Analysis of three-dimensional field distributions for focussed unapodized/apodized annular beams

by

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Abstract

The study of focal shift in focussed beams using unapodized apertures has
been well documented. However, not much work has been done on
apodized apertures. In this thesis we use a Fourier-Optic approach to
analyze the field distribution of a focussed beam around the region of
geometrical focus. The analytical formulation developed is general in
nature as it is valid for any arbitrary aperture functions. This is then
applied to some specific cases. Two cases of interest that are considered are
the unapodized and the Gaussian apodized annular apertures. In order to
study the intensity distributions around the geometrical focus, simulation
results are presented using closed form analytical expressions and
approximate integral forms. Specific emphasis is placed on the focal shift
in the two apertures and on the effect of changing various parameters. A
prognosis for future work using θ-Modulation on Gaussian apodized
annular apertures is also presented.
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1.0 Introduction

When a converging spherical wave is diffracted by a circular aperture, the standard theory of the focussing of light predicts that the intensity distribution in the focal region will be symmetrical about the focal plane [1]. However, some asymmetry was observed in the microwave regime [2-4] and later in investigations relating to focussed laser beams [5-6]. These discrepancies led to recent interest in re-examining the diffracted optical field in the region of focus [7-9]. One common conclusion in these investigations [2-9] is that the principal maximum of the intensity of the diffracted wave is not at the geometrical focus but located closer to the aperture. This effect is what has been referred to as the focal shift, and the discrepancy is there because the theory includes an inherent assumption that the Fresnel number of a system is large compared to unity [8].

Anticipating a higher concentration of energy for a Gaussian wave as compared to the plane wave, it is important to know the point at which maximum power is received in a finite receiver area. Also if given the
problem of illuminating a target with a laser beam, maximum illumination is obtained if the target is at the point of maximum concentration of laser power. Thus, in order to receive maximum possible power concentration on a moving target, the beam must be actively adjusted on it. However, for a distant target, if the Fresnel number of the beam aperture as observed from the target is small, active focusing may not be necessary. Over a considerable range of the target distance, the concentration of power on it may be more than adequate. From numerical analysis, the Fresnel number and other parameters to obtain a wide range in which the power concentration is above a certain minimum value can be determined [10].

If a central stop is introduced into the pupil of a lens, we produce a lens with an annular aperture. The use of annular apertures in optical systems has been known to produce a greater depth of focus.

Focal shift has been observed in the case of rectangular [5,9] and annular apertures [20]. When a monochromatic gaussian beam is focused by a thin lens, there exists a focal shift, which has been discussed for a circular [15-16], and rectangular apertures [21-23]. Recently some publications show the existence of focal shift in the case of truncated Gaussian beam through an annular aperture [21]. In this thesis, we concentrate on the passage of a plane wave through a Gaussian apodized annular aperture. Fig 1 illustrates the system under consideration. A plane wave is incident on the aperture-lens combination. The lens of focal length f is assumed to be thin.
F is the geometrical focal point of the lens at a distance f from the lens. P is the point of interest and is at a distance z from the focal point and at \( l = f + z \) away from the lens. \((x,y)\) and \((x_i,y_i)\) represent the coordinates of the aperture and the observation planes, respectively.

Chapter 2 shows the basic Fourier-Optic approach used to analyze the problem. Chapter 3 concentrates on calculating the field in the case of unapodized and Gaussian apodized annular apertures. Some curves have been plotted to compare the analytical and the numerical results.

After establishing the accuracy of the analytical results, chapter 4 shows the curves plotted using the analytical solution. Comparisons between the unapodized and Gaussian apodized annular apertures are made under a variety of conditions.

Chapter 5 presents conclusions and covers the future work that can be done in this area. Some preliminary results are presented in the case of '0'-modulation with the Gaussian apodized aperture.
2.0 Formalation

This chapter begins with an introduction of the parameters used and then goes on to show the basic Fourier-Optic technique used to solve the problem. It then arrives at a general equation which can be further solved for some particular cases (annular, circular, etc).

2.1 Parameters

The parameter ‘σ’ is the distance at which the amplitude is 1/e times the on-axis amplitude $E_o$ for a Gaussian beam [see Appendix A]. The parameter ‘σ’ is often called the beam radius or the “spot size”, and ‘2σ’ is the beam diameter.

‘N’, the Fresnel number, is defined as being equal to the number of Fresnel zones of the geometrically predicted wave at the horizontal plane, of the lens, perpendicular to the direction of propagation. Since the lens under consideration is assumed to be thin, the phase shift accrued as the wave travels from the midplane of the lens on to the front half of the lens, is not taken into consideration. $N$ is $r_0^2/\lambda f$, where $r_0$ is the lens aperture size [see Appendix B], $f$ is the focal length of the lens and $\lambda$ is the wavelength of light.
2.2 Analysis

Taking the system shown in Fig 1.0 into consideration, the amplitude distribution before the lens is given by [26]

\[ U_l(x,y)P(x,y), \]

where \( U_l(x,y) \) is the incident field and \( P(x,y) \) is the aperture function.

The amplitude distribution after the lens is given by

\[ U_l'(x,y)=U_l(x,y)P(x,y) \exp\left(-\frac{ik(x^2+y^2)}{2f}\right), \]

(2.2.1)

where \( k \) is the wave number of light in free space.

The convolution of the above amplitude (field) distribution with the free space impulse response \( h_l(x,y; l) \) gives the field \( U_{lP}(x_i,y_i; l) \) at point 'P', at a distance of 'l' away from the lens [20]:

\[ U_{lP}(x_i,y_i; l) = U_l'(x,y) \ast h_l(x,y; l) \]  \hspace{1cm} (2.2.2)

Here "\( \ast \)" denotes convolution operation and \( h_l(x,y; l) \) is given by [20]
$$h_l(x,y; l) = (-i)^l \exp(ikl) \exp\left(\frac{ik(x^2+y^2)}{2l}\right).$$  \hspace{1cm} (2.2.3)$$

Explicitly, (2.2.2) gives

$$U_{lp}(x_i, y_i; l) = \int \int U_l(x,y) \left(\frac{-1}{i\lambda l}\right) \exp(ikl) \exp\left(\frac{ik((x_i-x)^2+(y_i-y)^2)}{2l}\right) \, dx \, dy,$$

\hspace{1cm} (2.2.4)

and by squaring the exponent and re-arranging the similar terms together, we get

$$U_{lp}(x_i, y_i; l) = (-i)^l \exp(ikl) \int \int U_l(x,y) \exp\left(\frac{ik[(x_i^2+y_i^2)+(x^2+y^2)-(2xx_i+yy_i)]}{2l}\right) \, dx \, dy.$$

\hspace{1cm} (2.2.5)

Taking the terms which are constant with respect to the integration outside of the integral, we have
\[ U_{lP}(x_i, y_i; l) = \left( \frac{1}{i\lambda L} \right) \exp(ikl) \exp\left(\frac{-ik(x_i^2 + y_i^2)}{2l}\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_l(x, y) \exp\left(\frac{-ik[(x^2 + y^2) - 2(x x_i + y y_i)]}{2l}\right) \, dx \, dy \]  

