THERMAL DEFORMATIONS OF PLATES PRODUCED BY TEMPERATURE DISTRIBUTIONS SATISFYING POISSON'S EQUATION

By

Robert R. McWithey

Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

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II. TABLE OF CONTENTS

CHAPTER PAGE

III. SYMBOLS

$a_n(x,y)$, $b_n(x,y)$	coefficients of power series in z
a, b	plate planform dimensions
$A(x,y,z)$	heat generated at any point within the plate per unit time per unit volume
B_n	Bernoulli numbers
C_n, D_m	constants
E_n	Euler numbers
e	$\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz}$
$f(x,y,z)$	$-\dfrac{A(x,y,z)}{K}$
h	plate thickness
I_o	energy incident to plate surface $z = h/2$
K	thermal conductivity of plate material
m	summing integer in power series
n	summing integer in power series
P_o	$\frac{1}{2}\left[T(x,y,h/2) - T(x,y,-h/2)\right]$
T	temperature change from an unstressed and undeflected datum
T_o	$\frac{1}{2}\left[T(x,y,h/2) + T(x,y,-h/2)\right]$
U_o	strain energy per unit volume of plate
U	total strain energy of plate
u,v,w	components of deformation in the x, y, and z directions, respectively
u_o, v_o	components of deformation of the midplane of the plate in the x and y directions, respectively

x, y, z	rectangular coordinates
α	linear coefficient of thermal expansion of plate material
β	linear absorption coefficient
β_n	$(2^{2n} - 2)B_n$
$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$	shear strain components
$\epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}$	normal strain components
∇^4	$\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$
∇_1^2	$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$
∇^2	$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
μ, λ	Lame constants
$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$	normal stress components
φ_o	$\int_{-h/2}^{h/2} T\, dz$
φ_1	$\int_{-h/2}^{h/2} Tz\, dz$

IV. INTRODUCTION

Steady-state temperature distributions within homogeneous bodies in which the mode of heat transfer is by conduction, and in which heat is generated, are governed mathematically by Poisson's equation. If heat is not generated within the body, the governing equation becomes homogeneous and reduces to the familiar Laplace Equation of Heat Conduction. Solutions satisfying the Laplace equation of heat conduction are abundant in the literature (see ref. 1, for example) but very few exist which satisfy the Poisson equation of heat conduction.

Heat generation within bodies occurs as a result of many commonly encountered phenomena. For example, heat is generated within bodies carrying electrical current, absorbing radiation, or in which chemical reactions are taking place. The particular effect of the temperature distributions created by internal heat generation within plates became of interest with the advent of nuclear reactors and space travel. In nuclear reactors plates are used to absorb radiation emitted by the reactor. This radiation is capable of penetrating relatively large thicknesses of metallic material and requires that the internal plates be designed to allow for the thermal deformation and stresses resulting from the radiation absorption process. In space, thermal radiation causes heat generation within semitransparent materials. Inasmuch as materials of this type are used in plate configurations for protective shields, viewing ports, and optical devices, thermal deformations and stresses resulting from thermal radiation absorption are of interest.

The analysis presented herein outlines a method which may be used to determine plate deformation and stresses when the plate is subjected to internal heat generation. Several example problems are presented.

V. DERIVATION OF PLATE EQUATIONS

The small deflection plate equations containing terms involving the temperature distribution within the plate may be found in the literature (see, for example, refs. 2-4). The derivation included herein is presented for completeness and to present the plate equations in the form most suitable for determining plate deformation.

The energy per unit volume, U_o, in a homogeneous body with strains is given in reference 2 as

$$U_o = \frac{\lambda}{2} e^2 + \mu\left(\epsilon_{xx}{}^2 + \epsilon_{yy}{}^2 + \epsilon_{zz}{}^2 + \frac{\gamma_{xy}{}^2}{2} + \frac{\gamma_{yz}{}^2}{2} + \frac{\gamma_{zx}{}^2}{2}\right)$$

$$- (3\lambda + 2\mu)\alpha Te + \frac{3}{2}(3\lambda + 2\mu)(\alpha T)^2 \tag{1}$$

where

$$e = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} \tag{2}$$

The stresses associated with these strains are related to the strains by the usual equations resulting from Hooke's law and are given by

$$\left.\begin{aligned}
\sigma_{xx} &= \lambda e + 2\mu\epsilon_{xx} - (3\lambda + 2\mu)\alpha T \\
\sigma_{yy} &= \lambda e + 2\mu\epsilon_{yy} - (3\lambda + 2\mu)\alpha T \\
\sigma_{zz} &= \lambda e + 2\mu\epsilon_{zz} - (3\lambda + 2\mu)\alpha T \\
\sigma_{xy} &= \mu\gamma_{xy} \\
\sigma_{yz} &= \mu\gamma_{yz} \\
\sigma_{zx} &= \mu\gamma_{zx}
\end{aligned}\right\} \tag{3}$$

Inasmuch as a plate has, by definition, two characteristic dimensions much greater than the third characteristic dimension, it may be assumed that a state of plane stress exists within the plate if there are no loads on the large surfaces, and no in-plane forces acting at the boundaries which cause out-of-plane loading. Using the plane stress assumption and defining the plate coordinate system as shown in figure 1 gives:

$$\left.\begin{aligned} \sigma_{zz} &= 0 \\ \sigma_{xz} &= 0 \\ \sigma_{yz} &= 0 \end{aligned}\right\} \tag{4}$$

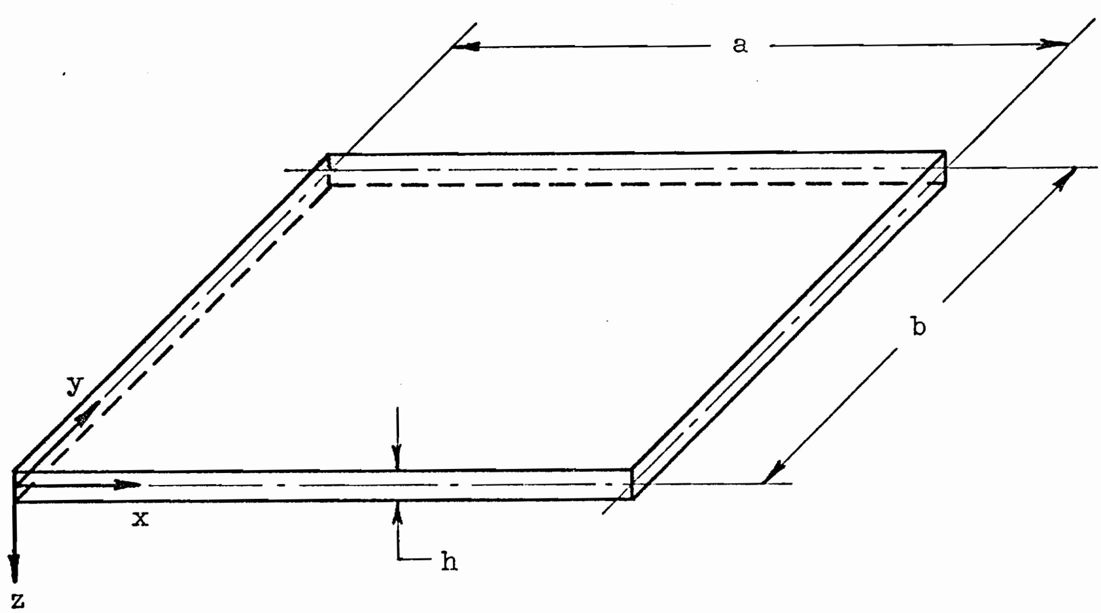

Figure 1.- Coordinate system associated with plate.

