MATRIX ANALYSIS OF RECTANGULAR RIGID FRAMES

by

William Paul Murden, Jr.

Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

MASTER OF SCIENCE

in

Applied Mechanics

APPROVED:

Director of Graduate Studies	Head of Department
Dean of Engineering	Supervisor

May 1951

Blacksburg, Virginia

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II. LIST OF SYMBOLS

Symbol	Definition
$[A]$	square or rectangular matrix
$\{A\}$	column matrix
a_{ij}	element in i^{th} row and j^{th} column of $[A]$
a_i	element in i^{th} row of $\{A\}$
Det $[A]$	determinant of $[A]$
H_i	algebraic summation of horizontal shears in top of columns of the i^{th} story; positive to right
K_{ij}	relative stiffness of member ij; equal to the moment of inertia divided by the length if the member is prismatic
ΣK_i	summation of the relative stiffnesses of all members meeting at joint i
L_i	height of i^{th} story
M_{ij}	bending moment at joint i in member ij; positive sense, clockwise on the member
M_i	algebraic summation of bending moments at joint i (i literal); or summation of end moments in columns of the i^{th} story (i numerical)
M'_{ij}	fixed-end moment at joint i in member ij
M'_i	algebraic summation of fixed-end moments at joint i (i literal); or summation of fixed-end moments in columns of i^{th} story (i numerical)

Symbol	Definition
norm $[A]$	square root of the sum of the squares of the elements of $[A]$
r	order of a square submatrix
s	order of a square submatrix
(r, s)	order of a rectangular matrix; r columns, s rows
θ_i	rotation, in radians, of joint i, positive sense, clockwise
ϕ_i	$2E\theta_i$, where E is Young's modulus of elasticity
Δ_i	lateral translation of joint i; in simple frames, positive to left; in complex frames, sign convention optional
α_i	$\frac{6E\Delta_i}{L_i}$
Σ	algebraic summation

Special matrices are defined when introduced.

III. INTRODUCTION

Any analysis of a statically indeterminate structure requires the solution of a number of simultaneous linear equations, the number depending upon the degree of indeterminacy. One of the most familiar forms for the expression of these equations, in the case of rigidly-connected plane frames, employs joint rotations and translations as redundants. Using these slope-deflection equations for a rectangular frame, it is necessary to determine the rotation of each joint, and each independent joint translation. The equations most commonly solved involve the equilibrium of bending moments at each rotating joint and the shear equilibrium of one free-body for each independent translation.

Matrix notation has long been used for the compact representation of systems of simultaneous linear equations. In the structural field, S. U. Benscoter(1)* has published matrix analyses of continuous beams, and Pei-Ping Chen(3) has discussed matrix analysis of pin-connected structures. Most engineers, including a majority of those who discussed the articles of Benscoter and Chen, have considered this compactness of notation to be the only advantage possessed by matrix methods. Actual solution of matrix equations requires the inversion of matrices, a time-consuming process when classical methods are used. Consequently, for matrix methods to be feasible, rapid routines for inversion are essential.

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* Numbers refer to Bibliography.

The purpose of the present thesis is to investigate the feasibility of matrix methods of analysis for rigidly-connected structural frames in one and two dimensions. Frames are analyzed by means of the slope-deflection equations, expressed in matrix notation. Solution of the matrix equations by alternative methods of inversion is investigated. The various routines for inversion are compared with each other and with more familiar methods of analysis, from the viewpoints of simplicity of operations, rapidity, and accuracy.

The value of matrix methods of analysis will be judged, not by student engineers, but by design engineers and engineering faculties. Both groups, the author feels, are intimately acquainted with common analytical procedures such as moment distribution[5], and the presentation of such methods in these pages would serve no useful purpose. If convinced of the utility of matrix methods, structural engineers would be willing to learn the fundamentals of matrix algebra. It would be audacious of the author to attempt to present these fundamentals in a paper of this nature when they are so thoroughly and adequately discussed in such texts as that of Frazer, Duncan, and Collar[8].

IV. THE REVIEW OF LITERATURE

S. U. Benscoter(1) broached the possibility of using matrix methods for the analysis of statically indeterminate structures. His article, "Matrix Analysis of Continuous Beams," appeared in the Transactions, American Society of Civil Engineers, in 1947, and was devoted exclusively to the analysis of continuous beams on non-settling supports. Benscoter discussed the matrix representation of the slope-deflection equations, method of three moments, method of three slopes, and moment distribution. In each form of analysis, the matrix equations included a square "stiffness" matrix, which had to be inverted to solve the equations.

Benscoter inverted the stiffness matrix by the classical method of determining the adjoint and dividing by the determinant of the matrix. This method of inversion, the only method presented in many texts(12), is no more economical of time than solution of the simultaneous equations by determinants, a notoriously time-consuming method. Benscoter suggested the possibility of decreasing the time required for inversion by using a matrix power series solution, but did not discuss convergence of the series.

Most of the discussers of Benscoter's article seemed to consider the matrix methods to be of academic interest only, because of the time required for matrix inversion. Benscoter, himself, concluded that the methods were practical only in the case of frames with a number of loadings exceeding the number of rotating joints.

Matrix methods of analysis have been applied to pin-connected structures by Pei-Ping Chen (3), whose article, "Matrix Analysis of Pin-Connected Structures," appeared in 1949 in the <u>Transactions, American Society of Civil Engineers.</u> Chen, like Benscoter, employed the classical method of inversion, and concluded that his methods were practical only for structures subject to the multiple loadings. The advantage of the matrix method, according to Chen, is that

" - - - the truss may be solved for a large variety of loadings with little additional labor, whereas standard methods generally require the repetition of almost all computations for each set of loads."*

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* Chen, Pei-Ping, "Matrix Analysis of Pin-Connected Structures," Transactions, American Society of Civil Engineers, vol 114 (1949), p 192.

V. THE INVESTIGATION

A. Object of the Investigation

WAS STORAGE CAPACITY

The only source of difficulty in solving the matrix form of the slope-deflection equations is the inversion of a "stiffness" matrix. Because of the special nature of this matrix, it was believed that approximate methods of inversion should yield results rapidly, with accuracy acceptable to the structural engineer. The investigation was initially intended to determine the suitability of such approximations, from the standpoint of accuracy coupled with economy of time.