(2.2.6)

Substituting (2.2.1) into (2.2.6), we get

\[ U_{lP}(x_i, y_i; l) = \left( \frac{1}{i\lambda L} \right) \exp(ikl) \exp\left(\frac{-ik(x_i^2 + y_i^2)}{2l}\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_l(x, y) \exp\left(\frac{-ik(x^2 + y^2)}{2\lambda f}\right) \exp\left(\frac{-ik[(x^2 + y^2) - 2(x x_i + y y_i)]}{2l}\right) \, dx \, dy . \]  

(2.2.7)

Putting similar terms together and on re-arranging, (2.2.7) becomes

\[ U_{lP}(x_i, y_i; l) = \left( \frac{1}{i\lambda L} \right) \exp(ikl) \exp\left(\frac{-ik(x_i^2 + y_i^2)}{2l}\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} U_l(x, y) \exp\left(\frac{-ik(1 - \frac{1}{L})(x^2 + y^2)}{2f}\right) \exp\left(\frac{-ik(x x_i + y y_i)}{l}\right) \, dx \, dy . \]  

(2.2.8)

Defining the Fourier transform as [26]
\begin{equation}
F\{f(x,y)\} = F(u,v) = \iint f(x,y) \exp(-i2\pi(ux+vy)) \, dx \, dy \tag{2.2.9}
\end{equation}

we can see a certain similarity between (2.2.8) and (2.2.9). Since the integral can be expressed as a Fourier transform, for simplifying the analysis, \(U_{I\,P}(x_i,y_i; \, l)\) can be written as, using (2.2.9),

\begin{align*}
U_{I\,P}(x_i,y_i; \, l) = & \frac{1}{i\lambda l} \exp(ikl) \exp\left(\frac{ik(x_i^2+y_i^2)}{2l}\right) \\
& \left\{ \mathcal{F}\left[U_I(x,y)P(x,y) \exp\left(\frac{ik[\frac{1}{l} - \frac{1}{f}]x^2 + y^2\right)\right] \right\} \bigg|_{u=x_i/\lambda l, \, v=y_i/\lambda l} . \tag{2.2.10}
\end{align*}

For circular symmetry, as is the case in our problem, \(U_{I\,P}(x_i,y_i; \, l)\) can be expressed in terms of Fourier Bessel transforms [28]:

\begin{align*}
U_{I\,P}(r^2; \, l) = & \frac{1}{i\lambda l} \exp(ikl) \exp\left(\frac{ikr^2}{2l}\right) B\left\{ U_I(r)P(r) \exp\left(\frac{ik[\frac{1}{l} - \frac{1}{f}]r^2\right)\right\} \bigg|_{\rho^2=u^2+v^2} , \tag{2.2.11}
\end{align*}

with \(r_i^2 = x_i^2 + y_i^2, \, r^2 = x^2 + y^2\),

where the Fourier Bessel Transform is defined as [26]

\begin{equation*}
B\{g(r)\} = G(\rho) = 2\pi \int_0^\infty r \, g(r) \, J_0(2\pi\rho r) \, dr .
\end{equation*}
We have arrived at the general equation (2.2.11) to calculate the field at a point P, at a distance \( l \) from the lens. By changing the functional forms of \( U_l(r) P(r) \), we can calculate the intensity for different apertures (circular, annular, etc).

In Chapter 3 we specifically solve the problem for unapodized and apodized Gaussian annular apertures using the general equation (2.2.11) derived in this chapter.
3.0 Analysis

In section 3.1 we calculate the field and the intensity distributions along z for the annular aperture, and in section 3.2 we do the same for a Gaussian apodized annular aperture. Fig 2.0 gives the figure for an annular aperture, showing that 'b' and 'a' are the outer and the inner diameters of the aperture, respectively. We assume that A is the amplitude of the incoming plane wave.

3.1 Unapodized Annular Aperture

The equation for an annular aperture is given by [20]

\[ P(x,y) = P(r) = \frac{\text{Circ}(L)}{R} - \frac{\text{Circ}(L_\eta)}{\eta R}, \]

(3.1.1)

where
3.0 Analysis

In section 3.1 we calculate the field and the intensity distributions along z for the an annular aperture, and in section 3.2 we do the same for a Gaussian apodized annular aperture. Fig 2.0 gives the figure for an annular aperture, showing that 'b' and 'a' are the outer and the inner diameters of the aperture, respectively. We assume that A is the amplitude of the incoming plane wave.

3.1 Unapodized Annular Aperture

The equation for an annular aperture is given by [20]

\[ P(x,y) = P(r) = \text{Circ}(\frac{r}{R}) - \text{Circ}(\frac{r}{\eta R}) \]  

(3.1.1)

where
\[
\text{Circ} \left( \frac{L}{R} \right) = 1 \text{ for } r < R, \text{ else } = 0,
\]
\[
\text{Circ} \left( \frac{L}{\eta R} \right) = 1 \text{ for } r < \eta R, \text{ else } = 0,
\]
(3.1.2)

and \( \eta \) is some constant smaller than unity.

If \( U_t(r) = A \), i.e., the amplitude of the incoming wave is \( A \), we substitute (3.1.1) in (2.2.11) to get the equation for the field distribution at a point \( P \) of an unapodized annular aperture:

\[
U_t (r; l) = \frac{1}{i \lambda l} \exp \left( ikl \right) \exp \left( \frac{ikr^2}{2l} \right) B \left\{ A P(r) \exp \left( \frac{-ik}{2f} \left| \frac{1}{f} \right|^2 \right) \right\}_{\rho^2 = u^2 + v^2 = \frac{l^2}{\lambda^2}}.
\]
(3.1.3)

On expanding the Fourier Bessel transform and expressing it in terms of an integral over \( 'r' \), we have

\[
U_t \left( r; l \right) = \frac{1}{i \lambda l} \exp \left( ikl \right) \exp \left( \frac{ikr^2}{2l} \right) 2\pi \int_0^\infty \left[ r A P(r) \exp \left( \frac{-ik}{2f} \left| \frac{1}{f} \right|^2 \right) J_0 \left( \frac{2\pi r m}{\lambda l} \right) \right] dr.
\]
(3.1.4)

Putting the functional form for \( P(r) \) given by (3.1.4) and modifying the limits of the integral according to (3.1.2), we obtain
\[ U_{l} P(r; l) = \frac{1}{i\lambda l} \exp(ikl) \exp\left(\frac{ikr^2}{2\lambda l}\right) 2\pi \int_{\eta R}^{R} r A \exp\left(-\frac{1}{2} \frac{1}{l} R^2 r^2\right) J_0\left(\frac{2\pi R r l}{\lambda l}\right) dr \]

(3.1.5)

where \( \eta \) is the obscuration ratio defined as

\[ \eta = \frac{a}{R}, \quad R = b \]

Changing the limits in (3.1.5) and expressing them in terms of \( \eta \) and correspondingly making the adjustment for \( r \), we get

\[ U_{l} P(r; l) = \frac{1}{i\lambda l} \exp(ikl) \exp\left(\frac{ikr^2}{2\lambda l}\right) 2\pi R^2 \int_{\eta}^{1} r A \exp\left(-\frac{1}{2} \frac{1}{l} R^2 r^2\right) J_0\left(\frac{2\pi R r l}{\lambda l}\right) dr \]

