Optimal Feedback Control for Nonlinear Discrete Systems and Applications to Optimal Control of Nonlinear Periodic Ordinary Differential Systems

by

Xiaohong Zhang

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APPROVED:

David L. Russell, Committee Chairman
Mathematics Department

John A. Burns
Mathematics Department

Terry L. Herdman
Mathematics Department

Kenneth B. Hannsgen
Mathematics Department

Eugene M. Cliff
Aerospace Engineering Department

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Abstract

This dissertation presents a discussion of the optimal feedback control for nonliner systems (both discrete and ODE) and nonquadratic cost functions in order to achieve improved performance and larger regions of asymptotic stability in the nonlinear system context.

The main work of this thesis is carried out in two parts; the first involves development of nonlinear, nonquadratic theory for nonlinear recursion equations and formulation, proof and application of the stable manifold theorem as it is required in this context in order to obtain the form of the optimal control law.

The second principal part of the dissertation is the development of nonlinear, nonquadratic theory as it relates to nonautonomous systems of a particular type; specifically periodic time varying systems with a fixed, time invariant critical point.

With All My Love

This Dissertation is Dedicated to the Memory of

MY MOTHER

ZHAORONG ZHANG

Who Has Been and Always Will Be

the Greatest Source of Inspiration in My Life

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To all the wonderful people of my life's journey, be merry! Be happy!

Contents

Nomenclature and Symbols

Symbol	Definition
$\mathbf{R}$	space of real numbers
$\mathbf{R}^n$	real n-dimensional space
$\mathbf{H}$	Hilbert space
$\mathbf{L}_n^2$	Lebesgue n-dimensional space
l^2	square summable sequence space
Z_α	a Banach space of sequences
C	space of continuous functions
C_m	space of continuously m-dimensional vector functions
N_u	a neighborhood of the origin in $\mathbf{R}^n$
$[0,T]$	closed interval
$(0,T)$	open interval
$x \in X$	x is an element of X
$\{x \in X \mid P(x)\}$	set of x in X with $P(x)$
$\dim x$	dimension of x
$\{x_k\}$	infinite sequence
$\\|x\\|$	norm of x on $\mathbf{R}$ if $x \in \mathbf{R}$ or on $\mathbf{R}^n$ if $x \in \mathbf{R}^n$
$\\|x\\|_X$	norm of x on X
$(x,y)_X$	inner product on X
Class C^n	n times continuously differentiable
lim	limit
sup	supremum
$\sum$	sum of
$\equiv$	equivalent to
$\rightarrow$	tends to
$a = o(b)$, as $c \rightarrow d$	$\lim_{c\rightarrow d} \frac{a}{b} = 0$
$a = O(b)$, as $c \rightarrow d$	$\lim_{c\rightarrow d} \frac{a}{b} =$ constant
I_n	n-dimension real identity matrix
matrix $A > 0$	real symmetric positive definite
A^*	transpose of A
A^{-1}	inverse of A

Chapter 1

Introduction and Literature Review

The linear - quadratic optimal control theory, introduced in 1960 by R. E. Kalman and R. S. Bucy [20], [23], has since that time enjoyed wide popularity as a mathematical framework in terms of which a wide variety of design objectives can be expressed and corresponding design techniques developed. This theory, further developed in an enormous number of journal articles and various books (see, e.g., [24], [45], [2]).

We will do a review here of this certain approach to the optimal control of finite dimensional autonomous systems of nonlinear differential equations near an equilibrium point. In particular, we are primarily concerned with the problem of closed loop control on the time interval $0 \leq t < \infty$. The reader may wish to consult [29], [2] and [12] for fundamentals of the theory of ordinary differential equations and control theory.

The problem is formulated in terms of a control system equation in $\mathbf{R}^n$, with ·

denoting $\frac{d}{dt}$,

$$\dot{x} = F(x, u),$$

and a performance integral,

$$J = \int_0^\infty G(x, u)\, dt,$$

which, for a given initial state $x(0) = x_0$, is to be minimized by the action of the control u. In fact, we ultimately seek an m-dimensional vector feedback control function of the state x, $u = u(x)$, which makes the integral as small as possible for all initial states x_0 near an equilibrium point in $\mathbf{R}^n$, which we may, without loss of generality, take to be $x = 0$. It is clear that we may also assume the equilibrium control to be $u = 0$ in $\mathbf{R}^m$.

Hence, for each feedback control function $u = u(x)$ we consider the autonomous differential equation

$$\dot{x} = F(x, u(x)) \tag{1.0.1}$$

with the corresponding solution $x = x(t, x_0)$, where $x(0, x_0) = x_0$ for an initial state x_0 near the equilibrium point in $\mathbf{R}^n$. Since we are interested in the dependence of the integral upon the initial state of (1.0.1) as well as upon the control function $u(x)$, we use the notation

$$J(x_0, u) = \int_0^\infty G(x(t, x_0), u(x(t, x_0)))\, dt. \tag{1.0.2}$$

As noted above, we assume that the origin is an equilibrium point of $F(x, u)$, which means

$$F(0,0) = 0,$$

and $F(x,u)$ and $G(x,u)$ are defined on some neighborhood of the origin in $\mathbf{R}^{n+m}$, both at least of class C^2 there, and can be represented in the form, wherein x^*, u^* denote the (row vector) transposes of the column vectors x, u, respectively,

$$F(x,u) = Ax + Bu + f(x,u), \tag{1.0.3}$$

$$G(x,u) = (x^*, u^*) \begin{pmatrix} W & R \\ R^* & U \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + g(x,u). \tag{1.0.4}$$

Here

$$A = \frac{\partial F}{\partial x}(0,0), \quad B = \frac{\partial F}{\partial u}(0,0),$$

A, B, W, U and R are real matrices and

$$G(0,0) \equiv 0, \quad \frac{\partial G}{\partial x}(0,0) = 0, \quad \frac{\partial G}{\partial u}(0,0) = 0,$$

and $f(x,u)$ and $g(x,u)$ are higher order remainder terms with

$$f(0,0) = 0, \quad \frac{\partial f}{\partial x}(0,0) = 0, \quad \frac{\partial f}{\partial u}(0,0) = 0,$$
$$g(0,0) = 0, \quad \frac{\partial g}{\partial x}(0,0) = 0, \quad \frac{\partial g}{\partial u}(0,0) = 0, \quad \frac{\partial^2 g}{\partial x^2}(0,0) = 0, \quad \frac{\partial^2 g}{\partial u^2}(0,0) = 0.$$

We assume that $\begin{pmatrix} W & R \\ R^* & U \end{pmatrix}$ is a real symmetric positive definite matrix. Also we assume that $F(x,u)$ is *linearly stabilizable*, that is, there exists a real matrix K for which $A + BK$ is a stability matrix [1].

[1] A real matrix is called a *stability matrix* if its eigenvalues all have negative real parts.

We consider the class of feedback controls which are of the form

$$u = u(x) = Kx + h(x), \tag{1.0.5}$$

where again $h(x)$ denotes the higher order remainder terms. The real matrices K are selected so that $u(x)$ *stabilizes* (1.0.1); that is, in

$$F(x, u(x)) = (A + BK)x + Bh(x) + f(x, u(x)),$$

$A + BK$ should be a stability matrix.

In 1969 D. L. Lukes [31], following an approach first developed by Kalman and Bucy [20], [23] for the linear, quadratic case, developed a quite complete theory for the case when $F(x,u)$ and $G(x,u)$ were at least of class C^2 near the origin in $\mathbf{R}^{n+m}$. The theory is even more complete when $F(x,u)$ and $G(x,u)$ are real analytic functions near the origin in $\mathbf{R}^{n+m}$; i.e., $f(x,u)$ and $g(x,u)$ in (1.0.3) - (1.0.4) are real convergent power series about the origin beginning with second and third order terms in (x,u), respectively, and every $h(x)$ in (1.0.5) is given by real power series converging about the origin and beginning with second order terms.

Note that one of our basic assumptions is that the Hessian matrix of $G(x,u)$ at the origin, $\begin{pmatrix} W & R \\ R^* & U \end{pmatrix}$, is positive definite. Hence $J(x_0, u) > 0$ near the origin and the integral is a composite indicator of the rate at which the feedback controls return the disturbed process to its equilibrium state and the control energy expended during the operation. This is the motivation for making the following technical definition.

Definition 1.1 *A C^1 stabilizing feedback control $\hat{u}(x) = \hat{K}x + \hat{h}(x)$ is called* optimal *for the process (1.0.1) with respect to the performance integral (1.0.2) if for every C^1 stabilizing feedback control $u(x) = Kx + h(x)$ there exists a neighborhood N_u of the origin in $\mathbf{R}^n$ in which*

$$J(x_0, \hat{u}) \leq J(x_0, u).$$

In [31] Lukes proves

Theorem 1.2 *For the C^2 stabilizable control process in $\mathbf{R}^n$*

$$\dot{x} = F(x, u) = Ax + Bu + f(x, u)$$

with performance integral

$$J(x_0, u) = \int_0^\infty G(x, u)\, dt = \int_0^\infty \left\{ (x^*, u^*) \begin{pmatrix} W & R \\ R^* & U \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + g(x, u) \right\} dt,$$

there exists an optimal C^1 stabilizing feedback control $\hat{u}$. The optimal control function $\hat{u}$ has the form (1.0.5), i.e.,

$$\hat{u}(x) = \hat{K}x + \hat{h}(x),$$

where $\hat{K}$ depends only on matrices A, B, W, R and U, solves the functional equation

$$\frac{\partial J}{\partial x}(x_0, \hat{u}) \frac{\partial F}{\partial u}(x_0, \hat{u}(x_0)) + \frac{\partial G}{\partial u}(x_0, \hat{u}(x_0)) = 0 \tag{1.0.6}$$

for all x_0 near the origin and is unique in that:

1. *$\hat{u}$ is the unique C^1 solution to (1.0.6);*

2. *$\hat{u}$ is the unique C^1 optimal stabilizing feedback control;*

3. *$\hat{u}$ synthesizes [2] the unique optimal open - loop control.*

Further, the minimal cost has the form

$$J(x_0, \hat{u}) = x_0{}^* P x_0 + \hat{\jmath}(x_0),$$

where in the lowest order term $P > 0$ depends only upon matrices A, B, W, U and R.

Before we proceed to the linear case, which Lukes also treated in [31], we require the following lemma.

Lemma 1.3 *In the collection of all positive definite real symmetric $n \times n$ matrices there exists a unique solution $P > 0$ to the quadratic matrix equation*

$$A^* P + PA + W - (PB + R)U^{-1}(B^* P + R^*) = 0. \qquad (1.0.7)$$

[2] That is, there exists an $\epsilon > 0$ and a neighborhood of the origin $\hat{N}$ such that for each $x_0 \in \hat{N}$, the response $\hat{x}(t)$ satisfies

$$\dot{\hat{x}} = F(\hat{x}, \hat{u}(\hat{x})), \;\; \hat{x}(0) = x_0, \;\; \hat{x}(t) \subseteq \hat{N} \;\; \text{for all} \;\; 0 \leq t < \infty,$$

and the corresponding control $\hat{u}^{\mathrm{op.}} = \hat{u}(\hat{x}(t))$ is the unique open-loop control achieving the minimum of $J(u) = \int_0^\infty G(x(t), u(t))\, dt$ among all measurable controls $u(t)$ on $0 \leq t < \infty$ with $\|u(t)\| \leq \epsilon$ and generating trajectories $x(t)$ satisfying $\dot{x} = F(x, u(t))$, $x(0) = x_0$, $x(t) \subseteq \hat{N}$ for all $0 \leq t < \infty$.

For the proof see [30], [42]. There the converse is true. That is, if equation (1.0.7) has a solution $P > 0$ for $\begin{pmatrix} W & R \\ R^* & U \end{pmatrix} > 0$, then that solution is the unique symmetric positive definite solution and the matrices A, B are a stabilizable pair.

Theorem 1.4 (Linear System) *For the special case of theorem (1.2) in which*

$$\dot{x} = Ax + Bu, \tag{1.0.8}$$

and

$$J(x_0, u) = \int_0^\infty (x^*, u^*) \begin{pmatrix} W & R \\ R^* & U \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} dt, \tag{1.0.9}$$

the optimal control is

$$\hat{u}(x) = \hat{K}x,$$

where

$$\hat{K} = -U^{-1}(R^* + B^*P).$$

Here $P > 0$ is the unique positive definite symmetric solution of the matrix equation (1.0.7). Further, $\hat{K}x$ is a global optimal control in the sense that in the definition of optimal feedback control we can take N_u to be all of $\mathbf{R}^{n+m}$ and in footnote 2 we may take $\epsilon = \infty$ and $\hat{N} = \mathbf{R}^n$. Finally,

$$J(x_0, \hat{u}) = {x_0}^* P x_0.$$

Hence in order to find the optimal control $\hat{u}(t)$ for linear system on $0 \leq t < \infty$ minimizing (1.0.9) it is sufficient to find the above described solution P of the matrix quadratic equation (1.0.7) and then solve

$$\dot{\hat{x}} = (A - BU^{-1}(R^* + B^*P))\hat{x},$$
$$\hat{x}(0) = x_0,$$

the closed loop system arising from use of

$$\hat{u}(t) = -U^{-1}(R^* + B^*P)\hat{x}(t), \tag{1.0.10}$$

in the original system (1.0.8) [45]. Finally, $\hat{u}(t)$ is determined from $\hat{x}(t)$ by (1.0.10).

Next we will set up the nonlinear problem for a perturbation analysis. In order to do this we will reformulate the solution to the linear system in term of a Hamiltonian system. First of all we need the following lemma which Lukes proved in [31].

Lemma 1.5 *Let $F(x,u)$ and $G(x,u)$ be as described above, then there exists a unique continuously differentiable solution $\hat{u}(x,p)$ to the equation*

$$p^*\frac{\partial F}{\partial u}(x,u) + \frac{\partial G}{\partial u}(x,u) = 0 \tag{1.0.11}$$

for (x,p) near the origin in $\mathbf{R}^{2n}$ such that $\hat{u}(0,0) = 0$. Furthermore,

$$\hat{u}(x,p) = -\frac{1}{2}U^{-1}\left(2R^*x + B^*p\right) + \hat{h}(x,p),$$

where $\hat{h}(x,p) = o(\|x\| + \|p\|)$ as $\|x\|, \|p\| \to 0$ in $\mathbf{R}^{2n}$. In the analytic case $\hat{h}(x,p)$

is a convergent power series about $(0,0)$ *beginning with terms of second degree in* (x,p).

Following Lukes [31], let us consider the quadratic form

$$2H(x,p) = x^*\hat{W}x + 2x^*\hat{R}p - p^*\hat{U}p$$

in the two real n-vectors x and p, where we define

$$\begin{aligned}
\hat{W} &= 2\left(W - RU^{-1}R^*\right), \\
\hat{U} &= \frac{1}{2}BU^{-1}B^*, \\
\hat{R} &= A^* - RU^{-1}B^*,
\end{aligned}$$

in terms of the data A, B, W, U and R from the linear problem. Then we may define a Hamiltonian system in $\mathbf{R}^{2n}$,

$$\begin{aligned}
\dot{x} &= \left(\frac{\partial H}{\partial p}(x,p)\right)^*, \\
\dot{p} &= -\left(\frac{\partial H}{\partial x}(x,p)\right)^*,
\end{aligned}$$

which can be written in the vector / matrix form

$$\dot{\begin{pmatrix} x \\ p \end{pmatrix}} = \begin{pmatrix} \hat{R}^* & -\hat{U} \\ -\hat{W} & -\hat{R} \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}. \tag{1.0.12}$$

If we define

$$M = \begin{pmatrix} I_n - 2QP & Q \\ 2P & -I_n \end{pmatrix}, \tag{1.0.13}$$

where $P > 0$ is the unique matrix solution of (1.0.7), the nonsingular real linear transformation

$$\begin{pmatrix} y \\ q \end{pmatrix} = M \begin{pmatrix} x \\ p \end{pmatrix},$$

transforms the linear Hamiltonian system (1.0.12) in $\mathbf{R}^{2n}$ into the block diagonalized system

$$\begin{pmatrix} \dot{y} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \hat{A} & 0 \\ 0 & -\hat{A}^* \end{pmatrix} \begin{pmatrix} y \\ q \end{pmatrix}, \tag{1.0.14}$$

where $\hat{A} = A + B\hat{K}$ is a stability matrix,

$$\hat{K} = -U^{-1}(B^*P + R^*)$$

and Q is the matrix solution of the equation

$$\hat{A}Q + Q\hat{A}^* = -\hat{U}.$$

The inverse of M is given by the formula

$$M^{-1} = \begin{pmatrix} I_n & Q \\ 2P & 2PQ - I_n \end{pmatrix}. \tag{1.0.15}$$

As a consequence of this diagonalization, Lukes [31] obtained the following theorem which restates the conclusions of Theorem 1.4 in a form which leads to a proof of Theorem 1.2 by a perturbation analysis.

Theorem 1.6 *For the linear Hamiltonian system (1.0.12) in* $\mathbf{R}^{2n}$ *there is a linear* n*-dimensional invariant manifold in which the origin is asymptotically stable. The manifold is described by the equation* $p = 2Px$*, where* P *is the unique positive definite symmetric solution of the matrix equation (1.0.7). Moreover, this manifold generates the optimal feedback control for the linear problem of Theorem 1.4. That is, if we define* $\hat{p}(x) = 2Px$*, then*

$$\hat{u}(x, \hat{p}(x)) = -\frac{1}{2}U^{-1}\left(2R^*x + B^*\hat{p}(x)\right) = -U^{-1}(R^* + B^*P)x = \hat{K}x \qquad (1.0.16)$$

is the optimal control.

The motion in the manifold projects as the optimal closed-loop motion; that is for any trajectory $\begin{pmatrix} x(t) \\ p(t) \end{pmatrix}$ *in the manifold,* $\dot{x} = (A + B\hat{K})x$.

It is the extension of the decoupling process carrying (1.0.12) into (1.0.14) which provides the key to the result of Theorem 1.2 for nonlinear nonquadratic problems. Let the function $\hat{u}(x,p)$ be defined in Lemma 1.5 and in terms of the given functions $F(x,u)$ and $G(x,u)$ in Theorem 1.2. We select the Hamiltonian

$$H(x,p) = p^*F(x, \hat{u}(x,p)) + G(x, \hat{u}(x,p))$$

and analyze the corresponding system of canonical differential equations by using (1.0.11),

$$\dot{x} \;=\; \left(\frac{\partial H}{\partial p}(x,p)\right)^*,$$

$$\dot{p} \;=\; -\left(\frac{\partial H}{\partial x}(x,p)\right)^*,$$

which can be rewritten as

$$\begin{aligned}
\dot{x} &= F(x,\hat{u}),\\
\dot{p} &= -\left[\left(\frac{\partial F}{\partial x}(x,\hat{u})\right)^* p + \left(\frac{\partial G}{\partial x}(x,\hat{u})\right)^*\right],
\end{aligned} \tag{1.0.17}$$

because $\hat{u}(x,p)$ satisfies (1.0.11).

By collecting the linear terms we have

$$\dot{\begin{pmatrix} x \\ p \end{pmatrix}} = \begin{pmatrix} \hat{R}^* & -\hat{U} \\ -\hat{W} & -\hat{R} \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix} + r \begin{pmatrix} x \\ p \end{pmatrix}, \tag{1.0.18}$$

where

$$r \begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} B\hat{h}(x,p) + f(x,\hat{u}) \\ -2\left[R\hat{h}(x,p) + g_x(x,\hat{u}) + f_x(x,\hat{u})p\right] \end{pmatrix}.$$

By (1.0.13) the change of variables $\begin{pmatrix} y \\ q \end{pmatrix} = M \begin{pmatrix} x \\ p \end{pmatrix}$ transforms the system into

$$\dot{\begin{pmatrix} y \\ q \end{pmatrix}} = \begin{pmatrix} \hat{A} & 0 \\ 0 & -\hat{A}^* \end{pmatrix} \begin{pmatrix} y \\ q \end{pmatrix} + r_M \begin{pmatrix} y \\ q \end{pmatrix}, \tag{1.0.19}$$

where

$$r_M \begin{pmatrix} y \\ q \end{pmatrix} = Mr\left(M^{-1}\begin{pmatrix} y \\ q \end{pmatrix}\right).$$

The following general theorem is a classical result, proved, e.g., in [12], [31]:

Theorem 1.7 (Stable Manifold Theorem) *Consider a system*

$$\dot{\xi} = \mathcal{F}(\xi),$$

$\mathcal{F}$ *is of class* C^2 *in* $\mathbf{R}^{2n}$ *and* $\mathcal{F}(0) = 0$. *With* $\xi = \begin{pmatrix} y \\ q \end{pmatrix}$, $\dim y = n$ *and* $\dim q = n$, *we assume this system takes the form*

$$\begin{pmatrix} \dot{y} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} A_1 & C \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} y \\ q \end{pmatrix} + \begin{pmatrix} f_1(y,q) \\ f_2(y,q) \end{pmatrix},$$

assume that A_1 *and* $-A_2$ *are stability matrices. Then there is a real* n*-dimensional invariant manifold* S *in which the origin is asymptotically stable. That is, if the manifold* S *is described by* $q = \Phi(y)$, $\Phi(y) = o(\|y\|)$ *as* $\|y\| \longrightarrow 0$, *such that* S *is invariant under the system* $\dot{\xi} = \mathcal{F}(\xi)$, *and all solutions* $\xi(t)$ *initiating at points* $\xi_0 = \begin{pmatrix} y_0 \\ \Phi(y_0) \end{pmatrix} \in S$ *with* $\|y_0\|$ *sufficiently small tend to* 0 *as* $t \longrightarrow \infty$.

Since the Hamiltonian system (1.0.17) takes the form (1.0.19) by collecting the linear terms and changing of variables we obtain

$$\begin{pmatrix} \dot{y} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} \hat{A} & 0 \\ 0 & -\hat{A}^* \end{pmatrix} \begin{pmatrix} y \\ q \end{pmatrix} + \begin{pmatrix} f_1(y,q) \\ f_2(y,q) \end{pmatrix}, \tag{1.0.20}$$

where $\hat{A} = A + B\hat{K}$, $\hat{K} = -U^{-1}(B^*P + R^*)$ and P is the unique positive definite symmetric solution of the matrix equation (1.0.7). Since $\hat{A}$ is a stability matrix, that is, $\hat{A}$ has all eigenvalues with negative real parts, then $-\hat{A}^*$ has all eigenvalues with positive real parts. Following Lukes, we apply the stable manifold theorem to the system (1.0.20) in the variables (y, q). As a consequence there exists a function Ψ such that $q = \Psi(y)$ describes the real n - dimensional invariant manifold S for

(1.0.20) in which the origin of in $\mathbf{R}^{2n}$ is asymptotically stable.

Since $\begin{pmatrix} y \\ q \end{pmatrix} = M \begin{pmatrix} x \\ p \end{pmatrix}$, where M is given in (1.0.13), and M^{-1} is given in (1.0.15) we have

$$p = 2Py + (2PQ - I_n)\Psi(y), \quad \text{and} \quad y = (I_n - 2QP)x + Qp.$$

Then there exists a function Φ such that $p = \Phi(x)$, and $p(t) = \Phi(x(t)) \longrightarrow 0$ as $t \longrightarrow \infty$. The optimal control (1.0.16) thus takes the following form:

$$\hat{u}(\hat{x}) \equiv \hat{u}(\hat{x}, \hat{p}(\hat{x})) = -\frac{1}{2}U^{-1}\left(2R^*\hat{x} + B^*\Phi(\hat{x})\right) + h(\hat{x}), \tag{1.0.21}$$

and the closed loop system is

$$\dot{\hat{x}} = F\left(\hat{x}, -\frac{1}{2}U^{-1}(2R^*\hat{x} + B^*\Phi(\hat{x})) + h(\hat{x})\right).$$

For initial states $\hat{x}(0) = x_0$ sufficiently close to 0, we will have $\lim_{t\to\infty} \hat{x}(t) = 0$.

Our first objective in this thesis will be to extend the above results to the general finite dimensional discrete control system

$$x_{k+1} = F(x_k, u_k), \quad k = 0, 1, 2, \ldots \tag{1.0.22}$$

with $F(0,0) = 0$ and a cost functional of the form

$$J = \sum_{k=0}^{\infty} G(x_k, u_k), \tag{1.0.23}$$

with

$$F(x_k, u_k) = Ax_k + Bu_k + f(x_k, u_k),\ k = 0, 1, 2, \dots, \tag{1.0.24}$$

$$G(x_k, u_k) = (x_k^*, u_k^*) \begin{pmatrix} W & R \\ R^* & U \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} + g(x_k, u_k),\ k = 0, 1, 2, \dots. \tag{1.0.25}$$

We will proceed in a manner parallel to Lukes treatment of nonlinear differential systems near the origin as we have described above. In later chapters we will study a nonautonomous system of a particular type – the periodic system

$$\dot{x} = F(t, x, u)$$

and a cost function

$$J = \int_0^\infty G(t, x, u)\, dt,$$

with

$$F(t + T, x, u) = F(t, x, u),$$

$$G(t + T, x, u) = G(t, x, u),$$

as an application of the results we obtain for (1.0.22) - (1.0.25). These results will depend on our being able to generate corresponding versions of the stable manifold theorem for use in these new situations.

Chapter 2

Optimal Control of General Finite Dimensional Discrete Systems Near an Equilibrium Point

2.1 Necessary Conditions for Optimal Control of General Finite Dimensional Discrete Systems; Solution by Decoupling in the Linear Case

Let us consider a general discrete (recursion) control system with state $x = \{x_k\} \in \mathbf{R}^n$ and control $u = \{u_k\} \in \mathbf{R}^m$:

$$x_{k+1} = F(x_k, u_k), \quad k = 0, 1, 2, \dots, \tag{2.1.1}$$

and a corresponding cost function, depending on the initial state x_0 and the applied control sequence $u = \{u_k\}$,

$$J(x_0, u) = \sum_{k=0}^{\infty} G(x_k, u_k). \tag{2.1.2}$$

We assume that the origin is an equilibrium point of $F(x, u)$, and $F : \mathbf{R}^{n+m} \to \mathbf{R}^n$

and $G : \mathbf{R}^{n+m} \to \mathbf{R}$ are at least twice continuously differentiable in some region containing the origin in $\mathbf{R}^{n+m}$ with

$$F(0,0) = 0, \;\; G(0,0) = 0, \;\; \frac{\partial G}{\partial x}(0,0) = 0, \;\; \frac{\partial G}{\partial u}(0,0) = 0.$$

Then $F(x,u)$ and $G(x,u)$ can be represented in the form

$$\begin{aligned} F(x_k, u_k) &= Ax_k + Bu_k + f(x_k, u_k), \;\; k = 0, 1, 2, \dots, &\quad (2.1.3)\\ G(x_k, u_k) &= (x_k{}^*, u_k{}^*) \begin{pmatrix} W & R \\ R^* & U \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} + g(x_k, u_k), \; k = 0, 1, 2, \dots, &\quad (2.1.4) \end{aligned}$$

where

$$A = \frac{\partial F}{\partial x}(0,0), \;\; B = \frac{\partial F}{\partial u}(0,0),$$

and A, B, W, U and R are real matrices and $f(x,u)$ and $g(x,u)$ are higher order terms, and in addition to the conditions

$$f(0,0) = 0, \;\; g(0,0) = 0, \;\; \frac{\partial g}{\partial x}(0,0) = 0, \;\; \frac{\partial g}{\partial u}(0,0) = 0,$$

and

$$\frac{\partial f}{\partial x}(0,0) = 0, \;\; \frac{\partial f}{\partial u}(0,0) = 0, \;\; \frac{\partial^2 g}{\partial x^2}(0,0) = 0, \;\; \frac{\partial^2 g}{\partial u^2}(0,0) = 0.$$

We assume that the matrix $\begin{pmatrix} W & R \\ R^* & U \end{pmatrix} > 0$ and that (2.1.1) is discrete stabilizable near $x = 0$, $u = 0$. That is, there is an $m \times n$ matrix K such that the matrix $A + BK$ is a discrete stability matrix, all of its eigenvalues have modulus less than one. We also assume that $A - BU^{-1}R^*$ is a nonsingular matrix. We seek an m-dimensional vector feedback control function of the state $x = \{x_k\}$, $u = \{u_k = u(x_k)\}$, which makes the cost function (2.1.2) as small as possible for all initial states x_0 near the origin in $\mathbf{R}^n$ and such that the origin is asymptotically stable.

Let us define admissible open loop controls as follows.

Definition 2.1 *An admissible open loop control $u = \{u_k\}$, for a given initial state x_0, is a control such that*

1. $\lim_{k \to \infty} \|u_k\| = 0$;
2. *if $x = \{x_k\}$ is the solution with x_0 as given, corresponding to the control $u = \{u_k\}$, then* $\lim_{k \to \infty} \|x_k\| = 0$;
3. $\sum_{k=0}^{\infty} G(x_k, u_k) < \infty$.

Also we need the following definition.

