A THEORETICAL AND EXPERIMENTAL INVESTIGATION INTO THE
NONLINEAR DYNAMICS OF FLOATING BODIES

by

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(ABSTRACT)

The nonlinear dynamic characteristics and stability of floating vehicles are investigated theoretically and experimentally. Mathematical models of such floating bodies are used to investigate their complicated motions in regular waves. In particular, we address the phenomenon of indirectly exciting the roll motion of a vessel due to nonlinear couplings of the heave, pitch, and roll modes. In the analytical approach the method of multiple scales is used to determine first-order approximations to the solutions, yielding a system of nonlinear first-order equations governing the modulation of the amplitudes and phases of the system. The fixed-point solutions of these equations are determined and their bifurcations are investigated. Hopf bifurcations are found. Numerical simulations are used to investigate the bifurcations of the ensuing limit cycles and how they produce chaos. Experiments are conducted with tanker and destroyer models. They demonstrate some of the nonlinear effects, such as the jump phenomenon, the subcritical instability, and the coexistence of multiple solutions. The experimental results are qualitatively in good agreement with the results predicted by the theory.
Coupling of the pitch and roll motions of a vessel when their frequencies are in the ratio of two-to-one is modeled by a two-degree-of-freedom system. The damping in the pitch mode is modeled by a linear viscous damping, whereas that of the roll mode is modeled by the sum of a linear viscous part and a quadratic viscous part. The effect of the quadratic damping is investigated when either mode is externally excited through a primary resonance. Force-response and frequency-response curves are generated. Coexistence of multiple solutions is found. The jump phenomenon continues to exist, whereas the saturation phenomenon ceases in the presence of quadratic damping. Hopf bifurcations are found. They indicate conditions for the nonexistence of steady-state periodic responses. Instead, the response is an amplitude- and phase-modulated motion consisting of both modes. Floquet theory is used to determine the stability of limit-cycle solutions. They undergo a pitchfork bifurcation followed by a cascade of period-doubling bifurcations, leading to chaos and hence chaotically modulated motions. When the roll mode is excited, the quadratic damping causes the region between the two Hopf bifurcation frequencies to shrink. However, the quadratic damping which may be introduced by attaching antirolling devices does not eliminate complicated motions completely in this region.

The dynamic stability and excessive motion of the roll mode of a vessel in following or head regular waves is investigated theoretically and experimentally. The motion is modeled by a three-degree-of-freedom system with quadratic and cubic nonlinearities. The heave and pitch modes are
linearized and their harmonic solutions are coupled into the roll mode. The resulting nonlinear ordinary-differential equation with time-varying coefficients is used to determine the stability of the roll mode for the case of principal parametric resonance. Experiments with a tanker model were conducted to validate the theory. They demonstrate the jump phenomenon and subcritical instability. They also reveal that the large-amplitude roll response depends not only on the encounter frequency but also on the position of the model relative to the waves.
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Chapter I

INTRODUCTION

1.1 Overview

The present work is a theoretical and experimental study of the nonlinear dynamic characteristics and stability of floating vehicles. To investigate the complicated responses of vehicles in regular waves, we modeled the vehicles by dissipative nonlinear dynamic systems subject to harmonic excitations. Two nonlinear mechanisms which cause large-amplitude motions are investigated. The first mechanism is internal or autoparametric resonance while the second is parametric resonance. They are exemplified by addressing the phenomenon of the indirect excitation of the roll motion of a vessel due to nonlinear couplings among its heave, pitch, and roll modes.
The complicated responses due to nonlinear interactions were investigated for a two-degree-of-freedom system modeling the pitch and roll motions of a vessel in the presence of a two-to-one internal resonance. These responses include periodic and periodically and chaotically modulated motions, coexistence of multiple motions, and jumps. Linear-plus-quadratic terms were used to model the roll damping and a linear term is used to model the pitch damping. Also, the dynamic stability and excessive motion of the roll mode due to an excitation of the pitch and heave modes (which excite the roll mode through parametric resonance) were studied. Because the physical models used in the present work are dissipative nonlinear dynamical systems subject to deterministic excitations, the results are applicable to many mechanical and structural systems.

Experiments were performed on models of actual vessels: a tanker and a destroyer. There were a number of previous experimental studies of the linear behavior and impulsive loading (such as slamming) of vessels. However, very few experiments have been conducted to investigate the nonlinear couplings among the modes of motion and the resultant extraordinary responses which cannot be evaluated by using the linear approach. The experiments demonstrate the jump phenomenon, the subcritical instability, and the coexistence of multiple motions, which are frequently observed in the response of many mechanical and structural systems. The experimental results are qualitatively in agreement with the results predicted by the theory. The experiments also reveal that large-amplitude responses depend not only
on the encounter frequency but also on the position of the model relative to the waves.

1.2 Motivation

1.2.1 Dynamics of Floating Systems

When some disturbances or excitation forces are imposed externally on a body immersed in a fluid or a body floating on the interface between two fluids, it may experience an oscillatory motion. The extent of the motion has important consequences on the safety, operability, and economical aspects of the floating structure or vehicle. There are many examples of floating systems, such as offshore oil drilling platforms, ships, buoys, and submersibles. Vessels not only cross the oceans but might have at-sea missions as well. The vessel is the best representative structure of all floating bodies because it possesses in the most complex way all the features that any one floating structure might have.

One of the objectives of the present work was to investigate the undesirable and potentially dangerous characteristics of the dynamics of a vessel. The results of the present work are applicable without considerable modifications to many other floating systems as well as to many mechanical and structural systems.
1.2.2 Resonances and Stability of Vibratory Systems

Vibrations in a mechanical or structural system may cause temporary or permanent malfunction due to excessive motions. It can create disturbance or discomfort, including noise. To predict and to control the vibration level, we need to understand the mechanisms that produce resonances, which in turn may produce large-amplitude vibrations. Moreover, the problems of the stability and probability of capsize are particularly critical for floating structures. The losses due to these are a worldwide safety concern. To predict and prevent such hazards, one needs to understand the mechanisms causing them, including the various types of resonances.

The phenomenon of resonance is the most important aspect of oscillations or vibrations. Resonant oscillations are forced responses (except for internal or autoparametric resonances) which occur for some specific relationships between the frequency of excitation and the natural frequencies of the system. Physically, resonances may cause large-amplitude responses to relatively small excitations. In turn, large-amplitude vibrations may cause hazardous consequences. In a structural system, large-amplitude vibrations produce large strains, which in turn cause large stresses, possibly exceeding the yield stress, the ultimate stress, or even the rupture stress. Even relatively small cyclic strains can cause fatigue. These problems can lead to catastrophic failure.
Many floating structures risk losing their stability by either collision, or grounding, or internal explosion, or excessive motions, such as heavy rolling. Such accidents are frequent enough in practice and become fatal when flooding of water-tight compartments follows. Loss of stability can happen due to energy transfer between the modes of motion if nonlinearities are present and various resonances of the internal, external, or parametric type occur. The present research is concerned with the dynamic stability and excessive motion of a vessel under the influence of various resonances.

1.3 Dynamic Characteristics of Floating Bodies

The subject of vehicle dynamics encompasses all aspects of a vehicle's motion: trajectory, speed, orientation, and maneuvering. It includes all forces and moments acting on the vehicle and the motions generated by them. The motions include the oscillations performed by the vessel, behaving as a perfectly rigid body, as it floats on the surface of the still or disturbed water. As a rigid body, its motion has six degrees of freedom, three of which are rotational motions about three mutually orthogonal axes: roll, pitch, and yaw, and the other three are translational motions along these axes: surge, sway, and heave. The most important and fundamental forms of these motions are rolling, pitching, and heaving. In reality, more general motions can be
observed comprising a combination of two or more of all six degrees of freedom of motion.

Although floating bodies may have existed ever since the appearance of mankind on earth, real progress in the scientific study of their motions had not been made until the modern ages. In the early 18th century, Leonard Euler (1707-1783) initiated work into the scientific foundation of the theory of oscillations of a floating system [27]. He developed a simple theory of motion in still water.

In the first half of the 19th century, ocean transportation was transformed drastically from wooden sailing vessels first to iron, and then to steel, steam-powered vehicles. With the advent of steam propulsion, for the first time in history, water-born vehicles were able to move directly to windward and face head waves. Problems of pitch and slamming in heavy weather, causing damage to the structures, became very acute, and thus the dynamic characteristics of the vessels became one of the main concerns of the designers.

During the missions of floating structures and vessels the personnel and crew have to perform complex functions and intricate maneuvers under frequently hostile environmental conditions. Excessive motion is unfavourable to the seaworthiness of a vessel, and may even lead to capsizing due to large inclinations in rolling. Damage to the hull or structure may be caused by the forces and moments associated with these motions. Shifting of cargo, flooding of the deck, seasickness, decrease in the accuracy of gunnery in warships, and
decrease in the forward speed due to an increase in the resistance and the worsening performance of the propellers are also detrimental to the operation of a vessel.

All floating structures are subject to motions due to the action of waves. The extent of motion, however, differs because vessels have a great diversity of hull forms, propulsion systems, types, and missions. Smoothness and smallness of the motions of a floating structure are just as important characteristics as the strength, controllability, and resistance, which all need to be considered during the design. It is the responsibility of a designer to investigate methods of decreasing the motions and eliminating their harmful consequences. To accomplish this, the designer must know the various factors affecting the motions and develop methods of assessing them. The engineer should be aware of the capabilities and limitations of the methods of analyses and should establish realistic specifications and criteria of admissible motion characteristics while the structure is still in the planning stage, and thus develop the most optimal design for decreasing the motions and preventing capsizing. Hence, it is of great significance to study every possible case of wave-structure interaction in order to determine the characteristics of the motions of a floating structure, especially those which are concerned with its safety.

The favorable dynamic characteristics of a vessel may be assured in various ways. First, one can apply special arrangements for reducing or even eliminating undesirable motions by using stabilizers. Second, for freight or
passenger transport vessels, one can distribute, within certain limits, the load so as to avoid potentially dangerous conditions, such as hydrostatic instability or certain ratios between the natural frequencies of the vessel. Finally, one can also reduce the motions by suitable changes in course and forward speed; that is, heading angle and encounter frequency. The latter two aspects are investigated in the present work.

1.4 Linear Approach

The usual approach in the analysis of the response of a floating system to waves is the application of linear theory to obtain the frequency-response operators for the six-degree-of-freedom rigid-body motion. The origin of the coordinate system is conventionally located at the center of gravity [108, 121]. In the linearized theory, the response of the body in moderate wave states is assumed to be small so that the six nonlinear coupled equations of motion with six unknowns reduce to two sets of three linear equations. For linear analyses, three different domains are used: time, frequency, and probability domains, and various transformations from one domain to another. The three domains provide three different ways of describing the same process; each has certain advantages.

The development of this conventional approach was based on two significant ideas that appeared during the 1950s. The first was the
development of a suitable stochastic model for irregular waves by Pierson in 1952 [98]. Pierson and St. Denis [99] showed the connection between Pierson's stochastic model for random waves and a similar model for the response of a vessel in such waves using linear operator theory. Their work paved the way to the practical application of linear theory for the design and operation of a vessel. The concept of linear superposition of Airy waves permitted the union of classical theory and the new stochastic model for the random wave field.

The second idea was the development of strip theory by Korvin-Kroukovsky [59] and Korvin-Kroukovsky and Jacobs [60]. Due to the slenderness of the body, the predominant fluid velocities are assumed to be in a transverse plane at each station along the length. Korvin-Kroukovsky's strip theory produced reasonably accurate predictions and provided a motivation for conducting research into numerical methods for computing the hydrodynamic coefficients of two-dimensional sections in or near the free surface. By 1960, a number of researchers [38, 100, 123] had developed practical methods for solving the two-dimensional boundary-value problem in the presence of the free surface for various section shapes using conformal mapping (multipole) methods.

According to strip theory, the oscillations of the vessel are presumed to generate negligible longitudinal velocity components with the exception of surge motion. Under this assumption, the hydrodynamic added mass and damping at each section could be determined by solving a two-dimensional
Cauchy problem, no-penetration of the body, and a linearized free-surface boundary condition. The total hydrodynamic added mass and damping for the body are then obtained by integrating the local section properties over the entire length of the body.

Salvensen, et al. [105] showed a number of predictions obtained with strip theory, which are in agreement with the results obtained with model tests for all degrees of freedom except surge. Later, Newman [86] developed a unified strip theory, which is more uniformly valid over the entire range of wavelengths than any prior strip theory.

Typically, strip theory is used to solve for the response of a vessel to regular waves in the frequency domain, which, in turn, is used to characterize the statistical properties of the response [128]. The normalized solution in the frequency domain produces the frequency-response operator which is used to calculate a real-valued transfer function. The transfer function is multiplied by the given wave spectral density to yield the response spectral density. This is the basic input-output model. For more details, the reader is referred to the reference [128].

Recently, various efforts have been made to develop full three-dimensional analytical models using distributed panels and to solve nonlinear problems. The current state of the theory of motion of floating systems aims at incorporating such nonlinear effects that cause complicated and dangerous consequences.
1.5 The Necessity of Nonlinear Formulations

The equations describing the motion of a vessel in waves are nonlinear differential equations. The nonlinearity may be due to the restoring forces, the damping, and the inertia. Linearization of these equations is adequate only if the response of the vessel to the waves is sufficiently small so that the method of superposition holds. Since the linear equations are usually much easier to solve than the more complete nonlinear equations, people start with the study of mathematically simpler systems which are adequately described by a set of linear differential equations.

Including the nonlinearities in the equations of motion produces phenomena that cannot be predicted using the linearized equations. For example, the frequencies of the system depend on the motion amplitude, multiple responses (and their associated "jump" phenomena) may exist for a given excitation, the responses to a deterministic excitation may be chaotic, the vessel may capsize, and interactions between the roll and pitch and heave modes may occur. The last two phenomena are discussed further below.

Although linear theories are approaching completeness, there remain many unsolved or inadequately solved problems in the nonlinear dynamics of floating systems. In the traditional study of the dynamics of floating bodies, the response is properly described by a system of linear equations for small motions. However, as the amplitude of oscillation increases, nonlinear effects must be considered.
Capsizing is a transition from one stable state of equilibrium to another, and is a strongly nonlinear event. It can happen due to excessive motions like heavy rolling and it is extremely sensitive to initial conditions. Capsizing is intrinsically a nonlinear phenomenon and hence cannot be addressed using linearized equations. Accordingly, the problem of determining the stability and safety of a floating structure demands a nonlinear approach.

The roll motion is a lightly damped response and the roll damping is dominated by viscous processes that are nonlinear functions of the roll amplitude. Hence, the roll is regarded as the mode of motion which shows nonlinear behavior most readily even in small beam waves [51]. The roll motion can occur even when it is not directly excited. Energy input to the pitch or heave mode may be fed into the roll mode due to the mechanism of nonlinear coupling between these modes. Such a phenomenon cannot be addressed using linearized equations.

Consequently, in the present work, in order to overcome the limitations of linear theory, we employ nonlinear equations to investigate the characteristics of the dynamic stability of a floating vessel by utilizing the knowledge and methods of modern nonlinear dynamics developed in the fields of many elastic and dynamic systems, such as elastic pendulums, beams, arches, composite plates, and shells. Moreover, we conduct experiments with tanker and destroyer models to validate the theoretical predictions.
1.6 Literature Review

1.6.1 Single-Degree-of-Freedom Systems

The most obvious way to approach a nonlinear problem is by using a time domain simulation. It is, however, time consuming and thus very expensive.

It was demonstrated successfully with time domain simulations that considering nonlinear Froude-Krilov and hydrostatic terms and linear hydrodynamic terms yields reasonable results [19, 26, 29]. In investigating the relative influence of various nonlinear terms, de Kat and Paulling [19], among others, showed that the nonlinear Froude-Krilov and hydrostatic terms are the dominant ones.

Fallon, et al. [29] and Growchowalski [39] studied the phenomenon of capsize and identified and classified the basic capsize processes and mechanisms. The mechanisms of capsize identified through both model tests and analytical investigations are: 1) pure loss of stability when the vessel is perched on a wave (the loss of stability occurring when a crest moves into an amidship position for nearly zero encounter frequency of following waves); 2) subharmonic resonance; 3) broaching (the large heading deviation due to breaking waves striking the model in series); 4) resonance excitation; 5) wave impacts due to steep or breaking waves; 6) water-on-deck; and 7) tripping over a submerged bulwark.
As an alternative to extensive nonlinear time domain simulations, Lyapunov stability criteria were suggested [65]. Odabasi [89] argued that the stability and capsize problem can be addressed using Lyapunov’s direct method. He also mentioned that some cases might be judged unacceptable when in fact the vessel possesses reasonable capsize safety because the Lyapunov direct method is conservative.

Considering a cubic restoring moment, Grim [37] indicated the possibility of unstable rolling motions in regular waves. Kaplan [55] studied rolling motions with quadratic damping using the method of equivalent linearization. Haddara [42], however, concluded that the method of equivalent linearization is not applicable to nonlinear problems in which the response loses stability, indicating a jump or energy transfer due to the role of nonlinear terms in the governing equations of motion. He used a modification of Struble’s method (e.g., Nayfeh [72]) to study the rolling motion, taking into account quadratic damping and a cubic restoring moment. He showed that unstable rolling motions can occur in irregular waves. Haddara [41] used the same method and concluded that unstable responses of sway, roll, and yaw can occur due to nonlinear couplings among them.

A considerable amount of work treated the roll motion by a single-degree-of-freedom equation. Blocki [9] studied the roll motion when it is parametrically excited by heave or pitch, and he investigated the possibility of capsizing. He also performed an experiment by placing a half cylinder in beam waves. Nayfeh and Khdeir [77, 78], Wright and Marshfield [134], Nayfeh
and Sanchez [84], Virgin [130], Falzarano [30], Cardo and Trincas [14], and Cardo, et al. [13] investigated the nonlinear roll response of ships to beam waves using perturbation methods, numerical computations, and analog-computer simulations. Falzarano studied complicated roll motions, including unbounded motions, by applying the geometric methods of nonlinear dynamics. Using digital- and analog-computer simulations, Nayfeh and Sanchez found complicated roll motions which cannot be obtained by linear theory. These complicated responses include the coexistence of multiple stable solutions, hysteretic loops, jumps between coexisting solutions, symmetry-breaking, period-doubling, and saddle-node bifurcations, chaotic responses, fractal boundaries in the domains of attraction, and unbounded solutions, such as capsizing. These responses were detected for the first time by Wellicome [133] in 1975. He publicized the results of analog-computer simulations in which a gap exists in the frequency-response curves where there are no steady-state motions. It was an indication of the existence of chaotic responses in physical systems, as reported later by, for example, Huberman and Crutchfield [48]. Following the work of Nayfeh and Khdeir [77, 78], Sanchez [106] determined regions in the parameter space (consisting of the excitation amplitude and frequency) where stable periodic and chaotic solutions coexist.
1.6.2 Coupled-Motion Systems

Froude [32] observed that a ship whose natural frequency in pitch is twice its natural frequency in roll has undesirable seakeeping characteristics (a condition of an internal or autoparametric resonance). Robb [103] reported records of full scale data which describe a phenomenon of continuous mutual energy transfer between the pitch and roll modes even though the wave state is relatively constant. For such a long period of time as one century, however, no further research on these phenomena was pursued. Paulling and Rosenberg [95] used a nonlinear set of equations to represent the coupled motion of a vessel, which is free to pitch and roll only. They neglected damping, the nonlinear effect of the roll mode on the pitch moment, and forcing terms, and derived the Mathieu equation, which they used to show that unstable roll motions can occur for certain frequency ratios. Kinney [57] added a linear damping term to the roll equation and essentially repeated the analysis of Paulling and Rosenberg. Kerwin [56] used the same Mathieu equation to include wave-excited roll motions.

Bass [5] investigated the influence of heave-induced parametric variations of the metacentric height, which is dependent on time and angle, on the response of a biased ship in large-amplitude beam waves. Blocki [9] added nonlinear damping and a nonlinear restoring moment to the roll equation and used it to investigate the probability of capsizing. Feat and Jones [31] used a simple Duffing-type oscillator with a softening cubic nonlinearity as a model.
of parametrically excited roll motions. This mathematical model was also used by Koch and Leven [58] to study the motion of a parametrically excited pendulum for small displacements.

Extending Blocki's work [9], Sanchez and Nayfeh [107] investigated the qualitative behaviour of rolling in longitudinal waves. They used an analytical-numerical technique based on the method of multiple scales to predict the qualitative changes taking place as one of the parameters is slowly changed. They confirmed their results, including complicated responses, by using both analog- and digital-computer simulations.

Paulling and Rosenberg, Kinney, Kerwin, Bass, Blocki, Ffei and Jones, and Sanchez and Nayfeh studied the case of parametrically excited roll motions in which energy is fed to the roll mode by the prescribed pitch or heave motion, or, equivalently, wave motion. Thus, their studies did not take into account the influence of the roll motion on the pitch and heave motions.

To explain the connection between the frequency ratio and the undesirable seakeeping characteristics, Nayfeh, Mook, and Marshall [80] used model equations that couple the pitch mode to the roll mode by including the dependence of the pitching moment on the roll orientation. Thus, the pitch (heave) motion is not prescribed but is coupled to the roll motion, and consequently, the pitch (heave) and roll motions are determined simultaneously as functions of a prescribed excitation. They found that the pitch (heave) motion exhibits a “saturation” phenomenon. They offered an explanation of the observation of Froude.

INTRODUCTION
Nayfeh and Mook [79] pointed out that the coupled pitch-roll problem is mathematically similar to that of describing the forced response of many elastic and dynamic systems, such as elastic pendulums, beams, arches, composite plates, and shells. All lead to systems of coupled, inhomogeneous ordinary-differential equations with quadratic nonlinearities. Steady-state solutions of such systems exhibit particularly complicated behavior when their linear undamped natural frequencies are commensurate; that is, when these systems possess internal (autoparametric) resonances.

Many authors [45, 46, 69, 113, 114] presented papers for the cases described above. Among others, Nayfeh, Mook, and Marshall [80] and Mook, Marshall, and Nayfeh [70] used the method of multiple scales to analyze a simple system of two coupled oscillators with quadratic nonlinearities as a model for the coupling of pitch and roll motions. They showed that the approach of Taylor-series expansions in deriving the governing equations can lead to physically unrealistic self-sustained oscillations; they obtained more realistic equations using an energy approach. They demonstrated the existence of a saturation phenomenon when \( \omega_2 \approx 2 \omega_1 \) and \( \Omega \approx \omega_2 \), where the \( \omega_n \) are the linear natural frequencies and \( \Omega \) is the excitation frequency. Moreover, when \( \omega_2 \approx 2 \omega_1 \) and \( \Omega \approx \omega_1 \), they showed that there are conditions for which stable periodic steady-state motions do not exist. Instead, there exist amplitude- and phase-modulated motions in which the energy is continuously exchanged between the two modes. Nayfeh [75] considered nonlinearly coupled roll and pitch motions in regular head waves in which the couplings
are primarily in the hydrostatic terms when the pitch frequency is approximately twice the roll frequency and the encounter frequency is near either the pitch or the roll natural frequency. He demonstrated the saturation phenomenon when the encounter frequency is near the pitch natural frequency and demonstrated the existence of a Hopf bifurcation. He showed that initially only the pitch mode was excited, but, as the wave steepness was increased, the pitch mode saturated and energy was transferred into the roll mode. He also found a Hopf bifurcation in the response when the encounter frequency is near the roll frequency.

One of the exciting new topics in mathematics and dynamics is chaos. Except for those mentioned above [77, 78, 84, 107, 130], little work has yet been published applying these new analytical perspectives to the responses of ships. Papoulis, et al. [94] applied chaotic theory to the behavior of floating vessels in single-point moorings and attempted to explain what has heretofore appeared to be random and disordered behavior.

1.6.3 Control Problems

Our experiments reveal that large-amplitude responses depend not only on the excitation frequency but also on the location of the model on the waves. These observations are consistent with experiments on the directional stability under a zero encounter frequency and are closely related to the broaching phenomenon.
When a vessel is in longitudinal waves, it can easily experience some control problems, such as broaching and loss of course stability, especially in following regular waves. These are very susceptible initiated by heeling of the body, which is the result of the roll motion being excited by the longitudinal waves. Wahab and Swaan [131] modeled the case of a zero encounter frequency by using linear equations with constant coefficients and considering Froude-Krill forces to be the only wave forces and moments. With this fairly simple model they were able to demonstrate the directional instability of a vessel with fixed controls when it is placed over one-half of the wave cycle, particularly when its center of gravity is placed near the steepest part of the wave and its stern is placed near the wave crest. This phenomenon has been familiar to operators historically. They were also able to demonstrate analytically that the probability of broaching was greater in steeper waves.

Eda and Crane [24] included hydrodynamic maneuvering coefficients obtained from model test experiments and horizontal plane wave exciting forces and moments obtained from a strip theory. They also considered regular waves with very low encounter frequencies; their predicted results of maneuvering performance in regular following or stern quartering waves were in reasonable agreement with model tests. Eda [23] extended his work and demonstrated that a vessel which is directionally stable in calm water will not be stable in regular following waves with zero frequency of encounter, a result similar to that of Wahab and Swaan [131]. Eda, et al. [25] studied extreme rolling motions that are occasionally experienced by high-speed crafts.
operating in waves under automatic steering control [122]. They showed that a significant coupling of roll and yaw motions can develop due to the asymmetry of the heeled underwater hull form.

Under similar circumstances, large-amplitude motions can occur not only in the heave and pitch modes but also in the roll mode. The latter phenomenon is caused by nonlinear couplings between the modes of motion, resulting in parametric resonance; it is investigated in the present work. Similar examples which may be related to the case of parametric resonance are the sloshing phenomenon in the fuel containers of aerospace vehicles, such as rockets and missiles, the liquid cargo tanks of liquid product carriers, and the water surface in the swimming pool of a large passenger vessel. The free surface of the liquid may oscillate due to parametric resonance when the systems are vertically excited. This phenomenon was studied by Nayfeh [74] and Nayfeh and Nayfeh [82].

Following Wahab and Swaan [131], Eda and Crane [24], Eda [23], Eda, et al. [25], and Taggart and Kobayashi [122], we study the effect of various positions of the center of gravity of the model on the waves. Using a circulating water channel, Renilson and Driscoll [102] investigated experimentally the locations of the model on the waves, which lead to broaching in the case of zero encounter frequency.
1.7 Methodology

Two different methods are available for studying the response of a floating system [66]: experimental and theoretical methods. Experimental methods can be divided into two subgroups depending on their scales: full scale and model tests.

Full scale tests are the closest approach to reality. However, they have a few disadvantages. They are expensive and difficult to implement, the environmental state is difficult to measure, and it is impossible to fully control the experiments. On the other hand, model tests are controlled experiments by which it is possible to model the desired conditions. They are relatively inexpensive and useful design tools. Nonetheless, there are limitations to model tests: scale effects with regard to viscous effects and the physical dimensions of model test facilities. The second limitation can be overcome by testing models in the natural environment; however, it is impossible to control the wave conditions.

Theoretical methods have been thus far oriented toward linear analyses; this is the main disadvantage of modern programs. It is, however, advantageous to use these theoretical methods. They are inexpensive and easy to use, allow fully controlled studies, and are suitable for use during design development because various alternatives can be compared easily.
In the present work, we perform model scale experiments and theoretical investigations utilizing some of the modern methods of nonlinear dynamics developed in other physical systems.

1.7.1 Theoretical Study

Two of the difficulties of nonlinear seakeeping are the treatment of fluid-body interactions and wave-induced loads. In the present work, we assume that the hydrodynamic loads can be represented as functions of the vessel state with the hydrodynamic coefficients obtained either from experiments or computational fluid dynamics. We use the Froude-Krilov assumption that the presence of the floating body does not affect the pressure field of the incident waves. In the present work, the Froude-Krilov loads are considered to be the only external excitations. It is a reasonable proposition when the wavelength is assumed to be larger than the characteristic length of the vessel.

The study of nonlinear systems is considerably more complicated than that of linear systems. This is because the principle of superposition, which holds for linear systems, does not hold for nonlinear systems and because, in general, closed-form solutions do not exist for nonlinear equations. As a result, special methods of approach have been proposed to obtain approximate solutions, which provide insight into the system behavior.
The treatment of nonlinear systems often requires entirely different methods of attack because the theory of nonlinear differential equations is not nearly as well developed as that of linear differential equations. There are two basic approaches to the study of nonlinear systems [79]. The first one is the qualitative approach, which is concerned with the general stability characteristics of a system in the neighbourhood of a known solution. The second one is the quantitative approach, which is concerned with determining approximate analytical or numerical solutions. A powerful approach for obtaining approximate solutions is the use of perturbation methods. If the perturbation terms, which are generally identified by means of a small parameter $\epsilon$, are relatively small, the system is said to be nearly linear or weakly nonlinear. For weakly nonlinear systems, a solution is commonly sought in the form of a power series in the small parameter $\epsilon$. There are a number of perturbation methods designed to produce approximate solutions to weakly nonlinear systems. Among others, the method of multiple scales and the method of averaging are most widely used. For more details about this subject, the reader is referred to references [72, 73, 79].

In the present work, the method of multiple scales is used to determine first-order approximations to the solutions of the governing equations. For the case of pitch-roll coupling in the presence of a two-to-one internal resonance, a set of four autonomous first-order nonlinear ordinary-differential equations are obtained (either in polar or Cartesian coordinates) that govern the modulation of the amplitudes and phases of the interacting modes.
Periodic solutions correspond to fixed points of the modulation equations. There are two possible solutions. First, the ever-existing linear solution. Second, a nonlinear solution which deviates from the linear solution beyond certain bifurcation values. The unstable fixed-point solutions correspond to unstable periodic solutions, which are not achievable in experiments.

