivatives which are defined over the commensurate period and are modulation frequency dependent, we can use the partial derivatives defined over \(T_s\) under the steady-state to make approximations:

\[
\frac{\partial F_{kN+M}}{\partial X_{mN+M}} \approx \frac{\partial F^ss_k}{\partial X^ss_m}, \tag{2.72}
\]

\[
\frac{\partial F_{kN+M}}{\partial U_M} \approx \frac{\partial F^ss_k}{\partial U_0}. \tag{2.73}
\]

Following the same idea presented for the PWM converters, it can be shown that when modulation frequency is much lower than the switching frequency, the above approximations are valid. Then we obtain an LTI small-signal model:

\[
\frac{d\hat{x}_k}{dt} + jk \omega \hat{x}_k = \sum_{m=-k}^{k} \frac{\partial F^ss_k}{\partial X^ss_m} \hat{x}_m + \frac{\partial F^ss_k}{\partial U_0} \hat{u}, \tag{2.74}
\]

\[k = 0, \pm 1, \pm 2, \ldots, \pm K.\]

Equation (2.74) is the general form of the extended describing function model. To complete this model, the output equations also need to be derived. If the nonlinear output of the switching converter is defined by

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\[ y = g(x, u, t) , \quad (2.75) \]

then under small-signal modulation, \( g(\cdot) \) can be also expanded into Fourier series over the commensurate period:

\[ y(t) = \sum_{l = -\infty}^{\infty} G_l(X, U) e^{j\omega_l t} . \quad (2.76) \]

The describing function term, \( G_l(\cdot) \) can then be linearized for \( l = kN + M \) to give

\[ Y_{kN+M} = \sum_{m = -K}^{K} \frac{\partial G_{kN+M}}{\partial X_{mN+M}} X_{mN+M} + \frac{\partial G_{kN+M}}{\partial U_M} U_M . \quad (2.77) \]

Similarly, by defining the slowly varying small-signal output

\[ \dot{y}_k = Y_{kN+M} e^{j\omega_m t} , \quad (2.78) \]

and by employing the following approximations when the modulation frequency is not high:

\[ \frac{\partial G_{kN+M}}{\partial X_{mN+M}} \approx \frac{\partial G^{ss}_k}{\partial X^{ss}_m} , \quad (2.79) \]

\[ \frac{\partial G_{kN+M}}{\partial U_M} \approx \frac{\partial G^{ss}_k}{\partial U_0} , \quad (2.80) \]

the approximate small-signal output equation is obtained:

\[ \dot{y}_k = \sum_{m = -K}^{K} \frac{\partial G^{ss}_k}{\partial X^{ss}_m} \dot{x}_m + \frac{\partial G^{ss}_k}{\partial U_0} \dot{u} . \quad (2.81) \]

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The steady-state multi-variable describing function term, $G^s_k(\cdot)$, is defined by:

$$G^s_k(x^{ss}, U_0) = \frac{1}{T_s} \int_0^{T_s} g(x^{ss}, U_0, t)e^{-jk\omega_0 t} dt . \quad (2.82)$$

Together, (2.74) and (2.81) defines the complete small-signal model for a general switching converter.

For dc/dc converter applications, we are usually interested in the average components of the output variables. In this case, $k$ equals zero in (2.81). For high frequency inverter applications, the amplitudes of the fundamental components of $y(t)$ is our concern, therefore $k$ equals one in (2.81).

Now we can conclude the basic formulation of the extended describing function method:

1) the small-signal model is defined by (2.74) and (2.81) which have LTI expression;

2) the limitation of the model is that the modulation frequency should be lower than the Nyquist frequency. Under this condition, the partial derivatives defined over the commensurate period can be approximated by their counterparts defined over one switching cycle;

3) for PWM converters, if only the average components are considered, the extended describing function model is equivalent to the state space averaging model;
4) by incorporating any harmonic terms, the proposed method can be used for modeling the converters with strongly oscillatory nature;

5) the complexity of Groves’ model has been greatly reduced. This will bring significant simplification to the implementation of the extended describing function method.

2.4.2. Systematic Implementation

The extended describing function method is not complete if there is no systematic procedure for calculating the model parameters (the partial derivatives of the describing functions). Recall that the multi-variable describing functions are defined at the steady-state operating point using the standard Fourier expansion:

\[ F_k(x^{ss}, U_0) = \frac{1}{T_i} \int_0^{T_i} f(x^{ss}(t), U_0, t) e^{-j\omega_0 t} dt, \quad (2.83) \]

where

\[ x^{ss}(t) = \sum_{m=-\infty}^{\infty} X_k^{ss} e^{jm\omega_0 t} dt. \quad (2.84) \]

Our objective is to evaluate the partial derivatives of \( F_k^{ss}(\cdot) \) at the operating point \( \{x^{ss}, U_0\} \). If \( F_k^{ss}(\cdot) \) could always be found analytically, the task of finding their partial derivatives would become trivial. To better visualize the difficulties of this problem when several harmonics have to be considered, let us examine the following example.
Figure 2.4 shows a full-wave rectifier driven by a resonant inductor current and
loaded with a dc voltage. It is a simplified circuit of an SRC. Now we consider the
rectifier as a nonlinear block: the inputs are the inductor current and the capacitor
voltage, and the output is the rectifier voltage seen by the resonant tank

\[ v_R = f(i, V_o) \quad . \] \hspace{1cm} (2.85)

It is clear that \( v_R \) is a square wave whose polarity is determined by the inductor
current direction and whose magnitude is determined by the capacitor voltage:

\[ v_R = V_o \text{sgn}(i) \quad . \] \hspace{1cm} (2.86)

Supposing that we need to find the fundamental component of \( v_R \):

\[ V_{R,1}(V_o, T_1, T_3) = \frac{1}{T_s} \int_0^{T_s} v_R(t) e^{-j\omega_0 t} dt \] \hspace{1cm} (2.87)

\[ = \frac{V_o}{T_s} \left[ \int_0^{T_1} e^{-j\omega_0 t} dt - \int_{T_1}^{T_3} e^{-j\omega_0 t} dt + \int_{T_3}^{T_s} e^{j\omega_0 t} dt \right] . \]

We can see that \( V_{R,1} \) is the function of the switching instants \( T_1 \) and \( T_3 \), and these two
instants are determined by the zero crossing of the inductor current, \( i(t) \).

To simplify the problem, we assume the inductor current can be approximated
by its fundamental plus the third harmonics:

\[ i(t) = I_1 e^{j\omega_0 t} + I_{-1} e^{-j\omega_0 t} + I_3 e^{j3\omega_0 t} + I_{-3} e^{-j3\omega_0 t} . \] \hspace{1cm} (2.88)
But even for this simplified equation, we cannot solve the switching instant

\[ i(T_1) = 0 \quad , \]

(2.89)

to get an analytical solution with the form:

\[ T_1 = h(I_1, I_{-1}, I_3, I_{-3}) \quad . \]

(2.90)

Up to here, we have demonstrated that generally it is very difficult to obtain the analytical solutions for the multi-variable describing functions, and to find their partial derivatives based on the analytical expressions.

Instead of solving the partial derivatives analytically, a numerical process is proposed below to perform the calculation. This process starts with a simulation to find the steady-state of the inductor current. Therefore, all of the Fourier components \( \{ I_k^{ss} \} \) can be calculated

\[ I_k^{ss} = \frac{1}{T_s} \int_0^{T_1} i^{ss}(\tau)e^{-jk\omega_1\tau} d\tau \quad , \]

(2.91)

and the switching instant \( \{ T_1, T_3 \} \) can be solved from

\[ i(t) = 0 \quad . \]

(2.92)

To find the partial derivative

\[ \frac{\partial V_{R,1}}{\partial I_m} = \frac{\dot{V}_{R,1}}{I_m} \quad , \]

(2.93)
we can perturb the \( m \text{th} \) Fourier component of the inductor current by adding a perturbed term to the time domain waveform:

\[
i(t) = \sum_{k = -\infty}^{\infty} I_k^{ss} e^{jk\omega_o t} + (I_m + \hat{I}_m) e^{jm\omega_o t} \\
= i^{ss}(t) + \hat{I}_m e^{jm\omega_o t}.
\]  \hspace{1cm} (2.94)

This perturbation will alter the switching instants from their steady-state values; therefore, the nonlinear output, \( v_R \), will also be changed, as shown in Fig. 2.4.

The partial derivative can then be evaluated as

\[
\frac{\partial V_{R,1}}{\partial I_m} = \frac{\hat{v}_{R,1}}{\hat{I}_m} = \frac{i}{i_m T_s} \int_0^{T_s} [f(i, V_o) - f(i^{ss}, V_o)] e^{-j\omega_o t} dt.
\]  \hspace{1cm} (2.96)

A numerical procedure is possible to calculate the partial derivatives using (2.96). However, we will show that the analytical expressions of the partial derivatives can be further derived provided the converter circuit is a piecewise linear network.

According to Fig. 2.4, if the perturbation level is small, the difference between the perturbed and unperturbed \( v_R(t) \) can be represented by two narrow pulses, then we have
\[
\frac{\partial V_{R,1}}{\partial I_m} = \frac{1}{i_m T_s} \left[ \int_{t_1}^{T_1} 2V_o e^{-j\omega_s t} dt + \int_{T_1}^{T_3} (-2V_o) e^{-j\omega_s t} dt \right] \tag{2.97}
\]
\[
= \frac{2V_o}{T_s} \left( \frac{f_1}{i_m} e^{-j\omega_s T_1} - \frac{f_3}{i_m} e^{-j\omega_s T_3} \right)
\]
\[
= \frac{2V_o}{T_s} \left( \frac{\partial T_1}{\partial I_m} e^{-j\omega_s T_1} - \frac{\partial T_3}{\partial I_m} e^{-j\omega_s T_3} \right).
\]

Now we can see that the partial derivative of the multi-variable describing function is determined by the perturbation of the switching instants. A systematic derivation will be provided below employing the piecewise linear description which is the best form to deal with the switching instants and their perturbations.

Assuming there are \( Q \) pieces of linear circuits in a switching cycle:

\[
\frac{dx}{dt} = A_x x + B_x u \quad , \tag{2.98a}
\]
\[
y = C_x x + D_x u \quad , \tag{2.98b}
\]
\[
T_{i-1} < t < T_i \quad , \tag{2.98c}
\]
\[
i = 1, \ldots, Q \quad .
\]

Notations \( \{T_i, i = 0, \ldots, Q\} \) will be used for the switching instants when steady-state has been reached.

By performing Fourier analysis on one switching cycle, the multi-variable describing functions are defined under steady-state operation:
Fig. 2.4. The full-wave rectifier of an SRC is taken as an example of the nonlinear function. The inputs of the nonlinear function are the resonant inductor current, \( i \), and the output capacitor voltage, \( V_o \). The output is the rectifier voltage seen by the resonant tank:

\[
v_R = f(i, V_o)
\]

If the inductor current is perturbed, \( v_R \) will also be perturbed.
\[ F_k^{ss} = \frac{1}{T_s} \sum_{i=1}^{Q} \int_{T_{i-1}}^{T_i} \left[ A_i x(t)^{ss} + B_i U_0 \right] e^{-j\omega t} \, dt, \quad (2.99) \]

\[ G_k^{ss} = \frac{1}{T_s} \sum_{i=1}^{Q} \int_{T_{i-1}}^{T_i} \left[ C_i x(t)^{ss} + D_i U_0 \right] e^{-j\omega t} \, dt. \quad (2.100) \]

where

\[ x^{ss}(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega t}. \quad (2.101) \]

The partial derivative of the describing function can then be found as

\[
\frac{\partial F_k^{ss}}{\partial X_m^{ss}} = \frac{1}{T_s} \sum_{i=1}^{Q} \left[ \left\{ A_i x^{ss}(T_i) + B_i U_0 \right\} e^{-j\omega T_i} \frac{T_i}{\partial X_m^{ss}} \right.

\left. -\left[ A_i x^{ss}(T_{i-1}) + B_i U_0 \right] e^{-j\omega T_{i-1}} \frac{T_{i-1}}{\partial X_m^{ss}} \right]

+ \int_{T_{i-1}}^{T_i} A_i \frac{\partial x^{ss}}{\partial X_m^{ss}} e^{-j\omega t} \, dt \right]. \quad (2.102) \]

By knowing that

\[
\frac{\partial x^{ss}}{\partial X_m^{ss}} = \frac{\partial}{\partial X_m^{ss}} \left\{ \sum_{k=-\infty}^{\infty} X_k e^{jk\omega t} \right\} = e^{jk\omega t}, \quad (2.103) \]

(2.102) can be deduced to the form:

\[
\frac{\partial F_k^{ss}}{\partial X_m^{ss}} = \sum_{i=1}^{Q} \left[ \Delta_i e^{-j\omega T_i} \frac{T_i}{\partial X_m^{ss}} + A_i \Gamma_i(k, m) \right], \quad (2.104) \]

\[ k = 0, \pm 1, \ldots, \pm K, \quad m = 0, \pm 1, \ldots, \pm K. \]
where

\[
\Delta_i = \left\{ \begin{array}{c}
\frac{1}{T_s} [(A_i - A_{i+1})x^{ss}(T_i) + (B_i - B_{i+1})U_0] & i = 1, 2, \ldots, Q - 1 \\
\frac{1}{T_s} [A_i x^{ss}(T_i) + B_i U_0] & i = Q
\end{array} \right. ,
\]

\[
\Gamma_i(k, m) = \frac{1}{T_s} \int_{T_{i-1}}^{T_i} e^{-j(k-n)\omega_0 t} \, dt .
\]

Similarly, we can find the other partial derivative terms

\[
\frac{\partial F_k^{ss}}{\partial U_0} = \sum_{i=1}^{\Omega} \left[ \Delta_i e^{-j\omega_0 T_i} \frac{\partial T_i}{\partial U_0} + B_i \Gamma_i(k, 0) \right] ,
\]

\[
\frac{\partial G_k^{ss}}{\partial x_m^{ss}} = \sum_{i=1}^{\Omega} \left[ \Lambda_i e^{-j\omega_0 T_i} \frac{\partial T_i}{\partial x_m^{ss}} + C_i \Gamma_i(k, m) \right] ,
\]

\[
\frac{\partial G_k^{ss}}{\partial U_0} = \sum_{i=1}^{\Omega} \left[ \Lambda_i e^{-j\omega_0 T_i} \frac{\partial T_i}{\partial U_0} + D_i \Gamma_i(k, 0) \right] ,
\]

\[
k = 0, \pm 1, \ldots, \pm K, \ \ m = 0, \pm 1, \ldots, \pm K ,
\]

where

\[
\Lambda_i = \left\{ \begin{array}{c}
\frac{1}{T_s} [(C_i - C_{i+1})x^{ss}(T_i) + (D_i - D_{i+1})U_0] & i = 1, \ldots, Q - 1 \\
\frac{1}{T_s} [C_i x^{ss}(T_i) + B_i U_0] & i = Q
\end{array} \right. .
\]

If the steady-state has been reached and \( \{T_i, x^{ss}(T_i)\} \) are known, then the coefficients \( \{\Delta_i, \Lambda_i, \Gamma_i(k, m)\} \) can be calculated. Up to here, we have provided analytical

2. Extended Describing Function Method
expressions for the partial derivatives of the multi-variable describing function terms.

The remaining unknowns are the partial derivatives of the switching instants, \( T_i \), with respect to \( X_{m}^{ss} \) and \( U_0 \).

By employing the switching boundary condition, implicit expressions of the switching instant, \( T_i \), can be found as:

\[
 h(T_i, X^{ss}, U_0) = a_i X^{ss}(T_i) + b_i U_0 + c_i T_i + d_i = 0 \quad , \quad (2.111)
\]

where

\[
x^{ss}(T_i) = \sum_{m=-\infty}^{\infty} X_{m}^{ss} e^{j m \omega_0 T_i} \quad , \quad (2.112)
\]

Using (2.111) and (2.112) then we can find the partial derivatives of the switching instant, \( T_i \), with respect to the Fourier component, \( X_{m}^{ss} \), and to the input, \( U_0 \):

\[
 \frac{\partial T_i}{\partial X_{m}^{ss}} = \frac{-a_i e^{j m \omega_0 T_i}}{a_i [A_i x^{ss}(T_i) + B_i U_0] + c_i} \quad , \quad (2.113)
\]

\[
 \frac{\partial T_i}{\partial U_0} = \frac{-b_i}{a_i [A_i x^{ss}(T_i) + B_i U_0] + c_i} \quad , \quad (2.114)
\]

\[
i = 1, 2, ..., Q, \quad m = 0, \pm 1, ..., \pm K \quad .
\]

This concludes the systematic implementation of the extended describing function method.

2. Extended Describing Function Method
2.4.3. Compare with Generalized Averaging Method

The generalized averaging method was proposed by Sanders et al. in [35]. The modeling procedure starts with Fourier expansion of the state variable in a moving window \((t - T_s, t]\):

\[
x(\tau) = \sum_{k = -\infty}^{\infty} < x >_k (t) e^{jk\omega_s \tau},
\]

where \(\tau \in (t - T_s, t]\).

The time-varying Fourier coefficient is defined as:

\[
< x >_k (t) = \frac{1}{T_s} \int_{t - T_s}^{t} x(\tau) e^{-jk\omega_s \tau} d\tau.
\]

When the system is under steady-state, \(< x >_k(t)\) is a constant which is the same result as the standard Fourier expansion. When the system is subject to modulation, \(< x >_k(t)\) becomes time-varying.

In [35], \(< x >_k(t)\) is called "index k average". The modeling idea is to use \{\(< x >_k(t), k = 0, \pm 1, \pm 2, \ldots\)\} as new state variables. This allows one to study the dynamics carried by these generalized average terms. It was shown in [35] that the derivative of the index k average can be represented by:

\[
\frac{d < x >_k(t)}{dt} = -jk\omega_s < x >_k(t) + \frac{1}{T_s} \int_{t - T_s}^{t} f(x, u, \tau) e^{-jk\omega_s \tau} d\tau.
\]
In order to make (2.117) a self-contained state-space model, the key issue is to evaluate the second term on the right hand side. This integral term has to be expressed as a function of the \( \{ < x >_k (t), k = 0, \pm 1, \pm 2, \ldots \} \). By substituting (2.115) into (2.117), we have:

\[
\frac{d < x >_k (t)}{dt} = - j k \omega_s < x >_k (t) + \frac{1}{T_s} \int_{1-\tau}^{1} f \left( \sum_k < x >_k (\tau) e^{j k \omega_s \tau}, u, \tau \right) e^{-j k \omega_s \tau} d\tau.
\]

(2.118)

Since (2.118) is an implicit nonlinear model defined by time variant integration, it is not trivial to linearize this model.

Instead of providing a systematic approach to realized (2.118), an SRC was taken as an example in [35] to demonstrate the application of the modeling concept. By assuming that the switching frequency is quite close to the resonant frequency, the tank current and voltage can then be approximated by their fundamental components, then the Fourier series in (2.115) can be truncated as:

\[
x(t) \approx < x >_1 (t) e^{j \omega_0 t} + < x >_{-1} (t) e^{-j \omega_0 t}.
\]

(2.119)

After truncating the Fourier series, an analytical expression under steady-state is used to replace the time-varying integral

\[
\frac{1}{T_s} \int_{1-\tau}^{1} f(x(\tau), u, \tau) e^{-j \omega_0 \tau} d\tau \approx \frac{1}{T_s} \int_{0}^{T_s} f(x^w, U_0, \tau) e^{-j \omega_0 \tau} d\tau
\]

\[
= F_1^w (< x >_{-1}, < x >_1, U_0),
\]

(2.120)
then an analytical model can be found to give:

\[ \frac{d < x >_1}{dt} = -j \omega_s < x >_1 + F_{11}^{ss}(< x >_1, < x >_{-1}, u). \quad (2.113) \]

This model has an autonomous form; it can be linearized to provide a small-signal model:

\[ \frac{d < \hat{x} >_1}{dt} + j \omega_s < \hat{x} >_1 = \sum_{m = \pm 1} \frac{\partial F_{11}^{ss}}{\partial < x >_m} < \hat{x} >_m + \frac{\partial F_{11}^{ss}}{\partial u} \hat{u}. \quad (2.122) \]

However, for the same circuit, if the switching frequency is not very close to the resonant frequency and the load is light (corresponding to low quality factor of the tank), the resonant waveforms can be highly distorted from pure sine waves, as shown in Fig. 2.5. In this case, it is questionable whether the state waveforms can be approximated by the sine waves at the switching frequency. Ideally, more harmonic terms should be taken into account to study their effects on the model accuracy. However, as we discussed in 2.3.2, when more than one harmonic terms are included, it is impossible to find analytical solutions (see Fig. 2.4). The work [35] did not offer a systematic approach to implement the modeling concept for general cases. While the concept is relatively familiar in nonlinear system analysis [1, 32], the difficult part consists in providing a systematic implementation.

Generally speaking, the simple sine wave approximation has very limited applications. Figure 2.6 shows a forward multi-resonant converter (MRC) and its typical waveforms. It can be seen that several waveforms are not quite sinusoidal
and more harmonics need to be taken for a reasonable approximation. Later in Chapter 5, we will use the MRCs to demonstrate that simply using the dc and the fundamental term will result in poor accuracy of the small-signal model.
Fig. 2.5. **The circuit diagram of an SRC and its waveforms.** When load is light and the switching frequency is not close to the resonant frequency, the resonant waveforms are not quite like sine waves. The approximation of using only the fundamental components becomes questionable.
Fig. 2.6. The circuit diagram of a forward multi-resonant converter (FMRC) and its typical waveforms. Some of the waveforms are dc biased and not quite like sine waves.
2.4. Computer Program

A computer program has been developed to implement the extended describing function method. Since the small-signal formulation needs the steady-state information of the converter for a given operating point, the program should also have capabilities to perform large-signal simulation and to find steady-state responses.

The input of the program is the nonlinear state equation of the converter in piecewise linear form associated with the switching boundary conditions. The large-signal simulation method is based on integrating the piecewise linear state equations, which is similar to the method used in COSMIR [31]. Newton's method [86] is built in the program as the steady-state finder.