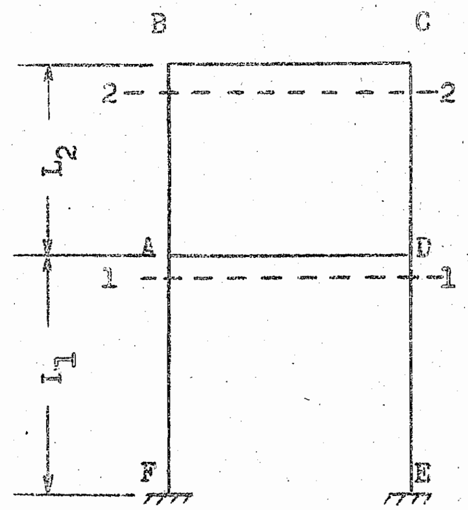
Early in the investigation, it became apparent that the power series approximations being studied would not be completely satisfactory. A search of the matrix literature indicated the existence of fairly rapid direct methods of inversion. Consequently, the scope of the investigation was broadened to consider the feasibility and comparative usefulness of direct, as well as of approximate, methods of inversion, as applied to the structural problem of plane rigid frame analysis.

B. Development of the Slope-Deflection Equations in Matrix Notation

A derivation of the scalar slope-deflection equations may be found in any text on statically indeterminate structures.[11] The notation used herein departs slightly from the most common notation, and the reader is referred to the list of symbols, page 5.

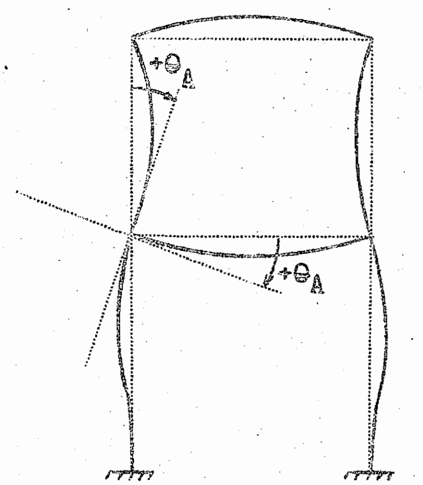
In the frame of Figure 1, the moment in member AB at joint A is given by:

$$M_{AB} = 2\, K_{AB}\, \phi_A + K_{AB}\, \phi_B + K_{AB}\, a_2 + M'_{AB} \text{ - - - - - - - - (1a)}$$



1(a)

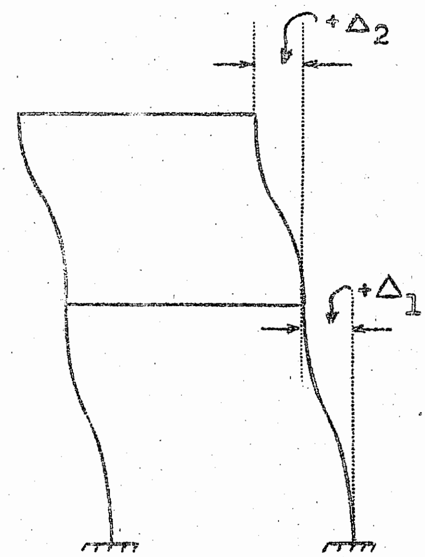
Unloaded Frame



1(b)

Joint Rotation Due to

Symmetrical Vertical Load



1(c)

Joint Translation Due to

Horizontal Loads at Joints

FIGURE 1

FRAME ILLUSTRATING SLOPE-DEFLECTION NOTATION

Similarly:

$$M_{AD} = 2K_{AD} \phi_A + K_{AD} \phi_D + M'_{AD} \qquad \text{- - - - - - - - (1b)}$$

$$M_{AF} = 2K_{AF} \phi_A + K_{AF} \alpha_1 + M'_{AF} \qquad \text{- - - - - - - - (1c)}$$

These three equations express the moments at joint A in the members intersecting at A, indirectly in terms of the joint rotations and translations and of the fixed-end moments at A.

For joint A to be in equilibrium, the algebraic summation of bending moments at A must equal zero. In terms of the modified slope-deflection redundants:

$$\Sigma M_A = 2\Sigma K_A \phi_A + K_{AB} \phi_B + K_{AD} \phi_D + K_{AF} \alpha_1 + K_{AB} \alpha_2 + M'_A = 0 \qquad \text{- - - - - - - - (1')}$$

or

$$2\Sigma K_A \phi_A + K_{AB} \phi_B + K_{AD} \phi_D + K_{AF} \alpha_1 + K_{AB} \alpha_2 = - M'_A \qquad \text{- - (1)}$$

Similarly, for equilibrium of joints B, C, and D, respectively:

$$K_{AB} \phi_A + 2\Sigma K_B \phi_B + K_{BC} \phi_C + K_{AB} \alpha_2 = - M'_B \qquad \text{- - - - - - - - (2)}$$

$$K_{BC} \phi_B + 2\Sigma K_C \phi_C + K_{CD} \phi_D + K_{CD} \alpha_2 = - M'_C \qquad \text{- - - - - - - - (3)}$$

$$K_{AD} \phi_A + K_{CD} \phi_C + 2\Sigma K_D \phi_D + K_{DE} \alpha_1 + K_{CD} \alpha_2 = - M'_D \qquad \text{- - (4)}$$

If the portion of the frame above section 1-1 is considered as a free-body, the sum of the horizontal forces on this portion must equal zero. The horizontal forces will consist of column shears in the first-story columns, and of any external horizontal loads or load components. The summation of the external horizontal forces above section 1-1 will be called the story shear, designated by H_1, and is considered positive when directed to the right. If the columns of the first story are of equal length, as in the frame of Figure 1, the sum of the column shears is equal to the sum of the column end moments divided by the column length. The equation of shear equilibrium then becomes:

$$\frac{\Sigma M_1}{L_1} = \frac{M_{AF} + M_{FA} + M_{DE} + M_{ED}}{L_1} = -H_1 \qquad \text{-------- (5')}$$