The stability of a fixed point depends on the real parts of the eigenvalues of the Jacobian matrix of the modulation equations. A given fixed point is asymptotically stable if and only if all the eigenvalues lie in the left half of the complex plane and unstable if at least one eigenvalue lies in the right half of the complex plane. If a pair of complex-conjugate eigenvalues have a positive real part, then the response may be an amplitude- and phase-modulated motion.

The stability of the periodic solutions of the modulation equations is determined using Floquet theory. If three Floquet multipliers (one of the Floquet multipliers is always $+1$ because the modulation equations are autonomous) lie inside the unit circle, the orbit is asymptotically stable. On the other hand, if as a control parameter is varied, one of the multipliers leaves the unit circle through $+1$, it implies either a cyclic-fold (tangent) or pitchfork (symmetry-breaking) bifurcation, and if it leaves the unit circle through $-1$, it implies a flip or period-doubling bifurcation.

The numerical simulations have been performed on an IBM 3090 digital supercomputer. The fixed-point solutions are verified and the interesting behaviors ranging from limit cycles to chaotic responses are found by
numerically integrating the amplitude- and phase-modulation equations using a 5th- and 6th-order Runge-Kutta-Verner algorithm available from the IMSL library. Stable and unstable limit cycles are calculated by using an efficient algorithm originally proposed by Aprille and Trick [2]. The fast Fourier transform (FFT), a spectral analysis technique, is used to search for cyclical patterns or periodicities in signals.

Next, we investigate bifurcations of the periodic solutions and the existence of chaotic attractors using the detuning of the excitation frequency as a control parameter. The bifurcation diagram shows the bifurcation points and the solutions in the various detuning intervals inside and around the two Hopf bifurcation points. Some of the phase portraits of the calculated attractors are shown. A cascade of period-doubling bifurcations is found, leading to chaos. Chaotic attractors are irregular in nature and sensitive to initial conditions. Poincaré sections show that a chaotic attractor does not lie on a simple geometrical object. It evolves continuously and smoothly while maintaining its structural characteristics. Near subcritical Hopf bifurcations, chaotic attractors exist and extend in two directions around the Hopf point.

1.7.2 Experimental Study

An apparatus was designed and built to restrict the model to pitch, heave, and roll only and to accurately measure these motions. The apparatus was fixed at the center of gravity of the model.
Among the many experimental difficulties, three can be singled out. First, the time needed to acquire an amount of data was about 1,000 times longer than that needed to acquire the same data in vibration experiments. Second, the required condition of the model was hard to secure in both senses of hydrostatics or geometry and of dynamic characteristics. And third, the same test condition was hard to maintain for the whole duration of a series of runs. These will be described in more detail in chapter IV.

The towing basin is approximately 30m x 1.6m x 2.1m. A plunger type wavemaker and a set of wave absorbers are placed in the basin. The combination of linear and rotational motion apparatus allows three degrees of freedom in total; that is, one translational motion: heave, and two rotational motions: pitch and roll. Various electronic instruments were used to monitor and acquire the digitized data of the responses of the model and other signals.

As a basic experiment, wave tests were done without a model to map out the frequencies and amplitudes of the wavemaker that produce plane waves in the towing basin.

Two series of experiments were conducted for the case of primary resonance of the pitch mode in the presence of a two-to-one autoparametric resonance of the pitch and roll modes using a wooden destroyer model. The main particulars of the model are LOA x B x D = 10'4" x 11.5" x 9" and its weight is 120 lbs.

In the first condition, the natural frequency of roll was 0.495 Hz and that of pitch was 0.910 Hz. Since the pitch mode was to be resonantly excited,
waves with frequencies around 0.910 Hz were generated. To obtain the
force-response curves, we swept up and down the amplitudes of the waves
within the limit for which regular plane waves exist in the basin.

Jump phenomena, subcritical instability, and coexistence of multiple
solutions were observed, in qualitative agreement with the theoretical results.
In this case, the pitch response did not exhibit saturation. Instead, the
amplitude of the pitch mode grew beyond a bifurcation point of wave
amplitude.

In the second case, the condition of two-to-one frequency ratio was further
tuned by setting the natural frequency of roll to be 0.455 Hz and that of pitch
to be 0.908 Hz. In this case, neither the jump phenomenon nor the subcritical
instability were observed readily. Coexistence of multiple solutions was
confirmed by changing the initial conditions.

Other experiments were conducted for the case of principal parametric
resonance of the roll mode using a wooden tanker model. The main
particulars of the model are $\text{LOA} \times B \times D = 7'6'' \times 13.5'' \times 8.5''$ and its weight
is 110 lbs. Waves with frequencies around 0.60 Hz were used. Since the roll
mode was to be excited by principal parametric resonance, the excitation
frequency was varied near twice the natural frequency in roll. The roll natural
frequency (0.32 Hz) of the model was adjusted to be approximately one-half
the frequency of the waves (0.60 Hz). To obtain the force-response curves, we
swept the wave amplitudes up and down within the limit for which regular
plane waves exist in the basin.

INTRODUCTION
The jump phenomenon, penetration phenomenon, subcritical instability, and coexistence of multiple solutions were observed, in qualitative agreement with the theoretical results. In this case, the pitch and heave responses showed linear behaviors even beyond a bifurcation point of wave amplitudes. It was also found that the occurrence of large-amplitude roll motions depends on the position of the model along the standing waves.
Chapter II

A THEORETICAL INVESTIGATION OF A
PARAMETRICALLY EXCITED ROLL MOTION

2.1 Statement of the Problem

One of the mechanisms causing large-amplitude roll motions is the direct excitation of the roll mode by beam waves. The problem can be mathematically modeled by a single-degree-of-freedom roll equation having nonlinearities, constant coefficients, and an external forcing term. This mechanism of roll motion is readily understandable because it deals with the roll motion to beam waves: that is, a transverse motion due to a transverse excitation.
The roll motion, which shows nonlinear behavior most readily even in small waves, can occur even when it is not directly excited. When a vessel is in either following or head waves, violent rolling motion can occur due to the parametric excitations that result from the heave-pitch-roll coupling. The input excitation energy to the pitch or heave mode may be fed into roll mode due to nonlinear coupling among these modes. Then, the vessel can exhibit a large-amplitude roll motion as well as heave and pitch motions.

This phenomenon is not very new. Although some real experiences of rolling in longitudinal waves were reported by crews, it has been believed for a long time that a vessel moving toward the oncoming waves from ahead of it would exhibit only the heave and pitch motions. In other words, a system excited by in-plane excitations will respond with in-plane modes of motion only, which is the linear result. The present chapter, however, reveals theoretically that it is not always true. Instead, a vessel encountering following or head waves may possess an out-of-plane response (roll), which is a critical mode to the safety, as well as the in-plane modes of motion (heave and pitch). Chapter V provides the experimental verification of this phenomenon, which is predicted theoretically in this chapter.

In the traditional study of the dynamics of vessels, the response is properly described by a system of linear equations for small motions. By linearization, the six nonlinear coupled equations of motion are reduced to two sets of three linear equations. In this linearized model, the out-of-plane modes (yaw, sway, and roll) are decoupled from the in-plane modes (pitch, heave,
and surge). This approach, therefore, neglects the sometimes pronounced effects of other modes due to nonlinear coupling. These coupling effects often take the form of a parametric resonance, which can lead to a particularly dangerous situation.

Such a possibility of large-amplitude roll motions and even capsizing due to the nonlinear interactions among the modes, which necessarily requiring a two- or multi-degree-of-freedom nonlinear formulation, began to be recognized since Froude’s observation [32]. Froude observed undesirable rolling motions due to the coupling effects between the pitch and roll modes in the middle of the 19th century. However, research on nonlinear interactions between the modes was not pursued until the 1950s.

Paulling and Rosenberg [95] attempted to address the indirect excitation of the roll mode due to energy transfer from either the heave or pitch mode, which are directly excited modes. In their analysis, the nonlinear roll equation is linearized (it contains a time-varying coefficient) by assuming harmonic pitch and heave motions. The resulting Mathieu (or Hill) equation is used to determine conditions for the stability of trivial solutions (no-roll motions). With this approach, however, the predicted roll angle grows exponentially with time, which is unrealistic.

Later, Kinney [57], Kerwin [56], Blocki [9], Bass [5], Feat and Jones [31], Sanchez and Nayfeh [107], and Sanchez [106] followed a reasoning similar to Paulling and Rosenberg [95], and derived linear and nonlinear Mathieu equations. They included linear and/or nonlinear damping terms and
nonlinear restoring moments into the Mathieu-equation-based roll equation. They studied the case of parametrically excited roll motions in which energy is fed to the roll mode by the prescribed pitch or heave motion, or, equivalently, wave motion.

Among others, Blocki [9] considered a ship with only two degrees of freedom (heave and roll). Such a restriction (the elimination of pitch) implies that the ship is symmetric with respect to the midship section (sometimes called fore-and-aft symmetry) and in a beam wave. Attempting to satisfy such limitations, Blocki used a simple model of a half cylinder in a beam wave in his experiments. In the analysis, he ignored the wave-induced roll moment, an obvious inconsistency but an approximation that is reasonable when the slope of the waves is small and the wavelength is large compared to the beam of the model. He attempted to address the parametrically excited roll motion in the presence of only the heave by assuming that pitch does not occur ($\dot{\theta} \approx 0$).

The aim of this chapter is to develop an improvement of the previous theoretical work. We describe the real situation more accurately than Blocki [9] and Sanchez and Nayfeh [107]. Specifically, we lift the restriction of fore-and-aft symmetry, add a third degree of freedom (pitch), and consider head and following waves both theoretically and experimentally. The heave and pitch motions are assumed to be independent of the roll motion, an assumption that was verified experimentally. They are considered simultaneously and assumed to be harmonic. Due to the heave-pitch-roll
coupling, the amplitudes and frequencies of the heave and pitch motions play the role of an effective amplitude and frequency of the parametric excitation. The parametric term in the roll equation basically accounts for the time-dependent variation of the metacentric height. We investigate the principal parametric resonance case in which the wave frequency is approximately twice the natural frequency in roll.

2.2 Equation of Motion

We choose the right-handed coordinate systems shown in Figure 2.1. Moreover, we use Euler angles to define the rotations of the body-fixed coordinate system (whose origin is at the center of gravity G of the ship) with respect to the inertial coordinate system. They are defined in the following sequence:

1. the yaw angle $\psi$ is the rotation about the initial position of the $z$ axis,

2. the pitch angle $\theta$ is the rotation about the new position of the $y$ axis, and

3. the roll angle $\phi$ is the rotation about the final position of the $x$ axis.

The motion is described by the generalized coordinates
\[ \{q\} = \begin{bmatrix} x \\ y \\ z \\ \phi \\ \theta \\ \psi \end{bmatrix} \]  \hspace{1cm} (2.1)

where \(x, y,\) and \(z\) are the components of a position vector \( \mathbf{R} \) with respect to the inertial coordinate system. Next, we denote the generalized velocities by

\[ \{\pi\} = \begin{bmatrix} u \\ v \\ w \\ p \\ q \\ r \end{bmatrix} \]  \hspace{1cm} (2.2)

where \(u, v,\) and \(w\) are the components of the translational velocity and \(p, q,\) and \(r\) are the components of the angular velocity; both sets are referred to the body-fixed coordinate system. The two coordinate systems are related by the following coordinate transformation representing the sequence of rotation

\(R_\psi \rightarrow R_\theta \rightarrow R_\phi\) (or \(3 \rightarrow 2 \rightarrow 1\) rotation):

\[ \{\pi\} = [\alpha]\{\dot{q}\} \]  \hspace{1cm} (2.3)

where
\[
\begin{bmatrix}
    \psi c \theta & s \psi c \theta & -s \theta & 0 & 0 & 0 \\
    \psi s \theta s \phi - s \psi c \phi & s \psi s \theta s \phi + c \psi c \phi & c \theta s \phi & 0 & 0 & 0 \\
    \psi s \theta c \phi + s \psi s \phi & s \psi s \theta c \phi - c \psi s \phi & c \theta c \phi & 0 & 0 & 0 \\
    0 & 0 & 0 & 1 & 0 & -s \theta \\
    0 & 0 & 0 & 0 & c \phi & c \theta s \phi \\
    0 & 0 & 0 & 0 & -s \phi & c \theta c \phi
\end{bmatrix}
\]  

(2.4)

where \(c\) and \(s\) stand for the sine and cosine functions, respectively.

Following Pauling and Rosenberg [95] and Blocki [9], we use the Newtonian approach. Then the equations of motion describing the dynamics of a rigid body are obtained by using Euler’s equations for the rotational motions (pitch, roll, and yaw) around the center of gravity and Newton’s second law for the translational motions (surge, heave, and sway). By this, the equations of motion describing the motion of a rigid body having six degrees of freedom are [35]

\[
\begin{align*}
    m(\ddot{u} + wq - vr) &= \vec{X} \\
    m(\ddot{v} + ur - wp) &= \vec{Y} \\
    m(\ddot{w} + vp - uq) &= \vec{Z} \\
    I_x \ddot{\phi} + (I_z - I_y)qr &= \vec{K} \\
    I_y \ddot{\phi} + (I_x - I_z)rp &= \vec{M} \\
    I_z \ddot{\phi} + (I_y - I_x)pq &= \vec{N}
\end{align*}
\]  

(2.5)
where \( m \) is the mass of the ship; \( I_x, I_y, \) and \( I_z \) are moments of inertia; and the terms on the right-hand side represent the hydrodynamic and external forces and moments acting on the body.

Motions of a general character are observed comprising one combination or another of all the six degrees of freedom of motion. However, it is known that the most important and fundamental motions that determine the safety of a vessel are the rolling, pitching, and heaving. Thus, Blocki [9] assumed the following order of magnitude of the different components of motion:

\[
\begin{align*}
    z &= 0(\varepsilon), \quad \theta = 0(\varepsilon), \quad \phi = 0(1), \quad x = 0(\varepsilon^2), \quad y = 0(\varepsilon^2), \quad \text{and} \quad \psi = 0(\varepsilon^2) \quad (2.6)
\end{align*}
\]

where \( \varepsilon \) is a small dimensionless parameter, \( z \) is the heave, \( \theta \) is the pitch, \( \phi \) is the roll, \( x \) is the surge, \( y \) is the sway, and \( \psi \) is the yaw. Based on these assumptions, the latter three displacements and their velocities and accelerations are set equal to zero. Then, the resultant equations of motion to \( 0(\varepsilon^2) \) become

\[
\begin{align*}
    m\ddot{z} &= \dddot{Z} \\
    I_y\dddot{\theta} &= \dddot{M} \\
    I_x\dddot{\phi} &= \dddot{K} \quad (2.7)
\end{align*}
\]

From the relation (2.3), we know that

\[
\begin{align*}
    w &= \dot{z} \cos \theta \cos \phi \\
    q &= \dot{\theta} \cos \phi \\
    \rho &= \dot{\phi} \quad (2.8)
\end{align*}
\]
Then the third to fifth equations of equations (2.5) become

\[ m\ddot{w} = \bar{Z} \]
\[ I_y (\ddot{\theta} \cos \phi - \dot{\phi} \dot{\theta} \sin \phi) - (I_x - I_y) \dot{\phi} \dot{\theta} \sin \phi = \bar{M} \]
\[ I_x \ddot{\phi} - (I_z - I_y) \dot{\theta}^2 \cos \phi \sin \phi = \bar{K} \] (2.9)

We assume here that the force \( \bar{Z} \) and the moments \( \bar{M} \) and \( \bar{K} \) in equations (2.9) are represented by

\[ \bar{Z} = Z(q_1, ..., q_9) + Z(t) \]
\[ \bar{M} = M(q_1, ..., q_9) + M(t) \]
\[ \bar{K} = K(q_1, ..., q_9) + K(t) \] (2.10)

where the symbols \( q_i \) (i = 1, 2, ..., 9) are introduced in such a way that they denote the displacements \{z, \theta, \phi\} for i = 1, 2, 3, their velocities \{\dot{z}, \dot{\theta}, \dot{\phi}\} for i = 4, 5, 6, and their accelerations \{\ddot{z}, \ddot{\theta}, \ddot{\phi}\} for i = 7, 8, 9. The force \( Z \) and moments \( M \) and \( K \) are assumed to be differentiable with respect to the \( q_i \) (i = 1, 2, ..., 9) and the external force \( Z(t) \) and moments \( M(t) \) and \( K(t) \) are explicit functions of time.

The force \( Z \) and moments \( M \) and \( K \) can be expanded in the neighborhood of the initial equilibrium state in Taylor series as...
\[ Z = Z_0 + \frac{\partial Z}{\partial q_i} q_i + \frac{1}{2} \frac{\partial^2 Z}{\partial q_i \partial q_j} q_i q_j + \ldots \]

\[ M = M_0 + \frac{\partial M}{\partial q_i} q_i + \frac{1}{2} \frac{\partial^2 M}{\partial q_i \partial q_j} q_i q_j + \ldots \]  \hspace{1cm} (2.11)

\[ K = K_0 + \frac{\partial K}{\partial q_i} q_i + \frac{1}{2} \frac{\partial^2 K}{\partial q_i \partial q_j} q_i q_j + \ldots \]

where the subscript \( o \) denotes the initial equilibrium position such that

\[ Z_0 = Z(0,0,...,0) = 0 \]
\[ M_0 = M(0,0,...,0) = 0 \]  \hspace{1cm} (2.12)
\[ K_0 = K(0,0,...,0) = 0 \]

Here we assume the functions \( Z, M, \) and \( K \) in the following form:

\[ Z = Z_1(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) + Z_2(\mathbf{q}_4, \mathbf{q}_5, \mathbf{q}_6) + Z_3(\mathbf{q}_7, \mathbf{q}_8, \mathbf{q}_9) \]

\[ M = M_1(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) + M_2(\mathbf{q}_4, \mathbf{q}_5, \mathbf{q}_6) + M_3(\mathbf{q}_7, \mathbf{q}_8, \mathbf{q}_9) \]  \hspace{1cm} (2.13)

\[ K = K_1(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3) + K_2(\mathbf{q}_4, \mathbf{q}_5, \mathbf{q}_6) + K_3(\mathbf{q}_7, \mathbf{q}_8, \mathbf{q}_9) \]

where \( Z_1, M_1, \) and \( K_1 \) account for the restoring forces and moments, \( Z_2, M_2, \) and \( K_2 \) account for the damping forces and moments, and \( Z_3, M_3, \) and \( K_3 \) account for the added mass and moments of inertia. This allows us to eliminate the cross couplings, such as static-kinematic, static-dynamic, and kinematic-dynamic. If we further assume lack of dynamic-dynamic couplings, then equations (2.13) are left with only the static-static and kinematic-kinematic couplings.
We substitute equations (2.11) into (2.10), then substitute the result into equation (2.9), neglect terms of order higher than \( O(\varepsilon) \), take into account the initial assumption (2.6), and obtain the equations of motion in the following form:

\[
\ddot{z} + 2\zeta_z \dot{z} + \omega_z^2 z = \bar{Z}(t) \tag{2.14}
\]

\[
\ddot{\theta} + 2\zeta_\theta \dot{\theta} + \omega_\theta^2 \theta = \bar{M}(t) \tag{2.15}
\]

\[
\ddot{\phi} + \omega_\phi^2 \phi + 2\mu_1 \dot{\phi} + 2\mu_3 \dot{\phi}^3 - \alpha_3 \phi^3 - \frac{1}{2} \left( K_{\phi z} \dot{\phi} z + K_{\phi \theta} \dot{\phi} \theta + K_{\phi \phi} \ddot{\phi} \phi + K_{\phi \dot{\phi}} \ddot{\phi} \dot{\phi} \right) = \bar{K}(t) \tag{2.16}
\]

where \( \zeta_z \) and \( \zeta_\theta \) are damping coefficients; \( \omega_z, \omega_\theta, \) and \( \omega_\phi \) are the natural frequencies; \( \mu_1 \) and \( \mu_3 \) are linear and cubic roll damping coefficients; \( \alpha_3 \) is a constant cubic "stiffness" coefficient; and \( K_{\phi z}, K_{\phi \theta}, K_{\phi \phi}, \) and \( K_{\phi \dot{\phi}} \) are constant coefficients of the quadratic coupling terms. Blocki [9] considered only heave and roll modes by assuming that there is no pitch \( (\theta \approx 0) \) and ignored the kinematic-kinematic coupling; thus, he had only one (static-static) quadratic term: \( K_{\phi z} \dot{\phi} z \). Here, we include additional static-static couplings between the pitch and roll modes as well as kinematic-kinematic couplings among all three modes. The set of equations (2.14)-(2.16) is considered as a mathematical model that accounts for the motion of a ship model supported on a sting with three degrees of freedom: roll \( (\phi) \), pitch \( (\theta) \), and heave \( (z) \).

We assume a simple harmonic wave excitation of the form...
\[ \ddot{Z}(t) = \bar{Z}_0 \cos \Omega t \]  
\[ \ddot{M}(t) = \bar{M}_0 \cos(\Omega t + \tau_d) \]

where \( \Omega \) is the encounter frequency, \( \tau_d \) is the phase delay of the pitch moment relative to the heave force, \( \bar{Z}_0 \) is a measure of the amplitude of the heave excitation force, and \( \bar{M}_0 \) is a measure of the amplitude of the pitch excitation moment. Here, \( \bar{Z}_0 \) and \( \bar{M}_0 \) are functions of the wave height as well as the position of the mass center on the wave. Since the sets (2.14) and (2.17) and (2.15) and (2.18) are uncoupled linear equations, their steady-state solutions can be expressed as

\[ z = a_z \cos(\Omega t + \tau_z) \]  
\[ \theta = a_\theta \cos(\Omega t + \tau_\theta) \]

where \( a_z, a_\theta, \tau_z, \) and \( \tau_\theta \) are constants.

We consider the case in which the ship is in longitudinal waves so that \( \bar{K}(t) = 0 \) in equation (2.16). Substituting (2.19) and (2.20) into (2.16), we obtain

\[ \ddot{\phi} + \omega_\phi^2 \phi + 2\mu_1 \dot{\phi} + 2\mu_3 \phi^3 - a_3 \phi^3 + \left[ f_1 \cos(\Omega t + \tau_2) + f_3 \cos(\Omega t + \tau_\theta) \right] \phi \]
\[ + \left[ f_2 \sin(\Omega t + \tau_2) + f_4 \sin(\Omega t + \tau_\theta) \right] \dot{\phi} = 0 \]

where
\[ f_1 = -\frac{1}{2} a_z K_{\phi z}, \quad f_2 = \frac{1}{2} \Omega a_z K_{\phi z}, \]
\[ f_3 = -\frac{1}{2} a_{\theta} K_{\phi \theta}, \quad f_4 = \frac{1}{2} \Omega a_{\theta} K_{\phi \theta} \]  

(2.22)

Equation (2.21) is a nonlinear equation with time-varying coefficients that includes the effect of the pitch and heave motions on the roll motion and thus describes reality more closely than the equations analyzed by Blocki [9] and Sanchez and Nayfeh [107].

2.3 Analysis

2.3.1 Method of Multiple Scales

An approximate analytical solution of equation (2.21) is obtained for small but finite amplitudes for the case of lightly damped vessels. The straightforward expansion shows that resonances occur when \( \Omega/\omega_\phi \approx 1, 2, 4 \), etc. The first two cases are known as the fundamental and principal parametric resonances, respectively. It was concluded by Blocki [9] that the most dangerous case is \( \Omega/\omega_\phi \approx 2 \). Sanchez and Nayfeh [107] presented a bifurcation diagram in terms of the frequency and amplitude of the excitation and showed that the principal resonance occurs at the smallest excitation amplitude.
We use the method of multiple scales to determine a first-order approximation to the solution of equation (2.21). We begin by assuming that an approximation to $\phi$ can be written in the following form:

$$\phi(t; \epsilon) \approx \epsilon \phi_1(T_0, T_1) + \epsilon^3 \phi_3(T_0, T_1)$$  \hspace{1cm} (2.23)

where $T_0 = t$ is a fast time scale, characterizing motions occurring at the frequencies $\Omega$ and $\omega_\phi$; $T_1 = \epsilon^2 t$ is a slow scale, characterizing the modulation of the amplitude and phase due to the nonlinearity, damping, and resonances; and $\epsilon$ is a dimensionless measure of the amplitude of the motion, which is used solely as a bookkeeping device. In writing expansion (2.23), we have "scaled" the amplitude of the response. The following analysis provides an approximation that is valid for $\epsilon << 1$ and, therefore, for small amplitudes of the motion. What the analysis does not provide is a definition of "small." It is not unusual for an analysis such as this one to still be quite accurate at amplitudes of 30 or 45 degrees. The time derivatives are transformed as

$$\frac{d}{dt} \approx D_0 + \epsilon^2 D_1$$  \hspace{1cm} (2.24)

$$\frac{d^2}{dt^2} \approx D_0^2 + 2\epsilon^2 D_0 D_1$$  \hspace{1cm} (2.25)

where

$$D_0 = \frac{\partial}{\partial T_0} \quad \text{and} \quad D_1 = \frac{\partial}{\partial T_1}$$  \hspace{1cm} (2.26)
and terms of $O(\varepsilon^3)$ have been neglected. Next we must scale the linear damping and forcing so that all damping and forcing as well as the static restoring moment interact at the same order. We put $\mu_i = \varepsilon^2 \hat{\mu}_i$, and $f_i = \varepsilon^2 \hat{f}_i$ for $i = 1, 2, 3, \text{and} \ 4$. The implication of the latter is that small-amplitude pitch and heave motions can produce large-amplitude rolling. In the experiments we observed that waves of very small amplitude did produce large-amplitude rolling, an observation that is consistent with this assumption.

Substituting these definitions and equations (2.23) - (2.25) into equation (2.21) and then equating coefficients of like powers of $\varepsilon$, we obtain

$$0(\varepsilon): \quad D_0^2 \phi_1 + \omega_0^2 \phi_1 = 0 \quad \quad (2.27)$$

$$0(\varepsilon^3): \quad D_0^2 \phi_3 + \omega_0^2 \phi_3 = - 2D_0 D_1 \phi_1 - 2\hat{\mu}_1 D_0 \phi_1 - 2\mu_3 (D_0 \phi_1)^3 + \omega_3 \phi_1^3$$
$$- \hat{f}_1 \cos(\Omega t + \tau_2) \phi_1 - \hat{f}_2 \sin(\Omega t + \tau_2) D_0 \phi_1$$
$$- \hat{f}_3 \cos(\Omega t + \tau_4) \phi_1 - \hat{f}_4 \sin(\Omega t + \tau_4) D_0 \phi_1 \quad (2.28)$$

The solution of equation (2.27) can be written as

$$\phi_1(T_0, T_1) = A(T_1) e^{i\omega_0 T_0} + \text{cc} \quad (2.29)$$

where cc stands for the complex conjugate of the preceding terms. The function $A(T_1)$ is an arbitrary complex function of $T_1$ at this level of approximation. It is determined by imposing the solvability condition at the next level of approximation.

Substituting equation (2.29) into equation (2.28) yields
\[ D_0^2 \phi_3 + \omega_0^2 \phi_3 = -2i\omega_\phi [A' + \hat{\mu}_1 A] e^{i\omega_\phi T_0} \]
\[ - 2\mu_3 (i\omega_\phi)^3 [A^3 e^{3i\omega_\phi T_0} - 3A^2 A e^{i\omega_\phi T_0}] \]
\[ + \alpha_3 [A^3 e^{3i\omega_\phi T_0} + 3A^2 A e^{i\omega_\phi T_0}] \]
\[ - \frac{1}{2} \hat{f}_1 [A e^{i(\Omega T_0 + \omega_\phi T_0 + \tau_Z)} + A e^{i(\Omega T_0 - \omega_\phi T_0 + \tau_Z)}] \]
\[ + \frac{1}{2} \hat{f}_2 [i\omega_\phi A e^{i(\Omega T_0 + \omega_\phi T_0 + \tau_Z)} - i\omega_\phi A e^{i(\Omega T_0 - \omega_\phi T_0 + \tau_Z)}] \]
\[ - \frac{1}{2} \hat{f}_3 [A e^{i(\Omega T_0 + \omega_\phi T_0 + \tau_\theta)} + A e^{i(\Omega T_0 - \omega_\phi T_0 + \tau_\theta)}] \]
\[ + \frac{1}{2} \hat{f}_4 [i\omega_\phi A e^{i(\Omega T_0 + \omega_\phi T_0 + \tau_\theta)} - i\omega_\phi A e^{i(\Omega T_0 - \omega_\phi T_0 + \tau_\theta)}] \]
\[ + cc \]

### 2.3.2 Modulation Equations

Because we are considering the principal parametric resonance corresponding to \(\Omega \approx 2\omega_\phi\), we introduce a detuning parameter \(\hat{\sigma}\) according to

\[ \Omega = 2\omega_\phi + \varepsilon^2 \hat{\sigma} \quad (2.31) \]

Then we substitute equation (2.31) into equation (2.30) and find that secular terms are eliminated from \(\phi_3\) if

\[ 2i(\hat{B} + \mu_1 B) + (6i\mu_3 \omega_\phi^2 - \frac{3\alpha_3}{\omega_\phi}) B^2 \bar{B} + \frac{f}{\omega_\phi} B e^{i\omega t + i\tau_\tau} = 0 \quad (2.32) \]
where the overdot denotes the derivative with respect to the original time \( t \),  
\( \sigma = \Omega - 2\omega_\phi \),  
\( B = \varepsilon A \), the unscaled amplitude (here we have eliminated \( \varepsilon \) by rewriting all the variables in their original form; \( \varepsilon \) is no longer needed), and

\[
f = \frac{1}{2} (f_1 e^{iz} - f_2 \phi e^{iz} + f_3 e^{iz} - f_4 \phi e^{iz})
\]
\[
= |f| e^{iz}
\]

(2.33)

Here \( f \) is an effective amplitude, due to the combined influence of heave and pitch, and is a complex function of \( K_{\phi z} \), \( K_{\phi \theta} \), \( K_{\phi z} \), \( K_{\phi \phi} \), \( a_z \), \( a_\theta \), \( \Omega \), \( \omega_\phi \), \( \tau_z \), and \( \tau_\theta \) [see equation (2.22)].