Definition 2.2 *A discrete stabilizing feedback control $\hat{u} = \{\hat{u}_k\}$,*

$$\hat{u}_k = \hat{K}\hat{x}_k + \hat{h}(\hat{x}_k), \quad k = 0, 1, 2, \ldots,$$

is called optimal *for the process (2.1.1) with respect to the cost function (2.1.2) if for every discrete stabilizing feedback control $u = \{u_k\}$,*

$$u_k = Kx_k + h(x_k), \quad k = 0, 1, 2, \ldots,$$

there exists a neighborhood N_u of the origin in $\mathbf{R}^n$ such that for $x_0 \in N_u$,

$$J(x_0, \hat{u}) \leq J(x_0, u).$$

Let us begin by looking at this problem from a variational point of view. We assume that $\hat{u} = \{\hat{u}_k\}$, $\hat{x} = \{\hat{x}_k\}$ form an optimal control trajectory pair. We consider a small variation about the optimal control

$$\tilde{u} = \{\tilde{u}_k = \hat{u}_k + \delta u_k,\ k = 0, 1, 2, \dots,\}$$

and we let

$$\tilde{x} = \{\tilde{x}_k = \hat{x}_k + \delta x_k,\ k = 0, 1, 2, \dots,\}$$

be the corresponding variation in the optimal trajectory. Since

$$\begin{aligned}\tilde{x}_{k+1} &= F(\tilde{x}_k, \tilde{u}_k) = F(\hat{x}_k + \delta x_k, \hat{u}_k + \delta u_k) \\ &= F(\hat{x}_k, \hat{u}_k) + \frac{\partial F}{\partial x}(\hat{x}_k, \hat{u}_k)\delta x_k + \frac{\partial F}{\partial u}(\hat{x}_k, \hat{u}_k)\delta u_k + (\text{higher order terms}),\end{aligned}$$

for $k = 0, 1, 2, \dots$, we have the first order variational equation

$$\delta x_{k+1} = A_k\, \delta x_k + B_k\, \delta u_k,\quad k = 0, 1, 2, \dots, \tag{2.1.5}$$

where

$$A_k = \frac{\partial F}{\partial x}(\hat{x}_k, \hat{u}_k) \text{ and } B_k = \frac{\partial F}{\partial u}(\hat{x}_k, \hat{u}_k),$$

and

$$\delta x_0 = 0$$

because $\tilde{x}_0 = x_0$, $\hat{x}_0 = x_0$ and $\tilde{x} = \hat{x}_0 + \delta x_0$. Since the cost function takes the form

$$\begin{aligned}
\tilde{J} &= \sum_{k=0}^{\infty} G(\tilde{x}_k, \tilde{u}_k) = \sum_{k=0}^{\infty} G(\hat{x}_k + \delta x_k, \hat{u}_k + \delta u_k) \\
&= \sum_{k=0}^{\infty} \left[G(\hat{x}_k, \hat{u}_k) + \frac{\partial G}{\partial x}(\hat{x}_k, \hat{u}_k)\delta x_k + \frac{\partial G}{\partial u}(\hat{x}_k, \hat{u}_k)\delta u_k + \text{(higher order terms)} \right],
\end{aligned}$$

and since the minimum cost $\hat{J} = \sum_{k=0}^{\infty} G(\hat{x}_k, \hat{u}_k)$ and $\tilde{J} = \hat{J} + \delta J$, the corresponding variation in the cost to the first order is

$$\delta J = \sum_{k=0}^{\infty} \left[\frac{\partial G}{\partial x}(\hat{x}_k, \hat{u}_k)\delta x_k + \frac{\partial G}{\partial u}(\hat{x}_k, \hat{u}_k)\delta u_k \right]. \tag{2.1.6}$$

Then a necessary condition for optimality is that $\delta J = 0$ for all possible variations $\delta u = \{\delta u_k\}$ and $\delta x = \{\delta x_k\}$ satisfying

$$\begin{aligned}
&\delta x_{k+1} = A_k \delta x_k + B_k \delta u_k, \quad k = 0, 1, 2, \dots, \\
&\delta x_0 = 0, \\
&\lim_{N \to \infty} \delta x_N = 0.
\end{aligned}$$

We introduce the adjoint equation

$${p_{k+1}}^* \frac{\partial F}{\partial x}(\hat{x}_k, \hat{u}_k) - {p_k}^* + \frac{\partial G}{\partial x}(\hat{x}_k, \hat{u}_k) = 0, \quad k = 0, 1, 2, \dots, \tag{2.1.7}$$

where $\{p_k\}$ are vectors in $\mathbf{R}^n$. Then by substituting (2.1.7) and (2.1.5) into (2.1.6) δJ becomes

$$\begin{aligned}
\delta J &= \sum_{k=0}^{\infty} \left[(-{p_{k+1}}^* A_k + {p_k}^*)\delta x_k + \frac{\partial G}{\partial u}(\hat{x}_k, \hat{u}_k)\delta u_k \right] \\
&= \sum_{k=0}^{\infty} (-{p_{k+1}}^* \delta x_{k+1} + {p_k}^* \delta x_k) + \sum_{k=0}^{\infty} \left[{p_{k+1}}^* B_k \delta u_k + \frac{\partial G}{\partial u}(\hat{x}_k, \hat{u}_k)\delta u_k \right].
\end{aligned}$$

Since $\delta x_0 = 0$ and $\delta x_N \to 0$, as $N \to \infty$, we have

$$\begin{aligned}
&\sum_{k=0}^{\infty}(-p_{k+1}{}^*\delta x_{k+1} + p_k{}^*\delta x_k) \\
= \quad &\lim_{N\to\infty}\sum_{k=0}^{N}(-p_{k+1}{}^*\delta x_{k+1} + p_k{}^*\delta x_k) = \lim_{N\to\infty}(-p_{N+1}^*\delta x_{N+1}) = 0,
\end{aligned}$$

if $\{p_k\}$ are bounded, in particular if, $\|p_N\| \to 0$, as $N \to \infty$. Hence we have

$$\delta J = \sum_{k=0}^{\infty}\left[p_{k+1}{}^*\frac{\partial F}{\partial u}(\hat{x}_k, \hat{u}_k) + \frac{\partial G}{\partial u}(\hat{x}_k, \hat{u}_k)\right]\delta u_k. \tag{2.1.8}$$

Now $\{\delta u_k\}$ are independent, so we obtain the necessary condition:

$$p_{k+1}{}^*\frac{\partial F}{\partial u}(\hat{x}_k, \hat{u}_k) + \frac{\partial G}{\partial u}(\hat{x}_k, \hat{u}_k) = 0, \;\; k = 0, 1, 2, \ldots. \tag{2.1.9}$$

If we define a function $\mathcal{H}$ as

$$\mathcal{H}(p, x, u) \equiv p^*\frac{\partial F}{\partial u}(x, u) + \frac{\partial G}{\partial u}(x, u),$$

then

$$\begin{aligned}
&\mathcal{H}(p_{k+1}, \hat{x}_k, \hat{u}_k) = 0, \\
&\frac{\partial \mathcal{H}}{\partial u}(p_{k+1}, \hat{x}_k, \hat{u}_k) = p_{k+1}{}^*\frac{\partial^2 F}{\partial u^2}(\hat{x}_k, \hat{u}_k) + \frac{\partial^2 G}{\partial u^2}(\hat{x}_k, \hat{u}_k).
\end{aligned}$$

Hence we have

$$\begin{aligned}
&\mathcal{H}(0,0,0) = 0, \\
&\frac{\partial \mathcal{H}}{\partial u}(0,0,0) = \frac{\partial^2 G}{\partial u^2}(0,0) = 2U > 0,
\end{aligned}$$

by the implicit function theorem [18], in a neighborhood of the origin in $\mathbf{R}^{2n}$ there exists a continuously differentiable function $\mathcal{K}$ such that $\hat{u}_k$ is given by the control law

$$\hat{u}_k = \mathcal{K}(\hat{x}_k, p_{k+1}), \quad k = 0, 1, 2, \ldots . \tag{2.1.10}$$

The state / adjoint system then takes the form

$$\begin{aligned} \hat{x}_{k+1} &= F(\hat{x}_k, \mathcal{K}(\hat{x}_k, p_{k+1})), \quad k = 0, 1, 2, \ldots, \\ p_{k+1}{}^* A_k - p_k{}^* &+ \frac{\partial G}{\partial x}(\hat{x}_k, \mathcal{K}(\hat{x}_k, p_{k+1})) = 0, \quad k = 0, 1, 2, \ldots . \end{aligned} \tag{2.1.11}$$

From (2.1.4) the partial derivatives of G respect to u and x at $(\hat{x}_k, \hat{u}_k)$, respectively, are

$$\frac{\partial G}{\partial u}(\hat{x}_k, \hat{u}_k) = 2(\hat{x}_k{}^*, \hat{u}_k{}^*) \begin{pmatrix} R \\ U \end{pmatrix} + \frac{\partial g}{\partial u}(\hat{x}_k, \hat{u}_k), \; k = 0, 1, 2, \ldots, \tag{2.1.12}$$

$$\frac{\partial G}{\partial x}(\hat{x}_k, \hat{u}_k) = 2(\hat{x}_k{}^*, \hat{u}_k{}^*) \begin{pmatrix} W \\ R^* \end{pmatrix} + \frac{\partial g}{\partial x}(\hat{x}_k, \hat{u}_k), \; k = 0, 1, 2, \ldots . \tag{2.1.13}$$

Substituting (2.1.12) into (2.1.9), the control law yields

$$p_{k+1}{}^* B_k + 2\hat{x}_k{}^* R + 2\hat{u}_k{}^* U + \frac{\partial g}{\partial u}(\hat{x}_k, \hat{u}_k) = 0, \; k = 0, 1, 2, \ldots . \tag{2.1.14}$$

Thus the optimal control can be expressed to the first order term and plus the higher order term in the form

$$\hat{u}_k = \mathcal{K}(\hat{x}_k, p_{k+1}) = -\frac{1}{2} U^{-1}(B_k^* p_{k+1} + 2R^* \hat{x}_k) + h(\hat{x}_k, p_{k+1}), \; k = 0, 1, 2, \ldots, \tag{2.1.15}$$

where we define $h(x,p)$, the higher order term, as

$$h(x,p) \equiv -\frac{1}{2}U^{-1}\frac{\partial g}{\partial u}(x,\mathcal{K}(x,p))^*$$

which has lowest term of degree two. If we substitute (2.1.13) into the adjoint equation (2.1.7), we have

$$A_k^* p_{k+1} - p_k + 2W\hat{x}_k + 2R\hat{u}_k + \frac{\partial g}{\partial x}(\hat{x}_k,\hat{u}_k)^* = 0, \;\; k = 0,1,2,\dots . \tag{2.1.16}$$

By using (2.1.15) for $\hat{u}_k$ and defining

$$r(x,p) \equiv \frac{\partial g}{\partial x}(x,\mathcal{K}(x,p))^* + 2Rh(x,p),$$

the adjoint equation (2.1.16) takes the form

$$(A_k^* - RU^{-1}B_k^*)p_{k+1} - p_k + 2(W - RU^{-1}R^*)\hat{x}_k + r(\hat{x}_k, p_{k+1}) = 0, \;\; k = 0,1,2,\dots ,$$

where $r(\hat{x}_k, p_{k+1})$ has lowest terms of second degree. Then the general discrete control system (2.1.1) with function $F(x,u)$ can be written as in (2.1.3) takes the following form by using (2.1.15):

$$\hat{x}_{k+1} = (A - BU^{-1}R^*)\hat{x}_k - \frac{1}{2}BU^{-1}B_k^* p_{k+1} + r_f(\hat{x}_k, p_{k+1}),$$

where

$$r_f(\hat{x}_k, p_{k+1}) \equiv Bh(\hat{x}_k, p_{k+1}) + f(\hat{x}_k, \mathcal{K}(\hat{x}_k, p_{k+1}))$$

includes all higher order terms. Hence the state / adjoint system has the form

$$\hat{x}_{k+1} = (A - BU^{-1}R^*)\hat{x}_k - \frac{1}{2}BU^{-1}B_k^*p_{k+1} + r_f(\hat{x}_k, p_{k+1}), \tag{2.1.17}$$

$$k = 0, 1, 2, \dots,$$

$$(A_k^* - RU^{-1}B_k^*)p_{k+1} - p_k + 2(W - RU^{-1}R^*)\hat{x}_k + r(\hat{x}_k, p_{k+1}) = 0, \tag{2.1.18}$$

$$k = 0, 1, 2, \dots .$$

In order to provide a background for this nonlinear problem let us consider the special case wherein (2.1.3) reduces to the linear vector recursion equation

$$x_{k+1} = A_k x_k + B_k u_k, \quad k = 0, 1, 2, \dots, N-1, \tag{2.1.19}$$

while the cost is just the quadratic functional

$$J(x_0, u_0, u_1, \dots, u_{N-1}) = \sum_{k=0}^{N-1} (x_k{}^*, u_k{}^*) \begin{pmatrix} W_k & R_k \\ R_k{}^* & U_k \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} + x_N{}^* W_N x_N, \tag{2.1.20}$$

wherein, as before, we assume the matrices

$$\mathcal{W}_k = \begin{pmatrix} W_k & R_k \\ R_k{}^* & U_k \end{pmatrix} > 0, \quad k = 0, 1, 2, \dots, N-1. \tag{2.1.21}$$

Then from a variational point of view (cf. (2.1.5) - (2.1.15)), the adjoint equation and final condition for the linear case have the following form (cf. (2.1.7)),

$$A_k{}^* p_{k+1} - p_k + 2(W_k \ R_k) \begin{pmatrix} \hat{x}_k \\ \hat{u}_k \end{pmatrix} = 0, \quad k = 0, 1, 2, \dots, N-1, \tag{2.1.22}$$

$$p_N = 2W_N \hat{x}_N, \tag{2.1.23}$$

and the necessary condition for optimality gives us (cf. (2.1.15))

$$\hat{u}_k = -\frac{1}{2} U_k{}^{-1} \left(2 R_k{}^* \hat{x}_k + B_k{}^* p_{k+1} \right), \quad k = 0, 1, 2, \ldots, N-1. \tag{2.1.24}$$

Now we study the same problem from the point of view of dynamic programming [17], [6].

It is clear that the optimal cost should be a quadratic form in x_0 ; call it $x_0{}^* P_0 x_0$. At each later stage, once u_k has been fixed, the *remaining cost after k steps* is

$$J(x_{k+1}, u_{k+1}, \ldots, u_{N-1}) = \sum_{l=k+1}^{N-1} (x_l^*, u_l^*) \begin{pmatrix} W_l & R_l \\ R_l^* & U_l \end{pmatrix} \begin{pmatrix} x_l \\ u_l \end{pmatrix} + x_N{}^* W_N x_N, \tag{2.1.25}$$

the minimal value of which will take the form $x_{k+1}{}^* P_{k+1} x_{k+1}$. When $k = N-1$ we will have $P_N = W_N$.

Let the minimizing control sequence be $\{\hat{u}_k,\ k = 0, 1, 2, \ldots, N-1\}$. The *principle of optimality* [7] then shows that the subsequence minimizing (2.1.25) is just $\{\hat{u}_l,\ l = k+1, \ldots, N-1\}$. At each stage we have

$$x_k{}^* P_k x_k = \min_{u_k} \left\{ (x_k{}^*, u_k{}^*) \begin{pmatrix} W_k & R_k \\ R_k{}^* & U_k \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} + x_{k+1}{}^* P_{k+1} x_{k+1} \right\}, \tag{2.1.26}$$

where in (2.1.19) applies to give the dependence of x_{k+1} on x_k and u_k. The condition that the optimal control $\hat{u}_k$ should realize the indicated minimum, assuming the optimal strategy has been pursued to yield the optimal $\hat{x}_k$, is

$$\frac{\partial}{\partial u} \left\{ (\hat{x}_k{}^*, u^*) \begin{pmatrix} W_k & R_k \\ R_k{}^* & U_k \end{pmatrix} \begin{pmatrix} \hat{x}_k \\ u \end{pmatrix} + (A_k \hat{x}_k + B_k u)^* P_{k+1} (A_k \hat{x}_k + B_k u) \right\} \Bigg|_{u = \hat{u}_k} = 0,$$

or

$$2(\hat{x}_k{}^*, \hat{u}_k{}^*)\begin{pmatrix} R_k \\ U_k \end{pmatrix} + 2(A_k\hat{x}_k + B_k\hat{u}_k)^* P_{k+1} B_k = 0,$$

giving

$$\hat{u}_k = -(U_k + B_k{}^* P_{k+1} B_k)^{-1}(R_k{}^* + B_k{}^* P_{k+1} A_k)\hat{x}_k \equiv K_k \hat{x}_k. \tag{2.1.27}$$

Thus the optimal $\hat{x}_k$ and $\hat{u}_k$ satisfy

$$\begin{aligned} \hat{x}_{k+1} &= (A_k + B_k K_k)\hat{x}_k, \quad k = 0, 1, 2, \ldots, N-1, \\ \hat{u}_k &= K_k \hat{x}_k, \qquad\qquad k = 0, 1, 2, \ldots, N-1, \end{aligned} \tag{2.1.28}$$

with K_k defined by (2.1.27). Then by using (2.1.26), we have

$$\begin{aligned} P_k &= W_k + R_k K_k + K_k^* R_k{}^* + K_k^* U_k K_k + (A_k + B_k K_k)^* P_{k+1}(A_k + B_k K_k), \\ &\qquad\qquad k = 0, 1, 2, \ldots, N-1, \qquad (2.1.29) \\ P_N &= W_N. \end{aligned}$$

The question now is to reconcile the expression (2.1.24) with (2.1.27). We rewrite (2.1.27) in the form

$$(U_k + B_k{}^* P_{k+1} B_k)\,\hat{u}_k = -\,(R_k{}^* + B_k{}^* P_{k+1} A_k)\,\hat{x}_k,$$

or

$$U_k \hat{u}_k = -\left[B_k{}^* P_{k+1}(B_k \hat{u}_k + A_k \hat{x}_k) + R_k{}^* \hat{x}_k\right],$$

so that

$$\hat{u}_k = -U_k{}^{-1}\left(R_k{}^*\hat{x}_k + B_k{}^*P_{k+1}\hat{x}_{k+1}\right). \tag{2.1.30}$$

Formula (2.1.30), along with (2.1.27), yields the useful identity

$$K_k\hat{x}_k = -U_k{}^{-1}\left(R_k{}^*\hat{x}_k + B_k{}^*P_{k+1}\hat{x}_{k+1}\right). \tag{2.1.31}$$

Comparison of (2.1.24) with (2.1.30) suggests that we should have

$$p_k = 2P_k\hat{x}_k, \quad k = 0, 1, 2, \ldots, N. \tag{2.1.32}$$

This agrees with (2.1.29), (2.1.22) and (2.1.23) as far as the case $k = N$ is concerned. There remains now the question of the consistency of the equations (2.1.29), (2.1.22) and (2.1.23) and the relationship (2.1.32). Substituting (2.1.30) and (2.1.32) into (2.1.22) and dividing by two we have

$$A_k{}^*P_{k+1}\hat{x}_{k+1} - P_k\hat{x}_k + (W_k \; R_k)\begin{pmatrix} \hat{x}_k \\ -U_k{}^{-1}(R_k{}^*\hat{x}_k + B_k{}^*P_{k+1}\hat{x}_{k+1}) \end{pmatrix} = 0. \tag{2.1.33}$$

We multiply this relationship, (2.1.33), on the left by $\hat{x}_k{}^*$, we get:

$$\begin{aligned} &\hat{x}_k{}^*A_k{}^*P_{k+1}(A_k + B_kK_k)\hat{x}_k - \hat{x}_k{}^*P_k\hat{x}_k + \hat{x}_k{}^*W_k\hat{x}_k \\ &-\hat{x}_k{}^*R_kU_k{}^{-1}(R_k{}^*\hat{x}_k + B_k{}^*P_{k+1}\hat{x}_{k+1}) = 0, \end{aligned} \tag{2.1.34}$$

and then by adding some terms, this equation becomes,

$$\hat{x}_k{}^*(A_k + B_kK_k)^*P_{k+1}(A_k + B_kK_k)\hat{x}_k - \hat{x}_k{}^*P_k\hat{x}_k + \hat{x}_k{}^*W_k\hat{x}_k + \hat{x}_k{}^*R_kK_k\hat{x}_k$$
$$+\hat{x}_k{}^*K_k^*R_k{}^*\hat{x}_k + \hat{x}_k{}^*K_k^*U_kK_k\hat{x}_k - \hat{x}_k{}^*R_kU_k{}^{-1}(R_k{}^*\hat{x}_k + B_k{}^*P_{k+1}\hat{x}_{k+1})$$

$$-\hat{x}_k{}^* K_k^* B_k{}^* P_{k+1} \hat{x}_{k+1} - \hat{x}_k{}^* R_k K_k \hat{x}_k - \hat{x}_k{}^* K_k^* R_k{}^* \hat{x}_k - \hat{x}_k{}^* K_k^* U_k K_k \hat{x}_k = 0.$$

Using the equivalent forms (2.1.30) and (2.1.31) of $K_k \hat{x}_k$ we verify that the last five terms of the above equation add to zero. Then we have

$$\hat{x}_k{}^* \left[(A_k + B_k K_k)^* P_{k+1} (A_k + B_k K_k) \right.$$
$$\left. - P_k + W_k + R_k K_k + K_k^* R_k{}^* + K_k^* U_k K_k \right] \hat{x}_k = 0,$$

which is then seen to be the same as (2.1.29) multiplied on the left by $\hat{x}_k{}^*$ and on the right by $\hat{x}_k$, and we conclude that (2.1.29), (2.1.22), (2.1.23) and (2.1.32) are, indeed, consistent. Up to this point all of this is standard (cf. [2], [6], e.g.).

But now let us proceed to interpret all of this in terms of decoupling, stable manifolds, etc., which does not appear to be developed in the existing literature. Let us note that the coupled state / adjoint equation system is, after substituting (2.1.24) into (2.1.19), (2.1.22),

$$\hat{x}_{k+1} = \left(A_k - B_k U_k{}^{-1} R_k{}^* \right) \hat{x}_k - \frac{1}{2} B_k U_k{}^{-1} B_k{}^* p_{k+1}, \qquad k = 0, 1, 2, \ldots, N-1, \tag{2.1.35}$$

$$\left(A_k{}^* - R_k U_k{}^{-1} B_k{}^* \right) p_{k+1} - p_k + 2 \left(W_k - R_k U_k{}^{-1} R_k{}^* \right) \hat{x}_k = 0, \qquad k = 0, 1, 2, \ldots, N-1. \tag{2.1.36}$$

In this system we make the transformation

$$p_{k+1} = q_{k+1} + 2 P_{k+1} \hat{x}_{k+1}, \quad k = 0, 1, 2, \ldots, N-1. \tag{2.1.37}$$

From the definition (2.1.27) of K_k, or from (2.1.31), one may verify the relationship

$$K_k = -U_k{}^{-1}\left[R_k{}^* + B_k{}^* P_{k+1}(A_k + B_k K_k)\right], \quad k = 0,1,2,\dots,N-1. \tag{2.1.38}$$

Let us suppose that the equation (2.1.35) transforms under (2.1.37) into an equation

$$\hat{x}_{k+1} = (A_k + B_k K_k)\hat{x}_k + V_k q_{k+1}, \quad k = 0,1,2,\dots,N-1. \tag{2.1.39}$$

If that is the case, then (2.1.37) takes the form

$$p_{k+1} = q_{k+1} + 2P_{k+1}\left[(A_k + B_k K_k)\hat{x}_k + V_k q_{k+1}\right], \quad k = 0,1,2,\dots,N-1. \tag{2.1.40}$$

Substituting the right hand side of (2.1.40) for p_{k+1} into (2.1.35) we have

$$\begin{aligned}
\hat{x}_{k+1} &= (A_k - B_k U_k{}^{-1} R_k{}^*)\hat{x}_k \\
&\quad - \frac{1}{2} B_k U_k{}^{-1} B_k{}^* \left[(I_n + 2P_{k+1}V_k)q_{k+1} + 2P_{k+1}(A_k + B_k K_k)\hat{x}_k\right] \\
&= \Big[A_k - B_k U_k{}^{-1} R_k{}^* - B_k U_k{}^{-1} B_k{}^* P_{k+1}(A_k + B_k K_k)\Big]\,\hat{x}_k \\
&\quad - \frac{1}{2} B_k U_k{}^{-1} B_k{}^*(I_n + 2P_{k+1}V_k)q_{k+1} \\
&= \Big\{A_k - B_k U_k{}^{-1}\left[R_k{}^* + B_k P_{k+1}(A_k + B_k K_k)\right]\Big\}\,\hat{x}_k \\
&\quad - \frac{1}{2} B_k U_k{}^{-1} B_k{}^*(I_n + 2P_{k+1}V_k)q_{k+1}.
\end{aligned}$$

Then, from (2.1.38)

$$\hat{x}_{k+1} = (A_k + B_k K_k)\hat{x}_k - \frac{1}{2} B_k U_k{}^{-1} B_k{}^*(I_n + 2P_{k+1}V_k)q_{k+1}. \tag{2.1.41}$$

Equation (2.1.41) agrees with (2.1.39) just in case, as we now specify,

$$V_k = -\frac{1}{2}\left(I_n + B_k U_k{}^{-1} B_k{}^* P_{k+1}\right)^{-1} B_k U_k{}^{-1} B_k{}^*. \tag{2.1.42}$$

Now substituting (2.1.37) for p_k, (2.1.40) for p_{k+1} into (2.1.36) we have

$$\begin{aligned}(A_k{}^* - R_k U_k{}^{-1} B_k{}^*)\,&\{q_{k+1} + 2P_{k+1}[(A_k + B_k K_k)\hat{x}_k + V_k q_{k+1}]\}\\ &-(q_k + 2P_k\hat{x}_k) + 2(W_k - R_k U_k{}^{-1} R_k{}^*)\hat{x}_k = 0,\end{aligned} \tag{2.1.43}$$

or

$$\begin{aligned}&(A_k{}^* - R_k U_k{}^{-1} B_k{}^*)q_{k+1} - q_k + 2(W_k - R_k U_k{}^{-1} R_k{}^* - P_k)\hat{x}_k\\ &+2(A_k{}^* - R_k U_k{}^{-1} B_k{}^*)P_{k+1}[(A_k + B_k K_k)\hat{x}_k + V_k q_{k+1}] = 0.\end{aligned} \tag{2.1.44}$$

Using (2.1.38) $-R_k U_k{}^{-1} = K_k^* + (A_k + B_k K_k)^* P_{k+1} B_k U_k{}^{-1}$, (2.1.44) yields

$$\begin{aligned}&\left[A_k{}^* + K_k^* B_k{}^* + (A_k + B_k K_k)^* P_{k+1} B_k U_k{}^{-1} B_k{}^*\right] q_{k+1} - q_k\\ &+2\left(W_k - R_k U_k{}^{-1} R_k{}^* - P_k\right)\hat{x}_k + 2[A_k{}^* + K_k^* B_k{}^*\\ &+(A_k + B_k K_k)^* P_{k+1} B_k U_k{}^{-1} B_k{}^*]P_{k+1}[(A_k + B_k K_k)\hat{x}_k + V_k q_{k+1}] = 0.\end{aligned} \tag{2.1.45}$$

After rearranging (2.1.45) we have

$$\begin{aligned}&(A_k + B_k K_k)^* q_{k+1} - q_k + 2[W_k - R_k U_k{}^{-1} R_k{}^* - P_k\\ &+(A_k + B_k K_k)^*(I_n + P_{k+1} B_k U_k{}^{-1} B_k{}^*)P_{k+1}(A_k + B_k K_k)]\hat{x}_k\\ &+2(A_k + B_k K_k)^*[(I_n + P_{k+1} B_k U_k{}^{-1} B_k{}^*)P_{k+1} V_k + \tfrac{1}{2}P_{k+1} B_k U_k{}^{-1} B_k{}^*]q_{k+1} = 0.\end{aligned} \tag{2.1.46}$$

Let us look at the coefficient of q_{k+1} in (2.1.46) first:

$$\begin{aligned}
&(I_n + P_{k+1}B_kU_k{}^{-1}B_k{}^*)P_{k+1}V_k + \tfrac{1}{2}P_{k+1}B_kU_k{}^{-1}B_k{}^* \\
&= P_{k+1}[(I_n + B_kU_k{}^{-1}B_k{}^*P_{k+1})V_k + \tfrac{1}{2}B_kU_k{}^{-1}B_k{}^*].
\end{aligned} \tag{2.1.47}$$

Using (2.1.42), (2.1.47) equals zero. The coefficient of $2\hat{x}_k{}^*$ in (2.1.46) is

$$\begin{aligned}
&(A_k + B_kK_k)^*(I_n + P_{k+1}B_kU_k{}^{-1}B_k{}^*)P_{k+1}(A_k + B_kK_k) \\
&\quad + W_k - R_kU_k{}^{-1}R_k{}^* - P_k \\
&= (A_k + B_kK_k)^*P_{k+1}(A_k + B_kK_k) + W_k - R_kU_k{}^{-1}R_k{}^* - P_k \\
&\quad + (A_k + B_kK_k)^*P_{k+1}B_kU_k{}^{-1}B_k{}^*P_{k+1}(A_k + B_kK_k).
\end{aligned} \tag{2.1.48}$$

After substituting (2.1.29) for P_k into this equation, (2.1.48) becomes:

$$\begin{aligned}
&-R_kK_k - K_k^*R_k{}^* - K_k^*U_kK_k - R_kU_k{}^{-1}R_k{}^* \\
&+(A_k + B_kK_k)^*P_{k+1}B_kU_k{}^{-1}B_k{}^*P_{k+1}(A_k + B_kK_k).
\end{aligned} \tag{2.1.49}$$

From (2.1.38) we see that $B_k{}^*P_{k+1}(A_k + B_kK_k) = -U_kK_k - R_k{}^*$ and substitute this into (2.1.49), we get,

$$-R_kK_k - K_k^*R_k{}^* - K_k^*U_kK_k - R_kU_k{}^{-1}R_k{}^* + (R_k + K_k^*U_k)U_k{}^{-1}(U_kK_k + R_k{}^*),$$

which must also equal zero. Therefore, (2.1.46), or (2.1.36) with (2.1.37), gives us

$$(A_k + B_kK_k)^*q_{k+1} - q_k = 0. \tag{2.1.50}$$

Combining (2.1.39) and (2.1.50), we see that we have the system

$$\hat{x}_{k+1} = (A_k + B_k K_k)\hat{x}_k + V_k q_{k+1}, \quad k = 0, 1, 2, \ldots, N-1, \tag{2.1.51}$$

$$(A_k + B_k K_k)^* q_{k+1} - q_k = 0, \quad k = 0, 1, 2, \ldots, N-1, \tag{2.1.52}$$

with V_k given by (2.1.42), in which the second equation is decoupled from the first equation. This system admits the special class of solutions for which $q_k \equiv 0$ and (2.1.37) and (2.1.39) agree with (2.1.32) and (2.1.28) for this special class of solutions, which coincide with the optimal solutions for the originally posed problem.

An important special case occurs when the matrices A_k, B_k, W_k of the original problem are constant matrices, A, B, W, and we let N in (2.1.19), (2.1.20) tend to ∞. In this case the optimal control law takes the form (cf. (2.1.27))

$$\hat{u}_k = -(U + B^* P B)^{-1}(R^* + B^* P A)\hat{x}_k \equiv K\hat{x}_k, \tag{2.1.53}$$

where P is the unique symmetric positive definite solution of

$$P = W + RK + K^* R^* + K^* U K + (A + BK)^* P (A + BK) \tag{2.1.54}$$

with K as in (2.1.53). The resulting optimal closed loop system then takes the form

$$\hat{x}_{k+1} = (A + BK)\hat{x}_k, \quad k = 0, 1, 2, \ldots . \tag{2.1.55}$$

It is readily established, via the discrete Liapounov theory [11], that the closed loop matrix $A + BK$ is a discrete stability matrix, i.e., all of its eigenvalues have moduli less than unity. Granting our present assumptions, to the effect that $A - BU^{-1}R^*$

is nonsingular, the matrix $A + BK$ is nonsingular which we will prove as follows.

Lemma 2.3 *Assume the matrices A, B, U and R are such that the matrix $A - BU^{-1}R^*$ is nonsingular. Then the optimal feedback matrix K is such that the matrix $A + BK$ is nonsingular.*

Proof. Suppose that the matrix $A + BK$ is singular. Then there is a non-zero vector x_0 such that

$$(A + BK)x_0 = 0.$$

Therefore, the optimal trajectory and control starting at x_0 must be such that

$$\hat{x}_k = (A + BK)^k x_0 = 0, \quad k = 1, 2, 3, \dots ,$$
$$\hat{u}_k = K\hat{x}_k = 0, \quad k = 1, 2, 3, \dots .$$

In particular,

$$\hat{x}_1 = Ax_0 + B\hat{u}_0 = 0. \tag{2.1.56}$$

From the principle of optimality, $\hat{u}_0$ must minimize

$$(x_0^*, \hat{u}_0^*)\begin{pmatrix} W & R \\ R^* & U \end{pmatrix}\begin{pmatrix} x_0 \\ \hat{u}_0 \end{pmatrix} + \hat{x}_1^* P \hat{x}_1 = {x_0}^* W x_0 + 2{x_0}^* R\hat{u}_0 + \hat{u}_0^* U \hat{u}_0 + \hat{x}_1^* P \hat{x}_1$$

subject to the condition $\hat{x}_1 = Ax_0 + B\hat{u}_0$. Hence

$$\frac{\partial}{\partial u_0}\left({x_0}^* W x_0 + 2{x_0}^* R u_0 + u_0^* U u_0 + (Ax_0 + Bu_0)^* P (Ax_0 + Bu_0)\right)\Big|_{u_0 = \hat{u}_0} = 0.$$

Therefore,

$$2x_0{}^*R + 2\hat{u}_0^*U + 2(Ax_0 + B\hat{u}_0)^*PB = 0,$$

by using (2.1.56) the above equation yields

$$2x_0{}^*R + 2\hat{u}_0^*U = 0,$$

which implies

$$\hat{u}_0 = U^{-1}R^*x_0,$$

hence we have

$$0 = \hat{x}_1 = (A - BU^{-1}R^*)x_0.$$

Since $x_0 \neq 0$, then $A - BU^{-1}R^*$ is singular, this is a contradiction to the assumption that $A - BU^{-1}R^*$ is nonsingular. Then it follows that the matrix $A + BK$ must be nonsingular if $A - BU^{-1}R^*$ is nonsingular.