(3.1.6)

The intensity distribution is then given by, \( I(r; l) = U_{l} P(r; l) \bar{U}_{l} P^{*}(r; l) \), i.e,

\[ I(r; l) = -\frac{1}{\lambda^2 l^2} A^2 4\pi^2 R^4 \left| \int_{\eta}^{1} r \exp\left(-\frac{1}{2} \frac{1}{l} R^2 r^2\right) J_0\left(\frac{2\pi R r l}{\lambda l}\right) dr \right|^2 \]

(3.1.7)

Since the region under consideration is around the geometrical focus for ease of calculation, \(' l ' \) is replaced by \(' z+f ' \) in (3.1.7), we then have
I(r_i, l) = \frac{1}{\lambda^2(z+f)^2} \cdot A^24\pi^2R^4 \left| \int_{\eta}^r \exp\left(-\frac{ik}{2}\left[\frac{1}{f} + \frac{1}{z+f}\right]R^2\tau^2\right) J_0\left(\frac{2\pi R \eta}{\lambda(z+f)}\right) d\tau \right|^2. \quad (3.1.8)

By putting \(r_i = 0\) and defining

\[ I_0 = \left(\frac{\pi R^2A}{\lambda f}\right)^2, \quad N (\text{Fresnel number}) = \frac{R^2}{\lambda f}, \quad \text{and} \quad \Delta z = \frac{Z}{f}, \quad (3.1.9) \]

the intensity distribution along the z axis is then given by [20]

\[ I(z, \eta) = I_0 \left(1 - \frac{\eta^2}{1 + \Delta z}\right)^2 \text{sinc}\left(\frac{\pi \Delta z N}{2(1+\Delta z)}(1-\eta^2)\right). \quad (3.1.10) \]

Note that there was a factor of two in the sinc function which was found to be present that had been overlooked in [20].

The intensity for this annular aperture is plotted in [Figs 3 - 5]. The graphs are plotted for different values of the Fresnel number and show the variations in the intensity with changing value of the opening, i.e., \(\eta\).

Figure 3 is plotted for \(N=5.0\), and the three plots represent the on-axis intensity along the z-axis. The intensity has been normalized with respect to the intensity at the focal point [\(I/I(r_i=0=\Delta Z)\)], for a circular aperture. Figure 4 is plotted correspondingly for \(N=0.5\) and Fig 5 for \(N=50.0\). The
three different values of obscuration ratio chosen were 0.0 (circular), 0.5, 0.8 and these results are consistent with those of Poon [20].

These curves clearly indicate that the point of maximum intensity is not at the geometrical focus but at a point closer to the lens, for systems with decreasing values of N. The shift towards the lens increases as we decrease the values of N and increase η. Also, for a given value of N the shift increases with increasing the values of the obscuration ratio. The increase in the value of 'η' signifies the decrease in the opening width of the annular aperture.

As we can see from the plots, when N is large as compared to unity the intensity distribution of light in the focal region is symmetrical about the focal plane and is concentrated at the focus, as the conventional theory of the focusing of light predicts, but at smaller values of N there is a marked deviation from the standard theory and the point of maximum intensity shifts towards the lens.
Unapodized Aperture

$N=5.0, \eta=0.0$

Varying $\eta$

\[ \eta=0.0 \quad \eta=0.5 \quad \eta=0.8 \]

Fig 3
Unapodized Aperture
N=0.5, \eta=0.0
Varying \eta

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure4}
\caption{Fig 4}
\end{figure}
Unapodized Aperture
N=50, \eta=0.0
Varying \eta

\[ \begin{align*}
\Delta z \\
\eta=0.0 & \quad \eta=0.5 & \quad \eta=0.8
\end{align*} \]

Fig 5
3.2 Gaussian Apodized Aperture

Defining \( U_f(x,y) \) for a Gaussian apodized annular aperture as

\[
U_f(x,y) = \exp\left(-\frac{(r-R_0)^2}{2\sigma^2}\right) \tag{3.2.1}
\]

and assuming unit amplitude light incidence, the equation for the field at a point \( P \), according to (2.2.11), will be modified as

\[
U_{fP}(r; I) = \frac{1}{i\lambda} \exp(ikl) \exp\left(\frac{ikr^2}{2l}\right) B \left\{ \exp\left(-\frac{(r-R_0)^2}{2\sigma^2}\right) \exp\left(\frac{ik}{2l}I - \frac{1}{2r^2}\right) \right\} \rho^2 = \frac{x^2 + y^2 - r^2}{\lambda} \tag{3.2.2}
\]

On expanding the Bessel Gaussian transformation and expressing it in terms of the integral over 'r', we get
$$U_1 p(r; l) = \frac{1}{i\lambda l} \exp(ikl) \exp\left(\frac{ikr^2}{2l}\right) 2\pi \int_0^r \exp\left(-\frac{(r-r_0)^2}{2\sigma^2}\right) \exp\left(\frac{ik\left[\frac{1}{l} - \frac{1}{f}\right]r^2}{2\sigma^2}\right) J_0\left(\frac{2\pi r l}{\lambda f}\right) dr$$.

(3.2.3)

Eqn (3.2.3) can be simulated numerically using the IMSL routines on the IBM 3090. But due to system constraints and the approximate nature of the calculation, it does not give a completely accurate result. In what follows we will solve (3.2.3) analytically.

Since the region of interest is around the geometrical focus, we replace 'l' by 'f + z' in (3.2.3), giving

$$U_1 p(r; f+z) = \frac{1}{i\lambda(f+z)} \exp(ik(f+z)) \exp\left(\frac{ikr^2}{2(f+z)}\right) 2\pi \int_0^r \exp\left(-\frac{(r-r_0)^2}{2\sigma^2}\right) \exp\left(\frac{ik\left[\frac{1}{f} - \frac{1}{f+z}\right]r^2}{2\sigma^2}\right) J_0\left(\frac{2\pi r f}{\lambda(f+z)}\right) dr$$.

(3.2.4)

We first evaluate the integral in (3.2.4), i.e.,

$$\int_0^r \exp\left(-\frac{(r-r_0)^2}{2\sigma^2}\right) \exp\left(\frac{ik\left[\frac{1}{f} - \frac{1}{f+z}\right]r^2}{2\sigma^2}\right) J_0\left(\frac{2\pi r f}{\lambda(f+z)}\right) dr$$.