Next, we express the function \( B \) in the following polar form:

\[
B = \frac{1}{2} ae^{i\beta}
\]

(2.34)

where \( a \) and \( \beta \) are the amplitude and phase of the response. It follows that

\[
\dot{a} = -\mu_1 a - \frac{3}{4} \mu_3 \omega_\phi^2 a^3 - \frac{1}{2} \frac{|f|}{\omega_\phi} a \sin \gamma
\]

(2.35)

\[
a \dot{\gamma} = a \sigma + \frac{3}{4} \frac{\sigma}{\omega_\phi} a^3 - \frac{|f|}{\omega_\phi} a \cos \gamma
\]

(2.36)

where

\[
\gamma = \sigma t - 2\beta + \tau_f
\]

(2.37)

Alternatively, we can write \( B \) in the Cartesian form
\[ B = \frac{1}{2} (p - iq)e^{\frac{1}{2} i(\sigma T_1 + \tau)} \]  \hspace{1cm} (2.38)

where the \( p \) and \( q \) are real. Substituting equation (2.38) into equation (2.32) and separating real and imaginary parts, we obtain the modulation equations in the Cartesian form

\[ \dot{p} = -\mu_1 p - \frac{1}{2} \sigma q - \frac{3}{4} \mu_3 \omega_\phi^2 (p^3 + pq^2) - \frac{3}{8} \frac{\sigma_3}{\omega_\phi} (p^2 q + q^3) - \frac{1}{2} \frac{|f|}{\omega_\phi} q \quad (2.39) \]

\[ \dot{q} = -\mu_1 q + \frac{1}{2} \sigma p - \frac{3}{4} \mu_3 \omega_\phi^2 (p^2 q + q^3) + \frac{3}{8} \frac{\sigma_3}{\omega_\phi} (p^3 + pq^2) - \frac{1}{2} \frac{|f|}{\omega_\phi} p \quad (2.40) \]

Comparing equations (2.34) and (2.38), we can obtain the following conversion relations between the polar coordinates (2.34) and the Cartesian coordinates (2.38):

\[ p = a \cos \frac{1}{2} \gamma \quad (2.41) \]

\[ q = a \sin \frac{1}{2} \gamma \quad (2.42) \]
2.3.3 Fixed Points

Periodic responses correspond to constant \( a \) and \( \gamma \) or fixed-point solutions of equations (2.35) and (2.36). To determine the fixed points, we let \( a = \gamma = 0 \). There exist two sets of fixed-point solutions. First,

\[
a = 0
\]  \hspace{1cm} (2.43)

is always a solution, which corresponds to the trivial solution of equation (2.16); that is,

\[
\phi(t) \approx 0
\]  \hspace{1cm} (2.44)

Second, when \( a \neq 0 \), manipulating equations (2.35) and (2.36) yields a set of algebraic equations which can be solved numerically to determine \( a \) and \( \gamma \). The equation for \( a \) is

\[
c_4 a^4 + 2c_2 a^2 + c_0 = 0
\]  \hspace{1cm} (2.45)

where

\[
c_4 = \frac{9}{4} \mu_3^2 \omega_\phi^4 + \frac{9}{16} \frac{x_3^2}{\omega_\phi^2}
\]

\[
c_2 = 3\mu_1 \mu_3 \omega_\phi^2 + \frac{3}{4} \frac{x_3}{\omega_\phi} \sigma
\]

\[
c_0 = 4\mu_1^2 + \sigma^2 - \frac{|f|^2}{\omega_\phi^2}
\]  \hspace{1cm} (2.46)
In this case, it follows from equations (2.34), (2.29), and (2.23) and from the relationship between $A$ and $B$ that

$$
\phi(t) \approx a \cos \left[ \frac{1}{2} (\Omega t - \gamma + \tau_p) \right] 
$$

(2.47)

where $a$ and $\gamma$ are given by equations (2.35) and (2.36) and we have made use of equation (2.31).

The above fixed-point solutions and hence periodic solutions of equation (2.21) can also be obtained in Cartesian coordinates. Again, we let $\dot{p} = \dot{q} = 0$ and obtain

$$
- \mu_1p - \frac{1}{2} (\sigma + \frac{|f|}{\omega_\phi})q - \frac{3}{4} \mu_2 \omega_\phi^2 (p^3 + pq^2) - \frac{3}{8} \frac{\alpha_3}{\omega_\phi} (p^2 q + q^3) = 0
$$

(2.48)

$$
\frac{1}{2} (\sigma - \frac{|f|}{\omega_\phi})p - \mu_1 q - \frac{3}{4} \mu_3 \omega_\phi^2 (p^2 q + q^3) + \frac{3}{8} \frac{\alpha_3}{\omega_\phi} (p^3 + pq^2) = 0
$$

(2.49)

Equations (2.48) and (2.49) are a set of coupled nonlinear algebraic equations to be solved for fixed-point solutions $(p, q)$. The fixed-point solutions obtained by this procedure are also verified by numerically integrating equations (2.35) and (2.36) using a 5th- and 6th-order Runge-Kutta-Verner’s scheme with double precision arithmetics on an IBM 3090 digital supercomputer.
2.4 Stability of Fixed Points

To determine the stability of the fixed points and hence the stability of the periodic responses, we rewrite equations (2.35) and (2.36) or (2.39) and (2.40) in vector form. Letting \( \tilde{p} = (a, \gamma)^T \) for equations (2.35) and (2.36) or \( \tilde{p} = (p, q)^T \) for equations (2.39) and (2.40) respectively, we rewrite them in the vector form

\[
\tilde{p}' = \mathbf{F}(\tilde{p}) \tag{2.50}
\]

where \( \mathbf{F} \) is a real two-dimensional vector function of the two-dimensional vector \( \tilde{p} \). If \( \tilde{p} = \tilde{p}_0 \) is a fixed point of equation (2.50), we examine its stability by superimposing on it a small disturbance \( \tilde{\zeta}(t) \) and obtain the perturbed equation

\[
\tilde{\zeta}' = \mathbf{F}(\tilde{p}_0 + \tilde{\zeta}) \tag{2.51}
\]

Expanding equation (2.51) for small \( \tilde{\zeta} \), using the fact that \( \mathbf{F}(\tilde{p}_0) = 0 \), and linearizing the resulting equation, we obtain the linear variational equation

\[
\tilde{\zeta}' = [\nabla \mathbf{F}(\tilde{p}_0)] \tilde{\zeta} \tag{2.52}
\]

or

\[
\tilde{\zeta}' = A \tilde{\zeta} \tag{2.53}
\]
where

$$A = \nabla F(p_0)$$  (2.54)

is the Jacobian matrix evaluated at a given fixed point $p_0$. The stability of a fixed point depends on the real parts of the eigenvalues of the Jacobian matrix; that is, the roots of

$$|A - \lambda I| = 0$$  (2.55)

A given fixed point is asymptotically stable if and only if both roots of equation (2.55) lie in the left half of the complex plane and unstable if at least one eigenvalue lies in the right half of the complex plane. The Jacobian matrix $A$ for the modulation equations (2.35) and (2.36) in polar form is

$$A = \begin{bmatrix}
-\mu_1 - \frac{9}{4} \mu_3 \omega^2_\phi a^2 - \frac{|f|}{2\omega_\phi} \sin \gamma - \frac{|f|}{2\omega_\phi} a \cos \gamma \\
\frac{3\alpha_3}{2\omega_\phi} a \\
\frac{|f|}{\omega_\phi} \sin \gamma
\end{bmatrix}$$  (2.56)

Equations (2.55) and (2.56) can be used to determine the stability of nontrivial fixed-point solutions. However, they cannot be used to determine the stability of the trivial fixed-point solutions because equation (2.56) was obtained after dividing both sides of equation (2.36) by $a$. 

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To determine the stability of both the trivial and nontrivial fixed-point solutions, we make use of the modulation equations (2.39) and (2.40) in Cartesian coordinates. The Jacobian matrix $A$ in this case is

$$A = \begin{bmatrix}
-\mu_1 - \frac{3}{4} \mu_3 \omega_\phi^2 (3p^2 + q^2) - \frac{3}{8} \frac{\sigma_\phi}{\omega_\phi} (2pq) - \frac{\sigma}{2} - \frac{|f|}{2\omega_\phi} - \frac{3}{4} \mu_3 \omega_\phi^2 (2pq) - \frac{3}{8} \frac{\sigma_\phi}{\omega_\phi} (p^2 + 3q^2) \\
\frac{\sigma}{2} - \frac{|f|}{2\omega_\phi} - \frac{3}{4} \mu_3 \omega_\phi^2 (2pq) + \frac{3}{8} \frac{\sigma_\phi}{\omega_\phi} (3p^2 + q^2) - \mu_1 - \frac{3}{4} \mu_3 \omega_\phi^2 (p^2 + 3q^2) + \frac{3}{8} \frac{\sigma_\phi}{\omega_\phi} (2pq)
\end{bmatrix}
$$

(2.57)

For the trivial solution, equation (2.57) becomes

$$A = \begin{bmatrix}
-\mu_1 - \frac{\sigma}{2} - \frac{|f|}{2\omega_\phi} \\
\frac{\sigma}{2} - \frac{|f|}{2\omega_\phi} - \mu_1
\end{bmatrix}
$$

(2.58)

Consequently, the eigenvalues of equation (2.58) are

$$\lambda = -\mu_1 \pm \frac{1}{2} \sqrt{\frac{|f|^2}{\omega_\phi^2} - \sigma^2}
$$

(2.59)

Equation (2.59) implies that, because $\mu_1 > 0$, the trivial fixed points are unstable when

$$|f| > \omega_\phi \sqrt{\sigma^2 + 4\mu_1^2} = f_c,
$$

(2.60)
and asymptotically stable if $|f| < |f_c|$. The condition that separates stable from unstable trivial fixed points is $|f| = f_c$, which can be expressed in terms of $\sigma$ as

$$\sigma^2 \leq \frac{|f|^2}{\omega_\phi^2} - 4\mu_1^2 = \sigma_c^2$$

(2.61)

For nontrivial fixed-point solutions, we first note from equation (2.46) that the coefficient $c_4$ is always nonnegative. Next, we draw from equation (2.45) the following conclusions:

1) one fixed-point solution exists if either

$$c_0c_4 \leq 0 \leq c_2$$

(2.62)

or

$$c_0c_4 < 0 \text{ and } c_2 < 0 \text{ and } c_0c_4 \leq c_2^2$$

(2.63)

2) two fixed-point solutions exist if

$$c_2 < 0 < c_0c_4 \text{ and } c_0c_4 \leq c_2^2$$

(2.64)

3) no fixed-point solution exists otherwise.

The stability of these fixed-point solutions is determined from equations (2.55) and (2.57).
2.5 Results and Discussion

Equations (2.35)-(2.37) and (2.43)-(2.47) yield two types of force-response curves, depending on the values of the parameters.

Figure 2.2 shows a typical supercritical-type force-response curve when \( \mu_1 = \mu_3 = 0.04, \alpha_3 = 1.0, \) and \( \sigma = 0.20. \) As the resultant forcing amplitude \( |f| \) is increased from 0, only the trivial solution exists and it is stable. As \( |f| \) increases beyond the bifurcation point \( |f| = 0.2155 (= |f|_c \) of equation (2.60)), the trivial solution becomes unstable and a nontrivial solution appears, which is stable. As \( |f| \) is increased further, the roll amplitude grows nonlinearly and monotonically.

Figure 2.3 shows a typical subcritical-type force-response curve; it displays some interesting features. The values of the parameters are the same as those in Figure 2.2 except that the sign of \( \sigma \) has been changed. As the effective amplitude \( |f| \) is increased from 0, only one solution is possible; the trivial solution, which is stable. As \( |f| \) passes 0.0957 (\( |f|_c \)) approximately, three solutions are possible, the trivial solution, which is stable, and two nontrivial solutions, the larger of which is stable and the smaller one is unstable. As \( |f| \) passes \( \zeta_2 = 0.2155 \) approximately (i.e., \( |f|_c \) of equation (2.60)), only two solutions are possible, the trivial solution, which is unstable, and a rather large-amplitude nontrivial solution, which is stable.

In an experiment, one would never see a motion corresponding to the lower-amplitude nontrivial solution in the range 0.0957 < \( |f| < 0.2155. \) For
$|f| > 0.2155$, one would always see a rolling motion. In the range $0.0957 < |f| < 0.2155$, one could expect to see one of two possible motions: either no roll at all (corresponding to the trivial solution), or a rather large-amplitude roll motion. The initial conditions, or external disturbances, determine which motion will develop. The existence of two stable responses to the same excitation is a characteristic of subcritical instability and has been observed in many other mechanical and structural systems. Some of such results can be found in the works of Hatwal, Mallik, and Ghosh [46], Sethna [113], and Sethna and Bajaj [114].

As the effective amplitude $|f|$ is slowly increased from 0, the rolling motion will not be excited until the bifurcation point $|f| = \zeta_2 = 0.2155$ is reached. As $|f|$ is increased further, the trivial solution becomes unstable and the roll motion suddenly occurs; that is, the roll amplitude suddenly jumps to a large value. From there, the "rolling" solution is the only stable response, and its amplitude increases nonlinearly and monotonically as $|f|$ is increased further.

As $|f|$ is decreased slowly from a large enough value, the magnitude of the large-amplitude roll motion decreases nonlinearly, and rolling continues even after the jump-up bifurcation point ($\zeta_2$) is passed. When $|f|$ reaches the second bifurcation point $|f| = \zeta_1 = 0.0957$, the large-amplitude roll motion suddenly disappears and the response is roll free; that is, the jump-down phenomenon occurs.
In the experiments presented in Chapter V, the supercritical-type force-response result was not obtained in spite of much efforts devoted to this case. In contrast, the subcritical-type force-response results were obtained in wide ranges of excitation amplitude and frequency.

Figures 2.4 (a) and (b) show two frequency-response curves for different values of the effective amplitudes $|r|$ of excitation. The values of the parameters are $\mu_1 = \mu_3 = 0.04$, $\omega_b = \alpha_3 = 1.0$ for both cases and $|r| = 0.20$ for (a) and $|r| = 0.15$ for (b). Figure 2.4 (a) shows that the trivial fixed-point solution is unstable when $-0.1834 = -|\sigma_c| \leq \sigma \leq |\sigma_c| = 0.1834$, where $\sigma_c$ is the critical value of the detuning parameter $\sigma$ defined in equation (2.61). The same is true in Figure 2.4 (b) in the range $-0.1269 = -|\sigma_c| \leq \sigma \leq |\sigma_c| = 0.1269$. Thus, we note that for values of $\sigma$ between the two bifurcation values ($-|\sigma_c|$ and $+|\sigma_c|$), the large-amplitude roll motion is the only stable response. Because the trivial solution is unstable in this region, the model will display violent rolling spontaneously even when the model is given zero initial conditions.

When $-1.0 \leq \sigma \leq -0.1834$ in Figure 2.4 (a) and $-0.8809 \leq \sigma \leq -0.1269$ in Figure 2.4 (b), there exist three fixed-point solutions: a stable large-amplitude roll, an unstable small-amplitude roll, and a stable no-roll motion. The initial conditions, or external disturbances, determine which motion will be exhibited by the ship. When proper external disturbances are imposed on the ship, the response will exhibit the jump-up or jump-down phenomena between the two stable large-amplitude and trivial
solutions. The unstable fixed-point solutions are not achievable in experiments, irrespective of whether they are trivial or nontrivial solutions.

When \( \sigma > 0.1834 \) in Figure 2.4 (a) and \( \sigma > 0.1269 \) and \( \sigma < -0.8809 \) in Figure 2.4 (b), the ship does not display any roll motion because the trivial no-roll motion is the only stable fixed-point solution in these regions.

We did not perform experiments to obtain frequency-response curves corresponding to Figures 2.4 (a) and (b) in which the excitation frequency is varied while the excitation amplitude is held constant. The reasons are discussed in Chapters IV and V.
Figure 2.1. Coordinate Systems
Figure 2.2. Force-Response Curve (supercritical type): $\mu_1 = \mu_2 = 0.04,$ $\sigma = 0.20,$ $\omega_\phi = 1, \alpha_2 = 1, |r|_e = 0.2155; \text{ stable (---),} \text{ unstable (----)}$
**Figure 2.3. Force-Response Curve (subcritical type):** \( \mu_1 = \mu_3 = 0.04, \sigma = -0.20, \omega_p = 1, \alpha_3 = 1, |F|/f_c = 0.2155; \) stable (---), unstable (----)
Figure 2.4. Frequency-Response Curves: (a) $|r| = 0.20$, (b) $|r| = 0.15$; $\mu_1 = \mu_3 = 0.04$, $\omega_0 = 1$, $\alpha_3 = 1$, stable (-----), unstable (----)
Chapter III

A THEORETICAL INVESTIGATION OF COUPLED PITCH AND ROLL MOTIONS IN THE PRESENCE OF INTERNAL RESONANCE

3.1 Statement of the Problem

In Chapter II, we discussed parametric excitations of the out-of-plane (roll) mode by an in-plane excitation (heave or pitch), or equivalently, longitudinal waves. In this chapter, we study another case of exciting the roll by longitudinal waves. The mechanism of coupling is two-to-one internal resonance in which the natural frequency in pitch is twice that in roll. The
present theoretical results are validated by the experiments discussed in Chapter VI.

The significance of internal resonance has been recognized recently in many mechanical and elastic systems. To model the mechanism of nonlinear interactions between modes, one needs to model the system by two or more nonlinearly coupled oscillators. There are a number of references dealing with physical two-degree-of-freedom systems. Among others, Nayfeh and Mook [79] discussed problems involving the forced responses of robots, elastic pendulums, beams and plates under static loadings, composite plates, arches, shells, and the sloshing of liquid gasoline in the fuel container of an airplane. All of these problems can be modeled by coupled, inhomogeneous ordinary-differential equations with quadratic nonlinearities. When these systems possess internal (or autoparametric) resonances, which may occur if the natural frequencies of the system are commensurate, their responses may exhibit extraordinarily complicated behaviors, which cannot be explained by linear formulations.

Interestingly, two-to-one autoparametric or internal resonances may strongly influence the dynamic behavior and stability of vessels [79]. A strong coupling of the involved modes of motion produced by internal or autoparametric resonance was first observed by Froude in 1863. He observed that a vessel whose linear undamped natural frequency in pitch is twice that in roll [32] has undesirable seakeeping characteristics. This observation was a manifestation of the two-to-one internal resonance [28, 43, 45, 85, 109]. In the
conventional linear approach for the analysis of the dynamics of vessels, the six nonlinear coupled equations of motion are reduced to two sets of three linear equations. This linearized theory is valid when the response is small and the wave state is moderate. In the linearized model, the yaw, sway, and roll modes are uncoupled from the pitch, heave, and surge modes because the nonlinear couplings between these two groups of modes are neglected. Consequently, the significance of the two-to-one frequency ratio between pitch and roll cannot be determined using linearized equations.

For a century after Froude, however, no further research on this phenomenon was pursued. In 1959, Paulling and Rosenberg [95] studied the coupled heave-roll motion of a vessel using a set of nonlinear ordinary-differential equations. They simplified the equations of motion by neglecting damping, any nonlinear coupling terms due to roll in the equation for the heave mode, and forcing terms. They obtained a single roll equation having the form of a simple linear Mathieu equation which contains a time-varying coefficient due to a simple harmonic motion of the heave mode. In this model, the heave influences the roll but the roll does not influence the heave. Experimentally, they tested the case of unstable rolling motion excited by the heave mode only. Their model was rigidly constrained to eliminate all of its degrees of freedom other than the roll and heave. Then, a forced heaving oscillation was impressed on the model which otherwise was sitting in calm water. This study has two principal shortcomings. First, due to the lack of consideration for damping and nonlinear coupling terms, the analytical
model was not capable of yielding realistic results. Second, in the experimental setup, the heave mode was given a prescribed motion and hence the effect of roll motion and waves generated thereafter on the heave mode are not taken into account.

Kinney [57], Kerwin [56], Blocki [9], and Sanchez and Nayfeh [107] also studied the response of the roll to longitudinal waves. Except for Blocki's study, all other studies are theoretical as discussed in Chapters I and II. In his experiment, Blocki considered coupling of the heave and roll modes. To achieve this, he considered a ship that possess fore and aft symmetry, which is placed in beam waves. As in the study of Paulling and Rosenberg, he studied the case of parametrically excited roll motions in which energy is fed to the roll mode by a prescribed heave or pitch motion, or equivalently, wave motion. His equations reduce essentially to a one-degree-of-freedom model governing only the roll mode.

To explain Froude's observation, Nayfeh, Mook, and Marshall [80] and Mook, Marshall, and Nayfeh [70] modeled the ship motion by two nonlinearly coupled equations involving the pitch and roll modes; they included the dependence of the pitching moment on the roll orientation. Thus, the pitch (heave) motion is not prescribed but is coupled to the roll motion, and consequently, the pitch (heave) and roll orientations are determined simultaneously as functions of a prescribed excitation. They clearly showed the significance of the frequency ratio in causing undesirable roll behaviors,
such as the “saturation” phenomenon. They offered an explanation of the observations of Froude.

Nayfeh [75] considered the nonlinearly coupled roll and pitch motions of a ship in regular head waves in which the couplings are primarily in the hydrostatic terms when the pitch frequency is approximately twice the roll frequency and the encounter frequency is near either the pitch or roll natural frequency. He demonstrated the saturation phenomenon when the encounter frequency is near the pitch natural frequency and demonstrated the existence of Hopf bifurcations.

In the present chapter, we extend the work of Nayfeh [75] by using a linear-plus-quadratic damping model for the roll motion and investigate the cases of primary resonances. The linear-plus-quadratic damping model has long been recognized by investigators to describe closely the dissipation of energy in the roll mode, but it was not used so far because of some analytical difficulties. We obtain various complicated responses, which are common features of the nonlinear dynamics of many mechanical and elastic systems. These responses include supercritical and subcritical instabilities, periodic motions, periodically and chaotically modulated motions, period-doubling bifurcations, and coexistence of multiple solutions and associated jumps. The quadratic damping eliminates the saturation phenomenon.

The theoretical predictions of this chapter are verified by the experiments described in Chapter VI.
3.2 Equations of Motion

We consider the response of a ship that is restricted to pitch and roll to a regular wave. We assume that the ship is laterally symmetric. We use the same coordinate system described in Chapter II: a body-fixed coordinate system $oxyz$ such that its origin $o$ is at the center of mass, the $x$-axis is positive toward the bow, the $y$-axis is positive toward starboard, and the $z$-axis is positive downward. The orientation of the ship with respect to an inertial frame $OXYZ$ is defined by the Euler angles $\phi$ and $\theta$ as follows: $\theta$ is a pitch rotation about the original $y$-axis, and $\phi$ is a roll rotation about the new $x$-axis. The components $p$ and $q$ of the angular velocity about the $x$- and $y$-axes are related to $\phi, \theta, \dot{\phi}$, and $\dot{\theta}$ by

$$p = \dot{\phi} \quad \text{and} \quad q = \dot{\theta} \cos \phi$$ (3.1)

The equations of motion can be written as

$$l_{xx}\ddot{\phi} - l_{xz}\dot{p}q = K + K_0 \cos \Omega t$$ (3.2)

$$l_{yy}\ddot{q} + l_{xz}\dot{p}^2 = M + M_0 \cos(\Omega t + \tau)$$ (3.3)

where $l_{xx}, l_{yy},$ and $l_{xz}$ are the moments and product of inertia, $\Omega$ is the encounter frequency, $K_0$ and $M_0$ are the amplitudes of the moments produced by the waves, and $\tau$ is a phase; they are assumed constants. We assume that the
hydrodynamic moments $K$ and $M$ are analytic functions of $\phi$ and $\theta$ and their derivatives. Following Nayfeh [75] and Nayfeh, Mook, and Marshall [81] and including a quadratic damping term in the roll equation, we obtain

$$\ddot{\phi} + \omega_1^2 \phi = \varepsilon \left[ -2\mu_1 \phi - \mu_3 \dot{\phi} \right] \phi \left| \dot{\phi} + \delta_1 \phi \theta + \delta_2 \phi \theta \right. \\
+ \delta_3 \theta \phi + \delta_4 \phi \theta + F_1 \cos \Omega t \right]$$

(3.4)

$$\ddot{\theta} + \omega_2^2 \theta = \varepsilon \left[ -2\mu_2 \dot{\theta} + \alpha_1 \phi^2 + \alpha_2 \phi \dot{\phi} + \alpha_3 \theta^2 \\
+ \alpha_4 \theta \phi + \alpha_5 \dot{\phi}^2 + \alpha_6 \dot{\theta}^2 + F_2 \cos(\Omega t + \tau) \right]$$

(3.5)

where $\varepsilon$ is a small dimensionless parameter that is introduced as a bookkeeping device in the perturbation analysis that follows.

3.3 Method of Multiple Scales

We use the method of multiple scales [72, 73] to determine a first-order approximation to the solutions of equations (3.4) and (3.5). We let

$$\phi(t; \varepsilon) = \phi_0(T_0, T_1) + \varepsilon \phi_1(T_0, T_1) + ...$$

(3.6)

$$\theta(t; \varepsilon) = \theta_0(T_0, T_1) + \varepsilon \theta_1(T_0, T_1) + ...$$

(3.7)

where $T_0 = t$ is a fast time scale, characterizing motions on the scales $\omega_1$ and $\Omega$; and $T_1 = \varepsilon t$ is a slow time scale, characterizing the modulation of the
amplitudes and phases of the motion. In terms of $T_0$ and $T_1$, the time derivatives are transformed into

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \ldots \quad \text{and} \quad \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \ldots \quad (3.8)$$

where $D_n = \partial / \partial T_n$.

Substituting equations (3.6)-(3.8) into equations (3.4) and (3.5) and equating coefficients of like powers of $\varepsilon$, we obtain

$0(\varepsilon^0)$:

$$D_0^2 \phi_0 + \omega_1^2 \phi_0 = 0 \quad (3.9)$$

$$D_0^2 \theta_0 + \omega_2^2 \theta_0 = 0 \quad (3.10)$$

$0(\varepsilon)$:

$$D_0^2 \phi_1 + \omega_1^2 \phi_1 = -2D_0 D_1 \phi_0 - 2\mu_1 D_0 \phi_0 - \mu_2 D_0 \phi_0 \mid D_0 \phi_0 \mid + \delta_1 \phi_0 \theta_0 + \delta_2 \phi_0 D_0^2 \theta_0 + \delta_3 \theta_0 D_0 \phi_0 + \delta_4 D_0 \phi_0 D_0 \theta_0 + F_1 \cos \Omega T_0 \quad (3.11)$$

$$D_0^2 \theta_1 + \omega_2^2 \theta_1 = -2D_0 D_1 \theta_0 - 2\mu_2 D_0 \theta_0 + \alpha_1 \phi_0^2 + \alpha_2 \phi_0 D_0^2 \phi_0 + \alpha_3 \theta_0^2 + \alpha_4 \theta_0 D_0^2 \theta_0 + \alpha_5 (D_0 \phi_0)^2 + \alpha_6 (D_0 \theta_0)^2 + F_2 \cos (\Omega T_0 + \tau) \quad (3.12)$$

The solutions of equations (3.9) and (3.10) can be expressed as

$$\phi_0 = A_1(T_1) e^{i\omega_1 T_0} + \overline{A_1(T_1)} e^{-i\omega_1 T_0} \quad (3.13)$$
\[ \theta_0 = A_2(T_1)e^{i\omega_2 T_0} + \overline{A}_2(T_1)e^{-i\omega_2 T_0} \] (3.14)

where \( A_1 \) and \( A_2 \) are unknown functions at this level of approximation. They are determined by imposing the solvability conditions at the next level of approximation. Alternatively, the solutions of equations (3.13) and (3.14) can be expressed as

\[ \phi_0 = a_1(T_1) \cos[\omega_1 T_0 + \beta_1(T_1)] \] (3.15)

\[ \theta_0 = a_2(T_1) \cos[\omega_2 T_0 + \beta_2(T_1)] \] (3.16)

where the \( a_n \) and \( \beta_n \) are the amplitudes and phases of the roll and pitch modes.

Comparing equations (3.13) and (3.14) with equations (3.15) and (3.16), we conclude that

\[ A_n(T_1) = \frac{1}{2} a_n(T_1)e^{i\beta_n(T_1)} \] (3.17)

Substituting equations (3.13) and (3.14) into equations (3.11) and (3.12) yields

\[ D_0^2 \phi_1 + \omega_1^2 \phi_1 = -2i\omega_1(A_1' + \mu_1 A_1)e^{i\omega_1 T_0} \]
\[ + (\delta_1 - \omega_2^2 \delta_2 - \omega_1 \omega_2 \delta_3 - \omega_1 \omega_2 \delta_4)A_2 A_1 e^{i(\omega_2 + \omega_1)T_0} \]
\[ + (\delta_1 - \omega_2^2 \delta_2 - \omega_1 \omega_2 \delta_3 + \omega_1 \omega_2 \delta_4)A \overline{A} e^{i(\omega_2 - \omega_1)T_0} \] (3.18)
\[ + \frac{1}{2} F_1 e^{i\Omega T_0} + cc + f \left[ i\omega_1 A_1 e^{i\omega_1 T_0} - i\omega_1 A \overline{A} e^{-i\omega_1 T_0} \right] \]
\[ D_0^2 \theta_1 + \omega_2^2 \theta_1 = -2i\omega_1 (A_1' + \mu_2 A_2)e^{i\omega_2 T_0} \\
+ (\alpha_1 - \omega_1^2 \alpha_2 - \omega_1^2 \alpha_5)A_1 e^{2i\omega_1 T_0} \\
+ (\alpha_3 - \omega_2^2 \alpha_4 - \omega_2^2 \alpha_6)A_2 e^{2i\omega_2 T_0} \\
+ (\alpha_1 - \omega_1^2 \alpha_2 + \omega_1^2 \alpha_5)A_1 \bar{A}_1 \\
+ (\alpha_3 - \omega_2^2 \alpha_4 + \omega_2^2 \alpha_6)A_2 \bar{A}_2 + \frac{1}{2} F_2 e^{i(\Omega T_0 + \tau)} + cc \] (3.19)

where the prime indicates the derivative with respect to \( T_1 \), \( \bar{A}_n \) is the complex conjugate of \( A_n \), and the function \( f \) accounts for the term \(-\mu_2 D_0 \phi_0 |D_0 \phi_0|\). Depending on the functions \( A_n \), particular solutions of equations (3.18) and (3.19) contain terms proportional to \( T_0 e^{\pm i\omega_n T_0} \) (i.e., secular terms). They also contain small-divisor terms if \( \Omega \approx \omega_1 \) or \( \Omega \approx \omega_2 \) (i.e., primary resonances of the pitch or roll mode) and/or if \( \omega_2 \approx 2\omega_1 \) (the frequency of pitch mode is approximately twice that of roll mode; i.e., two-to-one internal or autoparametric resonance).