The state / adjoint system (cf. (2.1.1) and (2.1.18)) now takes the form

$$\hat{x}_{k+1} = (A - BU^{-1}R^*)\hat{x}_k - \frac{1}{2}BU^{-1}B^*p_{k+1}, \ k = 0, 1, 2, \dots, \tag{2.1.57}$$

$$(A^* - RU^{-1}B^*)p_{k+1} - p_k + 2(W - RU^{-1}R^*)\hat{x}_k = 0, \ k = 0, 1, 2, \dots. \tag{2.1.58}$$

The same calculations as above in the time varying case show that the transformation

$$p = q + 2Px, \tag{2.1.59}$$

decouples this system to one of the form (cf. (2.1.51), (2.1.52))

$$\hat{x}_{k+1} = (A+BK)\hat{x}_k + V\left[(A+BK)^{-1}\right]^* q_k, \quad k = 0,1,2,\dots, \tag{2.1.60}$$

$$q_{k+1} = \left[(A+BK)^{-1}\right]^* q_k, \quad k = 0,1,2,\dots, \tag{2.1.61}$$

or

$$\begin{pmatrix} \hat{x}_{k+1} \\ q_{k+1} \end{pmatrix} = \begin{pmatrix} A+BK & V\left[(A+BK)^{-1}\right]^* \\ 0 & \left[(A+BK)^{-1}\right]^* \end{pmatrix} \begin{pmatrix} \hat{x}_k \\ q_k \end{pmatrix} \tag{2.1.62}$$

with (cf. (2.1.42))

$$V = -\frac{1}{2}\left(I_n + BU^{-1}B^*P\right)^{-1} BU^{-1}B^*, \tag{2.1.63}$$

and K is a solution of the following equation (cf. (2.1.38)):

$$K = -U^{-1}\left[R^* + B^*P(A+BK)\right], \quad k = 0,1,2,\dots,N-1, \tag{2.1.64}$$

or, equivalently, K is defined as in (2.1.53), i.e.,

$$K = -\left(U + B^*PB\right)^{-1}\left(R^* + B^*PA\right). \tag{2.1.65}$$

As already indicated, the matrix $A + BK$ is nonsingular and has eigenvalues with moduli less than one; correspondingly, $(A+BK)^{-1}$ exists and has eigenvalues with moduli greater than one. The stable subspace (manifold) in this case is just given by $q = 0$, corresponding to $p = 2Px$ in the original x, p coordinates.

Theorem 2.4 (Linear Discrete System) *For the linear discrete system with*

$x = \{x_k\}$ and $u = \{u_k\}$ is a control and trajectory pair,

$$x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, 2, \dots, \tag{2.1.66}$$

with cost function

$$J(x_0, u) = \sum_{k=0}^{\infty} ({x_k}^*, {u_k}^*) \begin{pmatrix} W & R \\ R^* & U \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix}, \tag{2.1.67}$$

the unique optimal control $\hat{u} = \{\hat{u}_k\}$ is given by

$$\hat{u}_k = \hat{u}(\hat{x}_k) = \hat{K}\hat{x}_k, \quad k = 0, 1, 2, \dots,$$

where

$$\hat{K} = -(U + B^*PB)^{-1}(R^* + B^*PA).$$

Here $P > 0$ is the unique positive definite symmetric solution of the matrix equation,

$$P = W + R\hat{K} + \hat{K}^*R^* + \hat{K}^*U\hat{K} + (A + B\hat{K})^*P(A + B\hat{K}). \tag{2.1.68}$$

Moreover, the minimal cost

$$\hat{J}(x_0) = {x_0}^*Px_0.$$

We have already shown, prior to the statement of this theorem, that the control $\hat{u}_k = \hat{K}\hat{x}_k$ stisfies the necessary conditions (2.1.19) - (2.1.24). The optimality and uniqueness will be proved later in this chapter.

Recall that for our original nonlinear problem we have the optimal control $\hat{u} = \{\hat{u}_k\}$

(cf. (2.1.15))

$$\hat{u}_k = -\frac{1}{2}U^{-1}\left(2R^*\hat{x}_k + B^*p_{k+1}\right) + h(\hat{x}_k, p_{k+1}),\ k = 0, 1, 2, \dots ; \tag{2.1.69}$$

where $\hat{x}_k$, p_k satisfy the state / adjoint system (cf. (2.1.17) and (2.1.18))

$$\hat{x}_{k+1} = (A - BU^{-1}R^*)\hat{x}_k - \frac{1}{2}BU^{-1}B^*p_{k+1} + r_f(\hat{x}_k, p_{k+1}),$$
$$k = 0, 1, 2, \dots ,$$
$$(A^* - RU^{-1}B^*)p_{k+1} - p_k + 2(W - RU^{-1}R^*)\hat{x}_k + r(\hat{x}_k, p_{k+1}) = 0,$$
$$k = 0, 1, 2, \dots .$$

If we make the same transformation (2.1.59), $p = q + 2Px$, then the state / adjoint system (cf. (2.1.60) and (2.1.61)) has the form

$$\hat{x}_{k+1} = (A + BK)\hat{x}_k + V\left[(A + BK)^{-1}\right]^* q_k + f_1(\hat{x}_k, q_k), \tag{2.1.70}$$
$$k = 0, 1, 2, \dots ,$$
$$q_{k+1} = \left[(A + BK)^{-1}\right]^* q_k + f_2(\hat{x}_k, q_k),\ k = 0, 1, 2, \dots , \tag{2.1.71}$$

where V is as same as in (2.1.63) and $f_1(x, q)$, $f_2(x, q)$ are higher order terms and continuously differentiable in some neighborhood of the origin in $\mathbf{R}^{2n}$. We will see in subsequent sections that extension of the feedback law (2.1.53) to generate optimal controls and trajectories for nonlinear systems depends on being able to extend the decoupling already accomplished in the linear part of (2.1.70), (2.1.71) to a complete decoupling of that system valid in a neighborhood of the origin in $\mathbf{R}^{2n}$.

Example. For the nonlinear discrete system

$$x_{k+1} = e^{x_k} - 1 + \sin(u_k), \quad k = 0, 1, 2, \dots, \tag{2.1.72}$$

we pose the problem, for a given x_0,

$$\min_{\{u_k\}} \sum_{k=0}^{\infty} \left[(1 - \cos x_k) + (1 - \cos u_k)\right], \tag{2.1.73}$$

the admissible controls $u = \{u_k\}$ being those for which $x = \{x_k\}$, x_k, u_k tend to 0 as $k \to \infty$ and the cost function (2.1.73) is finite.

This problem corresponds to (cf. (2.1.3), (2.1.4))

$$F(x_k, u_k) = e^{x_k} - 1 + \sin u_k, \quad k = 0, 1, 2, \dots, \tag{2.1.74}$$

$$G(x_k, u_k) = (1 - \cos x_k) + (1 - \cos u_k), \quad k = 0, 1, 2, \dots . \tag{2.1.75}$$

Then $F(0, 0) = 0$, and

$$\begin{aligned} A &= \frac{\partial F}{\partial x}(0,0) = 1, \quad B = \frac{\partial F}{\partial u}(0,0) = 1, \\ W &= \frac{1}{2}\frac{\partial^2 G}{\partial x^2}(0,0) = \frac{1}{2}, \quad U = \frac{1}{2}\frac{\partial^2 G}{\partial u^2}(0,0) = \frac{1}{2}, \quad R = \frac{\partial^2 G}{\partial x \partial u}(0,0) = 0. \end{aligned}$$

Thus we have

$$F(x_k, u_k) = x_k + u_k + f(x_k, u_k), \quad k = 0, 1, 2, \dots, \tag{2.1.76}$$

$$G(x_k, u_k) = (x_k, u_k) \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} + g(x_k, u_k), \quad k = 0, 1, 2, \dots , \tag{2.1.77}$$

where $f(x_k, u_k)$ and $g(x_k, u_k)$ represent the higher order terms.

In order to solve this problem we first need to set up the linearized system at the origin, which is

$$x_{k+1} = x_k + u_k, \quad k = 0, 1, 2, \dots , \tag{2.1.78}$$

and the cost function

$$G(x_k, u_k) = (x_k, u_k)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x_k \\ u_k \end{pmatrix}, \quad k = 0, 1, 2, \dots , \tag{2.1.79}$$

for $A = 1$, $B = 1$, $W = 1/2$, $U = 1/2$ and $R = 0$. Then the optimal control law (2.1.53) for this problem takes the form:

$$\hat{u}_k = -\left(\frac{1}{2} + P\right)^{-1} P\hat{x}_k = -\frac{2P}{1+2P}\hat{x}_k, \tag{2.1.80}$$

which corresponds to

$$K \equiv -\frac{2P}{1+2P}, \quad if \quad 1 + 2P \neq 0, \tag{2.1.81}$$

where P is the unique symmetric positive definite solution of (cf. (2.1.54))

$$P = \frac{1}{2} + \frac{1}{2}K^2 + (1+K)^2 P. \tag{2.1.82}$$

Substituting (2.1.81) into (2.1.82), we have

$$P = \frac{1}{2} + \frac{1}{2}\left(\frac{2P}{1+2P}\right)^2 + \left(1 - \frac{2P}{1+2P}\right)^2 P,$$

or

$$(1+2P)\left[(1+2P)\left(\frac{1}{2}-P\right)+P\right]=0.$$

Since $1+2P \neq 0$, this implies

$$(1+2P)\left(\frac{1}{2}-P\right)+P=0, \quad \text{or} \quad 4P^2-2P-1=0,$$

so that

$$P=\frac{1\pm\sqrt{5}}{4}. \tag{2.1.83}$$

Therefore, the positive definite solution of (2.1.82) is

$$P=\frac{1+\sqrt{5}}{4}. \tag{2.1.84}$$

When we substitute this into (2.1.81) we have

$$K=\frac{1-\sqrt{5}}{2}, \tag{2.1.85}$$

then the optimal control law is

$$\hat{u}_k=\frac{1-\sqrt{5}}{2}\hat{x}_k, \quad k=0,1,2,\ldots, \tag{2.1.86}$$

so that the resulting optimal closed loop system of the linearized system (2.1.78) has the form:

$$\hat{x}_{k+1}=\frac{3-\sqrt{5}}{2}\hat{x}_k, \quad k=0,1,2,\ldots. \tag{2.1.87}$$

(Note: $\left|\frac{3-\sqrt{5}}{2}\right|<1.$)

The state / adjoint system of the linearized system of this problem now takes the form (cf. (2.1.57) and (2.1.58)).

$$\hat{x}_{k+1} = \hat{x}_k - p_{k+1}, \quad k = 0, 1, 2, \dots ,$$
$$p_{k+1} - p_k + \hat{x}_k = 0, \quad k = 0, 1, 2, \dots . \tag{2.1.88}$$

Then the transformation

$$p = q + 2Px,$$

decouples the system (2.1.88) to the form (cf. (2.1.60) and (2.1.61))

$$\begin{aligned} \hat{x}_{k+1} &= (1+K)\hat{x}_k + V(1+K)^{-1}q_k, \quad k = 0, 1, 2, \dots , \\ q_{k+1} &= (1+K)^{-1}q_k, \quad k = 0, 1, 2, \dots , \end{aligned} \tag{2.1.89}$$

with (cf. (2.1.63))

$$V = -\frac{1}{2}(1+2P)^{-1}2 = -\frac{3-\sqrt{5}}{2}. \tag{2.1.90}$$

Substituting this into (2.1.89) we have

$$\hat{x}_{k+1} = \frac{3-\sqrt{5}}{2}\hat{x}_k - q_k, \quad k = 0, 1, 2, \dots . \tag{2.1.91}$$

So we have now the linearized system

$$\begin{aligned} \hat{x}_{k+1} &= \frac{3-\sqrt{5}}{2}\hat{x}_k - q_k, \quad k = 0, 1, 2, \dots , \\ q_{k+1} &= \frac{3+\sqrt{5}}{2}q_k, \quad k = 0, 1, 2, \dots , \end{aligned} \tag{2.1.92}$$

or

$$\begin{pmatrix} \hat{x}_{k+1} \\ q_{k+1} \end{pmatrix} = \begin{pmatrix} \frac{3-\sqrt{5}}{2} & -1 \\ 0 & \frac{3+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \hat{x}_k \\ q_k \end{pmatrix}, \quad k = 0, 1, 2, \dots . \tag{2.1.93}$$

It remains to solve the rest (the nonlinear part) of this problem, for which (2.1.70), (2.1.71) are the following:

$$\begin{pmatrix} \hat{x}_{k+1} \\ q_{k+1} \end{pmatrix} = \begin{pmatrix} \frac{3-\sqrt{5}}{2} & -1 \\ 0 & \frac{3+\sqrt{5}}{2} \end{pmatrix} \begin{pmatrix} \hat{x}_k \\ q_k \end{pmatrix} + \begin{pmatrix} f_1(\hat{x}_k, q_k) \\ f_2(\hat{x}_k, q_k) \end{pmatrix}, \; k = 0, 1, 2, \dots . \tag{2.1.94}$$

We will complete the analysis of this problem after we develop the optimal control theory for general nonlinear discrete systems.

2.2 The Stable Manifold Theorem for General Discrete Systems

Since the state / adjoint system of the general nonlinear discrete system with its adjoint equation can be assumed to have the following form (cf. (2.1.70) and (2.1.71)):

$$\begin{pmatrix} \hat{x}_{k+1} \\ q_{k+1} \end{pmatrix} = \begin{pmatrix} \hat{A} & C \\ 0 & (\hat{A}^{-1})^* \end{pmatrix} \begin{pmatrix} \hat{x}_k \\ q_k \end{pmatrix} + \begin{pmatrix} f_1(\hat{x}_k, q_k) \\ f_2(\hat{x}_k, q_k) \end{pmatrix}, \quad k = 0, 1, 2, \dots, \tag{2.2.1}$$

where $f_1(x,q)$, $f_2(x,q)$ are the higher order terms and are continuously differentiable functions in some neighborhood of the origin in $\mathbf{R}^{2n}$, comparable to the differential equation case (1.0.20) on page 13. The matrix $\hat{A} = A + BK$ is nonsingular and has all eigenvalues with modulus less than one; and $(\hat{A}^{-1})^*$ exists and has eigenvalues all with modulus greater than one. Our next task is to develop the stable manifold theorem for such nonlinear discrete systems.

Theorem 2.5 (Stable Manifold Theorem for the Discrete System)

Consider a system

$$\xi_{k+1} = \mathcal{F}(\xi_k), \quad k = 0, 1, 2, \dots, \tag{2.2.2}$$

with $\mathcal{F}$ of class C^2 in $\mathbf{R}^{2n}$ and $\mathcal{F}(0) = 0$. With $\xi = \{\xi\}$, $\xi_k = \begin{pmatrix} y_k \\ z_k \end{pmatrix}$, $\dim y_k = n$ *and* $\dim z_k = n$, *$k = 0, 1, 2, \dots,$ we further assume that the system (2.2.2) has the form:*

$$\begin{pmatrix} y_{k+1} \\ z_{k+1} \end{pmatrix} = \begin{pmatrix} A_1 & C \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} y_k \\ z_k \end{pmatrix} + \begin{pmatrix} f_1(y_k, z_k) \\ f_2(y_k, z_k) \end{pmatrix}, \quad k = 0, 1, 2, \dots, \tag{2.2.3}$$

with A_1 having eigenvalues with modulus less than one, and A_2 eigenvalues with modulus greater than one and

$$f_i(0,0)=0,\ \frac{\partial f_i}{\partial y}(0,0)=0,\ \frac{\partial f_i}{\partial z}(0,0)=0,\ i=1,2.$$

Then there is an n-dimensional invariant manifold S of class C^1 in which the origin is asymptotically stable. That is, if the manifold S is described, for $\|y\|$ sufficiently small, by $z=\Phi(y)$, $\Phi(y)=o(\|y\|)$, as $\|y\|\to 0$, then S is invariant under the system (2.2.2), and all solutions $\xi=\{\xi_k\}$ initiating at points $\xi_0=\begin{pmatrix} y_0 \\ \Phi(y_0)\end{pmatrix}\in S$, have the property $(\xi_k)\to 0$ as $k\to\infty$.

Proof. For $k=0,1,2,\dots$, if we set

$$\begin{pmatrix} y_k \\ z_k \end{pmatrix}=\begin{pmatrix} I_n & P \\ 0 & I_n \end{pmatrix}\begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} \tag{2.2.4}$$

where $P>0$ is the unique solution of $A_1P-PA_2+C=0$, then (2.2.3) will have the following form for all k:

$$\begin{pmatrix} \eta_{k+1} \\ \zeta_{k+1} \end{pmatrix}=\begin{pmatrix} I_n & -P \\ 0 & I_n \end{pmatrix}\begin{pmatrix} A_1 & C \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} I_n & P \\ 0 & I_n \end{pmatrix}\begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix}+\begin{pmatrix} r_1(\eta_k,\zeta_k) \\ r_2(\eta_k,\zeta_k) \end{pmatrix},$$

or

$$\begin{pmatrix} \eta_{k+1} \\ \zeta_{k+1} \end{pmatrix}=\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix}+\begin{pmatrix} r_1(\eta_k,\zeta_k) \\ r_2(\eta_k,\zeta_k) \end{pmatrix}, \tag{2.2.5}$$

where $r_1(\eta_k,\zeta_k)$, $r_2(\eta_k,\zeta_k)$ are also twice continuously differentiable functions of (η_k,ζ_k) in some neighborhood of the origin in $\mathbf{R}^{2n}$, and $r_i(\eta_k,\zeta_k)=O\left\|\begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix}\right\|^2$, as $\begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix}\to 0$, $i=1,2$.

For small $a \in R^n$ we seek a solution $\eta(a) = \{\eta_k(a)\}$, $\zeta(a) = \{\zeta_k(a)\}$ satisfying the conditions

$$\lim_{k\to\infty} \begin{pmatrix} \eta_k(a) \\ \zeta_k(a) \end{pmatrix} = 0, \tag{2.2.6}$$

$$\eta_0(a) = a. \tag{2.2.7}$$

To this end we consider the following equations:

$$\begin{aligned} \eta_k(a) &= A_1^k a + \sum_{j=0}^{k-1} A_1^{k-1-j} r_1(\eta_j(a), \zeta_j(a)), \quad k = 0, 1, 2, \dots, \\ \zeta_k(a) &= -\sum_{j=k}^{\infty} A_2^{k-1-j} r_2(\eta_j(a), \zeta_j(a)), \quad k = 0, 1, 2, \dots, \end{aligned} \tag{2.2.8}$$

and show that this system (2.2.8) is equivalent to (2.2.5), (2.2.6) and (2.2.7).

First of all we need to prove that the system (2.2.8) has a solution. Since A_1 has all eigenvalues with modulus less than one and A_2 has all eigenvalues with modulus greater than one, there exist M_1, $M_2 > 0$ and $0 \le \alpha_1 < 1$, $0 \le \alpha_2 < 1$, such that

$$\|A_1^k\| \le M_1 \alpha_1^k \quad \text{and} \quad \|A_2^{-k}\| \le M_2 \alpha_2^k.$$

Let us consider a space Z_α of sequences $\left\{ \begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix}, \ k = 0, 1, 2, \dots \right\}$ such that, with $\max\{\alpha_1, \alpha_2\} < \alpha < 1$, α fixed,

$$\left\| \left\{ \begin{matrix} \eta_k \\ \zeta_k \end{matrix} \right\} \right\|_\alpha \equiv \sup_{k=0,1,2,\dots} \left\| \begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} \alpha^{-k} \right\| < \infty. \tag{2.2.9}$$

It is easy to see that with the norm $\| \ \|_\alpha$, Z_α is a Banach space. We define Z_α^M to be the subset of Z_α consisting of sequences $\left\{ \begin{array}{c} \eta_k \\ \zeta_k \end{array} \right\}$ such that $\left\| \left\{ \begin{array}{c} \eta_k \\ \zeta_k \end{array} \right\} \right\|_\alpha \leq M$ for some $M > 0$. Let F_a: $Z_\alpha \longrightarrow Z_\alpha$ be defined by

$$
\begin{aligned}
\hat{\eta}_k &= A_1^k a + \sum_{j=0}^{k-1} A_1^{k-1-j} r_1(\eta_j, \zeta_j), \quad k = 0, 1, 2, \dots, \\
\hat{\zeta}_k &= -\sum_{j=k}^{\infty} A_2^{k-1-j} r_2(\eta_j, \zeta_j), \quad k = 0, 1, 2, \dots, \qquad (2.2.10)
\end{aligned}
$$

(cf. (2.2.8)). We want to show for sufficiently small a and some $M > 0$ that F_a maps Z_α^M into itself and is a contraction mapping in Z_α^M. Then by Banach's Fixed - Point Theorem [18], there exists a unique fixed point $\left\{ \begin{array}{c} \eta_k(a) \\ \zeta_k(a) \end{array} \right\}$ in Z_α^M which is the desired solution of (2.2.8).

To show that F_a maps Z_α^M into itself for small M, $M > 0$, and small a, we need to show that if $\left\{ \begin{array}{c} \eta_k \\ \zeta_k \end{array} \right\} \in Z_\alpha^M$, i.e., $\left\| \left\{ \begin{array}{c} \eta_k \\ \zeta_k \end{array} \right\} \right\|_\alpha \leq M$, then $\left\{ \begin{array}{c} \hat{\eta}_k \\ \hat{\zeta}_k \end{array} \right\} \in Z_\alpha^M$, i.e., that $\left\| \left\{ \begin{array}{c} \hat{\eta}_k \\ \hat{\zeta}_k \end{array} \right\} \right\|_\alpha \leq M$, where $\left\{ \begin{array}{c} \hat{\eta}_k \\ \hat{\zeta}_k \end{array} \right\}$ is defined by F_a in (2.2.10).

Let us fix $M > 0$. Since we can choose a very small, we assume $\|a\| \equiv \mu < M/4M_1$. From (2.2.10) we have

$$
\begin{aligned}
\|\{\hat{\eta}_k\}\|_\alpha &= \left\| \left\{ A_1^k a + \sum_{j=0}^{k-1} A_1^{k-1-j} r_1(\eta_j, \zeta_j) \right\} \right\|_\alpha \\
&= \sup_{k=0,1,2,\dots} \left\| \left(A_1^k a + \sum_{j=0}^{k-1} A_1^{k-1-j} r_1(\eta_j, \zeta_j) \right) \alpha^{-k} \right\| \\
&\leq \sup_{k=0,1,2,\dots} \left[\frac{\mu M_1 \alpha_1^k}{\alpha^k} + \sum_{j=0}^{k-1} M_1 \frac{\alpha_1^{k-1-j}}{\alpha^k} \|r_1(\eta_j, \zeta_j)\| \right]. \qquad (2.2.11)
\end{aligned}
$$

Since $r_1(\eta_j, \zeta_j) = O\left\|\begin{pmatrix} \eta_j \\ \zeta_j \end{pmatrix}\right\|^2$, as $\begin{pmatrix} \eta_j \\ \zeta_j \end{pmatrix} \to 0$, by taking M sufficiently small we may assume that $\|r_1(\eta_j, \zeta_j)\| \leq \epsilon_1 \left\|\begin{pmatrix} \eta_j \\ \zeta_j \end{pmatrix}\right\|$, $j = 0, 1, 2, \ldots$, for $\left\{\begin{matrix} \eta_k \\ \zeta_k \end{matrix}\right\} \in Z_\alpha^M$, for any given $\epsilon_1 > 0$. Thus for $j = 0, 1, 2, \ldots$, we have

$$\begin{aligned} \|r_1(\eta_j, \zeta_j)\| &\leq \epsilon_1 \left\|\begin{pmatrix} \eta_j \\ \zeta_j \end{pmatrix}\right\| = \epsilon_1 \alpha^j \left\|\begin{pmatrix} \eta_j \\ \zeta_j \end{pmatrix} \alpha^{-j}\right\| \\ &\leq \alpha^j \epsilon_1 \sup_{k=0,1,2,\ldots} \left\|\begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} \alpha^{-k}\right\| = \alpha^j \epsilon_1 \left\|\left\{\begin{matrix} \eta_k \\ \zeta_k \end{matrix}\right\}\right\|_\alpha \leq \epsilon_1 M \alpha^j. \end{aligned} \tag{2.2.12}$$

The inequality $\alpha_1 < \alpha$ gives us $\alpha_1/\alpha < 1$, and using (2.2.12), (2.2.11) yields

$$\begin{aligned} \|\{\hat{\eta}_k\}\|_\alpha &\leq \sup_{k=0,1,2,\ldots} \left[\frac{\mu M_1 \alpha_1^k}{\alpha^k} + \sum_{j=0}^{k-1} M_1 \frac{\alpha_1^{k-1-j}}{\alpha^k} \epsilon_1 M \alpha^j\right] \\ &= \sup_{k=0,1,2,\ldots} \left[\frac{\mu M_1 \alpha_1^k}{\alpha^k} + \frac{M_1 M \epsilon_1}{\alpha} \sum_{j=0}^{k-1} \left(\frac{\alpha_1}{\alpha}\right)^{k-1-j}\right] \\ &= \sup_{k=0,1,2,\ldots} \left[\frac{\mu M_1 \alpha_1^k}{\alpha^k} + M_1 M \epsilon_1 \frac{1 - \left(\frac{\alpha_1}{\alpha}\right)^k}{\alpha - \alpha_1}\right] \\ &\leq \mu M_1 + \frac{M_1 M \epsilon_1}{\alpha - \alpha_1} \leq \frac{M}{4} + \frac{M_1 M \epsilon_1}{\alpha - \alpha_1}. \end{aligned} \tag{2.2.13}$$

Assuming ϵ_1 chosen less than $(\alpha - \alpha_1)/4M_1$, (2.2.13) yields

$$\|\{\hat{\eta}_k\}\|_\alpha \leq \frac{M}{2}. \tag{2.2.14}$$

Similarly $r_2(\eta_j, \zeta_j) = O\left\|\begin{pmatrix} \eta_j \\ \zeta_j \end{pmatrix}\right\|^2$, as $\begin{pmatrix} \eta_j \\ \zeta_j \end{pmatrix} \to 0$. By taking M sufficiently small, given any $\epsilon_2 > 0$ we may assume that

$$\begin{aligned} \|r_2(\eta_j, \zeta_j)\| &\leq \epsilon_2 \left\|\begin{pmatrix} \eta_j \\ \zeta_j \end{pmatrix}\right\| \leq \epsilon_2 \alpha^j \sup_{k=0,1,2,\ldots} \left\|\begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} \alpha^{-k}\right\| \\ &= \epsilon_2 \alpha^j \left\|\left\{\begin{matrix} \eta_k \\ \zeta_k \end{matrix}\right\}\right\|_\alpha \leq \epsilon_2 M \alpha^j, \end{aligned} \tag{2.2.15}$$

for $\left\{ \begin{array}{c} \eta_k \\ \zeta_k \end{array} \right\} \in Z_\alpha^M$. Since $\alpha_2 < \alpha < 1$, $\alpha_2\alpha < 1$ and using (2.2.15), (2.2.10) also gives us

$$\begin{aligned}
\|\{\hat{\zeta}_k\}\|_\alpha &= \left\| \left\{ -\sum_{j=k}^{\infty} A_2^{k-1-j} r_2(\eta_j, \zeta_j) \right\} \right\|_\alpha \\
&= \sup_{k=0,1,2,\dots} \left\| \left(-\sum_{j=k}^{\infty} A_2^{k-1-j} r_2(\eta_j, \zeta_j) \right) \alpha^{-k} \right\| \\
&\leq \sup_{k=0,1,2,\dots} \sum_{j=k}^{\infty} M_2 \alpha_2^{j+1-k} \epsilon_2 M \alpha^{j-k} \\
&= \sup_{k=0,1,2,\dots} \frac{M_2 M \epsilon_2}{\alpha} \sum_{j=k}^{\infty} (\alpha_2\alpha)^{j+1-k} \\
&= \frac{M_2 M \epsilon_2 \alpha_2}{1-\alpha_2\alpha} \leq \frac{M}{2},
\end{aligned} \tag{2.2.16}$$

the last inequality of (2.2.16) following if we assume, as we may, that

$$\frac{M_2 \epsilon_2 \alpha_2}{1-\alpha_2\alpha} \leq \frac{1}{2}.$$

Combining this result (2.2.16) with (2.2.14) we see that for $\left\{ \begin{array}{c} \eta_k \\ \zeta_k \end{array} \right\} \in Z_\alpha^M$ we have $\left\| \left\{ \begin{array}{c} \hat{\eta}_k \\ \hat{\zeta}_k \end{array} \right\} \right\|_\alpha \leq M$, so that F_a: $Z_\alpha^M \to Z_\alpha^M$.