In the modified slope-deflection notation:

$$\Sigma M_1 = 3K_{AF}\,\phi_A + 3K_{DE}\,\phi_D + 2\Sigma K_1\,\alpha_1 + M'_1 = -H_1 L_1 \qquad \text{--- (5'')}$$

or

$$3K_{AF}\,\phi_A + 3K_{DE}\,\phi_D + 2\Sigma K_1\,\alpha_1 = -H_1 L_1 - M'_1 \qquad \text{------- (5)}$$

Similarly, for shear equilibrium of the portion of the frame above section 2-2:

$$3K_{AB}\,\phi_A + 3K_{AB}\,\phi_B + 3K_{CD}\,\phi_C + 3K_{CD}\,\phi_D + 2\Sigma K_2\,\alpha_2 =$$

$$-H_2 L_2 - M'_2 \qquad \text{------- (6)}$$

Equations (1) through (6) are solved simultaneously for the six redundants, which are then substituted into equations of the form of (1a), (1b), (1c), etc., to determine the final end moments in each member.

The modified slope-deflection equations, as presented above, are not the most general form, but are valid if all members of the frame are prismatic, or if an average stiffness value is assumed for each member.

In matrix notation, the above six equations may be represented by:

$$\begin{bmatrix} 2\Sigma K_A & K_{AB} & 0 & K_{AD} & K_{AF} & K_{AB} \\ K_{AB} & 2\Sigma K_B & K_{BC} & 0 & 0 & K_{AB} \\ 0 & K_{BC} & 2\Sigma K_C & K_{CD} & 0 & K_{CD} \\ K_{AD} & 0 & K_{CD} & 2\Sigma K_D & K_{DE} & K_{CD} \\ 3K_{AF} & 0 & 0 & 3K_{DE} & 2\Sigma K_1 & 0 \\ 3K_{AB} & 3K_{AB} & 3K_{CD} & 3K_{CD} & 0 & 2\Sigma K_2 \end{bmatrix} x \begin{bmatrix} \phi_A \\ \phi_B \\ \phi_C \\ \phi_d \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -M'_A \\ -M'_B \\ -M'_C \\ -M'_D \\ -M'_1-H_1L_1 \\ -M'_2-H_2L_2 \end{bmatrix}$$

or, in compact notation: $[K] \times \{\phi\} = \{-M'\}$ - - - - - - - - (7')

The single matrix equation is solved for $\{\phi\}$ by pre-multiplying both sides by $[K]^{-1}$, the inverse of $[K]$.

$$\{\phi\} = [K]^{-1} \times \{-M'\} \qquad \text{- - - - - - - (8)}$$

Once the column matrix $\{\phi\}$ has been determined, giving the joint rotations and translations, all end moments may be rapidly computed, using equations of the form of Equations (1a), (1b), etc. These computations may be expressed by four matrix equations, solving, in the first matrix equation, for the end moments to the right of each joint; in the second, for the end moments above each joint; and in the other two matrix equations, for the end moments to the left of and below the joints. If these moments be expressed as elements of column matrices $\{M_I\}$, $\{M_{II}\}$, $\{M_{III}\}$, $\{M_{IV}\}$, respectively, then, in the frame of Figure 1:

$$\begin{bmatrix} M_{AD} \\ M_{BC} \end{bmatrix} = \begin{bmatrix} 2K_{AD} & 0 & 0 & K_{AD} & 0 & 0 \\ 0 & 2K_{BC} & K_{BC} & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \phi_A \\ \phi_B \\ \phi_C \\ \phi_D \\ a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} M'_{AD} \\ M'_{BC} \end{bmatrix} \quad \text{- (9)}$$

or
$$\{M_I\} = [K_I] \times \{\phi\} + \{M'_I\} \quad \text{- - - - - - - - - (9')}$$

Similarly

$$\begin{bmatrix} M_{FA} \\ M_{AB} \\ M_{DC} \\ M_{ED} \end{bmatrix} = \begin{bmatrix} K_{AF} & 0 & 0 & 0 & K_{AF} & 0 \\ 2K_{AB} & K_{AB} & 0 & 0 & 0 & K_{AB} \\ 0 & 0 & K_{CD} & 2K_{CD} & 0 & K_{CD} \\ 0 & 0 & 0 & K_{DE} & K_{DE} & 0 \end{bmatrix} \times \begin{bmatrix} \phi_A \\ \phi_B \\ \phi_C \\ \phi_D \\ a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} M'_{FA} \\ M'_{AB} \\ M'_{DC} \\ M'_{ED} \end{bmatrix} \quad \text{- (10)}$$

or
$$\{M_{II}\} = [K_{II}] \times \{\phi\} + \{M'_{II}\} \quad \text{- - - - - - - (10')}$$

$$\begin{bmatrix} M_{CB} \\ M_{DA} \end{bmatrix} = \begin{bmatrix} 0 & K_{BC} & 2K_{BC} & 0 & 0 & 0 \\ K_{AD} & 0 & 0 & 2K_{AD} & 0 & 0 \end{bmatrix} \times \{\phi\} + \begin{bmatrix} M'_{CB} \\ M'_{DC} \end{bmatrix} \quad \text{— (11)}$$

$$\{M_{III}\} = [K_{III}] \times \{\phi\} + \{M'_{III}\} \quad \text{- - - - - - - (11')}$$

$$\begin{bmatrix} M_{AF} \\ M_{BA} \\ M_{CD} \\ M_{DE} \end{bmatrix} = \begin{bmatrix} 2K_{AF} & 0 & 0 & 0 & K_{AF} & 0 \\ K_{AB} & 2K_{AB} & 0 & 0 & 0 & K_{AB} \\ 0 & 0 & 2K_{CD} & K_{CD} & 0 & K_{CD} \\ 0 & 0 & 0 & 2K_{DE} & K_{DE} & 0 \end{bmatrix} \times \left\{ \phi \right\} + \begin{bmatrix} M'_{AF} \\ M'_{BA} \\ M'_{CD} \\ M'_{DE} \end{bmatrix} \quad \text{-- (12)}$$

$$\left\{ M_{IV} \right\} = \left[K_{IV} \right] \times \left\{ \phi \right\} + \left\{ M'_{IV} \right\} \quad \text{- - - - - - - - - (12')}$$

When the slope-deflection equations are written in matrix notation, we shall refer to the square matrix $\left[K \right]$, or any of the rectangular matrices designated by $\left[K \right]$ with a Roman subscript, as the stiffness matrix; the column matrix $\left\{ \phi \right\}$, as the rotation matrix; the column matrix $\left\{ - M' \right\}$ as the fixed-end moment matrix, and any of the column matrices $\left\{ M \right\}$ with a subscript as the end moment matrix.