To eliminate the secular and small-divisor terms, we first expand \( f(D_0 \phi_0) \) in a Fourier series as

\[ f = \sum_{n=-\infty}^{\infty} f_n(A_1, \bar{A}_1) e^{i\omega_1 T_0} \] (3.20)

where

\[ f_n(A_1, \bar{A}_1) = \frac{\omega_1}{2\pi} \int_{0}^{2\pi/\omega_1} f e^{-i\omega_1 \tau} d\tau \] (3.21)

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Consequently, the component of $f$ that produces a secular term is

$$
\frac{\omega_1}{2\pi} \int_0^{2\pi/\omega_1} f e^{-i\omega_1 T_0 d T_0} \quad (3.22)
$$

We analyze the case of primary resonance of the pitch mode in Section 3.4 and of primary resonance of the roll mode in Section 3.5.

### 3.4 Primary Resonance of the Pitch Mode

#### 3.4.1 Modulation Equations

To express the nearness of the resonances, we introduce two detuning parameters, $\sigma_1$ and $\sigma_2$, defined as

$$
\omega_2 = 2\omega_1 + \varepsilon \sigma_1 \quad \text{and} \quad \Omega = \omega_2 + \varepsilon \sigma_2 \quad (3.23a)
$$

Hence,

$$
\omega_2 T_0 = 2\omega_1 T_0 + \sigma_1 T_1 \quad \text{and} \quad \Omega T_0 = \omega_2 T_0 + \sigma_2 T_1 \quad (3.23b)
$$

Using equations (3.23) and (3.22) and eliminating the terms that produce secular terms from equations (3.18) and (3.19), we obtain
\[ 2i(A_1' + \mu_1 A_1) - 4\Lambda_1 A_2 \overline{A_1} \theta^{i\sigma_1 T_1} - \frac{1}{2\pi} \int_0^{2\pi/\omega_1} f e^{-i\omega_1 T_0} dT_0 = 0 \quad (3.24) \]

\[ 2i(A_2' + \mu_2 A_2) - 4\Lambda_2 A_1^2 e^{-i\sigma_1 T_1} - f_2 e^{i(\sigma_2 T_1 + \tau)} = 0 \quad (3.25) \]

where

\[ 4\omega_1 \Lambda_1 = \delta_1 - \omega_2^2 \delta_2 - \omega_1^2 \delta_3 + \omega_1 \omega_2 \delta_4 \quad (3.26) \]

\[ 4\omega_2 \Lambda_2 = \alpha_1 - \omega_1^2 (\alpha_2 + \alpha_5) \quad (3.27) \]

\[ \omega_2 f_2 = \frac{1}{2} F_2 \quad (3.28) \]

Nayfeh [75] and Nayfeh, Mook, and Marshall [80] concluded that \( \Lambda_1 \) and \( \Lambda_2 \) have the same sign; otherwise, the unforced ship would be self-oscillating, which is unrealistic due to dissipation.

Substituting equation (3.17) into equations (3.24) and (3.25), rewriting equation (3.15) as

\[ \phi_0 = a_1(T_1) \cos \chi_1, \quad \chi_1 = \omega_1 T_0 + \beta_1(T_1) \quad (3.29) \]

and separating real and imaginary parts, we obtain

\[ a_1' = -\mu_1 a_1 + \Lambda_1 a_1 a_2 \sin \gamma_1 - \frac{1}{2\pi\omega_1} \int_0^{2\pi} \sin \chi_1 f (-\omega_1 a_1 \sin \chi_1) d\chi_1 \quad (3.30) \]
\[ a_2' = -\mu_2 a_2 - \Lambda_2 a_1^2 \sin \gamma_1 + f_2 \sin \gamma_2 \]  
(3.31)

\[ a_1 \beta_1' = -\Lambda_1 a_1 a_2 \cos \gamma_1 - \frac{1}{2\pi \omega_1} \int_0^{2\pi} \cos \chi_1 f(-\omega_1 a_1 \sin \chi_1) d\chi_1 \]  
(3.32)

\[ a_2 \beta_2' = -\Lambda_2 a_1^2 \cos \gamma_1 - f_2 \cos \gamma_2 \]  
(3.33)

where

\[ \gamma_1 = \sigma_1 T_1 + \beta_2 - 2\beta_1 \quad \text{and} \quad \gamma_2 = \sigma_2 T_1 - \beta_2 + \tau \]  
(3.34)

To evaluate the last terms in equations (3.30) and (3.32), we replace the function \( f \) with \(-\mu_3 D_0 \phi_0 |D_0 \phi_0|\), use equation (3.29), and obtain

\[ -\frac{1}{2\pi \omega_1} \int_0^{2\pi} \sin \chi_1 (-\mu_3 D_0 \phi_0 |D_0 \phi_0|) d\chi_1 = -\frac{4\mu_3 \omega_1}{3\pi} a_1 |a_1| \]  
(3.35)

\[ -\frac{1}{2\pi \omega_1} \int_0^{2\pi} \cos \chi_1 (-\mu_3 D_0 \phi_0 |D_0 \phi_0|) d\chi_1 = 0 \]  
(3.36)

Equations (3.30)-(3.34) can be reduced to a generic system of equations by applying the following transformation of variables:

\[ a_1 = \hat{a}_1 / \sqrt{\Lambda_2} \, , \quad a_2 = \hat{a}_2 / \sqrt{\Lambda_1} \, , \quad T_1 = \hat{T}_1 / \sqrt{\Lambda_1} \]  
(3.37)
Substituting equation (3.37) into equations (3.30)-(3.34), using equations (3.35) and (3.36), and then deleting the hat from the equations, we obtain

\[ a_1' = -\mu_1 a_1 + a_1 a_2 \sin \gamma_1 - \frac{4\mu_3 \omega_1}{3\pi} a_1 |a_1| \tag{3.38} \]

\[ a_2' = -\mu_2 a_2 - a_1^2 \sin \gamma_1 + f_2 \sin \gamma_2 \tag{3.39} \]

\[ a_1 \beta_1' = -a_1 a_2 \cos \gamma_1 \tag{3.40} \]

\[ a_2 \beta_2' = -a_1^2 \cos \gamma_1 - f_2 \cos \gamma_2 \tag{3.41} \]

where

\[ \gamma_1 = \sigma_1 T_1 + \beta_2 - 2\beta_1 \quad \text{and} \quad \gamma_2 = \sigma_2 T_1 - \beta_2 + \tau \tag{3.42} \]

Alternatively, we can write the \( A_n \) in the Cartesian form

\[ A_n = \frac{1}{2} (p_n - iq_n)e^{i(v_n T_1 + \kappa_n \pi)} \tag{3.43} \]

where the \( p_n \) and \( q_n \) are real. Substituting equation (3.43) into equations (3.24) and (3.25) and separating real and imaginary parts, we obtain the following modulation equations in Cartesian form:

\[ p_1' = -\mu_1 p_1 - v_1 q_1 - \frac{4\mu_3 \omega_1}{3\pi} p_1 \sqrt{p_1^2 + q_1^2} + p_2 q_1 - p_1 q_2 \tag{3.44} \]
\[ q_1' = -\mu_1 q_1 + \nu_1 p_1 - \frac{4\mu_3 \omega_1}{3\pi} q_1 \sqrt{p_1^2 + q_1^2} + p_1 p_2 + q_1 q_2 \quad (3.45) \]

\[ p_2' = -\mu_2 p_2 - \nu_2 q_2 - 2p_1 q_1 \quad (3.46) \]

\[ q_2' = -\mu_2 q_2 + \nu_2 p_2 + p_1^2 - q_1^2 + f_2 \quad (3.47) \]

where

\[ \nu_1 = \frac{1}{2} (\sigma_1 + \sigma_2) \quad \text{and} \quad \nu_2 = \sigma_2 \quad (3.48) \]

We note that the modulation equations (3.44)-(3.47) are invariant under the transformation \((p_1, q_1, p_2, q_2) \Rightarrow (-p_1, -q_1, p_2, q_2)\). This implies that the projection of a solution of the modulation equations onto the \(p_1 - q_1\) plane remains unaffected by rotation through 180° around the origin. Hence, the fixed points, trajectories, limit cycles, and attractors occur in duplicate if they are not transformed onto themselves by the symmetry. For example, if there is an attractor (i.e., a long-time solution of the modulation equations), then, for the same bifurcation parameters another attractor can be obtained by just applying the above transformation, unless the attractor itself is symmetric. We will not mention the symmetric "duplicate" attractors unless they interact in an interesting manner.

Comparing equations (3.17) and (3.43), we obtain
\[ p_1 = a_1 \cos \left( \frac{1}{2} (\gamma_1 + \gamma_2) \right) \]  
\[ q_1 = a_1 \sin \left( \frac{1}{2} (\gamma_1 + \gamma_2) \right) \]  
\[ p_2 = a_2 \cos \gamma_2 \]  
\[ q_2 = a_2 \sin \gamma_2 \]  

3.4.2 Fixed Points

We investigate the fixed points of the modulation equations (3.30)-(3.34), which correspond to periodic responses of the ship. They correspond to \( a_1' = a_2' = 0 \) and \( \gamma_1' = \gamma_2' = 0 \). It follows from equation (3.42) that

\[ \beta_2' = \sigma_2 \quad \text{and} \quad \beta_1' = \frac{1}{2} (\sigma_1 + \sigma_2) \]  

Hence, the fixed points of equations (3.38)-(3.42) are given by

\[ \mu_1 a_1 = a_1 a_2 \sin \gamma_1 - \frac{4 \mu_2 \omega_1}{3 \pi} a_1 |a_1| \]  
\[ \mu_2 a_2 = -a_1^2 \sin \gamma_1 + f_2 \sin \gamma_2 \]  
\[ \frac{1}{2} (\sigma_1 + \sigma_2) a_1 = -a_1 a_2 \cos \gamma_1 \]
\[ \sigma_2 a_2 = -a_1^2 \cos \gamma_1 - f_2 \cos \gamma_2 \quad (3.57) \]

There are two possible solutions for equations (3.54)-(3.57). First,

\[ a_1 = 0 \quad \text{and} \quad a_2 = \frac{f_2}{\sqrt{\sigma_2^2 + \mu_2}} \quad (3.58) \]

and the response is given by

\[ \phi = 0 \quad \text{and} \quad \theta = a_2 \cos(\Omega t + \tau - \gamma_2) + \ldots \quad (3.59) \]

which is essentially the linear solution. Second,

\[ a_2 = \left[ \frac{1}{4} (\sigma_1 + \sigma_2)^2 + \left( \mu_1 + \frac{4\mu_2\omega_1}{3\pi} a_1 \right)^2 \right]^{1/2} \quad (3.60) \]

and \( a_1 \) is given by the algebraic equation

\[ a_1^4 + c_3 a_1^2 |a_1| + c_2 a_1^2 + c_1 |a_1| + c_0 = 0 \quad (3.61) \]

where

\[ c_0 = (\mu_2^2 + \sigma_2^2) \left[ \frac{1}{4} (\sigma_1 + \sigma_2)^2 + \mu_1^2 \right] - f_2^2 \]

\[ c_1 = (\mu_2^2 + \sigma_2^2) \frac{8\mu_1\mu_3\omega_1}{3\pi} \]
\[ c_2 = (\mu_2^2 + \sigma_2^2) \left( \frac{4 \mu_3 \omega_1}{3 \pi} \right)^2 + 2 \mu_1 \mu_2 - \sigma_2(\sigma_1 + \sigma_2) \]

\[ c_3 = \frac{8 \mu_2 \mu_3 \omega_1}{3 \pi} \]

The response in this case is given by

\[ \phi = a_1 \cos\left( \frac{1}{2} \Omega t + \tau - \frac{1}{2} \gamma_1 - \frac{1}{2} \gamma_2 \right) + \ldots \quad (3.62) \]

\[ \theta = a_2 \cos(\Omega t + \tau - \gamma_2) + \ldots \quad (3.63) \]

We note from equations (3.60) and (3.61) that if \( a_1 = 0 \), then \( c_0 = 0 \) and thus this second solution reduces to the first solution (3.58); that is, the linear solution.

When \( \mu_3 = 0 \), but \( a_1 \neq 0 \) equation (3.60) becomes

\[ a_2 = \left[ \frac{1}{4} (\sigma_1 + \sigma_2)^2 + \mu_1^2 \right]^{1/2} = a_2^* \quad (3.64) \]

which is independent of \( a_1 \) and \( f_2 \). Moreover, \( c_1 = c_3 = 0 \), and hence

\[ a_1 = \left[ \Gamma_1 \pm (\Gamma_2 - \Gamma_2^{1/2})^{1/2} \right]^{1/2} \quad (3.65) \]

where

\[ \Gamma_1 = \frac{1}{2} \sigma_2(\sigma_1 + \sigma_2) - \mu_1 \mu_2 \quad (3.66) \]
\[ \Gamma_2 = \sigma_2 \mu_1 + \frac{1}{2} \mu_2 (\sigma_1 + \sigma_2) \]  

Equations (3.64) and (3.65) are essentially the same as those obtained by Nayfeh [75]; they indicate the saturation phenomenon. Hence, the solution (3.64)-(3.67) is a special case of the second solution given by equations (3.60) and (3.61).

We note that the second solution (3.60)-(3.63) does not exhibit the saturation phenomenon unless \( \mu_3 = 0 \). Instead, the amplitude \( a_2 \) of the directly excited pitch mode as well as the amplitude \( a_1 \) of the roll mode vary as functions of \( f_2 \). We have developed a computer code to solve equation (3.61) for \( a_1 \). Then, \( a_2 \) is obtained from equation (3.60) and the corresponding \( \gamma_1 \) and \( \gamma_2 \) (\( \beta_1 \) and \( \beta_2 \) also) are obtained from equations (3.54)-(3.57). On the other hand, to obtain the fixed points from the Cartesian form, we let \( p'_n = q'_n = 0 \) in the modulation equations (3.44)-(3.47) and solve for the \( p_n \) and \( q_n \) in a manner similar to that used in the case of polar coordinates.

### 3.4.3 Stability of Fixed Points

The stability of the fixed points and hence the stability of the periodic responses are determined in the same manner as in Section 2.4. The modulation equations (3.38)-(3.42) or (3.44)-(3.47) are rewritten in the vector
form of equation (2.39). Then \( p = (a_1, a_2, \gamma_1, \gamma_2)^T \) for equations (3.38)-(3.42) and
\( \gamma = (p_1, q_1, p_2, q_2)^T \) for equations (3.44)-(3.47).

In this case, \( F \) in equation (2.39) is a real four-dimensional vector function of the four-dimensional vector \( \gamma \). The Jacobian matrix \( A \) for the set of equations (3.38)-(3.42) in polar coordinates is

\[
A = 
\begin{bmatrix}
-\mu_1 - \frac{8\mu_2\omega_1}{3\pi} |a_1| + a_2\sin\gamma_1 & a_1\sin\gamma_1 & 0 & a_1a_2\cos\gamma_1 \\
-2a_1\sin\gamma_1 & -\mu_2 & f_2\cos\gamma_2 & -a_1^2\cos\gamma_1 \\
\frac{2a_1}{a_2}\cos\gamma_1 & -\frac{a_1^2}{a_2^2}\cos\gamma_1 - \frac{f_2}{a_2}\cos\gamma_2 & -\frac{f_2}{a_2}\sin\gamma_2 & -\frac{a_1^2}{a_2}\sin\gamma_1 \\
-2\frac{a_1}{a_2}\cos\gamma_1 & \left(\frac{a_1^2}{a_2^2} + 2\right)\cos\gamma_1 + \frac{f_2}{a_2}\cos\gamma_2 & \frac{f_2}{a_2}\sin\gamma_2 & \left(\frac{a_1^2}{a_2} - 2a_2\right)\sin\gamma_1
\end{bmatrix}
\]

(3.68)

Similarly, the Jacobian matrix corresponding to equations (3.44)-(3.47) in Cartesian coordinates is
\[
A = \begin{bmatrix}
- \mu_1 - \frac{4 \mu_2 \omega_1}{3 \pi} \frac{2 p_1^2 + q_1^2}{\sqrt{p_1^2 + q_1^2}} - q_2 & - v_1 - \frac{4 \mu_2 \omega_1}{3 \pi} \frac{p_1 q_1}{\sqrt{p_1^2 + q_1^2}} + p_2 & q_1 - p_1 \\
v_1 - \frac{4 \mu_2 \omega_1}{3 \pi} \frac{p_1 q_1}{\sqrt{p_1^2 + q_1^2}} + p_2 & - \mu_1 - \frac{4 \mu_2 \omega_1}{3 \pi} \frac{p_1^2 + 2 q_1^2}{\sqrt{p_1^2 + q_1^2}} + q_2 & p_1 q_1 \\
- 2 q_1 & - 2 p_1 & - \mu_2 - v_2 \\
2 p_1 & - 2 q_1 & v_2 - \mu_2
\end{bmatrix}
\]

(3.69)

The stability of a fixed point depends on the real parts of the eigenvalues of the Jacobian matrix; that is, the roots of

\[|A - \lambda I| = 0\]

(3.70)

where

\[A = \nabla F(p_0)\]

(3.71)

is the Jacobian matrix evaluated at a given fixed point \(p_0\). A given fixed point is asymptotically stable if and only if all the \(\lambda\)'s lie in the left half of the complex plane and is unstable if at least one eigenvalue lies in the right half of the complex plane. If the real part of a pair of complex-conjugate eigenvalues is positive, the modulation equations possess either periodically or chaotically modulated solutions. Consequently, the response is an amplitude- and
phase-modulated combined pitch and roll motion, with the energy being continuously exchanged between the two modes.

3.4.4 Orbital Solutions

A Hopf bifurcation point is defined as a critical value of a control parameter at which a pair of complex-conjugate eigenvalues of the Jacobian matrix cross the imaginary axis into the right half of the complex plane with nonzero speed. This type of loss of stability fits the Hopf bifurcation theorem, according to which the modulation equations possess limit-cycle solutions near the bifurcation points. These limit cycles are found to undergo a sequence of period-doubling bifurcations, leading to chaos. We note again that fixed points of the modulation equations correspond to periodic oscillations of the pitch and roll modes. Moreover, limit-cycle solutions of the modulation equations correspond to two-period quasiperiodic pitch and roll motions. Furthermore, chaotic solutions of the modulation equations correspond to chaotically modulated motions.

To calculate the complicated motions, we perform computer simulations on the dynamical system governed by the autonomous modulation equations (3.44)-(3.47) rather than on the original equations of motion. We investigate the dynamics of the system in the region near the two Hopf bifurcation points and determine the stable and unstable limit cycles. It is computationally inefficient to find these solutions by conventional methods. Instead, we use

A THEORETICAL INVESTIGATION OF COUPLED PITCH AND ROLL MOTIONS IN THE PRESENCE OF INTERNAL RESONANCE
an algorithm originally proposed by Aprille and Trick [2] to locate the limit
cycles. By this, we eliminate the transient responses, latch onto a limit cycle,
and calculate its period. It uses a combination of a numerical-integration
scheme and a Newton-Raphson iteration procedure. This algorithm proved
efficient in reducing the computation time. It is sensitive to the critical guesses
and the step size of the integration when multiple solutions coexist. Using
different step sizes, we find that the algorithm sometimes converges to
different orbits for the same value of $\sigma_1$. Spectral analysis techniques are also
used to look for cyclical patterns or periodicities in signals. The algorithm
developed by Cooley and Tukey [16] and implemented by Singleton [116] is
used to compute the fast Fourier transform (FFT).

Combining this procedure and the stability analysis described in the next
section, we can investigate local bifurcations of the periodic solutions of the
modulation equations using any parameter, such as the detuning $\sigma_2$ of the
excitation frequency, as a control parameter.

3.4.5 Stability of Periodic Orbits

The stability of the periodic solutions of the modulation equations
(3.44)-(3.47) is determined using Floquet theory. To determine the stability of
a limit cycle

$$\chi(T_1) = \chi(T_1 + T)$$

(3.72)
with period $T$, where $\chi = [p_1, q_1, p_2, q_2]^T$, we superimpose on it a small disturbance $\chi(T_1)$ and obtain the perturbed equation

$$\ddot{\chi} + \dot{\chi} = F[\chi(T_1) + \chi(T_1)]$$

(3.73)

Expanding equation (3.73) in a Taylor series for small $\chi(T_1)$ and linearizing around the periodic orbit, we obtain the linear variational equation

$$\ddot{\chi} = A(T_1)\chi$$

(3.74)

where $A(T_1) = \{\nabla F[\chi(T_1)]\}$ is a $4 \times 4$ variational matrix with $T$-periodic elements. We let $\overline{\chi}(T_1)$ be the fundamental-matrix solution satisfying

$$\ddot{\overline{\chi}} = A(T_1)\overline{\chi}, \quad \overline{\chi}(0) = I$$

(3.75)

The Floquet multipliers are the eigenvalues of the monodromy matrix $\overline{\chi}(T)$. For the autonomous system of equations under consideration, one of the multipliers is always unity. If all the other three multipliers lie inside the unit circle, the orbit is asymptotically stable. On the other hand, if one of the multipliers leaves the unit circle through $+1$, then either a cyclic-fold (tangent) or a pitchfork (symmetry-breaking) bifurcation occurs; and if one of the Floquet multipliers leaves the unit circle through $-1$, then a flip or period-doubling bifurcation occurs. In our study, we found pitchfork and period-doubling bifurcations.
3.4.6 Numerical Results and Discussion

We implemented the computer simulations using an IBM 3090 digital supercomputer. The fixed-point solutions are verified and limit cycles and chaotic solutions are found by numerically integrating the autonomous amplitude- and phase-modulation equations (3.44)-(3.47) using a 5th- and 6th-order Runge-Kutta-Verner algorithm with double precision arithmetics. We examined bifurcations as we vary a bifurcation parameter, such as the detuning parameter $\sigma_2$, the excitation amplitude $f_2$, or the quadratic damping coefficient $\mu_3$ while all the other parameters (i.e., $\sigma_1, \mu_1, \mu_2$) are kept constant.

We point out here that the existence of real solutions for $a_1$ and $a_2$ can be determined by examining equations (3.60) and (3.61). From equation (3.60), we can say that $a_2$ is real if $a_1$ is real, where $a_1$ is determined from equation (3.61). Because equation (3.61) is a fourth-order algebraic equation, an analytic solution for $a_1$ is not readily available. To determine the region where $a_1$ is real, we use analytical continuations starting from $\mu_3 = 0$. Thus, we start by putting $\mu_3$ equal to zero and obtain equations (3.64)-(3.67).

For simplicity in further discussion, we define the following two critical values of $f_2$:

$$\zeta_1 = |\Gamma_2|$$

(3.76)

$$\zeta_2 = (\Gamma_1^2 + \Gamma_2^2)^{1/2} = a_2 \sqrt{\sigma_2^2 + \mu_2^2}$$

(3.77)
It is clear that $\zeta_1 \leq \zeta_2$. It follows from equation (3.65), that $f_2$ must be greater than or equal to $|\Gamma_2|$ (i.e., $f_2 \geq |\Gamma_2| = \zeta_1$) for $a_1$ to be real. Then, there are two possibilities, depending on whether $\Gamma_1$ is positive or negative. When $\Gamma_1 < 0$, there is only one real root of equation (3.65) when $f_2 \geq (\Gamma_1^2 + \Gamma_2^2)^{1/2} = \zeta_2$. When $\Gamma_1 > 0$, equation (3.65) has two real roots when $|\Gamma_2| \leq f_2 \leq (\Gamma_1^2 + \Gamma_2^2)^{1/2}$ or $\zeta_1 \leq f_2 \leq \zeta_2$ and one real root when $f_2 \geq (\Gamma_1^2 + \Gamma_2^2)^{1/2}$ or $f_2 \geq \zeta_2$.

3.4.6.1 Breaking of Saturation in Force-Response Curves

Figures 3.1 show the results for $\mu_3 = 0$ and $\mu_3 \neq 0$ when $\mu_1 = 0.2$, $\mu_2 = 0.5$, $\sigma_2 = 0.3$, $\sigma_1 = 0.1$, and thus $\Gamma_1 = -0.04$. Figure 3.1 (a) shows typical force-response curves of the supercritical-type in which $a_1$ and $a_2$ vary with $f_2$ for $\mu_3 = 0$ and $\mu_3 = 0.6$, a typical value of nonzero $\mu_3$. Since $\Gamma_1 < 0$ in this case, equation (3.65) has one stable real root for $f_2 \geq \zeta_2$ ($\zeta_2 \approx 0.1649$), irrespective of whether $\mu_3 = 0$ or $\mu_3 \neq 0$. When $f_2 \leq \zeta_2$, the responses are given by equation (3.59); their amplitudes are given by (3.58), which are independent of $\mu_3$. Thus, the response is linear in which the roll motion is not excited while the amplitude of the pitch mode is proportional to $f_2$. When $f_2 > \zeta_2$, the responses are given by equations (3.62) and (3.63); their amplitudes are given by (3.64) and (3.65) for $\mu_3 = 0$ and by (3.60) and (3.61) for $\mu_3 \neq 0$. We note that when $\mu_3 \neq 0$ the amplitude $a_2$ of the pitch mode no longer exhibits the saturation phenomenon which exists for $\mu_3 = 0$. Thus, the quadratic damping $\mu_3$ breaks the saturation. Instead, the amplitude $a_2$ of the pitch mode grows
nonlinearly rather than staying at a constant value $a_2^*$ as $f_2$ increases beyond $\zeta_2$. We note, however, that the slope of $a_2$ for $f_2 \geq \zeta_2$ is still much less than that corresponding to the linear case. With the introduction of the quadratic damping $\mu_3$, the rate of increase of the amplitude $a_i$ of the roll mode with $f_2$ is less than that in the case of $\mu_3 = 0$. Consequently, as $f_2$ increases beyond the bifurcation point $\zeta_2$, not all the extra energy input to the pitch mode is spilled over into the roll mode.

Figure 3.1 (b) shows variation of the amplitude $a_1$ of the roll mode with $f_2$ for different values of the quadratic roll damping coefficient $\mu_3$. We note that, for a given $f_2$, $a_1$ decreases as $\mu_3$ increases. While a ship is being operated in the ocean, the surface of the ship hull gets rougher and rougher due to peeling-off of paint or foul bottom. As a result, the magnitude of the quadratic damping coefficient will increase, producing a behavior as discussed here. Figure 3.1 (c) shows force-response curves for the pitch mode corresponding to Figure 3.1 (b). It is clear that the saturation phenomenon of the pitch mode no longer exists when $\mu_3$ is different from zero. Moreover, for a given $f_2$, the amplitude $a_2$ of the pitch mode increases with increasing quadratic roll damping coefficient $\mu_3$.

Figures 3.2 show typical force-response curves when $\Gamma_1 > 0$ ($\Gamma_1 = 0.2484$) when $\mu_1 = \mu_2 = 0.04$ and $\sigma_1 = \sigma_2 = 0.5$ for $\mu_3 = 0$ and $\mu_3 \neq 0$. Figure 3.2 (a) shows typical force-response curves of the subcritical-instability type for $\mu_3 = 0$ and $\mu_3 = 0.6$. Since $\Gamma_1 > 0$ in this case, equation (3.65) has only one stable real root for $f_2 \geq \zeta_2$ ($\zeta_2 \approx 0.2517$) for $\mu_3 = 0$ and $\mu_3 \neq 0$; two real roots for
\( \zeta_1 \leq f_2 \leq \zeta_2 \), where \( \zeta_1 \approx 0.04 \) when \( \mu_3 = 0 \) and \( \zeta_1 \approx 0.1020 \) when \( \mu_3 = 0.6 \). In the latter case, the larger root is stable whereas the smaller one is unstable. We note that the larger bifurcation value \( \zeta_2 \) is independent of the value of \( \mu_3 \), whereas the smaller bifurcation value \( \zeta_1 \) increases as \( \mu_3 \) increases. When \( f_2 \leq \zeta_1 \), there exists only one stable response given by equation (3.59), which is linear and consists of only the pitch mode. When \( \zeta_1 \leq f_2 \leq \zeta_2 \), two stable solutions coexist with an unstable solution; one of the stable responses is given by equation (3.59), and the other stable solution is given by equations (3.62) and (3.63). The response of the ship in this region depends on the initial conditions. When \( f_2 \geq \zeta_2 \), there exists only one stable response given again by equations (3.62) and (3.63). We note that the response of the roll mode exhibits the coexistence of two stable motions and the associated jump phenomenon for both \( \mu_3 = 0 \) and \( \mu_3 = 0.6 \). The saturation phenomenon of the response of the pitch mode, which exists when \( \mu_3 = 0 \), does not exist when \( \mu_3 = 0.6 \). When \( \mu_3 = 0.6 \), the amplitude \( a_2 \) of the pitch mode grows rather than remains constant as \( f_2 \) increases beyond \( \zeta_2 \). We refer to this phenomenon as “breaking of saturation”. We note, here again, that for a given \( f_2 \geq \zeta_1 \), \( a_2 \) increases as \( \mu_3 \) increases. As in Figure 3.1 (a), for a given \( f_2 \), \( a_1 \) decreases as \( \mu_3 \) increases.