To prove F_a is a contraction mapping in Z_α^M, we need to show that there exists $0 < \gamma < 1$ such that

$$\|\{F_a(\eta_k, \zeta_k) - F_a(\tilde{\eta}_k, \tilde{\zeta}_k)\}\|_\alpha \leq \gamma \left\| \left\{ \left(\begin{array}{c} \eta_k \\ \zeta_k \end{array} \right) - \left(\begin{array}{c} \tilde{\eta}_k \\ \tilde{\zeta}_k \end{array} \right) \right\} \right\|_\alpha,$$

for any $\left\{ \begin{array}{c} \eta_k \\ \zeta_k \end{array} \right\}, \left\{ \begin{array}{c} \tilde{\eta}_k \\ \tilde{\zeta}_k \end{array} \right\} \in Z_\alpha^M$. By the definition (2.2.10) of F_a we have

$$F_a(\eta_k, \zeta_k) - F_a(\tilde{\eta}_k, \tilde{\zeta}_k) = \left(\begin{array}{c} \sum\limits_{j=0}^{k-1} A_1^{k-1-j} \left[r_1(\eta_j, \zeta_j) - r_1(\tilde{\eta}_j, \tilde{\zeta}_j) \right] \\ -\sum\limits_{j=k}^{\infty} A_2^{k-1-j} \left[r_2(\eta_j, \zeta_j) - r_2(\tilde{\eta}_j, \tilde{\zeta}_j) \right] \end{array} \right), \; k = 0, 1, 2, \ldots . \tag{2.2.17}$$

Since r_1 and r_2 are of class C^2, by the Mean Value Theorem [18] in $\mathbf{R}^{2n}$ there exist $0 < \rho_i < 1$, $i = 1, 2$, and $\left\{ \left(\begin{array}{c} \bar{\eta}_k \\ \bar{\zeta}_k \end{array} \right)_i \right\} \in Z_\alpha^M$ such that

$$\left(\begin{array}{c} \bar{\eta}_j \\ \bar{\zeta}_j \end{array} \right)_i = \rho_i \left(\begin{array}{c} \eta_j \\ \zeta_j \end{array} \right) + (1 - \rho_i) \left(\begin{array}{c} \tilde{\eta}_j \\ \tilde{\zeta}_j \end{array} \right), \quad i = 1, 2, \quad j = 0, 1, 2, \ldots ,$$

and

$$r_i(\eta_j, \zeta_j) - r_i(\tilde{\eta}_j, \tilde{\zeta}_j) = Dr_i((\bar{\eta}_j, \bar{\zeta}_j)_i) \left[\left(\begin{array}{c} \eta_j \\ \zeta_j \end{array} \right) - \left(\begin{array}{c} \tilde{\eta}_j \\ \tilde{\zeta}_j \end{array} \right) \right], \quad i = 1, 2, \quad j = 0, 1, 2, \ldots , \tag{2.2.18}$$

where $Dr_i((\bar{\eta}_j, \bar{\zeta}_j)_i)$ denotes the gradient of r_i respect to η_j, ζ_j at $\left(\begin{array}{c} \bar{\eta}_j \\ \bar{\zeta}_j \end{array} \right)_i$, $i = 1, 2$. Let us take

$$L_1 < \frac{\alpha - \alpha_1}{2M_1 M}, \quad \text{and} \quad L_2 < \frac{1 - \alpha_2 \alpha}{2\alpha_2 M_2 M}.$$

Then since the $r_i(\eta, \zeta)$ are twice continuously differentiable with $Dr_i(0, 0) = 0$, we have for M small enough, and for $\left\{ \left(\begin{array}{c} \bar{\eta}_k \\ \bar{\zeta}_k \end{array} \right)_i \right\} \in Z_\alpha^M$,

$$\begin{aligned} \left\| Dr_i((\bar{\eta}_j, \bar{\zeta}_j)_i) \right\| &\leq L_i \left\| \left(\begin{array}{c} \bar{\eta}_j \\ \bar{\zeta}_j \end{array} \right)_i \right\| \leq L_i \alpha^j \sup_{k=0,1,2,\ldots} \left\| \left(\begin{array}{c} \bar{\eta}_k \\ \bar{\zeta}_k \end{array} \right)_i \alpha^{-k} \right\| \\ &= L_i \alpha^j \left\| \left\{ \left(\begin{array}{c} \bar{\eta}_k \\ \bar{\zeta}_k \end{array} \right)_i \right\} \right\|_\alpha \leq L_i M \alpha^j \leq L_i M, \quad i = 1, 2, \quad j = 0, 1, 2, \ldots . \end{aligned} \tag{2.2.19}$$

Hence we have,

$$\begin{aligned}
\left\| r_i(\eta_j,\zeta_j) - r_i(\tilde{\eta}_j,\tilde{\zeta}_j) \right\| &\le \left\| Dr_i((\bar{\eta}_j,\bar{\zeta}_j)_i) \right\| \left\| \begin{pmatrix} \eta_j \\ \zeta_j \end{pmatrix} - \begin{pmatrix} \tilde{\eta}_j \\ \tilde{\zeta}_j \end{pmatrix} \right\| \\
&\le L_i M \alpha^j \left\| \left[\begin{pmatrix} \eta_j \\ \zeta_j \end{pmatrix} - \begin{pmatrix} \tilde{\eta}_j \\ \tilde{\zeta}_j \end{pmatrix} \right] \alpha^{-j} \right\| \\
&\le L_i M \alpha^j \sup_{k=0,1,2,\ldots} \left\| \left[\begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} - \begin{pmatrix} \tilde{\eta}_k \\ \tilde{\zeta}_k \end{pmatrix} \right] \alpha^{-k} \right\| \\
&\le L_i M \alpha^j \left\| \left\{ \begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} - \begin{pmatrix} \tilde{\eta}_k \\ \tilde{\zeta}_k \end{pmatrix} \right\} \right\|_\alpha, \quad i = 1,2, \quad j = 0,1,2,\ldots . \qquad (2.2.20)
\end{aligned}$$

From (2.2.17) and using (2.2.20) we have

$$\begin{aligned}
&\left\| \left\{ \sum_{j=0}^{k-1} A_1^{k-1-j} \left[r_1(\eta_j,\zeta_j) - r_1(\tilde{\eta}_j,\tilde{\zeta}_j) \right] \right\} \right\|_\alpha \\
= &\sup_{k=0,1,2,\ldots} \left\| \left(\sum_{j=0}^{k-1} A_1^{k-1-j} \left[r_1(\eta_j,\zeta_j) - r_1(\tilde{\eta}_j,\tilde{\zeta}_j) \right] \right) \alpha^{-k} \right\| \\
\le &\sup_{k=0,1,2,\ldots} \frac{L_1 M_1 M}{\alpha} \left\| \left\{ \begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} - \begin{pmatrix} \tilde{\eta}_k \\ \tilde{\zeta}_k \end{pmatrix} \right\} \right\|_\alpha \sum_{j=0}^{k-1} \left(\frac{\alpha_1}{\alpha} \right)^{k-1-j} \\
\le &\frac{L_1 M_1 M}{\alpha - \alpha_1} \left\| \left\{ \begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} - \begin{pmatrix} \tilde{\eta}_k \\ \tilde{\zeta}_k \end{pmatrix} \right\} \right\|_\alpha \qquad (2.2.21)
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \left\{ -\sum_{j=k}^{\infty} A_2^{k-1-j} \left[r_2(\eta_j,\zeta_j) - r_2(\tilde{\eta}_j,\tilde{\zeta}_j) \right] \right\} \right\|_\alpha \\
= &\sup_{k=0,1,2,\ldots} \left\| \left(-\sum_{j=k}^{\infty} A_2^{k-1-j} \left[r_2(\eta_j,\zeta_j) - r_2(\tilde{\eta}_j,\tilde{\zeta}_j) \right] \right) \alpha^{-k} \right\| \\
\le &L_2 M_2 M \left\| \left\{ \begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} - \begin{pmatrix} \tilde{\eta}_k \\ \tilde{\zeta}_k \end{pmatrix} \right\} \right\|_\alpha \sup_{k=0,1,2,\ldots} \left\{ \sum_{j=k}^{\infty} \left(\alpha_2^{j+1-k} \alpha^{j-k} \right) \right\} \\
= &\frac{L_2 M_2 M}{\alpha} \left\| \left\{ \begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} - \begin{pmatrix} \tilde{\eta}_k \\ \tilde{\zeta}_k \end{pmatrix} \right\} \right\|_\alpha \sup_{k=0,1,2,\ldots} \left\{ \sum_{j=k}^{\infty} (\alpha_2 \alpha)^{j+1-k} \right\}
\end{aligned}$$

$$= \frac{L_2 M_2 M \alpha_2}{1-\alpha_2\alpha} \left\| \left\{ \begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} - \begin{pmatrix} \tilde{\eta}_k \\ \tilde{\zeta}_k \end{pmatrix} \right\} \right\|_\alpha . \tag{2.2.22}$$

Let

$$\gamma = \left(\frac{L_1 M_1 M}{\alpha - \alpha_1} + \frac{L_2 M_2 M \alpha_2}{1-\alpha_2 \alpha} \right).$$

Then $\gamma \in (0,1)$ by the way we chose L_1 and L_2. Hence in Z_α^M we have

$$\left\| \left\{ F_a(\eta_k, \zeta_k) - F_a(\tilde{\eta}_k, \tilde{\zeta}_k) \right\} \right\|_\alpha \leq \gamma \left\| \left\{ \begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} - \begin{pmatrix} \tilde{\eta}_k \\ \tilde{\zeta}_k \end{pmatrix} \right\} \right\|_\alpha ,$$

when $\left\{ \begin{matrix} \eta_k \\ \zeta_k \end{matrix} \right\}, \left\{ \begin{matrix} \tilde{\eta}_k \\ \tilde{\zeta}_k \end{matrix} \right\} \in Z_\alpha^M$. Therefore, F_a is a contraction mapping in Z_α^M, and we conclude that (2.2.8) has a unique solution $\left\{ \begin{matrix} \eta_k(a) \\ \zeta_k(a) \end{matrix} \right\} \in Z_\alpha^M$ for sufficiently small a.

We now show that a solution of (2.2.8) provides a solution of (2.2.5), (2.2.6), (2.2.7), and vice versa.

From (2.2.5) we have

$$\begin{aligned} \eta_k(a) &= A_1 \eta_{k-1}(a) + r_1(\eta_{k-1}(a), \zeta_{k-1}(a)) \\ &= A_1 \left[A_1 \eta_{k-2}(a) + r_1(\eta_{k-2}(a), \zeta_{k-2}(a)) \right] + r_1(\eta_{k-1}(a), \zeta_{k-1}(a)) \\ &= \cdots \\ &= A_1^k \eta_0(a) + A_1^{k-1} r_1(\eta_0(a), \zeta_0(a)) + \cdots \\ &\quad + A_1 r_1(\eta_{k-2}(a), \zeta_{k-2}(a)) + r_1(\eta_{k-1}(a), \zeta_{k-1}(a)), \end{aligned}$$

so that

$$\eta_k(a) = A_1^k a + \sum_{j=0}^{k-1} A_1^{k-1-j} r_1(\eta_j(a), \zeta_j(a)),$$

by using (2.2.7). Also, from (2.2.5), we have

$$\zeta_l(a) = A_2^{l-k}\zeta_k(a) + \sum_{j=k}^{l-1} A_2^{l-1-j} r_2(\eta_j(a), \zeta_j(a)); \tag{2.2.23}$$

multiplication of (2.2.23) by $(A_2^{l-k})^{-1}$ yields

$$\zeta_k(a) = (A_2^{-1})^{l-k}\zeta_l(a) - \sum_{j=k}^{l-1} A_2^{k-1-j} r_2(\eta_j(a), \zeta_j(a)). \tag{2.2.24}$$

If we let $l \to \infty$, then

$$\zeta_k(a) = -\sum_{j=k}^{\infty} A_2^{k-1-j} r_2(\eta_j(a), \zeta_j(a)).$$

Therefore, a solution of (2.2.5), (2.2.6) and (2.2.7) is a solution of (2.2.8).

Conversely, from the first equation of (2.2.8), we have

$$\begin{aligned}
\eta_{k+1}(a) &= A_1^{k+1} a + \sum_{j=0}^{k} A_1^{k-j} r_1(\eta_j(a), \zeta_j(a)) \\
&= A_1 \left[A_1^k a + \sum_{j=0}^{k-1} A_1^{k-j-1} r_1(\eta_j(a), \zeta_j(a)) \right] + r_1(\eta_k(a), \zeta_k(a)) \\
&= A_1 \eta_k(a) + r_1(\eta_k(a), \zeta_k(a))
\end{aligned}$$

and

$$\eta_0(a) = A_1^0 a = a.$$

From the second equation of (2.2.8),

$$\begin{aligned}\zeta_{k+1}(a) &= A_2\left[-\sum_{j=k}^{\infty} A_2^{k-1-j} r_2(\eta_j(a),\zeta_j(a))\right] + r_2(\eta_k(a),\zeta_k(a)) \\ &= A_2\zeta_k(a) + r_2(\eta_k(a),\zeta_k(a)),\end{aligned}$$

and

$$\lim_{k\to\infty}\eta_k(a) = \lim_{k\to\infty} A_1^k a + \lim_{k\to\infty}\sum_{j=0}^{k-1} A_1^{k-1-j} r_1(\eta_j(a),\zeta_j(a)) = 0,$$

because A_1 has all eigenvalues with modulus less than one and $r_1(\eta_j(a),\zeta_j(a)) = o\begin{pmatrix}\eta_j(a)\\ \zeta_j(a)\end{pmatrix}$, as $\begin{pmatrix}\eta_j(a)\\ \zeta_j(a)\end{pmatrix} \to 0$. Since A_2^{k-1-j} is bounded for each fixed k, and $r_2(\eta_j(a),\zeta_j(a)) = o\begin{pmatrix}\eta_j(a)\\ \zeta_j(a)\end{pmatrix}$, as $\begin{pmatrix}\eta_j(a)\\ \zeta_j(a)\end{pmatrix} \to 0$, becomes small when j is large, then we have,

$$\lim_{k\to\infty}\zeta_k(a) = -\lim_{k\to\infty}\sum_{j=k}^{\infty} A_2^{k-1-j} r_2(\eta_j(a),\zeta_j(a)) = 0.$$

Hence, a solution of (2.2.8) in Z_α^M is a solution of (2.2.5), (2.2.6) and (2.2.7). Thus we see that (2.2.5), (2.2.6), (2.2.7) and (2.2.8) are equivalent.

For $k = 0$, (2.2.8) gives us

$$\begin{aligned}\eta_0(a) &= a, \\ \zeta_0(a) &= -\sum_{j=0}^{\infty} A_2^{-1-j} r_2(\eta_j(a),\zeta_j(a)).\end{aligned} \tag{2.2.25}$$

If we define

$$\Psi(a) = \zeta_0(a) = -\sum_{j=0}^{\infty} A_2^{-1-j} r_2(\eta_j(a),\zeta_j(a)), \tag{2.2.26}$$

then initial conditions of the form $\begin{pmatrix} a \\ \Psi(a) \end{pmatrix} = \begin{pmatrix} \eta_0 \\ \Psi(\eta_0) \end{pmatrix}$ have the property that $\lim_{k\to\infty} \begin{pmatrix} \eta_k(a) \\ \zeta_k(a) \end{pmatrix} = 0$. This defines a manifold $\zeta = \Psi(\eta)$. The continuous differentiability of $\Psi(a)$ with respect to a for a near 0 follows from term by term differentiation of the right hand side of (2.2.26) together with (2.2.9) to establish the uniform convergence of the differentiated series.

Next we show that the manifold $S_0 : \ \zeta = \Psi(\eta)$ is positive invariant. Suppose that $\begin{Bmatrix} \eta_k(\eta_0) \\ \zeta_k(\eta_0) \end{Bmatrix}$ is a solution starting at $\begin{pmatrix} \eta_0 \\ \Psi(\eta_0) \end{pmatrix}$; to show that $\begin{pmatrix} \eta_k(\eta_0) \\ \zeta_k(\eta_0) \end{pmatrix} \in S_0$, $k = 0, 1, 2, \dots$, we have to show that $\zeta_k(\eta_0) = \Psi(\eta_k(\eta_0))$, $k = 0, 1, 2, \dots$. From (2.2.26):

$$\Psi(\eta_k(\eta_0)) = -\sum_{j=0}^{\infty} A_2^{-1-j} r_2(\eta_j(\eta_k(\eta_0)), \zeta_j(\eta_k(\eta_0))), \quad k = 0, 1, 2, \dots. \tag{2.2.27}$$

With $i = j + k$, $k = 0, 1, 2, \dots$, (2.2.27) has the form:

$$\begin{aligned} \Psi(\eta_k(\eta_0)) &= -\sum_{i=k}^{\infty} A_2^{k-1-i} r_2(\eta_{i-k}(\eta_k(\eta_0)), \zeta_{i-k}(\eta_k(\eta_0)) \\ &= -\sum_{i=k}^{\infty} A_2^{k-1-i} r_2(\eta_i(\eta_0), \zeta_i(\eta_0)) \\ &= \zeta_k(\eta_0), \end{aligned}$$

so that S_0 is positive invariant. Since

$$\begin{pmatrix} \eta_k \\ \zeta_k \end{pmatrix} = \begin{pmatrix} I_n & -P \\ 0 & I_n \end{pmatrix} \begin{pmatrix} y_k \\ z_k \end{pmatrix}, \quad k = 0, 1, 2, \dots$$

we have

$$\eta_k = y_k - Pz_k, \quad k = 0, 1, 2, \dots, \tag{2.2.28}$$

then the equation describing S_0 in the y, z vaviables becomes:

$$z = \Psi(y - Pz). \tag{2.2.29}$$

Clearly (2.2.29) can be equivalently written as

$$H(y, z) \equiv z - \Psi(y - Pz) = 0.$$

Then

$$\frac{\partial H}{\partial z} = I_n + \frac{\partial \Psi}{\partial \eta}(y - Pz)P$$

and, since $\frac{\partial \Psi}{\partial \eta}(y - Pz)P$ is small for (y, z) near $(0, 0)$, by the implicit function theorem there exists a continuously differentiable function Φ such that $z = \Phi(y)$ gives us the manifold S. Since S_0 is positive invariant for the system in the (η, ζ) variables, then S will be invariant for the same system expressed in the (y, z) variables and the theorem is proved.

Corollary 2.6 (for the Analytic Case) *The Stable Manifold Theorem for the Discrete System (Theorem 2.5) remains valid and the manifold S is analytic if '$\mathcal{F}$ of class C^2' is replaced by '$\mathcal{F}$ is an analytic function of ξ near the origin in $\mathbf{R}^{2n}$'.*

The proof follows from uniform convergence of (2.2.26) for a near 0 together with the standard result on analyticity of the uniform limit of analytic functions [31].

2.3 The Application of the Stable Manifold Theorem to the State / Adjoint System for a General Discrete System

Let us recall that the general nonlinear discrete system (2.1.1) with its adjoint equation can be written as (2.1.70) and (2.1.71) on page 37 by changing of variables and collecting the linear terms

$$\begin{aligned}\hat{x}_{k+1} &= (A+BK)\hat{x}_k + V\left[(A+BK)^{-1}\right]^* q_k + f_1(\hat{x}_k, q_k), \quad (2.3.1)\\ &\qquad k = 0,1,2,\dots,\\ q_{k+1} &= \left[(A+BK)^{-1}\right]^* q_k + f_2(\hat{x}_k, q_k),\ k = 0,1,2,\dots. \quad (2.3.2)\end{aligned}$$

Since the matrix $\hat{A} = A + BK$ is nonsingular and has all eigenvalues with modulus less than one; then correspondingly, the inverse matrix $\hat{A}^{-1} = (A+BK)^{-1}$ exists and has all eigenvalues with modulus greater than one. Applying the stable manifold theorem, Theorem 2.5, for the discrete system, there exists a function Φ such that $q = \Phi(\hat{x})$, with $\Phi(x) = o(\|x\|)$ in the variables $(\hat{x}, q)$, describes the n-dimensional invariant manifold S in which the origin is asymptotically stable. Solutions of the state / adjoint system (2.3.1) and (2.3.2) lying in S thus satisfy

$$\begin{aligned}\hat{x}_{k+1} &= (A+BK)\hat{x}_k + V\left[(A+BK)^{-1}\right]^* \Phi(\hat{x}_k) + f_1(\hat{x}_k, \Phi(\hat{x}_k)), \quad (2.3.3)\\ &\qquad k = 0,1,2,\dots,\\ q_{k+1} &= \left[(A+BK)^{-1}\right]^* \Phi(\hat{x}_k) + f_2(\hat{x}_k, \Phi(\hat{x}_k)),\ k = 0,1,2,\dots, \quad (2.3.4)\end{aligned}$$

where V is defined in (2.1.63) and K is defined in (2.1.64) or (2.1.65), i.e., K is a solution of the following equation,

$$K = -U^{-1}\left[R^* + B^*P(A + BK)\right], \tag{2.3.5}$$

or equivalently,

$$K = -\left(U + B^*PB\right)^{-1}\left(R^* + B^*PA\right). \tag{2.3.6}$$

From (2.1.59), $p = q + 2Px$, where P is the unique symmetric positive definite solution of (2.1.54); thus by using (2.3.3) and (2.3.4), we have

$$\begin{aligned}
p_{k+1} &= q_{k+1} + 2P\hat{x}_{k+1} \\
&= 2P(A+BK)\hat{x}_k + 2PV\left[(A+BK)^{-1}\right]^*\Phi(\hat{x}_k) \\
&\quad + \left[(A+BK)^{-1}\right]^*\Phi(\hat{x}_k) + 2Pf_1(\hat{x}_k,\Phi(\hat{x}_k)) + f_2(\hat{x}_k,\Phi(\hat{x}_k)) \\
&= 2P(A+BK)\hat{x}_k + (2PV + I_n)\left[(A+BK)^{-1}\right]^*\Phi(\hat{x}_k) \\
&\quad + 2Pf_1(\hat{x}_k,\Phi(\hat{x}_k)) + f_2(\hat{x}_k,\Phi(\hat{x}_k)), \\
&= 2P(A+BK)\hat{x}_k + r_p(\hat{x}_k), \\
&\equiv p(\hat{x}_k), \quad k = 0,1,2,\dots,
\end{aligned} \tag{2.3.7}$$

where the remainder term $r_p(x)$ defined as,

$$r_p(x) \equiv (2PV + I_n)\left[(A+BK)^{-1}\right]^*\Phi(x) + 2Pf_1(x,\Phi(x)) + f_2(x,\Phi(x)), \tag{2.3.8}$$

satisfies $r_p(x) = o(\|x\|)$, as $\|x\| \to 0$. From (2.1.69), the optimal control $\hat{u} = \{\hat{u}_k\}$ can be written as

$$\hat{u}_k = -\frac{1}{2}U^{-1}\left(2R^*\hat{x}_k + B^*p_{k+1}\right) + h(\hat{x}_k, p_{k+1}), \quad k = 0, 1, 2, \ldots. \tag{2.3.9}$$

Substituting (2.3.7) for p_{k+1} into (2.3.9), we have

$$\begin{aligned}\hat{u}_k &= -U^{-1}\left(R^*\hat{x}_k + B^*P(A+BK)\hat{x}_k\right) - \frac{1}{2}U^{-1}B^*r_p(\hat{x}_k) \\ &\quad + h(\hat{x}_k, 2P(A+BK)\hat{x}_k + r_p(\hat{x}_k)) \\ &= -U^{-1}\left(R^* + B^*P(A+BK)\right)\hat{x}_k + r_u(\hat{x}_k),\end{aligned} \tag{2.3.10}$$

where $r_u(x) = o(\|x\|)$, as $\|x\| \to 0$, is a higher order term because it is defined as

$$r_u(x) \equiv -\frac{1}{2}U^{-1}B^*r_p(x) + h(x, 2P(A+BK)x + r_p(x)). \tag{2.3.11}$$

If we compare the coefficient of the linear term in (2.3.10) with (2.3.5), we have

$$\hat{u}_k = K\hat{x}_k + r_u(\hat{x}_k). \tag{2.3.12}$$

Substituting (2.3.6) for K into the above equation, $\hat{u}_k$ has the following form:

$$\hat{u}_k = -\left(U + B^*PB\right)^{-1}\left(R^* + B^*PA\right)\hat{x}_k + r_u(\hat{x}_k), \quad k = 0, 1, 2, \ldots. \tag{2.3.13}$$

It is easy to see that the coefficient of the linear term of this nonlinear system is the same as in the linear discrete case (cf. (2.1.53)). The resulting optimal closed loop system then takes the form

$$\hat{x}_{k+1} = (A+BK)\hat{x}_k + r_f(\hat{x}_k), \quad k = 0, 1, 2, \ldots, \tag{2.3.14}$$

where we define $r_f(x)$ as

$$r_f(x) \equiv Br_u(x) + f(x, (A + BK)x + r_u(x)), \tag{2.3.15}$$

again a higher order term, that is $r_f(x) = o(\|x\|)$, as $\|x\| \to 0$. If the solution $\hat{x} = \{\hat{x}_k\}$ has initial states $x_0 \in S$, the n - dimensional invariant manifold, we will have $\lim_{k\to\infty} \|\hat{x}_k\| = 0$ by the stable manifold theorem. Hence we have the nonlinear optimal feedback control (2.3.13) and the closed loop system (2.3.14) for the general discrete system.

At this point we have shown only theoretically that the optimal control has the form $\hat{u} = \{\hat{u}_k\}$, $\hat{u}_k = \hat{u}(\hat{x}_k)$. Using this method to solve for the higher term of $\hat{u}$ appears rather impractical. We will use a different method to solve our previous example later.

2.4 The Existence and Uniqueness of the Optimal Control for General Discrete Systems

In this section we prove the existence and uniqueness of the optimal control of the discrete control system:

$$x_{k+1} = F(x_k, u_k) = Ax_k + Bu_k + f(x_k, u_k), \quad k = 0, 1, 2, \dots, \tag{2.4.1}$$

with cost function

$$\begin{aligned} J(x_0, u) &= \sum_{k=0}^{\infty} G(x_k, u_k) \\ &= \sum_{k=0}^{\infty} \left[(x_k{}^*, u_k{}^*) \begin{pmatrix} W & R \\ R^* & U \end{pmatrix} \begin{pmatrix} x_k \\ u_k \end{pmatrix} + g(x_k, u_k) \right], \end{aligned} \tag{2.4.2}$$

where $x = \{x_k\} \in \mathbf{R}^n$ and $u = \{u_k\} \in \mathbf{R}^m$, and A, B, W, U and R are real matrices and $f(x, u)$ and $g(x, u)$ are higher order terms as made precise earlier. We assume that $F(x, u)$, $G(x, u)$ are defined on some neighborhood of the origin in $\mathbf{R}^{n+m}$ and are at least of class C^2 there with

$$F(0,0) = 0, \;\; G(0,0) = 0, \;\; \frac{\partial G}{\partial x}(0,0) = 0, \;\; \frac{\partial G}{\partial u}(0,0) = 0.$$

Also we assume that $\begin{pmatrix} W & R \\ R^* & U \end{pmatrix}$ is a real symmetric positive definite matrix.

As in the differential equation case in Chapter 1, we consider the class of closed loop, i.e., feedback, controls for the discrete case which are of the form

$$u_k = u(x_k) = Kx_k + h(x_k), \quad k = 0, 1, 2, \dots, \tag{2.4.3}$$

where $h(x) = o(\|x\|)$, as $\|x\| \to 0$. The real matrices K are selected so that the matrix $A + BK$ should have all eigenvalues with modulus less than one. Then $x = 0$ is a locally asymptotically stable critical point for the nonlinear system

$$x_{k+1} = F(x_k, u(x_k)) = (A+BK)x_k + Bh(x_k) + f(x_k, u(x_k)), \; k = 0, 1, 2, \ldots . \quad (2.4.4)$$

Our fundamental hypothesis here is that such matrix K does exist, i.e., that the linear part of (2.4.1) is stabilizable. We refer to a control (2.4.3), with K as specified, as a *discrete stabilizing control* for the system (2.4.1).

Theorem 2.7 *For the discrete stabilizable control process (2.4.1) in* $\mathbf{R}^n$ *with cost function (2.4.2), there exists an optimal discrete stabilizing feedback control* $\hat{u} = \{\hat{u}_k\}$. *The optimal control solves the functional equation*

$$\frac{\partial \hat{J}}{\partial x}(\hat{x}_{k+1})\frac{\partial F}{\partial u}(\hat{x}_k, \hat{u}(\hat{x}_k)) + \frac{\partial G}{\partial u}(\hat{x}_k, \hat{u}(\hat{x}_k)) = 0, \quad k = 0, 1, 2, \ldots , \quad (2.4.5)$$

where we define $\hat{J}(x) \equiv J(x, \hat{u}(x))$, *if* $\hat{x} = \{\hat{x}_k\}$, $\hat{u} = \{\hat{u}_k = \hat{u}(\hat{x}_k)\}$, *are an optimal trajectory and control pair for the general discrete control system (2.4.1) and the cost function (2.4.2).* $\hat{J}(x_0)$ *is the minimal cost for all* x_0 *near the origin and is unique in that:*

1. $\hat{u} = \{\hat{u}_k\}$ *is the unique solution to (2.4.5);*

2. $\hat{u} = \{\hat{u}_k\}$ *is the unique optimal discrete stabilizing feedback control;*

3. $\hat{u} = \{\hat{u}_k\}$ *synthesizes the unique optimal open-loop control.*

Furthermore,

$$\hat{u}_k = \hat{K}\hat{x}_k + \hat{h}(\hat{x}_k), \quad k = 0, 1, 2, \dots ,$$

where the linear part is given by matrices $\hat{K}$ *and*

$$\hat{J}(x_0) = {x_0}^* P x_0 + \hat{j}(x_0),$$

here $P > 0$ *depending upon only the matrices* A*,* B*,* W*,* U*, and* R*, and* $\hat{j}(x)$ *is a function of class* C^2 *with* $\hat{j}(x) = o(\|x\|^2)$*, as* $\|x\| \to 0$*.*

We begin our proof of this theorem and the uniqueness of Theorem 2.4 on page 35 with several lemmas. Let us define P_K first as follows:

$$P_K = \sum_{k=0}^{\infty}((A+BK)^k)^*(W + RK + K^*R^* + K^*UK)(A+BK)^k, \tag{2.4.6}$$

then $P_K > 0$ since $W + RK + K^*R^* + K^*UK > 0$, which follows directly from $\begin{pmatrix} W & R \\ R^* & U \end{pmatrix} > 0$, and P_K satisfies the matrix equation (2.1.54) on page 32,

$$P_K = W + RK + K^*R^* + K^*UK + (A+BK)^*P_K(A+BK). \tag{2.4.7}$$

Lemma 2.8 *For each discrete stabilizing control* $u = \{u_k\}$*,*

$$u_k = u(x_k) = Kx_k + h(x_k), \quad k = 0, 1, 2, \dots ,$$

there exists a positive invariant neighborhood N_u *of the origin in* $\mathbf{R}^n$ *wherein the* *associated cost function takes the form*

$$J(x_0, \{u_k\}) = {x_0}^* P_K x_0 + j(x_0), \tag{2.4.8}$$

where P_K is given by (2.4.6) and $j(x)$ is twice continuously differentiable with $j(x) = o(\|x\|^2)$, as $\|x\| \to 0$. In N_u the functional equation

$$J\left(F(x_k, u_k), \{u_k\}_{j>k}\right) - J\left(x_k, \{u_k\}_{j\geq k}\right) + G(x_k, u_k) = 0, \ k = 0, 1, 2, \dots , \tag{2.4.9}$$

obtains.

Proof. Let $u = \{u_k\}$,

$$u_k = u(x_k) = Kx_k + h(x_k), \quad k = 0, 1, 2, \dots ,$$

be a discrete stabilizing control for the general discrete control system (2.4.1) and the cost function (2.4.2) with the corresponding trajectory $x = \{x_k\}$. If we define

$$N_u = \{x_k : {x_k}^* P_K x_k < \epsilon\}$$

for sufficiently small $\epsilon > 0$, where P_K is defined in (2.4.6), from (2.4.4) in N_u we have

$$\begin{aligned} {x_{k+1}}^* P_K x_{k+1} &= {x_k}^*(A + BK)^* P_K (A + BK) x_k \\ &\quad + \left(Bh(x_k) + f(x_k, u(x_k))\right)^* P_K (A + BK) x_k \\ &\quad + {x_k}^*(A + BK)^* P_K \left(Bh(x_k) + f(x_k, u(x_k))\right) \\ &\quad + \left(Bh(x_k) + f(x_k, u(x_k))\right)^* P_K \left(Bh(x_k) + f(x_k, u(x_k))\right), \end{aligned} \tag{2.4.10}$$

for $k = 0, 1, 2, \ldots$. By using (2.4.7) and defining the last three terms of (2.4.10) be $\bar{f}(x_k)$, a function of x_k of order higher than two, we obtain

$$x_{k+1}{}^* P_K x_{k+1} = x_k{}^* P_K x_k - x_k{}^* \left(W + RK + K^* R^* + K^* UK\right) x_k + \bar{f}(x_k). \quad (2.4.11)$$

Since $\bar{f}(x) = o(\|x\|^2)$ as $\|x\| \to 0$, and $W + RK + K^*R^* + K^*UK > 0$, $\bar{f}(x_k)$ is smaller than the quadratic term when x_k is small in (2.4.11), i.e.,

$$x_k{}^* \left(W + RK + K^* R^* + K^* UK\right) x_k - \bar{f}(x_k) > 0, \quad k = 0, 1, 2, \ldots ,$$

hence (2.4.11) yields

$$x_{k+1}{}^* P_K x_{k+1} \leq x_k{}^* P_K x_k, \quad k = 0, 1, 2, \ldots . \quad (2.4.12)$$

Therefore, each solution of (2.4.4) initiating at $x_0 \in N_u$ remains in N_u, i.e., N_u is invariant. It is easy to see that

$$x_k(x_0) = (A + BK)^k x_0 + (\text{higher terms}), \quad k = 0, 1, 2, \ldots , \quad (2.4.13)$$

and

$$u_k(x_0) = K(A + BK)^k x_0 + (\text{higher terms}), \quad k = 0, 1, 2, \ldots . \quad (2.4.14)$$

Since $u = \{u_k = Kx_k + h(x_k)\}$ is a discrete stabilizing control, the matrix $A + BK$ has all eigenvalues with modulus less than one, hence there exists $0 < \gamma < 1$ and positive numbers C_1, C_2 and C_3, such that for $k = 0, 1, 2, \ldots$, we have

$$\|x_k(x_0)\| \leq C_1 \gamma^k \|x_0\|,$$

$$\|u_k(x_0)\| \leq C_2\gamma^k\|x_0\|,$$

and

$$\|G(x_k, u(x_k))\| \leq C_3\gamma^{2k}\|x_0\|^2,$$

for all $x = \{x_k\}$, $x_k \in N_u$, $k = 0, 1, 2, \dots$, and hence

$$J(x_0, u) = \sum_{k=0}^{\infty} G(x_k, u(x_k)) < \infty.$$

By using (2.4.13) and (2.4.14) we see that, in N_u, the cost function takes the form

$$\begin{aligned}
& J(x_0, u) = \sum_{k=0}^{\infty} G(x_k, u_k) \\
= \; & \sum_{k=0}^{\infty} [x_k{}^* W x_k + u_k{}^* R^* x_k + x_k{}^* R u_k + u_k{}^* U u_k + g(x_k, u_k)] \\
= \; & x_0{}^* \left[\sum_{k=0}^{\infty} \left((A + BK)^k \right)^* (W + RK + K^*R^* + K^*UK)(A + BK)^k \right] x_0 \\
& + \sum_{k=0}^{\infty} g(x_k(x_0), u_k(x_0)) \\
\equiv \; & x_0{}^* P_K x_0 + j(x_0).
\end{aligned} \tag{2.4.15}$$

Since $G(x, u)$ is class of C^2,

$$\|g(x_k(x_0), u_k(x_0))\| \leq C_4(\|x_k(x_0)\|)^2\beta \leq C_4 C_1^2 \gamma^{2k} \beta \|x_0\|^2, \quad k = 0, 1, 2, \dots ,$$

where $\beta = o(\|x_0\|)$ as $\|x_0\| \to 0$, and C_4 is a positive number, we have

$$\|j(x_0)\| \leq \sum_{k=0}^{\infty} \|g(x_k(x_0), u_k(x_0)\| \leq C_4 C_1^2 \frac{\beta}{1 - \gamma^2} \|x_0\|^2.$$

Hence $j(x) = o(\|x\|^2)$, as $\|x\| \to 0$ in N_u.