Upon knowing the following information:
1) the steady-state operation:

\[ \{T_i, x^{ss}(T_i), i = 1, \ldots, Q \} \]

2) the piecewise linear description of the circuit

\[ \dot{x} = A_i x + B_i u \]

\[ y = C_i x + D_i u \]

\[ T_{i-1} < t < T_i \]

3) the switching boundary conditions
\[ h(T_i, x(T_i), u) = a_i x(T_i) + b_i u + c_i T_i + d_i = 0, \]

the small-signal model can be calculated using the extended describing function method developed in Section 2.3. The pole-zero distribution and the Bode plot of the transfer functions are then obtained.

For practical consideration, it is not necessary to calculate Fourier coefficients up to the \(kth\) harmonics for all of the state variables. Specifically, it is rational to take the dc components of the state variables associated with the low pass filter, while for certain resonant variables, the first few harmonic components may need to be taken into account for a good approximation. In this way, the dimension of the small-signal model is reduced to minimum, much less partial derivatives need to be calculated, and a lot of computational efforts can be saved.

The program provides a useful tool in small-signal analysis and control design. It is especially suitable to combine the small-signal routines with the circuit simulation package based on piecewise linear description, like COSMIR [31].

The application of the program will be demonstrated in the following chapters for different converter circuits. The details of the program, including the user’s manual and the source codes, can be found in Appendix A.
2.5. **Conclusions**

In this Chapter, the extended describing function method is proposed as a general modeling tool for switching converters based on modification of previous work [28]. The objectives are to reduce the model complexity and to simplify the implementation procedure. Under the condition that the modulation frequency is lower than the Nyquist frequency, it has been proved that the partial derivatives of the multiple variable describing functions defined over the commensurate period can be approximated by their counterparts defined over one switching cycle. For PWM converters, the approximate model is equivalent to the result of the state space averaging. The proposed method can also incorporate any harmonic components for generalized applications. By performing Fourier analysis over one switching cycle in instead of over the commensurate period \(T = NT_s = MT_m\), the extended describing function method provides much simplified models than Groves' method. The significance of the proposed method is that it greatly reduces the complexity of the implementation by avoiding the extensive simulation, yet it can provide useful small-signal models for general converter circuits.

Systematic implementation is presented for the extended describing function method. Comparing with the generalized average method, which depends on analytical solutions and only has limited applications, the proposed method is a complete approach which can be used for variant converter topologies and operating conditions.
A computer program has been developed to implement the extended describing function method. The program provides convenience for small-signal analysis and control design.
3. Modeling Series Resonant Converters

3.1. Introduction

In this chapter, the series resonant converter (SRC) and the LLC series resonant converter (LLC-SRC) will be employed as examples to demonstrate the modeling technique developed in Chapter 2. The extended describing function method allows us to study the effects of fundamental and higher order harmonics on the small-signal modeling, especially for the case where the resonant waveforms are heavily distorted from sine waves. As a result, continuous-time small-signal models accurate up to Nyquist frequency are obtained for the two series resonant converters. These models are particularly useful to study the interaction of the switching frequency and the tank natural resonant frequency. This interaction usually determines the high frequency dynamics of the SRCs, often referring to as the beat frequency dynamics.

A brief review of the previous work is given below for the efforts to provide continuous-time small-signal models for the SRCs. A small-signal model of
series-resonant converters with diode conduction angle control was proposed in [43]. The model is not accurate enough to predict the beat frequency dynamics of the SRCs and does not cover the more commonly used frequency control and the phase-shift control.

An approximated model based on intuition was proposed in [42]. The beat frequency dynamics were addressed for the first time as a result of the interaction of the switching frequency and the tank natural resonant frequency. Based on observation, the beat frequency dynamics were modeled by a double pole whose frequency location is determined by the difference of the switching frequency and the resonant frequency, i.e. $|F_s - F_o|$. Although this model provides good physical insight into the converter dynamics, it begins to lose accuracy for the operations in which the quality factor is low (corresponding to light load), or in which the switching frequency is close to the resonant frequency. Under such operating conditions, as it will be shown later, the beat frequency dynamics cannot be modeled as in [42].

A small-signal model based on sine wave approximation of the resonant waveforms was proposed in [34], and the low frequency dynamics of the SRC were studied without considering the beat frequency dynamics. This model has not been justified for the operations where the waveforms are not quite sinusoidal.

Other modeling work was mainly based on sampled-date analysis [44-47]; the applications are limited due to the discrete-time models and complicated expressions.
In this chapter, two useful serious resonant converters (SRCs) are analyzed for their small-signal dynamics. It will be demonstrated that the modeling tool developed in Chapter 2 is very useful for dealing with these circuits. The effects of incorporating higher order harmonics will be studied especially for those operating conditions where the resonant waveforms are heavily distorted from the pure sine waves. The small-signal models are verified experimentally. A complete small-signal equivalent circuit will also be derived for both SRC topologies to provide a useful tool for control-loop design.

3.2. Two Series Resonant Topologies

3.2.1. Basics of the Topologies

Two series resonant topologies will be analyzed, including the traditional series resonant converter (SRC) and a series resonant converter with an additional resonant inductor in parallel with the resonant capacitor (LLC-SRC). Although this chapter does not cover all of the resonant topologies, other circuits [59] can also be modeled using the methodology established in this work.

The circuit diagrams of the SRC and the LLC-SRC are shown in Fig. 3.1. It is easy to see that the LLC-SRC will reduce to SRC if the parallel resonant inductance, \( L_p \), is very large compared with the series resonant inductance, \( L_s \). The resonant frequency, \( F_o \), the characteristic impedance, \( Z_o \), and the quality factor, \( Q_o \), of the SRC and the LLC-SRC are defined as follows:
\[ F_o = \frac{1}{2\pi \sqrt{L_e C}} \]  
\[ Z_o = \sqrt{\frac{L_e}{C}} \]  
\[ Q_i = \frac{Z_o}{R} \]  
\[ L_e = \frac{L}{L_i L_p/(L_i + L_p)} \]  
\[ SRC \quad LLC - SRC \]  

By adding the additional inductor, \( L_p \), to the resonant tank, a parallel resonant frequency will be formed between \( L_p \) and \( C \):

\[ F_p = \frac{1}{2\pi \sqrt{L_p C}} = \frac{F_o}{\sqrt{1 + L_p/L_e}} \]  

This parallel resonant frequency is always lower than the resonant frequency \( F_o \). If the converter is operated at this resonant frequency, \( (F_p = F_o) \), the impedance of \( L_p \) and \( C \) will be infinity and no power can be delivered to the load. Therefore, the parallel resonance brings the conversion curves down to zero at \( F_p \), as shown in Fig. 3.2. Compared with the SRC when converter is operated below the resonant frequency, the LLC-SRC has a more reduced switching frequency range than the SRC does for the same line and load variation [58].
Fig. 3.1. The circuit diagrams of the two series resonant topologies.

Top: the traditional series resonant converter (SRC); and

Bottom: the series resonant converter with a third order tank (LLC-SRC).

3. Modeling Series Resonant Converters
Fig. 3.2. The voltage conversion ratio characteristics of the SRC and the LLC-SRC. A reduced switching frequency range can be achieved for the LLC-SRC when it is operated below the resonant frequency.
3.2.2. Piecewise Linear Models of the SRCs

The extended describing function method provides a systematic treatment for general switching converters and operating modes. For the SRC and the LLC-SRC, we will concentrate on the continuous inductor current mode because this mode will cover most of the useful operation regions of these converters. For the SRC, the switching frequency is restricted to

$$F_s > \frac{1}{2} F_o \quad .$$

For the LLC-SRC, the switching frequency is restricted to

$$F_o > F_s > F_p \quad ,$$

because the LLC-SRC has advantages only in this frequency range compared with the SRC; there is no need to operate this converter in other regions.

According to the circuit diagram of the SRC, the piecewise linear state equation can be written as:

$$\dot{x} = Ax + Bu \quad , \quad (3.6)$$

$$y = Cx + Du \quad , \quad (3.7)$$

where the state vector, the input vector, and the output variable are defined by:
\[ x = (i, v, v_C) \] (3.8)

\[ u = (v_g, \omega_s) \] (3.9)

\[ y = v_o \] (3.10)

The matrices in (3.6) and (3.7) represent the nonlinear and switching properties of the circuit:

\[
A = \begin{pmatrix}
-\frac{r_s}{L} - \frac{(r_c \parallel R)}{L} & -1 & -\frac{R \text{sgn}(i)}{(R + r_c)L} \\
\frac{1}{C} & 0 & 0 \\
\frac{R \text{sgn}(i)}{(R + r_c)C} & 0 & -\frac{1}{C_f(R + r_c)}
\end{pmatrix},
\] (3.11)

\[
B = \begin{pmatrix}
s(t) & \frac{\text{sgn}(i)(r_c \parallel R)}{L} \\
\frac{L}{L} & 0 & 0 \\
0 & -\frac{R}{(R + r_c)C_f}
\end{pmatrix},
\] (3.12)

\[
C = \begin{pmatrix}
\text{sgn}(i)(r_c \parallel R) & 0 & \frac{R}{R + r_c}
\end{pmatrix},
\] (3.13)

\[
D = [0 \ (r_c \parallel R)]
\] (3.14)

There are two switching elements in the matrices:

3. Modeling Series Resonant Converters
\[ sgn(i) = \begin{cases} 
1 & i \geq 0 \\
-1 & i < 0 
\end{cases}, \quad (3.15) \]

\[ s(t) = \begin{cases} 
1 & 0 \leq t < 0.5T_s \\
-1 & 0.5T_s < t \leq T_s 
\end{cases}. \quad (3.16) \]

The conduction loss of the switches and the loss of the resonant tank are modeled, approximately, by a lumped resistor, \( r_s \). The equivalent series resistor (esr) of the output capacitor is denoted as \( r_c \).

The switching boundary conditions can be easily derived. For example, if the SRC is operated below the resonant frequency, the typical waveforms will be shown as in Fig. 3.3(a). The boundary conditions at the switching instants \( T_1 \) and \( T_3 \) are given by

\[ i(T_1) = 0, \quad i(T_3) = 0, \quad (3.17) \]

These switching actions are caused by rectifier diode. The active switching actions occur at the instants \( T_2 \) and \( T_4 \), corresponding to the boundary conditions

\[ T_2 - 0.5T_s = 0, \quad T_4 - T_s = 0, \quad (3.18) \]

respectively.

The LLC-SRC has the same operating modes and switching boundary conditions. If the state vector of the LLC-SRC is defined as:

\[ x = (i_s, i_p, v, v_c)^T, \quad (3.19) \]
then the matrices of the piecewise linear state equation can be found:

\[
A = \begin{pmatrix}
-\frac{r_s - (r_c \parallel R)}{L_s} & 0 & -\frac{1}{L_s} & -\frac{R \text{sgn}(i)}{(R + r_c)L_s} \\
0 & 1 & 0 & 0 \\
\frac{1}{C} & -\frac{1}{C} & 0 & 0 \\
\frac{R \text{sgn}(i)}{(R + r_c)C} & 0 & 0 & -\frac{1}{C(R + r_c)}
\end{pmatrix}, \quad (3.20)
\]

\[
B = \begin{pmatrix}
\frac{s(t)}{L_s} & \text{sgn}(i)\frac{(r_c \parallel R)}{L_s} \\
0 & 0 \\
0 & 0 \\
0 & -\frac{R}{(R + r_c)C}
\end{pmatrix}, \quad (3.21)
\]

\[
C = \begin{pmatrix}
\text{sgn}(i)\frac{(r_c \parallel R)}{R} & 0 & 0 & \frac{R}{R + r_c}
\end{pmatrix}, \quad (3.22)
\]

\[
D = [0 \quad (r_c \parallel R)] \quad . \quad (3.23)
\]

When the SRC is operated above the resonant frequency, the typical resonant inductor waveform is shown in Fig. 3.3(b). From there, it is easy to determine the switching instants and the corresponding switching boundary conditions.
Fig. 3.3. Four piece of linear circuits can be identified according to the waveforms of the SRC:

(a): $F_s < F_o$;

(b): $F_s > F_o$. 

3. Modeling Series Resonant Converters
3.3. Small-Signal Analysis

3.3.1. Typical Modeling Procedures

Our modeling tool is the computer program introduced in Chapter 2. The details of the program can be found in Appendix A. Since the small-signal modeling method needs the steady-state information of the converter under a given operating condition, the program also needs to perform large-signal simulation. An acceleration algorithm based on Newton’s method [86] is also built in for finding the steady-state operation.

An SRC circuit is used to demonstrate the modeling procedures. The parameter values of this SRC are:

\[ L = 22.5 \mu H \quad C = 6.56 \mu F \]
\[ C_f = 16.75 \mu F \quad r_c = 0.212 \Omega \]
\[ F_o = 414.3 KHz \quad Z_o = 58.6 \Omega \]

A start up transient response of this converter is illustrated in Fig. 3.4. The transient response shows that the envelope of the resonant waveform carries the beat frequency dynamics. Since the program employs the similar time domain integration method as the one used in COSMIR [31], the simulation is much faster than commonly used simulation software like SPICE.

Based on the large-signal simulation, a steady-state algorithm can be executed to find the periodic operation to satisfy the condition
Fig. 3.4. The start up transient response of the SRC clearly shows the beat frequency dynamics carried by the envelope of the resonant waveform.
\[ \| x(t) - x(t + T_s) \| < \epsilon \quad , \] (3.24)

where \( \epsilon \) is a given small number; in our case \( \epsilon = 1.0 \times 10^{-8} \). The steady-state routine employs Newton's method for iteration until the tolerance is satisfied. The outputs of the steady-state routine are the state waveforms over one switching cycle, the switching instants, and the values of the state vector on the switching instants. For the operating condition

\[
V_s = 40.9V
\]

\[
F_s/F_o = 0.678
\]

\[
Q_s = 1.49 \quad ,
\]

the outputs of the steady-state program are shown in Fig. 3.5.

Upon knowing the steady-state waveforms, Fourier analysis can be performed by the program to calculate the steady-state spectrum of the state variables, as shown in Fig. 3.6. The spectrum clearly shows the amplitude of each harmonics. The values of the spectrum components will be used to evaluate the parameters of the small-signal model.

Taking the spectrum as a reference, we can take into account the first few harmonic components for small-signal modeling. Since different state variables have different harmonic constants, it does not make sense to take the fundamental and the
Fig. 3.5. The information of the steady-state operation can be obtained easily with the modeling program. The switching instants and related values of the state vector \( \{T_i, x(T_i), i = 1, 2, 3, 4\} \) are found to be:

\[
\{T_1, T_2, T_3, T_4\} = (1.16\, \mu s, 1.780\, \mu s, 2.89\, \mu s, 3.56\, \mu s)
\]

\[
\{x(t_1), x(T_2), x(T_3), x(T_4)\} = \begin{pmatrix}
0 & -0.3292 & 0 & 0.3292 \\
87.05 & 63.82 & -87.05 & -63.82 \\
25.24 & 25.23 & 25.24 & 25.22
\end{pmatrix}
\]

These values will be used by the small-signal routines to calculate model parameters.
Fig. 3.6. Fourier analysis can be performed by the modeling program for the steady-state waveforms. The spectrum can be a reference of taking appropriate number of harmonics for small-signal modeling. The values of the spectrum are also needed for evaluating the parameters of the small-signal model.
third harmonics for the output voltage (it only contains even number harmonics) or take the dc and the second harmonics for the resonant states (they only contain odd number harmonics). A harmonic table is designed to allow us to take specified harmonics for each state variable. This table is a matrix which has three rows (the dimension of the state space) and several columns, as shown in Table 3.1.

The elements in the harmonics table are ones and zeros, where one means taking the certain harmonic term and zero means not taking it. In this example, the fundamental and the third harmonics of the resonant current, the fundamental of the resonant voltage, and the dc component of the output capacitor voltage are taken into account. Therefore, the Fourier components of these harmonics form a new state vector of the small-signal model

\[ \hat{X} = (\hat{i}_{1,R}, \hat{i}_{1,I}, \hat{i}_{3,R}, \hat{i}_{3,I}, \hat{v}_{1,R}, \hat{v}_{1,I}, \hat{v}_{C_p,0})^T. \]  

(3.25)

Notice that the complex variables have been split into their real and imaginary parts (corresponding to the subscripts \( R \) and \( I \)) because these two parts are independent variables; thus, the system order is actually determined by the total dimension of \( \hat{X} \).

All of the procedures discussed above are the preparations for the small-signal modeling. Based on the results of the steady-state analysis and the Fourier analysis, the small-signal analysis routines using the extended describing function technique are invoked to calculate the model parameters. The small-signal model is obtained with a state space expression:
Table 3.1. The harmonic table of the SRC allows us to take specified harmonics for each state variable. Each state variable occupies one row of the table. The columns represent, sequentially, different harmonics, starting from dc. Element one indicates the corresponding harmonic term is taken into account, while zero means this term is not considered.

<table>
<thead>
<tr>
<th></th>
<th>dc</th>
<th>fundamental</th>
<th>2nd</th>
<th>3rd</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>(v)</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(v_{cf})</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
\[ \frac{d\hat{X}}{dt} = A_m\hat{X} + B_m\hat{u} \quad , \]  

(3.26)

\[ \hat{y}_o = C_m\hat{X} + D_m\hat{u} \quad , \]  

(3.27)

where \( \hat{X} \) is defined in (3.25), and the input and the output are defined by

\[ \hat{u} = (\hat{v}_o, \hat{\omega}_o)^T \quad , \]  

(3.28)

\[ \hat{y}_o = \hat{v}_o \quad . \]  

(3.29)

The matrices in (3.26) and (3.27) are formed by the partial derivatives of the describing function terms, which can be calculated according to the systematic implementation developed in Chapter 2. With the small-signal model available, the eigenvalues of the matrix, \( A_m \), or the system poles, can be calculated from

\[ \det(sI - A_m) = 0 \quad . \]  

(3.30)

The distribution of the eigenvalues over the complex plane is illustrated in Fig. 3.7. The low frequency pole of the output filter, \( p_1 \), and the beat frequency double poles, \( \{p_2, p_2^*\} \), can be clearly seen. The very high frequency poles are the model’s interpretations of the interaction between \( 3F_e \) and \( F_o \), and these poles do not have real physical meanings. They can be dropped out because of their little effects on the small-signal responses when \( f_m < F_x \).
Fig. 3.7. The distribution of the system poles on the complex plane. Located at low frequency, $p_1$ corresponds to the pole of the output low pass filter. The beat frequency double pole, $\{p_2, p_2^*\}$, is caused by the interaction between $F_s$ and $F_u$. The high frequency poles $\{p_3, p_3^*\}$ and $\{p_4, p_4^*\}$ have very little effects on the small-signal responses when $f_m < F_s$. 

3. Modeling Series Resonant Converters
The small-signal transfer functions and the Bode plots can also be easily obtained from the state space model; for this example, the control-to-output transfer function is given by:

\[
\frac{\hat{v}_o}{\omega_{SN}} = K \frac{(1 + s/\omega_z)}{(1 + s/\omega_{p1})(1 + s/\omega_{p2})(1 + s/\omega_{p2}^*)},
\]

(3.31)

where

\[
K = 79.4
\]

\[
\omega_z = 2.816 \times 10^5
\]

\[
\omega_{p1}, \omega_{p2}^* = (0.318 \pm j0.8107) \times 10^6
\]

The Bode plot is pictured in Fig. 3.8.

To summarize, the modeling tool provides a fairly comprehensive and automatic way to perform the small-signal analysis. The procedure is straightforward and all of the dynamic and steady-state information of the converter can be provided. In addition, the methodology that the computer program is based on is generally applicable to lots of switching converter topologies. All of these features make the modeling program a valuable tool for practical analyses and designs.
Fig. 3.8. The Bode plot of the control-to-output transfer function of the SRC. The beat frequency dynamics appear as a double pole located at \((F_o - F_s)\).
3.3.2. Spectral Analysis and Beat Frequency Dynamics

One of the advantages of the small-signal models obtained through extended describing function approach is that they provide valuable information for the control-loop design; the other advantage of these models is that they allow us to perform a detailed Fourier analysis of the side-band spectrum inside the circuit. The beat frequency phenomenon is a unique feature of the resonant converters, which was not carefully studied before. It will be very interesting to examine this phenomenon from the point of view of our modeling approach.

As we know, a resonant converter can be functionally divided into several subsystems, which include an inverter formed by active switches, a band pass filter or the resonant tank, a full wave rectifier, and a low pass filter, as shown in Fig. 3.9.