Figure 3.2 (b) shows variation of the amplitude \( a_1 \) of the roll mode with \( f_2 \) for different values of the quadratic roll damping coefficient \( \mu_3 \). We note that \( a_1 \) decreases as \( \mu_3 \) increases, as in Figure 3.1 (b). As \( \mu_3 \) increases, the bifurcation value \( \zeta_1 \) increases while \( \zeta_2 \) does not change. For example, \( \zeta_1 = \)
0.0715, 0.1020, and 0.1313 when \( \mu_3 = 0.3, 0.6, \) and 0.9, respectively. Hence, the overhang region between \( \zeta_1 \) and \( \zeta_2 \) decreases as \( \mu_3 \) increases. This indicates that the bifurcation values \( \zeta_1 \) and \( \zeta_2 \) can coalesce as \( \mu_3 \) increases, which results in the qualitative change of the responses from the subcritical-type into the supercritical-type. Then, when this happens, the response of the roll mode would no longer exhibit the coexistence of two stable motions and the subsequent jump phenomenon for large values of the quadratic roll damping coefficient \( \mu_3 \).

Figure 3.2 (c) shows the force-response curves of the pitch mode corresponding to Figure 3.2 (b). We note here again that the saturation phenomenon of the pitch mode no longer exists when \( \mu_3 \) is different from zero. Moreover, the pitch amplitude \( a_2 \) increases as \( \mu_2 \) increases.

We conclude from Figures 3.1 (a)-(c) and 3.2 (a)-(c) that the mechanism of energy transfer from the directly excited pitch mode to the indirectly excited roll mode becomes less effective as the quadratic damping coefficient \( \mu_3 \) of the roll mode increases. When \( \mu_3 \) is zero, all the extra energy into the pitch mode after saturation is transferred to the indirectly excited roll mode. When \( \mu_3 \) is different from zero, only part of the energy input into the pitch mode is transferred to the roll mode.
3.4.6.2 Effect of Quadratic Roll Damping on Frequency-Response Curves

In Figures 3.3, we show frequency-response curves when \( \mu_1 = \mu_2 = 0.02, \sigma_1 = 0.12, \) and \( f_2 = 0.1 \) for \( \mu_3 = 0 \) and \( \mu_3 \neq 0 \). Figure 3.3 (a) shows typical frequency-response curves when \( \mu_3 = 0 \) and \( \mu_3 = 0.6 \). We note that the reversed pitchfork bifurcation points are independent of the value of \( \mu_3 \), whereas the saddle-node bifurcation points move closer to the reversed pitchfork bifurcation points as \( \mu_3 \) increases. We also note that the jump phenomenon in the response of the roll mode exists for both \( \mu_3 = 0 \) and \( \mu_3 = 0.6 \). The interval \( -0.047 \leq \sigma_2 \leq -0.0127 \) between the Hopf bifurcation points when \( \mu_3 = 0 \) disappears or becomes indiscernible when \( \mu_3 = 0.6 \). In this interval, the real part of a pair of complex-conjugate roots of equation (3.70) is positive, and hence the response is an amplitude- and phase-modulated combined pitch and roll motion. The frequency-response curves are shifted slightly leftward with respect to \( \sigma_2 = 0 \) because the detuning \( \sigma_1 \) is positive. They would be shifted to the right if \( \sigma_1 \) is negative, and they would be symmetric with respect to \( \sigma_2 = 0 \) if \( \sigma_1 = 0 \); that is, the case of perfect tuning. The qualitative behavior of the solutions in the three cases is the same.

Figure 3.3 (b) shows variation of the amplitude \( a_1 \) of the roll mode with \( \sigma_2 \) for different values of \( \mu_3 \). We note that, for a given \( \sigma_2, a_1 \) decreases as \( \mu_3 \) increases, except near \( \sigma_2 = 0 \), where \( a_1 \) is almost independent of \( \mu_3 \). Moreover, the reversed pitchfork bifurcation points are independent of \( \mu_3 \) but the saddle-node bifurcation points move closer to \( \sigma_2 = 0 \) as \( \mu_3 \) increases. For
example, the left saddle-node bifurcation points are -0.9367, -0.7091, and -0.6066 whereas the right ones are 0.8729, 0.6442, and 0.5388 for \( \mu_3 = 0.3, 0.6, \) and 0.9, respectively. The reversed pitchfork bifurcation values are -0.5050 and 0.3950 for all values of \( \mu_3. \)

In Figure 3.3 (c), we show the frequency-response curves for the pitch mode corresponding to Figure 3.3 (b). We note that, for a given \( \sigma_2, a_2 \) increases as \( \mu_3 \) increases.

### 3.4.6.3 Hopf Bifurcation Region

Figure 3.4 shows the Hopf bifurcation curves in the parameter space \( \sigma_2 - \sigma_1 \) for different values of the quadratic roll damping parameter \( \mu_3 \) when \( \mu_1 = \mu_2 = 0.02 \) and \( f_2 = 0.1. \) In the outer region, the fixed points of the modulation equations are asymptotically stable and hence correspond to periodic motions. On the transition curves, a complex conjugate pair of eigenvalues of the Jacobian matrix cross the imaginary axis into the right half of the complex plane with nonzero speed. The modulation equations possess limit-cycle solutions near the bifurcation curves. The left ends of the curves are supercritical Hopf bifurcation boundaries while the right ends are subcritical ones. Between the two curves (i.e., in the inner region), oscillatory solutions, which may be either limit cycles or chaotic attractors, are found. The stable limit cycles near the supercritical Hopf bifurcation boundary have relatively small amplitudes when compared to those near the subcritical Hopf
bifurcation boundary. Moreover, limit cycles exist on both sides of the subcritical boundary. Hence, limit-cycle and fixed-point attractors coexist in some region on the right of the subcritical boundary; the initial conditions determine which of these attractors is attained. Figure 3.4 shows that the Hopf bifurcation curves move upward and approach each other as the quadratic roll damping coefficient $\mu_3$ increases. This implies that increasing $\mu_3$ causes the disappearance of aperiodic responses. Figure 3.5 shows typical time traces of amplitude- and phase- modulated pitch and roll motions inside the Hopf bifurcation frequencies.

3.4.6.4 Bifurcations of Orbital Solutions and Complicated Responses

Next, to investigate bifurcations of the periodic solutions of the modulation equations and the existence of chaotic attractors, we use the detuning parameter $\sigma_2$ of the excitation frequency as a control parameter. The parameters used in this investigation are $\mu_1 = \mu_2 = \mu_3 = 0.02$, $\sigma_1 = 0.12$, $f_2 = 0.1$, and $-0.04230 \leq \sigma_2 \leq -0.01747$. In Figure 3.6, we show the bifurcation diagram; it shows the bifurcation points and the solutions that we found in the various detuning intervals inside and around the two Hopf bifurcation points. Below the supercritical value $\sigma_2 = -0.04230$, only stable fixed points were found. This conclusion was verified by first finding attractor I at $\sigma_2 = -0.04$, and then sweeping $\sigma_2$ down toward the supercritical Hopf bifurcation point where attractor I disappeared.
Two-dimensional projections of attractor I in the state space are shown in Figures 3.7. As $\sigma_2$ is increased from the supercritical Hopf bifurcation frequency $\sigma_2 = -0.04230$, attractor I continues to exist but deforms smoothly until $\sigma_2 = -0.024496$, where it loses its stability and undergoes a sequence of period-doubling bifurcations. One of the Floquet multipliers leaves the unit circle through -1 at each bifurcation value. The period-multiplied attractor I evolves smoothly until $\sigma_2 = -0.02433950$, where it suddenly disappears. The bifurcation frequencies are summarized and the calculated Feigenbaum numbers are presented in Table 3.1.

<table>
<thead>
<tr>
<th>period</th>
<th>bif. freq. $\sigma_2$</th>
<th>Feigenbaum #</th>
</tr>
</thead>
<tbody>
<tr>
<td>2T</td>
<td>-0.02449600</td>
<td></td>
</tr>
<tr>
<td>4T</td>
<td>-0.02436250</td>
<td>8.1254</td>
</tr>
<tr>
<td>8T</td>
<td>-0.02434607</td>
<td>5.7852</td>
</tr>
<tr>
<td>16T</td>
<td>-0.02434323</td>
<td>5.2593</td>
</tr>
<tr>
<td>32T</td>
<td>-0.02434269</td>
<td></td>
</tr>
</tbody>
</table>
Table 3.2. Bifurcation Frequencies of Attractor II

<table>
<thead>
<tr>
<th>period</th>
<th>bif. freq. $\sigma_2$</th>
<th>Feigenbaum #</th>
</tr>
</thead>
<tbody>
<tr>
<td>2T</td>
<td>-0.023010</td>
<td></td>
</tr>
<tr>
<td>4T</td>
<td>-0.0228680</td>
<td>4.98246</td>
</tr>
<tr>
<td>8T</td>
<td>-0.0228395</td>
<td>4.75000</td>
</tr>
<tr>
<td>16T</td>
<td>-0.0228335</td>
<td>4.61538</td>
</tr>
<tr>
<td>32T</td>
<td>-0.0228322</td>
<td></td>
</tr>
</tbody>
</table>

Attractor II is born at $\sigma_2 = -0.030782$; it deforms smoothly as $\sigma_2$ is increased. It coexists up to $\sigma_2 = -0.02433950$ with the single-, double-, multiple-, or chaotic attractor I until $\sigma_2 \approx -0.02433950$, where attractor I suddenly disappears. Beyond this value of $\sigma_2$, only attractor II exists, maintains its symmetry, and deforms smoothly as $\sigma_2$ is increased to $\sigma_2 = -0.024095$. As $\sigma_2$ is increased further, the symmetry of attractor II is broken and two asymmetric conjugate attractors are born. These attractors deform further until they lose their stability at $\sigma_2 = -0.023010$ through a period-doubling bifurcation. Thereafter, they undergo a cascade of period-doubling bifurcations and eventually become chaotic. Their period-multiplying bifurcation frequencies are summarized in Table 3.2. The Feigenbaum number is also presented in Table 3.2; it converges to 4.66920. Two-dimensional projections of the phase portrait of attractor II are shown in Figures 3.8.
Figures 3.8 demonstrate the irregular nature of the chaotic attractor. The trajectories, therefore, converge toward some well-defined geometrical structure in the state space. The waveform is never periodic, nor almost periodic. In terms of the frequency content, the Fourier transform of the $q_1$ signal in Figure 3.8 (i) has a broadband character. The Poincaré section in Figure 3.8 (j) shows that the chaotic attractor does not lie on a simple geometrical object. It is known that a chaotic attractor is geometrically invariant and thus repeats its structure on even finer spatial scales. It evolves continuously and smoothly while maintaining its structural characteristics.

Near the subcritical Hopf bifurcation frequency, $\sigma_2 = -0.01747$, the chaotic attractor II exists and extends in two directions around the Hopf point. As $\sigma_2$ is increased above $\sigma_2 = -0.01747$, the chaotic attractor coexists with the stable fixed-point attractor until it disappears at $\sigma_2 = -0.01620$. In this overlapping region, the resultant motion tends to either the fixed-point attractor or attractor II depending on the initial conditions.

Previously, we noted that the modulation equations (3.44) - (3.47) are invariant under the transformation $(p_1, q_1, p_2, q_2) \leftrightarrow (-p_1, -q_1, p_2, q_2)$. This characteristic is shown in Figure 3.9. The second attractor was obtained by simply changing the signs of $(p_i, q_i)$ of the first one. Then it was verified by numerically integrating the modulation equations (3.44) - (3.47).

Figures 3.10 show typical responses of pitch ($\dot{\theta}$) and roll ($\dot{\phi}$) modes in the phase plane of the roll and pitch amplitudes ($a_1$ and $a_2$) and their time traces.
for $\sigma_1 = -0.0290$ which is inside the Hopf bifurcation region when
$\mu_1 = \mu_2 = \mu_3 = 0.02, \sigma_2 = 0.12$, and $f_2 = 0.1$.

3.5 Primary Resonance of the Roll Mode

3.5.1 Modulation Equations

To express the nearness of the resonances, we let, $\Omega = \omega_1 + \varepsilon \sigma_2$ and
$\omega_2 = 2\omega_1 + \varepsilon \sigma_1$. Then,

$$\Omega T_0 = \omega_1 T_0 + \sigma_2 T_1$$  \hspace{1cm} (3.78)

and

$$\omega_2 T_0 = 2\omega_1 T_0 + \sigma_1 T_1$$  \hspace{1cm} (3.79)

Using equations (3.78), (3.79), and (3.22) to eliminate the terms that produce
secular terms from equations (3.18) and (3.19), we obtain

$$2i(A'_1 + \mu_1 A_1) - 4\Lambda_1 A_2 \overline{A_1} e^{i\sigma_1 T_1} - f_1 e^{i\sigma_2 T_1} - \frac{1}{2\pi} \int_0^{2\pi/\omega_1} f e^{-i\omega_1 T_0} dT_0 = 0$$  \hspace{1cm} (3.80)

$$2i(A'_2 + \mu_2 A_2) - 4\Lambda_2 A_1^2 e^{-i\sigma_1 T_1} = 0$$  \hspace{1cm} (3.81)
where $\Lambda_1$ and $\Lambda_2$ are defined in equations (3.26) and (3.27) and

$$\omega_1 f_1 = \frac{1}{2} F_1 \quad (3.82)$$

Substituting equation (3.17) into equations (3.80) and (3.81), using equation (3.29), and separating real and imaginary parts, we obtain

$$a_1' = -\mu_1 a_1 + \Lambda_1 a_1 a_2 \sin \gamma_1 + f_1 \sin \gamma_2 - \frac{1}{2\pi \omega_1} \int_0^{2\pi} \sin \chi_1 f [ - \omega_1 a_1 \sin \chi_1 ] d\chi_1 \quad (3.83)$$

$$a_2' = -\mu_2 a_2 - \Lambda_2 a_1^2 \sin \gamma_1 \quad (3.84)$$

$$a_1 \beta_1' = -\Lambda_1 a_1 a_2 \cos \gamma_1 - f_1 \cos \gamma_2 - \frac{1}{2\pi \omega_1} \int_0^{2\pi} \cos \chi_1 f [ - \omega_1 a_1 \sin \chi_1 ] d\chi_1 \quad (3.85)$$

$$a_2 \beta_2' = -\Lambda_2 a_1^2 \cos \gamma_1 \quad (3.86)$$

where

$$\gamma_1 = \sigma_1 T_1 + \beta_2 - 2\beta_1 \quad \text{and} \quad \gamma_2 = \sigma_2 T_1 - \beta_1 \quad (3.87)$$

The values of the integrals in equations (3.83) and (3.85) are given in equations (3.35) and (3.36).
As in the preceding section, equations (3.83)-(3.87) can be reduced to a generic system of equations. To accomplish this, we apply the following transformation of variables:

\[ a_1 = \hat{a}_1/(\Lambda_1\Lambda_2)^{1/4}, \quad a_2 = \hat{a}_2(\Lambda_2/\Lambda_1)^{1/4}, \quad T_1 = \hat{T}_1/(\Lambda_1\Lambda_2)^{1/4} \] (3.88)

Substituting equation (3.88) into equations (3.83)-(3.87), using equations (3.35)-(3.36), and then deleting the hat from the equations, we obtain

\[ a_1' = -\mu_1 a_1 + a_1 a_2 \sin \gamma_1 + f_1 \sin \gamma_2 - \frac{4\mu_3 \omega_1}{3\pi} a_1 | a_1 | \] (3.89)

\[ a_2' = -\mu_2 a_2 - a_1^2 \sin \gamma_1 \] (3.90)

\[ a_1\beta_1' = -a_1 a_2 \cos \gamma_1 - f_1 \cos \gamma_2 \] (3.91)

\[ a_2\beta_2' = -a_1^2 \cos \gamma_1 \] (3.92)

where

\[ \gamma_1 = \sigma_1 T_1 + \beta_2 - 2\beta_1 \quad \text{and} \quad \gamma_2 = \sigma_2 T_1 - \beta_1 \] (3.93)
3.5.2 Fixed Points

Periodic solutions of (3.4) and (3.5) correspond to the fixed points of equations (3.89)-(3.93). They are obtained by setting \( a'_1 = a'_2 = 0 \) and \( \gamma'_1 = \gamma'_2 = 0 \). It follows from equation (3.93) that

\[
\beta'_1 = \sigma_2 \quad \text{and} \quad \beta'_2 = 2\sigma_2 - \sigma_1 \quad (3.94)
\]

Hence, the fixed points of equations (3.89)-(3.93) are given by the solutions of the following set of coupled nonlinear algebraic equations:

\[
\mu_1a_1 = a_1a_2 \sin \gamma_1 + f_1 \sin \gamma_2 - \frac{4\mu_3\omega_1}{3\pi} a_1 |a_1| \quad (3.95)
\]

\[
\mu_2a_2 = -a_1^2 \sin \gamma_1 \quad (3.96)
\]

\[
a_1\sigma_2 = -a_1a_2 \cos \gamma_1 - f_1 \cos \gamma_2 \quad (3.97)
\]

\[
(2\sigma_2 - \sigma_1)a_2 = -a_1^2 \cos \gamma_1 \quad (3.98)
\]

Equations (3.95)-(3.98) can be manipulated to yield the following polynomial equation for \( a_1 \):

\[
a_1^6 + d_5a_1^4 |a_1| + d_4a_1^4 + d_3a_1^2 |a_1| + d_2a_1^2 + d_1 |a_1| + d_0 = 0 \quad (3.99)
\]

where

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\[ d_0 = -\Gamma_2^{-2}, \quad d_1 = 0, \quad d_2 = \Gamma_2^{-2}, \quad d_3 = 2\Gamma_2^{-2} \frac{4\omega_1}{3\pi} \mu_3 \mu_1 \]

\[ d_4 = 2\mu_1 \mu_2 - 2\nu_1 \nu_2 + \Gamma_2^{-2} \left( \frac{4\omega_1}{3\pi} \mu_3 \right)^2, \quad d_5 = 2\Gamma_2^{-2} \frac{4\omega_1}{3\pi} \mu_3 \nu_2 \]

\[ \Gamma_n^{-2} = \mu_n^2 + \nu_n^2, \quad \nu_1 = \sigma_2, \quad \text{and} \quad \nu_2 = 2\sigma_2 - \sigma_1. \]

We note that if \( \mu_3 = 0 \), then \( d_3 = d_5 = 0 \) and \( d_4 = 2\mu_1 \mu_2 - 2\nu_1 \nu_2 \). It follows from equations (3.95)-(3.98) that \( a_2 \) is given by

\[ a_2 = \frac{a_1^2}{\Gamma_2} \quad (3.100) \]

The solutions of the sixth-order algebraic equation (3.99) are obtained numerically. Then, \( a_2 \) is calculated from equation (3.100). Finally, the corresponding phases \( \gamma_1 \) and \( \gamma_2 \) (\( \beta_1 \) and \( \beta_2 \) also) are obtained from equations (3.95)-(3.98).

We note that if \( \mu_2 = 0 \), then equation (3.99) reduces to the one obtained by Nayfeh [75].

### 3.5.3 Stability of Fixed Points

The stability of the fixed-point solutions are determined by the same procedure used in Section 3.3; that is, by investigating the eigenvalues of the following Jacobian matrix of equations (3.89)-(3.93):
\[ A = \begin{bmatrix}
-\mu_1 + a_2 \sin \gamma_1 - 2 \frac{4\omega_1}{3\pi} \mu_3 |a_1| & a_1 \sin \gamma_1 & f_1 \cos \gamma_2 & a_1 a_2 \cos \gamma_1 \\
-2a_1 \sin \gamma_1 & -\mu_2 & 0 & -a_1^2 \cos \gamma_1 \\
-\frac{f_1}{a_2^2} \cos \gamma_2 & \cos \gamma_1 & -\frac{f_1}{a_1} \sin \gamma_2 & -a_2 \sin \gamma_1 \\
-2 \frac{f_1}{a_1^2} \cos \gamma_2 - 2 \frac{a_1}{a_2} \cos \gamma_1 & \left(2 + \frac{a_1^2}{a_2^2}\right) \cos \gamma_1 & -2 \frac{f_1}{a_1} \sin \gamma_2 - \left(2a_2 - \frac{a_1^2}{a_2}\right) \sin \gamma_1 & \end{bmatrix} \]

Again, a given fixed point is asymptotically stable if and only if all the \( \lambda \)'s lie in the left half of the complex plane and is unstable if at least one eigenvalue lies in the right half of the complex plane. If a pair of complex-conjugate eigenvalues crosses the imaginary axis with nonzero speed, then we have a Hopf bifurcation. Near these bifurcation points, the response is an amplitude- and phase-modulated combined pitch and roll motion, with the energy being continuously exchanged between the two modes.

3.5.4 Numerical Results and Discussion

In Figures 3.11, we show the frequency-response curves when \( \mu_1 = \mu_2 = 0.08, \sigma_1 = 0, \) and \( f_1 = 0.08 \) for different values of \( \mu_3 \). The unstable
solutions are represented by broken lines and the stable solutions are marked by solid lines. Figures 3.11 (a) and (b) show variations of the roll amplitude $a_1$ and pitch amplitude $a_2$ with $\sigma_2$ for different values of $\mu_3$. As $\mu_3$ increases from zero to 0.3, the reversed pitchfork bifurcation points slightly change, whereas the saddle-node bifurcation points move towards $\sigma_2 = 0$. As $\mu_3$ is increased further to 0.6 and 0.9, the saddle-node bifurcation points disappear and the frequency-response curves become single-valued. Consequently, the jump phenomenon and subcritical instability disappear. Moreover, the Hopf bifurcation points approach each other as $\mu_3$ increases. For example, the Hopf bifurcation interval $-0.09995 \leq \sigma_2 \leq 0.09995$ for $\mu_3 = 0$ shrinks gradually to $-0.055 \leq \sigma_2 \leq 0.055$ for $\mu_3 = 0.9$.

We note that for a fixed value of $\sigma_2$, $a_1$ and $a_2$ decrease as $\mu_3$ increases. The curves of unstable fixed points around the region of perfect tuning, $\sigma_2 \approx 0$, converge in both modes. The introduction of quadratic roll damping $\mu_3$, by say attaching antirolling devices like bilge keels, causes the region between the two Hopf bifurcation frequencies (close to perfect resonance) to shrink. However, it does not eliminate complicated motions completely in this region.

Figures 3.12 show the frequency-response curves for the case in which the values of all the parameters are the same as those in Figures 3.11 except that $\sigma_1 = 0.2$. In this case, the curves are shifted slightly to the right and the peak amplitudes of the right branches of the roll mode are smaller than those of the left branches. The opposite occurs in the response of the pitch mode. If $\sigma_1$ is chosen to be negative, the frequency-response curves would be shifted to the
left. When $\sigma_1 = 0$, the case of perfect tuning, the curves would be symmetric with respect to $\sigma_2 = 0$. The qualitative behavior of the solutions in the three cases is the same.
Figure 3.1. (a) Force-Response Curves (Supercritical Type): for $\mu_3 = 0$ and 0.6; stable (---), unstable (-----)
Figure 3.1. (b)&(c) Force-Response Curves (Supercritical Type): variation of (b) roll amplitude $a_1$ and (c) pitch amplitude $a_2$ for different values of $\mu_3$; stable (---), unstable (----)
Figure 3.2. (a) Force-Response Curves (Subcritical Type): for $\mu_3 = 0$ and 0.6; stable (---), unstable (---)
Figure 3.2. (b)&(c) Force-Response Curves (Subcritical Type): variation of (b) roll amplitude $a_1$ and (c) pitch amplitude $a_2$ for different values of $\mu_3$; stable (---), unstable (----)
Figure 3.3. (a) Frequency-Response Curves: for $\mu_3 = 0$ and 0.6; stable (---), unstable (---)
Figure 3.3. (b)&(c) Frequency-Response Curves: variation of (b) roll amplitude $a_1$ and (c) pitch amplitude $a_2$, for different values of $\mu_3$; stable (-----), unstable (-----)

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Figure 3.4. Hopf Bifurcation Curves: for different values of $\mu_3$
Figure 3.5. Typical Time Traces of an Amplitude- and Phase- Modulated Pitch and Roll Motion
Figure 3.6. Bifurcation Diagram Inside the Hopf Bifurcation Frequencies
Figure 3.7. (a) Phase Portraits of Attractor I (1T-periodic): at $\sigma_2 = -0.040$

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Figure 3.7. (b) Phase portraits of attractor I (2T-periodic): at $\sigma_2 = -0.024363$
Figure 3.7. (c) Phase Portraits of Attractor I (4T-periodic): at $\sigma_2 = -0.02434610$
Figure 3.7. (d) Phase Portraits of Attractor 1 (chaotic): at 
$\sigma_2 = -0.0243420$
Figure 3.8. (a) Phase Portraits of Attractor II (symmetric): at $\sigma_2 = -0.02436250$
Figure 3.8. (b) Phase Portraits of Attractor II (asymmetric): at $\sigma_2 = -0.0240$
Figure 3.8. (c) Phase Portraits of Attractor II (2T-periodic): at $\sigma_2 = -0.023010$
Figure 3.8. (d)-(f) Phase Portraits of Attractor II (4T-periodic): at 
\[ \sigma_2 = -0.0228394 \]
Figure 3.8. (g)-(l) Phase Portraits of Attractor II (chaotic): at 
$\sigma_2 = -0.0190$
Figure 3.8. (i) Poincaré Sections of Attractor II (chaotic): at section \( q_2 = 0 \) for \( \sigma_2 = -0.0190 \)
Figure 3.9. (a)&(b) Reflected Phase Portraits of Attractor I: at 
$\sigma_2 = -0.0350$

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Figure 3.10. (a)&(b) Modulated Motion of Attractor II: (a) time traces (b) limit cycle at $\sigma_2 = -0.0290$
Figure 3.11. (a)&(b) Frequency-Response Curves ($\sigma_1 = 0$): variation of (a) roll amplitude $a_1$ and (b) pitch amplitude $a_2$ for different values of $\mu_3$; stable (---), unstable (----)
Figure 3.12. (a) & (b) Frequency-Response Curves ($\sigma_1 = 0.2$): variation of (a) roll amplitude $a_1$ and (b) pitch amplitude $a_2$, for different values of $\mu_3$; stable (---), unstable (----)

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Chapter IV

DESCRIPTION OF EXPERIMENTAL SETUP

In 1934, Albert Einstein said, “Pure logical thinking cannot yield us any knowledge of the empirical world; all knowledge of reality starts from experiment and ends with it.”

In 1500, Leonardo da Vinci said, “Remember, when dealing with water, to introduce first experience then reason.”

To verify the theoretical results predicted in Chapters II and III, we conducted a series of experiments in the ship dynamics laboratory of VPI & SU. While the theories developed in the previous chapters are applicable to many other mechanical and structural systems, the present experiments employed models of vessels to address the concern on the safety of human beings in practical senses. Two very different types of vessels were selected. The first model was of a tanker: its length-to-beam ratio is 7.65 and the shape of its midship section is nearly rectangular. The second one was of a
destroyer: its length-to-beam ratio is 10.78 and its sectional shape is round or nearly circular. More details describing the models are given in the following chapters.

Since its construction, the towing basin of VPI & SU has been used mainly for educational purposes. With the results contained in the present dissertation, the towing basin comes to fruition with its designed task, hard core research. Since the experiment was unprecedented in this facility, it was necessary to develop and establish procedures for the preparation and implementation of the experiments. While the idea and goal of the theoretical formulation were simple, experimental verification was difficult. It revealed lots of hidden barriers. It needed seemingly endless trials and errors to achieve the desired configuration. It is a general understanding that undertaking any experimental work requires much preaccumulated experience. It will be even more seriously true when such huge facilities as the towing tank and ship models are involved. The principal difficulties arose from the fact that absolutely no preaccumulated experiences were available in the ship dynamics laboratory of VPI & SU.

4.1 Experimental Facilities at VPI & SU

Photographs of the towing basin of VPI & SU are shown in Figure 4.1. The towing basin is approximately 30m x 2.1m x 1.6m. A plunger type wavemaker
made of a flat steel plate is installed at one end. A set of wave absorbers is placed behind the wavemaker and another set is placed at the other end of the basin. The slope angle of the latter one is adjusted to 10°, which was judged to be the most optimal to efficiently absorb waves having frequencies of concern to us. There is a towing carriage on the rails of the basin, which was used as a stationary mounting platform for the models in the present experiments. The models were supported in such a way that heave, pitch, and roll were allowed while surge, sway, and yaw were constrained.

4.2 Experimental Instruments and Apparatus

We designed and built a motion guidance system to constrain the model to three degrees of freedom. Its schematic is shown in Figure 4.2. Pitch and roll motions are possible up to ±22° each. The linear motion apparatus for heave is composed of a pair of 0.5" diameter hardened-and-ground steel rods and four supporting linear bearings mounted on a steel plate. The linear motion device is next linked to the rotational motion device at its bottom. The allowable magnitude of heave is ±6.0°. The gyro-like rotational motion apparatus for pitch and roll is made of an aluminum material. It is composed of two identical blocks using a pin-linkage system. It sits on the top of shim plates lying on the floor of the model. A variety of thicknesses of shim plates made of plexi-glass are ready to use to match the desired height of the center
of gravity. As a result, a combination of the linear and rotational motions, the guidance system allows a total of three degrees of freedom.