Applying the plane stress conditions (eq. (4)) in the stress-strain relations given in equation (3) and substituting the resulting expressions for the strains ϵ_{zz}, γ_{xz}, and γ_{yz} into the energy expression (eq. (1)) gives:

$$U_o = \frac{1}{2\mu + \lambda}\Bigg[2\mu(\mu + \lambda)(\epsilon_{xx}^2 + \epsilon_{yy}^2) + 2\epsilon_{xx}\epsilon_{yy}\lambda\mu - 2\mu\alpha T(3\lambda + 2\mu)(\epsilon_{xx} + \epsilon_{yy}) + \frac{\mu\gamma_{xy}^2}{2}(2\mu + \lambda) + 2\mu(3\lambda + 2\mu)(\alpha T)^2\Bigg] \tag{5}$$

It should be noted here that exact solutions to the plane stress problem are obtainable only when the steady-state temperature distribution is a function of z alone (see refs. 2 and 5). However, it is stated in reference 2 that if the body is very thin and its faces free of tractions, the assumption of plane stress will give close approximations to the exact solution for temperature distributions which are functions of all three coordinates. The usual strain-displacement relations for the strains are given by

$$\left.\begin{aligned} \epsilon_{xx} &= \frac{\partial u}{\partial x} \\ \epsilon_{yy} &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned}\right\} \tag{6}$$

where u and v are the displacements in the x and y directions, respectively.

If, in addition to the assumption of plane stress, the usual assumption is made that lines initially normal to the midplane of the plate remain normal to the midplane during loading, then the expressions for u and v may be written as:

$$\left.\begin{aligned} u &= u_o - z\,\frac{\partial w}{\partial x} \\ v &= v_o - z\,\frac{\partial w}{\partial y} \end{aligned}\right\} \qquad (7)$$

where u_o, v_o, and w are the displacements of the midplane of the plate in the x, y, and z directions, respectively. Equations (7) are the linear approximations for u and v and are the equations normally used in small deflection plate theory.

In order to express the energy per unit volume, U_o, in terms of the midplane deflections, equations (6) and (7) may be substituted into equation (5). This gives U_o as:

$$\begin{aligned} U_o = \frac{1}{2\mu + \lambda}\Bigg\{ & 2\mu(\mu + \lambda)\left[\left(\frac{\partial u_o}{\partial x}\right)^2 - 2z\,\frac{\partial^2 w}{\partial x^2}\frac{\partial u_o}{\partial x} + z^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial v_o}{\partial y}\right)^2\right. \\ & \left. - 2z\,\frac{\partial^2 w}{\partial y^2}\frac{\partial v_o}{\partial y} + z^2\left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] + 2\mu\lambda\left[\frac{\partial u_o}{\partial x}\frac{\partial v_o}{\partial y} - z\,\frac{\partial^2 w}{\partial x^2}\frac{\partial v_o}{\partial y} - z\,\frac{\partial^2 w}{\partial y^2}\frac{\partial u_o}{\partial x}\right. \\ & \left. + z^2\,\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right] - 2\mu\alpha(3\lambda + 2\mu)T\left[\frac{\partial u_o}{\partial x} - z\,\frac{\partial^2 w}{\partial x^2} + \frac{\partial v_o}{\partial y} - z\,\frac{\partial^2 w}{\partial y^2}\right] \\ & + \frac{\mu(2\mu + \lambda)}{2}\left[\left(\frac{\partial u_o}{\partial y}\right)^2 - 4z\,\frac{\partial^2 w}{\partial x\partial y}\frac{\partial u_o}{\partial y} + 2\,\frac{\partial u_o}{\partial y}\frac{\partial v_o}{\partial x} + 4z^2\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right. \\ & \left. - 4z\,\frac{\partial^2 w}{\partial x\partial y}\frac{\partial v_o}{\partial x} + \left(\frac{\partial v_o}{\partial x}\right)^2\right] + 2\mu\alpha^2T^2(3\lambda + 2\mu)\Bigg\} \end{aligned} \qquad (8)$$

The total strain energy, U, within the plate may thus be obtained by integrating U_o over the entire volume. Inasmuch as u_o, v_o, and w are independent of z, the integration of equation (8) with respect to z may be carried out immediately, thereby eliminating the variable z from the total energy expression. The resulting expression for the total energy is then given by

$$U = \frac{1}{2\mu + \lambda} \int_0^b \int_0^a \left\{ 2\mu(\mu + \lambda)\left[\left(\frac{\partial u_o}{\partial x}\right)^2 h + \frac{h^3}{12}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + h\left(\frac{\partial v_o}{\partial y}\right)^2 + \frac{h^3}{12}\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right] + 2\lambda\mu\left[h \frac{\partial u_o}{\partial x}\frac{\partial v_o}{\partial y} + \frac{h^3}{12}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right)\right] \right.$$
$$- 2\mu\alpha(3\lambda + 2\mu)\left[-\varphi_1 \frac{\partial^2 w}{\partial x^2} + \varphi_o \frac{\partial u_o}{\partial x} - \varphi_1 \frac{\partial^2 w}{\partial y^2} + \varphi_o \frac{\partial v_o}{\partial y}\right]$$
$$+ \frac{\mu(2\mu + \lambda)}{2}\left[h\left(\frac{\partial u_o}{\partial y}\right)^2 + 2h \frac{\partial u_o}{\partial y}\frac{\partial v_o}{\partial x} + \frac{h^3}{3}\left(\frac{\partial^2 w}{\partial x \partial y}\right) + h\left(\frac{\partial v_o}{\partial x}\right)^2\right]$$
$$\left. + 2\mu\alpha^2(3\lambda + 2\mu)\left[\int_{-h/2}^{h/2} T^2 \, dz\right]\right\} dx\, dy \tag{9}$$

where

$$\varphi_o = \int_{-h/2}^{h/2} T \, dz \tag{10}$$

and

$$\varphi_1 = \int_{-h/2}^{h/2} Tz \, dz \tag{11}$$

If the expressions φ_o and φ_1 are divided by the plate thickness, h, the resulting quantities, $\frac{\varphi_o}{h}$ and $\frac{\varphi_1}{h}$, are, respectively, the average temperature distribution through the plate thickness and the first moment of the temperature distribution with respect to the midplane of the plate. They are functions of the plate planform coordinates. Employing the Principle of Virtual Work gives the condition that

$$\delta U = 0 \tag{12}$$

With the use of equation (12), and using principles set forth in the Calculus of Variations, the linear governing differential equations and boundary conditions may be obtained for the plate. The governing differential equations are given by:

$$\nabla^4 w + \frac{6\alpha(3\lambda + 2\mu)}{(\mu + \lambda)h^3}(\nabla^2\varphi_1) = 0 \tag{13}$$

$$(2\mu + \lambda)\frac{\partial^2 u_o}{\partial y^2} + 4(\mu + \lambda)\frac{\partial^2 u_o}{\partial x^2} + (3\lambda + 2\mu)\left[\frac{\partial^2 v_o}{\partial x \partial y} - \frac{2\alpha}{h}\frac{\partial \varphi_o}{\partial x}\right] = 0 \tag{14}$$

$$(2\mu + \lambda)\frac{\partial^2 v_o}{\partial x^2} + 4(\mu + \lambda)\frac{\partial^2 v_o}{\partial y^2} + (3\lambda + 2\mu)\left[\frac{\partial^2 u_o}{\partial x \partial y} - \frac{2\alpha}{h}\frac{\partial \varphi_o}{\partial y}\right] = 0 \tag{15}$$

where the operator ∇^2 is defined as $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