If joint translation is absent or is negligible, the rotation matrix will not include terms α_1 and α_2, and the last two rows and columns of the stiffness matrix will disappear. Thus, the first r rows and columns of the stiffness matrix, where r is the number of rotating joints, give the stiffness matrix for the special case of zero joint translation. It will be noticed that this special stiffness matrix is symmetric if members are assumed to be prismatic. Inspection of Equation (7) shows that the element k_{ij}, in the i^{th} row and j^{th} column, is equal to the relative stiffness of member ij, being zero if member ij does not exist. Element $k_{ii} = 2\Sigma K_i$. Hence, these elements may be written directly from inspection of the frame.

The last s rows and columns of the stiffness matrix, where s is the number of independent joint translations, are necessitated by translation of the joints. These elements also may be written from inspection, provided each story has a constant column height, as in the frame of Figure 1. The coefficient of a_j in the i^{th} row, where $i \leq r$, is the relative stiffness of the column of the j^{th} story which meets at joint i. The coefficient of a_j in the principal diagonal is $2\Sigma K_j$. The coefficient of ϕ_i in row $r + j$ is three times the relative stiffness of the column of the j^{th} story meeting joint i, being zero if no such structural column exists.

If any story has columns of unequal lengths, or if vertical joint translations are possible, the analyst must use care in writing the equations of shear equilibrium to insure that the rows and columns of the stiffness matrix due to translation are correct. Also, when vertical translations must be considered, or when all horizontal translations are not in the same sense, care must be used in establishing a sign convention.

C. Solutions of the Slope-Deflection Equations in Matrix Form

1. Inversion by a Matrix Power Series Approximation

It will be noted that, except for the step of inverting the stiffness matrix, all steps of the solution involve only matrix multiplications and additions, which are readily and rapidly performed on a desk calculator of the Friden type. The inverse stiffness matrix, $[K]^{-1}$, is post-multiplied by the fixed-end moment matrix, $\{M'\}$ to obtain the rotation matrix $\{\emptyset\}$. Each end moment matrix, $\{M_i\}$, is then obtained by post-multiplying the stiffness matrix $[K_i]$ by $\{\emptyset\}$ and adding the fixed-end moment matrix $\{M'_i\}$.

Since matrix multiplication and addition are processes requiring no great degree of mathematical maturity, it has been suggested that a method of inversion requiring only these processes might bring the matrix methods within reach of the average engineer. Benscoter (1) suggests the possibility of using a matrix power series for inverting the stiffness matrix, as follows:

Let $[K] = [D] - [R]$ - - - - - - - - - - - - - - - - - (13)

where $[D]$ is the diagonal matrix corresponding to the principal diagonal of $[K]$, and $[R]$ is a remainder matrix.

Form $[Q] = [R][D]^{-1}$ - - - - - - - - - - - - - - - - - (14)

Then $[K]^{-1} = \{[D] - [R]\}^{-1}$ - - - - - - - - - - - - (15')

$= \{([I] - [R][D]^{-1})\,[D]\}^{-1}$ - - - - - (15'')

$$= [D]^{-1} \left\{[I] - [R][D]^{-1}\right\}^{-1} \quad --- (15''')$$

$$[K]^{-1} = [D]^{-1} \left\{[I] - [Q]\right\}^{-1} \quad ------ (15)$$

From analogy to the corresponding scalar power series, $\left\{[I] - [Q]\right\}^{-1}$ may be expressed as a matrix power series:

$$\left\{[I] - [Q]\right\}^{-1} = [I] + [Q] + [Q]^2 + [Q]^3 + \text{----} \quad (16)$$

provided the series converges. Benscoter suggests that it may be an easy matter to determine whether or not the series converges in any particular problem, but gives no criterion for investigating convergence.

Frazer, Duncan, and Collar [8] have shown that a necessary and sufficient condition for the convergence of the above series is that all latent roots of $[Q]$ be less than unity in absolute value. The necessity for investigating this condition would nullify any saving in time which might result from using the series approximation. Hotelling [10] has pointed out a sufficient, though not necessary, condition for the convergence of such a power series. He defines the norm of a real matrix (norm $[Q]$) as the square root of the sum of the squares of the matrix elements. If norm $[Q]$ is less than unity, the series converges.

Hotelling's norm test for convergence seems well-suited for practical use. Although the series may converge for norm $[Q]$ slightly in excess of unity, the present investigation has shown extremely slow convergence when the norm approaches unity. Consequently, a norm of unity would appear to represent the upper limit for practical application of the power series approximation.

Accepting unity as the upper permissible limit of norm $[Q]$, it is possible to investigate, at least approximately, the degree of complexity a frame may have without causing the power series to diverge. If the elements ϕ_i of the rotation matrix be replaced by $2\Sigma K_i \phi_i$, the principal diagonal elements of the stiffness matrix become unity. Then, following Benscoter's suggestion, $[K']$ may be expressed as $[I] - [R']$.

$$[Q] = [R'][I]^{-1} = [R'] \quad \text{-----------------} \quad (17)$$

Hence, the power series may be expected to converge if the sum of the squares of the non-diagonal elements of $[K']$ does not exceed unity, where $[K']$ is the matrix obtained by dividing the elements of each column of $[K]$ by the principal diagonal element of that column. That is: $k'_{ij} = k_{ij} / k_{jj}$

To interpret the above convergence condition in terms of a structural frame, assume that an average of three effective members meet at each joint, an effective member being one both ends of which are free to rotate; assume, also, that all members meeting at a joint are of equal stiffness. Three members per joint means that the auxiliary stiffness matrix $[K']$ will have three non-zero, non-diagonal elements in each row and column, when sidesway is neglected. If all members are of equal stiffness, each element k'_{ij} will equal 1/6. Then

$$\text{norm } [R'] = \frac{3n}{36}, \quad \text{--------------------} \quad (18)$$

where n is the order of the stiffness matrix. Thus, for ready convergence, i.e., for norm $[R']$ = 1, the stiffness matrix may be of the twelfth order only, limiting the use of this method to frames having a maximum of twelve rotating joints. This limitation is admittedly an approximation, but the existence of a limitation is undebatable.