The angular displacements in pitch and roll are measured by two Schaevitz R30D RVDTs, which are fixed to the shafts of the rotational motion apparatus. A photograph of the measurement instruments is shown in Figure 4.3. A schematic of the whole experimental setup is shown in Figure 4.4. The heave displacement is measured by a Schaevitz 5000HPD LVDT, which is mounted on the supporting plate of the heave staffs and in the middle of the two heave rods. The wave heights are measured by capacitance-type Davis WL/ WP 03 wave-level gauges. One of them is mounted ahead of the bow of the model and is aligned with the centerlines of both the model and the towing tank. Its location is moved around along the towing tank to measure the highest wave heights nearest to the bow of the model. The locations and amplitudes of the wave peaks depend on the wave frequencies. Another wave-level gauge is placed on the side of the midship section of the model to monitor any possible reflected waves from the sidewalls of the towing tank. To control and monitor the movement of the wavemaker, a 24 inch-long Temposonics DCTM-36 LVDT is mounted on top of the wavemaker along the direction of the driving piston. Its signal is fed back to the control box and into a Tektronix 2230 oscilloscope along with the signal from the Wavetek 275 function generator. The signal from the wavemaker contains high-frequency noise, which is approximately 180° out of phase with the signal from the function generator. The wavemaker is driven by an electric-hydraulic motor;
a feed-back control system is used to produce a constant movement amplitude.

A Fluke 8600A digital multimeter, an HP 3468A digital multimeter, and a Graphtec SR6335 strip-chart recorder are used to monitor the responses of the model. A Data Translation DT 2801 A/D converter is used for digitization, and data acquisition is done by a commercial software package called Labtech Notebook. They are installed on an IBM AT personal computer. Most of the analyses of the data is done by two microprocessors: an HP 3562A dynamic signal analyzer and a Rockland 5820B spectrum analyzer.

The essential sensors, such as the RVDTs, LVDTs, two wave-level gauges, and all the peripheral electronic instruments, were calibrated before being installed to check the linearities and to obtain scale factors for the physical quantities.

4.3 Basic Experiments

As a basic experiment, wave tests were conducted without a model to map out the region of the frequencies and amplitudes for which the wavemaker could produce smooth, regular, plane waves in the towing basin. The wave characteristic map is shown in Figure 4.5. From these tests, it was observed that operating the wavemaker at approximately 0.60 Hz produces waves of good quality for the widest range of amplitudes. The other frequencies, the
higher ones in particular, yielded rough waves with much smaller amplitudes by comparison. At the higher frequencies, longitudinal plane waves were contaminated by transverse cross waves very quickly.

Because the theoretical results in Chapter II indicate that large-amplitude waves are required to excite the roll mode indirectly through excitations of the heave and pitch modes in the absence of internal resonances, we selected excitation waves of frequencies around 0.60 Hz for these experiments. Because the roll mode is to be excited by a principal parametric resonance in which the excitation wave frequency is twice the natural frequency in roll, the natural frequency of roll of the model was adjusted to 0.31 Hz so that is approximately one-half the frequency of the excitation waves (0.60 Hz).

The second experiment was designed to ascertain the influence of a two-to-one internal resonance between either pitch and roll or heave and roll on the indirect excitation of the roll mode by either the pitch or heave mode. This demands that either the pitch or heave frequency of the model be adjusted to be twice the roll frequency and to be approximately about 0.6 Hz to be able to produce waves with large amplitudes. Unfortunately, the lowest pitch natural frequency achievable with the model was approximately 0.91 Hz and hence the roll natural frequency was adjusted to be approximately 0.46 Hz. Thus, the pitch mode was excited by waves having frequencies between 0.85 Hz and 0.95 Hz. These wave frequencies are much higher than, but the closest to, 0.60 Hz in the present experimental setup. However, the theoretical results of Chapter III indicate that the roll mode can be indirectly excited by
either the pitch or heave mode in the presence of a two-to-one internal resonance with relatively low-amplitude waves, and hence the low-amplitude waves generated at these frequencies are still plane waves.

4.4 Experimental Problems

4.4.1 Roll Center

To measure the rolling motion of a floating body, one needs to understand the notion of roll center. Unfortunately, this has been a vague and ill-defined concept [108]. A common notion regarding the roll center is that it is the elevation (the point along the axis of symmetry) of a body where sway motion is zero. However, there is no elevation where sway motion is zero. Thus the roll center is defined as the elevation that minimizes sway motions, and is in many cases well below the keel even for the largest vessels; traditionally it has been located near the waterline or center of gravity.

Another concept is the dynamic roll center which is, as a matter of convention and convenience, usually chosen to be the physical center of gravity or the vertically projected image of the center of gravity on the water plane [53, 104]. The dynamic roll center is defined as the principal axis for roll inertia that includes the effects of hydrodynamic added mass. This center always lies between the center of gravity and the center of buoyancy, and must
be used as the origin to determine the physical roll inertia, hydrodynamic coefficients and wave forcing function if a single-degree-of-freedom roll equation is to be written [53, 104]. It is noted that the elevation of the dynamic roll center itself is frequency dependent because of the frequency dependence of the hydrodynamic coefficients. Thus, in general, the dynamic roll center does not possess a constant elevation across a range of frequencies.

Among the definitions of a roll center offered above, the first one is no longer used in the analyses and remains for purely theoretical considerations. The second one is to be used in the future to study the responses of floating bodies both experimentally and analytically.

Since the center of roll motion is hard to determine and is not even fixed at a certain point of the floating body, it is not realistic to take the roll center as the reference point in the experiments using captive-type models. Instead, for practical reasons, the center of gravity is chosen after trade-offs to be the place at which the motion guidance system is affixed to allow the designated modes of motion of the model. The transducers are also installed at the center of gravity as a reference point to measure the magnitudes of motion. This is also a recommendation made by the International Towing Tank Conference (ITTC) [52].
4.4.2 Other Difficulties

One of the difficulties encountered was the time needed to carry out the experiments. In comparison with the time needed to conduct a usual vibration experiment, the present work needed 1,000 times more to acquire the same amount of data. The process of sweeping up and sweeping down the amplitudes of waves took long days and hours because the steady state for each step of response and excitation was required.

Another difficulty was that it was very hard to fix a required configuration involving both the hydrostatics or geometry and dynamic characteristics of the model because of the many parameters involved. They include the natural frequencies, trim, heel, static stability, designed waterline, weight, space for the location of the motion guidance system, and the location of the center of gravity. They need to be adjusted simultaneously and not one after the other. This required many careful iterations of a tedious and hard-to-handle process until the most optimal condition in every aspect is met. Moreover, because the natural frequencies of concern were very low (about half a Hertz), the time required to check them was lengthy. Thus it was in practice very difficult to obtain the desired configuration because of the time necessary and the weight and size of the model.

The third main difficulty is that it was almost impossible to maintain the same test condition for the whole duration of a given series of runs. This is because of the deterioration of the water quality and the continuous peeling
of the painted surface of the model. To maintain the same damping condition as possible, we painted the surface of the hull before each series of runs using epoxy-based enamel paint.

During the first stages of the research the goal was simply to verify the case in which the model would be allowed to move in two degrees of freedom; that is, roll and pitch. However, it was later decided, through a sequence of preliminary experiments, to include the heave mode, which is the most prominent motion of a floating body. Furthermore, for large wave amplitudes, even if the waves were longitudinal, it is irrational to ignore the side motions, such as sway, yaw, and possibly broaching, which will concurrently occur with roll motion in reality as pointed out by Eda, et al. [25] and Taggart and Kobayashi [122]. However, in the present experiments, these forces and moments were ignored.

4.5 Experimental Procedure

In the theory, the excitation amplitude and frequency are used as control parameters to investigate the behavior of the system. Either the excitation amplitude is varied slowly as the frequency is kept constant or the excitation frequency is varied slowly as the excitation amplitude is kept constant. In either case, the excitation and response of concern are the steady-state ones.
and in experiments it is required to wait for a while until a steady state is reached.

In usual nonlinear vibration experiments, the steady-state response can be achieved at most in several seconds for both cases and hence the excitation frequency as well as amplitude of the shaker can be slowly varied. This is possible because there is no reflection from the surroundings and there is no interaction between the components of the previous step and those at the next step while the control parameter is slowly varied.

In contrast, when we use water waves in the towing basin as external excitations, steady-state wave conditions in the basin require much longer time, such as several hours. It is due to the following reasons: first, in a fluid with viscosity like water, transient states need long times to disappear; second, since the water is contained in a space confined by the surrounding walls, the reflected waves from the four surrounding walls interact with the newly generated components. The second effect is even strong in the beginning stage of the parameter change. Consequently, it was impossible to vary the excitation frequency while keeping the wave amplitude constant; otherwise, it would have taken a very very long time to reach the steady state and the experimental condition would have become random and composed of a variety of frequencies and amplitudes. Accordingly, we restricted the experiment to the case of slowly varying the wave amplitude while keeping the frequency constant, thereby obtained only force-response curves.
The present experiments took a long time to conduct; the duration for just one case of tests was as long as 10 days in the most accurately tested case.

The primary findings of the present work are the subcritical instability and associated jump phenomena. Thus the experiments were focused on capturing the moments of critical wave amplitudes at which large-amplitude roll motions occurred and disappeared. To obtain the force-response curves, we swept up and down the wave amplitude, within the limit in which regular plane waves can be generated in the basin.

The scenario of the experiment was as follows:

1. We placed the model with zero initial heel angle in initially calm water along the centerline of the towing basin.

2. We set the smallest wave amplitude at a fixed frequency on the function generator.

3. We turned on the data acquisition instruments and function generator. (They might have already been on for warming up.)

4. We turned on the wavemaker (waves with the smallest amplitude started to travel toward the model).

5. We monitored the pitch, roll, and heave responses of the model using the spectrum analyzers, oscilloscopes, and strip chart recorder which were continuously drawing the incoming signals.

6. We waited until steady state was reached (it might have taken half an hour to several hours, depending on the excitation amplitude and frequency).

7. We judged whether the transient state was completely disappeared.
8. We recorded the data both from the spectrum analyzers and the personal computer.

9. We increased the wave amplitude slowly by changing the signal voltage on the function generator while monitoring all signals. This step took from 15 minutes to 2 hours, depending on the excitation amplitude and frequency. This step had to be implemented very carefully and slowly to minimize uncertainties and disturbances. This procedure was delicate because the response of the model depended on the initial conditions and sometimes it was not easy to keep similar conditions for the next step of wave amplitude when the domain of attraction of the response is relatively small.

10. We repeated the entire procedure from step 5 to step 9 for other values of the wave amplitude.

11. We judged whether the maximum allowable wave amplitude is reached so that no damage to the motion guidance system occurs.

12. We decreased the wave amplitude slowly in a manner similar to step 9 after arriving at the maximum available wave amplitude.

13. We repeated the entire procedure from step 5 through step 8, and step 12.

14. We turned off the wavemaker after reaching the smallest possible wave amplitude.

15. We repeated the entire procedure from step 2 through step 14 for other wave frequencies.
Steps 2 through 14 were used in typical sweep-up and sweep-down processes. The jump phenomenon, the subcritical instability, and the coexistence of multiple dynamic equilibrium solutions of the roll mode were observed. These results are in good qualitative agreement with the theoretical results predicted in Chapters II and III. If a large-amplitude roll motion did not occur spontaneously during the sweep-up/sweep-down process, various external disturbances were imposed on the model at each step of wave amplitude to obtain the hidden, coexisting large-amplitude roll motion.

Video-tape recordings and/or photographs were made of several tests.

4.6 Discrepancies between the Results of Theory and Experiments

In general, we found good qualitative agreement between the theoretical and experimental results. However, some discrepancies did exist.

We noted from the force-response curves that as the wave amplitude was increased, the magnitude of the large-amplitude roll motion remained constant, decreased, and even died out. Specifically, if the wave amplitude was increased continuously beyond the bifurcation value, the amplitude of the roll motion remained almost constant or even decreased instead of increasing monotonically as predicted by the theoretical approach. This was also observed by other investigators (e.g., see Dick, et al. [21]). In the free-model
scale experiments performed in the sea, they showed that when the model was at zero speed in head waves, the roll amplitude was almost constant at higher wave heights while the pitch amplitude increased linearly as the wave height increased.

The discrepancies may be due to one or more of the following three reasons. First, the hydrostatic characteristics of a model in waves can differ markedly from its characteristics in calm water. So, as the amplitudes of the heave and pitch increase, the draft or the immersed displacement of the hull at the maximum roll amplitude increases. This causes an increase in the restoring roll moment. As a result, the amplitude of the roll motion is blocked from increasing and even decreases at large wave amplitudes. Second, while the roll motion causes relatively small waves that reflect from the sidewalls of the basin, the heave and pitch motions generate relatively large waves that do reflect from the sidewalls and form transverse standing waves on both sides of the model. The crests of these waves are a little aft of the midship section of the model, and they always meet the model just before the maximum roll angle occurs. Consequently, the reflected waves also act to limit the roll motion. Third, because the model can pitch as well as heave, the difference in the phases of these two modes might cause the effective amplitude of the parametric excitation to decrease. It is a combination of the heave and pitch motions that produces the effective parametric excitation in the roll equation.

We note that, in the present work, both the analysis and experiment do not include the influence of the sway and yaw modes and their couplings with the
other modes. It is, however, worthwhile to consider the work of Eda, et al. [25] and Taggart and Kobayashi [122]. They showed that a significant coupling of the roll and yaw can develop due to the asymmetry of the underwater hull form of the heeled vessel. This is explained in the following way: the asymmetric form acts as a cambered low-aspect-ratio lifting body, which, together with the forward speed, produces sway forces and roll and yaw moments. When a vessel has relatively small values of metacentric height (GM), this can lead to dramatic increases in roll and yaw motions when the vessel is operating in waves. Thus, had we also considered the yaw and sway motions in the analysis, the roll amplitude would have still increased as the excitation amplitude increased. However, in an experiment in which yaw and sway motions are restricted, the reaction forces exerted on the vessel from the sides of the tank could decrease the roll motion.

The coupling between the pitch and yaw motions is another nonlinear effect which may need to be considered to understand the experimental results. It occurs when there is a drift angle between the vessel heading and the instantaneous forward speed vector.

The supercritical (or pitchfork) type of instability was not found. It is conjectured that either to use higher amplitude of waves or larger magnitude of damping or a different fluid medium other than water having larger density can yield the supercritical type of instability. These conditions are, however, considered hard to be met in reality.
Figure 4.1. Photographs of the Towing Basin at VPI & SU
Figure 4.2. Schematic of the Motion Guidance System
Figure 4.3. Photograph of the Measurement Instruments
Figure 4.4. Schematic of the Experimental Setup
Chapter V

AN EXPERIMENTAL INVESTIGATION OF COUPLED HEAVE, PITCH, AND ROLL MOTIONS

5.1 Overview of the Experiment

To verify the theoretical predictions of Chapter II, we performed a series of experiments in the ship dynamics laboratory at VPI & SU. The loss of dynamic stability and the resulting large-amplitude roll motion of a vessel in a head or following wave was studied experimentally using a model with three degrees of freedom; that is, roll, pitch, and heave. The case of principal parametric resonance of the roll mode was studied, where the natural frequency in roll is about one-half the frequency of the waves.
As described in Chapter IV, we conducted basic wave tests without a model to determine the region of frequencies and amplitudes of the wavemaker that produce plane waves in the towing basin. These tests revealed that operating the wavemaker around a frequency of 0.60 Hz produces plane waves for the widest range of amplitudes. Thus it was decided to use excitation waves with frequencies around 0.60 Hz.

With the model in place, the wavemaker was started at the lowest amplitude available. Then the amplitude was increased very slightly to the next step on the function generator while the behavior of the model in the waves and various signals were continuously monitored. A period ranging from half an hour to four hours, depending on the amplitude and frequency of the waves being generated, was required to achieve a steady state. After reaching the maximum amplitude of the waves available in the present experimental setup, the wave amplitude was slowly decreased in the same manner.

During this process, jump-up and jump-down phenomena were observed, and the range of wave amplitudes where roll motions exist was obtained. When jump up did not occur spontaneously, external disturbances of various kinds were imposed on the model at each different step of wave amplitude. These disturbances produced large-amplitude stable roll motions in many cases, an indication of the coexistence of multiple dynamic equilibria and a subcritical instability. The present experimental results showed good qualitative agreement with the theoretical predictions presented in Chapter II.
The experiments also revealed that the large-amplitude roll motion is dependent on the location of the model on the waves. Video-tape recordings and/or photographs were made during some of the tests.

Experiments in which the wave frequency was varied while the amplitude was fixed were not implemented because of the reasons described in Chapter IV. The supercritical (or pitchfork) type of instability was not obtained in spite of the extensive efforts devoted to this case.

5.2 Description of Model

There were 10 models ready to use at VPI & SU. Among them, only one turned out to be usable for the case of parametrically excited roll motion based on extensive tests of the natural frequency of each mode and checks of the dimensions and drafts of the models. The dimensions of the chosen wooden model of a tanker were approximately LxBxD = 223.5cm x 29.2cm x 19.1cm. Details of the principal particulars are presented in Table 5.1. Photographs of the model fully equipped with the apparatus and motion measuring transducers are shown in Figures 5.1 and 5.2. The weight of the model alone was approximately 30.5kg and the weight of the model and ballast was approximately 54.5kg. Because our interest was focused on the case of principal parametric resonance, we distributed weights inside the model so
that the natural frequency in roll (0.32 Hz approximately) was about one-half
the frequency of the waves (0.6 Hz approximately).

There is a towing carriage on the rails over the basin, which was used as
a stationary mounting platform for the model in the present experiment. The
model was mounted underneath the towing carriage and supported in such a
way that heave, pitch, and roll were allowed while surge, sway, and yaw were
constrained. Pitch and roll motions were possible up to $\pm 22^\circ$ each and heave
motion was possible up to $\pm 6.0$ inch.

More details of the experimental setup and procedure are described in
Chapter IV.

5.3 General Results

Figures 5.3 and 5.4 are photographs showing two views of the model
rolling in longitudinal regular plane waves. In Figure 5.4, the view is from the
absorber end of the towing basin. The rolling motion (model heeled to the
starboard) in longitudinal regular plane waves is clearly evident.

In the following figures, force-response curves are shown. The capital
letters are used to denote the sweep-up process of wave amplitude while the
lower-case letters are reserved for the sweep-down process. This practice is
followed all the way from Figure 5.5 through Figure 5.10 except Figure 5.6.
As the wave amplitude was slowly increased, rolling did not occur until the wave height reached a certain critical value ($\zeta_2$). Then, a large-amplitude roll motion occurred suddenly (the jump-up phenomenon). When we increased the wave amplitude continuously beyond the jump-up bifurcation value ($\zeta_2$), the roll amplitude remained almost constant or even decreased instead of increasing monotonically as predicted by the theory.

After reaching the maximum wave amplitude possible in our experimental setup, we slowly reduced the wave amplitude. The large roll amplitude continued at wave heights below the jump-up bifurcation value ($\zeta_2$). When the wave amplitude was decreased further to another critical value ($\zeta_1; \zeta_1 < \zeta_2$), the large roll amplitude suddenly died out (the jump-down phenomenon), and no discernible rolling existed below $\zeta_1$.

In the range of wave amplitudes between the jump-up ($\zeta_2$) and jump-down ($\zeta_1$) bifurcation values, there were two possible responses: a stable nonrolling motion and a stable large-amplitude roll motion. The latter was sometimes as large as $\pm 20^\circ$. When the model was not exhibiting any noticeable roll, some disturbances in the roll mode could cause a jump up to a large-amplitude steady-state roll motion anywhere between $\zeta_1$ and $\zeta_2$. The domain of attraction of the large roll motion amplitude increased as the wave amplitude was increased toward $\zeta_2$. This subcritical-type of instability was observed at all the locations of the model in the standing waves and over a wide range of wave frequencies.
The coexistence of two possible motions for the same wave pattern is further described in Figure 5.5 (a), which contains a typical force-response curve. The wave frequency was 0.58 Hz at location number 5 (refer to Figure 5.6). It illustrates the process of the present experiment. The sequence of events is marked by arrows from A to H; A → C → D → E → D → G → H → A. The wave amplitude marked at point H corresponds to \( \zeta_1 \), and the one at point C corresponds to \( \zeta_2 \). Just after the wave amplitude passed point C, a sudden jump up to point D occurred, which provided a moment of excitement for the observers on the scene. Thereafter, the trivial solution was unstable and only the large-amplitude roll motion was stable.

After arriving at point E, the wave amplitude was decreased slowly. The large roll amplitude continued to exist below point D, where the jump up occurred during the sweep up. When the wave amplitude was decreased slightly below point G, a sudden jump down to point H occurred and rolling stopped. Between H and C, two different stable motions coexist: a nonrolling motion and a large-amplitude roll motion. The motion that developed depended on the initial conditions. It was surprising to observe such a large roll amplitude in waves of such small amplitudes just before the jump down. These motions lasted as much as several hours in every case. We recall that the theory predicts that small pitch and heave motions can excite relatively large roll motions.

The responses in heave and pitch are nearly linear during the whole test, regardless of the magnitude of the roll response and the occurrence of jumps.
Figures 5.5 (b)-(d) show the fast Fourier transforms (FFT) and time traces corresponding to a few points on the curves in Figure 5.5 (a). Figure 5.5 (b) shows the FFT's and corresponding time signals at point B. The first harmonic components of all the responses of heave, pitch, and roll are at the same frequency as the wave. The roll was not discernible with the naked eye there. Figure 5.5 (c) corresponds to point D (i.e., just after the jump up occurred). The situation after rolling begins is described in Figure 5.5 (c). The roll response has the largest peak at one-half the frequency of the waves, which is a dramatic change from Figure 5.5 (b). The heave and pitch are at the same frequency as the waves. Figure 5.5 (d) shows plots of the responses at point H (i.e., just after the jump down occurred). The large peak at one-half the frequency of the waves disappeared and the slight rolling returned to the frequency of the waves. The magnitude of the roll motion decreased drastically and was not noticeable with the naked eye. The general characteristics of the plots in Figure 5.5 (d) are similar to those in Figure 5.5 (b). The responses between 0 and C, corresponding to the trivial roll motion, possess shapes similar to the plots of Figures 5.5 (b) and (d), and the responses between E and G are similar to those of Figure 5.5 (c).

Figures 5.5 (e) and (f) are parts of the long-time records for heave and roll responses recorded continuously on a strip chart recorder. Figure 5.5 (e) clearly shows the sudden jump up of the roll motion. After the jump, its magnitude remained almost constant while the heave amplitude varied as the amplitude of the wave was varied. Figure 5.5 (f) shows the jump down of the
roll motion to the trivial response. The small gap is due to an unintentional electric fluctuation that occurred for a very short period of time. It was recovered right away as shown there. This problem of electric fluctuations occurred often because the duration of one test was tremendously long, as much as several days. Thus, the whole process had to be monitored. The same problems sometimes occurred because of thunder storms and automatic cutoff of the electric motor due to overheating.

5.4 Effect of the Location of the Model in the Waves

5.4.1 Background

The theory predicts that the large-amplitude response is dependent on the excitation frequency. One of the significant findings of the present experiments is that the large-amplitude roll motion depends not only on the excitation frequency, but also on the location of the model on the waves.

Concerning these observations, we can identify a consistency with previous research conducted to study the directional stability and possibly broaching phenomenon under circumstances similar to the present experiments. Wahab and Swaan [131] used a linear formulation of the problem with constant coefficients and Froude-Krilov forces as the only wave forces and moments to study the zero-frequency-of-encounter case. With this fairly
simple model they demonstrated directional instability when a vessel with fixed controls is positioned anywhere over one-half of the wave cycle, but particularly when the center of gravity of the vessel is near the steepest part of the wave and the stern near the wave crest. This phenomenon is familiar to vessel operators. Wahab and Swaan also were able to demonstrate analytically that the probability of broaching was greater in steeper waves.

Eda and Crane [24] included hydrodynamic maneuvering coefficients obtained from experiments on models and horizontal plane-wave exciting forces and moments obtained from a strip theory. They also considered very low frequencies of encounter with regular waves and their predicted results of maneuvering performance in regular following or stern quartering waves were in reasonable agreement with model tests on a Series 60 hull. Eda [23] extended this work and demonstrated the same result as Wahab and Swaan; that is, a vessel which is directionally stable in calm water may not be stable in regular following waves with zero frequency of encounter. Renilson and Driscol [102] investigated experimentally the effect of the location of the model in the waves upon the possibility of broaching.

Regarding the problem of directional stability, we can find analogous cases in a generic manner. Most kinds of vehicles possess longitudinally asymmetric designs with respect to the middle section and are designed to make steady headway when moving forward. When automobiles and airplanes move backward, or are placed in winds blowing from their backs, they easily experience a tendency to run off the track, or to have side forces
exerted on them, or, in other words, course instability. In the same context, if a vessel encounters head waves, it may possess motion instability as is investigated in the present experiment. When a vessel encounters following waves (i.e., waves whose speed is faster or about the same as the ship and are coming from astern), the resultant situation is the same as those for automobiles and other vehicles described above. Hence, in such a situation, the ship can experience side forces, course instability, and possibly broaching in reality. In the present experiments, the condition that the model is located at a specific place along a standing wave may simulate the real situation of a ship navigating in following waves where their speeds are approximately the same so that the encounter frequency is about zero.

In similar circumstances, a vessel runs a great risk of experiencing large-amplitude motions not only in in-plane modes, such as heave and pitch, but also in out-of-plane modes, such as roll and yaw. The latter phenomenon is caused by the nonlinear interactions among the modes of motion and is the subject of the present investigation. We conducted experiments to show the dependence of the response on the location of the center of gravity of the model along a standing wave. These results are relevant to the questions of dynamic course instability and instability of motion of a ship when it is navigating with the same speed as the waves so that its encounter frequency is nearly zero. The jump-up and jump-down bifurcation values ($\zeta_2$ and $\zeta_1$) also varied with the different positions of the model in the waves.
These observations are consistent with the work of Renilson and Driscoll [102]. They concluded from free-model experiments conducted in the large circulating water channel (CWC) at the National Maritime Institute in Great Britain that the motion of a ship, the magnitude and direction of the side force exerted on it, and the possibility of its broaching while operating in slowly overtaking following or quartering regular waves are dependent on the longitudinal position of the ship in the wave system. They actually considered the case of zero frequency of encounter. They found that a longitudinal wave-induced force can lead a ship to a steady-state position relative to the waves; that is, a longitudinal equilibrium position in the stationary waves. At this point, the ship and the "following" waves are moving at the same speed. They showed that the response of a model depends on its position in the wave and the effects of heel angle due to rolling motion can lead to course instability, such as broaching. Their results were obtained using a model constrained in yaw, sway, and surge but allowed to heave, pitch, and roll, which is the same condition as the present experiments.

5.4.2 Results and Discussion

We placed the model at various positions along a standing wave with a frequency of 0.60 Hz. The relative locations are numbered 1-8 in Figure 5.6 for a typical standing wave. The node is numbered 1, the antinode is 5, and so on. Changing the location of the model along the wavelength changes the
phase between the pitch and heave motions and, hence, changes the effective
amplitude of the parametric excitation of the roll mode.

Figures 5.7 (a)-(c) show the responses of the heave, roll, and pitch modes
with the location of the model as a parameter when the wave frequency is 0.60
Hz. Figures 5.8 (a)-(h) show the responses of heave, roll, and pitch in one set
at each different location for the same standing wave of frequency 0.60 Hz.

The heave motions shown in Figure 5.7 (a) possess a systematic order:
the heave amplitude is proportional to the wave amplitude at a specific
location along a standing wave, which is the linear response as expected. In
other words, the heave amplitude is largest at the antinode (location number
5) where the wave amplitude is the largest (marks E and e along the curve), the
heave amplitude is smallest at the node (location number 1) where the wave
amplitude is the smallest (marks A and a), and the heave amplitudes at other
locations can be arranged in proper downward order from antinode to node
according to the wave amplitudes at specific locations.

The pitch motions, shown in Figure 5.7 (c), also demonstrate a systematic
order: the pitch amplitude is proportional to the wave slope at each specific
location along a standing wave, which is also expected from linear theory.
They are generally opposite in order to the heave responses. The pitch
amplitude is largest at the node (location number 1) where the wave slope is
the largest (marks A and a), the pitch amplitude is smallest at the antinode
(location number 5) where the wave slope is the smallest (marks E and e), and
the pitch amplitudes at other locations can be arranged in proper downward order from node to antinode according to the wave slopes at specific locations.

In Figures 5.7 (a) and (c), the heave and pitch responses clearly exhibit linear behavior, regardless of the existence of jump phenomena and large-amplitude roll motions for all the cases. These results confirm the assumption made in the analysis of Chapter II that the roll does not significantly influence the pitch and heave for the case considered here. Thus the present experimental results agree with the analysis of Chapter II.

Figure 5.7 (b), which shows a variety of responses in roll, demonstrates the coexistence of multiple stable responses. Some cases in Figure 5.7 (b) initially did not show the jump up, but when externally disturbed yielded stable large-amplitude roll motions.

The effective amplitude of the parametric excitation of the roll mode is produced by the approximately equally combined role of the heave and pitch and hence varies with the location of the center of gravity in the wave when the wave frequency is fixed. This result is in contrast to the case of varying the wave frequency in which the pitch mode played a more distinct role than the heave mode in yielding the effective amplitude of the parametric excitation.