If we define $J(x_0) \equiv J(x_0, \{u(x_k(x_0))\})$, for $x_0 \in N_u$, for arbitrary, then it is easy to see that

$$J(x_k) = G(x_k, u_k) + J(x_{k+1}), \quad k = 0, 1, 2, \ldots. \tag{2.4.16}$$

Therefore, we have (2.4.9) in N_u for $x = \{x_k\}$, $u = \{u_k\}$, $u_k = u(x_k)$, $k = 0, 1, 2, \ldots$ satisfying (2.4.1).

Lemma 2.9 *There exists a unique solution $\hat{u} = \{\hat{u}_k\}$, $\hat{u}_k = \hat{u}(x_k, p_{k+1})$, $k = 0, 1, 2, \ldots$, to the equation*

$$p_{k+1}{}^* \frac{\partial F}{\partial u}(x_k, u_k) + \frac{\partial G}{\partial u}(x_k, u_k) = 0, \quad k = 0, 1, 2, \ldots, \tag{2.4.17}$$

near the origin in $\mathbf{R}^{2n}$for which $\hat{u}(0,0) = 0$. Furthermore, this unique solution can be written as

$$\hat{u}_k = \hat{u}(x_k, p_{k+1}) = -\frac{1}{2}U^{-1}(2R^* x_k + B^* p_{k+1}) + \hat{h}(x_k, p_{k+1}), \quad k = 0, 1, 2, \ldots, \tag{2.4.18}$$

where $\hat{h}(x_k, p_{k+1})$ is a function in $\mathbf{R}^{2n}$ such that $\hat{h}(x,p) = o(\|x\| + \|p\|)$, as $\|x\| \to 0$ and $\|p\| \to 0$.

Proof. From (2.4.1) and (2.4.2) we have for $k = 0, 1, 2, \ldots$,

$$\begin{aligned} & p_{k+1}{}^* F(x_k, u_k) + G(x_k, u_k) \\ = \; & p_{k+1}{}^*(Ax_k + Bu_k + f(x_k, u_k)) + x_k{}^* W x_k + 2x_k{}^* R u_k + u_k{}^* U u_k + g(x_k, u_k), \end{aligned}$$

and it is easy to see that at the point $x_k = p_{k+1} = 0$, $u_k = 0$,

$$p_{k+1}{}^* \frac{\partial F}{\partial u}(x_k, u_k) + \frac{\partial G}{\partial u}(x_k, u_k) = 0, \tag{2.4.19}$$

and

$$\frac{\partial}{\partial u}\left[p_{k+1}{}^* \frac{\partial F}{\partial u}(x_k, u_k) + \frac{\partial G}{\partial u}(x_k, u_k)\right]^* = 2U > 0. \tag{2.4.20}$$

Since $f(x,u)$ and $g(x,u)$ are at least of class C^2, if we define

$$r(x_k, p_{k+1}, u_k) \equiv p_{k+1}{}^* \frac{\partial f}{\partial u}(x_k, u_k) + \frac{\partial g}{\partial u}(x_k, u_k)$$

then $r(x,p,u)$ is at least of class C^1 and $r(x,p,u) = o(\|x\| + \|p\| + \|u\|)$, as $\|x\| \to 0$, $\|p\| \to 0$ and $\|u\| \to 0$, we can compute the linear terms of $\hat{u}(x_k, p_{k+1})$ from (2.4.19) to get

$$p_{k+1}{}^* B + 2x_k{}^* R + 2u_k{}^*(x_k, p_{k+1})U + r(x_k, p_{k+1}, u_k) = 0.$$

Application of the implicit function theorem, there exists a unique solution $\hat{u} = \{\hat{u}_k\}$, $\hat{u}_k = \hat{u}(x_k, p_{k+1})$, $k = 0, 1, 2, \ldots$, which shows that for (2.4.19)

$$\hat{u}(x_k, p_{k+1}) = -\frac{1}{2}U^{-1}(2R^* x_k + B^* p_{k+1}) + \hat{h}(x_k, p_{k+1}),$$

where $\hat{h}(x_k, p_{k+1}) = -\frac{1}{2}U^{-1} r(x_k, p_{k+1})$ is of class C^1 and $\hat{h}(x,p) = o(\|x\| + \|p\|)$, as $\|x\| \to 0$ and $\|p\| \to 0$. Therefore, we have (2.4.18) and the lemma is proved.

Lemma 2.10 *Suppose there exists a discrete stabilizing feedback control* $\hat{u} = \{\hat{u}_k\}$,

$$\hat{u}_k = \hat{u}(x_k) = \hat{K} x_k + \hat{h}(x_k), \quad k = 0, 1, 2, \ldots,$$

for the discrete control system (2.4.1) and the cost function (2.4.2), which satisfy the nonlinear functional equation

$$\frac{\partial \hat{J}}{\partial x}(F(x_k, \hat{u}(x_k)))\frac{\partial F}{\partial u}(x_k, \hat{u}(x_k)) + \frac{\partial G}{\partial u}(x_k, \hat{u}(x_k)) = 0, \tag{2.4.21}$$

where we define

$$\hat{J}(x) \equiv J(x, \hat{u}(x)),$$

for all $x = \{x_k\}$ near the origin in $\mathbf{R}^n$. Then:

1. *$\hat{u} = \{\hat{u}_k\}$ is the unique solution to (2.4.21) near the origin in $\mathbf{R}^m$;*

2. *$\hat{u} = \{\hat{u}_k\}$ is the unique optimal discrete stabilizing feedback control;*

3. *$\hat{u} = \{\hat{u}_k\}$ synthesizes the unique optimal open-loop control.*

Furthermore,

$$\hat{K} = -(U + B^*PB)^{-1}(R^* + B^*PA)$$

and

$$\hat{J}(x_k) = x_k{}^*Px_k + \hat{\jmath}(x_k),$$

where $P > 0$ is the unique solution of (2.1.68) and $\hat{\jmath}(x)$ is twice continuously differentiable with $\hat{\jmath}(x) = o(\|x\|^2)$, as $\|x\| \to 0$, which described in Lemma 2.8.

Proof. Let us consider the real valued function defined near the origin in $\mathbf{R}^{n+m}$,

$$Q(x, u) = \hat{J}(F(x, u)) - \hat{J}(x) + G(x, u), \tag{2.4.22}$$

where we define

$$\hat{J}(x) = J(x, \hat{u}(x)).$$

By Lemma 2.8, for $x = \{x_k\}$ near the origin in $\mathbf{R}^n$, we have

$$Q(x_k, \hat{u}(x_k)) \equiv 0, \quad k = 0, 1, 2, \dots, \tag{2.4.23}$$

and our hypothesis asserts that

$$\frac{\partial Q}{\partial u}(x_k, \hat{u}(x_k)) \equiv 0, \quad k = 0, 1, 2, \dots, \tag{2.4.24}$$

near the origin in $\mathbf{R}^n$. Compute the Hessian

$$\frac{\partial^2 Q}{\partial u^2}(0, 0) = 2U > 0. \tag{2.4.25}$$

Hence there exists an $\epsilon > 0$, such that

$$0 = Q(x_k, \hat{u}(x_k)) \leq Q(x_k, \bar{u}_k), \quad k = 0, 1, 2, \dots;$$

that is

$$\begin{aligned} 0 &= \hat{J}(F(x_k, \hat{u}(x_k))) - \hat{J}(x_k) + G(x_k, \hat{u}(x_k)) \\ &\leq \hat{J}(F(x_k, \bar{u}_k)) - \hat{J}(x_k) + G(x_k, \bar{u}_k), \quad k = 0, 1, 2, \dots, \end{aligned}$$

provided $\|x_k\| < \epsilon$ and $\|\bar{u}_k\| < \epsilon$, $k = 0, 1, 2, \dots$, for $\bar{u} = \{\bar{u}_k\}$. Moreover, strict inequality holds for $\bar{u} \neq \hat{u}(x)$. We take ϵ sufficiently small so that $G(x, u) \geq 0$.

Now let $\bar{u} \not\equiv \hat{u}$, i.e., $\bar{u}_k \neq \hat{u}_k$ for some $k \in \{0, 1, 2, \dots\}$, be a discrete stabilizing feedback control for (2.4.1) and let $\bar{N}$ be a neighborhood of the origin in $\mathbf{R}^n$ such that $\|x_k\| \leq \epsilon$ and $\|\bar{u}(x_k)\| \leq \epsilon$, $k = 0, 1, 2, \dots,$ for a sufficiently small $\epsilon > 0$, in $\bar{N}$ and each response $\hat{x} = \{\hat{x}_k\}$ or $\bar{x} = \{\bar{x}_k\}$ to the corresponding feedback control $\hat{u} = \{\hat{u}_k\}$ or $\bar{u} = \{\bar{u}_k\}$, respectively, which initiates in some neighborhood $N \subseteq \bar{N}$ of the origin remains in $\bar{N}$. Hence for all initial $x_0 \in \bar{N}$,

$$
\begin{aligned}
0 &\leq \sum_{k=0}^{\infty} \left[\hat{J}(F(\bar{x}_k, \bar{u}(\bar{x}_k))) - \hat{J}(\bar{x}_k) + G(\bar{x}_k, \bar{u}(\bar{x}_k))\right] \\
&= \sum_{k=0}^{\infty} \left[\hat{J}(\bar{x}_{k+1}) - \hat{J}(\bar{x}_k)\right] + \sum_{k=0}^{\infty} G(\bar{x}_k, \bar{u}(\bar{x}_k)) \\
&= -\hat{J}(x_0) + \sum_{k=0}^{\infty} G(\bar{x}_k, \bar{u}(\bar{x}_k)),
\end{aligned}
$$

where is

$$\hat{J}(x_0) \leq J(x_0, \{\bar{u}_k\}),$$

and strict inequality holds provided $\bar{u}(x_0) \neq \hat{u}(x_0)$. Therefore, $\hat{u} = \{\hat{u}_k\}$ is the unique optimal feedback control and the unique solution of the function equation (2.4.21). Now choose a neighborhood $\hat{N} \subseteq \bar{N}$ of the origin which is positive invariant for the responses to the optimal control $\hat{u} = \{\hat{u}_k\}$. Choose $\hat{N}$ so small that $\|\hat{x}_k\| \leq \epsilon$, $k = 0, 1, 2, \dots,$ for all initial conditions in $\hat{N}$. Now let x_0 be an arbitrary fixed initial condition in $\hat{N}$ and consider any open-loop control $\tilde{u} = \{\tilde{u}_k\}$ satisfying the conditions that $\|\tilde{u}_k\| \leq \epsilon$, $k = 0, 1, 2, \dots,$ and the response $\tilde{x} = \{\tilde{x}_k\}$, for each $\tilde{x}_k \in \hat{N}$. There is no loss in assuming that

$$J(x_0, \tilde{u}) = \sum_{k=0}^{\infty} G(\tilde{x}_k, \tilde{u}_k) < \infty$$

since $G(\tilde{x}_k, \tilde{u}_k) \geq 0$. Then as above, $Q(x, \hat{u}) \leq Q(x, \tilde{u})$, we have

$$\begin{aligned} 0 &= \hat{J}(F(x_k, \hat{u}(x_k))) - \hat{J}(x_k) + G(x_k, \hat{u}(x_k)) \\ &\leq \hat{J}(F(x_k, \tilde{u}(x_k))) - \hat{J}(x_k) + G(x_k, \tilde{u}(x_k)), \quad k = 0, 1, 2, \dots, \end{aligned}$$

with strict inequality holding where $\tilde{u} \neq \hat{u}$. If $\tilde{u}_k = \tilde{u}(\tilde{x}_k) = \hat{u}(\tilde{x}_k)$, $k = 0, 1, 2, \dots$, then $\tilde{u}(\tilde{x}_k) = \hat{u}(\tilde{x}_k) = \hat{K}\tilde{x}_k + \hat{h}(\tilde{x}_k)$, and $\tilde{x}_k = (A + B\hat{K})^k x_0 +$ (higher terms), which implies $\tilde{x}_k = \hat{x}_k$, hence $\tilde{u}_k = \hat{u}(\hat{x}_k)$, $k = 0, 1, 2, \dots$. Now assume that $\tilde{u}_k \neq \hat{u}(\tilde{x}_k)$, for some $k \in \{0, 1, 2, \dots\}$, then

$$0 < \sum_{k=0}^{\infty} \left[\hat{J}(F(\tilde{x}_k, \tilde{u}(\tilde{x}_k))) - \hat{J}(\tilde{x}_k) + G(\tilde{x}_k, \tilde{u}(\tilde{x}_k)) \right],$$

or

$$0 < -\hat{J}(x_0) + J(x_0, \{\tilde{u}_k\}),$$

which is

$$\hat{J}(x_0) < J(x_0, \tilde{u}).$$

Thus $\hat{u} = \{\hat{u}_k = \hat{u}(\hat{x}_k)\}$ is the unique optimal open-loop control for x_0 with the required constraints.

By Lemma 2.8, we have

$$\hat{J}(\hat{x}_{k+1}) = \hat{x}_{k+1}{}^* P_K \hat{x}_{k+1} + \hat{j}(\hat{x}_{k+1}), \tag{2.4.26}$$

where P_K is defined in (2.4.6). So that along (2.4.1) we get

$$\begin{aligned}\frac{\partial \hat{J}}{\partial x}(F(\hat{x}_k, \hat{u}(\hat{x}_k))) &= 2\hat{x}_{k+1}{}^* P_K + (\text{higher terms in } \hat{x}_k) \\ &= 2(A\hat{x}_k + B\hat{u}(\hat{x}_k))^* P_K + (\text{higher terms in } \hat{x}_k). \end{aligned} \tag{2.4.27}$$

From (2.4.21) and (2.4.27) for $\hat{J}_x(F(\hat{x}_k, \hat{u}(\hat{x}_k)))$ by using Lemma 2.9, we have

$$\hat{u}(\hat{x}_k) = -\frac{1}{2}U^{-1}\left[2R^*\hat{x}_k + 2B^*P_K(A\hat{x}_k + B\hat{u}(\hat{x}_k))\right] + (\text{higher terms in } \hat{x}_k),$$

which yields

$$\hat{u}(\hat{x}_k) = -(U + B^*P_KB)^{-1}(R^* + B^*P_KA)\hat{x}_k + (\text{higher terms in } \hat{x}_k). \tag{2.4.28}$$

Also by Lemma 2.8,

$$\hat{J}(F(\hat{x}_k, \hat{u}(\hat{x}_k))) - \hat{J}(\hat{x}_k) + G(\hat{x}_k, \hat{u}(\hat{x}_k)) = 0$$

for $\hat{x} = \{\hat{x}_k\}$ with $\|\hat{x}_k\|$, $k = 0, 1, 2, \ldots$, small. Expanding the left-hand side by using (2.4.26) and using the expansion of $\hat{u}(\hat{x}_k)$ in (2.4.28) by $\hat{u}(\hat{x}_k) = \hat{K}\hat{x}_k + \hat{h}(\hat{x}_k)$ to collect quadratic terms in $\hat{x}_k$ produces the equation,

$$P_K = W + RK + K^*R^* + K^*UK + (A + BK)^*P_K(A + BK),$$

which is the equation (2.1.68). But $P_K > 0$ by Lemma 2.8 and hence, by the uniqueness of the solution, $P_K = P$ and $\hat{J}(\hat{x}_k) = \hat{x}_k{}^*P\hat{x}_k + \hat{\jmath}(\hat{x}_k)$.

Proof of Optimality and Uniqueness in Theorem 2.4 (Linear Discrete

System). Set

$$\hat{K} = -(U + B^*PB)^{-1}(R^* + B^*PA),$$

where $P > 0$, is the unique solution of (2.1.68). Let

$$\hat{u}_k = \hat{u}(\hat{x}_k) = \hat{K}\hat{x}_k, \quad k = 0, 1, 2, \dots ,$$

then the optimal closed loop system of (2.1.66) takes the form:

$$\hat{x}_{k+1} = (A + B\hat{K})\hat{x}_k, \quad k = 0, 1, 2, \dots .$$

By the discrete Liapounov theory [11], $(A + B\hat{K})$ is a discrete stability matrix. Also for $\hat{u}_k = \hat{K}\hat{x}_k$, $k = 0, 1, 2, \dots$, in terms of its associated quadratic form we can write quadratic matrix equation (2.4.9) as

$$\hat{x}_{k+1}{}^*P\hat{x}_{k+1} - \hat{x}_k{}^*P\hat{x}_k + G(\hat{x}_k, \hat{u}(\hat{x}_k)) = 0, \tag{2.4.29}$$

by using Lemma 2.8, along the trajectory (2.1.66) and x_0 is any initial condition in $\mathbf{R}^n$. If we define $\hat{A} = A + BK$, and since $\hat{x}_{k+1} = \hat{A}^{k+1}x_0$, (2.4.29) yields

$$x_0{}^*(\hat{A}^k)^*(\hat{A}^*P\hat{A} - P)\hat{A}^kx_0 + G(\hat{x}_k, \hat{u}(\hat{x}_k)) = 0. \tag{2.4.30}$$

By using (2.1.68) we have

$$x_0{}^*(\hat{A}^k)^*(W + R\hat{K} + \hat{K}^*R^* + \hat{K}^*U\hat{K})\hat{A}^kx_0 + G(\hat{x}_k, \hat{u}(\hat{x}_k)) = 0, \tag{2.4.31}$$

hence

$$x_0{}^* \left[\sum_{k=0}^{\infty} (\hat{A}^k)^* (W + R\hat{K} + \hat{K}^* R^* + \hat{K}^* U \hat{K}) \hat{A}^k \right] x_0 + \sum_{k=0}^{\infty} G(\hat{x}_k, \hat{u}(\hat{x}_k)) = 0. \quad (2.4.32)$$

This gives us the following equation by (2.4.6) for P_K:

$$-x_0{}^* P_K x_0 + \hat{J}(x_0) = 0,$$

or

$$\hat{J}(x_0) = x_0{}^* P_K x_0.$$

Since $P_K > 0$ is defined by (2.4.6) and satisfies the matrix equation (2.4.7), by the uniqueness of the solution we have $P_K = P$ and hence $\hat{J}(x_0) = x_0{}^* P x_0$. It is now a simple matter to verify that $\hat{u} = \{\hat{u}_k\}$ satisfies the functional equation (2.4.21) in Lemma 2.10. Since $\frac{\partial \hat{J}}{\partial x}(\hat{x}_{k+1}) = 2\hat{x}_{k+1}{}^* P$, then

$$\begin{aligned} \frac{\partial \hat{J}}{\partial x}(\hat{x}_{k+1}) \frac{\partial F}{\partial u}(\hat{x}_k, \hat{u}(\hat{x}_k)) + \frac{\partial G}{\partial u}(\hat{x}_k, \hat{u}(\hat{x}_k)) &= 2\hat{x}_{k+1}{}^* PB + 2\hat{x}_k{}^* R + 2\hat{u}_k{}^* U \\ &= 2\hat{x}_k{}^* (A^* PB + R) + 2\hat{u}_k{}^* (U + B^* PB), \qquad (2.4.33) \end{aligned}$$

by using $\hat{u}_k = -(U + B^* PB)^{-1} (R^* + B^* PA) \hat{x}_k$, we can see that (2.4.33) equals to zero. Hence we have $\hat{u} = \{\hat{u}_k\}$ satisfies the functional equation (2.4.21) and the proof of optimality and uniqueness is completed of the control $\hat{u}_k(x_k)$ obtained in Section 2.1 from the necessary conditions is completed.

Proof of Theorem 2.7. To prove this theorem it is sufficient to establish the existence of a discrete stabilizing control which solves the functional equation (2.4.5) or (2.4.15) occurring in Theorem 2.7 and Lemma 2.10. The remaining conclusions of the theorem then follow as a corollary to the lemma.

We define $\hat{u}(\hat{x}_k) = \hat{u}(\hat{x}_k, p(\hat{x}_k))$ about the origin in $\mathbf{R}^n$. Recall that $p = q + 2Px$, where P is the unique symmetric positive definite solution of (2.1.54), $q = \{q_k\}$ describes the manifold S discussed in Theorem 2.5 and $\hat{u}_k = \hat{u}(\hat{x}_k, p_{k+1})$, $k = 0, 1, 2, \dots$, was defined in Lemma 2.9. Since the motion of (2.2.1) on S about the origin in $\mathbf{R}^n$ satisfies

$$\begin{aligned}
&\hat{x}_{k+1} = F(\hat{x}_k, \hat{u}(\hat{x}_k)), \quad k = 0, 1, 2, \dots, \\
&A_k^* p_{k+1} - p_k + \frac{\partial G}{\partial x}(\hat{x}_k, \hat{u}(\hat{x}_k))^* = 0, \quad k = 0, 1, 2, \dots,
\end{aligned} \tag{2.4.34}$$

where $A_k = \frac{\partial F}{\partial x}(\hat{x}_k, \hat{u}(\hat{x}_k))$, $\hat{x}_k = \hat{x}_k(x_0)$ with $\|x_0\|$ small. The cost function

$$\sum_{k=0}^{\infty} G(\hat{x}_k, \hat{u}(\hat{x}_k)) = \sum_{k=0}^{\infty} G(\hat{x}_k, \hat{u}(\hat{x}_k, p(\hat{x}_k))) < \infty$$

in a neighborhood of the origin in $\mathbf{R}^n$ by the estimate

$$\begin{aligned}
\|\hat{x}_k\| &\leq C_1 \gamma^k \|x_0\|, \\
\|\hat{u}_k\| &\leq C_2 \gamma^k \|x_0\|, \\
\|p_{k+1}\| \leq \|\Phi(\hat{x}_k)\| + 2\|P\|\|\hat{x}_k\| &\leq C\|\hat{x}_k\| \leq CC_1\gamma^k\|x_0\|,
\end{aligned}$$

where C_1, C_2 and C are positive numbers and $0 < \gamma < 1$. Let us use the equation (2.4.17) for $\hat{u}(\hat{x}_k, p_{k+1})$

$$p_{k+1}{}^* \frac{\partial F}{\partial u}(\hat{x}_k, \hat{u}(\hat{x}_k, p_{k+1})) + \frac{\partial G}{\partial u}(\hat{x}_k, \hat{u}(\hat{x}_k, p_{k+1})) = 0, \tag{2.4.35}$$

and the system (2.4.33) for the motion on S. Now for $k = 0, 1, 2, \ldots$, let us compute

$$\begin{aligned}
\frac{\partial \hat{J}}{\partial \hat{x}_{k+1}}(\hat{x}_{k+1}) &= \sum_{j=k+1}^{\infty} \left[\frac{\partial G}{\partial x}(\hat{x}_j, \hat{u}_j)\frac{\partial \hat{x}_j}{\partial \hat{x}_{k+1}} + \frac{\partial G}{\partial u}(\hat{x}_j, \hat{u}_j)\frac{\partial \hat{u}_j}{\partial \hat{x}_{k+1}}\right] \\
&= \sum_{j=k+1}^{\infty} \left[(p_j^* - p_{j+1}^* A_j)\frac{\partial \hat{x}_j}{\partial \hat{x}_{k+1}} + \frac{\partial G}{\partial u}(\hat{x}_j, \hat{u}_j)\frac{\partial \hat{u}_j}{\partial \hat{x}_{k+1}}\right] \\
&= \sum_{j=k+1}^{\infty} \left[p_j^* \frac{\partial \hat{x}_j}{\partial \hat{x}_{k+1}} - p_{j+1}^* \frac{\partial \hat{x}_{j+1}}{\partial \hat{x}_{k+1}}\right] \\
&\quad + \sum_{j=k+1}^{\infty} \left[p_{j+1}^* \left(\frac{\partial \hat{x}_{j+1}}{\partial \hat{x}_{k+1}} - A_k \frac{\partial \hat{x}_j}{\partial \hat{x}_{k+1}}\right) + \frac{\partial G}{\partial u}(\hat{x}_j, \hat{u}_j)\frac{\partial \hat{u}_j}{\partial \hat{x}_{k+1}}\right]. \qquad (2.4.36)
\end{aligned}$$

Since

$$\frac{\partial \hat{x}_{k+1}}{\partial \hat{x}_{k+1}} = I_n \text{ and } \lim_{j\to\infty} p_{j+1}^* \frac{\partial \hat{x}_{j+1}}{\partial \hat{x}_{k+1}} = 0,$$

then the equation (2.4.36) yields

$$\begin{aligned}
\frac{\partial \hat{J}}{\partial \hat{x}_{k+1}}(\hat{x}_{k+1}) &= p_{k+1}^* + \sum_{j=k+1}^{\infty} p_{j+1}^* \left(\frac{\partial \hat{x}_{j+1}}{\partial \hat{x}_{k+1}} - A_j \frac{\partial \hat{x}_j}{\partial \hat{x}_{k+1}} - B_j \frac{\partial \hat{u}_j}{\partial \hat{x}_{k+1}}\right) \\
&\quad + \sum_{j=k+1}^{\infty} \left(p_{j+1}^* B_j \frac{\partial \hat{u}_j}{\partial \hat{x}_{k+1}} + \frac{\partial G}{\partial u}(\hat{x}_j, \hat{u}_j)\frac{\partial \hat{u}_j}{\partial \hat{x}_{k+1}}\right) = p_{k+1}^*, \qquad (2.4.37)
\end{aligned}$$

where $B_j = \frac{\partial F}{\partial u}(\hat{x}_j, \hat{u}_j)$. Thus, $\frac{\partial \hat{J}}{\partial x}(\hat{x}_{k+1}) = \bar{p}(\hat{x}_{k+1})$ for $\|\hat{x}_{k+1}\|$ small, and hence by (2.4.35) or (2.4.17)

$$\frac{\partial \hat{J}}{\partial x}(\hat{x}_{k+1})\frac{\partial F}{\partial u}(\hat{x}_k, u(\hat{x}_k)) + \frac{\partial G}{\partial u}(\hat{x}_k, u(\hat{x}_k)) = 0, \quad k = 0, 1, 2, \ldots,$$

which proves $\hat{u}(\hat{x}_k)$ solves (2.4.5). In perticular the above equation is also true when $\hat{x}_{k+1}$ is replaced by x_0.

Now let us summarize what we have shown. We have shown in Lemma 2.8 that each discrete stabilizing control $u_k = Kx_k + h(x_k)$ has an associated cost of the form (2.4.8). We have seen in Lemma 2.9 that (2.4.17), which has the same form as (2.4.5) except that $\frac{\partial \hat{J}}{\partial x}(\hat{x}_{k+1})$ is replaced by p_{k+1}, has the unique solution (2.4.18). We have seen in Lemma 2.10 that if (2.4.5), equivalently (2.4.21), has a solution $\hat{u}(x_k)$ of the form indicated there, that $\hat{u}(x_k)$ is the unique optimal control near $x = 0$, $u = 0$. Finally we have shown here, with use of the stable manifold theorem to give (cf. (2.3.7)) $p_{k+1} = p(\hat{x}_k)$, that $\hat{u}_k = \hat{u}(\hat{x}_k) = \hat{u}(\hat{x}_k, p(\hat{x}_k))$ satisfies (2.4.35), equivalently (2.4.17), and that $p_{k+1} = p(\hat{x}_k) = \frac{\partial \hat{J}}{\partial x}(\hat{x}_{k+1}) = \frac{\partial \hat{J}}{\partial x}(F(\hat{x}_k, \hat{u}(\hat{x}_k))$, so that $\hat{u}(x_k)$ does, in fact, satisfy (2.4.21), assuring its optimality and uniqueness. Thus the proof of Theorem 2.7 is complete.