The input voltage of the resonant tank, $v_{AB}$, is a square wave, which contains rich harmonics. When the switching frequency or the line voltage is under small-signal modulation, sidebands will appear for the fundamental, third, and other higher harmonics of $v_{AB}$, as shown in Fig. 3.10. When the perturbed signal passes through the band pass filter, different harmonic components will be attenuated correspondingly. As an example, Fig. 3.10 shows that the third harmonics are more heavily attenuated than the fundamental because $F_s$ is closer to the resonant frequency, $F_o$, than $3F_s$ is. It is interesting to see from Fig. 3.10 that when the modulation frequency is sweeping up, the sideband, $i_1$, will have different amplitude, especially when
Fig. 3.9. A block diagram of the SRC shows that this circuit contains the following subsystems: an inverter to generate square wave voltage, a band pass filter whose output is the resonant current \( i \), a diode bridge that rectifies the tank current to provide \( i_R \), and an output filter that removes the switching frequency ripples of \( i_R \) to provide output voltage.
Fig. 3.10. Small-signal frequency analysis of the SRC. When $f_m$ sweeps up and equals $(F_o - F_s)$, the sideband $\hat{i}_1$ has its maximum amplitude. The sideband of the third harmonics, $\hat{i}_3$, is much smaller than $\hat{i}_1$ and has very small contribution to the circuit output. The extended describing function method can calculate every sideband spectrum inside the circuit when the modulation frequency is sweeping.
\[ f_m = F_o - F_s \]  

\( \hat{i}_1 \) reaches the maximum value because it is exactly aligned to the resonant peak of the band pass filter. Meanwhile, the sideband of the third order harmonic, \( \hat{i}_3 \), does not have such an opportunity if \( f_m < F_s \). Therefore, \( \hat{i}_3 \) is much smaller in amplitude compared with \( \hat{i}_1 \). After the resonant current is rectified by the diode bridge, frequency shift occurs because the rectifier output current, \( i_R \), does not contain odd number harmonics. Conceptually, the most difficult part is to provide a clear picture of where \( \hat{i}_{R,0} \) comes from. Since the rectifier functions as a nonlinear block, it mixes different frequency components of its input, \( i \), to provide its output, \( i_R \). The rectifier cannot be simply treated as a frequency shifter, which moves the fundamental components to zero frequency and the 3rd harmonics to twice the switching frequency, and so on. For the same reasons as those discussed in section 2.5, the analytical relations do not exist for the coupling between the different frequency components. The best thing we can get is to use our modeling approach to actually calculate the sideband spectrum, \( \hat{i}_{R,0} \), for different modulation frequencies. The results of \( \hat{i}_{R,0}, \hat{i}_1, \) and \( \hat{i}_3 \) vs modulation frequency are calculated using the small-signal model. As shown in Fig. 3.10, \( \hat{i}_{R,0} \) is basically contributed by \( \hat{i}_1 \), but slightly coupled with \( \hat{i}_3 \).

The small-signal output voltage, \( \hat{v}_o \), is obtained by low pass filtering the sideband, \( \hat{i}_{R,0} \), and the actual responses vs the modulation frequency are also shown in Fig. 3.10. Notice that the linear scales are used for both the frequency axis and amplitude axes in Fig. 3.10. If log scales are used for both axis (the format of Bode
plot), the small-signal response of $v_o$ will look more familiar to us as in Fig. 3.8.

To summarize, the difficulties of analyzing the small-signal responses of the nonlinear switching circuits have been demonstrated. This will help us to appreciate the value of our modeling tool for its capability to present a clear picture of frequency analysis and the capability to capture the dynamics of every variable inside the circuit. As a good example, we have demonstrated that our modeling tool is very suitable to provide details to study the beat frequency dynamics.

It also needs to be pointed out that although the visual explanation of the beat frequency dynamics helps us to understand the circuit physics, it is not quantitative and cannot explain the situation when the switching frequency is very close to the resonant frequency. For those cases, the small-signal responses have to be studied with the systematic method.

### 3.3.3. Effects of Higher Order Harmonics

It is known that the resonant tank of the SRC is a good band pass filter when the quality factor, $Q$, is high (see Fig. 3.2). As we discussed in 3.3.2, if the converter is operated close to the resonant frequency, the fundamental components will have a very small attenuation, while the higher order (the 3rd, 5th, and so on) harmonics are basically filtered out. In this case, the resonant waveforms are almost pure sinusoids, as shown in Fig. 3.11(a).

To quantify the energy carried by the higher order harmonics, we can define the total harmonic distortion
Fig. 3.11  The waveforms of the resonant tank under different operating conditions.

(a) switching near the resonant frequency:

\[ \frac{F_s}{F_o} = 0.98 \quad Q_s = 2 \quad ; \]

(b) switching lower than the resonant frequency:

\[ \frac{F_s}{F_o} = 0.55 \quad Q_s = 1.05 \quad ; \]

(c) switching higher than the resonant frequency:

\[ \frac{F_s}{F_o} = 2 \quad Q_s = 1.0 \quad . \]
\[ THD = \sqrt{\sum_{k=2}^{\infty} \frac{|W_k|^2}{\sum_{i=1}^{\infty} |W_i|^2}}. \] (3.33)

where the \( \{W_k\} \) are the Fourier components of the given waveform. If the resonant waveform has relatively strong higher order (3rd, 5th and so on) harmonics, its \( THD \) will be high, and the waveform will be more distorted from the pure sine wave (at switching frequency).

For the operating condition where \( F_s \) is close to \( F_o \), the resonant waveforms have very low \( THD \) and the higher order harmonics are negligible. However, when the switching frequency is close to half of the resonant frequency, or when \( F_s \) is much higher than \( F_o \), the selectivity of the bandpass filter becomes not very good, especially when the quality factor, \( Q_s \), is low (corresponding to light load operation, as shown in Fig. 3.2). In such situations, the resonant waveforms contain relatively strong higher order harmonics and the waveforms are not quite like sinusoids any more, as shown in Fig. 3.11(b) and (c).

In the previous work [34, 36], the SRC model was derived based on the assumption that the resonant waveforms can be well approximated by their fundamental components. Since the assumption is only valid for a quite narrow operating region, the application of the model is also limited. Based on the above discussion, it is very natural for us to ask whether we need to take into account the effects of higher order harmonics for small-signal modeling when the resonant waveforms have relatively high \( THD \). This question could not be answered before, because there was
no theoretical method to study the effects of the higher order harmonics.

The modeling approach developed in Chapter 2 makes this kind of investigation possible. It is a straightforward procedure to compare the predictions of the small-signal models with different number of harmonics incorporated. Since the resonant waveforms are symmetrical, they do not contain even number harmonics, but the fundamental, the third, the fifth components and so on. We will use notations SSM-1, SSM-13, and SSM-135 to represent the small-signal models using only the fundamentals, the fundamental plus the third components, and fundamental plus the third as well as the fifth components, respectively. The output capacitor voltage will be approximated by its dc component for all cases.

Figure 3.12 shows the comparison of the SSM-1 and the SSM-13 for operating point near the resonant frequency (the same condition as in Fig. 3.11(a)). The predictions of the two models are almost identical. This is quite understandable because the THD is very small and the higher order harmonics are negligible.

The predictions of the SSM-1, the SSM-13, and the SSM-135 are shown in Fig. 3.13 for the operating condition defined in Fig. 3.11(b). Very surprisingly, we find that the effect of adding the third and the fifth harmonics to the model is very small. This fact can be explained by the observation that the coupling between the higher order harmonic sidebands and the sideband of the rectifier output, \( i_{R,0} \), is very weak. Therefore, in evaluating the small-signal output responses, the dominant contributor is the sideband of the fundamental inductor current, even though the resonant waveforms have severe distortions.
Fig. 3.12  Predicted control-to-output transfer functions of the SRC when the converter is operated near the resonant frequency. The predictions of the SSM-1 and the SSM-13 are almost identical. There is no need to add the third harmonics.
Fig. 3.13  Predicted control-to-output transfer functions of the SRC when the converter is operated close to one half the resonant frequency and the load is light. The resonant waveforms have high THD values (Fig. 4.4. (b)), but the effect of adding higher order harmonics is very small.
Fig. 3.14  Predicted control-to-output transfer functions of the SRC when the converter is switched at twice the resonant frequency. The resonant waveforms have high THD values (Fig. 4.4. (c)), but the effect of adding higher order harmonics is very small.
Fig. 3.15  The waveforms and the predicted control-to-output transfer functions of the LLC-SRC when the converter is switched lower than the resonant frequency and the load is light. The resonant waveforms have high THD values, but the effect of adding higher order harmonics is very small.
Figure 3.14 compares the predictions of the SSM-1 and the SSM-13 for the operating point defined in Fig 3.11(c), where the switching frequency is twice the resonant frequency and the $THD$ values of the resonant waveforms are also high. It is also true for this case that adding the third harmonics to the model has very little effect. Furthermore, the similar study has been done for the LLC-SRC. Figure 3.15 shows that for the operating condition where the waveform distortion is severe, very little difference exists for the predictions of the SSM-1 and SSM-13.

The above results reveal that the SSM-1 is not just a model which can only be used when waveforms are quit sinusoidal, but a model which can be used to capture the converter dynamics over the most of the useful operating regions of the SRCs, even though the higher order harmonics are relatively strong in large-signal sense for certain operations.

### 3.3.4. Predictions of the Small-Signal Model

The variation of the beat frequency dynamics of the SRCs vs the changing of the operating point will be discussed in this subsection using the small-signal models obtained through the extended describing function method.

A group of Bode plots of the SRC are shown in a three dimensional fashion in Fig. 3.16, where the switching frequency is changing from below the resonant frequency to above the resonant frequency, and the load is kept constant.
Fig. 3.16  Bode plots of the SRC with fixed load ($Q_s = 4.5$) and different switching frequencies ($0.6 \leq F_{SN} \leq 1.4$).
First, we can observe the gain of the transfer function varying as the switching frequency changes. The low frequency gain is proportional to the slope of the voltage conversion ratio curve (see Fig. 3.2). When the operating point moves close to the resonant frequency, the slope of the curve gets flat and the low frequency gain drops. Right on the resonant frequency, the gain is zero. When the operating point moves across the peak of the conversion ratio curves, the slope becomes negative and the phase of the Bode plot has a 180° jump.

Second, a single pole of the output filter and a double pole of the beat frequency dynamics can be clearly seen when the switching frequency is away from the resonant frequency. As the operating point is getting closer to the resonant frequency, the output filter pole is moving to a higher frequency, and the beat frequency poles are moving to a lower frequency, and with less peaking. Up to a certain point, the beat frequency poles split and one of them merges with the output filter pole. As the switching frequency keeps increasing and crosses the resonant frequency, the newly formed double pole splits and the output filter pole and the beat frequency double pole can be seen again.

If the SRC is operated at a fixed frequency close to the resonant frequency and the load is varying, the three-dimensional Bode plots can be obtained, as shown in Fig. 3.17. From there, we can see that if the quality factor of the resonant tank is high, the resonant peak is more clearly defined, so the beat frequency dynamics are easier to be seen. Under lighter load condition, the beat frequency poles get more damping and they may split and merge with the output filter pole.
Fig. 3.17  Bode plots of the SRC with fixed switching frequency ($F_{SN} = 1.1$) and different loads ($1.5 \leq Q_s \leq 15$), where

$$Q_s = \frac{Z_o}{R}$$
Based on above observations, a V-shaped operating region can be defined around the resonant frequency, as shown in Fig. 3.18.

Outside the V-region, there is a double pole corresponding to the beat frequency dynamics located approximately at $|F_s - F_o|$. This double pole is separated with the output filter single pole. In [42], Vorperian proposed an approximate model for the SRC. By comparing his model with our model, Fig. 3.19 shows that both models agree pretty well outside the V-region, but Vorperian's model begins to lose accuracy when the operating point approaches the boundary of the V-region. On the boundary, the beat frequency double pole begins to split.

When the operating point moves into the V-region, the beat frequency double pole splits and one of them merges with the output filter pole. The dynamics are determined by a double pole located at the output filter corner frequency:

$$f_c = \frac{1}{2\pi R C_f}.$$  \hspace{1cm} (3.34)

The approximate model in [42] cannot predict the dynamics in this situation because it assumes that the beat frequency double pole will keep moving to a very low frequency according to $|F_s - F_o|$ and split into two real poles. A comparison between our model and the approximate model is shown in Fig. 3.20.
Fig. 3.18  The operating region around the resonant peak is defined (V-region) in which the beat frequency poles split and one of them merges with the output filter pole.
Fig. 3.19  Comparison of the extended describing function model and the approximate model [42]:  
(a) outside the V-region;  
(b) on the boundary of the V-region.
Fig. 3.20  Comparison of the extended describing function model and the approximate model [42]. Inside the V-region, the approximated model cannot properly predict the dynamics.
3.4. Experimental Verification

Experimental circuits were built for measuring the small-signal responses of the SRC and the LLC-SRC. The parameters of the test SRC are:

\[
\begin{align*}
L &= 22.5 \mu H \\
C &= 6.56 \text{nF} \\
C_f &= 16.75 \mu F \\
r_c &= 0.212 \Omega \\
F_o &= 414.3 \text{kHz} \\
Z_o &= 58.6 \Omega
\end{align*}
\]

The conduction losses of the power switches and of the resonant tank are modeled by a lumped resistor, \( r_s \), which is in series with the resonant inductor, as shown in Fig. 4.1 (a). For our test circuit, \( r_s \) is estimated to have a value of 2\( \Omega \).

The LLC-SRC has exactly the same parameter values as the SRC, but with an additional resonant inductor, \( L_p \), which is in parallel to the resonant capacitor:

\[
L_p = 62.8 \mu H
\]

The small-signal transfer functions were measured using the network analyzer (HP-4194). The predictions of the small-signal models are compared with the measurement data.

Figure 3.21 shows the predicted and measured control-to-output transfer functions of the SRC. The operating point is defined in Fig. 3.11(b), where the resonant waveforms have severe distortions. As we can see, the agreement between the prediction (of the SSM-1) and the experimental data is very well up to the Nyquist

3. Modeling Series Resonant Converters
frequency. For this critical operating condition, Fig. 3.21 confirms the extended describing function method developed in Chapter 2 and the conclusion drawn in the previous section that the SSM-1 is adequate for modeling SRCs even when *THD* is high. A similar result of the SRC is obtained in Fig. 3.22, where the *THD* is smaller. Fig. 3.23 shows the comparison between the prediction and the test results for the LLC-SRC; the agreement is also very good.

When the switching frequency is close to the resonant frequency, the SRC behaves like a second order system. The predicted and the measured control-to-output transfer functions are shown in Fig. 3.24.

The small-signal models are also used to predict other transfer functions of the SRC with the experimental data taken from [41]. The results are shown in Fig. 3.25. All of these results confirms the small-signal models based on the extended describing function method.
Fig. 3.21  The control-to-output transfer function of the SRC. The operating condition causes severe waveform distortion (Fig. 4.4 (b)). However, the small-signal model (SSM-1) is well supported by the test data up to the switching frequency.
Fig. 3.22  Predicted and measured control-to-output transfer functions of the SRC have good agreement up to switching frequency. The waveforms have less distortion.
Fig. 3.23  The control-to-output transfer function of the LLC-SRC. The small-signal model (SSM-1) is well supported by the test data up to the switching frequency.
Fig. 3.24   The predicted and measured control-to-output transfer function of the SRC. When the switching frequency is close to the resonant frequency, the SRC behaves like a second order system.
Fig. 3.25  Predicted and measured audiosusceptibility and the input impedance of the SRC. The experimental data are taken from [41], where

\[ L = 197 \mu H \quad C = 51nF \quad C_f = 32\mu F \quad , \]

\[ F_o = 50.2kHz \quad Z_o = 62.2\Omega \quad . \]
3.5. *Equivalent Circuit Model*

By employing the extended describing function technique, we have proven that the higher order harmonics have little effects on the small-signal model of the SRC. The models using only the fundamental components to represent the tank variables (SSM-1) are very capable of capturing the circuit dynamics over a wide operating region.

In this section, we will follow the principles of the extended describing function method to derive a complete small-signal circuit model for the LLC-SRC. Similar derivation can be found in [34-36] for the SRC, but the derivation has not been done before for the LLC-SRC. Since the SRC is just a special case of the LLC-SRC, the derived model will cover both topologies. Instead of just presenting the modeling concept as in the previous work [34-36], we will provide a complete small-signal model which can be used as a convenient tool for practical designs.

The block diagram of the LLC-SRC is shown in Fig. 3.26. There are two nonlinear blocks in the LLC-SRC: one is the inverter which takes the dc input and generates high frequency square wave; another one is the rectifier which changes the high frequency ac current back to dc current. In between the inverter and the rectifier is the resonant tank which functions as a band pass filter. As we pointed out before, when the switching frequency is close to the resonant frequency, the band pass filter will allow fundamental harmonics to pass through, and the higher order harmonics
Fig. 3.26 The block diagram of the LLC-SRC. In between the inverter and the rectifier, the dominant frequency components are the fundamentals at the switching frequency. Outside the two nonlinear blocks, the dc components are dominant.
at $3F_s$, $5F_s$, and so on will be heavily attenuated.

For the modeling purpose, we assume that in the resonant tank, the fundamental components are dominant and in the output filter, the dc components are dominant. In order to model the inverter, we need to find the fundamental components of its output voltage. By assuming that the inverter can perform phase-shift control, and the duty-cycle of the quasi-square wave is denoted by $d$, we have

$$v_{AB,1} = \frac{1}{T_s} \int_0^{T_s} v_{AB}(t)e^{-j\omega t} dt = \frac{4}{\pi} v_s \sin \left( \frac{\pi d}{2} \right) .$$

Since we have assumed that the resonant inductor current, $i_s$, can be approximated by the fundamental

$$i_s(t) = i_{s,1}(t)e^{j\omega t} + i_{s,-1}(t)e^{-j\omega t} ,$$

we can find the average input current drawn by the power stage from the source:

$$i_{g,0} = \frac{1}{T_s} \int_0^{T_s} i_s(t) dt = \frac{2}{\pi} \sin \left( \frac{\pi d}{2} \right) \Re(i_{s,1}) ,$$

where $\Re(i_{s,1})$ represent the real part of the fundamental amplitude of the tank current.

Only the real part of $i_{s,1}$ can draw power from the source because it is in phase with the tank voltage. The average input current is directly dependent on the duty cycle, and $i_{g,0}$ is indirectly dependent on the switching frequency because $\Re(i_{s,1})$ is controlled by $\omega_s$. Up to here, we can see that the inverter can be represented by its extended describing function model, as shown in Fig. 3.27.
**Fig. 3.27**  The inverter and its model. The duty cycle of the quasi-square wave, $v_{AB}(t)$, is denoted by $d$, and

$$k(d) = \frac{2}{\pi} \sin \left( \frac{\pi}{2} d \right).$$
Since the resonant tank is loaded by the voltage sink, $v_R(t)$, we need to find its fundamental components. According to Fig. 3.28, $v_R(t)$ is a square wave, whose amplitude is $v_o$. This square wave is in phase with the tank current, $i_s(t)$. Under the assumption that $i_s(t)$ is a sinusoid, we can find

$$v_{R,1} = \frac{1}{T_s} \int_0^{T_s} v_o \text{sgn}(i_s)e^{-j\omega_s t} \, dt = \frac{4}{\pi} v_o \frac{i_{s,1}}{\| i_{s,1} \|},$$

(3.38)

where $\| \cdot \|$ represents the magnitude of a complex variable. The average component of the rectified tank current can also be found as

$$i_{R,0} = \frac{1}{T_s} \int_0^{T_s} |i_s(t)| \, dt = \frac{2}{\pi} \| i_{s,1} \|.$$

(3.39)

At this point, the rectifier can be characterized by its model, as shown in Fig. 3.28.

For a resonant inductor,

$$L \frac{di}{dt} = v,$$

(3.40)

if we assume the current and the voltage across it can be approximated by

$$i(t) = i_1(t)e^{j\omega t} + i_{-1}(t)e^{-j\omega t},$$

(3.41)

$$v(t) = v_1(t)e^{j\omega t} + v_{-1}(t)e^{-j\omega t},$$

(3.42)

then we have
Fig. 3.28  The nonlinear rectifier and its model.

\[ v_{R,1} = \frac{4}{\pi} \frac{v_o}{v} \frac{i_{s,1}}{i_{s,1}} \]

\[ i_{R,0} = \frac{2}{\pi} || i_{s,1} || \]
\[
L \left( \frac{di_i}{dt} + j\omega_i i_i \right) e^{j\omega_i t} + L \left( \frac{di_{i-1}}{dt} - j\omega_i i_{i-1} \right) e^{-j\omega_i t} = v_i e^{j\omega_i t} + v_{i-1} e^{-j\omega_i t} .
\]

By performing harmonic balance, a modulation equation can be obtained

\[
L \frac{di_i}{dt} + Z_i i_i = v_i , \tag{3.44a}
\]

where

\[
Z_i = j\omega_i L . \tag{3.44b}
\]

According to the modulation equation, an equivalent circuit model of the resonant inductor can be drawn, as shown in Fig. 3.29. When the system reaches the steady-state, the envelope terms will not change with time; therefore the modulation equation will reduce to standard phasor equation:

\[
Z_i I_i = V_i . \tag{3.45}
\]

When the system is under modulation, the envelope terms fluctuate around the steady-state level with the modulation frequency.

In a similar way, the resonant capacitor has its modulation equation

\[
C \frac{dv_i}{dt} + G_i v_i = i_i , \tag{3.46a}
\]

where

\[
G_i = j\omega_i C . \tag{3.46b}
\]
The equivalent circuit model of the resonant inductor governs the dynamics of the envelope when circuit is under modulation, where $Z_i = j\omega_i L$. With modulation free, the envelope is flat and the model reduces to the standard phasor model.
and the equivalent circuit models are shown in Fig. 3.30.

By substituting the models of the nonlinear blocks and the resonant components, we obtain a large-signal equivalent circuit model of the LLC-SRC, as shown in Fig. 3.31. This model is an autonomous system corresponding to the modulation equation derived in Section 2.3.

Approximate steady-state solutions can be found from the large-signal equivalent circuit model by letting all of the inductors short circuit and the capacitors open circuit.

By doing this, we can first find an equivalent load resistance seen by the resonant tank from (3.28) and (3.29):

$$\frac{V_{R,1}}{I_{S,1}} = \frac{8}{\pi^2} R = R_c \quad .$$  \hspace{1cm} (3.47)

Then it is easy to solve this phasor circuit to give the steady-state solutions:

$$I_{S,1} = \frac{V_{AB,1}}{Z_s + r_s + R_c + Z_p \| (1/G)} \quad ,$$ \hspace{1cm} (3.48)

$$V_1 = I_{S,1} [Z_p \| (1/G)] \quad ,$$ \hspace{1cm} (3.49)

$$I_{P,1} = V_1/Z_p \quad ,$$ \hspace{1cm} (3.50)

$$V_o = I_{R,0} R = \frac{2}{\pi} |I_{S,1}| R \quad .$$ \hspace{1cm} (3.51)
Fig. 3.30  The equivalent circuit model of the resonant capacitor, where

\[ G_1 = j\omega_1 C \]
Fig. 3.31  Large-signal equivalent circuit model of the LLC-SRC, where

\[ Z_s = j\omega L_s, \quad Z_p = j\omega L_p, \quad G = j\omega C \]

The models of the nonlinear inverter and the rectifier are defined in Fig. 3.27 and Fig. 3.28, respectively.
From these steady-state solutions, approximate dc characteristics of the LLC-SRC can then be derived. For example, the voltage conversion ratio can be found as:

\[
M = \frac{V_o}{V_s} = \frac{R_q}{\sqrt{R_q^2 + X^2}},
\]

(3.52)

where

\[
R_q = R_c + r_s,
\]

(3.52a)

\[
X = \omega_L + \frac{\omega_p L_p}{1 - \omega^2 L_p C}.
\]

(3.52b)

By perturbing the large-signal equivalent circuit around the steady-state and linearizing the results, an equivalent small-signal circuit model can be synthesized.