Figures 5.7 (a)-(c) explain the above as follows. Location number 2 produced the seventh largest heave amplitude out of the eight locations considered, the second largest pitch amplitude, and the largest roll amplitude in general. Location number 1 (the node) produced the smallest heave, the largest pitch, and the second largest roll amplitude. Location number 5 (the
antinode) produced the largest heave, the smallest pitch, and the third largest roll amplitude in general. Location number 6 produced the second largest heave, the seventh largest pitch, and the smallest roll amplitude, and so on. Thus, we conclude that both the heave and pitch modes participate in yielding the effective amplitude of parametric excitation of the roll mode. The roles of heave and pitch modes seem to be about equal. Hence, the present study significantly extends the work of Blocki [9] and Sanchez and Nayfeh [107]. In a real environment, the ship will necessarily experience pitch; therefore, the pitch mode should be included along with the heave in investigating the parametric resonance of the roll mode.

The nontrivial roll amplitudes are about the same for all eight locations considered in the present experiments. The largest rms (root mean square) amplitude of roll was approximately 9° and the smallest was 7° before the roll amplitude began to decrease at higher wave amplitudes. In other tests, at a wave frequency of 0.80 Hz, the roll amplitude reached 20° approximately.

Figures 5.8 (a)-(h) show the responses in heave, pitch, and roll for each location of the center of gravity in one set as the wave amplitude was varied. The center of gravity of the model was placed at location numbers 1-8 (refer to Figure 5.6) along a standing wave with a fixed wave frequency of 0.60 Hz. The large-amplitude roll motions exist at all the considered locations except number 3.

At location number 3, no roll motion was observed although a sufficiently long duration (of the order of hours) was allowed at each step of wave
amplitude to achieve steady state. No external disturbances were imposed in this case, and a subcritical instability might existed beyond the limits of the experiment. Comparing this case with the other cases in its neighborhood, we expect that rolling would have developed if external disturbances had been imposed on the model during the sweep-up/down procedure.

Jump up occurred for location numbers 1-5, as illustrated in Figures 5.8 (a)-(e). These correspond to the conditions in which the center of gravity of the model was located on the front part (between the node and the antinode) of the standing waves so that the model was positioned “downhill” on the waves. In these cases, jump down followed during the process of sweep down of wave amplitude. Location numbers 6-8 (Figures 5.8 (f)-(h), respectively) yielded no spontaneous jump up to large-amplitude rolling. These correspond to the situation in which the center of gravity of the model was located on the rear part (between the antinode and the node) of the standing waves so that the model was positioned “uphill” on the waves. External disturbances were used at each step of wave amplitude during the process of sweep down until the jump up to the large-amplitude roll motion was obtained. If the jump up was developed, then the jump down to the no-roll motion followed.

Figure 5.7 (b) shows that the earliest jump up occurred at location number 4 (mark D) and the latest jump up occurred at location number 1 (mark A) while the latest jump down occurred at location number 5 (mark e) and the earliest jump down occurred at location number 8 (node h). In other words, the smallest wave amplitude that produces the jump up corresponds to the
center of gravity of the model being at location number 4. The smallest wave 
amplitude for which the jump down occurred corresponds to the center of 
gravity being at location number 5. The upper-branch roll motions for location 
numbers 6, 7, and 8, which do not show jump up but show jump down only, 
were obtained by imposing external disturbances at each step.

The domains of attraction for downhill positions of the model were larger 
than those for uphill positions. The farther the center of gravity was behind the 
antinode, the harder it was to obtain the upper-branch roll motion.

The largest overhang range of wave amplitudes was found for location 
number 5 (the antinode, marks E and e); that is, the large-amplitude roll motion 
coexists with the trivial roll motion along the widest range of wave amplitudes.

Comparing these experimental results with the analytically predicted 
subcritical-type force-response curve, we see that their upper-branches show 
the largest similarity at location number 2. In other cases, moderate 
similarities are found. Generally, after the jump up, the roll amplitude 
remained nearly constant for a while and then decreased as the wave 
amplitude continued to increase. The range where the roll had a constant 
amplitude varied with the position of the mass center on the wave.

Because the wave frequency 0.60 Hz is slightly smaller than twice the roll 
natural frequency (2 x 0.32 Hz), the detuning parameter $\sigma$ is negative. The 
experimentally obtained force-response curves are of the subcritical-type, as 
predicted by the theory in Chapter II (refer to Figure 2.2). We could not 
experimentally produce a supercritical-type instability.
5.5 Effects of Excitation Frequency

From the results of the tests described in Section 5.4, we know that the model exhibits large-amplitude roll motions and the jump phenomena readily at positions number 5 and 4 (refer to Figure 5.6).

To investigate the effects of the wave frequency, we chose position number 4 for each different wave. The model was moved to a new position so that the center of gravity of the model was located at position 4 relative to the different waves.

Figures 5.9 (a)-(c) illustrate the effect of varying the wave frequency when the model was located at position 4 (refer to Figure 5.6). Figures 5.10 (a)-(h) show the heave, roll, and pitch responses for different wave frequencies when the model was located at position 4.

Figures 5.9 (a) and (c) show the responses of heave and pitch, respectively; they exhibit linear behavior, regardless of the existence of large-amplitude roll motions for all the wave frequencies with the exception of 0.90 Hz and 1.00 Hz. As the wave frequency was increased from 0.50 Hz to 0.80 Hz, the maximum roll amplitude increased, and the pitch amplitude also increased whereas the heave amplitude did not.

We note that the heave motions shown in Figure 5.9 (a) do not seem to possess a systematic order because the response curves lie in a mixed way with disorderly slopes. The pitch motions shown in Figure 5.9 (c) are qualitatively different. The pitch response for the lowest excitation frequency
0.50 Hz (mark A along the curve) has the smallest slope except for 1.0 Hz (mark I). As the wave frequency was increased, the slopes of the pitch amplitude were larger for frequencies between 0.50 Hz and 0.80 Hz (marks A through G). As a result, the pitch response curves are ordered in such a way that a curve corresponding to a higher wave frequency lies above a curve corresponding to a lower wave frequency, which is a linear result.

Figure 5.9 (b), which shows a variety of roll responses, demonstrates the coexistence of multiple stable responses. Some cases in Figure 5.9 (b) did not exhibit a spontaneous jump up, but when externally disturbed, they displayed a stable large-amplitude roll motion. This implies that the roll response depends on the initial conditions in this overhang region. It is worthwhile to focus on the maximum roll amplitude observed at each wave frequency. The maximum rms roll amplitude was approximately 14° rms (root mean square) for the wave frequency 0.80 Hz and was as small as 3° rms for the wave frequency 0.55 Hz; they were the largest and smallest roll amplitudes. The wave frequencies 0.80 Hz and 0.55 Hz were the highest and lowest wave frequencies that yielded multiple stable roll responses. Within these two frequencies, as the wave frequency was increased, the roll amplitude increased. The difference between the two rms roll amplitudes was as large as 11° approximately.

While we cannot find any direct relation between the heave and roll motions because of the disorderliness in heave, we can identify a close relation between the pitch response and the large-amplitude roll motion. It
appears that the pitch was the primary cause of parametric excitation due to the following observations. First, the larger the pitch slope was, the larger the roll amplitude was for various wave frequencies between 0.55 Hz and 0.80 Hz; the roll response correlated with the pitch motion rather than the heave motion. This implies that the role of the pitch mode is more significant than the role of the heave mode in producing the effective amplitude of the parametric excitation.

Second, comparing the responses for the excitation frequencies 0.60 Hz and 0.70 Hz in Figures 5.9 (a)-(c), we observe that the roll amplitudes at the wave frequency 0.60 Hz are smaller than those at 0.70 Hz when the pitch amplitudes are smaller and the heave amplitudes are larger than those at 0.70 Hz. In turn, the roll amplitudes at the wave frequency 0.70 Hz are larger than those at 0.60 Hz when the pitch amplitudes are larger and the heave amplitudes are smaller than those at 0.60 Hz. Thus, the roll response correlates with the pitch mode rather than the heave mode.

Moreover, at the wave frequency 0.80 Hz, the roll exhibits the largest response when the pitch amplitudes are the largest and yet the heave amplitudes are the smallest among all the considered excitation frequencies. In other words, the largest roll amplitudes occur when the pitch amplitudes are the largest and the heave amplitudes are the smallest. Thus, we can draw a conclusion that the pitch plays a more significant role than the heave in producing the effective amplitude of the parametric excitation.
Finally, further support of this conclusion can be drawn from the roll, heave, and pitch responses at 0.55 Hz and 0.57 Hz. At these frequencies the roll has the smallest rms amplitudes (approximately 3°); the heave has intermediate amplitudes; and the pitch has the smallest amplitudes. At the wave frequencies 0.50 Hz and 1.0 Hz, no nontrivial roll responses were observed.

Consequently, the nontrivial roll amplitude is affected by the pitch more than the heave. In some sense, it can be argued that the heave has a small effect on the parametric excitation of the roll mode in this case. Here we point out that Blocki [9] considered only the heave-roll coupling or pitch-roll coupling and Sanchez and Nayfeh [107] studied the parametric excitation of the roll mode by the heave mode only.

When the wave frequency increases, the wavelength becomes shorter and the wave slope increases for the same wave amplitude. The larger the wave slope is, the larger the pitch response is according to linear theory and the present experimental observations. Thus, the pitch amplitudes are the largest for 0.80 Hz which is the highest wave frequency, and they are the smallest for 0.55 Hz which is the lowest wave frequency considered here. We note that the pitch mode plays a more dominant role in yielding the effective amplitude of the parametric excitation of the roll mode. Therefore, for the highest wave frequency (0.80 Hz), both the roll and pitch amplitudes are the largest; for the lowest wave frequency (0.55 Hz), both the roll and pitch amplitudes are the smallest.
Figures 5.10 (a)-(h) show variation of the heave, pitch, and roll amplitudes with wave amplitude for different wave frequencies. The center of gravity of the model was located at position 4 for each tested wave frequency (refer to Figure 5.6). Large-amplitude roll motions existed for excitation frequencies between 0.55 Hz and 0.80 Hz. Even when we imposed external disturbances on the model during the process of sweep up and sweep down of wave amplitude, we did not observe large-amplitude roll motions for the wave frequencies 0.50 Hz (the lowest border of frequency), 0.90 Hz (the highest border of frequency), and 1.0 Hz.

We observed the jump-up phenomenon for the excitation frequencies 0.60 Hz and 0.70 Hz, as illustrated in Figures 5.10 (d) and (e), respectively. In these cases, jump down followed during the sweep-down process. Among the considered cases possessing large-amplitude roll motions, the excitation frequencies 0.55 Hz, 0.57 Hz, and 0.80 Hz (Figures 5.10 (b),(c), and (f), respectively) did not exhibit a spontaneous jump up to the large-amplitude roll motion. External disturbances were imposed in these cases at each level of wave amplitude during the process of sweep down to obtain the upper-branch roll responses, and then the jump down to the trivial roll motion followed.

Figure 5.9 (b) shows that the earliest jump up to large-amplitude roll motions was obtained for a wave frequency of 0.60 Hz (mark E) and the latest jump up was obtained for 0.70 Hz (mark F), while the latest jump down to trivial roll motions was obtained for a wave frequency of 0.80 Hz (mark g) and the earliest jump down was obtained for 0.55 Hz (mark c). In other words, the
smallest wave amplitude at which the jump up occurred was when the wave frequency was 0.60 Hz, and the smallest wave amplitude for which the jump down occurred was when the wave frequency was 0.80 Hz. The upper-branch roll motions for the excitation frequencies 0.55 Hz, 0.57 Hz, and 0.80 Hz, which did not exhibit a spontaneous jump up but exhibited jump down only, were obtained by imposing external disturbances at each step of wave amplitude during the process of sweeping down the wave amplitude. The farther the excitation frequency from 0.60 Hz (approximately twice the roll natural frequency) was, the more various external disturbances we had to impose to obtain large-amplitude roll motions. In other words, the larger the magnitude of the detuning frequency was, the harder it was to obtain the upper-branch roll motion. Thus, the domains of attraction for the more exactly tuned cases, such as the excitation frequencies 0.60 Hz and 0.70 Hz, were larger than those for the detuned cases, such as 0.55 Hz and 0.80 Hz. The largest overhang range of wave amplitudes was found for the wave frequency 0.57 Hz (marks D and d) for which the large-amplitude roll motion coexisted with the trivial roll motion along the widest range of wave amplitudes.

The shape of the analytically predicted subcritical-type force-response curve is qualitatively similar to the experimentally obtained one for the excitation frequency 0.57 Hz. Moderate similarities are found for 0.55 Hz and 0.70 Hz. At the excitation frequencies 0.60 Hz and 0.80 Hz, the magnitude of the roll motion remained relatively constant during the sweeping up and sweeping down of the wave amplitude.
As the wave amplitude was increased, the roll amplitude decreased for all the considered excitation frequencies (Figures 5.10 (b)-(f)) and even became trivial for the excitation frequencies 0.55 Hz and 0.57 Hz (Figures 5.10 (b) and (c)), respectively.

Extensive efforts were made to obtain force-response curves of the supercritical-type as predicted by the theory. The theory predicts force-response curves of the supercritical-type for wave frequencies equal and greater than 0.70 Hz because the detuning parameter $\sigma$ is positive, matching Figure 2.1. Such curves were not obtained in the present experiment. However, we can make some suggestions: first, use a model with larger damping; second, use a fluid medium whose density is higher than water; third, have better wavemaking systems to be able to generate smooth regular plane waves at higher amplitudes. This may require a model basin of better quality and a wave absorber of higher efficiency.

5.6 Miscellaneous Observations

Figures 5.11 and 5.12 show variations of the critical wave amplitudes for jump up/down with wave frequency and location of the model on the waves, respectively. Above the jump-up values, the trivial roll response is unstable and large-amplitude roll motions are the only stable responses. In the intervals between the jump-up and jump-down values, two stable responses
(i.e., the trivial and the large-amplitude roll responses) coexist. Below the jump-down values, the trivial roll motions are the only stable responses.

If the jump-up wave amplitude is not marked, it means the jump up to large-amplitude roll motions did not occur during the process of sweep up. In such cases, external disturbances were imposed at each step during the sweep-down process, then the jump down followed and its wave amplitude was recorded, as shown in Figures 5.11 and 5.12.

We also observed a penetration phenomenon in the case of a nonstationary excitation. When we increased the wave amplitude fast enough, the jump-up phenomenon occurred at a wave amplitude noticeably higher than the quasi-steady jump-up bifurcation value \( \zeta_2 \). The penetration of trivial roll motions into the regime where trivial motions are unstable is due to the nonstationarity of the wave excitation. The higher the sweep rate was, the deeper the penetration was.

This phenomenon was identified in some preliminary tests performed before the actual experiments of the long-time lasting sweep-up and sweep-down processes of wave amplitudes. Thus the tests were performed rather quickly and hence the amount of data was not sufficient to produce a resultant plot to describe this observation. Instead, a typical force-response curve theoretically produced by Nayfeh and Asfar [76] is shown in Figure 5.13. The observations were in qualitative agreement with this result.

In the present experiments, capsizing never seemed imminent in any of the tested cases. This can be attributed to having sufficient metacentric height
(GM) or area under the curve of restoring moment arm (GZ curve). The effect of metacentric height is not simple because it affects both the natural roll frequency and the static stability. In wave conditions up to the limit corresponding to regular plane waves in the towing basin at VPI & SU, although the model did not show capsizing in high waves, sometimes the water regularly swept the deck. In full scale tests, it is possible that in such a condition the full scale vessel would founder due to progressive flooding.
<table>
<thead>
<tr>
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<th>Value</th>
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</tr>
<tr>
<td>LBP</td>
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<tr>
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<tr>
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<tr>
<td>$C_{B_{(w)}}$</td>
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</tr>
<tr>
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</tr>
<tr>
<td>L/B</td>
<td>7.65</td>
</tr>
</tbody>
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Chapter VI

AN EXPERIMENTAL INVESTIGATION OF COUPLED HEAVE, PITCH, AND ROLL MOTIONS IN THE PRESENCE OF INTERNAL RESONANCE

6.1 Overview of the Experiment

In Chapter III, we demonstrated theoretically that the nonlinear couplings arising from a two-to-one internal or autoparametric resonance produce strong interactions between the interacting modes. To verify the theoretical predictions of Chapter III, we performed two series of model experiments in the ship dynamics laboratory at VPI & SU. To achieve the condition of two-to-one internal resonance, we adjusted the pitch natural frequency of the
model to be approximately twice that of its roll natural frequency. The model was allowed to move freely in three degrees of freedom (i.e., roll, pitch, and heave) and was placed in a head or following wave. Waves with frequencies near the pitch natural frequency were generated, which produced the primary resonance of the pitch mode.

Under these circumstances, the linear approach predicts that only the in-plane modes of motion (e.g., heave and pitch) are excited. However, the nonlinear analysis presented in Chapter III predicts that the two-to-one internal resonance may cause a strong interaction between the in- and out-of-plane (roll) modes of motion of the vessel. Thus, the roll mode, which is not directly excited, can exhibit a large-amplitude motion as a result of the loss of stability of the in-plane motion, thereby transferring part of its energy into the roll mode.

Based on the basic wave tests described in Chapter IV, it was desirable to use waves having frequencies in the neighborhood of 0.60 Hz because the wavemaker produces plane waves for the widest range of amplitudes around this frequency in the towing basin. Unfortunately, the lowest heave and pitch natural frequencies achievable with the ballasted model were approximately 1.26 Hz and 0.91 Hz, respectively. Since the pitch natural frequency 0.91 Hz was closer to 0.60 Hz than the heave natural frequency, the roll natural frequency was adjusted to 0.46 Hz, which was approximately one-half the pitch natural frequency.
The pitch mode was excited by waves having frequencies between 0.85 Hz and 0.95 Hz. These wave frequencies are admittedly higher than 0.60 Hz, but they were the best obtainable given the model and the present experimental setup. However, the theoretical results of Chapter III do indicate that the roll mode can be indirectly excited by either the pitch or heave mode in the presence of a two-to-one internal resonance with relatively low-amplitude waves. Fortunately, low-amplitude waves generated at these frequencies are plane waves.

Basically, the experiments were run in this manner. With the model in place, the wavemaker was started at the lowest amplitude available. Then, the amplitude was increased and decreased very slightly to the next step on the function generator in a manner similar to the sweep-up and sweep-down process described in Chapter IV. During this process, jump-up and jump-down phenomena were observed. When jump up did not occur spontaneously, external disturbances of various kinds were imposed on the model at each different step of wave amplitude. These disturbances often produced stable large-amplitude roll motions, an indication of the coexistence of multiple dynamic equilibria for the case of subcritical instability. The pitch responses were neither linear nor saturated, but, rather, nonlinear and/or sometimes nearly saturated. In general, heave responses exhibited linear behavior. These results are in good qualitative agreement with the theoretical results predicted in Chapter III. In a few instances, the responses appeared to be due to only the principal parametric resonance and the influence of the
two-to-one internal resonance was not significant. Video-tape recordings and/or photographs were made during the tests.

Experiments in which the wave frequency is varied while the amplitude is fixed were not implemented; the reasons for this are described in Chapter IV. The supercritical type of instability was not obtained in spite of extensive efforts devoted to this case.

6.2 Description of the Model

Finding an appropriate type and size of model for this experiment turned out to be more of a problem than anticipated. VPI & SU lacks facilities and skilled woodworking technicians for fabricating a desired model. After contacting several marine organizations, we obtained two destroyer models on loan from the Carderock Division of the Naval Surface Warfare Research Center (formerly, the David Taylor Naval Ship Research and Development Center or David Taylor Research Center).

Photographs of the model used in this experiment are shown in Figures 6.1 and 6.2. They clearly demonstrate the rolling motion of the model in longitudinal waves. The model, Hull No. 4794, is a U.S. Navy destroyer hull type similar in form to a contemporary LAMPS capable KNOX Class ASW frigate. The model was constructed and utilized by the Carderock Division of the Naval Surface Warfare Research Center for resistance and seakeeping
studies during the 1960's. Because of its age, a data sheet with normal model
details and specifications (i.e., scale ratio, full scale dimensions, appendage
locations, moments of inertia) was not available. However, the outer shape of
the model hull is exact except for the rudder, sonar dome, bilge keels, and
propeller. Its principal particulars, as measured, are listed in Table 6.1.

The model is constructed from common pine. In particular, the interior is
hollowed out to a large extent. This feature was extremely desirable for this
experiment because additional weight could be distributed to obtain the
necessary pitch and roll moments of inertia. Since the model was some 30-40
years old, the exterior paint was cracked and flaking in some areas. Prior to
the first series of tests, the model was repainted from the DWL down to the
keel using an epoxy-based paint. Unfortunately, during the first test series, the
new paint began to bubble and separate from the surface, indicating
incomplete surface preparation. To avoid further deterioration and possible
bias to the experimental data, we re-sanded the model, sealed it with a water
proof sanding sealer, and re-painted it for the second series of tests. It was
felt that any possible changes in damping and inertia would be minimized by
this extra effort in model preparation.

The measured natural frequencies of the "bare hull" of the unballasted
model were 1.65 Hz in pitch, 1.45 Hz in heave, and 1.40 Hz in roll. Since the
VPI & SU towing basin produced the best smooth regular plane waves at
around 0.60 Hz, the most desired two-to-one autoparametric condition of
natural frequencies for a model would be a pitch or heave frequency of 0.60
Hz and a roll frequency of 0.30 Hz. To utilize the better condition of waves, we made attempts at ballasting our model to its design waterline and simultaneously lowering either the natural frequency of heave from 1.45 Hz to 0.60 Hz or that of pitch from 1.65 Hz to 0.60 Hz. The lowest heave natural frequency achievable with the model at the designed waterline was 1.26 Hz and that of pitch was 0.91 Hz. Thus, it was decided to adjust the frequency ratio between the pitch and roll to approximately two-to-one by setting the roll natural frequency to be around one-half the pitch natural frequency. In the first series of tests, the roll natural frequency was set at 0.495 Hz so that $\sigma_r = -0.08$ Hz in equations (3.23 a) and (3.23 b). In the second series, the roll natural frequency was set at 0.460 Hz so that $\sigma_r = -0.01$ Hz.

It was a painstaking process because, if one characteristic parameter was varied, all the others varied simultaneously, and the center of gravity of the total weights and model had to be remeasured after taking the heavy ballasted model out of the basin. It required many iterations of careful weight adjustment and distribution. After achieving the desired pitch and roll natural frequencies by preliminary tests, we decided to configure the added weight in the following manner. Three compartments were built inside the model: two as far forward as possible and one aft. These compartments were filled with 20-25 lbs. of leadshot, fore and aft each. A 5 lb. lead weight was placed aft at the top of the stern deck, and another 5 lb. weight was placed forward. This 5 lb. lead weight was fitted through a short piece of machine threaded rod which vertically spanned the distance from the keel to the main deck in the
forecasted area. This allowed the vertical adjustment of this 5 lb. weight without affecting the longitudinal distribution of weight in the model. In effect, the vertical center of gravity (VCG) could be varied without changing the longitudinal center of gravity (LCG) and total displacement. Therefore, detuning between the pitch and roll natural frequencies from the two-to-one ratio could be accomplished for a given displacement.

We first checked roughly by a stop watch the natural frequency of each mode of the model floating on the calm water of the towing basin. Next, we mounted the model underneath the towing carriage and connect the cables of the motion measuring transducers with the data acquisition instruments and dynamic spectrum analyzers, such as HP 3562 A and Rockland 5820 B. Then, we gave the model initial displacements to generate free oscillations of the model and analyzed these data by using the dynamic spectrum analyzers. This procedure was repeated until the desired condition of the model, such as natural frequencies, trim, and designed waterline, was reached.

The parameters used in the test series A and B are listed in Table 6.2. The parameters \( \omega_{zn} \), \( \omega_{bn} \), and \( \omega_{rn} \) denote the natural frequencies of heave, pitch, and roll, respectively. The ratio of the natural frequencies of pitch and roll is represented by \( \frac{\omega_{bn}}{\omega_{rn}} \). This ratio was adjusted to be approximately two-to-one. The parameter \( \Omega \) represents the excitation frequency, and the parameters \( \sigma_1 \) and \( \sigma_2 \) represent the detuning parameters defined in equations (3.23 a) and (3.23 b). The frequency \( \Omega \) of the excitation wave was set near 0.910 Hz in order to resonantly excite the pitch mode.
6.3 General Results

Typical force-response curves of the subcritical type for various magnitudes of quadratic roll damping \( \phi | \dot{\phi} | \) are shown in Figure 6.3. These force-response curves are the ones obtained in Chapter III. It is shown that the pitch mode will be completely saturated beyond a certain bifurcation value of the excitation wave amplitude, \( \zeta_2 \), in the presence of only linear damping; that is, the amplitude of the directly excited pitch mode is constant. The additional excitation energy will then be transferred to the indirectly excited roll mode, thereby producing a large-amplitude roll motion.

When the quadratic damping term, \( \phi | \dot{\phi} | \), is included in the analysis, the saturation phenomenon of the pitch mode is broken. The pitch amplitude grows linearly until the bifurcation point \( \zeta_2 \) of excitation amplitude is reached. Then the pitch amplitude neither grows linearly nor shows saturation. Instead, it grows nonlinearly, thereby deviating from the saturated pitch response predicted in the case of linear roll damping. For more details, the reader is referred to Chapter III.

Figure 6.4 shows a typical force-response curve of the subcritical type obtained from the present experiments. These experimental results are consistent with the results predicted in Chapter III. Comparison between Figures 6.3 and 6.4 indicates many resemblances: (i) the subcritical instability, (ii) the coexistence of large-amplitude and trivial roll motions, (iii) the
associated jump phenomena in the range between the two bifurcation points \( \zeta_1 \) and \( \zeta_2 \) of the excitation amplitude, as shown in Figure 6.3, and the points L and C in Figure 6.4, (iv) the breaking of the saturation phenomenon, and (v) the nonlinear growth of the directly excited pitch mode.

The following description of Figure 6.4 requires the reader to refer to Figures 6.5(a) through 6.5(h) and Figures 6.6(a) through 6.6(h). Figure 6.4 is the result of test 3, which exhibited the subcritical instability most clearly. Figures 6.5(a) through 6.5(h) are parts of the long-time records of the analog signals of the pitch and roll responses recorded on a strip chart recorder continuously for 10 days. Figures 6.6(a) through 6.6(h) show the spectra and time traces corresponding to a few points on the curves of Figure 6.4. In Figure 6.4, the sweep-up process is defined as the amplitude of excitation is increased from point A to point H, and the sweep-down process as the amplitude of excitation is decreased from point H to points L and A.

The sequence of events is marked by arrows and capital letters from A to L: \( A \rightarrow B \rightarrow \ldots \rightarrow G \rightarrow H \rightarrow G \rightarrow I \rightarrow \ldots \rightarrow K \rightarrow L \rightarrow A. \) As the excitation amplitude was increased slowly from A to C, only the trivial roll motion existed. The wave amplitude at point L corresponds to \( \zeta_1 \), and at point C it corresponds to \( \zeta_2 \) in Figure 6.3. Figure 6.5(a) shows the continuous time records of the analog signals of the pitch and roll modes around point B of Figure 6.4. Figure 6.6(a) shows typical plots of the spectra and time traces of the pitch, roll, heave, and wave corresponding to point B of Figure 6.4.
A jump-up phenomenon was observed as the amplitude of excitation exceeded a critical value, denoted by point C in Figure 6.4. This jump led to a large-amplitude roll motion. The point of jump up is marked as point D in Figure 6.4. This jump-up phenomenon is clearly shown in the continuous long-time records displayed in Figures 6.5(b) and 6.6(b). To ascertain the stability of the large-amplitude roll response, we forced the model to have the trivial roll response. However, as the constraint was removed from the model, the roll response returned to the large-amplitude motion. This is represented in Figure 6.5(b) by the small gap in between the large-amplitude roll responses. The time traces and the spectra of the roll, pitch, and heave motions and the excitation wave are shown in Figures 6.6(a) and 6.6(b) corresponding to points B and D of Figure 6.4, respectively. It is very clear from these figures that there are no significant changes in the pitch, heave, and excitation wave signals. However, the roll signal displays drastic changes: first, the roll amplitude has increased to a very large value (also compare the scales of the plots); second, in the FFT results, the largest peak of the roll appears at the subharmonic of order one-half while the largest peaks of the pitch and heave are at the wave frequency.

When the excitation amplitude was increased further from point D to E, the amplitude of roll did not increase, but it did decrease, which is inconsistent with the theoretical result displayed in Figure 6.3. This discrepancy is also evident in Figures 6.5(c) and 6.6(c).
The amplitude of the large roll motion, displayed in Figure 6.4, continued to decrease from E to F and G as the excitation amplitude was increased further. Beyond point F, the large-amplitude roll motion dropped to the trivial motion at point G. In Figure 6.5(d) this behavior was well recorded continuously, and is also evident on comparing Figure 6.6(c) for E and Figure 6.6(d) for G. We note that we had to vary the scales considerably for the roll motion due to its large variations, while we did not need to change the scales for the pitch, heave, and wave because they only varied slightly. The reactions to imposing various external disturbances on the model are also shown in Figure 6.5(d). All of our efforts aimed at reviving the large-amplitude roll motion by imposing various disturbances were of no use because the roll disappeared after 10-20 cycles of such temporary heelings. Consequently, the trivial roll seemed to be the only stable motion and the large-amplitude roll motion no longer existed when the excitation amplitude was larger than that at point G.

The excitation wave amplitude was further increased very slowly up to point H, and the large-amplitude roll motion did not occur, although various external disturbances were imposed very often on the model. Figure 6.6(e) shows that the roll amplitude is a trivial one at point H. The time traces of the roll and wave contain more irregularities and their spectra are noisier than those at point G. The pitch and heave modes have almost the same behaviors as those for point G. The irregularity of the roll response, although its amplitude is negligibly small, was the result of the wave becoming rough.
because the smooth regular plane waves were contaminated by cross waves at these relatively large wave amplitudes. Thus the wave amplitude for point H was concluded to be the maximum wave amplitude attainable for that wave frequency in the present experimental setup of VPI & SU.