The plate boundary conditions are given by:

$$\frac{\partial^2 w}{\partial x \partial y}\delta w = 0 \tag{16}$$

at the four corners of the plate; at $x = 0$ and $x = a$:

$$\left[2(\mu + \lambda)\frac{\partial^2 w}{\partial x^2} + \lambda\frac{\partial^2 w}{\partial y^2} + \alpha\varphi_1\left(\frac{12}{h^3}\right)(3\lambda + 2\mu)\right]\delta\frac{\partial w}{\partial x} = 0 \tag{17}$$

$$\left[2(\mu + \lambda)\frac{\partial^3 w}{\partial x^3} + (4\mu + 3\lambda)\frac{\partial^3 w}{\partial y^2 \partial x} + \frac{12\alpha}{h^3}(3\lambda + 2\mu)\frac{\partial \varphi_1}{\partial x}\right]\delta w = 0 \tag{18}$$

$$\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right)\delta v_o = 0 \tag{19}$$

$$\left[2(\mu + \lambda)\frac{\partial u_o}{\partial x} + \lambda\,\frac{\partial v_o}{\partial y} - \frac{\alpha\varphi_o}{h}(3\lambda + 2\mu)\right]\delta u_o = 0 \tag{20}$$

and at $y = 0$ and $y = b$:

$$\left[2(\mu + \lambda)\frac{\partial^2 w}{\partial y^2} + \lambda\,\frac{\partial^2 w}{\partial x^2} + \alpha\varphi_1\left(\frac{12}{h^3}\right)(3\lambda + 2\mu)\right]\delta\,\frac{\partial w}{\partial y} = 0 \tag{21}$$

$$\left[2(\mu + \lambda)\frac{\partial^3 w}{\partial y^3} + (4\mu + 3\lambda)\frac{\partial^3 w}{\partial x^2 \partial y} + \frac{12\alpha}{h^3}(3\lambda + 2\mu)\frac{\partial \varphi_1}{\partial y}\right]\delta w = 0 \tag{22}$$

$$\left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right)\delta u_o = 0 \tag{23}$$

$$\left[2(\mu + \lambda)\frac{\partial v_o}{\partial y} + \lambda\,\frac{\partial u_o}{\partial x} - \frac{\alpha\varphi_o}{h}(3\lambda + 2\mu)\right]\delta v_o = 0 \tag{24}$$

Equation (16) represents the condition at the corners of the plate which requires either that the deflection, w, at the corners of the plate be zero or that the concentrated forces at the corners, which are proportional to $\dfrac{\partial^2 w}{\partial x \partial y}$ (see ref. 2, page 389) be zero. Equations (17) and (21) indicate that either the slope of the out-of-plane deformation in the direction normal to the edges of the plate, or the moments along the edges of the plate must be zero. Equations (18) and (22) indicate that either the total shearing forces or the deflection, w, at the edges of the plate must be zero.

Equations (19) and (23) require that either the deflections, u_o or v_o, or the midsurface shearing strains be zero at the edges of the plate. Finally, equations (20) and (24) require that either the deflections, u_o or v_o, or the thrust forces along the edges of the plate be zero. Thus, after φ_o and φ_1 are determined from the given temperature distribution $T(x,y,z)$, equations (13) through (24) may be used to determine the corresponding plate deformation.

It may be seen from the governing equations and the boundary conditions that the displacement w is uncoupled from the displacements u_o and v_o. Equations (14) and (15) may be uncoupled to give:

$$\nabla^4 v_o - \frac{(3\lambda + 2\mu)\alpha}{2h(\mu + \lambda)} \frac{\partial(\nabla^2 \varphi_o)}{\partial y} = 0 \tag{25}$$

$$\nabla^4 u_o - \frac{(3\lambda + 2\mu)\alpha}{2h(\mu + \lambda)} \frac{\partial(\nabla^2 \varphi_o)}{\partial x} = 0 \tag{26}$$

Solutions for u_o and v_o obtained by the use of the uncoupled governing equations (eqs. (25) and (26)) are not valid solutions to the plate equations unless they also satisfy the governing equations given by equations (14) and (15).

VI. STEADY-STATE TEMPERATURE DISTRIBUTIONS PRODUCED BY INTERNAL HEATING OF A PLATE

A. General Solution of Governing Equation

The plate equations derived in the previous section were obtained using the tacit assumptions that the temperature distributions throughout the plate are independent of the plate deformations; and that these temperature distributions represent the local temperature changes within the plate that cause the plate to deform from its initial unstressed, undeflected state.

If it is assumed that this local temperature change is caused by heat conduction within the plate, then the resulting governing equation on the steady-state temperature of the plate is given by the Poisson equation as (see ref. 1, page 9):

$$\nabla_1^2 T = -\frac{A(x,y,z)}{K} \tag{27}$$

where the operator ∇_1^2 is defined as $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$; $A(x,y,z)$ is the heat generated at any point within the plate per unit time per unit volume; and K is the thermal conductivity of the plate material. The function, A, and the constant, K, are both assumed to be known.

Solutions of Poisson's equation may be found by means of the Green's function technique (see ref. 6, page 791). However, application of this method of solution to boundary value problems is difficult inasmuch as the solution involves a surface and volume integral over the body geometry. Because of this complexity, solutions to heat conduction problems in which heat is generated within the solid are virtually nonexistant.

For solid bodies, such as plates, in which one of the characteristic dimensions is small as compared with the other two characteristic dimensions, the temperature distribution within the body is relatively unaffected by the temperature boundary conditions along the edges of the body. Temperature distributions, which must satisfy only the temperature boundary conditions on the large surfaces of the body, may therefore be determined in terms of these two boundary conditions and the internal heating function.

A method of determining this temperature distribution may be formulated by expressing both the temperature and the right-hand side of equation (27) in terms of power series in z. This gives:

$$T = \sum_{n=0,1,2,3}^{\infty} b_n(x,y)z^n \tag{28}$$

and

$$-\frac{A(x,y,z)}{K} = f(x,y,z) = \sum_{n=0}^{\infty} a_n(x,y)z^n \tag{29}$$

where the functions $a_n(x,y)$ are assumed known inasmuch as $f(x,y,z)$ is known. Substituting equations (28) and (29) into equation (27) and equating the coefficients of like powers of z gives the recurrence relation

$$a_n = \nabla^2 b_n + (n + 1)(n + 2)b_{(n+2)} \tag{30}$$

With the use of this recurrence relation, the coefficients of z^n in equation (28) may be expressed in terms of b_o, b_1, and a_n. After performing this manipulation, equation (28) becomes:

$$T = \sum_{n=0,2,4}^{\infty} (-1)^{\frac{n}{2}} \frac{1}{n!} \nabla^n b_o z^n + \sum_{n=0,2,4}^{\infty} z^{n+2} \sum_{m=0,2,4}^{n} (-1)^{\frac{m}{2}} \frac{(n-m)!}{(n+2)!} \nabla^m a_{(n-m)}$$

$$+ \sum_{n=1,3,5}^{\infty} (-1)^{\frac{n-1}{2}} \frac{1}{n!} \nabla^{(n-1)} b_1 z^n$$

$$+ \sum_{n=1,3,5}^{\infty} z^{n+2} \sum_{m=1,3,5}^{n} (-1)^{\frac{m-1}{2}} \frac{(n-m+1)!}{(n+2)!} \nabla^{(m-1)} a_{(n-m+1)} \tag{31}$$

where b_o and b_1 are, respectively, the temperature distribution and temperature gradient in the z direction at the midplane ($z=0$). If the functions b_o and b_1 are known, the temperature distribution within the plate is determined by equation (31). However, the functions b_o and b_1 are not usually known apriori and must be determined in terms of temperature boundary conditions. Chapter VI B presents a method of determining b_o and b_1 in terms of specified temperature distributions over the top and bottom surfaces of the plate.

B. General Solution of Governing Equation With Specified Temperatures at the Plate Boundaries

In order to determine b_o and b_1 in terms of specified boundary temperatures, $T(x,y,h/2)$ and $T(x,y,-h/2)$, it is convenient to define the specified boundary temperatures in terms of two boundary temperature functions, T_o and P_o, such that

$$T_o = \frac{1}{2}\left[T(x,y,h/2) + T(x,y,-h/2)\right] \tag{32}$$

$$P_o = \frac{1}{2}\left[T(x,y,h/2) - T(x,y,-h/2)\right] \tag{33}$$

Expressing T_o and P_o in terms of b_o, b_1, and a_n by the use of equation (31) gives:

$$T_o = T_o' + \sum_{n=0,2,4}^{\infty} \left(\frac{h}{2}\right)^{n+2} \sum_{m=0,2,4}^{n} (-1)^{\frac{m}{2}} \frac{(n-m)!}{(n+2)!} \nabla^m a_{(n-m)} \tag{34}$$

$$P_o = P_o' + \sum_{n=1,3,5}^{\infty} \left(\frac{h}{2}\right)^{n+2} \sum_{m=1,3,5}^{n} (-1)^{\frac{m-1}{2}} \frac{(n-m+1)!}{(n+2)!} \nabla^{(m-1)} a_{(n-m+1)} \tag{35}$$

where

$$T_o' = \sum_{n=0,2,4}^{\infty} (-1)^{\frac{n}{2}} \frac{1}{n!} \nabla^n b_o \left(\frac{h}{2}\right)^n$$

and

$$P_o' = \sum_{n=1,3,5}^{\infty} (-1)^{\frac{n-1}{2}} \nabla^{(n-1)} b_1 \frac{1}{n!} \left(\frac{h}{2}\right)^n$$