(3.2.5)

Crossmultiplying and putting

20
\[
\left[ \frac{1}{(f+z) \cdot f} \right] = \frac{z}{f(f+z)} \quad ,
\]

(3.2.5) becomes
\[
\int_0^\infty r \exp\left(\frac{-(r-r_0)^2}{2\sigma^2}\right) \exp\left(\frac{ik}{2f(f+z)} r^2\right) J_0\left(\frac{2\pi r f}{\lambda(f+z)}\right) dr \quad .
\]  

(3.2.6)

Defining
\[N_G = \frac{r_0^2}{\lambda f} \quad , \quad \Delta z = \frac{z}{f} \quad , \quad \text{and putting } k = \frac{2\pi}{\lambda} \quad ,
\]

We get, on rearranging the terms and taking 'r' outside the parentheses:
\[
\int_0^\infty r \exp\left(\frac{-(r-r_0)^2}{2\sigma^2}\right) \exp\left(\frac{i\pi}{\lambda f^2} \left[\frac{z}{(1+\Delta z)}\right] r^2\right) J_0\left(\frac{2\pi r f N_G}{\lambda f(1+\Delta z)}\right) dr \quad ,
\]

(3.2.7)

where \(N_G\) represents the Fresnel number associated with the Gaussian apodized annulus. (3.2.7) can then be written as
\[
\int_0^\infty r \exp\left(\frac{-(r-r_0)^2}{2\sigma^2}\right) \exp\left(\frac{i\pi}{r_0^2(1+\Delta z)} N_G\right) J_0\left(\frac{2\pi r f N_G}{r_0^2(1+\Delta z)}\right) dr \quad ,
\]

(3.2.8)

and on squaring the exponent, we get
\[
\int_0^r \left[ r \exp\left(-\frac{(r^2+r_0^2-2rr_0)}{2\sigma^2}\right) \exp\left(\frac{\imath \pi N_G \Delta z}{r_0^2(1+\Delta z)}\right) \right] r_0 \left( \frac{2\pi r_0 N_G}{r_0^2(1+\Delta z)} \right) \, dr \qquad .
\] (3.2.9)

Combining the coefficients of the common term in the exponential together, we get

\[
\exp\left(-\frac{r_0^2}{2\sigma^2}\right) \int_0^r \left\{ r \exp\left(\frac{r^2+r_0^2-2rr_0}{2\sigma^2} + \frac{\imath \pi N_G \Delta z}{r_0^2(1+\Delta z)}\right) \right\} \exp\left(\frac{r_0 \theta}{\sigma^2}\right) J_0\left(\frac{2\pi r_0 N_G}{r_0^2(1+\Delta z)}\right) \, dr \qquad .
\] (3.2.10)

For ease of analysis we, define the following parameters:

\[
\tilde{\mu} = \left(1 - \frac{\imath \pi N_G \Delta z}{2\sigma^2(1+\Delta z)}\right), \quad \nu = \left(\frac{r_0}{\sigma^2}\right) \quad \text{and} \quad \rho = \left(\frac{2\pi r_0 N_G}{r_0^2(1+\Delta z)}\right)
\]

in order to re-express the integral in a simple form. We have, from (3.2.10),

\[
\exp\left(-\frac{r_0^2}{2\sigma^2}\right) \int_0^r \left\{ r \exp(-\tilde{\mu}r^2) \exp(2\nu r) J_0(\rho r) \right\} \, dr \qquad .
\] (3.2.11)

The different exponentials can be grouped together to give a single exponential term.

Combining \(\exp(-\tilde{\mu}r^2)\exp(2\nu r)\) to form a complete square, we get
\[ \exp\left(-\frac{\pi^2}{2\sigma^2}\right) \exp\left(\frac{\nu^2}{\mu}\right) \int_0^\infty r \exp\left(\frac{\mu}{\mu} - \frac{\nu^2}{\mu}\right) J_0(pr) \, dr \quad \ldots \quad (3.2.12) \]

Now, the general form for a Hankel Transform of order '0' is given by [27]

\[ \mathcal{H}\{g(r)\} = \int_0^\infty r \ g(r) \ J_0(pr) \, dr \quad \ldots \quad (3.2.13) \]

Using (3.2.13) in (3.2.12), we get

\[ \exp\left(-\frac{\pi^2}{2\sigma^2}\right) \exp\left(\frac{\nu^2}{\mu}\right) \mathcal{H}\left\{ \exp\left(\frac{\mu}{\mu} - \frac{\nu^2}{\mu}\right) \right\} \quad \ldots \quad (3.2.14) \]

The term \( \exp\left(\frac{\mu}{\mu} - \frac{\nu^2}{\mu}\right) \) can be expressed as a convolution between the exponential term and a delta function, namely,

\[ \exp\left(\frac{\mu}{\mu} - \frac{\nu^2}{\mu}\right) = \exp(-\mu r^2) * \delta\left(\frac{r - \nu}{\mu}\right) \quad \ldots \quad (3.2.15) \]

In writing (3.2.15), since the phase shift is small, we have assumed \( \mu = |\mu| \).

Now, putting (3.2.15) in (3.2.14) and expressing it as a Hankel Transform of the function, we have
$$\exp\left(\frac{-\mathbf{r}_0^2}{2\sigma^2}\right) \exp\left(\frac{\mathbf{r}^2}{\mu}\right) \mathcal{H}\left\{\exp(-\mu r^2) \ast \delta\left(r - \frac{\mathbf{r}}{\mu}\right)\right\} \quad . \tag{3.2.16}$$

Since the Hankel Transform of a convolution is the product of the Hankel Transform of the individual terms, (3.2.16) can be further written as

$$\exp\left(\frac{-\mathbf{r}_0^2}{2\sigma^2}\right) \exp\left(\frac{\mathbf{r}^2}{\mu}\right) \mathcal{H}\left\{\exp(-\mu r^2)\right\} \mathcal{H}\left\{\delta\left(r - \frac{\mathbf{r}}{\mu}\right)\right\} \quad . \tag{3.2.17}$$

From the tables in [26], we get the transforms of each term

$$\mathcal{H}\left\{\exp(-\mu r^2)\right\} = \frac{1}{2\mu} \exp\left(-\frac{\mathbf{r}^2}{4\mu}\right) \quad \text{and} \quad \mathcal{H}\left\{\delta\left(r - \frac{\mathbf{r}}{\mu}\right)\right\} = \frac{\mathbf{r}}{\mu} J_0\left(\frac{\mathbf{r}2\pi\rho}{\mu}\right) \quad . \tag{3.2.18}$$

Putting (3.2.18) in (3.2.17), we then have

$$\exp\left(\frac{-\mathbf{r}_0^2}{2\sigma^2}\right) \exp\left(\frac{\mathbf{r}^2}{\mu}\right) \frac{1}{2\mu} \exp\left(-\frac{\mathbf{r}^2}{4\mu}\right) \frac{\mathbf{r}}{\mu} J_0\left(\frac{\mathbf{r}2\pi\rho}{\mu}\right) \quad . \tag{3.2.19}$$

Finally putting (3.2.19) in (3.2.4) we get the equation of the field at a point $P$ at distance $l$ from the lens.
\[
U_{f,p}(r; f+z) = \frac{1}{i\lambda(f+z)} \exp(ik(f+z)) \exp\left(\frac{ikr^2}{2(f+z)}\right) \frac{1}{2\mu} \exp\left(-\frac{r^2}{4\mu}\right) J_0\left(\frac{2\pi r G}{\sqrt{r_0^2(1+\Delta z)}}\right) \nu \frac{2\pi r G}{\sqrt{r_0^2(1+\Delta z)}}^2
\]

(3.2.20)

(3.2.20) represents the field due to a plane wave illumination of a gaussian apodized annular aperture at a point 'P', which is at a distance 'z' from the focus.