After reaching the maximum possible wave amplitude in the present experimental setup, we reduced slowly the wave amplitude from point H to point G. When the excitation amplitude was reduced slightly below point G, a large-amplitude roll occurred, corresponding to point I. This implies that the large-amplitude roll is the only stable motion and the no-roll motion is an unstable one below point G. Figure 6.5(e) demonstrates well such a behavior.

As the wave amplitude was decreased further to point J, the response became similar to that at points D and E and the amplitudes and shapes of the signals were between them. Figure 6.5(f) demonstrates that the roll amplitude increased gradually as the wave and pitch amplitudes decreased slowly from point I to point J. Figure 6.6(f) clearly shows that the large-amplitude roll motion was recovered and that the highest peak of the roll is at the subharmonic of order one-half.

As the wave amplitude was further reduced from point J to point K, the large-amplitude roll motion continued even after passing the wave amplitude where the jump up had occurred. This is the subcritical instability. The large-amplitude roll motion increased gradually and then decreased; in this interval, the shape of the experimental force-response curve resembles that predicted by the theory (Figure 6.3). Figure 6.5(g) shows a gradual increase
in the amplitude of the large roll motion. It also shows a premature jump down. To investigate whether the large-amplitude roll motion lost its stability and did no longer exist below that wave amplitude, we imposed various external disturbances on the model and finally retained the stable large-amplitude roll motion. Hence, it can be concluded that the previous jump down was due to an unknown disturbance imposed on the system and two dynamically stable roll motions still coexisted at this same wave amplitude. These motions are the trivial and the large-amplitude responses. The response depends on the external disturbances, or, equivalently, the initial conditions. Figure 6.6(g) shows the results at point K where the amplitude of roll motion is the largest among the cases of points considered so far along the curve. We also note that at point K the roll amplitude is very large, up to $\pm 18^\circ$, while the amplitudes of the pitch, heave, and wave are much smaller than those at points D and J. If one observed the model for conditions corresponding to point K, one would be surprised at the behavior of the model exhibiting such a large roll amplitude in response to very small wave amplitudes.

When the excitation amplitude was slightly decreased below point K, a dramatic change of roll response occurred. A sudden disappearance of the large-amplitude roll motion took place. Thus a jump-down phenomenon from a stable large-amplitude response to a stable trivial response (from point K to point L) of the roll mode was observed. Figure 6.5(h) clearly shows the record of such a catastrophic change of the roll motion. No discernible roll motion
existed below the jump-down bifurcation amplitude. No large-amplitude roll motion was attained in spite of so many different trials of imposing external disturbances below point L. Figure 6.6(h) shows that, at point L, the roll is very small and the largest peak of the roll spectrum is at the first harmonic. Compared to Figure 6.6(g) corresponding to point K, the roll response has changed drastically both in magnitude and shape while the pitch, heave, and wave are almost the same as those shown in Figure 6.6(g) for point K.

It was hard to predict when the jump up and jump down of roll would occur. They occurred all of a sudden and those moments passed so quickly that we were unable to acquire the digitized data. By a good chance, the moment of jump down was captured in a digitized form and its time traces and spectra for pitch and roll are presented in Figure 6.6(i). It shows that the roll motion changed very sharply while the pitch motion exhibited very insignificant variation. Note that the size of the scale for pitch is very small and thus the time trace of the pitch is significantly magnified compared to the roll. The instant of jump up would yield similar plots except that the right- and left-sides of the time traces would be reversed.

During the entire processes of sweep up and sweep down, the heave exhibited a simple linear behavior, irrespective of whether the roll response is complicated or not. However, the pitch exhibited neither a linear nor a saturated behavior, but a nonlinear response, as shown in Figure 6.4. This is in accordance with the theory presented in Chapter III. This result implies that as the wave energy was fed into the pitch mode, the pitch response grew
linearly with wave amplitude whereas the roll remained very small. As the wave amplitude increased beyond a certain critical value, the pitch response grew nonlinearly with wave amplitude, with its growth rate being much smaller than the linear rate, whereas the roll mode acquired a large-amplitude motion. Thus, beyond the critical wave amplitude, a significant portion of the input energy was spilled over into the roll mode while the small remaining portion increased the amplitude of the pitch mode slightly above the saturated one.

The energy transfer from the pitch mode to the roll mode is due to the nonlinear couplings between these modes, resulting from the two-to-one internal resonance. Here we note that, because of the strong coupling due to the two-to-one internal resonance, the large-amplitude roll motion occurred with wave amplitudes of the order of 0.2 - 0.8 inches (rms), which are much smaller than those (order of 1.0 - 4.0 inches (rms)) required to excite the roll motion in the absence of the internal resonance (Chapter V). In contrast with this, there seems to be no noticeable interaction between the heave and roll modes. The two modes might be coupled nonlinearly, however, there is no mechanism to strengthen such nonlinear couplings. Thus the energy fed into the heave mode is not transferable to the roll mode, or, vice versa, because of the lack of any internal resonances.

In the excitation range corresponding to points A, L, K, B, C, and D, Figure 6.4 bears some similarity to that of Figure 6.3. In other words, the experimental results of Figure 6.4 are in good qualitative agreement with the theoretical results of Figure 6.3 when the excitation amplitude is relatively
small. However, for excitation amplitudes higher than point D, the experimental results deviate from the theoretical results, the roll amplitude decreases rather than increases, and eventually the large-amplitude roll motion even disappears beyond point G. Such discrepancies are conjectured to be due to the combined following three reasons: (i) the difference in the hydrodynamic and hydrostatic characteristics in the presence and absence of waves, (ii) the unobserved, but possible waves reflected from the sidewalls of the basin, and (iii) the reaction forces resulting from the motion guidance system designed to block side motions that would occur in a completely free model. This is mainly due to the lack of consideration of yaw and sway modes and their couplings both in the analysis and following experiments.

6.4 Results of Tests: Series A

Figures 6.7 and 6.8 show all the results obtained in the tests of Series A (refer to Table 6.2). The results of test 1 are omitted here because test 1 was conducted too quickly and its results were rather rough compared to tests 2 and 3. The results of test 4 are also omitted from Figures 6.7 and 6.8 because test 4 did not produce any large-amplitude roll motion.

Figure 6.7 shows all the roll responses in one plot for comparison. The capital letters are used to denote the sweep-up process while the lower cases
are reserved for the sweep-down process. This practice is followed also in Figure 6.9.

The excitation frequency 0.873 Hz yielded roll amplitudes smaller than 0.90 Hz. The range of wave amplitude producing large-amplitude roll motions for 0.873 Hz is much smaller and thus the domain of attraction of the large-amplitude roll motion is much smaller than that for 0.90 Hz. Such results were expected because the excitation frequency 0.873 Hz is farther away from the pitch natural frequency 0.910 Hz of the model than 0.90 Hz and hence the magnitude of the detuning frequency for 0.873 Hz is larger than for 0.90 Hz.

Figures 6.8(a)-(d) show all the responses in heave, pitch, and roll for tests 2, 3, 5, and 6. Figure 6.8(a) shows the two different coexisting roll responses for the excitation frequency 0.873 Hz. The large-amplitude roll motion was obtained by imposing external disturbances on the model. Without disturbances, only the trivial roll motion was obtained during slow variations of sweep up and sweep down of the excitation amplitude. In this case, both the heave and pitch motions are linear and thus the influence of the two-to-one internal resonance seems to be less significant. The existence of the large-amplitude roll motion is conjectured to be the result of principal parametric resonance of the roll mode. For more details about this case, the reader is referred to Chapter V.

Figures 6.8(c) and (d) show the results for the same excitation frequency of 0.90 Hz. They are meaningful because the results of Figure 6.8(d) were for the same conditions as those of Figure 6.8(c). This ascertains the repeatability
of the present experiments. Test 2 shown in Figure 6.8(c) was implemented for three days while test 3 shown in Figure 6.8(d) was for ten days. Test 3 was conducted much more accurately than test 2. Compared to test 2, test 3 has more data points and much longer time of duration, three to five hours were used at each step of excitation amplitude. Since enough time was allowed to achieve steady state, the results are expected to be more accurate. Figures 6.8(c) and (d) show that the heave response is linear while the pitch response is neither linear nor saturated, but, is nonlinear as was predicted by the theory.

### 6.5 Results of Tests: Series B

Figure 6.9 shows all the roll responses in one plot for comparison. The excitation frequency 0.921 Hz yielded smaller roll amplitude than 0.898 Hz. The range of wave amplitudes producing the large-amplitude roll motion for 0.921 Hz is much smaller and thus the domain of attraction of the large-amplitude roll motion is much smaller than those for 0.898 Hz. For both excitation frequencies, the large-amplitude roll motions were obtained by imposing external disturbances on the model. Since we tuned the natural frequencies of pitch and roll more exactly than those for Series A, we expected to obtain the large-amplitude roll motion more easily. However, the stable large-amplitude roll motion was harder to obtain than in Series A either by
spontaneous jump up or by imposing external disturbances on the model. More than 20 different excitation frequencies, each with 5 different wave amplitudes and 5 different model locations along the basin were tested. However, all of the 500 different cases did not produce the large-amplitude roll motion except for the cases described above. As examples of such cases, the responses for 0.911 Hz and 0.919 Hz are included in Figure 6.9. They show only the trivial responses. In tests 1 and 2, the spontaneous jump-up phenomenon was not observed while the subcritical instability and coexistence of dynamically stable multiple motions were found by imposing external disturbances.

Figures 6.10(a) and (b) show enlarged plots of the time traces for the trivial and large-amplitude roll motions. We note that the roll signals shown in (a) and (b) have entirely different shapes and magnitudes; we also note that the scales of the roll are very different in (a) and (b). However, the pitch, heave, and wave signals have almost the same shapes and amplitudes between (a) and (b).

Figures 6.11(a)-(d) show typical results of the spectra and time traces of the pitch, roll, heave, and wave for an excitation frequency of 0.921 Hz. Figure 6.11(a) shows the trivial roll motion and Figure 6.11(b) shows the coexisting large-amplitude roll motion. The wave amplitudes are almost the same for both cases and so are the other signals of pitch and heave. Only the roll responses exhibit a drastic change in their shapes and amplitudes. Figure 6.11(c) shows the responses to a higher wave amplitude. Comparing Figures
6.11 (b) and (c), we conclude that as the excitation amplitude was increased, the amplitude of large roll motion decreased. Figure 6.11(d) shows that the large-amplitude roll motion disappeared as the excitation amplitude was increased further.

Figures 6.12(a) and (b) show all the responses of heave, pitch, and roll for the excitation frequencies 0.898 Hz and 0.921 Hz, respectively. In Figure 6.12(a), because the heave and pitch responses are nearly linear, the large-amplitude roll motion seems to have been caused by the principal parametric resonance as in Figure 6.8(b). The influence of the two-to-one internal resonance does not seem to be very significant in this case. Figure 6.12(b) shows the pitch response, which is nearly saturated, as was predicted in Chapter III.
Table 6.1. Principal Particulars of the Destroyer Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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</thead>
<tbody>
<tr>
<td>LOA</td>
<td>10'4&quot;</td>
</tr>
<tr>
<td>$B_{max}$</td>
<td>11.5&quot;</td>
</tr>
<tr>
<td>$D$</td>
<td>9&quot;</td>
</tr>
<tr>
<td>Displacement</td>
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</tr>
<tr>
<td>$d_o$</td>
<td>4&quot;</td>
</tr>
<tr>
<td>$d_t$</td>
<td>4&quot;</td>
</tr>
<tr>
<td>$d_m$</td>
<td>4&quot;</td>
</tr>
<tr>
<td>Trim</td>
<td>0&quot;</td>
</tr>
<tr>
<td>L/B</td>
<td>10.78</td>
</tr>
<tr>
<td></td>
<td>Series A</td>
</tr>
<tr>
<td>-------------------</td>
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</tr>
<tr>
<td>$\omega_{zn}$</td>
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<tr>
<td>$\omega_{\theta n}$</td>
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<tr>
<td>$\omega_{\phi n}$</td>
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<td>$\omega_{\theta n}/\omega_{\phi n}$</td>
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<tr>
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<td></td>
<td>$\sigma_2$</td>
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<td>test 6</td>
<td>$\Omega$</td>
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<tr>
<td></td>
<td>$\sigma_2$</td>
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P1 & Hv Amp.s are enlarged by 2 & 10 to fit to scale of Roll Amp.
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Chapter VII

CONCLUSIONS AND RECOMMENDATIONS

7.1 Summary and Conclusions

To design more comfortable and safe vessels, one must understand the complicated dynamics of a vessel moving in a general environment. Included among the important dynamic parameters are the ratios of natural frequencies and the nonlinear interactions among the hydrostatic and hydrodynamic forces and moments. The goal of the present effort was to contribute some basic insight toward such an understanding.
A Theoretical Investigation of a Parametrically Excited Roll Motion

It has been demonstrated theoretically in Chapter II that a vessel in a head or following wave can spontaneously develop severe rolling motions. The input energy into the pitch and heave modes by the wave excitation may be transferred into the roll mode by means of a nonlinear coupling among these modes. To predict the roll motion, earlier investigators used a single-degree-of-freedom roll equation and neglected the sometimes pronounced effects of other modes due to nonlinear couplings. These coupling effects often take the form of a parametric resonance, which can lead to a particularly dangerous situation. The nonlinear roll equation was linearized by assuming harmonic pitch and heave motions, and the resulting Mathieu (or Hill) equation was used to determine the conditions for the stability of trivial solutions. With this procedure, however, the predicted roll angle grows exponentially with time, which is unrealistic.

To investigate the loss of dynamic stability and the development of large-amplitude rolling motions of a vessel, we began with a dynamic system of three degrees of freedom. Both the pitch and heave modes were used to determine the effective amplitude of the parametric excitation of the roll mode. In the equation for the roll, the kinematic-kinematic nonlinear coupling terms among the three modes were included as well as the static-static terms. Thus,
the present approach is closer to reality when the longitudinal asymmetry of a vessel with respect to the midship section is taken into account.

The case of principal parametric resonance was considered; the frequency of the wave excitation is approximately twice the natural frequency of the roll mode.

In the analysis, the equations for the heave and pitch modes were linearized, and their harmonic solutions were substituted into the equation for the roll mode. Due to the heave-pitch-roll coupling, the amplitudes and frequencies of the heave and pitch motions combine to play the role of an effective amplitude and frequency of the parametric excitation. The parametric term in the roll equation basically accounts for the time-dependent variation of the metacentric height. The resulting nonlinear ordinary-differential equation with time-varying coefficients governs the motion of the roll mode.

The method of multiple scales was used to determine a first-order approximation to the solution. From the solvability condition, two equations were obtained to describe the modulation of the amplitude and phase of the approximate solution. These modulation equations were used to determine the fixed-point solutions and their stability. These solutions were verified by numerically integrating the modulation equations using a 5th- and 6th-order Runge-Kutta-Verner scheme.

Supercritical- and subcritical-type force-response curves were obtained. The latter shows the coexistence of multiple stable solutions, a frequent feature of nonlinear dynamics. Jump phenomena were also found. In the

CONCLUSIONS AND RECOMMENDATIONS
region between the two jump-up and jump-down points, two different responses were possible depending on the initial conditions.

The present theoretical results of Chapter II are validated by the experiments presented in Chapter V.

7.1.2 A Theoretical Investigation of Coupled Pitch and Roll Motions in the Presence of Internal Resonance

It has been believed for a long time that the linear-plus-quadratic model could adequately describe the hydrodynamic damping of the roll motion. However, many investigators avoided using this model because of the difficulties in the analyses. In Chapter III, a quadratic nonlinear damping model is introduced into the equation of the roll mode. We investigated the non linearly coupled pitch and roll response of a vessel in regular waves when the natural frequency in pitch is twice that of roll (a condition of a two-to-one internal or autoparametric resonance). The method of multiple scales was used to derive four first-order autonomous ordinary-differential equations for the modulation of the amplitudes and phases of the pitch and roll modes when either mode is excited. The modulation equations were used to determine the influence of the quadratic nonlinear damping on the periodic responses and their stability.

When the encounter frequency is near the pitch natural frequency, the jump phenomenon exists for both zero and nonzero quadratic roll damping (
\( \mu_3 \) is the coefficient of the quadratic roll damping. The saturation phenomenon is broken if a quadratic roll damping term is introduced. This implies that there is no critical value of the excitation amplitude beyond which all of the extra energy input to the pitch mode is spilled over into the roll mode. The amplitude of the roll mode decreases while that of the pitch mode increases as the magnitude of the quadratic roll damping \( \mu_3 \) increases. In the subcritical case of force-response and frequency-response curves, the overhang regions narrow down as \( \mu_3 \) increases. The fixed points of the modulation equations undergo a Hopf bifurcation as one of the control parameters is varied. Between the Hopf bifurcation points, the response is an amplitude- and phase-modulated motion consisting of both the pitch and roll modes. The modulation may be periodic or chaotic. The Hopf bifurcation can be either subcritical or supercritical.

The bifurcated periodic solutions were found by a numerical algorithm and their stability were analyzed using Floquet theory. The limit cycles deform and lose stability by either pitchfork or period-doubling bifurcations as either the encounter frequency or the excitation amplitude is varied. The pitchfork bifurcation breaks the symmetry of the limit cycle. The period-doubling bifurcations culminate in chaos. We characterized the different possible solutions (limit cycles and chaotic attractors) using phase portraits, Poincaré sections, fast Fourier transforms, and time traces. The chaotic solutions exhibit very irregular behavior with broadband spectra.
When the encounter frequency is near the roll natural frequency, the amplitudes of both the roll mode and the pitch mode decrease as the magnitude of the quadratic roll damping coefficient $\mu_3$ increases. Again, Hopf bifurcations occur as either the encounter frequency or excitation amplitude is varied.

The present theoretical results of Chapter III are validated by the experiments presented in Chapter VI.

### 7.1.3 An Experimental Investigation of Coupled Heave, Pitch, and Roll Motions

To validate the theoretical predictions presented in Chapter II, we conducted experiments in the ship dynamics laboratory at VPI & SU and the results are presented in Chapter V. We investigated the dynamic stability and excessive rolling motions of a vessel in a head or following wave.

The model was placed longitudinally in the towing basin to eliminate the possibility of any external excitation in roll and hence to produce a pure parametric excitation by regular longitudinal plane waves. We used a tanker model which was constrained in surge, sway, and yaw, but allowed to heave, pitch, and roll freely. From the basic wave tests, it was decided to use wave frequencies around 0.60 Hz, which produced plane waves for the widest range of amplitudes. The model was ballasted so that the natural frequency in roll was about one-half the wave frequency to produce a principal parametric
resonance. The wave amplitude was slowly varied while the wave frequency was kept fixed.

A subcritical instability and its accompanying jump phenomena were found in the experiments. The observed amplitude of the roll motion was sometimes as large as \( \pm 20^\circ \). At times when the model was not exhibiting any noticeable roll motion, some external disturbances of the roll mode caused a jump up to a stable large-amplitude roll motion, implying that the roll response depends on the initial conditions. The large-amplitude roll motion was clear from observations, time traces, and spectra.

It was observed that the occurrence of large-amplitude roll motions depends on the location of the model along the waves. These results can address the cases of dynamic course instability, possible broaching, and instability of the motion of a vessel when it is moving in following waves with the same speed as the waves so that its encounter frequency is nearly zero. It was also noted that the large-amplitude roll motion occurred more easily when the center of gravity of the model was located at the front part of the standing wave; that is, between the node and antinode so that the model was positioned downhill in the waves.

In producing an effective amplitude of the parametric excitation of the roll mode, it was observed (i) from the experiments of varying the locations of the model on the waves that both the heave and pitch motions contributed approximately equally and (ii) from the experiments of varying the excitation frequencies that the pitch motion played a more dominant role than the heave.
Sometimes, the heave motion seemed to have little influence on the parametric excitation of the roll mode. Hence, we can conclude from these two results that the pitch mode plays a more significant role than the heave in exciting the roll mode. In the real situation, the vessel will necessarily experience pitch motion and thus the pitch mode as well as the heave mode should be included in investigating the parametric resonance of the roll mode.

The penetration phenomenon associated with nonstationarity was also observed. The penetration of trivial roll motions into the regime where trivial motions are unstable is due to the nonstationarity of the wave excitation.

It was noted that as the wave amplitude was increased, the amplitude of the large roll motion decreased and even disappeared in some cases. Moreover, we were unable to produce a supercritical-type instability in spite of the extensive efforts devoted to this case. Except for these, we observed good qualitative agreement between the theoretical and experimental results.

However, there also exist several discrepancies which lie beyond the controllability of the experiments. The discrepancies are conjectured to be due to one or more of the following three reasons: the difference between the hydrodynamic and hydrostatic characteristics in and out of waves, the unobserved, but possible waves reflected from the sidewalls of the basin, and the role of interacting phases between the pitch and heave modes in yielding the effective parametric excitation of the roll mode.
An Experimental Investigation of Coupled Heave, Pitch, and Roll Motions in the Presence of Internal Resonance

It was demonstrated experimentally using a destroyer model that a vessel in head or following waves can possess severe roll motions. The energy directly input to the pitch mode by the excitation wave can be transferred to the roll mode by means of nonlinear couplings between these modes. Strong interactions between the pitch and roll modes may occur when their natural frequencies are approximately two-to-one (i.e., two-to-one internal or autoparametric resonance).

This study validated the theoretical results of Chapter III. The loss of dynamic stability and the possibility of large-amplitude roll motions were investigated. While the mathematical model used in Chapter III was for a system of two degrees of freedom allowing the pitch and roll only, it was decided to include the heave mode together with the pitch and roll modes in the experiments to make the motion of the model more natural. The present experimental results were compared with the theoretical results of Chapter III; they were in good qualitative agreement.

The model was placed longitudinally in the towing basin of VPI & SU. It was constrained in surge, sway, and yaw but allowed to heave, pitch, and roll freely. Its natural frequencies of pitch and roll were adjusted to be in the ratio of nearly two-to-one. Two series of tests were conducted: Series A with relatively large detuning between the natural frequencies of the pitch and roll
modes; and Series B with a very small detuning between them, close to perfect tuning. The longitudinal waves were generated with frequencies near the pitch natural frequency. The wave amplitudes were slowly varied while the wave frequency was held constant.

Force-response curves of the subcritical type were obtained. The subcritical instability and coexistence of large-amplitude and no-roll motions and associated jump phenomena were observed. The large-amplitude roll motion developed through a spontaneous jump up in general. When there was no spontaneous jump, it was obtained by frequently imposing various external disturbances on the model. The amplitude of the large roll motion was sometimes as large as $\pm 18^\circ$. The large-amplitude roll motion was evident by observations, continuous records of analog signals, time traces, and spectra.

The pitch mode exhibited neither linear nor saturated responses, but nonlinear or sometimes nearly saturated behaviors. The heave responses exhibited linear behaviors in general. These results are consistent with the theoretical predictions of Chapter III in which the effect of quadratic roll damping was taken into account. The responses were sometimes conjectured to occur due to only the principal parametric resonance and the influence of the two-to-one internal resonance was not significant.

It was noted that as the excitation amplitude increased, the amplitude of the large roll motion decreased and even disappeared very often. It was proposed to regard this phenomenon as a kind of supercritical-type bifurcation. This needs further investigations in the future. Since the natural
frequencies of Series B were close to perfect tuning, it was expected to obtain the large-amplitude roll motion more easily. However, the stable large-amplitude roll motion was harder to obtain than in Series A. Finally, the supercritical-type instability was not observed in spite of the much effort devoted to this case. Except for these, the present experimental results are in qualitative agreement with the theoretical results described in Chapter III.

7.2 Contributions

The modern theory of nonlinear dynamics has been applied primarily to mechanical, structural, and electrical systems. This is due to the fact that problems in these disciplines are relatively easy to simplify, to model in a suitable physical scale, and hence to validate the results quickly. However, the number of other applications of nonlinear dynamics is immense. The unification of the treatment for phenomena originating from diverse fields, such as physics, chemistry, astronomy, meteorology, biology, and economics not only provides common tools of analysis, but also reveals the presence of universal features shared by these systems. Although their apparent shapes are incomparably unique, the imbedding principle may be the same by the laws of nature.

The present study is a modest contribution to the nonlinear analysis of ship dynamics problems and to the experimental verification of some of these
solutions. The present work, especially the experiments, demonstrates that the modern theory of nonlinear dynamics can be used to analyze the complicated behaviors of floating systems. Furthermore, the experiments are a benchmark work in this discipline because of their pioneering nature. We provide the first laboratory demonstration since W. Froude in 1863. They will serve as a basis for understanding some nonlinear phenomena.

Finally, this work also qualitatively validates many of the previously published theoretical predictions of the responses of floating vessels. These results may form the basis for developing effective motion stabilizers and methods for controlling undesirable motions. This work will hopefully contribute to the design of floating structures that possess favorable dynamic characteristics and to the determination of conditions under which floating structures can be operated safely.

7.3 Concluding Remarks

It can be pointed out that the conventional approach to the study of the dynamic characteristics of floating systems involves an enormous amount of time and effort in evaluating the hydrodynamic effects and yet does not yield good results when nonlinear effects are significant. The present analytical approach adopts the modern theory of nonlinear dynamics used in mechanical and structural systems, such as beams, arches, and shells. The idea is the
modal interaction via nonlinear couplings between the modes involved and hence is quite a new approach in the study of the response of floating systems. We note that, in the present work, very reasonable analytical results are obtained efficiently by using a perturbation method to solve a system of nonlinear equations in spite of the simplifying assumptions. They are qualitatively in good agreement with experimental results. We also note that the potentially dangerous conditions and phenomena are explained clearly by the analyses and demonstrated impressively by the experiments.

The present results can be used to change the operating procedures of floating systems according to the frequency components of waves obtained from the wave climatology along the route or the area of operation. Thus to assure more favorable and safer dynamic characteristics of floating systems, it is necessary not only to have a design concept that avoids resonances, but also to regulate the distribution of loading so as to avoid any potentially dangerous conditions, such as hydrostatic instability or certain ratios between natural frequencies of various modes of motion. As a result, a measure of confidence or a clear warning regarding the safety of floating systems can be established; it can be utilized even during the conceptual and preliminary stages of the actual design. The present work is an effort to establish a basis for such a long-range goal by investigating frequently occurring phenomena which are potentially dangerous.
7.4 Recommendations for Future Research

There are still some open problems in the dynamics of floating bodies. With few exceptions, all linear problems are nearly solved. Clearly, future research into the dynamics of a floating structure lies in the nonlinear area. The nonlinear theory of motion plays a significant part in roll analyses, particularly when determining the safety of a floating structure with regard to capsizing, which essentially involves large-amplitude responses. Experimental methods for studying the motion of floating systems as well as methods of processing the obtained data need to be improved. Specifically, these include the introduction of newly developed measurement methods like ultrasonic testing, laser Doppler velocimetry, or other optical systems. Finally, the application of motion stabilizers appears to be the most effective method of reducing motions and thus it needs further theoretical and experimental investigation. We note that the more comprehensive the studied motion is, the more complex the theory becomes. As in many engineering problems, however, approximate solutions can be obtained while avoiding more complex theoretical methods.

By extending the present methods and results, further endeavors will be made toward the current trend of practical research: first, improved criteria for the operability of a floating system, such as capability functions which express human and equipment operability during various activities; second, guidelines such as scalar indices to be used by designers in the conceptual
and preliminary design stages together with regression models for resistance and powering; and third, optimal actions, such as avoidance and change of speed and heading due to the encountered circumstances and resulting responses of the vessel. These will come true when we know the short-term and long-term statistics, and the wave climatology distributed along the route or the area of operation. Then, we will be able to develop a design of favorable dynamic characteristics.

The present study demonstrates theoretically and experimentally that complicated vessel motions may occur in regular waves due to nonlinear interactions among the in-plane (heave and pitch) and out-of-plane (roll) modes of motion. Although we have made a start in the investigation of these complicated behaviors, more work lies ahead. We conclude this dissertation by suggesting some future endeavors. Specifically, the following topics are directly related to the present study:

1. Investigate the nonlinear responses of vessels to random excitations or irregular waves.

2. Investigate motion-control problems, including modal interactions, because they are very practical and useful in determining the safety of a vessel.

3. Develop a motion guidance system that allows more degrees of freedom.
4. Investigate the influence of quadratic roll damping experimentally. This can be accomplished by using different sizes of appendages.

5. Investigate the influence of different loading conditions. Having a shallow draft can lead a floating structure to capsize.

6. Apply the developed theory and obtained results to other types of floating structures having a variety of different designs.
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IL GEUN OH was born in Jinhae city, Kyungnam, Korea on April 27, 1958 as the first son of a sincere engineer, Chul Soo Oh, and Chan Ja Chung, the first grandson of a businessman-landowner, Jin Sun Oh, and the first grandson-in-law of a patriot, Woo Chae Chung, who devoted himself to the independence of the country and was honored by the Republic of Korea.

He spent his beautiful boyhood as a rural boy in Naju, Junnam, running and jumping in the forests and fields. At age 10, he left home by himself for the capital of the province in pursuit of a better education. Through the schooling years in Kwangju Jeil highschool, he learned the honor, justice, and patriotism which guided his later life.

After graduating from Seoul National University with majoring in naval architecture and ocean engineering in 1981, where he learned what a gentleman and intelligentsia should be, he joined the graduate school of the same University. Shortly after that, he served in the military for a year and half, working in the division of the staffs of strategies and operations at the headquarters of the Korean Navy, and translated two history books written by the Department of the Navy, U.S.A.

He obtained a master of science degree in 1984 and worked in the Korea Shipbuilding and Engineering Corporation in Seoul until he came to the
Virginia Polytechnic Institute and State University in 1986. He joined the Nonlinear Dynamics Group of the Department of Engineering Science and Mechanics where he was faced with many exciting new concepts and ideas. After obtaining his Ph. D. degree, he hopes to find a way to contribute to his country and people and he is working towards this goal.