It should be noted that equations (34) and (35) establish relationships between T_o and b_o independent of b_1, and between P_o and b_1 independent of b_o. Expressions for b_o and b_1 may be obtained in terms of T_o' and P_o', respectively, as outlined in the appendix. These expressions are given by:

$$b_o = \sum_{n=0,2,4}^{\infty} \frac{E_n}{n!} \left(\frac{h}{2}\right)^n \nabla^n T_o' \tag{36}$$

$$b_1 = \frac{2P_o'}{h} + \sum_{n=3,5,7}^{\infty} \frac{\beta\left(\frac{n-1}{2}\right)}{(n-1)!}\left(\frac{h}{2}\right)^{n-2} \nabla^{(n-1)} P_o' \tag{37}$$

where the constants E_n are the Euler numbers, and the constants $\beta\left(\frac{n-1}{2}\right)$ are related to the Bernoulli numbers.

When the solutions for b_o and b_1 are substituted into equation (31) the resulting temperature distribution will satisfy the governing Poisson equation (eq. (27)) and the specified temperature boundary conditions on the large surfaces of the plate.

C. Determination of General Expressions for the Plate Temperature Parameters

Expressions for the plate temperature parameters, φ_o and φ_1, may be obtained in terms of b_o and b_1 by substituting equation (31) into equations (10) and (11) and performing the indicated integrations. The resulting expressions for φ_o and φ_1 are given by:

$$\varphi_o = \varphi_o' + 2 \sum_{n=0,2,4}^{\infty} \frac{\left(\frac{h}{2}\right)^{n+3}}{n+3} \sum_{m=0,2,4}^{n} (-1)^{\frac{m}{2}} \frac{(n-m)!}{(n+2)!} \nabla^m a_{(n-m)} \tag{38}$$

$$\varphi_1 = \varphi_1' + 2 \sum_{n=1,3,5}^{\infty} \frac{\left(\frac{h}{2}\right)^{n+4}}{n+4} \sum_{m=1,3,5}^{n} (-1)^{\frac{m-1}{2}} \frac{(n-m+1)!}{(n+2)!} \nabla^{(m-1)} a_{(n-m+1)} \tag{39}$$

where

$$\varphi_o' = 2 \sum_{n=0,2,4}^{\infty} (-1)^{\frac{n}{2}} \frac{\nabla^n b_o}{(n+1)!}\left(\frac{h}{2}\right)^{n+1} \tag{40}$$

and

$$\varphi_1' = 2 \sum_{n=1,3,5}^{\infty} (-1)^{\frac{n-1}{2}} \frac{\nabla^{(n-1)} b_1}{(n+2)(n!)}\left(\frac{h}{2}\right)^{n+2} \tag{41}$$

The functions φ_o and φ_1 may thus be determined from equations (38) and (39) after obtaining expressions for b_o and b_1.

D. Temperature Distributions in Which $\nabla^2 T(x,y,h/2)$, $\nabla^2 T(x,y,-h/2)$, and $\nabla^2 a_n$ are Zero

The general solution for the temperature distribution presented in Chapter VI B imposed no restrictions on either the temperature distribution at the boundaries or the heating function. If the boundary temperature distributions and heating functions are restricted to the general class of functions which satisfy

$$\left.\begin{aligned} \nabla^2 T(x,y,h/2) &= 0 \\ \nabla^2 T(x,y,-h/2) &= 0 \\ \nabla^2 a_n &= 0 \end{aligned}\right\} \tag{42}$$

then the equations presented in Chapter IV B are greatly simplified and the resulting temperature distributions are more readily attainable.

The physical interpretation of the first two expressions of equation (42) is obvious. A physical interpretation of the third expression may be obtained after examination of equation (29). Imposing the restriction on a_n, given in equations (42), on equation (29) results in the requirement that

$$\nabla^2 A(x,y,z) = 0 \tag{43}$$

where ∇^2 is $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. This requires that the heating function, A, be a harmonic function with respect to the coordinates x and y.

The affect of these restrictions on the equations presented in Chapter IV B may be seen after the operator ∇^2 is applied to equations (32), (33), (34), and (35), and imposing the restrictions of equation (42). This procedure results in the following requirements:

$$\left.\begin{aligned} \nabla^2 T_o &= 0 \\ \nabla^2 P_o &= 0 \\ \nabla^2 T_o' &= 0 \\ \nabla^2 P_o' &= 0 \end{aligned}\right\} \tag{44}$$

After examination of the definitions of T_o' and P_o', it may be seen that the last two expressions of equations (44) further require that

$$\left.\begin{aligned} \nabla^2 b_o &= 0 \\ \nabla^2 b_1 &= 0 \end{aligned}\right\} \tag{45}$$

Substitution of equations (44) and the last expression in equations (42) into equation (31) gives the temperature distribution within the plate as

$$T = b_o + b_1 z + \sum_{n=0}^{\infty} \frac{a_n z^{n+2}}{(n + 1)(n + 2)} \tag{46}$$

The expressions for b_o and b_1 may be obtained from equations (36) and (37) after imposing the conditions of equations (45). The functions b_o and b_1 are then given by:

$$\left.\begin{aligned} b_o &= T_o' \\ b_1 &= \frac{2P_o'}{h} \end{aligned}\right\} \tag{47}$$

Expressing T_o' and P_o' in terms of T_o, P_o, and a_n from equations (34) and (35) gives b_o and b_1 as:

$$\left.\begin{aligned} b_o &= T_o - \sum_{n=0,2,4}^{\infty} \frac{\left(\frac{h}{2}\right)^{n+2} a_n}{(n + 2)(n + 1)} \\ b_1 &= \frac{2P_o}{h} - \sum_{n=1,3,5}^{\infty} \frac{\left(\frac{h}{2}\right)^{n+1} a_n}{(n + 2)(n + 1)} \end{aligned}\right\} \tag{48}$$

Applying these results to the plate temperature parameters φ_o and φ_1, given by equations (38) and (39) gives:

$$\left.\begin{aligned}
\varphi_o &= b_o h + 2 \sum_{n=0,2,4}^{\infty} \frac{\left(\frac{h}{2}\right)^{n+3} a_n}{(n+3)(n+2)(n+1)} \\
\varphi_1 &= \frac{b_1 h^3}{12} + 2 \sum_{n=1,3,5}^{\infty} \frac{\left(\frac{h}{2}\right)^{n+4} a_n}{(n+4)(n+2)(n+1)}
\end{aligned}\right\} \qquad (49)$$

Note from equations (49) that both $\nabla^2\varphi_o$ and $\nabla^2\varphi_1$ are zero.

In Chapter VII C, an example plate deformation problem is presented in which the plate temperature parameters are of the form given in equations (49).

VII. EXAMPLE PROBLEMS OF PLATE DEFORMATION DUE TO INTERNAL HEATING

A. Out-of-Plane Deformation With Homogeneous Boundary Conditions on w

It has been shown (see ref. 2, page 391) that if the plate boundary conditions on the out-of-plane deformation, w, are homogeneous, the solution for w is identical with that for the same plate with a uniform temperature distribution and a transverse loading proportional to $\nabla^2\varphi_1$. A method for determining out-of-plane deformation with homogeneous boundary conditions on w and an arbitrary transverse loading is given in reference 2, section 12.4(d). The quantity $\nabla^2\varphi_1$ is determined by the procedure outlined in Chapter VI.