The field intensity is

\[
I(r; f+z) = U_{f,p}(r; f+z) \cdot U_{f,p}^*(r; f+z)
\]

(3.2.21)

it is given explicitly by

\[
I(r; f+z) = \left(\frac{2\pi N_G}{1+\Delta z}\right)^2 \exp\left(-\frac{r^2}{2\sigma^2}\right) \exp\left(\frac{\nu^2}{\mu}\right) \frac{1}{2\mu} \exp\left(-\frac{\rho^2}{4\mu}\right) J_0\left(\frac{\nu \rho}{\mu}\right) \nu \frac{2\pi \rho G}{\sqrt{r_0^2(1+\Delta z)}}^2
\]

(3.2.22)

This analytical solution has been plotted against the numerically calculated values obtained from (3.2.3). The two sets of plots match very well in general because both represent the same complex field intensity at the point of interest and hence verifying the accuracy of the analytical result.
However, in certain cases where ‘σ’ or ‘N’ has high values there is a slight variation. That is due to the fact that numerically plotted graphs are an approximation. The integral is calculated using the QDAGS routine in the IMSL library. For that reason we chose to work with the analytical solution which gives the exact interpretation of the complex field intensity around the geometrical focus.

Figures (6-13) are plotted as a comparison between the analytical and the numerical graphs.

The plots are better matched for smaller values of σ, because there is a closer approximation to the actual apodization, and they also match better at points further away from the focus.

Figures (6-8) are plotted for N=0.5 and σ=.1 for different values of Δz, giving the profile along r1. Fig 9 is plotted for N=5.0 and σ = 0.08 and Fig 10 is plotted for N=50 and σ = 0.08. Figures 11-13 plot the on-axis intensity as a function of Δz. Fig 11 is plotted for N=0.5, σ = 0.04, Fig 12 for σ = 0.1, N = 0.5 and Fig 13 for N=5.0, σ = 0.1 along the Δz axis at r1=0.0.

All these figures reiterate the accuracy of the analytical result. The intensity has been normalized with the intensity at the focal point. i.e. I/I(r1=0=ΔZ).
Gaussian apodized Aperture

$N = 0.5, \Delta z = -0.5$

$\sigma = 0.1$

---

Normalized Intensity

$\text{Analytical} \quad \text{Numerical}$

**Fig 6**
Gaussian apodized Aperture

N=0.5, Δz =0.0
σ=0.1

Fig 7
Gaussian apodized Aperture
N=0.5, Δz =0.5
σ=0.1

Fig 8
Gaussian apodized Aperture
N=5.0, ∆z = 0.25
σ=0.08

Fig 9
Gaussian Apodized Aperture

N=50, Δz=0.0
σ=0.08

Fig 10
Gaussian Apodized Aperture

$N = 0.5, \quad r_i = 0.0$

$\sigma = 0.04$

![Graph showing normalized intensity vs $\Delta z$. The graph compares analytical and numerical results. The analytical data is represented by a solid line, while the numerical data is represented by a dashed line. The x-axis represents $\Delta z$ ranging from -1.0 to 1.0, and the y-axis represents normalized intensity ranging from 17 to 0.]

**Fig 11**
Gaussian Apodized Aperture

\[ N=0.5, \quad \eta=0.0 \]
\[ \sigma=0.1 \]

Fig 12
Gaussian Apodized Aperture

\[ N = 5.0, \quad \tau = 0.0 \]
\[ \sigma = 0.1 \]

Normalized Intensity

\[ \Delta z \]

Analytical \hspace{1cm} Numerical

**Fig 13**
4.0 Results and Discussions

The following section presents some of the plots followed by a brief discussion of the results presented.

4.1 Gaussian Apodized Aperture

Now that we are confident with the analytical solution (demonstrated in the last section), we use that result to plot figs. 14-40.

The following graphs (figs. 14-40) show the focal shift in the case of the Gaussian apodized annular aperture. In this also the shift increases on reducing the value of ‘N’ or reducing the spot size ‘σ’ of the gaussian aperture.

As N increases, the peak intensity along z sharpens indicating a
concentration of energy in a very small area. That may be in some case disadvantageous because a little deviation from the area rapidly decreases the received energy.

The maximum encircled energy for large \( N \) may be higher than that of small \( N \) for small radii circles, but for large radii circles it is higher for small \( N \).

The systems with small values of \( N \) offer a greater depth of focus indicating an advantage over larger \( N \) in remote photography where focussing is impractical or distance to object cannot be estimated.

Figure 14 is plotted with different \( N \), so as to observe the focal shift with changing \( N \). Figure 15 is plotted with varying \( \sigma \) to observe the focal shift.

Figure 16 represents the 3-D simulation for \( N=1.0 \) and \( \sigma = 0.06 \), figures 17-18 are the cross-sections along \( \Delta z \) and \( r_i \) respectively. Similarly, Figure 19 is the 3-D simulation for \( N=2.0 \) and \( \sigma=0.06 \) and Figure 20 is the 3-D simulation for \( N=50 \) and \( \sigma=0.06 \) with figs. 21-22 illustrating cross-sections along \( \Delta z \) and \( r_i \) respectively. Figure 23 is the 3-D simulation for \( N=0.5, \sigma=0.1 \). Figure 24 is the 3-D simulation for \( N=0.1, \sigma=0.1 \), while figures 25-26 are the cross-sections along \( \Delta z \) and \( r_i \) respectively. Figures 27-28 are the 3-D simulations for \( \sigma = 0.1 \) with \( N=2 \) and 5 respectively, while figures 29-30 are the cross-
sections along $\Delta z$ and $r_i$ for $N=5$ and $\sigma=.1$. Figures 31-32 are the 3-D simulations for $\sigma = 0.1$ with $N=7$. and 50 respectively, while figures 33-34 are the cross-sections along $\Delta z$ and $r_i$ for $N=50$. Figures 35-40 are the 3-D distributions and cross-sections along $r_i$ and $\Delta z$ for $N=5.0$ and $\sigma=.05,.01$ respectively.

From figures 29-31, and 35-40, we can see that as $\sigma$ gets smaller the diffraction in the beam reduces, due to system constraints we could not simulate for $\sigma$ smaller than .01, but for a very small value of $\sigma$ we should get a diffraction-free beam.

As we can see from the graphs the irradiance distribution is given by (3.2.4) & (3.2.21)

$$I(r_i; f+z) = \frac{1}{\lambda(f+z)} \int_0^\infty r \exp\left[\frac{-(r-r_0)^2}{2\sigma^2}\right] \exp(\phi(r)) J_0\left(\frac{2\pi r_i r}{\lambda(f+z)}\right) dr^2$$

$$\phi(r) = \frac{\pi[1 - \frac{f}{1}]r^2}{\lambda l} = \frac{N_G \Delta z}{r_0^2(1+\Delta z)}$$

where

is the defocus aberration.