Now let us solve our previous example by using the following equations in Lemma 2.8 and Theorem 2.7,

$$J(F(\hat{x}_k, \hat{u}_k), \{\hat{u}_k\}) - J(\hat{x}_k, \{\hat{u}_k\}) + G(\hat{x}_k, \hat{u}_k) = 0, \ k = 0, 1, 2, \dots, \tag{2.4.38}$$

$$\frac{\partial \hat{J}}{\partial x}(F(\hat{x}_k, \hat{u}(\hat{x}_k)))\frac{\partial F}{\partial u}(\hat{x}_k, \hat{u}(\hat{x}_k)) + \frac{\partial G}{\partial u}(\hat{x}_k, \hat{u}(\hat{x}_k)) = 0, \ k = 0, 1, 2, \dots. \tag{2.4.39}$$

Since the optimal linear feedback law of this problem as we solved earlier is (cf. (2.1.87))

$$\hat{x}_{k+1} = K\hat{x}_k, \quad k = 0, 1, 2, \dots,$$

and the minimum cost as we proved in this section is

$$\hat{J}(\hat{x}_k) = \hat{x}_k{}^* P \hat{x}_k, \quad k = 0, 1, 2, \dots,$$

where $K = \frac{1-\sqrt{5}}{2} \approx -0.6180$, $P = \frac{1+\sqrt{5}}{4} \approx 0.8090$, we can expand the optimal feedback control $\hat{u}(\hat{x}_k)$ and the minimum cost $\hat{J}(\hat{x}_k)$ in power series about the origin:

$$\hat{u}_k = \hat{u}(\hat{x}_k) = K\hat{x}_k + h_2\hat{x}_k{}^2 + h_3\hat{x}_k{}^3 + h_4\hat{x}_k{}^4 + \cdots, \quad k = 0, 1, 2, \ldots, \tag{2.4.40}$$

$$\hat{J}(\hat{x}_k) = P\hat{x}_k{}^2 + j_3\hat{x}_k{}^3 + j_4\hat{x}_k{}^4 + \cdots, \quad k = 0, 1, 2, \ldots. \tag{2.4.41}$$

We will compute all terms of order less than four in the power series expansion of the optimal feedback control (2.4.40) and in the power series expansion of the minimum cost (2.4.41). Substitute (2.4.40) into $F(\hat{x}_k, \hat{u}_k)$, $G(\hat{x}_k, \hat{u}_k)$, $\frac{\partial F}{\partial u}(\hat{x}_k, \hat{u}_k)$ and $\frac{\partial G}{\partial u}(\hat{x}_k, \hat{u}_k)$, we will have the following:

$$\begin{aligned}
\frac{\partial G}{\partial u}(\hat{x}_k, \hat{u}_k) &= \sin \hat{u}_k \\
&= K\hat{x}_k + h_2\hat{x}_k{}^2 + \left(h_3 - \frac{1}{3!}K^3\right)\hat{x}_k{}^3 + \left(h_4 - \frac{1}{2}K^2h_2\right)\hat{x}_k{}^4 + \cdots, && (2.4.42) \\
\frac{\partial F}{\partial u}(\hat{x}_k, \hat{u}_k) &= \cos \hat{u}_k \\
&= 1 - \frac{1}{2}K^2\hat{x}_k{}^2 - Kh_2\hat{x}_k{}^3 - \left(\frac{1}{2}h_2^2 + Kh_3 - \frac{1}{4!}K^4\right)\hat{x}_k{}^4 + \cdots, && (2.4.43) \\
F(\hat{x}_k, \hat{u}_k) &= e^{\hat{x}_k} - 1 + \sin \hat{u}_k \\
&= (1 + K)\hat{x}_k + \left(\frac{1}{2} + h_2\right)\hat{x}_k{}^2 + \left[\frac{1}{3!}(1 - K^3) + h_3\right]\hat{x}_k{}^3 \\
&\quad + \left(\frac{1}{4!} - \frac{1}{2}K^2h_2 + h_4\right)\hat{x}_k{}^4 + \cdots, && (2.4.44) \\
G(\hat{x}_k, \hat{u}_k) &= (1 - \cos \hat{x}_k) + (1 - \sin \hat{u}_k) \\
&= \frac{1}{2}(1 + K^2)\hat{x}_k{}^2 + Kh_2\hat{x}_k{}^3 - \left[\frac{1}{4!}(1 + K^4) - \frac{1}{2}h_2^2 - Kh_3\right]\hat{x}_k{}^4 + \cdots. && (2.4.45)
\end{aligned}$$

Using (2.4.44), (2.4.41) gives us

$$\begin{aligned}
\hat{J}(F(\hat{x}_k, \hat{u}(\hat{x}_k))) &= PF^2(\hat{x}_k, \hat{u}(\hat{x}_k)) + j_3F^3(\hat{x}_k, \hat{u}(\hat{x}_k)) + j_4F^4(\hat{x}_k, \hat{u}(\hat{x}_k)) + \cdots \\
&= P(1 + K)^2\hat{x}_k{}^2 + [P(1 + K)(1 + 2h_2) + (1 + K)^3j_3]\hat{x}_k{}^3
\end{aligned}$$

$$+\left[\frac{1}{3}P(1+K)(1-K^3)+2P(1+K)h_3+P\left(\frac{1}{2}+h_2\right)^2\right.$$
$$\left.+3(1+K)^2\left(\frac{1}{2}+h_2\right)j_3+(1+K)^4j_4\right]\hat{x}_k{}^4+\cdots. \quad (2.4.46)$$

Substituting (2.4.46), (2.4.41) and (2.4.45) into (2.4.38) will have the following:

$$\left\{P(1+K)^2-P+\frac{1}{2}(1+K^2)\right\}\hat{x}_k{}^2+$$
$$\{P(1+K)(1+2h_2+[(1+K)^3-1]j_3+Kh_2\}\hat{x}_k{}^3+$$
$$\left\{P\left(\frac{1}{2}+h_2\right)^2+P(1+K)\left[\frac{1}{3}(1-K^3)+2h_3\right]+3(1+K)^2\left(\frac{1}{2}+h_2\right)j_3+\right.$$
$$\left.[(1+K)^4-1]j_4-\frac{1}{4!}(1+K^4)+\frac{1}{2}h_2^2+Kh_3\right\}\hat{x}_k{}^4+\cdots=0. \quad (2.4.47)$$

And also by using (2.4.44) for $F(\hat{x}_k,\hat{u}_k)$ we will have

$$\frac{\partial\hat{J}}{\partial x}(F(\hat{x}_k,\hat{u}(\hat{x}_k)))=2PF(\hat{x}_k,\hat{u}(\hat{x}_k))+3j_3F^2(\hat{x}_k,\hat{u}(\hat{x}_k))+4j_4F^3(\hat{x}_k,\hat{u}(\hat{x}_k))+\cdots$$
$$=2P(1+K)\hat{x}_k+[P(1+2h_2)+3(1+K)^2j_3]\hat{x}_k{}^2$$
$$+\left[\frac{1}{3}P(1-K^3)+2Ph_3+3(1+K)(1+2h_2)j_3+4(1+K)^3j_4\right]\hat{x}_k{}^3$$
$$+\left[P\left(\frac{1}{12}-K^2h_2+2h_4\right)+(1+K)[(1-K)^3+6h_3]j_3\right.$$
$$\left.+3\left(\frac{1}{2}+h_2\right)^2j_3+6(1+K)^2(1+2h_2)j_4\right]\hat{x}_k{}^4+\cdots. \quad (2.4.48)$$

Substitute (2.4.48), (2.4.43) and (2.4.42) into (2.4.39) giving,

$$[2P(1+K)+K]\hat{x}_k+[P+(2P+1)h_2+3(1+K)^2j_3]\hat{x}_k{}^2+$$
$$\left\{P\left[\frac{1}{3}(1-K^3)+2h_3\right]+3(1+K)(1+2h_2)j_3+4(1+K)^3j_4-P(1+K)K^2+\right.$$
$$\left.\left(h_3-\frac{1}{3!}K^3\right)\right\}\hat{x}_k{}^3+\left\{P\left(\frac{1}{12}-K^2h_2+2h_4\right)+(1+K)[(1-K)^3+6h_3]j_3+\right.$$
$$3\left(\frac{1}{2}+h_2\right)^2j_3+6(1+K)^2(1+2h_2)j_4-\frac{1}{2}K^2[P(1+2h_2)+3(1+K)^2j_3]-$$

$$2P(1+K)Kh_2 + \left(h_4 - \frac{1}{2}K^2h_2\right)\Big\} \hat{x}_k{}^4 + \cdots = 0. \tag{2.4.49}$$

If we set each coefficient of $\hat{x}_k$, $\hat{x}_k{}^2$, $\hat{x}_k{}^3$ and $\hat{x}_k{}^4$ in (2.4.47) and (2.4.49) equal to zero, then the computation of successively higher order terms of the optimal feedback control and the minimum cost reduces to solving the following successively higher order system of nonlinear algebraic equations,

$$P(1+K)^2 - P + \frac{1}{2}(1+K^2) = 0, \tag{2.4.50}$$

$$P(1+K)(1+2h_2) + [(1+K)^3 - 1]j_3 + Kh_2 = 0, \tag{2.4.51}$$

$$\begin{aligned} &P\left(\frac{1}{2}+h_2\right)^2 + P(1+K)\left[\frac{1}{3}(1-K^3)+2h_3\right] + 3(1+K)^2\left(\frac{1}{2}+h_2\right)j_3 \\ &\quad + [(1+K)^4-1]j_4 - \frac{1}{4!}(1+K^4) + \frac{1}{2}h_2^2 + Kh_3 = 0, \end{aligned} \tag{2.4.52}$$

$$2P(1+K) + K = 0, \tag{2.4.53}$$

$$P + (2P+1)h_2 + 3(1+K)^2 j_3 = 0, \tag{2.4.54}$$

$$\begin{aligned} &P\left[\frac{1}{3}(1-K^3)+2h_3\right] + 3(1+K)(1+2h_2)j_3 + 4(1+K)^3 j_4 \\ &\quad - P(1+K)K^2 + \left(h_3 - \frac{1}{3!}K^3\right) = 0, \end{aligned} \tag{2.4.55}$$

$$\begin{aligned} &P\left(\frac{1}{12} - K^2h_2 + 2h_4\right) + (1+K)[(1-K)^3 + 6h_3]j_3 + 3\left(\frac{1}{2}+h_2\right)^2 j_3 \\ &\quad + 6(1+K)^2(1+2h_2)j_4 - \frac{1}{2}K^2[P(1+2h_2) + 3(1+K)^2 j_3] \\ &\quad - 2P(1+K)Kh_2 + h_4 - \frac{1}{2}K^2h_2 = 0. \end{aligned} \tag{2.4.56}$$

Since (2.4.50) and (2.4.53) are equivalent to (2.1.82) and (2.1.81) on page 39 respectively, we have five equations (2.4.51), (2.4.52), (2.4.54), (2.4.55) and (2.4.56) and five unknowns. If we rearrange these five equations by using (2.4.50) and (2.4.53) and solve them in the order (2.4.51), (2.4.54), (2.4.52), (2.4.55) and (2.4.56), we get

$$
\begin{aligned}
j_3 &= -\frac{P(1+K)}{(1+K)^3-1} \approx 0.3273, \\
h_2 &= -\frac{P+3(1+K)^2 j_3}{2P+1} \approx -0.3637, \\
j_4 &= -\frac{1}{(1+K)^4-1}\left[\frac{1}{3}P(1+K)(1-K^3) - \frac{1}{4!}(1+K^4) + P\left(\frac{1}{2}+h_2\right)^2\right. \\
&\quad \left. +\frac{1}{2}h_2^2 + 3(1+K)^2\left(\frac{1}{2}+h_2\right)j_3\right] \approx 0.1842, \\
h_3 &= -\frac{1}{2P+1}\left[\frac{1}{3}P(1-K^3) - \frac{1}{3!}K^3 - P(1+K)K^2 + 3(1+K)(1+2h_2)j_3\right. \\
&\quad \left. +4(1+K)^3 j_4\right] \approx -0.1520, \\
h_4 &= -\frac{1}{2P+1}\left\{\frac{P}{12}(1-6K^2) - \left[2P(2K+1)+\frac{1}{2}K\right]Kh_2\right. \\
&\quad +6(1+K)^2(1+2h_2)j_4 + \left[(1+K)(1-K)^3 - \frac{3}{2}(1+K)^2K^2\right. \\
&\quad \left.\left. +\,3\left(\frac{1}{2}+h_2\right)^2 + 6(1+K)h_3\right]j_3\right\} \approx -0.1981.
\end{aligned}
$$

Therefore, the optimal feedback control for this example is given by

$$\hat{u}_k = \hat{u}(\hat{x}_k) \approx -0.6180\hat{x}_k - 0.3637{\hat{x}_k}^2 - 0.1520{\hat{x}_k}^3 - 0.1981{\hat{x}_k}^4 + \cdots, \qquad (2.4.57)$$

and the minimum cost is

$$\hat{J}(\hat{x}_k) \approx 0.8090{\hat{x}_k}^2 + 0.3273{\hat{x}_k}^3 + 0.1842{\hat{x}_k}^4 + \cdots. \qquad (2.4.58)$$

Chapter 3

Optimal Feedback Control of Periodic Systems

3.1 Modifications to Chapter 2 to Cover the Case of an Infinite Dimensional Control

We again consider the discrete control system (2.1.1) with cost function of the form (2.1.2) on page 16 but we allow the control u to be infinite dimensional; specifically we suppose $u \in \mathbf{H}$, a real Hilbert space. All symbols used will be as in Sections 2.1 - 2.4 except that some obvious re-interpretations must be made.

Thus we suppose $F : \mathbf{R}^n \times \mathbf{H} \to \mathbf{R}^n$ is continuously differentiable, as before, with $F(0,0) = 0$, but differentiability with respect to u is now to be understood in the Frechet sense [48]. The matrices A and W introduced in Section 2.1 remain matrices here but B, U and R are now to be understood as operators:

$$B = \frac{\partial F}{\partial u}(0,0) : \mathbf{H} \to \mathbf{R}^n, \tag{3.1.1}$$

where $\frac{\partial F}{\partial u}$ denotes the (bounded) Jacobian operator defined in the Frechet sense;

$$U : \mathbf{H} \to \mathbf{H} \tag{3.1.2}$$

is a bounded positive self - adjoint operator on $\mathbf{H}$ with $(u, Uu) \geq b\|u\|_H^2$ for some $b > 0$ and

$$R : \mathbf{R}^n \to \mathbf{H} \tag{3.1.3}$$

is a linear operator defined on a finite dimensional space, and therefore necessarily bounded, but with range in the (possibly) infinite dimensional space $\mathbf{H}$.

The work carried out in Chapter 2 remains entirely valid in this infinite dimensional context; it is not necessary to repeat the computations or re-prove the theorems. However, we will make some remarks here to cover questions which might naturally arise.

We note that the closed loop system matrix $A + BK$ remains exactly that, an $n \times n$ matrix, even though $B : \mathbf{H} \to \mathbf{R}^n$ and $K : \mathbf{R}^n \to \mathbf{H}$ are operators; the latter should now be referred to as the feedback operator. Note that equations like (2.1.64) are now operator equations.

In the finite dimensional context we have used transposed vectors to indicate linear functionals. In the case where u is infinite dimensional an expression $u^* Mv$, where v is a vector, M an operator or matrix, should be understood as $(Mv, u)_H$. It should be noted that (2.1.14) and similar equations are now linear functional identities.

When we come to the use of the implicit function theorem following (2.1.9) we now require that theorem in a Hilbert space setting; the reader is referred to [48].

The stable manifold theorem of Section 2.2 does not need to be changed at all because it does not involve u in any explicit way; both y and z in (2.2.3), etc. remain n - dimensional vectors. It is not until we come to Section 2.3 and equations such as (2.3.9), (2.3.10), etc. that infinite dimensional vectors enter the picture again.

With infinite demensional controls now admitted into the class for which the results of Chapter 2 remain valid, we are in a position to reinterpret certain continuous processes in a discrete setting by taking the discrete controls u_k to be restrictions of a continuously defined control function $u(t)$ to successive time intervals $[t_k, t_{k+1}]$. This is developed in more detail in the following sections.

3.2 The Linear T - Periodic (Continuous) Case as an Instance of the Linear Discrete Case with Control in $\mathbf{L}^2_m[0,T]$

Let us consider a linear T - periodic control system consisting of a system of linear differential equation

$$\dot{x} = A(t)x + B(t)u, \quad t \in [0,\infty), \tag{3.2.1}$$

for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, where $A(t)$ and $B(t)$ are piecewise continuous and T - periodic, i.e.,

$$A(t+T) = A(t), \quad B(t+T) = B(t),$$

for some period $T > 0$. With (3.2.1) we consider a cost function of the form

$$J(x_0, u) = \int_0^\infty G(t, x(t), u(t))\, dt, \tag{3.2.2}$$

where $G(t,x,u)$ is given in the following (quadratic) form

$$G(t,x,u) = (x^*, u^*) \begin{pmatrix} W(t) & R(t) \\ R(t)^* & U(t) \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}, \tag{3.2.3}$$

with $G(t,x,u)$ T - periodic, that is

$$G(t+T, x, u) = G(t,x,u).$$

Here the matrix $\mathcal{W}(t) = \begin{pmatrix} W(t) & R(t) \\ R(t)^* & U(t) \end{pmatrix}$ is required to be symmetric and uniformly positive definite on $[0,T]$, hence, by periodicity, for $t \in [0,\infty)$. We seek a control in periodic state feedback form,

$$u(t) = u(t, x(t)), \;\; u(t+T, x) = u(t, x),$$

which makes the cost function (3.2.2) as small as possible for all initial states in $\mathbf{R}^n$. Let us define u_k to be the element of $\mathbf{L}^2_m[0,T]$ such that

$$u_k(t) = u(t+kT), \;\; k = 0, 1, 2, \dots, \;\; t \in [0, T]. \tag{3.2.4}$$

If $\Phi(t,\tau)$ is the fundamental matrix solution of the equation

$$\dot{\Phi} = A(t)\Phi \;\; \text{with} \;\; \Phi(\tau,\tau) = I_n, \tag{3.2.5}$$

then the solution of (3.2.1) at $t = (k+1)T$ obeys the relation

$$x((k+1)T) = \Phi((k+1)T, kT)x(kT) + \int_{kT}^{(k+1)T} \Phi((k+1)T, \tau)B(\tau)u(\tau)\, d\tau. \tag{3.2.6}$$

From periodicity we have

$$\Phi((k+1)T, kT) = \Phi(T, 0), \;\; k = 0, 1, 2, \dots, \tag{3.2.7}$$

$$\Phi((k+1)T, \tau) = \Phi(T, \tau - kT), \;\; k = 0, 1, 2, \dots. \tag{3.2.8}$$

Let us set

$$\mathcal{A} \equiv \Phi(T, 0) \tag{3.2.9}$$

and also define for $k = 0, 1, 2, \dots,$

$$x_k = x(kT) \in \mathbf{R}^n. \tag{3.2.10}$$

Let $\tau = s + kT$, $\tau - kT = s$; then $\tau \in [kT, (k+1)T]$, $s \in [0, T]$. Using this change of variable in the integral in (3.2.6), we have

$$\begin{aligned} & \int_{kT}^{(k+1)T} \Phi((k+1)T, \tau) B(\tau) u(\tau)\, d\tau \\ = & \int_0^T \Phi((k+1)T, s + kT) B(s + kT) u(s + kT)\, ds \\ = & \int_0^T \Phi(T, s) B(s) u_k(s)\, ds, \quad k = 0, 1, 2, \dots . \end{aligned} \tag{3.2.11}$$

If we define a linear operator $\mathcal{B}$: $\mathbf{L}_m^2[0, T] \to \mathbf{R}^n$ by

$$\mathcal{B} u_k \equiv \int_0^T \Phi(T, s) B(s) u_k(s)\, ds, \tag{3.2.12}$$

then, since in (3.2.12) everything except u_k is independent of k, (3.2.6) yields

$$x_{k+1} = \mathcal{A} x_k + \mathcal{B} u_k \tag{3.2.13}$$

with $\mathcal{A}$ and $\mathcal{B}$ defined by (3.2.9) and (3.2.12) for $k = 0, 1, 2, \dots$. This has the linear discrete control system form with state $x = \{x_k\}$ in the finite dimensional space $\mathbf{R}^n$ and control $u = \{u_k\} \in \mathbf{L}_m^2[0, T]$.

Next we rewrite the cost function (3.2.2) in a corresponding discrete form. First we can write the cost function (3.2.2) as follows:

$$\begin{aligned} & \int_0^\infty (x(t)^*, u(t)^*) \begin{pmatrix} W(t) & R(t) \\ R(t)^* & U(t) \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} dt \\ = & \sum_{k=0}^{\infty} \int_{kT}^{(k+1)T} (x(t)^*, u(t)^*) \begin{pmatrix} W(t) & R(t) \\ R(t)^* & U(t) \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} dt \end{aligned}$$

$$= \sum_{k=0}^{\infty} \int_0^T (x(t+kT)^*, u(t+kT)^*) \begin{pmatrix} W(t+kT) & R(t+kT) \\ R(t+kT)^* & U(t+kT) \end{pmatrix} \begin{pmatrix} x(t+kT) \\ u(t+kT) \end{pmatrix} dt$$

$$= \sum_{k=0}^{\infty} \int_0^T (x(t+kT)^*, u_k(t)^*) \begin{pmatrix} W(t) & R(t) \\ R(t)^* & U(t) \end{pmatrix} \begin{pmatrix} x(t+kT) \\ u_k(t) \end{pmatrix} dt. \qquad (3.2.14)$$

For $k = 0, 1, 2, \ldots$, $t \in [0, T]$, we have,

$$\begin{aligned} x(t+kT) &= \Phi(t+kT, kT)x(kT) + \int_0^t \Phi(t+kT, s+kT)B(s+kT)u(s+kT)\, ds \\ &= \Phi(t,0)x_k + \int_0^t \Phi(t,s)B(s)u_k(s)\, ds. \end{aligned} \qquad (3.2.15)$$

If we define

$$\Phi(t) \equiv \Phi(t,0), \qquad (3.2.16)$$

$$(\theta u_k)(t) \equiv \int_0^t \Phi(t,s)B(s)u_k(s)\, ds, \qquad (3.2.17)$$

where $(\theta u_k)(t)$: $\mathbf{L}^2_m[0,T] \to \mathbf{L}^2_n[0,T]$, then for $t \in [0,T]$, $x(t+kT)$ can be written as a function of x_k and u_k:

$$x(t+kT) = \Phi(t)x_k + (\theta u_k)(t), \quad k = 0, 1, 2, \ldots. \qquad (3.2.18)$$

Note that $(\theta u_k)(T) = \mathcal{B}u_k$. If we substitute (3.2.18) for $x(t+kT)$ in (3.2.14), then the integral in (3.2.14) becomes

$$\begin{aligned} &\int_0^T (x(t+kT)^*, u_k(t)^*) \begin{pmatrix} W(t) & R(t) \\ R(t)^* & U(t) \end{pmatrix} \begin{pmatrix} x(t+kT) \\ u_k(t) \end{pmatrix} dt \\ = &\int_0^T \{(\Phi(t)x_k + (\theta u_k)(t))^* W(t) (\Phi(t)x_k + (\theta u_k)(t)) + u_k(t)^* U(t) u_k(t) \\ &+ (\Phi(t)x_k + (\theta u_k)(t))^* R(t)u_k(t) + u_k(t)^* R(t)^* (\Phi(t)x_k + (\theta u_k)(t))\}\, dt \end{aligned}$$

$$= \int_0^T \{x_k{}^*\Phi(t)^*W(t)\Phi(t)x_k + 2x_k{}^*\Phi(t)^*R(t)u_k(t) + 2x_k{}^*\Phi(t)^*W(t)(\theta u_k)(t)$$
$$+ (\theta u_k)(t)^*W(t)(\theta u_k)(t) + 2(\theta u_k)(t)^*R(t)u_k(t) + u_k(t)^*U(t)u_k(t)\}\ dt,$$
$$k = 0, 1, 2, \dots . \tag{3.2.19}$$

Let us redefine the first two terms in (3.2.19), respectively, as follows:

$$\int_0^T x_k{}^*\Phi(t)^*W(t)\Phi(t)x_k\,dt \equiv x_k{}^*\mathcal{W}x_k, \quad k = 0, 1, 2, \dots , \tag{3.2.20}$$
$$\int_0^T x_k{}^*\Phi(t)^*R(t)u_k(t)\,dt = \int_0^T x_k{}^*\tilde{R}(t)u_k(t)\,dt, \quad k = 0, 1, 2, \dots , \tag{3.2.21}$$

where $\tilde{R}(t)$ is defined as

$$\tilde{R}(t) \equiv \Phi(t)^*R(t). \tag{3.2.22}$$

If we substitute (3.2.17) for $(\theta u_k)(t)$ into the third term of the last expression in (3.2.19), we get

$$\begin{aligned}
&\int_0^T x_k{}^*\Phi(t)^*W(t)(\theta u_k)(t)\,dt \\
&= \int_0^T x_k{}^*\Phi(t)^*W(t)\left(\int_0^t \Phi(t,s)B(s)u_k(s)\,ds\right)\,dt \\
&= \int_0^T \int_0^t x_k{}^*\Phi(t)^*W(t)\Phi(t,s)B(s)u_k(s)\,ds\,dt \\
&= \int_0^T \int_s^T x_k{}^*\Phi(t)^*W(t)\Phi(t,s)B(s)u_k(s)\,dt\,ds \\
&= \int_0^T x_k{}^* \int_s^T \Phi(t)^*W(t)\Phi(t,s)\,dt\,B(s)u_k(s)\,ds \\
&= \int_0^T x_k{}^*\hat{R}(s)u_k(s)\,ds, \quad k = 0, 1, 2, \dots ,
\end{aligned} \tag{3.2.23}$$

where $\hat{R}(s)$ is defined as

$$\hat{R}(s) \equiv \int_s^T \Phi(t)^* W(t) \Phi(t,s)\, dt\, B(s). \tag{3.2.24}$$

Let us combine the second and third term in (3.2.19), i.e., (3.2.21) and (3.2.23), and redesignate the sum as

$$\int_0^T x_k{}^* \tilde{R}(t) u_k(t)\, dt + \int_0^T x_k{}^* \hat{R}(t) u_k(t)\, dt \equiv (x_k, \mathcal{R} u_k)_{R^n}, \quad k = 0, 1, 2, \ldots. \tag{3.2.25}$$

Then, substituting (3.2.17) for $(\theta u_k)(t)$ in the fourth term of (3.2.19), we have

$$\begin{aligned}
\int_0^T (\theta u_k)(t)^* W(t) (\theta u_k)(t)\, dt &= \int_0^T (\theta u_k)(r)^* W(r) (\theta u_k)(r)\, dr \\
&= \int_0^T \left(\int_0^r \Phi(r,s) B(s) u_k(s)\, ds \right)^* W(r) \left(\int_0^r \Phi(r,t) B(t) u_k(t)\, dt \right) dr \\
&= \int_0^T \int_0^T u_k(s)^* B(s)^* \int_{\max\{t,s\}}^T \Phi(r,s)^* W(r) \Phi(r,t)\, dr B(t) u_k(t)\, ds\, dt \\
&= \int_0^T \int_0^T u_k(s)^* \tilde{U}(t,s) u_k(t)\, ds\, dt, \quad k = 0, 1, 2, \ldots,
\end{aligned} \tag{3.2.26}$$

where we define

$$\tilde{U}(t,s) \equiv B(s)^* \int_{\max\{t,s\}}^T \Phi(r,s)^* W(r) \Phi(r,t)\, dr B(t). \tag{3.2.27}$$

Clearly $\tilde{U}(t,s) = \tilde{U}(s,t)^*$. By performing the same substitution in the fifth term of (3.2.19), we get

$$\begin{aligned}
&2 \int_0^T (\theta u_k)(t)^* R(t) u_k(t)\, dt \\
= \; &2 \int_0^T \left(\int_0^t \Phi(t,s) B(s) u_k(s)\, ds \right)^* R(t) u_k(t)\, dt
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^T \int_0^t u_k(s)^* B(s)^* \Phi(t,s)^* R(t) u_k(t)\, ds\, dt \\
&\qquad + \int_0^T \int_0^s u_k(t)^* B(t)^* \Phi(s,t)^* R(s) u_k(s)\, dt\, ds \\
&= \int_0^T \int_0^t u_k(s)^* B(s)^* \Phi(t,s)^* R(t) u_k(t)\, ds\, dt \\
&\qquad + \int_0^T \int_t^T u_k(t)^* B(t)^* \Phi(s,t)^* R(s) u_k(s)\, ds\, dt \\
&= \int_0^T \int_0^t u_k(s)^* B(s)^* \Phi(t,s)^* R(t) u_k(t)\, ds\, dt \\
&\qquad + \int_0^T \int_t^T u_k(s)^* R(s)^* \Phi(s,t) B(t) u_k(t)\, ds\, dt \\
&= \int_0^T \int_0^T u_k(s)^* \hat{U}(t,s) u_k(t)\, ds\, dt, \quad k = 0,1,2,\ldots, \qquad (3.2.28)
\end{aligned}
$$

where $\hat{U}(t,s)$ is defined as

$$
\hat{U}(t,s) \equiv \begin{cases} B(s)^* \Phi(t,s)^* R(t), & \text{if } s \le t, \\ R(s)^* \Phi(s,t) B(t), & \text{if } s \ge t, \end{cases} \qquad (3.2.29)
$$

and $\hat{U}(t,s) = \hat{U}(s,t)^*$. If we combine (3.2.26), the fourth term, (3.2.28), the fifth term and the last term, $u_k(t)^* U(t) u_k(t)$, in (3.2.19), we will have

$$
\begin{aligned}
&\int_0^T \int_0^T u_k(s)^* \tilde{U}(t,s) u_k(t)\, ds\, dt + \int_0^T \int_0^T u_k(s)^* \hat{U}(t,s) u_k(t)\, ds\, dt \\
&\quad + \int_0^T u_k(t)^* U(t) u_k(t)\, dt \equiv (u_k, \mathcal{U} u_k)_{L^2_m[0,T]}, \quad k = 0,1,2,\ldots, \qquad (3.2.30)
\end{aligned}
$$

which is a positive quadratic form for $u_k \in \mathbf{L}^2_m[0,T]$. Then, by substituting (3.2.20), (3.2.25) and (3.2.30) into (3.2.19), we have

$$
\begin{aligned}
&\int_0^T (x(t+kT)^*, u_k(t)^*) \begin{pmatrix} W(t) & R(t) \\ R(t)^* & U(t) \end{pmatrix} \begin{pmatrix} x(t+kT) \\ u_k(t) \end{pmatrix} dt \\
&= x_k{}^* \mathcal{W} x_k + 2(x_k, \mathcal{R} u_k)_{R^n} + (u_k, \mathcal{U} u_k)_{L^2_m[0,T]}. \qquad (3.2.31)
\end{aligned}
$$

Hence with the control $u = \{u_k\}$ the total cost (3.2.14) becomes

$$J(x_0, u) = \sum_{k=0}^{\infty} \left\{ x_k{}^* \mathcal{W} x_k + 2(x_k, \mathcal{R} u_k)_{R^n} + (u_k, \mathcal{U} u_k)_{L^2_m[0,T]} \right\}. \tag{3.2.32}$$

Therefore, the linear continuous T - periodic quadratic optimization problem, consisting of the constraint equation

$$\dot{x} = A(t)x + B(t)u,$$

and the cost functional

$$\begin{aligned} J(x_0, u) &= \int_0^\infty G(t, x(t), u(t))\, dt, \\ &= \int_0^\infty (x(t)^*, u(t)^*) \begin{pmatrix} W(t) & R(t) \\ R(t)^* & U(t) \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} dt \end{aligned}$$

can be written as the linear discrete quadratic optimization problem consisting of the equation (3.2.13) and the cost functional (3.2.32),

$$\begin{aligned} x_{k+1} &= \mathcal{A} x_k + \mathcal{B} u_k, \\ J(x_0, u) &= \sum_{k=0}^{\infty} \left\{ x_k{}^* \mathcal{W} x_k + 2(x_k, \mathcal{R} u_k)_{R^n} + (u_k, \mathcal{U} u_k)_{L^2_m[0,T]} \right\}, \end{aligned}$$

for control and trajectory pair $u = \{u_k\}$ and $x = \{x_k\}$. The original continuous periodic optimal control problem now becomes the discrete optimal control problem

$$\begin{aligned} \text{minimize :} \quad J(x_0, u) &= \sum_{k=0}^{\infty} \mathcal{G}(x_k, u_k) \\ &= \sum_{k=0}^{\infty} \left\{ x_k{}^* \mathcal{W} x_k + 2(x_k, \mathcal{R} u_k)_{R^n} + (u_k, \mathcal{U} u_k)_{L^2_m[0,T]} \right\}, \end{aligned} \tag{3.2.33}$$

subject to the constraint that $x = \{x_k\}$ and $u = \{u_k\}$ should satisfy (3.2.13). Here

$u_k \in \mathbf{L}^2_m[0,T]$, $k = 0,1,2,\dots$, and $\mathcal{W}$, $\mathcal{R}$ and $\mathcal{U}$ are defined as in (3.2.20), (3.2.25) and (3.2.30). Thus we may consider the linear continuous T - periodic case as an instance of the linear discrete case with control in $\mathbf{L}^2_m[0,T]$.

Since we have

$$\mathcal{G}(x_k, u_k) = x_k{}^*\mathcal{W}x_k + 2(x_k, \mathcal{R}u_k)_{R^n} + (u_k, \mathcal{U}u_k)_{L^2_m[0,T]}, \quad k = 0,1,2,\dots, \tag{3.2.34}$$

in the cost function (3.2.33) the partial derivatives of $\mathcal{G}$ with respect to u and x at (x_k,u_k) have the form

$$\frac{\partial \mathcal{G}}{\partial u}(x_k, u_k) = 2x_k{}^*\mathcal{R} + 2u_k{}^*\mathcal{U}, \tag{3.2.35}$$

$$\frac{\partial \mathcal{G}}{\partial x}(x_k, u_k) = 2x_k{}^*\mathcal{W} + 2u_k{}^*\mathcal{R}^*, \tag{3.2.36}$$

respectively, which are in the same form as the discrete case except $\mathcal{W}$, $\mathcal{R}$ and $\mathcal{U}$ are now defined as in (3.2.20), (3.2.25) and (3.2.30). Thus we may use all the results obtained in Chapter 2 for the present linear continuous T - periodic case. Therefore, by using Theorem 2.4 on page 35 there exists a unique optimal control trajectory sequence pair $\hat{x} = \{\hat{x}_k\}$ and $\hat{u} = \{\hat{u}_k\}$, for the system (3.2.13) and (3.2.32) and the optimal control law takes the form

$$\hat{u}_k(t) = \left(-(\mathcal{U} + \mathcal{B}^*\mathcal{P}\mathcal{B})^{-1}(\mathcal{R}^* + \mathcal{B}^*\mathcal{P}\mathcal{A})\hat{x}_k\right)(t) \equiv (\mathcal{K}\hat{x}_k)(t), \quad k = 0,1,2,\dots, \tag{3.2.37}$$

for $t \in [0,T]$, where $\mathcal{P}$ is the unique symmetric positive definite solution of the algebraic Riccati equation obtained from

$$\mathcal{P} = \mathcal{W} + \mathcal{R}\mathcal{K} + \mathcal{K}^*\mathcal{R}^* + \mathcal{K}^*\mathcal{U}\mathcal{K} + (\mathcal{A} + \mathcal{B}\mathcal{K})^*\mathcal{P}(\mathcal{A} + \mathcal{B}\mathcal{K}), \tag{3.2.38}$$

with $\mathcal{K}$ expressed in terms of $\mathcal{P}$ as in (3.2.37).

Let us rename this $\mathcal{P}$ in (3.2.38) as $\mathcal{P}(0)$ and $\mathcal{A}$, $\mathcal{B}$, $\mathcal{W}$, $\mathcal{R}$, $\mathcal{U}$ as $\mathcal{A}(0)$, $\mathcal{B}(0)$, $\mathcal{W}(0)$, $\mathcal{R}(0)$, $\mathcal{U}(0)$ since we have the initial state $x_0 \in \mathbf{R}^n$ when $t = 0$.