For many plate problems the temperature distribution may be such that the out-of-plane deformation is zero. It is therefore of interest to determine the conditions on the boundary temperatures and heating function which result in zero out-of-plane deformation. The obvious requirement for zero out-of-plane deformation, when the boundary conditions on w are homogeneous, is that

$$\nabla^2\varphi_1 = 0 \tag{50}$$

From equation (39) it may be seen that, in general, the requirements that $\nabla^2\varphi_1$ vanish are

$$\nabla^2 b_1 = 0 \tag{51}$$

and

$$\nabla^2 a_n = 0 \tag{52}$$

These requirements, when put in terms of the boundary temperature functions (eqs. (34) and (35)), put no restrictions on T_o and only impose the requirement that (see eq. (35))

$$\nabla^2 P_o = 0 \tag{53}$$

Thus, if $\nabla^2 a_n$ and $\nabla^2 P_o$ are both zero, no out-of-plane deformation will occur when there are homogeneous boundary conditions on w.

B. Temperature Distributions and Plate Deformations Caused by Constant Heat Generation Within the Plate

An example in which, A, the heat generation per unit time per unit volume, may be considered constant occurs during the curing process of a concrete slab. In this case the coefficients, a_n, for the series defined by equation (29) would be

$$a_o = -\frac{A}{K}$$

and (54)

$$a_n = 0 \quad \text{for} \quad n > 0$$

The temperature distribution given by equation (31) thus reduces to

$$T = \sum_{n=0,2,4}^{\infty} (-1)^{\frac{n}{2}} \frac{\nabla^n b_o z^n}{n!} + \sum_{n=1,3,5}^{\infty} (-1)^{\frac{n-1}{2}} \frac{\nabla^{(n-1)} b_1 z^n}{n!} - \frac{Az^2}{2K} \tag{55}$$

The relationships between T_o and b_o, and P_o and b_1 are obtained from equations (34) and (35) and are given by

$$\begin{aligned} T_o &= T_o' - \frac{h^2 A}{8K} \\ P_o &= P_o' \end{aligned} \tag{56}$$

The plate temperature parameters are obtained from equations (38) and (39) and are given by

$$\left.\begin{aligned} \varphi_o &= \varphi_o' - \frac{h^3 A}{24K} \\ \varphi_1 &= \varphi_1' \end{aligned}\right\} \tag{57}$$

Equations (56) and (57) indicate that the plate temperature parameter φ_o is the only plate temperature parameter affected during constant heat generation. Thus, constant internal heat generation does not affect the out-of-plane plate deformation, w, inasmuch as φ_o does not appear in the plate equations for the w deflection (see Chapter V). A discussion of the deformations u_o and v_o is given in Chapter VII C.

If it is assumed that $\nabla^2 T_o$ and $\nabla^2 P_o$ are zero, then $\nabla^2 b_o$ and $\nabla^2 b_1$ are also zero (see Chapter VI D), and the plate temperature distribution is obtained from equations (46) and (48) as

$$T = \frac{A}{2K}\left(\frac{h^2}{4} - z^2\right) + \frac{2P_o z}{h} + T_o \tag{58}$$

The corresponding plate temperature parameters are obtained from equations (48) and (49) and are given by

$$\varphi_o = T_o h + \frac{h^3 A}{12K} \tag{59}$$

and

$$\varphi_1 = \frac{P_o h^2}{6} \tag{60}$$

C. Temperature Distributions and Plate Deformations Resulting From Absorption of Electromagnetic Radiation

1. General solution for temperature distributions with specified temperatures at the plate boundaries

As a further example, assume that some form of electromagnetic radiation is being absorbed within the plate according to the Lambert Law of Absorption (see refs. 7 and 8). Then

$$A(x,y,z) = \beta I_o(x,y)e^{-\beta\left(\frac{h}{2} - z\right)} \tag{61}$$

where I_o is the energy incident and normal to the plane surface $z = \frac{h}{2}$ (this assumes no reflective losses), and β is the linear absorption coefficient. The function $f(x,y,z)$, as defined in equation (29), is given by

$$f(x,y,z) = -\frac{\beta I_o(x,y)}{K} e^{-\beta\left(\frac{h}{2} - z\right)} \tag{62}$$

from which

$$\sum_{n=0}^{\infty} a_n z^n = -\frac{\beta I_o(x,y)}{K} e^{-\beta\left(\frac{h}{2} - z\right)} \tag{63}$$

Expanding the right side of equation (63) in a power series in z and equating like powers of z gives the expression for a_n as

$$a_n(x,y) = -\frac{I_o(x,y)e^{-\frac{\beta h}{2}}\beta^{n+1}}{Kn!} \tag{64}$$

Substitution of equation (64) into equation (31) gives the temperature equation as:

$$T = \sum_{n=0,2,4}^{\infty} (-1)^{\frac{n}{2}} \frac{\nabla^n b_o}{n!} z^n + \sum_{n=1,3,5}^{\infty} (-1)^{\frac{n-1}{2}} \frac{\nabla^{(n-1)} b_1 z^n}{n!}$$

$$- \sum_{n=0,2,4}^{\infty} (-1)^{\frac{n}{2}} \frac{\nabla^n I_o e^{-\frac{\beta h}{2}}}{K\beta^{(n+1)}} \left[e^{\beta z} - \sum_{m=0,1,2,3}^{n+1} \frac{(\beta z)^m}{m!} \right] \tag{65}$$

In order to satisfy the specified temperature boundary conditions, the functions b_o and b_1 in equation (65) must be determined in terms of T_o' and P_o' as is outlined in Chapter VI B. The functions T_o' and P_o' are found by solving equations (65) for $T(x,y,h/2)$ and $T(x,y,-h/2)$, and substituting the resulting expressions into equations (32) and (33). This gives T_o and P_o as

$$T_o = T_o' - \frac{1}{2} \sum_{n=0,2,4}^{\infty} (-1)^{\frac{n}{2}} \frac{\nabla^n I_o}{K\beta^{(n+1)}} \left[-e^{-\beta h} + 1 - 2e^{-\frac{\beta h}{2}} \sum_{m=0,2,4}^{n} \frac{1}{m!} \left(\frac{\beta h}{2} \right)^m \right] \tag{66}$$

and

$$P_o = P_o' - \frac{1}{2} \sum_{n=0,2,4}^{\infty} (-1)^{\frac{n}{2}} \frac{\nabla^n I_o}{K\beta^{(n+1)}} \left[-e^{-\beta h} + 1 - 2e^{-\frac{\beta h}{2}} \sum_{m=1,3,5}^{n+1} \frac{1}{m!} \left(\frac{\beta h}{2} \right)^m \right] \tag{67}$$

Equations (66) and (67) may be solved directly for T_o' and P_o'.

Appropriately integrating equation (65) for φ_o and φ_1 gives:

$$\varphi_o = \varphi_o' - 2 \sum_{n=0,2,4}^{\infty} (-1)^{\frac{n}{2}} \frac{\nabla^n I_o e^{-\frac{\beta h}{2}}}{K\beta^{(n+2)}} \left[\sinh \frac{\beta h}{2} - \sum_{m=0,2,4}^{n} \left(\frac{\beta h}{2}\right)^{m+1} \frac{1}{(m+1)!} \right] \tag{68}$$

$$\varphi_1 = \varphi_1' - 2 \sum_{n=0,2,4}^{\infty} (-1)^{\frac{n}{2}} \frac{\nabla^n I_o e^{-\frac{\beta h}{2}}}{K\beta^{(n+3)}} \left[\frac{\beta h}{2} \cosh \frac{\beta h}{2} - \sinh \frac{\beta h}{2} - \sum_{m=1,3,5}^{n+1} \left(\frac{\beta h}{2}\right)^{m+2} \frac{1}{(m+2)m!} \right] \tag{69}$$

Equations (68) and (69) are the general expressions for φ_o and φ_1 in terms of b_o and b_1. The expressions for φ_o and φ_1 for specified temperatures at the boundaries are obtained by expressing the functions b_o and b_1 in terms of T_o' and P_o' as outlined in Chapter VI.