Since the commonly used circuit simulator cannot take complex numbers, we need to split the variables in the resonant tank into their real parts and imaginary parts, which correspond to the sine part and the cosine part of the fundamental components. We will use the following notations to represent complex variables:

\[
i_i = i_s + j i_c,
\]

(3.53)

\[
v_i = v_s + j v_c,
\]

(3.54)

where the subscripts \(s\) and \(c\) correspond to the sine and cosine parts of the sinusoidal variable, respectively. The modulation equation for the resonant inductor

\[
L \frac{di_i}{dt} + j \omega_L i_i = v_i,
\]

(3.55)
then becomes

\[ L \frac{di_s}{dt} - \omega_s L i_c = v_s , \tag{3.56} \]

\[ L \frac{di_c}{dt} + \omega_s L i_s = v_r . \tag{3.57} \]

Perturbing these equations and linearizing the results, we have

\[ L \frac{d\hat{e}_s}{dt} - \hat{e}_s = \hat{v}_s , \tag{3.58a} \]

\[ L \frac{d\hat{e}_c}{dt} + \hat{e}_c = \hat{v}_c , \tag{3.58b} \]

where

\[ \hat{e}_s = (\omega_s L) \hat{i}_c + (\omega_s L) \hat{\gamma}_S \hat{f}_{SN} , \tag{3.58c} \]

\[ \hat{e}_c = (\omega_s L) \hat{i}_s + (\omega_s L) \hat{\gamma}_S \hat{f}_{SN} , \tag{3.58d} \]

\[ \hat{f}_{SN} = \hat{\omega}_s / \omega_o . \tag{3.58e} \]

Similarly, the small-signal model of the resonant capacitor splits into

\[ C \frac{d\hat{\psi}_s}{dt} - \hat{j}_s = \hat{i}_s , \tag{3.59a} \]

\[ C \frac{d\hat{\psi}_c}{dt} + \hat{j}_c = \hat{i}_c , \tag{3.59b} \]

where

\[ \hat{j}_s = (\omega_s C) \hat{\psi}_c + (\omega_s C) V_c \hat{f}_{SN} , \tag{3.59c} \]
\[ \hat{j}_s = (\omega_s C) \hat{v}_s + (\omega_s C)V_L \hat{I}_{sv} \quad . \] (3.59d)

The small-signal equivalent circuit models of the resonant inductor and the resonant capacitor are shown in Fig. 3.32.

Then we need to perturb the models of the nonlinear blocks. The small-signal model of the inverter can be found by perturbing (3.35) and (3.37), to give:

\[ \hat{i}_{s,0} = k_i \hat{i}_{ss} + J_d \hat{d} \quad , \] (3.60)

\[ \hat{v}_{Ap,1} = k_v \hat{v}_g + E_d \hat{d} \quad . \] (3.61)

where

\[ \hat{i}_{ss} = Re(\hat{i}_{s,1}) \quad , \] (3.62a)

\[ k_i = \frac{2}{\pi} \sin \left( \frac{\pi}{2} D \right) \quad , \] (3.62b)

\[ J_d = \cos \left( \frac{\pi}{2} D \right) Re(I_{s,1}) \quad , \] (3.62c)

\[ k_v = 2k_i \quad , \] (3.62d)

\[ E_d = 2V_g \cos \left( \frac{\pi}{2} D \right) \quad . \] (3.62e)

The small signal model of the rectifier can be found by perturbing (3.38) and (3.39), to give:

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Fig. 3.32  The small-signal equivalent circuit models of the resonant inductor and the resonant capacitor, where

\[
\hat{e}_s = (\omega_L)\hat{i}_c + (\omega_o L)\gamma_c \hat{f}_{FN},
\]

\[
\hat{e}_c = (\omega_L)\hat{i}_s + (\omega_o L)\gamma_s \hat{f}_{FN},
\]

\[
\hat{j}_s = (\omega_s C)\hat{v}_c + (\omega_o L)\nu_c \hat{f}_{FN},
\]

\[
\hat{j}_c = (\omega_s C)\hat{v}_s + (\omega_o L)\nu_s \hat{f}_{FN}.
\]
\[ \hat{i}_{R,0} = k_s \hat{i}_{ss} + k_c \hat{i}_{sc} \quad , \]  
\[ \hat{v}_{R,1} = (2k_s \hat{v}_s + R_s \hat{i}_{ss}) + j(2k_c \hat{v}_s + R_c \hat{i}_{sc}) \quad , \]  

where

\[ \hat{i}_{ss} = Re(\hat{i}_{s,1}) \quad , \]  
\[ \hat{i}_{sc} = Im(\hat{i}_{s,1}) \quad , \]  
\[ k_s = \frac{2}{\pi} \frac{R_q}{\sqrt{R_q^2 + X^2}} \quad , \]  
\[ k_c = \frac{2}{\pi} \frac{-X}{\sqrt{R_q^2 + X^2}} \quad , \]  
\[ R_s = r_s + R_c \frac{X^2}{\sqrt{R_q^2 + X^2}} \quad , \]  
\[ R_c = r_s + R_c \frac{R_q^2}{\sqrt{R_q^2 + X^2}} \quad . \]  

By substituting the small-signal models into the large-signal model and splitting the real part and the imaginary part, we get a small-signal circuit model as shown in Fig. 3.33.

This equivalent circuit model can be easily implemented in the circuit simulator like SPICE or SABER. It is very convenient to use this model for practical design. A SPICE node list can be found in Appendix B.
Fig. 3.33 Complete small-signal circuit model. This model can be easily implemented in the popular circuit simulator like SPICE or SABER. The SPICE node list can be found in Appendix B.
3.6. Conclusions

In this chapter, the extended describing function method developed in Chapter 2 was applied to two series resonant topologies.

By showing a typical modeling procedure, the capabilities of our model tool were demonstrated for providing comprehensive information related to the dynamic analysis of a switching converter (large-signal responses, steady-state responses, continuous-time small-signal models, Bode plots, pole/zero distribution, etc.).

The effect of adding higher order harmonics to the small-signal model was studied. It has been found that even for the operating conditions under which the resonant waveforms have severe distortion from the sinusoids, the effects of incorporating higher order harmonics are very small. This is an important finding because it justifies the small-signal model based on using only the fundamental components (SSM-1) over the entire operating region where the inductor current is in continuous mode.

The dynamic behavior of the SRCs is investigated for both the low frequency and the beat frequency dynamics. It was found that the beat frequency double pole will split and one of them will merge with the output filter pole when switching frequency is close to the resonant frequency. Corresponding to this phenomenon, a V-shaped operating region is defined. It is shown that the approximate model in [42] becomes inaccurate in this V-region.
The small-signal analyses were verified experimentally. The models provided by our modeling method are accurate up to Nyquist frequency.

A complete equivalent small-signal circuit model was derived for the LLC-SRC which also covers the SRC as its special case. This model can be easily implemented in popular circuit simulators like SPICE or SABER, to provide a useful tool for practical designs.
4. Modeling Parallel Resonant Converters

4.1. Introduction

In this chapter, two parallel resonant topologies are employed to further demonstrate the extended describing function modeling approach developed in Chapter 2. Continuous-time small-signal models accurate to the switching frequency are obtained for both topologies. Regarding previous modeling efforts for the parallel resonant converters (PRCs), most of them were based on sampled-data analysis with discrete-time models as the end results [41,44,46,47]. A continuous-time small-signal model was presented in [42] based on high Q approximation and intuitive analysis. Approximated transfer function was provided in closed form, but the model cannot accurately predict the beat frequency dynamics when the switching frequency is close to the resonant frequency.
By employing the small-signal model provided by the extended describing function method, the effect of incorporating higher order harmonics will be studied for the operating conditions when resonant waveforms are severely distorted from sine waves. The analysis of the dynamics vs the operating point variation will be presented. A complete small-signal equivalent circuit model will also be derived for both topologies to provide a useful tool for practical designs. The small-signal analyses are confirmed experimentally.

4.2. Two Parallel Resonant Topologies

4.2.1. Basics of the Topologies

Two parallel resonant topologies will be analyzed, including the traditional parallel resonant converter (PRC) and a parallel resonant converter with an additional resonant capacitor in series with the resonant inductor (LCC-PRC).

The circuit diagrams of the PRC and the LCC-PRC are shown in Fig. 4.1. We can see that the LCC-PRC will reduce to the PRC if the series resonant capacitor, $C_s$, is very large compared with the parallel resonant capacitor, $C_p$. In fact, for isolated PRC applications, a very large $C_s$ is usually used as a dc blocking capacitor. Reducing its capacitance and letting it join the resonance, we get an LCC-PRC. The resonant frequency, $F_r$, the characteristic impedance, $Z_o$, and the quality factor, $Q_p$, of the PRC
Fig. 4.1. The circuit diagrams of the parallel resonant topologies:

(a) the traditional parallel resonant converter (PRC); and

(b) the parallel resonant converter with a third order tank (LCC-PRC).
and the LCC-PRC are defined as follows:

\[ F_o = \frac{1}{2\pi \sqrt{L_x C}} , \quad (4.1) \]

\[ Z_o = \sqrt{\frac{L_e}{C}} , \quad (4.2) \]

\[ Q_p = \frac{R}{Z_e} , \quad (4.3) \]

where

\[ C_e = \begin{cases} C & \text{PRC} \\ \frac{C_s C_p}{C_s + C_p} & \text{LCC - PRC} \end{cases} , \quad (4.4) \]

When the LCC-PRC is operated under heavy load and the switching frequency is close to the series resonant frequency formed by \( L \) and \( C_s \), most of the tank energy will be delivered to the load instead of circulating through the parallel capacitor, \( C_p \), and the circuit will behave like an SRC. When the load is light, the LCC-PRC is just a PRC with the output voltage divided by the two tank capacitors. Therefore, an LCC-PRC is sometimes called a series-parallel resonant converter (SPRC). Compared with the PRC, the LCC-PRC has improved dc characteristics, shown in Fig. 4.2, which can reduce the current stress of the resonant tank by operating at lower values of \( Q_p \) for the same load, line, and the switching frequency ranges [53].
Fig. 4.2. The voltage conversion ratio characteristics of the PRC and the LCC-PRC. The LCC-PRC can be operated at lower $Q$ values to reduce the current stress of the resonant tank [53].
4.2.2. Piecewise Linear Models

The extended describing function method provides a systematic treatment for general switching converters and operating modes. In this work, we will concentrate on the most useful operating regions of the PRCs and the LLC-PRCs, where the resonant capacitor voltage is in continuous voltage mode and the switching frequency is higher than one half of the resonant frequency \( F_s > 0.5F_o \).

The piecewise linear state equation is found according to the circuit diagram and the operating mode of the PRC:

\[
\dot{x} = Ax + Bu , \\
y = Cx + Du ,
\]

where the state vector, input vector, and the output variable are defined by:

\[
x = (i, v, i_{f}, v_{j})^T ,
\]

\[
u = (v_g, i_o)^T ,
\]

\[
y = v_o .
\]

The matrices in (4.5) and (4.6) represent the nonlinear and switching properties of the circuit:
\[
A = \begin{bmatrix}
-\frac{r_s}{L} & -1 & 0 & 0 \\
\frac{1}{L} & 0 & -\frac{\text{sgn}(v)}{C} & 0 \\
\frac{\text{sgn}(v)}{L_f} & \frac{-r_c \| R}{L_f} & \frac{-R}{(R + r_c)L_f} & 0 \\
0 & \frac{R}{(R + r_c)C_f} & \frac{-1}{(R + r_c)C_f} & 0
\end{bmatrix},
\]

(4.10)

\[
B = \begin{bmatrix}
\frac{s(t)}{L} & 0 \\
0 & 0 \\
0 & 0 \\
0 & \frac{-R}{(R + r_c)C_f}
\end{bmatrix},
\]

(4.11)

\[
C = \begin{bmatrix}
0 & 0 & r_c \| R \\
0 & \frac{R}{R + r_c}
\end{bmatrix},
\]

(4.12)

\[
D = [0 \quad r_c \| R].
\]

(4.13)

There are two switching elements in the matrices. One of them represents the switching action of the diode:

\[
\text{sgn}(v) = \begin{cases}
1 & v \geq 0 \\
-1 & v < 0
\end{cases};
\]

(4.14)

the other one represents the switching action of the active inverter bridge:

\[
s(t) = \begin{cases}
1 & 0 \leq t < 0.5T_s \\
-1 & 0.5T_s \leq t < T_s
\end{cases}.
\]

(4.15)
The switching boundary conditions can be easily determined according to the typical waveform shown in Fig. 4.3.

The boundary conditions at the switching instants $T_1$ and $T_3$ are given by

$$ v(t) \bigg|_{t = T_1} = 0, \quad v(t) \bigg|_{t = T_3} = 0. \quad (4.16) $$

The active switching actions occur at the instants $T_2$ and $T_4$:

$$ (t - 0.5T_2) \bigg|_{t = T_2} = 0, \quad (4.17) $$

$$ (t - T_4) \bigg|_{t = T_4} = 0. \quad (4.18) $$

The operating modes and switching boundary conditions are also suitable for the LCC-PRC. By defining the state vector

$$ x = (i, v_s, v_p, i_t, v_c)^T, $$

the piecewise linear matrices can be obtained in a similar way:
Fig. 4.3. Four pieces of linear circuits can be identified according to the waveforms of the PRC.
\[ A = \begin{pmatrix}
-\frac{r_s}{L} & -1 & -1 & 0 & 0 \\
L & L & 0 & 0 \\
\frac{1}{C_s} & 0 & 0 & 0 & 0 \\
\frac{1}{C_p} & 0 & 0 & \frac{-\text{sgn}(v)}{C_r} & 0 \\
0 & 0 & \frac{\text{sgn}(v)}{L_f} & \frac{-r_c\parallel R}{L_f} & \frac{-R}{(R + r_c)L_f} \\
0 & 0 & 0 & \frac{R}{(R + r_c)C_f} & \frac{-1}{(R + r_c)C_f}
\end{pmatrix}, \quad (4.19) \]

\[ B = \begin{pmatrix}
\frac{s(t)}{L} & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & \frac{-R}{(R + r_c)C_f}
\end{pmatrix}, \quad (4.20) \]

\[ C = \begin{bmatrix}
0 & 0 & 0 & r_c\parallel R & \frac{R}{(R + r_c)}
\end{bmatrix}, \quad (4.21) \]

\[ D = [0 \ r_c\parallel R]. \quad (4.22) \]

### 4.3. Small-Signal Analysis

By employing the modeling tool based on the extended describing function method, the small-signal analysis of both PRC topologies can be performed in a systematic way. The modeling program provides all of the information related to
the small-signal analysis, including the large-signal response, the steady-state response, the continuous-time small-signal model in state space expression, and the Bode plots as well as the pole/zero distribution.

One of the advantages of our modeling method is that it can calculate the sideband spectrum associated with different harmonic components of any circuit variable. This allows us to perform detailed frequency domain analysis to track the propagation of the small-signal sidebands from the control input to the converter output. Figure 4.4 illustrates this process conceptually.

Similar to the discussion mode for the SRCs, a PRC can also be viewed as a system containing several blocks; they are the inverter, the band pass filter, the rectifier, and the low pass filter, as shown in Fig. 4.4. When this system is subject to modulation, the sidebands of the inverter output will go through the band pass filter, the rectifier, and the output filter to provide the small-signal output voltage. We can see that the \( \hat{v}_1 \) will have its maximum amplitude when \( f_m = F_o - F_s \). The peaking of \( \hat{v}_1 \) will penetrate the rectifier and the low pass filter and appear in output responses. This is the so-called beat frequency phenomenon.

Because of the existence of the band pass filter, the sideband of the third harmonics of the resonant voltage, \( \hat{v}_3 \), gets more attenuation than \( \hat{v}_1 \) does, if the switching frequency is around the resonant frequency. It is very hard to evaluate the contributions of \( \{ \hat{v}_1, \hat{v}_3, \ldots \} \) to \( \hat{v}_{r,0} \) because the nonlinear rectifier combines all harmonic components. However, the extended describing function method can solve this problem straightforwardly.
Fig. 4.4. **Block diagram of the PRC and the small-signal frequency analysis.** The bandpass filter attenuates the sidebands of $v_{AB}$ accordingly. The rectifier processes nonlinear coupling between the different harmonics of its input. The small-signal output comes from $\hat{v}_{R,0}$ through the output filter. When

$$f_m = F_o - F_1,$$

$\hat{v}_1$ has its maximum amplitude, which is the beat-frequency phenomenon.
In the remaining part of this section, we will use the small-signal models provided by our modeling tool to study the effects of higher order harmonics and the variation of the dynamics vs the operating point.

4.3.1. Effects of Higher Order Harmonics

The extended describing function method provides a theoretical approach to evaluating the effects of incorporating different numbers of harmonics. The approach is straightforward in that it compares the predictions of the small-signal models using different approximations of the resonant waveforms:

1) only fundamental components (SSM-1);
2) the fundamental and the third harmonics (SSM-13); and
3) the fundamental plus the third and fifth harmonics (SSM-135).

The output choke current and the output capacitor voltage will be approximated by their dc components for all cases.

Figure 4.5 shows the waveforms of the PRC when it is operated near the resonant frequency. The resonant waveforms have very low THD values. The predicted control-to-output transfer functions of SSM-1, SSM-13, and SSM-135 are also shown in Fig. 4.5. These small-signal Bode plots are almost identical.

When the PRC is operated at twice the resonant frequency and the load is heavy (Fig. 4.6), the resonant inductor current becomes a triangle waveform and the THD
Fig. 4.5  The waveform and the predicted control-to-output transfer functions of the PRC.

The operation is near the resonant frequency. The small-signal models incorporating different number of harmonics give almost identical results.
**Fig. 4.6** The waveform and the predicted control-to-output transfer functions of the PRC.

The converter is operated at twice the resonant frequency and the $Q_p$ is low.
is high; that means that a quite portion of energy is carried by the higher order harmonics. Nevertheless, the predictions of the SSM-1, the SSM-13, and the SSM-135 are basically the same.

In Fig. 4.7, the PRC is operated near the half of the resonant frequency under low Q condition. The resonant waveforms are more distorted than in the previous case and have high THD values. Comparing the models, we can see that the predictions of SSM-13 and the SSM-135 are identical, but there is a small discrepancy between SSM-1 and SSM-13. According to the frequency domain analysis, we know that in this case, the fundamental sideband of the capacitor voltage, \( \hat{v}_1 \), is the major contributor to the small-signal output, but the sideband of the third harmonics, \( \hat{v}_3 \), also has a small contribution.

For the LCC-PRC, we have basically the same observation that the effects of adding higher order harmonics to the small-signal model are very small. Based on these facts, we can conclude that SSM-1 can be employed to capture the dynamics for both converters over the operating regions where the switching frequency is beyond half of the resonant frequency and the resonant capacitor voltage is in the continuous mode.

4.3.2. Predictions of the Small-Signal Model

In this subsection, the dynamics of the PRCs and the LCC-PRCs will be discussed using the small-signal model provided by our modeling tool. Special considerations are given to the beat frequency dynamics as the operating condition vary.
Fig. 4.7  The waveform and the predicted control-to-output transfer functions of the PRC.

The operation is close to half of the resonant frequency and the $Q_p$ is low. The

effects of incorporating higher order harmonics are small.
Figure 4.8 illustrates the dynamic behavior of the LCC-PRC using a group of Bode plots presented in a three dimensional fashion. The Bode plots are control-to-output transfer functions generated for a fixed load ($Q_p = 1$) and different switching frequencies ($0.5 \leq F_s/F_o \leq 1.5$). The following features can be observed from these Bode plots:

- the low-frequency gain of the transfer function is varying as the operating point moves along the dc conversion curve; this gain is proportional to the slope of the dc conversion ratio curve;

- the damping factor of the low-frequency poles, which are contributed by the output filter, is related to the load sensitivity of the converter at the given operating point. When $F_s < 0.8F_o$, the dc conversion ratio curves have very small separations for different $Q_p$, the resonant tank is not load sensitive (or it has low output impedance); therefore, the low pass filter poles have a peak. When $F_s > 0.8F_o$, the conversion ratio has larger $\frac{\partial T}{\partial Q_p}$ values, the converter is more load sensitive and the output filter poles have more damping;

- the high-frequency dynamics are dominated by beat frequency double poles. These poles move toward the lower frequency as the operating point moves close to the resonant peak. For this particular case, the minimum beat frequency is still much higher than the output filter corner frequency.

In [42], Vorperian proposed an approximated small-signal model based on intuition. His model gives the beat frequency as the difference between the switching
Fig. 4.8 The control-to-output transfer function of the LCC-PRC under fixed load ($Q_p = 1$) and different switching frequencies ($0.5 \leq F_{SN} \leq 1.5$).
frequency and the resonant frequency, $|F_o - F_s|$. Figure 4.9 compares his model with our model; it can be seen that both models agree well when the switching frequency is away from the resonant frequency. But when $F_s$ is close to $F_o$, the beat frequency never decreases according to $|F_s - F_o|$. 

Figure 4.10 shows the control-to-output transfer functions of the LCC-PRC operated under fixed switching frequency ($F_s/F_o = 1$) and different loads. It can be seen that the beat frequency is not affected if the load is changing.