The irradiance distribution as a function of $z$ is asymmetric about the focal plane $z=f$. This is because the inverse square dependence on $z$ in front of
the integral increases the irradiance for $z<f$ and decreases it for $z>f$. Also the defocus aberration of (4.2.2) is asymmetric about the focal point [10], \textit{i.e.},

$$\varphi(r, l = f+z) \neq \varphi(r, l = f-z)$$

(4.2.3)

The asymmetry is also due to the spread of the irradiance distribution, dependent on the argument of $J_0$ in (4.2.1) \textit{i.e.}, on the scale of the Hankel transform, increases or decreases accordingly as $l > f$ or $l < f$, respectively.

When $N_0$ is large as compared to unity, the intensity is maximum around the region of geometrical focus and is symmetrical about it as predicted by the standard theory of light focussing, but it deviates from the theory when $N$ is small. As we can see from (4.2.2) the effect of inverse square dependence on $z$ is reduced for large values of $N$.

For a given value of $N$ the shift towards the lens is more with a decrease in the value of the spot size $\sigma$, and for a given value of the spot size the shift is more with a decrease in the value of the Fresnel number.

Using the Fourier-Optic approach we have derived an exact expression for the axial irradiance of a Gaussian apodized annular aperture valid for all axial points. This expression represents the interference of waves originating at the inner and outer boundaries of the annular aperture. The resulting amplitude depends on the distance between the inner and outer edges of the aperture and the Fresnel number. The phase difference depends on defocus.

At the geometric focus Huygen's spherical wavelets arrive in phase and
therefore they interfere constructively producing a large irradiance. At any other point there is some destructive interference and therefore the irradiance is reduced on account of it. However, at a point closer to the aperture there is an increase in the irradiance because of the inverse square dependence of the wavelets on the distance from the aperture. Thus at any point of observation the inverse square dependence and the destructive interference both are present with the net result that the principal maximum of axial irradiance does not necessarily occur at the geometric focus.

If 'z' represents the longitudinal defocus, then in systems with large Fresnel numbers a small amount of defocus corresponds to a large amount of defocus aberration. Hence, at points closer to the aperture, any gain in axial irradiance due to the inverse square law is very small as compared to the loss resulting from destructive interference. Hence the principal maximum of the axial irradiance occurs at the geometric focus, and is symmetric in the region around the focal plane.

In systems with a small value of N, the depth of focus is large. A significant amount of defocus aberration is there for points which are some distance from the focus. Therefore any loss in axial irradiance due to destructive interference may be smaller than the gain due to the inverse square law, resulting in a higher axial irradiance at the points closer to the aperture than at the focus. The irradiance distribution due to the defocus aberration and the inverse square law, is in general asymmetric.
Gaussian Apodized Aperture
Varying N, \( n_1 = 0.0 \)
\( \sigma = 0.1 \)

![Graph showing normalized intensity vs. \( \Delta z \) for different values of N.](image)

Fig 14
Gaussian Apodized Aperture

$N = 0.5, \quad r_i = 0.0$

Varying $\sigma$

**Fig 15**
Gaussian Apodized Aperture

$N=1.0, \quad \sigma=0.06$

Fig 16
Gaussian Apodized Aperture

\[ N=1.0, \quad \sigma=0.06 \]
\[ r_i=0.0 \]

Fig 17
Gaussian Apodized Aperture

$N=1.0, \quad \sigma=0.06$

$\Delta z=-0.5, 0.0, 0.5$

---

**Fig 18**
Gaussian Apodized Aperture

$N = 2.0, \quad \sigma = 0.06$
Gaussian Apodized Aperture

N=50, \quad \sigma=0.06

Fig 20
Gaussian Apodized Aperture

$N=50, \quad \sigma=0.06$

$r_i=0.0$

Fig 21
Gaussian Apodized Aperture

$N=50, \quad \sigma=0.06$

$\Delta z=0.0$

Fig 22
Gaussian Apodized Aperture

N=0.5, \( \sigma=0.1 \)

Fig 23
Gaussian Apodized Aperture

N=1.0, \( \sigma=0.1 \)

Fig 24
Gaussian Apodized Aperture

$N=1.0, \quad \sigma=0.1$

$\Delta z$
Gaussian Apodized Aperture

N=1.0, \( \sigma=0.1 \)
\( \Delta z=-0.5, 0.0, 0.5 \)

Normalized Intensity

Fig 26
Gaussian Apodized Aperture

$N=2.0, \quad \sigma=0.1$

Fig 27
Gaussian Apodized Aperture

$N=5.0, \quad \sigma=0.1$

Fig 28
Gaussian Apodized Aperture
N=5.0, \( \sigma=0.1 \)
\( r_i=0.0 \)

Normalized Intensity

\( \Delta z \)

**Fig 29**
Gaussian Apodized Aperture

$N=5.0, \quad \sigma=0.1$

$z=-0.5, 0.0, 0.5$

Fig 30
Gaussian Apodized Aperture

$N=7.0, \quad \sigma=0.1$

Fig 31
Gaussian Apodized Aperture

N=50, \( \sigma=0.1 \)

Fig 32
Gaussian Apodized Aperture

$N=50, \quad \sigma=0.1$

$\Delta z$

**Fig 33**
Gaussian Apodized Aperture

$N=50, \quad \sigma=0.1$

$\Delta z=0.0$

Fig 34
Gaussian Apodized Aperture

$N = 5.0, \quad \sigma = 0.05$

Fig 35
Gaussian Apodized Aperture

$N=5.0, \quad \sigma=0.05$

$r_i=0.0$

Fig 36
Gaussian Apodized Aperture

N=5.0, \quad \sigma=0.05

\Delta z=-0.5, 0.0, 0.5

Normalized Intensity

\Delta z

-0.5 \quad 0.0 \quad 0.5

Fig 37
Gaussian Apodized Aperture

$N = 5.0, \quad \sigma = 0.01$

Fig 38
Gaussian Apodized Aperture

$N=5.0$, $\sigma=0.01$

$r_0=0.0$

Fig 39
Gaussian Apodized Aperture

N=5.0, \( \sigma=0.01 \)

\( z=1.0, 1.5, 2.0 \)

\[ \Delta z \]

1.0 \[ \quad \] 1.5 \[ \quad \] 2.0

Fig 40
4.2 Unapodized vs Gaussian

The following graphs (figures 41-50), are drawn as a comparison between the unapodized and Gaussian apodized annular apertures.

To make a suitable comparison, the area under the annular regions were made equal so as to ensure that both apertures received equal energy.

Figure 41 represents the peak intensity, \( (I(z,\eta))_{\text{max}} \) (3.1.10) for annular and \( I(r_1, f+z)_{\text{max}} \) (3.2.22) for Gaussian apodized) vs \( N \). Figure 42 represents \( \Delta z_{\text{max}} \) (the distance away from \( f \), where the intensity is maximum) vs \( N \). Figure 43-50 show the comparison between the two for some particular values.

The cross-section along \( r_1 \) is drawn at \( \Delta z=0.0 \), while the cross-section along \( \Delta z \) is drawn at \( r_1=0.0 \). Figures 43-48 are plotted for \( \sigma=0.224, \eta=0.8 \) and \( N=1, 5, \ldots, 50 \).
10. Fig 49-50 are plotted for $\sigma = 0.112, \eta = 0.9$ and $N = 10$.