If we start the problem at some time $\tau \in [0, T]$ with an initial state x_τ, the cost function (3.2.2) becomes

$$\begin{aligned} J(x_\tau, u) &= \int_\tau^\infty G(t, x(t), u(t))\, dt, \\ &= \int_0^\infty G(t+\tau, x(t+\tau), u(t+\tau))\, dt \end{aligned} \tag{3.2.39}$$

where $G(t, x, u)$ is T - periodic, $G(t+T, x, u) = G(t, x, u)$ and $G(t, x, u)$ is given in (3.2.3). If we consider a control $u(t+\tau) \in \mathbf{L}^2_m[0, \infty]$, then $u(t+\tau+kT) \in \mathbf{L}^2_m[0, T]$. Let us define u_k to be the element of $\mathbf{L}^2_m[0, T]$, such that

$$u_k(t) = u(t+\tau+kT), \quad k = 0, 1, 2, \dots, \quad t \in [0, T] \tag{3.2.40}$$

and set

$$\Phi_\tau(t, s) \equiv \Phi(t+\tau, s+\tau), \tag{3.2.41}$$

where $\Phi(t, s)$ is the fundamental matrix defined in (3.2.5) and define

$$\mathcal{A}(\tau) \equiv \Phi_\tau(T, 0), \tag{3.2.42}$$

$$\mathcal{B}(\tau)u_k \equiv \int_0^T \Phi_\tau(T,s)B(s+\tau)u_k(s)\,ds \tag{3.2.43}$$

and let

$$x_k = x(\tau + kT) \in \mathbf{R}^n, \quad k = 0,1,2,\ldots. \tag{3.2.44}$$

Performing the same operations as in (3.2.6) - (3.2.12), we have

$$x_{k+1} = \mathcal{A}(\tau)x_k + \mathcal{B}(\tau)u_k, \quad k = 0,1,2,\ldots. \tag{3.2.45}$$

Also if we define

$$\Phi_\tau(t) \equiv \Phi_\tau(t,0), \tag{3.2.46}$$

$$(\theta u_k)_\tau(t) \equiv \int_0^t \Phi_\tau(t,s)B(s+\tau)u_k(s)\,ds, \tag{3.2.47}$$

then the cost function (3.2.39) for $u = \{u_k\}$, has the form:

$$J(x_\tau,u) = \sum_{k=0}^{\infty}\left\{x_k{}^*\mathcal{W}(\tau)x_k + 2(x_k,\mathcal{R}(\tau)u_k)_{R^n} + (u_k,\mathcal{U}(\tau)u_k)_{L^2_m[0,T]}\right\}, \tag{3.2.48}$$

(cf. (3.2.14) - (3.2.32)), where $\mathcal{W}(\tau)$, $\mathcal{R}(\tau)$ and $\mathcal{U}(\tau)$ are defined as

$$x_k{}^*\mathcal{W}(\tau)x_k \equiv \int_0^T x_k{}^*\Phi_\tau(t)W(t+\tau)\Phi_\tau(t)x_k\,dt, \tag{3.2.49}$$

$$(x_k,\mathcal{R}(\tau)u_k)_{R^n} \equiv \int_0^T x_k{}^*\left(\tilde{R}_\tau(t)+\hat{R}_\tau(t)\right)u_k(t)\,dt, \tag{3.2.50}$$

$$\tilde{R}_\tau(t) = \Phi_\tau(t)^*R(t+\tau), \tag{3.2.51}$$

$$\hat{R}_\tau(t) = \int_t^T \Phi_\tau(s)^*W(s+\tau)\Phi_\tau(s,t)\,dsB(t+\tau), \tag{3.2.52}$$

$$(u_k,\mathcal{U}(\tau)u_k)_{L^2_m[0,T]} \equiv \int_0^T\int_0^T u_k(s)^*\left(\tilde{U}_\tau(t,s)+\hat{U}_\tau(t,s)\right)u_k(t)\,ds\,dt$$

$$+\int_0^T u_k(t)^* U(t+\tau) u_k(t)\,dt, \tag{3.2.53}$$

$$\tilde{U}_\tau(t,s) \equiv B(s+\tau)^* \int_{\max\{t,s\}}^T \Phi_\tau(r,s)^* W(r+\tau)\Phi_\tau(r,t)\,dr B(t+\tau), \tag{3.2.54}$$

$$\hat{U}_\tau(t,s) \equiv \begin{cases} B(s+\tau)^*\Phi_\tau(t,s)^* R(t+\tau), & \text{if } s \le t, \\ R(s+\tau)^*\Phi_\tau(s,t) B(t+\tau), & \text{if } s \ge t, \end{cases} \tag{3.2.55}$$

for $k = 0, 1, 2, \ldots$. Hence the optimal control $\hat{u}(t) = \{\hat{u}_k(t)\}$ can be written as (cf.(3.2.37)):

$$\begin{aligned} \hat{u}_k(t) &= \left(-(\mathcal{U}(\tau) + \mathcal{B}(\tau)^*\mathcal{P}(\tau)\mathcal{B}(\tau))^{-1}(\mathcal{R}(\tau)^* + \mathcal{B}(\tau)^*\mathcal{P}(\tau)\mathcal{A}(\tau))\hat{x}_k\right)(t) \\ &\equiv (\mathcal{K}_\tau \hat{x}_k)(t), \quad k = 0, 1, 2, \ldots, \end{aligned} \tag{3.2.56}$$

where $\mathcal{P}(\tau)$ is the unique symmetric positive definite solution of the algebraic Riccati equation (cf. (3.2.38))

$$\begin{aligned} \mathcal{P}(\tau) &= \mathcal{W}(\tau) + \mathcal{R}(\tau)\mathcal{K}_\tau + \mathcal{K}_\tau^*\mathcal{R}(\tau)^* + \mathcal{K}_\tau^*\mathcal{U}(\tau)\mathcal{K}_\tau \\ &\quad + (\mathcal{A}(\tau) + \mathcal{B}(\tau)\mathcal{K}_\tau)^*\mathcal{P}(\tau)(\mathcal{A}(\tau) + \mathcal{B}(\tau)\mathcal{K}_\tau). \end{aligned} \tag{3.2.57}$$

So the minimal achievable value of $J(x_\tau, u)$ is $x_\tau^*\mathcal{P}(\tau)x_\tau$ and for each initial state $x_0 \in \mathbf{R}^n$, when $t = 0$, the minimal achievable value of $J(x_0, u)$ is $x_0{}^*\mathcal{P}(0)x_0$.

Now, the Principal of Optimality [11] shows that the segment of the optimal control $\hat{u}$ on $[0, T]$ must have the property that it minimizes

$$\int_0^T G(t, x, u)\,dt + x(T)^*\mathcal{P}(T)x(T), \tag{3.2.58}$$

subject to the constraint $x(0) = x_0$ and

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{3.2.59}$$

where $x(T)^*\mathcal{P}(T)x(T)$ is the optimal cost starting with $x(T)$ at time T.

This problem is now in the standard LQG form for a finite interval [45]. Then $\hat{u}(t)$, $t \in [0, T]$ must be given by [45]

$$\hat{u}(t) = -\frac{1}{2}U(t)^{-1}(2R(t)^* + B(t)^*Q(t))\hat{x}(t), \tag{3.2.60}$$

where $Q(t)$ satisfies the matrix Riccati differential equation

$$\begin{aligned} &\dot{Q}(t) + A(t)^*Q(t)A(t) + W(t) \\ &- (R(t)^* + B(t)^*Q(t))^* U(t)^{-1} (R(t)^* + B(t)^*Q(t)) = 0 \end{aligned} \tag{3.2.61}$$

with

$$Q(T) = \mathcal{P}(T).$$

We also know that the cost from a given time $\tau \in [0, T]$, i.e.,

$$\int_\tau^\infty G(t, \hat{x}(t), \hat{u}(t))\, dt = \int_\tau^T G(t, \hat{x}(t), \hat{u}(t))\, dt + \hat{x}(T)^*\mathcal{P}(T)\hat{x}(T) \tag{3.2.62}$$

is $\hat{x}(\tau)^*Q(\tau)\hat{x}(\tau)$. But this cost must also be $\hat{x}(\tau)^*\mathcal{P}(\tau)\hat{x}(\tau)$. Since this is true for all possible values of $\hat{x}(\tau)$ and $Q(\tau)$, $\mathcal{P}(\tau)$ are both symmetric, we conclude

$$Q(\tau) \equiv \mathcal{P}(\tau).$$

But $\mathcal{P}(\tau)$ is periodic,

$$\mathcal{P}(0) = \mathcal{P}(T), \quad \mathcal{P}(\tau + T) = \mathcal{P}(\tau).$$

If follows that for the periodic linear quadratic problem we have the theorem

Theorem 3.1 [1] *Suppose the T - periodic system*

$$\dot{x} = A(t)x + B(t)u \tag{3.2.63}$$

is stabilizable in the sense that there is a T periodic matrix $K(t)$ such that all solutions of

$$\dot{x} = (A(t) + B(t)K(t))\, x \tag{3.2.64}$$

tend to zero as $t \to \infty$, and suppose the pair $(W(t), A(t))$ is observable, i.e., there is no non-zero solution $\zeta(t)$ of

$$\dot{\zeta}(t) = A(t)\zeta(t) \tag{3.2.65}$$

such that $W(t)\zeta(t) \equiv 0$, $t \in [0, \infty)$. Then the periodic linear quadratic optimal control problem

$$\min \int_0^\infty G(t, x(t), u(t))\, dt,$$

where $G(t, x, u)$ is defined in (3.2.3), subject to $x(0) = x_0$, $\dot{x}(t) = A(t)x(t) + B(t)u(t)$,

[1] We are happy to acknowledge a private conmunication to D. L. Russell from G. Schmidt of McGill University, Montreal in connection with this proof of Theorem 3.1.

has the unique solution $\hat{u}(t)$, $\hat{x}(t)$, where

$$\begin{aligned}
\hat{u}(t) &\equiv -\frac{1}{2}U(t)^{-1}\left(2R(t)^* + B(t)^*Q(t)\right)\hat{x}(t) \equiv \hat{K}(t)\hat{x}(t), && (3.2.66)\\
\dot{\hat{x}}(t) &= \left[A(t) - \frac{1}{2}B(t)U(t)^{-1}(2R(t)^* + B(t)^*Q(t)\right]\hat{x}(t), && (3.2.67)\\
\hat{x}(0) &= x_0. && (3.2.68)
\end{aligned}$$

Here $Q(t)$ is the unique symmetric positive definite T - periodic solution (i.e. $Q(t+T) = Q(t)$) of the matrix Riccati differential equation

$$\begin{aligned}
&\dot{Q}(t) + A(t)^*Q(t) + Q(t)A(t) + W(t)\\
&- \left(R(t)^* + B(t)^*Q(t)\right)^* U(t)\left(R(t) + B(t)^*Q(t)\right) = 0. \qquad (3.2.69)
\end{aligned}$$

Moreover, $\hat{K}(t)$ is a periodic stabilizing feed back matrix; all solutions of (3.2.66) - (3.2.68) tend to zero as $t \to \infty$ and the optimal cost $J(x_0, \hat{u})$ is $x_0{}^*Q(0)x_0$.

3.3 The Nonlinear Periodic System as a Nonlinear Discrete System

Let us consider a nonlinear T - periodic control system, with $\cdot$ denoting $\frac{d}{dt}$,

$$\dot{x} = F(t, x, u), \quad t \in [0, \infty), \quad x \in \mathbf{R}^n, \quad u \in \mathbf{R}^m, \tag{3.3.1}$$

with

$$F(t + T, x, u) = F(t, x, u),$$

for some period $T > 0$, for all values of (t, x, u) under consideration. We assume that F and its first and second order partial derivatioves with respect to x and u are continuous functions of (t, x, u) for t as indicated and for (x, u) in a neighborhood of the origin in $\mathbf{R}^{n+m}$. Without loss of generatlity, we can assume that the system has $x = 0$, $u = 0$ as a critical point; i.e.,

$$F(t, 0, 0) = 0, \quad t \in [0, \infty).$$

If we define

$$A(t) = \frac{\partial F}{\partial x}(t, 0, 0), \quad B(t) = \frac{\partial F}{\partial u}(t, 0, 0)$$

then $F(t, x, u)$ can be written as

$$F(t, x, u) = A(t)x + B(t)u + f(t, x, u), \tag{3.3.2}$$

where $f(t,x,u)$ is the higher order remainder terms and is continuously differentiable with respect to x, u with

$$f(t,0,0)=0,\ \frac{\partial f}{\partial x}(t,0,0)=0,\ \frac{\partial f}{\partial u}(t,0,0)=0.$$

With this system we consider a cost functional of the form

$$J(x_0,u)=\int_0^\infty G(t,x(t),u(t))\,dt \tag{3.3.3}$$

with G is also T - periodic in t, i.e.,

$$G(t+T,x,u)=G(t,x,u),$$

and $G(t,x,u)$ twice continuously differentiable on a domain in $[0,\infty)\times\mathbf{R}^n\times\mathbf{R}^m$. We further suppose that $G(t,x,u)$ takes the form

$$G(t,x,u)=(x^*,u^*)\begin{pmatrix} W(t) & R(t)\\ R^*(t) & U(t)\end{pmatrix}\begin{pmatrix} x\\ u\end{pmatrix}+g(t,x,u), \tag{3.3.4}$$

where $g(t,x,u)$ is the higher order remaninder terms and is continuously differentiable with $g(t,0,0)=0$, and its first and second order partial derivatives with respect to x and u also vanish at $(t,0,0)$.

We seek an m - dimensional vector feedback function of the state $x(t)$, $u(t)=u(x(t))$, which makes the cost function (3.3.3) as small as possible for all initial states near an equilibrium point for (3.3.1) in $\mathbf{R}^n\times\mathbf{R}^m$.

Clearly the matrix $\mathcal{W}(t)=\begin{pmatrix} W(t) & R(t)\\ R^*(t) & U(t)\end{pmatrix}$ in (3.3.4) also T - periodic; we will assume in addition that it is positive definite symmetric, uniformly with respect to t. From the periodicity property it is enough to assume these uniformity properties are valid for $t \in [0, T]$ in order to conclude that they are also valid for $t \in [0, \infty)$. We define the space of admissible controls to be $l^2(C_m[0, \infty))$ which is a Banach space defined as follows

$$l^2(C_m[0,\infty)) = \left\{ u \in C_m[0,\infty) \,\middle|\, \sum_{k=0}^{\infty} \sup_{t \in [(k-1)T, kT]} \|u(t)\|^2 < \infty \right\}.$$

Then

$$u_k(t) \equiv u(t + kT) \in C_m[0, T), \quad k = 0, 1, 2, \dots .$$

Hence the solution of (3.3.1) with $F(t, x, u)$ in (3.3.2), $x(t)$, at $t = t + kT$ can be written as

$$\begin{aligned} x(t + kT) &= \Phi(t + kT, kT)x(kT) + \int_{kT}^{t+kT} \Phi(t + kT, \tau)B(\tau)u(\tau)\, d\tau \\ &\quad + \int_{kT}^{t+kT} \Phi(t + kT, \tau) f(\tau, x(\tau), u(\tau))\, d\tau, \end{aligned} \tag{3.3.5}$$

where $\Phi(t, \tau)$ is the fundamental matrix defined in (3.2.5). Let us define

$$x_k = x(kT) \in \mathbf{R}^n, \quad k = 0, 1, 2, \dots .$$

By the periodicity of $\Phi(t, \tau)$, $B(t)$ and $f(t, x, u)$, and with the change of variable: $\tau = s + kT$, for $k = 0, 1, 2, \dots$, we have (3.3.5) as follows:

$$\begin{aligned} x(t + kT) &= \Phi(t, 0)x_k + \int_0^t \Phi(t, s)B(s)u(s)\, ds \\ &\quad + \int_0^t \Phi(t, s) f(s, x(s + kT), u_k(s))\, ds. \end{aligned} \tag{3.3.6}$$

If we define $\Phi(t) = \Phi(t,0)$ and $(\theta u_k)(t) = \int_0^t \Phi(t,s)B(s)u_k(s)\,ds$, just as in (3.2.16) and (3.2.17), then we can solve (3.3.6) for $x(t+kT)$ on $[0,T]$ as an integral equation depending parametrically on x_k and $u_k(\cdot)$, i.e.,

$$\begin{aligned} x(t+kT) &= \Phi(t)x_k + (\theta u_k)(t) + x^N(t,x_k,u_k), \\ &= x^L(t,x_k,u_k) + x^N(t,x_k,u_k), \quad k = 0,1,2,\dots, \end{aligned} \tag{3.3.7}$$

where $x^N(t,x_k,u_k) = o(\|x_k\| + \|u_k\|)$ uniformly for $t \in [0,T]$, and $x^L(t,x_k,u_k) \equiv \Phi(t)x_k + (\theta u_k)(t)$ is the linear part for $x(t+kT)$. Hence we can define $x(t+kT)$ as a function which depends only on x_k and u_k,

$$\tilde{x}(t,x_k,u_k) \equiv x(t+kT), \quad k = 0,1,2,\dots. \tag{3.3.8}$$

If we define $\mathcal{A} = \Phi(T,0)$, $\mathcal{B}u_k = \int_0^T \Phi(T,s)B(s)u_k(s)\,ds$, as same as in (3.2.9) and (3.2.12) on page 87, when $t = T$, from (3.3.6) we have

$$x_{k+1} = \mathcal{A}x_k + \mathcal{B}u_k + \int_0^T \Phi(T,s)f(s,x(s+kT),u_k(s))\,ds, \quad k = 0,1,2,\dots. \tag{3.3.9}$$

From (3.3.8), we can define $\int_0^T \Phi(T,s)f(s,x(s+kT),u_k(s))\,ds$ as a function of x_k and u_k as follows:

$$\tilde{f}(x_k,u_k) \equiv \int_0^T \Phi(T,s)f(s,x(s+kT),u_k(s))\,ds, \quad k = 0,1,2,\dots. \tag{3.3.10}$$

Therefore, the nonlinear T - periodic control system (3.3.1) can be written as

$$x_{k+1} = \tilde{F}(x_k,u_k) = \mathcal{A}x_k + \mathcal{B}u_k + \tilde{f}(x_k,u_k), \quad k = 0,1,2,\dots, \tag{3.3.11}$$

with $\tilde{f}(x,u) = O(\|x\|^2 + \|u\|^2)$ as $\|x\|$, $\|u\| \to 0$.

Now let us consider the cost function (3.3.3) with $G(t,x,u)$ is defined in (3.3.4). First of all we can write the part $\int_0^\infty g(t,x(t),u(t))\,dt$ as follows,

$$\begin{aligned}
&\int_0^\infty g(t,x(t),u(t))\,dt = \sum_{k=0}^{\infty}\int_{kT}^{(k+1)T} g(t,x(t),u(t))\,dt \\
= \;& \sum_{k=0}^{\infty}\int_0^T g(s+kT, x(s+kT), u(s+kT))\,ds = \sum_{k=0}^{\infty}\int_0^T g(s, x(s+kT), u_k(s))\,ds \\
= \;& \sum_{k=0}^{\infty}\int_0^T g(s, \tilde{x}(s,x_k,u_k), u_k(s))\,ds,
\end{aligned}$$

by using (3.3.8) for $x(s+kT)$, which shows that the higher order term $\int_0^\infty g(t,x(t),u(t))\,dt$ in the cost function (3.3.3) can be written as a function of x_k and u_k. Let

$$g_1(x_k,u_k) \equiv \int_0^T g(s,\tilde{x}(s,x_k,u_k),u_k(s))\,ds, \quad k = 0,1,2,\ldots, \tag{3.3.12}$$

then $\int_0^\infty g(t,x(t),u(t))\,dt$ in the cost function (3.3.3) becomes

$$\int_0^\infty g(t,x(t),u(t))\,dt = \sum_{k=0}^{\infty} g_1(x_k,u_k). \tag{3.3.13}$$

If we write the total cost function (3.3.3) as

$$\int_0^\infty G(t,x(t),u(t))\,dt = \sum_{k=0}^{\infty}\int_{kT}^{(k+1)T} G(t,x(t),u(t))\,dt, \tag{3.3.14}$$

then by the periodicity of the system (3.3.1) and $G(t,x,u)$ in the cost function and use (3.3.12), we have

$$
\begin{aligned}
&\int_{kT}^{(k+1)T} G(t, x(t), u(t))\, dt \\
= &\int_{kT}^{(k+1)T} (x(t)^*, u(t)^*) \begin{pmatrix} W(t) & R(t) \\ R(t)^* & U(t) \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} dt + \int_{kT}^{(k+1)T} g(t, x(t), u(t))\, dt \\
= &\int_0^T (x(t+kT)^*, u_k(t)^*) \begin{pmatrix} W(t) & R(t) \\ R(t)^* & U(t) \end{pmatrix} \begin{pmatrix} x(t+kT) \\ u_k(t) \end{pmatrix} dt + g_1(x_k, u_k), \\
&\qquad\qquad\qquad\qquad k = 0, 1, 2, \ldots . \qquad (3.3.15)
\end{aligned}
$$

Substitute (3.3.7) for $x(t+kT)$, (3.3.15) yields

$$
\begin{aligned}
&\int_{kT}^{(k+1)T} G(t, x(t), u(t))\, dt \\
= &\int_0^T \left((x^L(t, x_k, u_k) + x^N(t, x_k, u_k))^*, u_k(t)^*\right) \begin{pmatrix} W(t) & R(t) \\ R(t)^* & U(t) \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} x^L(t, x_k, u_k) + x^N(t, x_k, u_k) \\ u_k(t) \end{pmatrix} dt + g_1(x_k, u_k) \\
= &\int_0^T \left(x^L(t, x_k, u_k)^*, u_k(t)^*\right) \begin{pmatrix} W(t) & R(t) \\ R(t)^* & U(t) \end{pmatrix} \begin{pmatrix} x^L(t, x_k, u_k) \\ u_k(t) \end{pmatrix} dt \\
&\quad + g_0(x_k, u_k) + g_1(x_k, u_k), \quad k = 0, 1, 2, \ldots , \qquad (3.3.16)
\end{aligned}
$$

where $g_0(x, u) = o(\|x\|^2 + \|u\|^2)$ uniformly for $t \in [0, T]$ and $g_1(x, u)$ is defined as in (3.3.12). Since the linear part in (3.3.7) for $x(t+kT)$ is $x^L(t, x_k, u_k) = \Phi(t)x_k + (\theta u_k)(t)$, which is as the same as (3.2.18) with $(\theta u_k)(t)$ defined as in (3.2.17), by using the result in section 2.2 for the linear T - periodic case and defining

$$
\tilde{g}(x_k, u_k) \equiv g_0(x_k, u_k) + g_1(x_k, u_k), \quad k = 0, 1, 2, \ldots , \qquad (3.3.17)
$$

we see that the total cost (3.3.3) for nonlinear T - periodic case takes the form

$$J(x_0,u) = \sum_{k=0}^{\infty} \left\{ x_k{}^* \mathcal{W} x_k + 2(x_k, \mathcal{R} u_k)_{R^n} + (u_k, \mathcal{U} u_k)_{L^2_m[0,T]} + \tilde{g}(x_k,u_k) \right\}. \quad (3.3.18)$$

We define

$$\tilde{G}(x_k,u_k) = x_k{}^* \mathcal{W} x_k + 2(x_k, \mathcal{R} u_k)_{R^n} + (u_k, \mathcal{U} u_k)_{L^2_m[0,T]} + \tilde{g}(x_k,u_k), \quad (3.3.19)$$

for $k = 0,1,2,\dots$, with $\tilde{g}(x,u) = o(\|x\|^2 + \|u\|^2)$ as $\|x\|$, $\|u\| \to 0$, and where $\mathcal{W}$, $\mathcal{R}$ and $\mathcal{U}$ are defined as in (3.2.20), (3.2.25) and (3.2.30), respectively. Therefore, the nonlinear T - periodic control system (3.3.1) and the cost function (3.3.3),

$$\dot{x} = F(t,x,u) = A(t)x + B(t)u + f(t,x,u), \quad (3.3.20)$$

$$J(x_0,u) = \int_0^{\infty} G(t,x(t),u(t))\, dt, \quad (3.3.21)$$

with $G(t,x,u)$ as defined in (3.3.4) are equivalent to a nonlinear discrete system (3.3.11) and the cost function (3.3.18) for $x = \{x_k\}$, $u = \{u_k\}$:

$$x_{k+1} = \tilde{F}(x_k,u_k) = \mathcal{A}x_k + \mathcal{B}u_k + \tilde{f}(x_k,u_k), \quad k = 0,1,2,\dots, \quad (3.3.22)$$

$$J(x_0,u) = \sum_{k=0}^{\infty} \tilde{G}(x_k,u_k), \quad (3.3.23)$$

with $\tilde{G}(x,u)$ is defined in (3.3.19).

Now we can use all the results developed in the previous chapter for general discrete systems. If we let $\hat{x} = \{\hat{x}_k\}$ and $\hat{u} = \{\hat{u}_k\}$ be the optimal control trajectory sequence pair, then by using the stable manifold theorem for general discrete systems the optimal control $\hat{u} = \{\hat{u}_k\}$ can be written as

$$\hat{u}_k(t) = \left(-(\mathcal{U} + \mathcal{B}^*\mathcal{P}\mathcal{B})^{-1}(\mathcal{R}^* + \mathcal{B}^*\mathcal{P}\mathcal{A})\hat{x}_k + r_u(\hat{x}_k)\right)(t) \equiv (\mathcal{K}\hat{x}_k)(t) + (r_u(\hat{x}_k))(t), \tag{3.3.24}$$

for $k = 0, 1, 2, \dots$, where

$$\hat{x}_{k+1} = \mathcal{A}\hat{x}_k + \mathcal{B}\hat{x}_k + \tilde{f}(\hat{x}_k, \hat{u}_k), \quad k = 0, 1, 2, \dots , \tag{3.3.25}$$

for $t \in [0, T]$, where $r_u(\hat{x}_k)$ is the higher order term defined in (2.3.11) on page 58, and $\mathcal{P}$ is the unique symmetric positive definite solution of the equation (3.2.38) on page 94. Since the linear part of the nonlinear periodic systems (3.3.20) and (3.3.21) as a nonlinear discrete system (3.3.22) and (3.3.23), consists of the equations (3.2.1) and (3.2.2) on page 85 of the linear discrete optimization problem (3.2.13) and (3.2.33) on page 92, the quadratic part of the function $\tilde{G}$ in (3.3.19) can be separated out by writing

$$\tilde{G}(\hat{x}_k, \hat{u}_k) = \mathcal{G}(\hat{x}_k, \hat{u}_k) + \tilde{g}(\hat{x}_k, \hat{u}_k), \quad k = 0, 1, 2, \dots , \tag{3.3.26}$$

where $\mathcal{G}(\hat{x}_k, \hat{u}_k)$ consists of quadratic terms which agree with (3.2.34) on page 93 in the linear case. Hence by using the result of Section 2.2, we also have

$$\hat{u}_k(t) = K(t)\hat{x}_k + (r_u(\hat{x}_k))(t), \quad k = 0, 1, 2, \dots , \tag{3.3.27}$$

or

$$\hat{u}(t + kT) = K(t)\hat{x}(kT) + r_u((\hat{x}_k))(t). \tag{3.3.28}$$

From the result in the previous chapter for the discrete case, we know that the minimal cost $\hat{J}(x_0)$ can be written as

$$\hat{J}(x_0) = x_0{}^*\mathcal{P}x_0 + \hat{\jmath}(x_0), \tag{3.3.29}$$

where $\mathcal{P}$ is the unique symmetric positive definite solution of the algebraic Riccati equation (3.2.38) on page 94 and $\hat{\jmath}(x)$ is a function of class C^2 with $\hat{\jmath}(x) = o(\|x\|^2)$, as $\|x\| \to 0$.

With this we have a beginning for the proof of

Theorem 3.2 *Consider the periodic nonlinear system*

$$\dot{x} = F(t, x, u) = A(t)x + B(t)u + f(t, x, u) \tag{3.3.30}$$

as described above, and the problem

$$\min_u \int_0^\infty G(t, x(t), u(t))\, dt \tag{3.3.31}$$

where $x(t)$, $u(t)$ together satisfy (3.3.30) and $x(0) = x_0$. Then, for sufficiently small x_0 the optimal control $\hat{u}(t)$ yielding the optimal solution $\hat{x}(t)$ is characterized by a nonlinear T - periodic feedback relation

$$\hat{u}(t) = k(t, \hat{x}(t)) = K(t)\hat{x}(t) + \tilde{k}(t, \hat{x}(t)) \tag{3.3.32}$$

where

$$K(t) = -\frac{1}{2}U(t)^{-1}\left(B(t)^*Q(t) + 2R(t)^*\right) \tag{3.3.33}$$

and $\tilde{k}(t, x)$ is the higher order remainder terms and is continuously differentiable *with* $k(t,0) \equiv 0$, $\frac{\partial k}{\partial x}(t,0) \equiv 0$. *Moreover,* $Q(t)$ *is the unique symmetric positive definite* T *- periodic solution of the matrix Reccati differential equation*

$$\begin{aligned} &\dot{Q}(t) + A(t)^* Q(t) + Q(t)A(t) + W(t) \\ &- (Q(t)B(t) + R(t))\, U(t)^{-1} (B(t)^* Q(t) + R(t)^*) = 0. \end{aligned} \tag{3.3.34}$$

The optimal cost has the form

$$\hat{J}(x_0) = {x_0}^* \mathcal{P} x_0 + j(x_0) = {x_0}^* Q(0) x_0 + j(x_0) \tag{3.3.35}$$

where $\mathcal{P}$ *is the unique symmetric positive definite solution of the algebraic Riccati equation (3.2.38) and* j *is twice continuous differentiable with* $j(0) = 0$, $\frac{\partial j}{\partial x}(0) = 0$, $\frac{\partial^2 j}{\partial x^2}(0) = 0$.

Proof. The basic idea of the proof is to combine our knowledge of the form of the optimal cost $\hat{J}(x_0)$ obtained from the discrete theory with easily obtained results for optimal control of differential equations on a finite interval.

Applying the principle of optimality the original continuous periodic nonlinear optimal control problem

$$\text{minimize}: \int_0^\infty G(t, x(t), u(t))\, dt \tag{3.3.36}$$

is, taking the periodicity of the system and cost functional into account, the same as

$$\text{minimize}: \int_0^T G(t, x(t), u(t))\, dt + \hat{J}(x(T)), \tag{3.3.37}$$

where $\hat{J}$ has been defined in (3.3.35). Let us suppose $\hat{x}(t)$ and $\hat{u}(t)$ to be the unique solution of the nonlinear T - periodic control system (3.3.1) and (3.3.3). Then from the equivalence of the discrete and continuous formulations we have the minimum cost

$$\hat{J}(x_0) = \int_0^\infty G(t, \hat{x}(t), \hat{u}(t))\, dt. \tag{3.3.38}$$

where

$$\dot{\hat{x}} = F(t, \hat{x}(t), \hat{u}(t)), \tag{3.3.39}$$

$$\hat{x}(0) = x_0, \tag{3.3.40}$$

where $\hat{x}(t)$ and $\hat{u}(t)$ are the optimal control trajectory pair. To solve (3.3.37) we introduce the adjoint equation with boundary condition:

$$\dot{p} = -\frac{\partial F}{\partial x}(t, x(t), u(t))^* p - \frac{\partial G}{\partial x}(t, x(t), u(t))^*, \tag{3.3.41}$$

$$p(T) = \frac{1}{2}\frac{\partial \hat{J}}{\partial x}(x(T))^*. \tag{3.3.42}$$

We also need the following lemma (cf. Lemma 1.5 on page 8)

Lemma 3.3 *Let $F(t, x, u)$ and $G(t, x, u)$ be as described above, then there exists a unique continuously differentiable solution $\bar{u}(t, x, u)$ to the equation*

$$p(t)^* \frac{\partial F}{\partial u}(t, x(t), u(t)) + \frac{\partial G}{\partial u}(t, x(t), u(t)) = 0, \tag{3.3.43}$$

for (x, p) near the origin in $\mathbf{R}^{2n}$ such that $\bar{u}(t, 0, 0) = 0$, for all $t \in [0, \infty)$. Further-

more,

$$\hat{u}(t) \equiv \bar{u}(t, x(t), p(t)) = -\frac{1}{2}U(t)^{-1}(2R(t)^*x(t) + B(t)^*p(t)) + \hat{h}(t, x(t), p(t)), \tag{3.3.44}$$

where $\hat{h}(t, x(t), p(t)) = o(\|x\| + \|p\|)$ *as* $\|x\|$, $\|p\| \to 0$ *in* $\mathbf{R}^n$.