An interesting phenomenon is revealed by examining Fig. 4.11, where the converter is operated at a fixed operating point on the resonant peak, but the ratio of the output choke inductance to the resonant tank inductance varies. As we can see, when output inductance increases, the beat frequency first moves toward lower frequencies. But when the inductance ratio becomes larger than ten, the beat frequency stops decreasing. This implies that the beat frequency dynamics can be affected by the output filter design. For practical designs, it is often desirable to maintain the output choke current in a continuous mode under a certain light load condition. That means the output filter inductance is usually much larger than the resonant inductance.

However, if there is a need to speed up the transient responses, the beat frequency can be pushed to a higher frequency by reducing the output filter inductance. By doing this, the control-loop may have a higher cross-over frequency before the beat frequency double pole brings in 180° phase lag.
Fig. 4.9  Comparison of the approximate model in [42] with the extended describing function model (PRC control-to-output transfer function).
(a) When $F_s$ is away from $F_o$, both models agree well;
(b) When $F_s$ is close to $F_o$, approximate model loses its accuracy.
Fig. 4.10  The control-to-output transfer function of the LCC-PRC under fixed frequency ($F_{sw} = 1$) and different loads ($0.5 \leq Q_p \leq 5$), where

$$Q_p = \frac{R}{Z_o}.$$
Fig. 4.11  The control-to-output transfer function of the LCC-PRC under fixed operating point ($F_{SN} = 0.98$, $Q_p = 1$) but with different output choke inductances ($L_p/L = 1, 2, \ldots 100$).
To summarize, the dynamics of the LCC-PRC are determined by two low-frequency poles contributed by the output filter and two high frequency poles caused by the interaction of the switching frequency and the natural resonant frequency. It is shown that the beat frequency is affected not only by the operating point but also by the output filter design. Not as implied by the approximated model in [42], the beat frequency does not decrease according to $|F_o - F_s|$ when the converter is operated close to the resonant frequency. Instead, the minimum value of the beat frequency is always higher than the output filter corner frequency. To optimize the power stage design, the operating points can be selected near the resonant peak to minimize the reactive power, without fearing that the beat frequency reduces to a very low value.

4.4. Equivalent Circuit Model

The analysis results in the last section have justified the small-signal models based on using the fundamental components to approximate the resonant waveforms (SSM-1). In this section, the equivalent circuit of SSM-1 will be derived for the LCC-PRC because it covers the PRC as its special case. Similar derivation can be found in [34, 36] for the SRC, but the model has never been verified for the operations where the resonant waveforms are severely distorted. The derivation of SSM-1 for the LCC-PRC has not been done before.
Employing the same approach in Section 3.6, a large-signal equivalent circuit can be derived for the LCC-PRC, as shown in Fig. 4.12, where the impedance terms are defined as:

\[ Z = j\omega_c L \quad , \quad (4.23a) \]
\[ G_s = j\omega_c C_s \quad , \quad (4.23b) \]
\[ G_p = j\omega_c C_p \quad , \quad (4.23c) \]

and the inverter model is the same as that defined in Fig. 3.27.

Different from the series resonant topologies, the rectifier of the LCC-PRC is voltage-fed current loaded type. This rectifier can be modeled by finding its

\[ v_{R,0} = \frac{i}{T_s} \int_0^{T_s} |v_p(t)| \, dt = \frac{2}{\pi} \| v_{p,1} \| \quad , \quad (4.24a) \]

and the fundamental components of its input current:

\[ i_{R,1} = \frac{1}{T_s} \int_0^{T_s} i_{L_s \text{sgn}(v_p)} e^{-j\omega t} \, dt = \frac{4}{\pi} i_o \| v_{p,1} \| \quad . \quad (4.24b) \]

Therefore, the rectifier can be characterized by its model as shown in Fig. 4.13.

To linearize the model in Fig. 4.12, we need to find the steady-state solutions, \( \{I, V_{s,1}, V_{p,1}, I_o, V_{C_s}\} \), for a given operating point, \( \{V_s, D, \omega_s\} \). This can be done by letting all of the inductors short circuit and all of the capacitors open circuit. According to (4.24), we can find the equivalent load resistance seen by the resonant tank:

4. Modeling Parallel Resonant Converters
Fig. 4.12  The equivalent circuit model of the LCC-PRC. where

\[ Z = j\omega L, \quad G_s = j\omega C_s, \quad G_p = j\omega C_p. \]

The inverter model is derived in defined in Fig. 3.27, where

\[ i_{g,0} = \frac{2}{\pi} \sin \left( \frac{\pi}{2} d \right) R_s(i_i) \]

\[ v_{AB,1} = \frac{4}{\pi} \sin \left( \frac{\pi}{2} d \right) v_g \]

and \( d \) is the duty cycle of the inverter output voltage, \( v_{AB}(t) \).
**Fig. 4.13**  The nonlinear rectifier and its model, where

\[
i_{R,1} = \frac{4}{\pi} \cdot \frac{v_{p,1}}{v_{R,1}}
\]

\[
v_{R,0} = \frac{2}{\pi} \cdot v_{R,1}
\]
\[
\frac{V_{p,1}}{I_{R,1}} = \frac{\pi^2}{8} R = R_e. \tag{4.25}
\]

Then it is easy to solve the phasor circuit and find the steady-state solutions:

\[
I_1 = \frac{V_{AB,1}}{r_s + Z + 1/G_{C_i} + (1/G_{C_p}) \parallel R_e}, \tag{4.26a}
\]

\[
V_{s,1} = I_1 \left(\frac{1}{G_{C_i}}\right), \tag{4.26b}
\]

\[
V_{p,1} = I_1 \left[\left(\frac{1}{G_{C_p}}\right) \parallel R_e\right], \tag{4.26c}
\]

\[
V_0 = V_{R,0} = \frac{2}{\pi} \| V_{p,1} \|, \tag{4.26d}
\]

\[
I_0 = \frac{V_0}{R}. \tag{4.26e}
\]

The dc characteristics of the resonant converter can be derived from the steady-state solutions. For example, the voltage conversion ratio is found to be:

\[
M = \left(\frac{V_0}{V_g}\right) = \frac{\omega_s C_s R}{\sqrt{(1 - \omega_s^2 L C_s)^2 + [\omega_s R_s (C_p + C_s) (1 - \omega_s^2 L C_s)]^2}}. \tag{4.27}
\]

The same result based on fundamental approximation was provided in [53].

To synthesize the small-signal circuit model, we need to perturb the rectifier model (4.24); the results are given by
\[
\dot{v}_{R,0} = k_s \dot{v}_{ps} + k_c \dot{v}_{pc}
\]  
(4.28)

\[
\hat{i}_{R,1} = (2k_s \hat{i}_o + g_{ps} \hat{v}_{ps}) + j(2k_c \hat{i}_o + g_{pc} \hat{v}_{pc})
\]  
(4.29)

where

\[
\dot{v}_{ps} = Re(\hat{v}_{p,1}), \quad \dot{v}_{pc} = Im(\hat{v}_{p,1})
\]  
(4.30)

\[
k_s = \frac{2 \text{Re}(V_{p,1})}{\pi \| V_p \|}
\]  
(4.31)

\[
k_c = \frac{2 \text{Im}(V_{p,1})}{\pi \| V_{p,1} \|}
\]  
(4.32)

\[
g_{ps} = \frac{1}{R_c \alpha^2 + \beta^2}
\]  
(4.33)

\[
g_{pc} = \frac{1}{R_c \alpha^2 + \beta^2}
\]  
(4.34)

\[
\alpha = 1 - \omega_s^2 L C_s - R_c r_s \omega_s^2 C_p C_s
\]  
(4.35)

\[
\beta = R_c \omega_s (C_p + C_s) \left( 1 - \omega_s^2 L C_s + \frac{r_i C_i}{R_c C_p} \right)
\]  
(4.36)

Similar to Section 3.6, by substituting the small-signal models of the inverter, the rectifier, and the resonant components (Fig. 3.32) into the large-signal model, a small-signal equivalent circuit can be synthesized as shown in Fig. 4.14.

In the small-signal circuit, the resonant tank is split into the real part and the imaginary part, and they talk to the output filter through the controlled sources. The resistor, \( r_s \), represents the power loss of the resonant tank. The resistors parallel to

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the capacitor, \( C_p \), are not physically in the circuit; they are the small-signal resistances. This circuit model can be easily implemented in circuit simulators like SPICE or SABER. A SPICE node list can be found in Appendix B.

### 4.5. Experimental Verification

A high-frequency full-bridge LCC resonant converter was built and measured to verify the small-signal models provided by the extended describing function method. The circuit parameters were:

\[
L = 36.3\mu H \quad C_s = 1.23nF
\]

\[
C_p = 0.93nF \quad L_f = 37.1\mu H
\]

\[
C_f = 1.19\mu F \quad r_c = 0.973\Omega
\]

\[
F_o = 1.15MHz \quad Z_o = 262\Omega.
\]

The frequency-control transfer functions are shown in Fig. 4.15, where the model predictions match the measurement data very well up to the switching frequency, which is one-half of the output ripple frequency or the Nyquist frequency. Figure 4.15 also shows that the small-signal models are valid for low Q operating condition (in fact when \( Q_p = 0.19 \), the load is so heavy that the resonant capacitor voltage is in discontinuous mode), and the switching frequency can be very close to
the resonant frequency.

Figure 4.16 shows the predicted and measured transfer functions with the duty-ratio as the control variable. These transfer functions also have low-frequency dynamics and high-frequency dynamics. The low-frequency dynamics are contributed by the output filter which is heavily damped by the output impedance of the resonant tank. The output filter poles are usually well separated. The high-frequency dynamics are the result of the interaction of the switching frequency and the resonant frequency; usually, a double-pole is observed at the beat frequency. The high-frequency dynamics are correctly predicted by the models.

The line-to-output transfer function is shown in Fig. 4.17. Besides the good match between the model predictions and the measurement data, the dynamic pattern of audio susceptibility is quite similar to the control-to-output transfer functions.

The predicted and measured output impedances are shown in Fig. 4.18. The magnitude at the low-frequency end is determined by the load resistance and the output resistance of the resonant tank, while the magnitude at the high-frequency end is determined by the esr of the output capacitor.

Figures 4.19 and 4.20 show the predicted and measured small-signal transfer functions of the PRCs. The measurement data are taken from [41], where the circuit parameters are:
Fig. 4.14. Complete small-signal circuit model of the LCC-PRC. This model can be easily implemented in a circuit simulator like SPICE or SABER to help practical design. The SPICE node list can be found in Appendix B.
Fig. 4.15  Frequency-to-Output Transfer Functions of the LCC-PRC. The switching frequency is the control variable. The duty-ratio is not modulated.
Fig. 4.16  Duty-Ratio-to-Output Transfer Functions of the LCC-PRC. The duty-ratio is the control variable. The switching frequency is not modulated.
Fig. 4.17 Audio Susceptibility of the LCC-PRC.
Fig. 4.18 Output Impedance of the LCC-PRC.
Fig. 4.19  Control-to-Output Transfer Functions of the PRC. The switching frequency is the control variable. Measurement data are taken from [41].
Fig. 4.20  Line-to-Output Transfer Functions of the PRC. Measurement data are taken from [41].
\[ C = 470nF \quad L = 36\mu H \quad , \]

\[ L_f = 1.35mH \quad C_f = 32\mu F \quad , \]

\[ F_o = 38.7KHz \quad Z_o = 8.75\Omega \quad . \]

All of the measurement results support the small-signal models provided by the extended describing function technique.

4.6. Conclusions

In this chapter, two parallel-resonant topologies were modeled using the extended describing function method developed in Chapter 2. The effect of incorporating higher order harmonics was studied when the converters are operated not close to the resonant frequency and the resonant waveforms have high THD values. It has been found that the small-signal model using only the fundamental and average components (SSM-1) has the capability to capture the system dynamics for the operating region when the switching frequency is beyond half of the resonant frequency and the resonant voltage is in the continuous mode.

The dynamics of the LCC-PRCs are investigated. It was found that the beat frequency will never go below the output filter corner frequency, and the minimum value of the beat frequency is related to the output filter design.
An equivalent circuit of the SSM-1 was derived for the LCC-PRC. This model also covers the PRC as its special case. This equivalent circuit model is very easy to implement in circuit simulation software like SPICE or SABER, to provide an useful design tool.

The experimental results show that the model predictions agree with the measurement data up to the switching frequency. The high frequency dynamics around the beat frequency are accurately modeled.
5. Modeling Multi-Resonant Converters

5.1. Introduction

In this chapter, two multi-resonant topologies are taken as examples to demonstrate the extended describing function method. These two topologies are forward multi-resonant converter (FMRC) and single-ended-parallel multi-resonant converter (SEP-MRC). Belonging to the family of zero-voltage switching multi-resonant converters (ZVS-MRCs) [71-77], these two isolated topologies have advantages of absorbing the major circuit parasitics (the output capacitance of the MOSFETs, the junction capacitance of the rectifier diodes, and the leakage inductance of the transformer) and of allowing all of the semiconductor devices to operate under ZVS condition. The FMRC and the SEP-MRC are especially suitable for distributed power applications, in which small volume, high-frequency operation, and low EMI are required.
Despite the advantages of these converters, they are very complex circuits because of their structure and operation. Both circuits have third order resonant tanks; the FMRC has six energy-storage elements in its power stage, and the SEP-MRC has eight. It was shown in [74,75] that under constant off-time control, several operating modes exist for these circuits with respect to different loads and switching frequencies, and it is impossible to obtain dc characteristics analytically.

The FMRC and the SEP-MRC are similar to the resonant converters in the following aspects: they have strong oscillatory nature, and the dc output power is delivered mainly by the resonant tank through the switching frequency harmonics. These features prevent us from directly applying the state space averaging [1-5] and the circuit averaging [6, 68] techniques to the circuits for small-signal analysis.

The FMRC and the SEP-MRC are more complex circuits than the resonant converters regarding their dynamic behavior. First, the resonant variables have dc and strong harmonic components which are coupled together to determine the dynamic responses, whereas for resonant converters, only the fundamental components are dominant for the resonant waveforms. Second, the MRCs have time-variant resonant tank, and two resonant frequencies exist in one switching cycle, corresponding to the on/off status of the active switch. The resonant tank is not a clearly defined band pass filter as in the case of the resonant converters.

The extended describing function method is applied to these challenging circuits. The modeling problem will be treated systematically, starting with the piecewise linear descriptions of the FMRC and the SEP-MRC. We will see that it
is not sufficient to use only the dc and the fundamental components to model these circuits. The higher order harmonics have to be taken into account, and their effects to the small-signal models are discussed. The small-signal responses of both the MRCs are presented using the Bode plots and pole-zero distributions. The FMRC and the SEP-MRC are compared from the dynamic characteristics point of view. The small-signal analyses are verified experimentally, and the predictions are confirmed to be accurate up to the Nyquist frequency.

5.2. Two Multi-Resonant Topologies

5.2.1. Basics of the Topologies

The circuit diagram of the FMRC and its typical waveforms are shown in Fig. 5.1 [74]. The resonant tank is formed by L, C_s, and C_d. The magnetizing inductance is denoted by L_m. The conduction loss of the resonant tank is modeled by a lumped resistor, r_s. The core loss of the transformer is modeled by r_m. The equivalent series resistor (esr) of the output capacitor is denoted by r_c.

The circuit diagram and the typical waveforms of the SEP-MRC are illustrated in Fig. 5.2 [75]. This circuit looks like a class-E derived dc-dc converter, but the two converters differ essentially. In the SEP-MRC, C_b functions as a dc blocking capacitor which is not tuned to form series resonance with the inductor, L, as in the case of
class-E amplifier [78]. The SEP-MRC is a current-fed topology with an input choke, \( L_g \). The resonant tank contains \( L, C_s, \) and \( C_d \). The parasitic resistance, \( r_g \), represents the copper loss of the input choke, and the resistances, \( r_s, r_m, \) and \( r_c \), have the same meanings as in the FMRC.

The FMRC and the SEP-MRC have very similar operation. For both circuits, the active switch has a fixed turn-off time, and the control is realized by adjusting the turn-on time. During the turn-on period, a resonance is formed by \( L \) and \( C_d \):

\[
F_{on} = \frac{1}{2\pi \sqrt{LC_d}} ,
\]  

(5.1)

and during the turn-off period, a faster resonance is formed by \( L, C_s, \) and \( C_d \):

\[
F_{off} = \frac{1}{2\pi \sqrt{LC_sC_d/(C_s + C_d)}} .
\]  

(5.2)

Usually, the MRCs are operated in between the two resonant frequencies:

\[
F_{on} < f_s < F_{off} .
\]  

(5.3)

With proper design, there will be enough energy in the resonant inductor to drive the MOSFET voltage, \( v_s \), to zero before the turn-on signal arrives. Therefore, zero-voltage turn-on of the active switch can be achieved.

The resonant frequency and the characteristic impedance of both MRCs are defined by
Fig. 5.1. The forward multi-resonant converter (FMRC) and its typical waveforms.
Fig. 5.2. The single-ended-parallel multi-resonant converter (SEP-MRC) and its

    typical waveforms.
\[ F_o = \frac{1}{2\pi\sqrt{LC}} \quad , \quad (5.4) \]

\[ Z_o = \sqrt{\frac{L}{C_s}} \quad . \quad (5.5) \]

A performance comparison of the FMRC and the SEP-MRC can be found in [75] based on the dc analysis. There has been no report on small-signal analysis of these circuits.

### 5.2.2. Piecewise Linear Models

According to Fig. 5.1, if we define the state vector as

\[ x = (i, v, i_m, v_d, i_o, v_f)^T \quad , \quad (5.6) \]

then we can derive the piecewise linear state equation for the FMRC:

\[ \frac{dx}{dt} = Ax + Bv_g \quad , \quad (5.7) \]

\[ v_o = Cx + Dv_g \quad , \quad (5.8) \]

where
\[
A = \begin{pmatrix}
\frac{-r_s}{L} & \frac{-1}{L} & 0 & \frac{-n}{L} & 0 & 0 \\
\bar{S}_a & 0 & 0 & 0 & 0 & 0 \\
\frac{\bar{S}_a}{C_s} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{n}{L_m} & 0 & 0 \\
\frac{n}{C_d} & 0 & \frac{-n}{C_d} & \frac{-n^2}{C_d} & \frac{-S_d}{C_d} & 0 \\
0 & 0 & 0 & \frac{S_d}{L_f} & \frac{-r}{L_f} & \frac{-k}{L_f} \\
0 & 0 & 0 & 0 & \frac{k}{C_f} & \frac{-k}{C_f R} \\
\end{pmatrix}, \qquad (5.9)
\]

\[
B = \left[ \begin{array}{cccccc}
\frac{1}{L} & 0 & 0 & 0 & 0 & 0 \\
\end{array} \right]^T, \qquad (5.10)
\]

\[
C = \begin{bmatrix}
0 & 0 & 0 & 0 & r & k
\end{bmatrix}, \qquad (5.11)
\]

\[
D = [0] , \qquad (5.12)
\]

and

\[
r = R \parallel r_c , \qquad (5.13)
\]

\[
k = \frac{R}{R + r_c} , \qquad (5.14)
\]

\[
n = \frac{N_p}{N_s} = \text{turns ratio} . \qquad (5.15)
\]

There are two switching elements in the A matrix:
\[
\bar{S}_a = \begin{cases} 
0 & \text{MOSFET or its body diode on} \\
1 & \text{otherwise}
\end{cases}
\] (5.16)

\[
S_d = \begin{cases} 
1 & v_d > 0 \\
0 & v_d \leq 0
\end{cases}
\] (5.17)

where \( \bar{S}_a \) and \( S_d \) represent the switching actions of the active switch and the rectifier diodes, respectively. The switching boundary conditions can also be determined according to the zero crossing of \( v_d \) and \( v_s \) as shown in Fig. 5.1:

\[
v_d(t) \bigg|_{t = T_1, T_3} = 0
\] (5.18)

\[
v_s(t) \bigg|_{t = T_2} = 0
\] (5.19)

Notice that in Fig. 5.1, \( T_0 = 0 \) and \( T_2 = T_s \), and there are four pieces of linear circuits in each switching cycle. In [74], it was shown that when operating condition varies, \( v_s(t) \) may touch zero earlier than \( v_d(t) \) does, and the different zero crossing sequence corresponds to different operating mode. Nevertheless, these variations will not affect the unified treatment of the modeling problem using the extended describing function method.

Similarly, if we define the state vector for the SEP-MRC,

\[
x = (i_g, v_s, i, v_d, i_m, i_o, v_j)^T
\] (5.20)

the piecewise linear state equation can be found as
\[
\frac{dx}{dt} = Ax + Bv_g , \quad (5.21)
\]
\[
v_o = Cx + Dv_g , \quad (5.22)
\]

where
\[
A = \begin{bmatrix}
-\frac{r_s}{L_g} & -\frac{1}{L_g} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\overline{S}_a & 0 & \overline{S}_a & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{L} & -\frac{r_s}{L} & -\frac{1}{L} & -\frac{1}{L} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{C_s} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{C_d} & 0 & -\frac{1}{C_d} & -\frac{1}{C_d} & \overline{S}_d & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{L_m} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{S_d}{nL_f} & 0 & -\frac{r}{L_f} & -\frac{k}{L_f} & \frac{k}{C_f} & -\frac{k}{RC_f}
\end{bmatrix}, \quad (5.23)
\]
\[
B = \begin{bmatrix}
\frac{1}{L_g} \\
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0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
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\[
D = [0] . \quad (5.26)
\]

The switching element $\overline{S}_a$ has the same definition as (5.16). The switching action of the full-wave rectifier is represented by
\[ S_d = \begin{cases} 1 & v_d \geq 0 \\ -1 & v_d < 0 \end{cases} \quad (5.27) \]

The switching boundary conditions are the same as those of the FMRC. It needs to be pointed out that the extended describing function method is suitable to combine with circuit simulators using piecewise linear description for switching converters, like COSMIR [31]. By doing so, the generation of the state equations will become automatic.