If four members of equal stiffness meet at each joint, sixteen rotating joints are permissible for certain convergence, since four non-zero elements, each equal to 1/8, will appear in each row of the remainder matrix $[R']$. Since, in a rectangular frame, no more than four members may meet at any joint, sixteen rotating joints are all that can be analyzed by Benscoter's method without modifications. In a practical frame, of course, all members will not be of equal stiffness. The effect of unequal stiffnesses is to reduce the permissible order of the stiffness matrix. That such is true is evident from an inspection of the auxiliary stiffness matrix $[K']$. It is seen that the sum of the non-diagonal elements of each row or column is 1/2, if sidesway is neglected and if all members are effective, regardless of the relative magnitudes of the individual member stiffnesses. The sum of the squares of the elements, and therefore the norm, is a minimum when all elements are equal, which is to say, when all stiffnesses are equal.

Similarly, it can be shown that the power series will converge for any frame of not more than four rotating joints, if joint translation is negligible. The upper limiting value of non-diagonal elements of the auxiliary stiffness matrix is 1/2, indicating one member meeting at each joint. In such a case, norm $[R']$ = $\frac{n}{4}$, indicating that the order n may be four for norm $[R']$ = 1.

When joint translation is considered, definite numerical limits may not be placed upon the order of the stiffness matrix, since the effect of the additional elements due to translation is a function of the geometry of the structure. In general, however, the addition of a row and column to the stiffness matrix, necessitated by joint translation, appears to increase norm $[R']$ more than the addition of a row and column necessitated by an additional rotating joint. Qualitatively, this effect is apparent, since consideration of translation adds non-zero elements to the original stiffness matrix $[K]$ without a corresponding increase in the principal diagonal elements.

It should be noted that the remainder matrix $[R']$, obtained from the auxiliary stiffness matrix $[K']$, is identical with the matrix $[Q]$, obtained from the original stiffness matrix $[K]$. Consequently, the above discussion of convergence applies directly to Benscoter's suggested method of inversion.

A slight variation of the above method permits its extension to frames of higher order. Instead of expressing the auxiliary stiffness matrix $[K']$ as the difference between the unit matrix and a remainder matrix, we may consider it as the difference between a triangular matrix and a remainder matrix. That is

$$[K'] = [T] - [R'] \quad \text{(19)}$$

Here, $[T]$ is the triangular matrix obtained by eliminating from $[K']$ all elements above the principal diagonal. The inverse of a triangular matrix is easily obtained,[8] although not so readily as the inverse of a diagonal matrix. Then

$$[Q'] = [R'][T]^{-1} \quad \text{--------------------(20)}$$

$$[K']^{-1} = [T]^{-1}\left\{[I] - [Q']\right\}^{-1} \quad \text{--------------(21)}$$

The power series approximation for $\left\{[I] - [Q']\right\}^{-1}$ may again be used, as in Equation (16), provided the series converges.

Assuming three members of equal stiffness meeting at each joint, and neglecting sidesway, each non-zero, non-diagonal element of $[K']$ will equal 1/6. Each diagonal element of $[T]^{-1}$ will be unity, and the non-diagonal elements will be powers of 1/6, the first powers being negative. Postmultiplication of $[R']$ by $[T]^{-1}$ will have the effect of decreasing norm $[R']$. That is, norm $[Q'] <$ norm $[R']$. Consequently, if norm $[R']$ does not exceed unity, the power series will converge. Since the non-zero elements of $[R']$ are each 1/6 , then $[R']$ may have 36 non-zero elements. Therefore, $[K']$ may have 72 non-zero, non-diagonal elements, or may be of the 24th order. Thus, 24 rotating joints are permissible, or somewhat fewer if sidesway is considered. This order, as before, should be considered as a rather rough approximation.

Since Benscoter originally suggested use of the power series approximation for the analysis of continuous beams, it is worth while to investigate convergence in that special one-dimensional case. Each interior joint of a continuous beam is effective, and two members meet at each effective joint. Consequently, if all members are of equal stiffness, each non-zero, non-diagonal element of the auxiliary stiffness matrix $[K']$ equals 1/4. The first and last rows and columns of $[K']$ contain one non-zero, non-diagonal element each, all other rows and columns containing two such elements each. Hence, if $[K']$ is expressed as $[I] - [R']$, norm $[R'] = 1$ when the order of the stiffness matrix is nine. Thus, using Benscoter's suggested method without modification, the matrix power series converges for continuous beams with ten spans, or fewer if members differ appreciably in stiffness.

If the power series converges slowly, a method suggested by Frazer, Duncan and Collar [8] will accelerate the convergence. If $[C_o]$ is an approximate inverse of $[K]$, the approximation may be improved by computing

$$[C_1] = [C_o]\{[2I] - [K][C_o]\}$$

$$[C_2] = [C_1]\{[2I] - [K][C_1]\}$$

$$[C_{m+1}] = [C_m]\{[2I] - [K][C_m]\} \quad \text{-------------- (22)}$$

This sequence converges to $[K]^{-1}$ if the latent roots of $[I] - [K][C_o]$ are all less than unity in absolute value, and, therefore, if norm $\{[I] - [K][C_o]\} < 1$. If $[C_o]$ is an approximate inverse of the auxiliary stiffness matrix $[K']$, obtained by the power series approximation, then

$$[C_o] = [I] + [Q] + [Q]^2 + \text{----} + [Q]^n$$

$$[K][C_o] = \{[I] - [Q]\}[C_o] = [I] - [Q]^{n+1}$$

$$[2I] - [K][C_o] = [I] + [Q]^{n+1}$$

$$[C_1] = \{[I] + [Q] + [Q]^2 + \cdots + [Q]^n\}\{[I] + [Q]^{n+1}\}$$

$$[C_1] = [I] + [Q] + [Q]^2 + \cdots\cdots + [Q]^{2n+1} \quad (23)$$

Thus, one cycle of the above correction method is equivalent to doubling the number of terms of the power series. Actually, if the computations are performed using a constant number of computer places, as is common, the accuracy obtained from one cycle of correction is better than that obtained by doubling the number of terms of the series, since there is less opportunity for making computational errors and introducing errors of rounding.