The graphs (figures 43-50) show that along $r$, the shape of both apodized and Gaussian apodized is the same, while along $\Delta z$ the side lobes present in the unapodized aperture are not found in the Gaussian apodized aperture. The peak intensity is much higher in the case of the Gaussian apodized aperture possibly because of the absence of side-lobes which are very much prevalent in the case of the unapodized aperture. The intensity may be more concentrated in the case of the Gaussian beams.

The shift in the point of maximum intensity is much less in the case of the Gaussian aperture and for very small values of $N$ the peak intensity is less than that of the unapodized beams, as can be seen in figures 48-50.

For smaller values of $\sigma$ or $N$, the number of side-lobes for the unapodized aperture along $\Delta z$, are reduced, thereby reducing the difference in peak intensity.
Gaussian Apodized vs Unapodized Apertures
\( \sigma=0.1, \eta=0.9 \)

\( I(n, f+z)_{max} \)

Gaussian apodized  \quad \text{Unapodized}  \\

Fig 41
Gaussian Apodized vs Unapodized Apertures

$\sigma=0.1, \eta=0.9$

Fig 42
Gaussian Apodized vs Unapodized Apertures

$\sigma=0.224$, $\eta=0.8$, $N=1.0$

$\Delta z$

Gaussian apodized --- Unapodized 

Fig 43
Gaussian Apodized vs Unapodized Apertures

$\sigma=0.224$, $\eta=0.8$, $N=1.0$

$\Delta z=0.0$

Fig 44
Gaussian Apodized vs Unapodized Apertures

$\sigma=0.224$, $\eta=0.8$, $N=5.0$

$\eta_i=0.0$

![Graph showing intensity vs $\Delta z$ with Gaussian apodized and Unapodized lines.]

Fig 45
Gaussian Apodized vs Unapodized Apertures

$\sigma=0.224$, $\eta=0.8$, $N=5.0$

$\Delta z=0.0$

Fig 46
Gaussian Apodized vs Unapodized Apertures

$\sigma=0.224$, $\eta=0.8$, $N=10.0$

$r_1=0.0$

Fig 47
Gaussian Apodized vs Unapodized Apertures

\[ \sigma = 0.224, \eta = 0.8, N = 10.0 \]
\[ \Delta z = 0.0 \]
Gaussian Apodized vs Unapodized Apertures

\[ \sigma = 0.112, \eta = 0.9, N = 10.0 \]

\[ r_l = 0.0 \]

Fig 49
Gaussian Apodized vs Unapodized Apertures

$\sigma=0.112, \eta=0.9, N=10.0$

$\Delta z=0.0$

Gaussian apodized $\quad\quad\quad$ Unapodized

Fig 50
5.0 Conclusions and Future Work

In this chapter we present a prognosis for future work using \( \theta \)-Modulation on Gaussian apodized apertures followed by conclusion.

\( \theta \)-Modulation

The same Fourier-Optic approach can be used to analyze the field in case some \( \theta \) modulation is introduced within the Gaussian apodized annular aperture.

The effect of \( \theta \) modulation can be modelled as

\[
U_I (r,\theta)P(r,\theta) = \cos(n\theta) \exp \left( \frac{-(r-r_0)^2}{2\sigma^2} \right).
\]  \hspace{1cm} (5.1.0)

Putting (2.2.10) here for convenience, we have
\[ U_{lP}(x_i, y_i; l) = \frac{1}{i\lambda l} \exp(ikl) \exp\left(\frac{ik(x_i^2+y_i^2)}{2l}\right) \mathcal{F}\left[U_l(x, y)P(x, y) \exp\left(\frac{ik\left[\frac{1}{2l} - \frac{1}{l}\right]}{f^2}(x^2+y^2)\right)\right]_{u=x/l, v=y/l} . \tag{5.1.1} \]

Changing to polar coordinates, we get

\[ U_{lP}(r, \theta) = \frac{1}{i\lambda l} \exp(ikl) \exp\left(\frac{ik(r^2)}{2l}\right) \mathcal{F}\left[U_l(r, \theta)P(r, \theta) \exp\left(\frac{ik\left[\frac{1}{2l} - \frac{1}{l}\right]}{f^2}(r^2)\right)\right] , \tag{5.1.2} \]

where \( x = r\cos\theta, y = r\sin\theta, u = \rho\cos\phi, v = \rho\sin\phi \).

Explicitly the above equation gives

\[ U_{lP}(r, \theta) = \frac{1}{i\lambda l} \exp(ikl) \exp\left(\frac{ik(r^2)}{2l}\right) \int_0^{2\pi} \int_0^{\infty} \left[U_l(r, \theta)P(r, \theta) \exp\left(\frac{ik\left[\frac{1}{2l} - \frac{1}{l}\right]}{f^2}(r^2)\right) \right. \nonumber \]
\[ \left. \exp\left(-i2\pi\rho\cos(\theta-\phi)\right) rd\theta dr \right] . \tag{5.1.3} \]

On substituting (5.1.0) in (5.1.3), we have

\[ U_{lP}(r, \theta) = \frac{1}{i\lambda l} \exp(ikl) \exp\left(\frac{ik(r_i^2)}{2l}\right) \int_0^{\infty} \int_0^{2\pi} \cos(n\theta) \exp\left(-\frac{(r-r_0)^2}{2\sigma^2}\right) \nonumber \]
\[ \exp\left(-i2\pi\rho\cos(\theta-\phi)\right) rd\theta dr . \tag{5.1.3} \]
Since
\[ \int_0^\infty \exp \left( i \beta \cos(x) \right) \cos(nx) \, dx = i^n \pi J_n(\beta) \]  \hspace{1cm} (5.1.4)

or
\[ \int_0^\infty \exp \left( i 2 \pi r \cos(\theta - \phi) \right) \cos(n\theta) \, d\theta = 2i^n \pi J_n(2\pi r) \]  \hspace{1cm} (5.1.5)

Putting (5.1.6) in (5.1.4), we get the field in terms of Bessel functions:

\[ U_{LP}(r, \theta) = \frac{1}{i \lambda l} \exp(ikl) \exp \left( \frac{-ik(r_0)^2}{2l} \right) \int_0^\infty \exp \left( \frac{-kr^2}{2l} \right) \exp \left( \frac{ik}{2l} \left( 1 - \frac{1}{l} \right) r^2 \right) 2i^n \pi J_n(2\pi r) \, r \, dr \]  \hspace{1cm} (5.1.6)

Following the analysis from (3.2.10)-(3.2.17), and putting \( n=1, \cos(\theta) \) as a simple example, we have

\[ \exp \left( -\frac{r_0^2}{2\sigma^2} \right) \exp \left( \frac{\nu^2}{\mu} \right) \mathcal{H}' \left[ \exp(\mu r^2) \right] \mathcal{H}' \left[ \delta \left( \frac{r - \nu}{\mu} \right) \right] \]  \hspace{1cm} (5.1.7)

where \( \mathcal{H}' \) is the Hankel Transform of the first order, the general form of which is given as

\[ \mathcal{H}' \{ g(r) \} = \int_0^\infty r \, g(r) \, J_1(pr) \, dr \]  \hspace{1cm} (5.1.8)

We can get the Hankel Transform from [28], hence we get
\[
U_{\ell p}(r_i; f+z) = \frac{1}{i\lambda(f+z)} \exp\left(ik(f+z)\right) \exp\left(\frac{i kr^2}{2(f+z)}\right) 2\pi \left[I_0\left(\frac{\rho^2}{8\mu}\right) - I_1\left(\frac{\rho^2}{8\mu}\right)\right] \\
\exp\left(-\frac{r_0^2}{2\sigma^2}\right) \exp\left(\frac{\nu^2}{\mu}\right) \frac{\pi}{8\mu^{3/2}} \exp\left(\frac{\rho^2}{8\mu}\right) \nu J_1\left(\frac{\nu_2\pi\rho}{\mu}\right)
\]
\]

(5.1.10)

Where \(I_0\) and \(I_1\) are Modified Bessel Transforms of the first kind order zero and one respectively.