### 5.3. Small-Signal Analysis

#### 5.3.1. Harmonic Analysis

In order to apply the extended describing function method to the FMRC and the SEP-MRC, it is very important to properly choose certain harmonic components as the new state variables of the models.

It is relatively straightforward to take the average terms to approximate the output filter variables including the output choke current and the output capacitor voltage. Since the magnetizing inductance is usually much larger than the resonant inductance for both the FMRC and the SEP-MRC, the magnetizing current can also be adequately represented by its average component. For the SEP-MRC, the
switching frequency ripples are quite small on the input choke current and on the dc blocking capacitor voltage; therefore, these two state variables can be approximated by their average components, too.

All of the remaining state variables belong to the resonant tank; they are the resonant capacitor voltages, \( v_r \) and \( v_d \), and the resonant inductor current, \( i \). Since the resonant tank is not a band pass filter and the tank variables have dc biases, the dynamic features of the MRCs are different from that of the resonant converters. To properly approximate the resonant variables using certain harmonic terms, we can employ the Fourier spectrum of the steady-state waveforms as our reference.

Figure 5.3 shows the steady-state waveforms of the FMRC and the spectrum. As we can see, the resonant inductor current, \( i \), contains dc bias and strong fundamental component. Most of the energy is carried by the first three Fourier components, including the dc term. Regarding the resonant capacitor voltage, \( v_r \), the amplitudes of the fourth and higher harmonics are very small. Hence, the dc up to the third harmonic terms can approximate \( v_r \) very well. The resonant voltage, \( v_d \), is very close to a pure sine wave. This voltage is across the transformer winding and does not contain dc bias because the flux balance of the magnetizing inductance requires so.

However, the average of \( v_d \) may not be zero when the circuit is subject to the modulation because the FMRC is a single-ended topology. For example, a time-domain response of the FMRC is shown in Fig. 5.4 where a sinusoidal modulation
Fig. 5.3. The steady-state waveforms of the FMRC and the Fourier spectrum.
Fig. 5.4. The time-domain response of the FMRC when it is subject to small-signal modulation ($f_m = F_s / 5$). The average component of the resonant voltage, $v_{r,0}$, fluctuates with the modulation frequency. This average term should be taken into account for the dynamic analysis.
is applied to the control input with \( f_m = F_i/5 \). It can be clearly seen that the averages of \( v_d \) and \( i_m (L_m di_m/dt = n V_d) \) fluctuate with the modulation frequency. Therefore, for modeling the dynamic behavior of the FMRC, the average of \( v_d \) has to be taken into account. This issue will be further addressed by comparing the small-signal frequency responses with and without considering the average of \( v_d \).

The resonant waveforms of the SEP-MRC and the spectrum are shown in Fig. 5.5. Similar to the FMRC, the resonant voltage \( v_d \) is very close to a pure sine wave and has no dc bias under steady-state because of the flux balance requirement. The resonant inductor current, \( i \), contains a strong fundamental term and a small second order harmonic term. The harmonics with even higher order are quite negligible. The average of \( i \) is zero under steady-state operation due to the existence of the dc blocking capacitor. For the same reasons discussed for the FMRC, the average terms of \( v_d \) and \( i \) need to be taken into account for modeling the dynamics of the SEP-MRC. Besides the strong dc and the fundamental components, the resonant capacitor voltage \( v_c \) also contains relatively strong second order harmonic term. The even higher harmonics have very small amplitudes.

5.3.2. Effects of Higher Order Harmonics

For the convenience to compare the small-signal models incorporating different harmonic terms, we use the name \( SS01 \) for the small-signal model in which the resonant tank variables are approximated by their average and fundamental components. Similarly, the name \( SS012 \) stands for the model using the average up to the
Fig. 5.5. The steady-state waveforms of the SEP-MRC and the Fourier spectrum.
second order harmonic components to approximate the tank variables; and so on.

Figure 5.6 shows the predictions of SS01 up to SS01234 of the FMRC. As we can see, the predictions of SS0123 and SS01234 are almost identical. This indicates that for the FMRC, it is not necessary to take into account the 4th and even higher harmonic terms. On the other hand, if the third harmonic terms are dropped, the difference begins to appear between SS0123 and SS012. The difference increases when the second harmonic terms are also dropped out.

Up to this point, we can see that the SS0123 is adequate for the FMRC. In fact, the small-signal response will not change if we further remove the third order harmonics of $i$ and the third as well as the second harmonics of $v_d$ because these components are very small on the steady-state spectrum of the tank variables, as discussed in 5.3.1..

If we exclude the average component of $v_d$, which can be denoted by $v_{d,0}$, the small-signal response will change a lot. Figure 5.7 illustrates the results where the $SS0123 \backslash v_{d,0}$ represents the model without considering $v_{d,0}$. As we can see, $v_{d,0}$ has contribution to the dynamics around $10-20 KHz$. For single-ended FMRC, $v_{d,0}$ should be included in the model even it is zero under steady-state operation.

If the average magnetizing current, $i_{m,0}$, is removed from the model, we can observe the similar change in small-signal responses, as shown in Fig. 5.8. These results indicate that the parallel resonance between $C_d$ and $L_m$ has impact on the dynamics of the FMRC.
Fig. 5.6. Control-to-output transfer functions of the FMRC. The small-signal models contain different harmonic components.
Fig. 5.7. Control-to-output transfer functions of the FMRC with and without considering $v_{d,0}$.
Fig. 5.8. Control-to-output transfer functions of the FMRC with and without
considering $i_{m,c}$. 

5. Modeling Multi-Resonant Converters
Figure 5.9 compares the small-signal models of the SEP-MRC which contain different Fourier components. We can see that SS012 is good enough to capture the circuit dynamics, and SS0123 is not necessary. Since the second harmonic component of $v_d$ is very small according to the steady-state spectrum, we can remove $v_{d,2}$ without degrading the model accuracy. However, further removing the second harmonic terms of $i$ and $v_s$ will cause different predictions. Very similar to the FMRC, if the average term $v_{d,0}$ is excluded, the high frequency response will change as illustrated in Fig. 5.10. Because of dc blocking capacitor and the full-wave rectifier, the SEP-MRC has more symmetrical operation than the FMRC does. Therefore, the SEP-MRC can be modeled using the dc up to the second order harmonic terms whereas the FMRC model needs to include the third harmonic component.

The consideration of selecting the proper harmonic terms for both MRCs can be summarized as the follows (see also Table 5.1):

1) unlike in the modeling of the resonant converters, the harmonic terms beyond the fundamental components have to be taken into account for modeling the FMRC and the SEP-MRC;

2) the average components of the resonant variables need to be included in the model, even though they may be zeros under steady-state;

3) for the FMRC, SS0123 is the adequate model. The model accuracy cannot be improved by adding more harmonic terms;

4) for the SEP-MRC, SS012 is the proper model; and
Fig. 5.9.  

Control-to-output transfer functions of the SEP-MRC. The small-signal models contain different harmonic components.
Fig. 5.10. Control-to-output transfer functions of the SEP-MRC with and without considering $v_{d,0}$.
Table 5.1 Harmonic tables of the FMRC and the SEP-MRC. For example, the row corresponding to the resonant voltage $v_d$ means that the average and the fundamental components are taken into account for $v_d$.

### FMRC

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5) the Fourier spectrum of the steady-state waveforms provides a good reference on how to select the model variables.

5.3.3. Predictions of the Small-Signal Models

The control-to-output transfer function of the FMRC and the SEP-MRC can be evaluated by means of the small-signal models based on the extended describing function method. As we pointed out before, these circuits feature high order, strong oscillation, and time-variant resonant tank. Therefore it is very difficult to provide good physical explanations for the complex dynamic behavior of the FMRC and the SEP-MRC.

A control-to-output transfer function of the FMRC is shown in Fig. 5.11. The pole-zero distribution is also illustrated in the picture. There are six poles and three zeros in this transfer function. As we know, the double pole located at the lowest frequency ($p_1$ and its complex conjugate) corresponds to the output filter dynamics. The single real zero is the $esr$ zero of the output capacitor. However, the remaining two double poles and one double zero do not have a clear physical explanation.

For the SEP-MRC, the pole-zero distribution of a control-to-output transfer function is shown in Fig. 5.12. This transfer function has an even more complex pattern than that of the FMRC: it has eight poles and six zeroes. Among the six zeroes, four of them are right half plane zeros. As shown in Fig. 5.12, the control-to-output transfer function has a large phase drop which is over 1080° because of these poles and zeroes. The non-minimum phase property is caused by charging and discharging
Fig. 5.11. The predicted control-to-output transfer function of the FMRC and the pole-zero distribution.
Fig. 5.12. The predicted control-to-output transfer function of the SEP-MRC and the pole-zero distribution.
the input choke and the dc blocking capacitor. These operations cause delay between
the control action and the output response. When output needs more energy to
compensate the load change, the active switch has to be turned on for a longer time
to first pump the energy into the input choke; after the active switch is turned off,
the energy in the choke is used to charge the dc blocking capacitor. Then until the
next turn on of the active switch, the energy stored in the dc blocking capacitor can
be delivered to the output. This situation is similar to the non-minimum phase
operation of the PWM boost converter, but here the phenomenon is more complex.

In [75], a comparison of the FMRC and the SEP-MRC was made from the dc
analysis point of view. It was concluded that both circuits are very close in their
steady-state performance. However, the results presented in Fig. 5.11 and Fig. 5.12
clearly show that the FMRC has better dynamic characteristics than the SEP-MRC
does. Due to the right half plane zeros, the SEP-MRC has to use a much lower
control-loop gain than that of the FMRC (for similar applications). The non-minimum
phase property of the SEP-MRC severely limited the speed of the transient response
unless other control schemes are employed to improve the dynamic characteristics.

5.4. Experimental Verification

To verify the small-signal analysis experimentally, a test FMRC was built with
the following parameters (see Fig. 5.1):

\[ L = 45 \mu H, \quad C_s = 48.2 nF \]

\[ L_m = 139 \mu H, \quad C_d = 661 nF \]

\[ L_f = 82 \mu H, \quad C_f = 34 \mu F \]

\[ \frac{N_p}{N_s} = 2, \quad r_s = 0.63 \Omega \]

\[ r_c = 206 m\Omega, \quad r_m = 7.3 K\Omega \]

\[ F_o = \frac{1}{2\pi \sqrt{L C_s}} = 108.1 KHz \]

\[ Z_o = \sqrt{\frac{L}{C_s}} = 30.6 \Omega \]

where \( r_s \) is a lumped resistance for modeling the conductor loss of the resonant tank and \( r_m \) is for modeling the core loss of the transformer.

A test SEP-MRC was also built which has the following parameter values:

\[ L_g = 635 \mu H \quad C_p = 2.1 \mu F \]

\[ \frac{N_p}{N_s} = 4 \]

\[ L = 51.5 \mu H \]

\[ C_s = 53.2 nF \]

\[ C_D = 107.5 nF \]

\[ r_g = 0.28 \Omega \]

\[ r_s = 0.6 \Omega \]
\[ L_m = 137.6 \mu H \quad r_m = 2.0K \Omega \]

\[ L_f = 63 \mu H \quad C_f = 22 \mu F \]

\[ r_c = 195m\Omega \]

\[ F_c = 96.2KHz \quad Z_o = 31.1\Omega \]

The control circuits of both test converters have the same configuration, which contains a voltage-controlled oscillator (VCO) to implement the constant off-time variable on-time control. The schematic of the control circuit is shown in Fig. 5.13, where the arrangement is for open-loop measurement of the control-to-output transfer functions. The injected modulation signal (coming from the network analyzer HP4194) is fed into a buffer operational amplifier. A transistor is used to construct a voltage controlled resistor to vary the RC constant of the turn-on time. The RC constant of the turn-off time is fixed. The measured VCO characteristics are shown in Fig. 5.14. The VCO gain is calculated by linear regression of the test data, to give:

\[ G_{VCO} = \frac{f_s}{\hat{v}_o} = 99.88(KHz/Volt) = 116dB(Rad/Sec/Volt) \]

As indicated in 5.3.2, the model SS0123 will be used to predict the small-signal responses of the FMRC, and the model SS012 will be used for the SEP-MRC.

Figure 5.15 shows the predicted and measured control-to-output transfer functions of the FMRC for different operating conditions. The agreement between the prediction and the measurement is quite good up to the Nyquist frequency.
Fig. 5.13. The voltage-controlled oscillator (VCO) implements the constant off-time variable on-time control for both the FMRC and the SEP-MRC.
Fig. 5.14. The measured VCO characteristics and the VCO gain.
Fig. 5.15. The predicted and measured control-to-output transfer functions of the FMRC.
Fig. 5.16. The predicted and measured transfer functions of the SEP-MRC:
Top: control-to-output transfer function; and
Bottom: Line-to-
output transfer function.
Fig. 5.17. The predicted and measured small-signal terminal impedances:

*Top:* output impedance of the FMRC; and

*Bottom:* input impedance of the SEP-MRC.
The predicted and measured transfer functions of the SEP-MRC are illustrated in Fig. 5.16. For both the control-to-output transfer function and the audiosusceptibility, the model predictions match the experimental data very well.

The predicted and measured small-signal terminal impedances are shown in Fig. 5.17. Good agreement between the analysis and the experiments are also obtained.

5.5. Conclusions

The small-signal analyses of two multi-resonant topologies, the FMRC and the SEP-MRC, are presented to demonstrate the extended describing function technique. Although these converters have very complex structures and operation, they can be treated in a systematic way using our modeling method. To properly choose the Fourier components for each state variable, we can use the spectrum of the steady-state waveforms as a reference. The average components of all the resonant variables have to be taken into account even though some of them are zeros under steady-state operation. It has been shown that for modeling the FMRC, the dc up to the third order harmonic components are required; and for the SEP-MRC, the dc, the fundamental, and the second harmonic components are sufficient for a good model.
The control-to-output transfer functions of both MRCs have complex patterns. For the FMRC, the transfer function has six poles and three zeros. For the SEP-MRC, the numbers of poles and zeros are eight and six, respectively. It has been revealed that the SEP-MRC has non-minimum phase property: four of the zeros are located on the right half plane. This has a strong impact on the control-loop design of the SEP-MRC.

The analyses have been verified experimentally. The small-signal models are well supported by the measurement data up to the Nyquist frequency. The results clearly demonstrate that the extended describing function method is suitable for modeling the multi-resonant converters.
6. Conclusions

In recent years, lot of new switching converter topologies have been invented aiming at more compact, more efficient, and more reliable power conversion. The technological innovation brings increasing demand for optimizing the dynamic performance of the switching regulators in various applications. There has been a strong desire to develop a generalized small-signal modeling technique to meet this demand.

It is well known that the state space averaging method is very effective for modeling PWM converters. However, resonant technique has been used in many converters to achieve soft-switching, and these converters cannot be treated by averaging method. As a general modeling approach, sampled-data method can be applied to the resonant type of converters; but in practical design, the discrete time small-signal models are not well appreciated because they are not supported by the commonly used measurement scheme and they are not consistent with the popular analog sensing and compensation circuitry.
It is a challenging task to provide continuous-time small-signal models for the switching converters with strongly oscillatory nature. The models should be capable of predicting the circuit dynamics when the modulation frequency is close to the Nyquist frequency (or one half of the output ripple frequency). This involves to capture the high frequency dynamics caused by the interaction of the circuit natural frequency and the switching frequency. To achieve this, we need to take into account not only the average components of the circuit variables but also their harmonic amplitudes (at multiplies of the switching frequency).

An effort was made in [35] to extend the state space averaging technique for modeling the resonant converters. A generalized averaging concept was proposed to model a resonant converter. When the resonant waveforms can be approximated by sinusoids (at switching frequency), the small-signal model can be derived. However, this procedure cannot be carried out when more harmonic terms are needed to approximate the resonant waveforms because the derivation depends on analytical solutions. No systematic approach was suggested in [35] to implement the concept for general applications.

J. Grovs proposed a modeling approach which imitates the network analyzer to calculate the small-signal spectrum when a given converter is subject to small-signal modulation [28]. If the modulation frequency is commensurate with the switching frequency, then a commensurate cycle (usually includes hundreds of switching cycles) exists and the circuit has periodic trajectory over this long period. By employing harmonic balance over the commensurate period and using the piecewise
linear description of the switching circuit, a complex algorithm is derived for general modeling problem. The execution of the method depends upon knowing the steady-state responses of the converter over the commensurate period under every modulation frequency. Generally, providing such responses requires performing large-signal simulation. Therefore, the modeling procedure is complicated and time consuming.

The extended describing function method is proposed as a general modeling approach based on the modification of Groves' method. The circuit nonlinearities are modeled by the multiple variable describing functions which are the Fourier coefficients of the nonlinear function. The parameters of the small-signal model are the partial derivatives of the describing functions. It has been shown that if the modulation frequency is low (compared with the switching frequency), the model parameters defined over the long commensurate period can be approximated by their counterparts defined over one switching cycle. By doing this, the complexity of the small-signal model is greatly reduced and the implementation procedure is significantly simplified. For PWM converters, the derived model is equivalent to the state space averaging model. The proposed method can incorporate any Fourier components to ensure good model accuracy; therefore, it can be used for modeling various converter topologies under all kinds of operating conditions. The execution of the proposed method only depends on the steady-state response over one switching cycle and the extensive simulation effect in [28] is avoided. Compared with the
generalized averaging method [35], the proposed method offers advantages in systematic implementation in which a complete derivation is provided ready for coding into a computer program.

A computer program has been developed to implement the extended describing function method. This modeling tool is able to provide comprehensive information related to the dynamic analysis of a switching converter, which includes large-signal responses, steady-state responses, continuous-time small-signal models, and pole-zero distributions. It is very convenient to use this modeling tool for practical designs.

Two series resonant topologies (SRC and LLC-SRC) and two parallel resonant topologies (PRC and LCC-PRC) are taken as examples to demonstrate the extended describing function method. These converters have a strongly oscillatory nature, and therefore they cannot be modeled by the averaging techniques. The Fourier components of fundamental and higher harmonics are presumably required for modeling these circuits.

It has been found that even for the operating conditions under which the resonant waveforms have severe distortion from sinusoids, the effects of incorporating higher order harmonic terms (beyond the fundamental one) are very small. This is an important finding because it justifies the small-signal model based on using only the fundamental components over the most useful operating regions of these converters.

Based on the approximation of using the fundamental components, complete equivalent small-signal circuit models can be realized using network synthesis
technique for the LLC-SRC and the LCC-PRC, which cover the SRC and the PRC as their special cases. These models can be easily implemented in popular circuit simulators, like SPICE or SABER, and can be used for practical designs.

The dynamic behavior of SRCs is investigated for both the low frequency and the beat frequency dynamics. It is found that the beat frequency double pole will split, and one of them will merge with the output filter pole when the switching frequency is close to the resonant frequency. Corresponding to this phenomenon, a V-shaped operating region is defined. It is shown that the approximate model in [42] becomes inaccurate in this V-region.

The dynamics of the PRCs are investigated. It was shown that the beat frequency will never go below the output filter corner frequency, and the minimum value of the beat frequency is related to the output filter design.

To further demonstrate the capability of the proposed modeling technique, the small-signal analyses are presented for two multi-resonant topologies (the FMRC and the SEP-MRC). Although these converters have complex structure and operation, they can be treated systematically. The steady-state spectrum is employed as the reference to choose the harmonic components for each state variable. It is shown that the average components of all the resonant variables have to be taken into account, even though they may be zeros under steady-state operation. For modeling FMRC, the dc up to the third order harmonic components are necessary, and for the SEP-MRC, the dc, the fundamental, and the second harmonic components are sufficient for good model accuracy.

6. Conclusions
The small-signal control characteristics of both MRCs have been evaluated. For the FMRC, the typical transfer function has six poles and three zeros. For the SEP-MRC, the numbers of poles and zeros are eight and six, respectively. It has been revealed that the SEP-MRC has non-minimum phase property; four of the zeros are located on the right half plane. This has strong impact on the control-loop design and severely limits the transient response speed of the SEP-MRC.

All of the small-signal analyses for the resonant and multi-resonant converters are verified experimentally. The predictions of the small-signal models are well supported by the measurement data up to the Nyquist frequency.
Appendix A. Computer Program for Implementing the Extended Describing Function Method

A.1. Introduction

A computer program has been developed to implement the extended describing function method proposed in Chapter 2. In Appendix A, a user's manual and complete source codes of the program are provided. The program is developed under MATLAB environment, and is executable on both the IBM compatible personal computers and the SUN work stations. It is easy to transplant the program into C language. It is especially attractive to combine this program with the switching circuit simulator COSMIR [31] because both software packages are based on piecewise linear description of the converters.

The program is relatively small, it has a total of about 500 lines. The structure of the program is pretty simple, which can be illustrated in Fig. A1. The main program contains several major function blocks which are invoked according to the following sequence. The large-signal simulation can be performed by calling the function
LARGE whose outputs are the transient response. For a given converter, the piecewise linear state equations and the switching boundary conditions are defined in the function TOPO. The function STEADY can be invoked based on calling LARGE to find the steady-state operation which includes the switching instants and the state variables on these instants:

\[ \{T_i, x(T_i), i = 1, \ldots, Q \} \]

This information is then passed to the function SMALL which implements the systematic modeling algorithms derived in Chapter 2. The output of SMALL are the small-signal model parameters in a state space representation (matrices A, B, C, and D). The post processing subroutines follow the function SMALL to generate the Bode plot and the pole-zero distribution using the matrices of the small-signal model.

In A.2, the user’s manual is provided to demonstrate the procedures of using the program to analyze a converter circuit. A.3 of this appendix provides complete source codes of the program. There are plenty comments in the source codes to make them easy-to-read and self-explanatory.
Fig. A1. The structure of the computer program for implementing the extended describing function method.