2. Temperature distributions in which $\nabla^2 T(x,y,h/2)$, $\nabla^2 T(x,y,-h/2)$, and $\nabla^2 a_n$ are zero

If the conditions on the boundary temperatures and heating function are such that

$$\left.\begin{aligned} \nabla^2 T(x,y,h/2) &= 0 \\ \nabla^2 T(x,y,-h/2) &= 0 \\ \nabla^2 a_n &= 0 \end{aligned}\right\} \tag{70}$$

then, as shown in Chapter VI D:

$$\left.\begin{aligned} \nabla^2 T_o &= 0 \\ \nabla^2 P_o &= 0 \\ \nabla^2 T_o' &= 0 \\ \nabla^2 P_o' &= 0 \end{aligned}\right\} \tag{71}$$

$$\left.\begin{aligned} \nabla^2 b_o &= 0 \\ \nabla^2 b_1 &= 0 \\ \nabla^2 \varphi_o &= 0 \\ \nabla^2 \varphi_1 &= 0 \end{aligned}\right\} \tag{71}$$

Furthermore, it may be seen from equation (64) that the restriction on a_n requires that

$$\nabla^2 I_o = 0 \tag{72}$$

Substituting equations (71) and (72) into equations (65) through (69) gives:

$$T = b_o + b_1 z - \frac{I_o e^{-\frac{\beta h}{2}}}{K\beta}\left(e^{\beta z} - 1 - \beta z\right) \tag{73}$$

$$T_o = b_o - \frac{I_o}{2K\beta}\left(e^{-\beta h} + 1 - 2e^{-\frac{\beta h}{2}}\right) \tag{74}$$

$$P_o = \frac{b_1 h}{2} - \frac{I_o}{2K\beta}\left(-e^{-\beta h} + 1 - \beta h e^{-\frac{\beta h}{2}}\right) \tag{75}$$

$$\varphi_o = b_o h - \frac{2I_o e^{-\frac{\beta h}{2}}}{K\beta^2}\left(\sinh\frac{\beta h}{2} - \frac{\beta h}{2}\right) \tag{76}$$

$$\varphi_1 = \frac{b_1 h^3}{12} - \frac{2I_o e^{-\frac{\beta h}{2}}}{K\beta^3}\left[\frac{\beta h}{2}\cosh\frac{\beta h}{2} - \sinh\frac{\beta h}{2} - \frac{(\beta h)^3}{24}\right] \tag{77}$$

Solving equations (74) and (75) for b_o and b_1, respectively, and substituting the resulting expressions into equations (76) and (77) gives φ_o and φ_1 in terms of the temperature boundary functions. These expressions are given by:

$$\left.\begin{aligned}
\varphi_0 &= T_0 h + \frac{I_0 e^{-\frac{\beta h}{2}}}{K\beta}\left[h \cosh \frac{\beta h}{2} - \frac{2}{\beta} \sinh \frac{\beta h}{2}\right] \\
\varphi_1 &= \frac{h^2 P_0}{6} + \frac{h^2 I_0 e^{-\frac{\beta h}{2}}}{6K\beta}\left[\sinh \frac{\beta h}{2} - \frac{\beta h}{2}\right] \\
&\quad - \frac{2 I_0 e^{-\frac{\beta h}{2}}}{K\beta^3}\left[\frac{\beta h}{2} \cosh \frac{\beta h}{2} - \sinh \frac{\beta h}{2} - \frac{(\beta h)^3}{24}\right]
\end{aligned}\right\} \qquad (78)$$

3. Example problem of plate deformations resulting from radiation absorption

As an example of plate deformation resulting from electromagnetic radiation absorption, it is assumed that the equations describing the plate temperature parameters are given by equations (78) where T_0 and P_0 are equal to $\frac{T_m}{2}$. These conditions on T_0 and P_0 would result from temperature distributions over the top and bottom surfaces of the plate given by:

$$\left.\begin{aligned}
T(x,y,h/2) &= T_m(x,y) \\
T(x,y,-h/2) &= 0
\end{aligned}\right\} \qquad (79)$$

Furthermore, it is assumed that the plate boundary conditions are such that the edges of the plate are restrained in a manner which prohibits their deformation in the z direction and allows the edges to move freely in the x and y directions. The latter assumption is made inasmuch as the plate theory presented herein does not include the effect of in-plane loads on deformation in the z direction; and any edge restraint on the u_0 and v_0 deformations would, of course, result in large in-plane loads at the edges of the plate, and probably buckling of the plate.

The plate governing equations and boundary conditions resulting from these assumptions are obtained from equations (13) through (24). The governing equations are:

$$\nabla^4 w = 0 \tag{80}$$

$$(2\mu + \lambda)\frac{\partial^2 u_o}{\partial y^2} + 4(\mu + \lambda)\frac{\partial^2 u_o}{\partial x^2} + (3\lambda + 2\mu)\left[\frac{\partial^2 v_o}{\partial x \partial y} - \frac{2\alpha}{h}\frac{\partial \varphi_o}{\partial x}\right] = 0 \tag{81}$$

$$(2\mu + \lambda)\frac{\partial^2 v_o}{\partial x^2} + 4(\mu + \lambda)\frac{\partial^2 v_o}{\partial y^2} + (3\lambda + 2\mu)\left[\frac{\partial^2 u_o}{\partial x \partial y} - \frac{2\alpha}{h}\frac{\partial \varphi_o}{\partial y}\right] = 0 \tag{82}$$

The boundary conditions at $x = 0$ and $x = a$ are given by:

$$w = 0 \tag{83}$$

$$2(\mu + \lambda)\frac{\partial^2 w}{\partial x^2} + \frac{12\alpha\varphi_1}{h^3}(3\lambda + 2\mu) = 0 \tag{84}$$

$$\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} = 0 \tag{85}$$

$$2(\mu + \lambda)\frac{\partial u_o}{\partial x} + \lambda\frac{\partial v_o}{\partial y} - \frac{\alpha\varphi_o}{h}(3\lambda + 2\mu) = 0 \tag{86}$$

The boundary conditions at $y = 0$ and $y = b$ are given by:

$$w = 0 \tag{87}$$

$$2(\mu + \lambda)\frac{\partial^2 w}{\partial y^2} + \frac{12\alpha\varphi_1}{h^3}(3\lambda + 2\mu) = 0 \tag{88}$$

$$\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x} = 0 \tag{89}$$

$$2(\mu + \lambda)\frac{\partial v_o}{\partial y} + \lambda \frac{\partial u_o}{\partial x} - \frac{\alpha\varphi_o}{h}(3\lambda + 2\mu) = 0 \tag{90}$$

An expression which satisfies the governing equation on w and the boundary conditions given by equations (84) and (88) is given by:

$$\nabla^2 w = - \frac{6\alpha\varphi_1(3\lambda + 2\mu)}{h^3(\mu + \lambda)} \tag{91}$$

when the w deflection is constant at the edges of the plate. The deformation, w, may be found by first expressing w and the right side of equation (91) in terms of a double sine series. The desired solution for w is then found by imposing the operator ∇^2 on the series for w and equating coefficients of like terms in the series for $\nabla^2 w$ and $-\frac{6\alpha\varphi_1(3\lambda + 2\mu)}{h^3(\mu + \lambda)}$. A solution for the w deformation in which φ_1 is constant (i.e., T_m and I_o are constant) over the plate planform is given by:

$$w = \frac{96\alpha\varphi_1(3\lambda + 2\mu)}{h^3\pi^4(\mu + \lambda)} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}}{mn\left[\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2\right]} \tag{92}$$

where φ_1 is given by equation (78). This type of solution automatically satisfies the zero deflection condition on w at the edges of the plate.

Solutions for u_o and v_o may be obtained by expressing u_o and v_o as

$$u_o = \frac{\alpha}{h} \int \varphi_o \, dx \tag{93}$$

$$v_o = \frac{\alpha}{h} \int \varphi_o \, dy \tag{94}$$

These expressions satisfy both the governing equations on u_o and v_o when $\nabla^2\varphi_o = 0$, and the free edge boundary conditions imposed by equations (86) and (90). The midsurface shearing strain boundary conditions given by equations (85) and (89) are not satisfied, however, unless

$$\int \frac{\partial\varphi_o}{\partial y} \, dx + \int \frac{\partial\varphi_o}{\partial x} \, dy = 0 \tag{95}$$

at the boundaries of the plate. Expressions for φ_o satisfying equation (95) in no way contradict the requirement that $\nabla^2\varphi_o = 0$ inasmuch as

$$\frac{\partial^2}{\partial x \partial y}\left[\int \frac{\partial\varphi_o}{\partial y} \, dx + \int \frac{\partial\varphi_o}{\partial x} \, dy\right] = \nabla^2\varphi_o \tag{96}$$

Thus, equation (95) merely imposes a further restriction on φ_o.