The improvement method requires two matrix multiplications per cycle. Consequently, it saves time only if the initial approximate inverse includes at least three terms of the power series. It proves most valuable when the power series approximation is used for inverting matrices of high order, where a large number of terms of the power series may be necessary to insure reasonable accuracy.

A well-known direct method of inversion, based upon the Cayley-Hamilton equation,(2) requires only n-1 matrix multiplications, where n is the order of the matrix to be inverted. Hence, the power series method is difficult to justify if the number of terms required exceeds the order of the matrix. Such has proved to be the case for many stiffness matrices considered in the course of the present investigation.

2. Inversion by Dwyer's Method

Paul Dwyer (6, 7) has discussed a modified form of the Doolittle method for the solution of simultaneous equations for which the matrix of coefficients is symmetric, and the adaption of this method to the determination of the inverse of a symmetric matrix. The method is direct, reasonably rapid, and compact in form.

The original problem is the solution of the system of equations

$$\sum_{i=1}^{n} a_{ij}\, x_i = a_{n+1,\, j} \quad (j = 1,2,\ldots, n) \text{------} \quad (24)$$

These equations are solved by introducing auxiliary equations, each of which has one variable less than the preceding equation, due to the successive elimination of $x_1, x_2, \ldots, x_{n-1}$. A final auxiliary equation in x_n alone is solved, and the other x_i obtained by substituting into the auxiliary equations in inverse order. This method is illustrated in Table 1.

In the first n lines of Table 1, the n equations of (24) are written. The first equation of (24) is written in line n + 1, and divided by its leading coefficient a_{11} to obtain

$$\sum_{i=1}^{n} b_{i1}\, x_i = b_{n+1,\, i} \qquad \text{where } b_{i1} = \frac{a_{i1}}{a_{11}} \text{------} \quad (25)$$

Then form

$$\sum_{i=1}^{n} a_{i2.1}\, x_i = a_{n+1,\, 2.1} \quad \text{where } a_{i2.1} = a_{i2} - a_{i1}b_{21} \quad \text{---} \; (26)$$

$$\sum_{i=1}^{n} b_{i2.1}\, x_i = b_{n+1,2.1} \quad \text{where } b_{i2.1} = \frac{a_{i2.1}}{a_{22.1}} \quad \text{------} \; (26')$$

In general:

$$\sum_{i=1}^{n} a_{ij.j-1}\, x_i = a_{n+1,\, j.j-1} \quad (j = 1, 2, \ldots, n) \text{-------} \; (27)$$

and

$$\sum_{i=1}^{n} a_{ij.j-1}\, x_i = b_{n+1,\, j.j-1} \quad \text{----------------} \; (27')$$

where

$$a_{ij.j-1} = a_{ij} - a_{i1}\, b_{j1} - a_{i2.1}\, b_{j2.1} - \circ \circ \circ - a_{i,j-1.j-2}b_{j,\, j-1.j2} \quad \text{--} \; (27'')$$

and

$$b_{ij.j-1} = \frac{a_{ij.\, j-1}}{a_{jj.\, j-1}} \quad \text{--------------------} \; (27''')$$

Table 1, due to Dwyer, [7] illustrates the solution of the following system of equations:

$$1.0000\, x_1 + 0.4000\, x_2 + 0.5000\, x_3 + 0.6000\, x_4 = 0.2000$$

$$0.4000\, x_1 + 1.0000\, x_2 + 0.3000\, x_3 + 0.4000\, x_4 = 0.4000$$

$$0.5000\, x_1 + 0.3000\, x_2 + 1.0000\, x_3 + 0.2000\, x_4 = 0.6000$$

$$0.6000\, x_1 + 0.4000\, x_2 + 0.2000\, x_3 + 1.0000\, x_4 = 0.8000$$

Only the diagonal and super-diagonal elements of the matrix of coefficients are shown in Table 1, to represent a symmetric matrix.

To determine the inverse of a symmetric matrix, the column matrix $\{a\}$ is replaced by the columns of the unit matrix in succession, n sets of equations being solved simulteneously. This method will be illustrated in a later section of the paper, when frames which have been analyzed are presented.

TABLE 1

DWYER'S METHOD FOR SOLVING SIMULTANEOUS EQUATIONS

	x_1	x_2	x_3	x_4	a
a_{i1}	1.0000	0.4000	0.5000	0.6000	0.2000
a_{i2}		1.0000	0.3000	0.4000	0.4000
a_{i3}			1.000	0.2000	0.6000
a_{i4}				1.0000	0.8000
a_{i1}	1.0000	0.4000	0.5000	0.6000	0.2000
b_{i1}	1.0000	0.4000	0.5000	0.6000	0.2000
$a_{i2.1}$		0.8400	0.1000	0.1600	0.3200
$b_{i2.1}$		1.0000	0.1190	0.1905	0.3810
$a_{i3.2}$			0.7381	-0.1190	0.4619
$b_{i3.2}$			1.0000	-0.1612	0.6258
$a_{i4.3}$				0.5903	0.6935
$b_{i4.3}$				1.0000	1.1748

Hence from $b_{i4.3}$ $x_4 = 1.1748$

from $b_{i3.2}$ $x_3 = 0.6258 - (-0.1612)\ x_4$

from $b_{i2.1}$ $x_2 = 0.3810 - (0.1905)\ x_4 - (0.1190)\ x_3$

from b_{i1} $x_1 = 0.2000 - (0.6000)\ x_4 - (0.5000)\ x_3$
$- (0.4000)\ x_2$

Dwyer's method yields the inverse of a symmetric matrix rapidly, the accuracy of the inverse depending upon the number of computer places used in the computation. Since, as has been mentioned, the stiffness matrix is symmetric if joint translation is neglected, Dwyer's method is directly applicable to the analysis of a great number of practical frames. Cornish[4] has pointed out that joint translations due to a symmetry of vertical loads usually have a negligible effect upon final end moments, when all loads are considered. Consequently, frames not subject to lateral loads, and particularly low building frames, for which wind loads may be negligible, may be analyzed neglecting sidesway, to take full advantage of the rapidity of Dwyer's inversion method.