A series resonant converter (SRC) is used to demonstrate the modeling procedures. The circuit diagram of the SRC is shown in Fig. 3.1. The parameter values of the SRC are:

\[
L = 22.5\mu H \quad C = 6.56nF \\
C_f = 16.75\mu F \quad r_c = 0.212\Omega \\
F_o = 414.3KHz \quad Z_o = 58.6
\]

Since the program has a relatively simple structure, it is easy to execute it by following the procedures described below:

Step 1. Create a user's subdirectory and copy all of the files provided in A.3 to the subdirectory;

Step 2. Under the user's subdirectory, edit the file `topo.m` to define the circuit parameters, the switching boundary conditions, the operating condition, and piecewise state equations. If the SRC circuit is operated below the resonant frequency in continuous tank current mode, then there will be four linear circuits in each switching period, as shown in Fig. 3.3. The example file (`topo.m`) can be found in A.3 which completely defines the SRC under analysis.
Step 3. Under the user’s subdirectory, enter *MATLAB* and wait for the prompt `>>` appear on the screen;

Step 4. To run the program, enter *main*. The main menu will be displayed on the screen:

1) *Large-signal simulation*

2) *Steady-state analysis*

3) *Small-signal analysis*

4) *Post Processing*

5) *Exit*

Step 5. To perform large-signal simulation, enter 1. The transient responses will be displayed on the screen, as shown in Fig. 3.4. The program then returns to the main menu.

Step 6. To find the steady-state response, enter 2. The steady-state subroutine will be invoked and after Newton’s iteration converges, the steady-state waveforms, the steady-state initial conditions, the switching instants, and the values of the state vector on the switching instants will also be displayed on screen, as shown in Fig. 3.5. For this example, the operating condition is defined as
\[ V_g = 40.9V \]
\[ F_s/F_o = 0.678 \]
\[ Q_s = 1.49 \]

The program then returns to the main menu;

Step 7. To perform small-signal analysis, enter 3. Upon knowing the steady-state waveforms, Fourier analysis will be performed first to calculate the steady-state spectrum of the state variables, as shown in Fig. 3.6. The spectrum clearly shows the amplitude of each harmonics. The harmonic table can be defined according to the steady-state spectrum, as shown in Table 3.1. In this example, the fundamental and the third harmonics of the resonant current, the fundamental of the resonant voltage, and the dc component of the output capacitor voltage are taken into account. The small-signal model is obtained with a state space expression:

\[ \frac{d\hat{X}}{dt} = A_m\hat{X} + B_m\hat{u} \]

\[ \hat{y}_o = C_n\hat{X} + D_m\hat{u} \]

The matrices are calculated and displayed on the screen:
\[
A_n = \begin{pmatrix}
-3.507e + 05 & 1.481e + 06 & 2.858e + 05 & 1.127e + 05 & -4.444e + 04 & 1.404e + 00 & -2.593e + 04 \\
-2.047e + 06 & -9.041e + 05 & 6.787e + 05 & 2.677e + 05 & 1.245e + 00 & -4.444e + 04 & 1.092e + 04 \\
2.857e + 05 & 6.789e + 05 & -9.168e + 05 & 5.024e + 06 & -1.417e + 00 & -3.367e + 00 & 3.438e + 03 \\
1.125e + 05 & 2.673e + 05 & -5.564e + 06 & -3.380e + 05 & -5.579e - 01 & -1.325e + 00 & -8.727e + 03 \\
1.524e + 08 & 0.000e + 00 & 3.110e - 07 & 0.000e + 00 & 0.006e + 00 & 1.764e + 06 & 0.000e + 00 \\
0.006e + 00 & 1.524e + 08 & 0.000e + 00 & -1.676e - 08 & -1.764e + 06 & 0.000e + 00 & 0.000e + 00 \\
6.968e + 04 & -2.934e + 04 & -9.232e + 03 & 2.345e + 04 & 2.781e - 07 & 6.608e - 07 & -1.509e + 03 \\
\end{pmatrix}
\]

\[
B_n = \begin{pmatrix}
-3.365e - 11 & -3.584e - 01 \\
-2.829e + 04 & -3.848e - 01 \\
-2.171e - 11 & -9.081e - 02 \\
-9.431e + 03 & 4.486e - 01 \\
0.000e + 00 & -3.325e + 01 \\
0.000e + 00 & 3.097e + 01 \\
0.006e + 00 & 0.000e + 00 \\
\end{pmatrix}
\]

\[
\]

\[
D_n = (0 & 0)
\]

The program then go back to the main menu;

Step 8. To generate the small-signal Bode plot and the pole-zero distribution, enter 4. The results will be displayed on the screen, as illustrated in Fig. 3.7 and 3.8;

Step 9. To analyze another converter or the same converter under a different operation condition, edit the file topo.m and repeat steps 2-8.

### A.3. Complete Source Codes

In A.3, complete source codes of the program are provided. For each subroutine,
a header is written which describes the name, the function, the input variables, the output variables, and the lower level subroutines called. Detailed comments can be found in the source codes for the explanation of procedures and the definitions of every variables. The source codes are easy to read and self-explanatory.

There are total 12 files which include the main program and the all subroutines. These files have same extension name ".m". The list of these files are given below:

1    main.m
2    large.m
3    steady.m
4    small.m
5    topo.m
6    get_dim.m
7    fourier.m
8    ptxu.m
9    Delta.m
10   Gama.m
11   Lambda.m
12   un_map.m

The source codes are provided according to the sequence of the above list.

% %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Name: main.m
% Function:main program
% Input:none
% Output:large signal responses
% steady-state responses

Appendix A.
% small-signal model (bode-plot, pole-zero)
% Calling: topo.m ------- define the converter
% large.m ------- large-signal simulation
% steady.m ------- find steady-state operation
% spec_ss ------- find spectrum of steady-state
% waveforms
% small.m ------- find small-signal model
% my_plot.m ------- generate Bode-plot

Initialization

call topo.m to get operating conditions and the circuit parameters: [switching, num_mode, dim_out, harm_tbl, x0, U0, Ts, ...
A, B, C, D, Ab, Bb, Cb, Db] = topo(1, 0, 0, 0);
% operating conditions:
% x0 -------- initial value of the state vector
% U0 -------- given input vector
% Ts -------- switching period
[dim_ss, nx] = size(harm_tbl);
dim_in = length(U0);
% The following parameters of the problem are defined:
% num_mode ----- number of the linear circuits in one switching cycle
% dim_ss ------- dimension of the state vector
% dim_in ------- dimension of the input vector
% dim_out ------- dimension of the output vector
% dim_X ------- dimension of the small-signal model
% harm_tbl ----- define harmonic components taken as model variables,
% where 1 = yes, 0 = no (see Chapter 3, Table 3.1)
% For example:
% %
% dc    fundamental
% harm_tbl = [ 0 1 ] -- 1st state variable
% 0 1      -- 2nd state variable
% 1 0      -- 3rd state variable
%
Main Options

format compact
i=0;
while i < 5
% pop up main menu
Large-Signal Analysis
if i == 1,
n_cycle = 5; % number of cycles for simulation
% call large-signal simulator
[xend, T, x_T, Time, x] ...
    = large(n_cycle, x0, dim_ss, Ts, U0, num_mode);
% returned variables:
% xend --- end value of the state vector: xend=x(n_cycle*Ts)
% T ------ switching instants of the last cycle
% x_T ---- state vectors on the instants T
% Time---- time points for the whole simulation
% x ------- state vectors on all of the time points

% generate transient waveforms
clg;
subplot(211); plot(Time, x(1, :)); title('inductor current');
xlabel('time'); grid;
subplot(212); plot(Time, x(2, :)); title('output voltage');
xlabel('time'); grid;
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%% Steady-State Analysis %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if i == 2,
    err_level = 1.0e-5;  % error control for Newton's iteration
    iteration = 10;      % max. number of iteration
    % get initial condition
    n_ic = input('1. x0  2. last simulation  ');
    if n_ic == 1
        x0 = x0;
    else
        x0 = xend;
    end
    % call steady-state finder
    [x_ss, x_T_ss, T_ss] = steady(iteration, x0, dim_ss, ...
                                   Tn, U0, num_mode, err_level);
% returned steady-state variables:
% x_ss -------- initial condition
% T_ss -------- switching instants in one switching cycle
% x_T_ss ------ state vector on the switching instants T_ss
% simulate another cycle to generate steady-state waveforms
n_cycle = 1;
[xend, T, x_T, Time, x] ...  
    = large(n_cycle, x_ss, dim_ss, Ts, U0, num_mode);
% calculate the spectrum of the steady-state waveforms
[Freq, Spec] = spec_ss(x_ss, U0, T_ss, x_T_ss,...
                       Tn, num_mode, dim_ss);
% plot steady-state waveforms and the spectra
clg;
subplot(221); plot(Time, x(1, :)); title('inductor current');
xlabel('time'); grid;
subplot(222); plot(Time, x(2, :)); title('output voltage');
xlabel('time'); grid;
subplot(223), bar(Freq, Spec(1, :), '-');
ylabel('spectrum'); grid; xlabel('harmonics');
subplot(224), bar(Freq, Spec(2, :), '-');
ylabel('spectrum'); grid; xlabel('harmonics');
end

%-------------------------------------------------------------
% Small-Signal Analysis %-------------------------------------------------------------
if i == 3,
    % call small-signal function
    [Am, Bm, Cm, Dm] = small(dim_in, dim_out, Ts, U0, x_sm, ...
                           x_T_ss, T_ss, num_mode, harm_tbl);
    % returned results:
    % Am, Bm, Cm, Dm ---- small-signal model matrices
    i = 4;
end

%-------------------------------------------------------------
% Post Processing %-------------------------------------------------------------
if i == 4,
    % define Bode plot parameters
    fst = 10.01; % start frequency
    fend = 1 / Ts; % end frequency
    n = 100; % number of points on the Bode plot
    Title = 'SRC control-to-output';
    % choose input source
    ui = input('From input: 1. Vg 2. Fs '); % choose output variable
    uo = input('To output: 1. Vo '); % mshift = input(' mag_shift = ');
    % generate Bode plot
    wst = log10(fst*2*pi);
    wend = log10(fend*2*pi);
    w = logspace(wst, wend, n);
    f = w / (2 * pi);
    [mag, phase] = bode(Am, Bm, Cm(uo, :), Dm(uo, :), ui, w);
    mag(:)=20*log10(mag(:)) + mshift;
    clg
    semilogx(f, mag, f, phase/4.5); title(Title);
    xlabel('Hz');
    ylabel('Mag (dB)  Phase(Degree)/4.5 ')
    grid
    pause;
    % generate pole-zero distribution
    [Z, P, K] = ss2zp(Am, Bm, Cm, Dm, ui);
    ws = 2 * pi / Ts;
    Zr = real(Z) / ws;
    Zi = imag(Z) / ws;
    Pr = real(P) / ws;
    Pi = imag(P) / ws;
    % plot poles and zeros
    clg;
    plot(Zr, Zi, 'o', Pr, Pi, 'x'); grid;
    xy = [-2, 0.5, -1.25, 1.25];
    axis(xy);
    axis('square');
title('Pole-Zero Distribution');
xlabel(' Real [Fs]'); ylabel(' Imag [Fs]');
end
end % end of while-loop
return

% Name: large.m
% Function: large-signal simulation
% Input: n_cycle ------ # of cycles of simulation
% x0 ------------ initial condition
% dim_ss ------ dimension of state space
% Ts ------------ switching period
% u ------------ input vector
% num_mode ---- # of topological mode
%
% Output: xend ------ end value of state: x(n_cycle*Ts)
% T ---------- switching instants of the last cycle
% x_T ---------- state vector on T
% Time -------- all of the time points of simulation
% x ----------- all of the state vector of simulation
%
% Calling: topo.m -------- define the converter and checking the
% switching boundary condition
%
function [xend, T, x_T, Time, x] ...
    = large(n_cycle, x0, dim_ss, Ts, u, num_mode);
    % initialization
    Tend = Ts * (n_cycle + 0.01); % define end-simulation time
    cur_mode = 1; % set initial mode = 1
    h = Ts / 50; % fixed-step size
    h_min = 1.0e-6 * Ts; % min step-size
    x_old = x0;
    h_old = h;
    k = 1;
    t = 0;
    x(:, 1) = x0; % first point of time domain response
    Time(1) = 0;
    % call topo.m to get (A, B, C, D) matrices of current mode
    % switching, num_mode, dim_out, harm_tbl, xx0, UU0, Ts, ...
    A, B, C, D, Ab, Bb,Cb, Db] = topo(cur_mode, x0, u, t);
    % find Phi & Ksi matrices
    rank_A = rank(A);
    if rank_A == dim_ss,
        [Phi] = expm(A * h);
        [Ksi] = inv(A) * (Phi - eye(dim_ss)) * B;
    end
else,
    [Phi, Ksi] = c2d(A, B, h);
end

% main loop for simulation
while t < Tend,
    x_new = Phi * x_old + Ksi * u;  % integrate one step
    if t+h > Tend
        cross_tend = 1;
    end
% check switching boundary conditions
    [switching, num_mode, dim_out, harm_tbl, xx0, uu0, Ts, ...]
    A, B, C, D, Ab, Bb, Cb, Db] = topo(cur_mode, x_new, u, t+h);
    if switching == 1,  % not crossing the boundary
        % update time and state
        k = k + 1;
        t = t + h;
        x(:, k) = x_new;
        x_old = x_new;
        Time(k) = t;
    elseif h < h_min,  % switching instant is found
        % update time and state
        t = t + h;
        k = k + 1;
        Time(k) = t;
        x(:, k) = x_new;
        x_old = x_new;
        % recording switching instant and state vector
        T(cur_mode) = t;
        x_T(:, cur_mode) = x_new;
        h = h_old;
        % switch to next mode
        cur_mode = cur_mode + 1;
        if cur_mode > num_mode,
            cur_mode = 1;
            current_cycle = round(t / Ts)
        end
% get {A, B, C, D} matrices for next mode
    [switching, num_mode, dim_out, harm_tbl, xx0, uu0, Ts, ...]
    A, B, C, D, Ab, Bb, Cb, Db] = topo(cur_mode, x, u, t);
% find Phi & Ksi matices for the new mode
    rank_A = rank(A);
    if rank_A == dim_ss,
        [Phi] = expm(A * h);
        [Ksi] = inv(A) * (Phi - eye(dim_ss)) * B;
    else,
        [Phi, Ksi] = c2d(A, B, h);
    end

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elseif switching == -1,  % cross switching boundary
    h = h * .5;  % reduce step-size by half
    % find Phi & Ksi matrices for the new stepsize
    rank_A = rank(A);
    if rank_A == dim_ss,
        [Phi] = expm(A * h);
        [Ksi] = inv(A) * (Phi - eye(dim_ss)) * B;
    else,
        [Phi, Ksi] = c2d(A, B, h);
    end

end  % end of t-loop
% define the end-value of the state vector
xend = x_T(:, num_mode);
return

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Name: steady.m
% Function: steady-state finder
% Input: iteration --------- max # of iteration
% x_guess ----------- initial guess of steady-state
% dim_ss ----------- dimension of state space
% Ts ----------------- switching period
% u -------------- input vector
% num_mode ----------- # of topological mode
% err_level ----------- tolerance for Newton's iteration
%
% Output: ss ----------- steady-state value of state vector
% T_sss------------- switching instants in SS cycle
% x_T_ss ----------- state vector on T_sss
%
% Calling: large.m ------ large-simulation
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function  [x_ss, x_T_ss, T_ss] = steady(iteration, x_guess, ...
    dim_ss, Ts, u, num_mode, err_level);
% initialization
    k = 1;
    steady_state = 0;
    n_cycle = 1;
    % main loop of iteration
    while k < iteration+1 & steady_state == 0,
        % simulate one cycle
            [x_ss, T_ss, x_T_ss, Time, x] = ...  
            large(n_cycle, x_guess, dim_ss, Ts, u, num_mode );
        % check steady-state error with tolerance
            x_diff = x_ss - x_guess;

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error = x_diff' * x_diff
if error < err_level,  % steady-state reached
    steady_state = 1;
else  % steady-state not reached
    % find Jacobian matrix
    x_new = x_guess;
    for i = 1:dim_ss,
        x_new(i) = x_guess(i) * 1.05;
        [x_update, T, x_T, Time, x] = ... large(n_cycle, x_new, dim_ss, Ts, u, num_mode);
        Jacob(:, i) = (x_update - x_ss) / ...
                    (x_new(i) - x_guess(i));
    end
    Jacob = Jacob - eye(dim_ss);
    Jacob = inv(Jacob);
    % modify the guess of initial conditions
    x_guess = x_guess - Jacob * x_diff;
    k = k + 1;
end  % end of Jacobian
end  % end of iteration
return

% Name: small.m
% Function: small-signal modeling
% Input: dim_in ------ dimension of input vector
%        dim_out ------ dimension of output vector
%        Ts --------- switching period
%        u --------- input vector
%        x_ss ------ steady-state initial condition
%        T_ss ------ switching instants in one SS cycle
%        x_T ------- state vector on T_ss
%        num_mode ---- # of topological mode in one cycle
%        harm_tbl ---- harmonic table (see Chap.3 for details)
% Output:{A, B, C, D} - small-signal model matrices
%
% Calling: get_dim ------ processing harmonic table to get
%                dimension of the small-signal model
%                fourier ------ Fourier expansion to find SS values
%                of new state variables
%                ptxu ------- find partial derivatives of the switching
%                instants with respect to new state
%                variables and the input variables
%                topo ------- provide piecewise linear state matrices
%                for a specific mode

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% delta -------- find delta (see Chap. 2 Eq. (2.105))
% gama -------- find gama (see Chap. 2 Eq. (2.106))
% lambda ------ find lambda (see Chap. 2 Eq. (2.110))
% un_map --- define parameters of a model variable

function [A, B, C, D] = small(dim_in, dim_out, Ts, u, x_ss, ...
    x_T, T_ss, num_mode, harm_tbl)
% processing harmonic table to get dimension of the model
[Hindex, dim_X, dim_ss, dim_high] = get_dim(harm_tbl);
% Hindex: a mapping table for each model variable.

% from from 1 = sine
% which which -1 = cosine
% state harmonics 0 = dc
% variable
% Hindex = [ 1 1 1 1 1 1 2nd model variable
% 1 1 -1 2nd model variable
% 2 1 1 3rd model variable
% 2 1 -1 4th model variable
% 3 0 0 5th model variable

% Fourier expansion to find SS values of new state variables
[X] = fourier(x_ss, u, T_ss, x_T, Ts, num_mode, Hindex, dim_X, ...
    dim_ss, dim_high)
% find partial derivatives of the switching instants with respect
% to new state variables and the input variables
[ptpx, ptpu] = ptvx(x_ss, x_T, T_ss, dim_ss, dim_X, dim_in, ...
    dim_high, Ts, u, Hindex, num_mode); 

ws = 2 * pi / Ts;
N = num_mode;
% find small-signal model A matrix (see Chap. 2 Eq. (2.104))
for i = 1:dim_X,
    [i1, i2, i3] = un_map(i, Hindex);
    for j = 1:dim_X,
        [j1, j2, j3] = un_map(j, Hindex);
        if (j == i+3) & i3 == 1, % real part
            A(i, j) = i2 * ws;
        elseif (j == i+3) & i3 == -1, % image part
            A(i, j) = -i2 * ws;
        else % DC
            A(i, j) = 0;
        end
    end
end

for l = 1:N,
    [switching, num_mode, dim_out, harm_tbl, x0, U0, ... 
    Ts, A_pl, B_pl, C_pl, E_pl, Ab, Bb, Cb, Db] ... 
    = topo(l, x_ss, u, 0);
    delta = Delta(l, i1, A_pl, B_pl, x_T, u, Ts, num_mode);
    gama = Gama(l, i2, j2, i3, j3, T.ss, Ts, num_mode);
    if i3 == 1, % real

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\[ H = \cos(i2 \times ws \times T_{ss}(l)) \]

else if \( i3 = -1 \), % imag part
\[ H = -\sin(i2 \times ws \times T_{ss}(l)) \]

else if \( i3 = 0 \), % dc
\[ H = 1 \]
end
\[ A(i, j) = A(i, j) + \text{delta} \times H \times \text{ptpx}(i, j) + ... \]
\[ A_{pl}(i1, j1) \times gama; \]
end % end N loop
end % end of j loop
i
end % end of i loop
A
% find small-signal model B matrix (see Chap. 2 Eq. (2.107))
for \( i = 1: \text{dim}_X \),
\[ \text{[i1, i2, i3]} = \text{un_map}(i, \text{Hindex}); \]
for \( j = 1: \text{dim}_\text{in} \),
\[ B(i, j) = 0; \]
for \( l = 1:N \),
\[ \text{[switching, num_mode, dim_out, harmTbl, x0, U0, ...} \]
\[ \text{Ts, A_{pl}, B_{pl}, C_{pl}, E_{pl}, Ab, Bb, Cb, Db]} \ldots \]
\[ = \text{topo}(l, x_{ss}, u, 0); \]
\[ \text{delta} = \text{Delta}(l, i1, A_{pl}, B_{pl}, x_T, u, Ts, \text{num_mode}); \]
\[ gama = \text{Gama}(l, i2, 0, i3, 0, T_{ss}, Ts, \text{num_mode}); \]
if \( i3 = 1 \), % real
\[ H = \cos(i2 \times ws \times T_{ss}(l)); \]
else if \( i3 = -1 \), % imag part
\[ H = -\sin(i2 \times ws \times T_{ss}(l)); \]
else if \( i3 = 0 \), % dc
\[ H = 1; \]
end
\[ B(i, j) = B(i, j) + \text{delta} \times H \times \text{ptpu}(l, j) + ... \]
\[ B_{pl}(i1, j1) \times gama; \]
end
end
\[ B(i, \text{dim}_\text{in}+1) = i3 \times i2 \times X(i+i3); \]
end
\[ B \]
% find small-signal model C matrix (see Chap. 2 Eq. (2.108))
for \( i = 1: \text{dim}_\text{out} \),
for \( j = 1: \text{dim}_X \),
\[ \text{[j1, j2, j3]} = \text{un_map}(j, \text{Hindex}); \]
\[ C(i, j) = 0; \]
for \( l = 1:N \),
\[ \text{[switching, num_mode, dim_out, harmTbl, x0, U0, ...} \]
\[ \text{Ts, A_{pl}, B_{pl}, C_{pl}, E_{pl}, Ab, Bb, Cb, Db]} \ldots \]
\[ = \text{topo}(l, x_{ss}, u, 0); \]
\[ \text{lambda} = \text{Lambda}(l, i, C_{pl}, E_{pl}, x_T, u, Ts, \text{num_mode}); \]
\[ gama = \text{Gama}(l, 0, j2, 0, j3, T_{ss}, Ts, \text{num_mode}); \]
C(i,j) = C(i,j) + lambda * ptxp(I, j) + ...
    C_pl(i, j) * gama;
end
end
end