Since \(I(r; f+z) = U_{\ell p}(r_i; f+z) U_{\ell p}^*(r_i; f+z)\),

we get

\[
I(r_i; f+z) = \frac{N_G}{(1+\Delta z)} 2\pi \left[I_0\left(\frac{\rho^2}{8\mu}\right) - I_1\left(\frac{\rho^2}{8\mu}\right)\right] \exp\left(-\frac{r_0^2}{2\sigma^2}\right) \exp\left(\frac{\nu^2}{\mu}\right) \frac{\pi}{8\mu^{3/2}} \exp\left(\frac{\rho^2}{8\mu}\right) \nu J_1\left(\frac{\nu_2\pi\rho}{\mu}\right) \right]^2
\]

(5.1.11)

Fig 51 represents the 3-D plot for \(N=0.5\) and \(\sigma=0.1\). Fig 52-53 are the cross-sections along \(r_i\) and \(\Delta z\), respectively. Similarly figures 54-56 are for \(N=5.0\), \(\sigma=0.1\), figures 57-59 for \(N=5.0\), \(\sigma=.08\), figures 60-62 for \(N=5.0\), \(\sigma=.06\) and figures 63-66 are for \(N=5.0\), \(\sigma=.038\).

The cross-section along the \(\Delta z\) axis was taken at the value of \(r_i\) where the intensity is maximum, since at \(r_i=0\), the intensity is equal to 0.
\( \theta \)-Modulation

\( N=0.5, \sigma=0.1 \)

Fig 51
$\theta$-Modulation

$N = 0.5, \sigma = 0.1$

$t_i = 0.06$

Fig 52
θ-Modulation

N=0.5, σ=0.1
Δz=-0.50

Fig 53
θ-Modulation

N=5.0, σ=0.1

Fig 54
θ-Modulation

N=5.0, σ=0.1
r_i=0.06

Fig 55
$\theta$-Modulation

$N=5.0$, $\sigma=0.1$

$\Delta z=0.00$

Fig 56
0-Modulation

N=5.0, \( \sigma=0.08 \)

Fig 57
$\theta$-Modulation

$N=5.0$, $\sigma=0.08$

$r_c=0.06$

Fig 58
$\theta$-Modulation

$N=5.0, \sigma=0.08$

$\Delta z=0.0$

Fig 59
$\theta$-Modulation

$N = 5.0, \sigma = 0.06$

Fig 60
\( \theta \)-Modulation

\( N=5.0, \sigma=0.06 \)

\( r_i=0.06 \)

Fig 61
θ-Modulation

N=5.0, σ=0.06
Δz=0.0

Fig 62
9–Modulation
N=5.0, σ=0.038

Fig 63
0-Modulation
N=5.0, σ=0.038
r_i=0.145

Fig 64
\theta-Modulation

N=5.0, \sigma=0.038
\tau_0=0.06

Fig 65
θ-Modulation

$N=5.0$, $\sigma=0.038$

$\Delta z=-0.25$

Fig 66
Conclusions

In this thesis we have developed an analytical solution for the three-dimensional field distributions for focussed annular apertures using the Fourier-Optic approach. The analysis developed is general in nature and can be applied to any arbitrary pupil function. The two cases considered in this thesis are the unapodized and the Gaussian apodized annular apertures.

Focal shift was observed in both the cases, with the shift in the case of unapodized aperture being more. For systems with Gaussian apodized annular apertures, the shift increases with decreasing values of $N$ and $\sigma$. Also the diffracted beam profile seems to be approaching close to being diffraction free as we reduce the value of $\sigma$.

Finally, we present a prognosis for future work using $\theta$-Modulation on a Gaussian apodized annular aperture. Preliminary results are presented.
Gaussian Profile

Fig 67
Appendix A: Gaussian Beam

In figure 67 we see the amplitude of the Gaussian beam is given by

\[ E(x) = E_0 \exp \left( \frac{x^2}{\sigma^2} \right) \]

where \( \sigma \) is the beam radius which is the width at the point where the amplitude is \( E_0/e \).

The intensity is given by

\[ I = |E|^2 = E_0^2 \exp \left( -2 \frac{x^2}{\sigma^2} \right) \]
Zone Plates

Fig 68
Appendix B: Fresnel Number

In figure 68 the wavefront on the lens surface has been divided into a number of annular regions. The boundaries of the various regions correspond to the intersections of the wavefront with a series of spheres centered at P of radius \( f + \lambda/2, f + 2\lambda/2, f + 3\lambda/2 \) and so forth. These are called the Fresnel or half-period zones. If there are \( N \) Fresnel zones, then the radius of the \( N \)th sphere will be given by \( f + N\lambda/2 \).

From the figure we can see that

\[
(f + N\lambda/2)^2 = r_0^2 + f^2
\]

where \( r_0 \) is the radius of the aperture, \( \lambda \) is the wavelength of the incident light and \( f \) is the focal length of the lens.

On squaring the left hand side, we get

\[
(f^2 + N\lambda f + [N\lambda/2]^2) = r_0^2 + f^2
\]
Now $[N\lambda/2]^2 \ll N\lambda f$; hence it can be neglected in comparison to the other term. We get, then,

$$N\lambda f \equiv r_0^2,$$

or

$$N \equiv r_0^2/\lambda f,$$

where $N$ is commonly known as the Fresnel number.
References


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Vita

Shaleen J. Bhabu was born in the wee hours of January 31’ 1966 in the sprawling metropolis of New Delhi, India. Trekking and mountaineering being his two greatest passions, he hiked extensively in the Himalayas. At a very early age he chose two diametrically opposite career choices. Not having the means to become a cowboy in India and fate dealing a cruel blow in his choice to become a pilot (bad eyesight), he chose engineering. Following the natural path of least resistance, he chose Electrical Engineering (since a lot of people in the family were Mechanical Engrs. he did not want too many academic questions being asked). He did his engineering in Birla Institute of Technology, where he spent some of the most exciting years of his life. Having travelled extensively all over India, he decided to explore the world and also of course to satiate his thirst for knowledge. He bravely ventured west and chose Va Tech for its proximity to the mountains and a good Electrical Engineering discipline. Having had his thirst quenched for the time being and his bank balance being reduced to the lower of the double digit values he decided to work in the capitalist world, naturally to gain more experience. He plans to return to India and start his own factory.

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