For example, if φ_o is of the form

$$\varphi_o = f(x)g(y) \tag{97}$$

all solutions satisfying $\nabla^2\varphi_o = 0$ also satisfy equation (95) with the exception of solutions of the forms and

$$\left.\begin{aligned} \varphi_o &= c_1 x \\ \varphi_o &= c_1 y \\ \varphi_o &= c_1 xy \end{aligned}\right\} \tag{98}$$

where c_1 is an arbitrary constant (not equal to zero). In performing the integrations for u_o and v_o indicated by equations (93) and (94), the arbitrary constants may be assumed zero, thus referencing the u_o and v_o deformations to the origin of the coordinate system. For the case in which both T_m and I_o are constant, φ_o is also constant, and u_o and v_o are given by:

$$\left.\begin{aligned} u_o &= \frac{\alpha\varphi_o x}{h} \\ v_o &= \frac{\alpha\varphi_o y}{h} \end{aligned}\right\} \tag{99}$$

where φ_o is given by equation (78).

The plate temperature distribution for this particular problem may be obtained from equation (73) by obtaining the appropriate expressions for b_o and b_1. The expressions for b_o and b_1 are obtained from equations (74) and (75), respectively, and are given by:

$$b_o = \frac{T_m}{2} + \frac{I_o}{2K\beta}\left[e^{-\beta h} + 1 - 2e^{-\frac{\beta h}{2}}\right] \tag{100}$$

$$b_1 = \frac{T_m}{h} + \frac{I_o}{hK\beta}\left[-e^{-\beta h} + 1 - \beta h e^{-\frac{\beta h}{2}}\right] \tag{101}$$

Substitution of equations (100) and (101) into equation (73) gives the temperature distribution within the plate as:

$$T = T_m\left(\frac{1}{2} + \frac{z}{h}\right) + \frac{I_o}{K\beta}\left[\frac{e^{-\beta h}}{2} + \frac{1}{2} + \frac{z}{h}(1 - e^{-\beta h}) - e^{-\beta\left(\frac{h}{2} - z\right)}\right] \tag{102}$$

which, of course, satisfies the governing equation (eq. (27)) and the boundary conditions at $z = \frac{h}{2}$ and $z = -\frac{h}{2}$. If the magnitude of β approaches infinity, which is the case for materials in which the incident radiation is absorbed essentially at the surface of the material $\left(z = \frac{h}{2}\right)$, then, from equation (102), the temperature distribution within the plate is given by:

$$T = T_m\left(\frac{1}{2} + \frac{z}{h}\right) \tag{103}$$

which is the linear temperature distribution that would be expected in a plate with boundary conditions as stated for this problem. Similarly, if β is zero, which is the case in which no incident radiation is absorbed by the plate $(A = 0)$, the temperature distribution again reduces to equation (103).

VIII. RESULTS AND DISCUSSION

It has been shown that the determination of small-deflection plate deformations caused by thermal stresses requires the determination of two temperature parameters denoted herein as φ_o and φ_1. The parameter φ_o is proportional to the average temperature distribution through the plate thickness, and appears only in the plate deformation equations determining the in-plane plate deformations u_o and v_o. The parameter φ_1 is proportional to the moment of the temperature distribution with respect to the midplane of the plate, and appears only in the plate deformation equations determining the out-of-plane deformation w.

The temperature distribution within the plate, which determines the quantities φ_o and φ_1, is entirely dependent upon the method of heat transfer within the plate, the distribution and magnitude of the heat sources within the plate, and the temperature boundary conditions. If conduction is the mode of heat transfer, the governing equation on the temperature, for steady-state conditions, is Poisson's equation. Inasmuch as Poisson's equation is an elliptic equation, the temperature distribution is uniquely determined when either the temperature distribution at the boundaries, the temperature gradient normal to the boundaries, or a linear combination of these two quantities is specified over the entire surface (see ref. 6, Chapter 6). In the temperature analysis presented herein, the general equation for the temperature distribution is given in terms of the functions b_o and b_1 which are, respectively, the midplane temperature and temperature gradient in the thickness direction. These functions are dependent upon the temperature boundary conditions over the entire surface of the plate. However, it is assumed that the temperature boundary conditions at the edges of the plate will have little effect on the overall temperature distribution and, therefore, these temperature boundary conditions may be neglected. An estimate of the magnitude of error in the temperature distribution resulting from this assumption is possible after cursory comparison between the actual temperature boundary conditions at the edges of the plate and those calculated from the analysis.

After eliminating the necessity for satisfying temperature boundary conditions at the edges of the plate, it is apparent that the functions b_o and b_1 need only satisfy the temperature boundary conditions over the large surfaces of the plate. A general solution for the temperature distribution, which is valid for any type of internal heat generation, is presented for the case in which the temperature distributions over the large surfaces of the plate are known. This solution is used to determine general expressions for the temperature distribution in plates which are subjected to either constant heat generation or radiation absorption throughout the volume of the plate. The results indicate that constant heat generation within the plate does not affect the out-of-plane plate deformation, and the out-of-plane deformation is only dependent on the boundary temperatures. This is not surprising after examination of the equation for φ_1. From this equation, it may be seen that any heat generating function which is symmetric with respect to the midplane of the plate will contribute nothing to the moment of the temperature distribution, upon which w is dependent. For the case of radiation absorption, the heat generating function is not symmetric with respect to the midplane, and the resulting out-of-plane deformation is dependent on the intensity of the incoming radiation and the absorption characteristics of the material. It is also shown that if the plate boundary conditions on w are homogeneous, the out-of-plane deformation is zero when both the temperature difference between the upper and lower surfaces of the plate and the planform distribution of the heating function satisfy the two-dimensional Laplace equation in terms of the coordinates x and y.

Solutions for the plate deformations u_o and v_o must satisfy two coupled governing equations and coupled boundary conditions on u_o and v_o. The method that is presented for the determination of these deformations requires that $\nabla^2\varphi_o$ be zero, and that the plate boundary conditions allow expansion in the plane of the plate. These boundary conditions were selected to minimize the magnitude of the in-plane loads caused by the thermal expansion of the plate. For problems in which the in-plane loadings are of sufficient magnitude to cause out-of-plane deformation, the plate theory presented herein is no longer valid for the determination of the plate deformations. Discussions of this problem are given in reference 2, Chapter 13.

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XI. VITA

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Robert A. McWithey

XII. APPENDIX

The temperature distribution within the plate as defined by equation (31) is dependent on the unknown midplane temperature functions b_o and b_1. Equations (34) and (35) relate b_o and b_1 to the known boundary temperature functions and heat generating function but are not directly solvable for b_o and b_1. Expressions for b_o and b_1 may be found in terms of the known functions T_o' and P_o', however, by appropriately reversing the series equations for T_o' and P_o'. For example, a set of equations involving T_o' and its derivatives may be obtained in the form:

$$\left.\begin{aligned}
T_o' &= b_o - \frac{\nabla^2 b_o}{2!}\left(\frac{h}{2}\right)^2 + \frac{\nabla^4 b_o}{4!}\left(\frac{h}{2}\right)^4 - \frac{\nabla^6 b_o}{6!}\left(\frac{h}{2}\right)^6 + \frac{\nabla^8 b_o}{8!}\left(\frac{h}{2}\right)^8 - \cdots \\
\frac{C_2}{2!}\left(\frac{h}{2}\right)^2 \nabla^2 T_o' &= C_2\frac{\nabla^2 b_o}{2!}\left(\frac{h}{2}\right)^2 - \frac{C_2}{2!}\frac{\nabla^4 b_o}{2!}\left(\frac{h}{2}\right)^4 + \frac{C_2}{2!}\frac{\nabla^6 b_o}{4!}\left(\frac{h}{2}\right)^6 - \frac{C_2}{2!}\frac{\nabla^8 b_o}{6!}\left(\frac{h}{2}\right)^8 + \cdots \\
\frac{C_4}{4!}\left(\frac{h}{2}\right)^4 \nabla^4 T_o' &= C_4\frac{\nabla^4 b_o}{4!}\left(\frac{h}{2}\right)^4 - \frac{C_4}{4!}\frac{\nabla^6 b_o}{2!}\left(\frac{h}{2}\right)^6 + \frac{C_4}{4!}\frac{\nabla^8 b_o}{4!}\left(\frac{h}{2}\right)^8 - \cdots \\
\frac{C_6}{6!}\left(\frac{h}{2}\right)^6 \nabla^6 T_o' &= C_6\frac{\nabla^6 b_o}{6!}\left(\frac{h}{2}\right)^6 - \frac{C_6}{6!}\frac{\nabla^8 b_o}{2!}\left(\frac{h}{2}\right)^8 + \cdots \\
&\cdots\cdots\cdots\cdots
\end{aligned}\right\} \quad \text{(A-1)}$$

where C_n are arbitrary constants. If these constants satisfy the set of equations given by:

$$\left.\begin{aligned}
&C_2 - 1 = 0 \\
&C_4 - \frac{C_2(4)(3)}{2!} + 1 = 0 \\
&C_6 - \frac{C_4(6)(5)}{2!} + \frac{C_2(6)(5)}{2!} - 1 = 0 \\
&C_8 - \frac{C_6(8)(7)}{2!} + \frac{C_4(8)(7)(6)(5)}{4!} - \frac{C_2(8)(7)}{2!} + 1 = 0 \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}\right\} \qquad \text{(A-2)}$$

then the summation of equations (A-1) is simply

$$b_o = \sum_{n=0,2,4}^{\infty} \frac{C_n}{n!}\left(\frac{h}{2}\right)^n \nabla^n T_o' \qquad \text{(A-3)}$$

Equations (A-2) are of the form presented in reference 9, series 1145, and it is immediately apparent that equations (A-2) are satisfied if:

$$C_n = E_n \qquad \text{(A-4)}$$

where the constants E_n are Euler numbers and are defined in reference 9, pages 238 and 239. Thus, equation (A-3) becomes:

$$b_o = \sum_{n=0,2,4}^{\infty} \frac{E_n}{n!}\left(\frac{h}{2}\right)^n \nabla^n T_o' \qquad \text{(A-5)}$$

In determining b_1 in terms of P_o', a set of equations may be obtained in the form

$$\left.\begin{aligned}
P_o' &= \frac{b_1 h}{2} - \frac{\nabla^2 b_1}{3!}\left(\frac{h}{2}\right)^3 + \frac{\nabla^4 b_1}{5!}\left(\frac{h}{2}\right)^5 - \frac{\nabla^6 b_1}{7!}\left(\frac{h}{2}\right)^7 + \frac{\nabla^8 b_1}{9!}\left(\frac{h}{2}\right)^9 - \cdots \\
\frac{D_1}{2!}\left(\frac{h}{2}\right)^2 \nabla^2 P_o' &= D_1 \frac{\nabla^2 b_1}{2!}\left(\frac{h}{2}\right)^3 - D_1 \frac{\nabla^4 b_1}{2!3!}\left(\frac{h}{2}\right)^5 + D_1 \frac{\nabla^6 b_1}{2!5!}\left(\frac{h}{2}\right)^7 - D_1 \frac{\nabla^8 b_1}{2!7!}\left(\frac{h}{2}\right)^9 + \cdots \\
\frac{D_2}{4!}\left(\frac{h}{2}\right)^4 \nabla^4 P_o' &= D_2 \frac{\nabla^4 b_1}{4!}\left(\frac{h}{2}\right)^5 - D_2 \frac{\nabla^6 b_1}{4!3!}\left(\frac{h}{2}\right)^7 + D_2 \frac{\nabla^8 b_1}{4!5!}\left(\frac{h}{2}\right)^9 - \cdots \\
\frac{D_3}{6!}\left(\frac{h}{2}\right)^6 \nabla^6 P_o' &= D_3 \frac{\nabla^6 b_1}{6!}\left(\frac{h}{2}\right)^7 - D_3 \frac{\nabla^8 b_1}{6!3!}\left(\frac{h}{2}\right)^9 + \cdots \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}\right\} \qquad (A\text{-}6)$$

where the constants D_m are arbitrary constants. If these constants satisfy the set of equations given by

$$\left.\begin{aligned}
&\frac{D_1}{2!} - \frac{1}{3!} = 0 \\
&\frac{D_2}{4!} - \frac{D_1}{2!3!} + \frac{1}{5!} = 0 \\
&\frac{D_3}{6!} - \frac{D_2}{4!3!} + \frac{D_1}{2!5!} - \frac{1}{7!} = 0 \\
&\frac{D_4}{8!} - \frac{D_3}{6!3!} + \frac{D_2}{4!5!} - \frac{D_1}{2!7!} + \frac{1}{9!} = 0 \\
&\cdots\cdots\cdots\cdots
\end{aligned}\right\} \qquad \text{(A-7)}$$

then the summation of equations (A-6) is simply

$$\frac{b_1 h}{2} = P_0' + \sum_{n=3,5,7}^{\infty} \frac{D_{\frac{n-1}{2}}}{(n-1)!}\left(\frac{h}{2}\right)^{n-1} \nabla^{(n-1)} P_0' \qquad \text{(A-8)}$$

Using equation 1135 of reference 9 along with the identities

$$\left.\begin{aligned}
&\beta_n = (2^{2n} - 2)B_n && \text{for all } n \\
&B_n(1) = 0 && \text{for all } n \\
&B_n\left(\tfrac{1}{2}\right) = 0 && \text{for odd } n
\end{aligned}\right\} \qquad \text{(A-9)}$$

which are listed respectively in equations 1130, 1136, and 1142 of reference 9, it may be shown that, for odd values of $n \geq 3$

$$0 = -\frac{1}{n!} + \frac{\beta_1}{2!(n-2)!} - \frac{\beta_2}{4!(n-4)!} + \cdots - \frac{(-1)^{\frac{n-1}{2}} \beta_{\frac{n-1}{2}}}{(n-1)!} \qquad \text{(A-10)}$$

Equations obtained from (A-10) have the same form as equations (A-7). Equations (A-7) are therefore satisfied if

$$D_{\frac{n-1}{2}} = \beta_{\frac{n-1}{2}} \qquad \text{(A-11)}$$

where the constants $\beta_{\frac{n-1}{2}}$ are related to the Bernoulli numbers as indicated by equation (A-9). Thus equation (A-8) becomes

$$b_1 = \frac{2P_o'}{h} + \sum_{n=3,5,7}^{\infty} \frac{\beta_{\left(\frac{n-1}{2}\right)}}{(n - 1)!} \left(\frac{h}{2}\right)^{n-2} \nabla^{(n-1)} P_o' \qquad \text{(A-12)}$$

THERMAL DEFORMATIONS OF

PLATES PRODUCED BY TEMPERATURE DISTRIBUTIONS

SATISFYING POISSON'S EQUATION

By

Robert R. McWithey

ABSTRACT

Small-deflection plate equations are presented in terms of the midplane plate deformations and the temperature distribution within the plate, which is assumed independent of the plate deformation. The plate boundary conditions are presented in a general form and are suitable for solutions involving either fixed, free, or hinged edge conditions.

The temperature distribution within the plate is assumed to be governed by Poisson's equation and a specified temperature distribution over the surfaces of the plate. Solutions for the temperature distribution are given in terms of a power series with respect to the plate thickness coordinate, the coefficients of which are dependent on the midplane temperature distribution and the midplane temperature gradient in the plate thickness direction.

Out-of-plane plate deformations are discussed for plates with fixed edges. Discussions of plate deformations are also presented in which the temperature distributions result from constant heat generation within the plate and from radiation absorption.