C
% find small-signal model D matrix (see Chap.2 Eq.(2.109))
for i = 1:dim_out,
    for j = 1:dim_in,
        D(i, j) = 0;
        for l = 1 : N,
            [switching, num_mode, dim_out, harm_tbl, x0, U0,...
                Ts, A_pl, B_pl, C_pl, E_pl, Ab, Bb, Cb, Db] ...
                = topo(I, x_ss, u, 0);
            lambda = Lambda(I, l, c_pl, E_pl, x_T, u, Ts, num_mode);
            gama = Gama(I, 0, 0, 0, 0, T_ss, Ts, num_mode);
            D(i, j) = D(i, j) + lambda * ptpu(l, j) + ...
                E_pl(i, j) * gama;
        end
    end
D(i, dim_in+1) = 0;
end
D
return

******************************************************************************
% Name: topo.m
% Function:define converter circuit, operating condition,
% and switching boundary condition
% Input: cur_mode ------- current topological mode
% x --------------- current state vector
% u --------------- current input vector
% t --------------- current time
% Output: switching ----- 1 = not cross switching boundary
% -1 = cross switching boundary
% num_mode ------- # of modes in one cycle
% dim_out ------- dim. of output vector
% harm_tbl ------ harmonic table (see Chap.3)
% x0 -------------- initial condition
% U0 -------------- given input vector
% Ts -------------- switching period
% A, B, C, D ---- state matrices of current mode
% Ab, Bb, Cb, Db---- boundary matrices of current mode
% Calling: ncne
%
******************************************************************************

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function [switching, num_mode, dim_out, harm_tbl, x0, U0, Ts, ... 
A, B, C, D, Ab, Bb, Cb, Db] = topo(cur_mode, x, u, t)

% define # of mode and dimension of output
num_mode = 4;
dim_out = 1;
% define harmonic table
%     dc  fndmntl  2nd    3rd
harm_tbl = [0 1 0 1;  % i -- 1st state
            0 1 0 0;  % v -- 2nd state
            1 0 0 0];  % vo - 3rd state
% define initial condition
x0 = [0.3292;
      -6.3223;
      25.2262];
% define operating condition
Zo = 58.6;
P0 = 414.3e3;
Vg = 40.9;
V0 = [Vg];
Qs = 1.49;
R = Zo / Qs;
Fsn = 0.678;
Fs = Fsn * P0;
Ts = 1/Fs;
% define circuit parameters
Lr = 22.e-6;
Cr = 6.56e-9;
Cf = 16.75e-6;
rc = 0.212;
rs = 5;
k = R / (R+rc);
r = k * rc;
% generate ramp signal
t_ramp = t;
    while t_ramp > Ts,
        t_ramp = t_ramp - Ts;
    end
if u == 0
u = U0;
x = x;
end
% define switching boundary conditions
if cur_mode == 1,
    Si = 1;  % rectifier polarity
    Sg = 1;  % active bridge polarity
    % set crossing boundary flag
    if x(1) < 0.0,
        switching = -1;
    end

else
    switching = 1;
end

% switching boundary condition:
%    Ab * x + Bb * u + Cb * t + Db < 0
Ab = [1 0 0];
Bb = [0];
Cb = 0;
Db = 0;
elseif cur_mode == 2,
    Si=-1;
    Sg=1;
    if t_ramp > 0.5 * Ts;
        switching = -1;
    else
        switching = 1;
    end
Ab = [0 0 0];
Bb = [0];
Cb = 0;
Db = 0;
elseif cur_mode == 3,
    Si=-1;
    Sg=-1;
    if x(1) > 0,
        switching = -1;
    else
        switching = 1;
    end
Ab = [-1 0 0];
Bb = [0];
Cb = 0;
Db = 0;
elseif cur_mode == 4,
    Si=1;
    Sg=-1;
    if t_ramp < 0.5 * Ts,
        switching = -1;
    else
        switching = 1;
    end
Ab = [0 0 0];
Bb = [0];
Cb = 0;
Db = 0;
end

Appendix A.
% define piecewise linear state equations
A = [(-rs-r)/Lr  -1/Lr  -k*Si/Lr;
    1/Cr  0  0;
    k*Si/Cf  0  -k/R/Cf];
B = [Sg/Lr;
    0;
    0];
C = [Si*r  0  k];
D = [0];
return

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Name: get_dim.m
% Function: processing harmonic table to get dimensions of the
% small-signal model
% Input: harm_tbl ---- harmonic table (see Chpa.3 for details)
% Output: Hindex ------ harmonic index. A table defines all
% of the model variable (see comments
% of small.m)
% dim_X ------- dimension of the small-signal model
% dim_ss ------ dimension of state space
% dim_high ---- the highest harmonics being taken
% Calling: none
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [Hindex, dim_X, dim_ss, dim_high] = get_dim(harm_tbl);
    [dim_ss, dim_high] = size(harm_tbl);
    k = 1;
    for i = 1 : dim_ss,
        for j = 1: dim_high,
            if harm_tbl(i, j) ~= 0
                if j == 1
                    Hindex(k, 1) = i;
                    Hindex(k, 2) = 0;
                    Hindex(k, 3) = 0;
                    k = k + 1;
                else
                    Hindex(k, 1) = i;
                    Hindex(k, 2) = j-1;
                    Hindex(k, 3) = 1;
                end
            % 1 for cos -1 for sin
            k = k + 1;
            Hindex(k, 1) = i;
            Hindex(k, 2) = j-1;
            Hindex(k, 3) = -1;
        end
    end
end

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\[
\begin{align*}
\text{k} &= \text{k} + 1; \\
\text{end} \\
\text{end} \\
\text{end} \\
\text{dim}_X &= \text{k} - 1; \\
\text{return}
\end{align*}
\]

% Name: fourier.m
% Function: find steady-state values of the model variables
% through Fourier expansion
% Input:
x0 --------- steady-state initial condition
% u ------------ given input vector
% T ----------- switching instants in one cycle
% x_T --------- state vectors on T
% Ts ----------- switching period
% dim_in ------ dimension of input vector
% dim_out ------ dimension of output vector
% Ts ----------- switching period
% num_mode ------ # of topological mode in one cycle
% Index ------- harmonic index (see comments of small.m)
% dim_X ------- dimension of the small-signal model
% dim_ss ------- dimension of the state-space
% dim_high ------ the highest harmonics being taken
%
% Output:
X ------- steady-state values of the model variables
%
% Calling:
topo ------ provide piecewise linear state matrices
% un_map --- define parameters of a model variable
% %
% function [X] = Fourier(x0, u, T, x_T, Ts, num_mode, ...)
% \hspace{1cm} \text{Hindex}, \text{dim}_X, \text{dim}_ss, \text{dim}_high)
\]

\[
\begin{align*}
\text{ws} &= \text{sqrt}(-1) \times 2 \times \pi / \text{Ts}; \\
\text{sum} &= \text{zeros}(...); \\
\text{for } \text{l} &= 1: \text{num}_\text{mode} \\
\text{if } \text{l} &= 1, \\
\text{T1} &= 0; \\
\text{T2} &= \text{T}(1); \\
\text{cur}_\text{mode} &= 1; \\
\text{x}_\text{old} &= \text{x0}; \\
\text{else} \\
\text{T1} &= \text{T}(\text{l}-1); \\
\text{T2} &= \text{T}(1); \\
\text{cur}_\text{mode} &= 1; \\
\text{x}_\text{old} &= \text{x}_\text{T}(::, \text{l}-1); \\
\text{end}
\end{align*}
\]
M = round(100 * (T2 - T1) / Ts); % number of steps
h = (T2 - T1) / M;
[switching, num_mode, dim_out, harm_tbl, x0, U0, ... 
Ts, Ai, Bi, Ci, Di, Ab, Bb,Cb,Db] ...
= topo(cur_mode, x_T(1), u, T(1));
rank_Ai = rank(Ai);
if rank_Ai == dim_ss,
[Phi] = expm(Ai * h);
[Ksi] = inv(Ai) * (Phi - eye(dim_ss)) * Bi;
else,
[Phi, Ksi] = c2d(Ai, Bi, h);
end
for m = 0:M,
t = T1 + m * h;
for j = 1:dim_high,
exp_old = exp(-(j-1) * ws * t);
if m == 0 | m == M,
sum(:, j) = sum(:, j) + h * (x_old * exp_old)/2;
else
sum(:, j) = sum(:, j) + h * x_old * exp_old;
end
end
x_old = Phi * x_old + Ksi * u;
end % end of for loop
end % end of for loop
for k = 1:dim_X,
[k1, k2, k3] = un_map(k, Hindex); 
if k3 == 0,
X(k) = sum(k1, i);
elseif k3 == 1, % cos terms
X(k) = real(sum(k1, k2+1));
elseif k3 == -1, % sin terms
X(k) = imag(sum(k1, k2+1));
end
end % end of for loop
X = X / Ts;
return

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Name: ptxu.m
% Function: find partial derivatives of the switching instants
% with respect to model variables and input variables
% (see Chap.2 Eq.(2.113-114))
% Input x --------- steady-state initial condition
% T --------- switching instants in one SS cycle
% x_T --------- state vector or T_ss
% dim_ss ---- dimension of the state-space
% dim_X ------ dimension of the small-signal model

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%  dim_in  ----  dimension of input vector
%  dim_high  ----  the highest harmonics being taken
%  Ts  ----  switching period
%  u  ----  input vector
%  Hindex  ----  harmonic index (see comments of small.m)
%  num_mode  ----  # of topological mode in one cycle
%
%  Output: partial derivatives
%
%  Calling: topo  ----  provide piecewise linear state matrices
%  for a specific mode
%  un_map  ----  define parameters of a model variable
%
function [ptpx, ptpu] = Ptxu(x, x_T, T, dim_ss, dim_X, dim_in, ...
  dim_high, Ts, u, Hindex, num_mode)

  t = 0;
  for i = 1:num_mode,
    cur_mode = i;
    [switching, num_mode, dim_out, harm_tbl, x0, U0, Ts, ...]
    A, B, C, D, Ab, Bb, Cb, Db = topo(cur_mode, x, u, t);
    phpt = Ab * (A * x_T(:, i) + B * u) + Cb;
    for j = 1:dim_X,
      if phpt == 0,
        ptpx(i, j) = 0;
      else
        [j1, j2, j3] = un_map(j, Hindex);
        if j3 == 0,
          pxpX = 1;
        elseif j3 == 1,
          pxpX = 2 * cos(t * j2 * 2 * pi / Ts);
        elseif j3 == -1,
          pxpX = -2 * sin(t * j2 * 2 * pi / Tsi);
        end
        ptpx(i, j) = -Ab(j1) * pxpX / phpt;
      end
    end
  end
  for j = 1:dim_in,
    if phpt == 0,
      ptpu(i, j) = 0;
    else
      ptpu(i, j) = -Bb(j) / phpt;
    end
  end
end
return

Appendix A.
function [delta] = Delta(l, ix, A_pl, B_pl, x_T, u, Ts, num_mode);
    if l < num_mode,
        [switching, num_mode, dim_out, harm_tbl, x0, U0, Ts, ...]
        A, B, C, E, Ab, Bb, Cb, Db] = topo(l+1, x_T(1), u, 0);
        delta = ( (A_pl(ix, :) - A(ix, :)) * x_T(:, l) + ...)
                 (B_pl(ix, :) - B(ix, :)) * u ) / Ts;
    else
        delta = ( A_pl(ix, :) * x_T(:, l) + ...)
                 B_pl(ix, :) * u ) / Ts;
    end
    return

function [gama] = Gama(l, i2, j2, i3, j3, T, Ts, num_mode)
if l == 1
  if T(num_mode) < Ts,
    T_zml = T(num_mode) - Ts;
  else
    T_zml = 0;
  end
else
  T_zml = T(1-l);
end
alpha = i2 * 2 * pi / Ts;
beta = j2 * 2 * pi / Ts;
apb = alpha + beta;
amb = alpha - beta;
if i3 == 1 & j3 == 1,
  if i2 == j2,
    gama = ( (T(1) - T_zml) + ...
             (sin(apb*T(1)) - sin(apb*T_zml)) / apb ) / Ts;
  else
    gama = ( ( sin(apb * T(1)) - sin(apb * T_zml) ) / apb ...
            + ( sin(amb * T(1)) - sin(amb * T_zml) ) / amb ) / Ts;
  end
elseif i3 == 1 & j3 == -1,
  if i2 == j2,
    gama = ( cos(apb * T(1)) - cos(apb * T_zml) ) / apb / Ts;
  else
    gama = ( (cos(apb * T(1)) - cos(apb * T_zml) ) / apb ...
            + (-cos(amb * T(1)) + cos(amb * T_zml) i) / amb ) / Ts;
  end
elseif i3 == 1 & j3 == 0,
  gama = ( sin(alpha * T(1)) - sin(alpha * T_zml) ) * 2 ...
        / Ts / alpha;
elseif i3 == -1 & j3 == 1,
  if i2 == j2,
    gama = ( cos(apb * T(1)) - cos(apb * T_zml) ) / apb / Ts;
  else
    gama = ( (cos(apb * T(1)) - cos(apb * T_zml) ) / apb ...
            + (cos(amb * T(1)) - cos(amb * T_zml) ) / amb ) / Ts;
  end
elseif i3 == -1 & j3 == -1,
  if i2 == j2,
    gama = (T(1) - T_zml - (sin(apb*T(1)) - sin(apb*T_zml))...
             / apb) / Ts;
  else
    gama = - ( ( sin(apb * T(1)) - sin(apb * T_zml) ) / apb ...
               - ( sin(amb * T(1)) - sin(amb * T_zml) ) / amb ) / Ts;
  end
elseif i3 == -1 & j3 == 0,
  gama = ( cos(alpha * T(1)) - cos(alpha * T_zml) ) * 2 ...
         / Ts / alpha;

Appendix A.
else if i3 == 0 & j3 == 1,
gama = ( \sin(b \cdot T(l)) - \sin(b \cdot T_{lml}) ) ...
    / T_s / b;
else if i3 == 0 & j3 == -1,
gama = ( \cos(b \cdot T(l)) - \cos(b \cdot T_{lml}) ) ...
    / T_s / b;
else if i3 == 0 & j3 == 0,
gama = (T(l) - T_{lml}) / T_s;
end
return

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Name: Lambda.m
% Function: find lambda (see Chap.2 Eq.(2.110))
% Input1 -------- lth mode
% i- ------- ith input variable
% C_pl, E_pl- piecewise linear state matrices of lth
% mode
% x_T ------- state vector on T_ss
% u ------- input vector
% Ts -------- switching period
% num_mode -- # of topological mode in one cycle
%
% Output: Lambda
%
% Calling: topo ------- provide piecewise linear state matrices
% for a specific mode
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
function [lambda] = Lambda(l, i, C_pl, E_pl, x_T, u, Ts, num_mode);
if l < num_mode
    [switching, num_mode, dim_out, harm_tbl, x0, U0, Ts, ...
    A, B, C, E, Ab, Bb, Cb, Db] = topo(l+1, x_T(l), u, 0);
    lambda = ( (C_pl(i, :) - C(, :) ) * x_T(:, l) + ...
                (E_pl(i, :) - E(, :) ) * u ) / T_s;
else
    lambda = ( C_pl(i, :) * x_T(:, l) + E_pl(i, :) * u ) / T_s;
end
return

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Name: un_map.m
% Function: define parameters of a model variable
% Inputm -------- mth model variable
% Hindex ------ Harmonic index (see comments of small.m)
%
% Output: i -- the model variable comes from ith state
function [i, j, k] = un_map(m, Hindex);
i = Hindex(m, 1);
j = Hindex(m, 2);
k = Hindex(m, 3);
return
Appendix B. SPICE Node Lists of the Small-Signal Equivalent Circuit Models of Resonant Converters

Appendix B provides SPICE node lists of the small-signal equivalent circuit models of the LLC-SRC and the LCC-PRC. These circuit models are also suitable for the SRC and the PRC, because the SRC and the PRC can be treated as special cases of the LLC-SRC (when \( L_p \gg L_s \)) and the LCC-PRC (when \( C_s \gg C_p \)), respectively.

**B.1. SPICE Codes for LLC-SRC**

Based on the complete small-signal circuit model derived in Chapter 3 (Fig. 3.33), the SPICE node list can be written. The circuit parameters and the operating conditions are defined in the node list.
*Small-Signal Circuit Model of LLC-SRC

*Circuit parameters:
* Ls=22.5uH, Lp=62.8uH, C=6.56nF,
* Cf=16.8uF, rs=0.5 Ohm, rc=0.112 Ohm
* Operating point: Vg=40V, Qs=2.13, Fs/Fo=0.747, D=1.0

* Four Input Signals : Vg, d, fsn, Io
Vg  40 0  ac  0
Rx1 40 0  1k
Vd  50 0  ac  0
Rx0 50 0  1K
Vfsn 60 0  ac  1
Rx2 60 0  1k
Io  0 30  ac  0

* Output #1: Output Voltage -------- Vo = V(30)
* Output #2: Averaged Input Current --- Ig = v(70)
Fig 0  70  Vpiss 0.637
Rig 70 0  1

* The Upper Part of Resonant Tank (SINE)
Es  1 0  60 0  426.1
EKv 2 1  40 0  1.273
Ed  3 2  50 0  0.0
Vpiss 3 4  dc  0
Hzs 5 4  Vpisc 55.6
E2x 5 6  31 0  0.316
Cs  6 7  6.56nF
Vpips 6 9  DC  0
Lps 9 10  62.8uH
Hzp 11 10  VPips 142.3

* The Lower Part of Resonant Tank (COSINE)
Ec  0 13  60 0  10.9
VPisc 13 14  dc  0
Hzc 15 14  Vpiss -46.4
E2kc 15 16  31 0  1.234
Cc  16 17  6.56nF
VPipc 16 19  DC  0
Lpc 19 20  62.8uH
Hzp2 20 21  VPips 142.3
Epc 21 17  60 0  -27.3
Gc2 16 17  6 7  0.0149
Gjc 16 17  60 0  1.58
Rc  17 18  1.68
Lsc 18 0  22.5uH
B.2. SPICE Codes for LCC-PRC

Based on the complete small-signal circuit model derived in Chapter 4 (Fig. 4.14), the SPICE node list can be written. The circuit parameters and the operating conditions are defined in the node list.

```
* Small-Signal Circuit Model of LCC
* The CFT parameters:
  * L=36.3uH, Cs=1.23nF, Cp=0.93nF,
  * Lf=37.1uH, Cf=1.19uF, rc=0.973 Ohm
* Operating point:
  * D=1.0, Fs/Fo=0.97, Qp=0.33, Vg=81.7V
* The Injected Signals
Vg  20  0   ac   0
Rx1 20  0   1k
Vfsm 30  0   ac   1
Rx2 30  0   1k
Is  0  17   ac   0
* Sample Input Current --- v(40)
FIn  0  40   Vpis  0.637
RIn  40  0   1
```
* The Upper Part of Resonant Tank (SINE)
E_s  1  0  30  0  -78.9
E_{Kv} 2  1  20  0  1.27
V_{Pis} 2  3  dc  0
L_{s}  3  4  36.3uH
r_{ss}  4  5  78.5
H_{ZLs} 6  5  V_{Pic}  254
C_{ss}  6  7  1.23nF
G_{s1}  7  6  13  12  0.0086
G_{Jss} 7  6  30  0  -0.531
C_{ps}  7  0  0.93nF
R_{gps} 7  0  130
G_{sc}  0  7  0  13  0.0029
F_{2ks} 7  0  V_{Pio}  0.538
G_{Jps} 0  7  30  0  -0.317

* The Lower Part of Resonant Tank (COSINE)
E_{c} 0  8  30  0  -135
V_{Pic} 9  8  dc  0
L_{c} 10  9  36.3uH
r_{sc} 11  10  78.5
H_{ZLc} 12  11  V_{Pic}  254
C_{sc} 13  12  1.23nF
G_{s2} 13  12  6  7  0.0086
G_{Jsc} 13  12  30  0  -0.311
C_{pc} 0  13  0.93nF
R_{gpc} 0  13  596
G_{gca} 13  0  7  0  -0.0101
F_{2kc} 0  13  V_{Pio} -1.15
G_{Jpc} 13  0  30  0  -0.148

* The Output Low Pass Filter
E_{kc} 14  0  0  13  -0.577
E_{ks} 15  14  7  0  0.269
V_{pio} 15  16  DC  0
L_{f} 16  17  37.1uH
r_{c} 17  18  0.973
C_{f} 18  0  1.19uF
R 17  0  87.4
.ac dec 20 1KHz 1.0e6Hz
.print ac vdb(17) vp(17)
.probe
.END
References


References


Vita

Eric Xian-Qing Yang was born in Beijing, China. He received the B.S. and M.S. degrees from Tsinghua University, Beijing, China, in 1982 and 1984, respectively. From 1984 to 1988, he was employed as a lecturer in the department of Electrical Engineering at same university, where he taught undergraduate courses including circuit theory, electronics, and signal processing. He also engaged in the research of developing medical ultrasonic instruments based on micro-processors.

In 1989, he joined the Virginia Power Electronics Center at Virginia Polytechnic Institute and State University. As a graduate research assistant, he is actively involved in power electronics research in the areas of modeling, control, dc-dc and ac-dc converter design, high-frequency converters, and computer-aided design for switching power circuits.

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