3. Inversion by Partitioning

When a frame composed of prismatic or symmetrically haunched members is subject to joint translation, the stiffness matrix is non-symmetric, but has a symmetric submatrix. This submatrix may be inverted readily by Dwyer's method. Inversion of the non-symmetric stiffness matrix follows fairly readily, if the number of joint translations is comparatively small, by use of the method of partitioning or of submatrices, as described by Frazer, Duncan, and Collar.[8]

Given the non-symmetric matrix $[K]$, of order n, having a symmetric submatrix $[K_{11}]$, of order r, partition $[K]$ as follows:

$$[K] = \left[\begin{array}{c|c} K_{11} & K_{12} \\ (r,\ r) & (r,\ s) \\ \hline K_{21} & K_{22} \\ (s,\ r) & (s,\ s) \end{array}\right] \qquad \text{where } r + s = n \quad ------ \quad (28)$$

Similarly, partition $\left[\beta\right] \equiv \left[K\right]^{-1}$:

$$\left[\beta\right] = \left[\begin{array}{c|c} \beta_{11} & \beta_{12} \\ (r, r) & (r, s) \\ \hline \beta_{21} & \beta_{22} \\ (s, r) & (s, s) \end{array}\right] \qquad \text{--------- (29)}$$

Since $\left[\beta\right]\left[K\right] = \left[I_n\right]$, --------- (30)

$$\left[\beta_{11}\right]\left[K_{11}\right] + \left[\beta_{12}\right]\left[K_{21}\right] = \left[I_r\right] \qquad \text{--------- (31)}$$

$$\left[\beta_{11}\right]\left[K_{12}\right] + \left[\beta_{12}\right]\left[K_{22}\right] = \left[0\right] \qquad \text{--------- (32)}$$

$$\left[\beta_{21}\right]\left[K_{11}\right] + \left[\beta_{22}\right]\left[K_{21}\right] = \left[0\right] \qquad \text{--------- (33)}$$

$$\left[\beta_{21}\right]\left[K_{12}\right] + \left[\beta_{22}\right]\left[K_{22}\right] = \left[I_s\right] \qquad \text{--------- (34)}$$

Introduce $\left[X\right] \equiv \left[K_{11}\right]^{-1}\left[K_{12}\right]$ --------- (35a)

$$\left[Y\right] \equiv \left[K_{21}\right]\left[K_{11}\right]^{-1} \qquad \text{--------- (35b)}$$

$$\left[\theta\right] \equiv \left[K_{22}\right] - \left[Y\right]\left[K_{12}\right] \qquad \text{--------- (35c)}$$

Then:

$$\left[\beta_{11}\right] = \left[K_{11}\right]^{-1} + \left[X\right]\left[\theta\right]^{-1}\left[Y\right] \qquad \text{--------- (31')}$$

$$\left[\beta_{12}\right] = -\left[X\right]\left[\theta\right]^{-1} \qquad \text{--------- (32')}$$

$$\left[\beta_{21}\right] = -\left[\theta\right]^{-1}\left[Y\right] \qquad \text{--------- (33')}$$

$$\left[\beta_{22}\right] = \left[\theta\right]^{-1} \qquad \text{--------- (34')}$$

Thus, $[\beta]$ is completely determined if $[K_{11}]^{-1}$ and $[\theta]^{-1}$ exist. They will exist if the matrix $[K]$ is non-singular, and if $[K]$ has no zero elements in the principal diagonal. The stiffness matrix always meets these requirements.

The numerical work is most conveniently performed by using the following tabular form:

	K_{21}	K_{22}
$X = K_{11}^{-1}\ K_{12}$	K_{11}^{-1}	K_{12}
θ^{-1}	$Y = K_{21}\ K_{11}^{-1}$	$\theta = K_{22} - Y K_{12}$

$$[K]^{-1} = \left[\begin{array}{c:c} K_{11}^{-1} + X\,\theta^{-1}\,Y & -X\,\theta^{-1} \\ \hdashline -\theta^{-1}\,Y & \theta^{-1} \end{array}\right]$$

Inversion of $[\theta]$, which, in general, is non-symmetric, is rapid only if $[\theta]$ is of low order. Since a matrix of the second order is readily inverted, the partitioning process may be used repeatedly so that, in each cycle, $[\theta]$ is of the second order. The order of $[\theta]$ is equal to the number of independent joint translations. Hence, the method of partitioning becomes tedious if a large number of translations must be considered, unless a rapid method for inverting a non-symmetric matrix is available.

4. Inversion by Zurmuhl's Method

Rudolph Zurmuhl[16] has discussed a method of matrix inversion which is similar to Dwyer's method in that it solves a system of simultaneous linear equations by the introduction of auxiliary equations featuring a decreasing number of variables, these auxiliary equations being solved in reverse order. The method is more general than Dwyer's, however, in that is is applicable to the inversion of non-symmetric matrices.

The original problem is the solution of the system of simultaneous linear equations expressed by the matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{bmatrix} \quad \text{---------} (36)$$

or $$[A]\{X\} = \{a\} \quad \text{---------} (36')$$

Equation (36) may be transferred into

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & a_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & a_n \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} \quad \text{-----} (37)$$

or $$[A']\{X'\} = \{0\} \quad \text{---------} (37')$$

$[A]$ is considered as the product of two triangular matrices $[C]$ and $[B]$. Similarly, $[A']$ is the product of $[C]$ and $[B']$.

$$[A] = [C]\,[B] \qquad \text{-------- (38)}$$

$$[A'] = [C]\,[B'] \qquad \text{-------- (38')}$$

Here,

$$[C] = \begin{bmatrix} c_{11} & 0 & 0 & \cdots & 0 \\ c_{21} & c_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nn} \end{bmatrix} \qquad \text{-------- (39)}$$

$$[B] = \begin{bmatrix} 1 & b_{12} & \cdots & b_{1n} \\ 0 & 1 & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \qquad \text{-------- (40)}$$

$$[B'] = \begin{bmatrix} 1 & b_{12} & \cdots & b_{1n} & b_1 \\ 0 & 1 & \cdots & b_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 1 & b_n \end{bmatrix} \qquad \text{-------- (40')}$$