The proof of this lemma will be similarly to the proof of Lemma 1.5, see [31].

Then the state / adjoint system with the optimal control trajectory $\hat{x}(t)$ and $\hat{u}(t)$ has the form

$$\dot{\hat{x}} = F(t, \hat{x}(t), \hat{u}(t)), \tag{3.3.45}$$

$$\dot{p} = -\frac{\partial F}{\partial x}(t, \hat{x}(t), \hat{u}(t))^*p - \frac{\partial G}{\partial x}(t, \hat{x}(t), \hat{u}(t))^*, \tag{3.3.46}$$

with the boundary conditions

$$\hat{x}(0) = x_0 \tag{3.3.47}$$

$$p(T) = \frac{1}{2}\frac{\partial \hat{J}}{\partial x}(\hat{x}(T))^*. \tag{3.3.48}$$

By using the above lemma, Lemma 3.3, we have the optimal control $\hat{u}(t)$ in terms of the optimal trajectory $\hat{x}(t)$ and the solution, $p(t)$, of the adjoint equation as follows:

$$\begin{aligned} \hat{u}(t) &= \bar{u}(t, \hat{x}(t), p(t)) \\ &= -\frac{1}{2}U(t)^{-1}(2R(t)^*\hat{x}(t) + B(t)^*p(t)) + \hat{h}(t, \hat{x}(t), p(t)). \end{aligned} \tag{3.3.49}$$

For any τ, $0 < \tau < T$, let us consider the initial condition at $t = \tau$:

$$p(\tau) = p_\tau,$$

$$\hat{x}(\tau) = \hat{x}_\tau.$$

If we solve forward on $[\tau, T]$, we will have $p(T)$, $\hat{x}(T)$. Let us call this solution $p(t, p_\tau, \hat{x}_\tau)$ and $\hat{x}(t, p_\tau, \hat{x}_\tau)$ on $[\tau, T]$. At time $t = T$ we must satisfy the terminal condition corresponding to (3.3.42), i.e.,

$$p(T, p_\tau, \hat{x}_\tau)^* = \frac{1}{2}\frac{\partial J}{\partial x}(\hat{x}(T, p_\tau, \hat{x}_\tau)). \tag{3.3.50}$$

Let us define

$$\phi(q, y) = p(T, q, y)^* - \frac{1}{2}\frac{\partial \hat{J}}{\partial x}(\hat{x}(T, q, y)), \tag{3.3.51}$$

then

$$\phi(p_\tau, \hat{x}_\tau) = 0$$

is a set of n equations which we intend to solve for p_τ in terms of $\hat{x}_\tau$ so that the maximum principle will give the optimal control $\hat{u}(t)$ in terms of $\hat{x}(t)$ alone, rather than in terms of $\hat{x}(t)$ and $p(t)$ as in (3.3.49). For $p_\tau = 0$, $\hat{x}_\tau = 0$, we observe $p(T, 0, 0) = 0$ and $\hat{x}(T, 0, 0) = 0$, so that we have $\frac{\partial \hat{J}}{\partial x}(0) = 0$ from (3.3.51). We wish to determine p_τ in terms of $\hat{x}_\tau$ so that $\phi(p_\tau(\hat{x}_\tau), \hat{x}_\tau) \equiv 0$ for $\hat{x}_\tau$ near the origin. By the implicit function theorem [18] we can do this if $\frac{\partial \phi}{\partial p_\tau}(0, 0)$ is non-singular.

Let us consider the linear case first. The variational equation of the nonlinear system based on $p(t) \equiv 0$, $\hat{x}(t) \equiv 0$, and the quadratic part of the cost function, yields a linear quadratic control problem for which we have the same two point boundary value problem but with $F(t, x(t), u(t))$ (cf. (3.3.2)) and $G(t, x(t), u(t))$ (cf. (3.3.4)) replaced by just their linear part and quadratic parts; viz:

$$F^L(t, x(t), u(t)) = A(t)x(t) + B(t)u(t), \tag{3.3.52}$$

$$G^L(t, x(t), u(t)) = (x(t)^*, u(t)^*) \begin{pmatrix} W(t) & R(t) \\ R(t)^* & U(t) \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}. \tag{3.3.53}$$

If $\hat{x}(t)$ and $\hat{u}(t)$ is the unique optimal control trajectory pair of the nonlinear system (3.3.1) and (3.3.3), then we can write

$$\hat{x}(t) = \hat{x}^L(t) + \hat{x}^N(t),$$

$$\hat{u}(t) = \hat{u}^L(t) + \hat{u}^N(t),$$

where $\hat{x}^L(t)$, $\hat{u}^L(t)$ are the linear part of $\hat{x}(t)$, $\hat{u}(t)$, and $\hat{x}^N(t)$, $\hat{u}^N(t)$ are the higher order remainder terms of $\hat{x}(t)$, $\hat{u}(t)$, respectively. Hence it is easy to see that $x^L(t)$ and $u^L(t)$ is the unique optimal control trajectory pair for the linear system, then the linear T - periodic optimal control system and optimal cost can be written as

$$\dot{\hat{x}}^L = A(t)\hat{x}^L + B(t)\hat{u}^L, \tag{3.3.54}$$

$$\hat{J}^L(x_0) = \int_0^\infty G^L(t, \hat{x}^L(t), \hat{u}^L(t))\, dt. \tag{3.3.55}$$

By Theorem 2.1, p. 197, in [45], the optimal control for this problem is characterized on the interval $[0, T]$ by

$$\hat{u}^L(t) = -\frac{1}{2}U(t)^{-1}[B(t)^* p^L(t) + 2R(t)^* \hat{x}^L(t)], \tag{3.3.56}$$

where $\hat{x}^L(t)$ and $p^L(t)$ satisfy the linear two point boundary value problem consisting of the equations (cf. (3.3.45) and (3.3.46))

$$\dot{\hat{x}}^L = A(t)\hat{x}^L(t) + B(t)\hat{u}^L(t), \tag{3.3.57}$$

$$\dot{p}^L = -A(t)^* p^L(t) - 2W(t)\hat{x}^L(t) - 2R(t)\hat{u}^L(t) \tag{3.3.58}$$

and the boundary conditions (cf. (3.3.47) and (3.3.48))

$$\hat{x}^L(0) = x_0, \tag{3.3.59}$$

$$p^L(T) = \mathcal{P}(T)\hat{x}^L(T), \tag{3.3.60}$$

where $p^L(t)$ is the solution of the linear system and hence it is the linear part of the solution of the nonlinear system (3.3.46) with the boundary condition (3.3.48), i.e. if $p(t)$ is the solution of nonlinear system (3.3.46) with the boundary condition (3.3.48) then we can write $p(t)$ as

$$p(t) = p^L(t) + p^N(t),$$

where $p^N(t)$ is the nonlinear remainder term. Let us substitute the optimal control (3.3.56) into (3.3.57) and (3.3.58), we have

$$\dot{\hat{x}}^L = \left(A(t) - B(t)U(t)^{-1}R(t)^*\right)\hat{x}^L(t) - \frac{1}{2}B(t)U(t)^{-1}B(t)^* p^L(t), \tag{3.3.61}$$

$$\dot{p}^L = 2\left(R(t)U(t)^{-1}R(t)^* - W(t)\right)\hat{x}^L + \left(-A(t)^* + R(t)U(t)^{-1}B(t)^*\right)p^L(t). \tag{3.3.62}$$

Let us define

$$C(t) \equiv A(t) - B(t)U(t)^{-1}R(t)^*, \tag{3.3.63}$$

$$D(t) \equiv -\frac{1}{2}B(t)U(t)^{-1}B(t)^*, \tag{3.3.64}$$

$$E(t) \equiv 2(R(t)U(t)^{-1}R(t)^* - W(t)), \tag{3.3.65}$$

then

$$\begin{pmatrix} \dot{\hat{x}}^L \\ p^L \end{pmatrix} = \begin{pmatrix} C(t) & D(t) \\ E(t) & -C(t)^* \end{pmatrix} \begin{pmatrix} \hat{x}^L \\ p^L \end{pmatrix}. \tag{3.3.66}$$

If we make the change of variables

$$\hat{x}^L = \hat{y}, \tag{3.3.67}$$

$$p^L = q + Q(t)\hat{x}^L, \tag{3.3.68}$$

we obtain the decoupled system

$$\begin{pmatrix} \dot{\hat{y}} \\ q \end{pmatrix} = \begin{pmatrix} C(t) + D(t)Q(t) & D(t) \\ 0 & -(C(t) + D(t)Q(t))^* \end{pmatrix} \begin{pmatrix} \hat{y} \\ q \end{pmatrix}, \tag{3.3.69}$$

where $Q(t)$ satisfies the equation

$$\dot{Q}(t) + Q(t)C(t) + Q(t)D(t)Q(t) + C(t)^*Q(t) - E(t) = 0.$$

Thus $Q(t)$ satisfies the matrix Riccati differential equation:

$$\begin{aligned} &\dot{Q}(t) + A(t)^*Q(t) + Q(t)A(t) + W(t) \\ &- (Q(t)B(t) + R(t))\, U(t)^{-1}\, (B(t)^*Q(t) + R(t)^*) = 0, \end{aligned} \tag{3.3.70}$$

with

$$Q(T) = Q(0) = \mathcal{P}. \tag{3.3.71}$$

If we let

$$\tilde{C}(t) \equiv C(t) + D(t)Q(t), \tag{3.3.72}$$

then

$$\begin{pmatrix} \dot{\hat{y}} \\ q \end{pmatrix} = \begin{pmatrix} \tilde{C}(t) & D(t) \\ 0 & -\tilde{C}(t)^* \end{pmatrix} \begin{pmatrix} \hat{y} \\ q \end{pmatrix}, \tag{3.3.73}$$

and

$$q(t) = \Phi(t,\tau)q(\tau), \tag{3.3.74}$$

where $\Phi(t,\tau)$ is the fundamental matrix for

$$\dot{q} = -\tilde{C}(t)^* q. \tag{3.3.75}$$

Then $\Phi(t,\tau)$ is nonsingular, and

$$\frac{\partial q(T)}{\partial q(\tau)} = \Phi(T,\tau). \tag{3.3.76}$$

Also we can write

$$\hat{y}(t) = \Psi(t,\tau)\hat{y}(\tau) + \int_\tau^t \Psi(t,s)D(s)q(s)\,ds, \tag{3.3.77}$$

where $\Psi(t,\tau)$ is the fundamental matrix for

$$\dot{y} = \tilde{C}(t)y. \tag{3.3.78}$$

Hence

$$\begin{aligned}
\hat{y}(T) &= \Psi(T,\tau)\hat{y}(\tau) + \int_\tau^T \Psi(T,s)D(s)q(s)\,ds \\
&= \Psi(T,\tau)\hat{y}(\tau) + \int_\tau^T \Psi(T,s)D(s)\Phi(s,\tau)\,dsq(\tau) \\
&= \Psi(T,\tau)\hat{y}(\tau) + \Delta(T,\tau)q(\tau), \qquad (3.3.79)
\end{aligned}$$

where $\Delta(T,\tau)$ is defined as

$$\Delta(T,\tau) \equiv \int_\tau^T \Psi(T,s)D(s)\Phi(s,\tau)\,ds. \qquad (3.3.80)$$

Substituting (3.3.74) for $q(T)$ into (3.3.68) we have

$$\begin{aligned}
p^L(T) &= q(T) + Q(T)\hat{x}^L(T) \\
&= \Phi(T,\tau)q(\tau) + Q(T)\hat{x}^L(T) \\
&= \Phi(T,\tau)(p^L(\tau) - Q(t)\hat{x}^L(\tau)) + Q(T)\hat{x}^L(T) \\
&= \Phi(T,\tau)p^L(\tau) + Q(T)\hat{x}^L(T) - \Phi(T,\tau)Q(t)\hat{x}^L(\tau). \qquad (3.3.81)
\end{aligned}$$

Also substituting (3.3.68) for $q(\tau)$ into (3.3.79) we see that $\hat{x}^L(T)$ has the following form:

$$\begin{aligned}
\hat{x}^L(T) &= \Psi(T,\tau)\hat{x}^L(\tau) + \Delta(T,\tau)(p^L(\tau) - Q(\tau)\hat{x}^L(\tau)) \\
&= (\Psi(T,\tau) - \Delta(T,\tau)Q(\tau))\hat{x}^L(\tau) + \Delta(T,\tau)p^L(\tau). \qquad (3.3.82)
\end{aligned}$$

Hence (3.3.81) implies:

$$p^L(T) = \Phi(T,\tau)p^L(\tau) - \Phi(T,\tau)Q(t)\hat{x}^L(\tau)$$

$$
\begin{aligned}
&+Q(T)\left[(\Psi(T,\tau)-\Delta(T,\tau)Q(\tau))\hat{x}^L(\tau)+\Delta(T,\tau)p^L(\tau)\right] \\
= \quad &(\Phi(T,\tau)+Q(T)\Delta(T,\tau))p^L(\tau) \\
&+\left[Q(T)(\Psi(T,\tau)-\Delta(T,\tau)Q(\tau))-\Phi(T,\tau)Q(t)\right]\hat{x}^L(\tau). \quad (3.3.83)
\end{aligned}
$$

Therefore, we have

$$
\frac{\partial p^L(T)}{\partial p_\tau} = \Phi(T,\tau) + Q(T)\Delta(T,\tau). \tag{3.3.84}
$$

From (3.3.82), the partial derivative of $\hat{x}^L(T)$ respect to p_τ takes the form:

$$
\frac{\partial \hat{x}^L(T)}{\partial p_\tau} = \Delta(T,\tau). \tag{3.3.85}
$$

Hence

$$
\frac{\partial p^L(T)}{\partial p_\tau} = \Phi(T,\tau) + Q(T)\frac{\partial \hat{x}^L(T)}{\partial p_\tau}. \tag{3.3.86}
$$

Since for the nonlinear system we defined the function $\phi(q,y)$ in (3.3.51), i.e.

$$
\phi(q,y) = p(T,q,y)^* - \frac{1}{2}\frac{\partial \hat{J}}{\partial x}(\hat{x}(T,q,y)),
$$

then $\phi(q,y)$ can be separated as

$$
\phi(q,y) = \phi^L(q,y) + \phi^N(q,y),
$$

where $\phi^L(q,y)$ is the linear part and $\phi^N(q,y)$ is the higher order remainder terms. It is easy to see that the linear part of the function $\phi(q,y)$, $\phi^L(q,y)$, satisfies (3.3.51) also for linear system, i.e.,

$$
\phi^L(q,y) = p^L(T,q,y)^* - \frac{1}{2}\frac{\partial \hat{J}^L}{\partial x}(\hat{x}^L(T,q,y)),
$$

and the partial derivative of ϕ^L with respect to p_τ at point $(0,0)$ has the following form:

$$\frac{\partial \phi^L}{\partial p_\tau}(0,0) = \frac{\partial p^L(T)}{\partial p_\tau} - \frac{1}{2}\frac{\partial^2 \hat{J}^L}{\partial x^2}(0)\frac{\partial \hat{x}^L(T)}{\partial p_\tau}, \tag{3.3.87}$$

where the optimal cost for the linear quadratic problem (3.3.54), (3.3.55) starting with $\hat{x}^L(T)$ at time T is

$$\hat{J}^L(\hat{x}^L(T)) = \hat{x}^L(T)^* \mathcal{P}(T)\hat{x}^L(T). \tag{3.3.88}$$

Now the unique positive definite solution $Q(t)$ of the matrix Riccati differential equation (3.3.70) at time T satisfies the condition

$$Q(T) = \mathcal{P}(T). \tag{3.3.89}$$

Hence

$$\frac{\partial^2 \hat{J}^L}{\partial x^2}(0) = 2Q(T). \tag{3.3.90}$$

If we substitute (3.3.86) and (3.3.90) into (3.3.87), $\frac{\partial \phi^L}{\partial p_\tau}$ will have the form:

$$\begin{aligned} \frac{\partial \phi^L}{\partial p_\tau}(0,0) &= \Phi(T,\tau) + Q(T)\frac{\partial \hat{x}^L(T)}{\partial p_\tau} - Q(T)\frac{\partial \hat{x}^L(T)}{\partial p_\tau} \\ &= \Phi(T,\tau). \end{aligned} \tag{3.3.91}$$

Since $\phi^L(T,\tau)$ is a fundamental matrix solution of (3.3.75), so it is nonsingular. Therefore,

$$\frac{\partial \phi}{\partial p_\tau}(0,0) = \Phi(T,\tau) + \text{higher order terms}, \tag{3.3.92}$$

so that $\frac{\partial \phi}{\partial p_\tau}(0,0)$ is nonsingular. By the implicit function theorem [18], there exists a continuously differentiable function λ such that

$$p_\tau = \lambda(\tau, \hat{x}_\tau),$$

or

$$p(\tau) = \lambda(\tau, \hat{x}(\tau)). \tag{3.3.93}$$

Since τ was chosen arbitrary in $[0, T]$, then (3.3.93) is also true for any $t \in [0, T]$, thus we have

$$p(t) = \lambda(\tau, \hat{x}(t)), \quad t \in [0, T]. \tag{3.3.94}$$

Now we have $p(t)$ is a function of $\hat{x}(t)$ then let us substitute (3.3.94) for $p(t)$ into (3.3.49),

$$\begin{aligned} \hat{u}(t) &= \bar{u}(t, \hat{x}(t), p(t)) = \bar{u}(t, \hat{x}(t), \lambda(t, \hat{x}(t))) \\ &= -\frac{1}{2}U^{-1}(t)[2R(t)^*\hat{x}(t) + B(t)^*\lambda(t, \hat{x}(t))] + \hat{h}(t, \hat{x}(t), \lambda(t, \hat{x}(t))). \end{aligned} \tag{3.3.95}$$

Therefore, the optimal control $\hat{u}(t)$ now is given only in terms of $\hat{x}(t)$ alone and we can write it as

$$\hat{u}(t) = K(t)\hat{x}(t) + \hat{k}(t, \hat{x}(t)), \tag{3.3.96}$$

where $\hat{k}(t, \hat{x}(t))$ is the higher term and $\hat{k}(t, 0) \equiv 0$, $\frac{\partial \hat{k}}{\partial x}(t, 0) \equiv 0$. Now the only thing left for this theorem which we need to prove is that $K(t)$ in (3.3.96) will have the form (3.3.33), i.e.,

$$K(t) = -\frac{1}{2}U(t)^{-1}\left(B(t)^*Q(t) + 2R(t)^*\right).$$

Since $\hat{J}(0,\hat{x}(0)) = \hat{J}(x_0)$ is the minimum cost starting at x_0, when $t = 0$, and $\hat{J}(T,\hat{x}(T)) = \hat{J}(\hat{x}(T))$ is the minimum cost starting at $\hat{x}(T)$, when $t = T$, and (cf. (3.3.48))

$$\frac{\partial \hat{J}}{\partial x}(\hat{x}(T)) = 2p(T)^*$$

and the general argument in [31], then when $t = \tau \in [0, T]$, we have

$$\hat{J}(\tau, 0) = 0, \tag{3.3.97}$$

$$\frac{\partial \hat{J}}{\partial x}(\tau, \hat{x}(\tau)) = 2p(\tau)^* = 2\lambda(\tau, \hat{x}(\tau))^*, \tag{3.3.98}$$

with

$$p(\tau)^* = \lambda(\tau, \hat{x}(\tau))^* = Q(\tau)\hat{x}(\tau) + o\|\hat{x}(\tau)\|, \quad \|\hat{x}(\tau)\| \to 0.$$

Thus the optimal cost can be written as

$$\hat{J}(\tau, \hat{x}(\tau)) = \hat{x}(\tau)^*Q(\tau)\hat{x}(\tau) + \text{higher order terms}.$$

Then the partial derivative of $\hat{J}(\tau, \hat{x}(\tau))$ with respect to x is

$$\frac{\partial \hat{J}}{\partial x}(\tau, \hat{x}(\tau)) = 2Q(\tau)\hat{x}(\tau) + \text{higher order terms}. \tag{3.3.99}$$

It then follows that $K(t)$ in (3.3.96) can be written as

$$K(t) = -\frac{1}{2}U(t)^{-1}(2R(t)^* + B(t)^*Q(t)),$$

which is as same as in (3.3.33). Hence the theorem is proved.

Chapter 4

Conclusions and Future Research

"In today's rapidly progressing science and technology, the field of control theory is at the forefront of the creative interplay of mathematics, engineering, and computer science. Drawing upon these disciplines, control theory brings powerful theoretical results to bear on advanced technologies. As a foundation of control systems engineering, it is at the heart of the new industrial revolution involving automation, computers, and robotics." These successes of control theory attracted many new participants, including mathematicians fascinated by the associated research opportunities. As new tools were introduced in late sixties, "the theory widened and became diverse, to the extent that a 'Tower of Babel' phenomenon has arisen: specialists in one branch may experience diffculties in understanding the work of experts in another. However, to this day the linear - quadratic regulator theory postulated by Kalman plays the role of a universal language accessible to all branches" [15].

Following Kalman, Lukes [31] original work was carried out in the context of autonomous nonlinear ordinary differential equations control systems in the vicinity of a critical point. Much of the work of this thesis consists of variations on his original theme. Throughout this description we refer to the standard linear quadratic theory as the LQ theory and to extensions of that theory to cover nonlinear systems and / or nonquadratic cost functions, in the spirit of Lukes development, as NLNQ (nonlinear, nonquadratic, or perhaps, to the extent that it involves perturbations about the LQ cases, nearly linear, nearly quadratic) theory.

We have extended Lukes' results to time invariant nonlinear discrete systems and we have, in turn, applied those results to get nonlinear control laws for periodic ordinary differential equations systems. The most elementary part of the work concerns the form which the NLNQ theory takes in a setting corresponding to the finite interval LQ case, developed here in Section 3.3. We have shown that the extension from LQ to NLNQ in that case simply involves the implicit funciton theorem in a quite specific setting, the desired nonsingular Jacobian being derived from the LQ theory. The main contribution of this thesis is the development of a setting within which the stable manifold theorem of nonlinear ordinary differential equations theory can be constractively extended to discrete systems and the development of NLNQ theory as it relates to nonautonomous systems of particular type; specifically periodic systems.

Future research directions include development of NLNQ theory in the context of stabilization of autonomous systems with reference to invariant sets other than critical points; e.g., periodic solutions such as occur in the theory of self-excited oscillations. This appears to introduce several essential complications over an beyond those encountered in the present work but the form which the eventual theory is likely to take is emerging. We have some conjectures and speculations as to the form which NLNQ theory may take with reference to arbitary compact invariant sets of an autonomous system of differential equations but so far these are, indeed, speculative.

Many optimal control problems can be solved numerically. The power series approximations Lukes stressed, and which we followed to solve our example is limited to analytic systems. Some modified proofs of the stable manifold theorem, both in the context of nonlinear ordinary differential equations and in the context of nonlinear recursion equations appears to be particularly well adapted to numerical approximation of NLNQ feedback control laws by spline, or other finite element, methods. These are discussed in the article [46], which is related to the work of this thesis.

"Control problems will continue to provide a rich source of very complex mathematical problems. In addition to the payoffs through applications to modern technology, the solution of these problems special efforts will be needed to ensure the continued development of control sciences in ways which take full advantage of present and future opportunities"[15]. If in writing this thesis we succeed in encouraging others to improve upon and extend our work we will have achieved the main purpose of our present efforts.

Bibliography

[1] E. G. Al'Brekht, "On the Optimal Stabilization of Nonlinear Systems", *J. Appl. Math. Mech.*, 25(1961), pp. 1254-1266.

[2] B. D. O. Anderson and J. B. Moore, *Linear Optimal Control*, Prentice - Hall, Inc., Englewood Cliffs, N.J., 1971.

[3] R. Aris, *Discrete Dynamic Programming*, Blaisdell Publishing Co., New York, 1964.

[4] M. Athans and P. L. Falb, "Time - Optimal Control for a Class of Nonlinear Systems", *IEEE Trans. Automatic Control*, AC - 8, 378(1963).

[5] M. Athans and P. L. Falb, *Optimal Control*, McGraw -Hill Book Co., New York, 1966.

[6] M. Aoki, *Optimal control and System Theory in Dynamic Economic Analysis*, North - Holland Publishing Co., New York, 1976.

[7] R. Bellman, *Dynamic Programming*, Princeton University Press, Princeton, New Jersey, 1957.

[8] R. Bellman, "Dynamic Programming, Terminal Control and General Nonlinear Control Processes", *Proc. Nat. Acad. Sci. U. S.*, 43(1957), pp. 927-930.

[9] R. Bellman and S. Dreyfus, *Applied Dynamic Programming*, Princeton Univ. Press, Princeton, New Jersey, 1962.

[10] R. Bellman and R. Kalaba, "Dynamic Programming and Feedback Control," *Proc. IFAC. Congress*, Moscow, 1(1960), pp. 460-464.

[11] G. C. Chow, *Analysis and Control of Dynamic Economic Systems*, John Wiley & Sons, Inc., New York, 1975.

[12] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw - Hill, New York, 1955.

[13] J. Cronin, *Differential Equations*, Marcel Dekker, Inc., New York, 1980.

[14] S. E. Dreyfus, *Dynamic Programming and the Calculus of Variations*, Academic Press Inc., New York, 1965.

[15] W. H. Flemin, *Future Directions in Control Theory: A Mathematical Perspective*, Society for Industrial and Applied Mathematics, Philadelphia, 1988.

[16] J. W. Grizzle, "Feedback Linearization of Discrete - Time Systems", in: A. Bensoussan, J. L. Lions, Eds., *Analysis and Optimization of Systems*, Springer, Berlin - New York, 1986.

[17] G. Hadley, *Nonlinear and Dynamic Programming*, Addison - Wesley Publishing Co., Reading, Mass., 1964.

[18] E. Hewitt and K. Stromberg, *Real and Abstract Analysis*, Springer - Verlay Berlin Heidelberg, 1965.

[19] B. Jakubczyk, "Feedback Linearization of Discrete - Time Systems", Systems & Contro Letters, North - Holland, 9(1987), pp. 441 - 419.

[20] R. E. Kalman, "Contributions to the Theory of Optimal Control", *Bol. Soc. Mat. Mexicana*, 5(1960), pp. 102-119.

[21] R. E. Kalman, "Mathematical Description of Linear Dynamical Systems", *SIAM J. Control*, 1(1963), pp. 152-192.

[22] R. E. Kalman, "When is a Linear Control System Optimal?" *Trans. ASME, Ser. D, J. Basic Eng.*, 86(1964).

[23] R. E. Kalman and R. S. Bucy, "New Results in Linear Prediction and Filtering Theory," *J. Basic Eng., Trans. ASME, Ser. D*, 83(1961), pp. 95-100.

[24] R. E. Kalman, P. L. Falb and M. A. Arbib, *Topics in Mathematical Systems Theory*, McGraw - Hill Book Co., New York, 1969.

[25] R. E. Kalman, Y. Ho and K. S. Narendra, "Controllability of Linear Dynamical Systems", *Contributions to Differential Equations*, Vol. 1, John Wiley & Sons, Inc., New York, 1963, pp.189-213.

[26] J. P. LaSalle, "Complete Stability of a Nonlinear Control System", *Proc. Nat. Acad. Sci.*, 48(1962), pp. 600-603.

[27] J. P. LaSalle and S. Lefschetz, *Stability by Lyapunov's Direct Method*, Academic Press, New York, 1961.

[28] E. B. Lee and L. Markus, "Optimal Control of Nonlinear Processes", *Arch. Rat. Mech. Anal.*, 8(1961), pp. 36-58.

[29] E. B. Lee and L. Markus, *Foundations of Optimal Control Theory*, John Wiley & Sons, Inc., New York, 1967.

[30] D. L. Lukes, "Stabilizability and Optimal Control," *Funkcialaj Ekvacioj*, 11(1968), pp 39-50.

[31] D. L. Lukes, "Optimal Regulation of Nonlinear Dynamical Systems," *SIAM J. Control*, 7(1969), pp. 75-100.

[32] O. C. Mangasarian, "Sufficient Conditions for the Optimal Control of Nonlinear Systems", *J. Soc. Ind. Appl. Math., Ser. A, Control*, 4(1966), No. 1, pp. 139-152.

[33] L. Markus and E. B. Lee, "On Necessary and Sufficient Conditions for Time Optimal Control of Second - Order Nonlinear Systems", *Proc. IFAC Congress, Basel*, 1-6(1963), p.416.

[34] M. J. McCann, "Introduction to Variational Methods for Optimal Control", *Trans. Soc. Instr. Tech.*, 13(1961), pp. 232-237.

[35] C. W. Merriam III, *Optimization Theory of the Design of Feedback Control Systems*, McGraw - Hill, New York, 1964.

[36] R. K. Miller and A. N. Michel, *Ordinary Differential Equations*, Academic Press, New York, 1982.

[37] J. B. Moore and P. M. Colebatch, "Extensions of Quadratic Minimisation Theory: the Discrete - Time Case", Proc. IEE, Vol 117, No. 1, 1(1970), pp. 219 - 222.

[38] A. Nagata, S. Kodama and S. Kumagai, "Time - Optimal Discrete Control System with Bounded State Variable", *IEEE Trans. Automatic Control*, No. 2, 10(April 1965), pp. 155-164.

[39] R. S. Neiswander and R. H. MacNeal, "Optimization of Nonlinear Control Systems by Means of Non - linear Feedbacks", *Trans. AIEE, II*, 72(1953), pp.262-272.

[40] M. N. Oguztoreli, "A Time Optimal Control Problem for Systems Described by Differential - Difference Equations", *J. Soc. Ind. Appl. Math., Ser. A, Control*, No. 3, 1(1963), pp. 290-310.

[41] H. Osborn, "On the Foundations of Dynamic Programming", *J. Math. Mech.*, 8(1959), pp. 867-872.

[42] J. E. Potter, "Matrix Quadratic Solutions", *J. SIAM Appl. Math.*, 14(1966), pp. 496-501.

[43] H. L. Royden, *Real Analysis*, Macmillan Publishing Co., New York, 1988.

[44] D. L. Russell, *Calculus of Variations and Control Theory*, Academic Press, Inc., New York, 1976.

[45] D. L. Russell, *Mathematics of Finite Dimensional Control Systems*, Marcel Dekker, Inc., New York, 1979.

[46] D. L. Russell and X. Zhang, "Extending Linear - Quadratic Optimal Control Laws to Nonlinear Systems and / or Non - Quadratic Cost Criteria", *Differential Equations and Geometric Dynamics: Control Science and Dynamical Systems*, Marcel Dekker, Inc., New York, 1993.

[47] A. P. Sage and C. C. White III, *Optimum Systems Control*, Prentice - Hall Inc., Englewood Cliffs, N. J., 1977.

[48] J. T. Schwartz, *Non - Linear Functional Analysis*, Gordon and Breach Science Publishers Inc., New York, 1969.

[49] L. M. Silverman and B. D. O. Anderson, "Controllability, Observability and Stability of Linear Systems", *SIAM J. Control*, 6(1968), No. 1, pp.121-130.

[50] R. L. Stratonovich, "Most Recent Development of Dynamic Programming Techniques and Their Application to Optimal System Designs", *Proc. Second IFAC Congress Basel*, 1963.

[51] R. A. Struble, *Nonlinear Differential Equations*, McGraw - Hill Book Company, Inc., New York, 1962.

[52] A. E. Taylor and D. C. Lay, *Introduction to Functional Analysis*, Robert E. Krieger Publishing Co., Malabar, Florida, 1986.

[53] C. H. Wilts, *Principles of Feedback Control*, Addison - Wesley Publishing Co., Reading, Massachusetts, 1960.

Vita

Xiaohong Zhang was born in Beijing, China on May 8, 1959, daughter of Zhaorong Zhang and Fuchen Zhang. She was educated in the public schools of her hometown. She received her Bachelor of Science in Mathematics from Sichuan University, Chengdu, Sichuan, China, in 1982. She worked for Chinese Academy of Sciences as a research associate from 1982 to 1985. She came to the United States in 1985 and graduated from West Virginia University where in 1987 she received her M. S. degree in Mathematics. She received the Ph. D. degree in Mathematics from Virginia Polytechnic Institute and State University in 1993.

Xiaohong Zhang

Xiaohong Zhang