Since $\quad \text{Det}\,[A] = \text{Det}\,[C]\ \text{Det}\,[B]$, and $\text{Det}\,[B] = 1$,

$$\text{Det}\,[A] = \text{Det}\,[C] = c_{11}\ c_{22}\ c_{33}\ \cdots\ c_{nn} \qquad \text{--- (41)}$$

Since $[A]$ is assumed to be non-singular, Det $[A] \neq 0$.

$$[A'] \; \{X'\} = [C] \; [B'] \; \{X'\} = \{0\}$$

Hence: $[B'] \; \{X'\} = \{0\}$

or $[B] \; \{X\} = \{b\}$ - - - - - - - - (42')

Expanded,

$$\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ 0 & 0 & \cdots & \cdots \\ 0 & 0 & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix} \qquad \text{- - - - - - (42)}$$

Since $[A] = [C] \; [B]$,

$$a_{ij} = \sum_{k=1}^{n} c_{ik} \; b_{kj} \, , \qquad \text{- - - - - - - - (43)}$$

But since both $[B]$ and $[C]$ are triangular,

$$\text{maximum} \quad c_{ik} = c_{ii} \qquad \text{- - - - - - - - (43')}$$

$$\text{maximum} \quad b_{kj} = b_{jj} \qquad \text{- - - - - - - - (43'')}$$

Computing a_{ij} starting with the elements of the first column, and proceeding from the first to the last row, then in each of the equations of (43) one new unknown element b_{ij} or c_{ij} is introduced.

If $i < j$, $b_{ij} = \frac{1}{c_{i1}} \left(a_{ij} - c_{i1} \; b_{ij} - \cdots - c_{i,i-1} \; b_{i-1,j} \right.$ - - (44a)

If $i \geqq j$, $c_{ij} = a_{ij} - c_{i1} \; b_{ij} - c_{i2} \; b_{2j} - \cdots - c_{i,j-1} \; b_{j-1,j}$ - (44b)

Table 2, due to Zurmuhl, shows the method of application of Equations (44). To apply Equation (44) to the columns of a_i and b_i, assume

$$a_i = a_{i,n+1} \qquad b_i = b_{i,n+1}$$

For the first column of $[C]$

$$c_{i1} = a_{i1} \quad (i = 1, 2, \ldots, n, n+1) \qquad \text{--------- (45a)}$$

For the first row of $[B']$,

$$b_{1j} = \frac{a_{1j}}{a_{11}} \quad (j = 1, 2, \ldots, n, n+1) \qquad \text{-------- (45b)}$$

TABLE 2

ZURMUHL'S METHOD FOR SOLVING SIMULTANEOUS EQUATIONS

	j=1	2	3	...	n	a_i	
i=1	a_{11}	a_{12}	a_{13}	...	a_{1n}	a_1	
2	a_{21}	a_{22}	a_{23}	...	a_{2n}	a_2	
3	a_{31}	a_{32}	a_{33}	...	a_{3n}	a_3	
...	...	...	...	...	...	...	
n	a_{n1}	a_{n2}	a_{n3}	...	a_{nn}	a_n	
i=1	c_{11}	b_{12}	b_{13}	...	b_{1n}	b_1	x_1
2	c_{21}	c_{22}	b_{23}	...	b_{2n}	b_2	x_2
...	...	...	...	...	...	...	...
n	c_{n1}	c_{n2}	c_{n3}	...	c_{nn}	b_n	x_n
	j=1	2	3	...	n	b_i	x_i

In Table 2, matrices $[C]$ and $[B']$ are combined in one square, the principal diagonal including only the elements of $[C]$, since each diagonal element of $[B']$ equals unity. This square matrix may be called the $[CB']$ matrix.

After the $[CB']$ matrix has been obtained, the equations of (42) are solved in reverse order for x_i. Since $b_{ii} = 1$,

$$x_n = b_n \qquad \text{--------- (46a)}$$

$$x_i = b_i - x_n\, b_{in} - x_{n-1}\, b_{i,n-1} - \cdots - x_{i+1}\, b_{i,i+1} \qquad \text{--- (46b)}$$

If matrix $[A]$ is symmetric,

$$a_{ij} = a_{ji} \qquad \text{--------- (47a)}$$

$$c_{i1} = a_{i1} \qquad \text{--------- (47b)}$$

$$b_{1i} = \frac{a_{1i}}{c_{11}} = \frac{a_{i1}}{c_{11}} = \frac{c_{i1}}{c_{11}} \qquad \text{--------- (47c)}$$

$$c_{12} = a_{12} - c_{i1}\, b_{12} = a_{i2} - \frac{c_{i1}\, c_{21}}{c_{11}} \qquad \text{--------- (47d)}$$

$$b_{2i} = \frac{1}{c_{22}}\left(a_{2i} - c_{21}\, b_{1i}\right) = \frac{c_{i2}}{c_{22}} \qquad \text{--------- (47e)}$$

Hence, by induction,

$$b_{ji} = \frac{c_{ij}}{c_{jj}} \qquad \text{--------- (47f)}$$

The computation, in the symmetric case, may be performed exactly as when using Dwyer's method. Consequently, the above derivation of Zurmuhl's method serves also as a justification of Dwyer's method.

TABLE 3

ILLUSTRATION OF ZURMUHL'S METHOD

$$43.57\ x_1 - 23.34\ x_2 - 51.65\ x_3 + 19.44\ x_4 = 0.7362$$

$$62.89\ x_1 + 84.97\ x_2 + 21.84\ x_3 - 39.35\ x_4 = 1.1872$$

$$37.48\ x_1 + 93.24\ x_2 + 84.39\ x_3 + 26.75\ x_4 = 0.3875$$

$$-19.37\ x_1 + 54.38\ x_2 + 14.59\ x_3 + 62.85\ x_4 = 0.5738$$

$j = 1$	2	3	4	$a_i.10^2$	$x.10^2$
43.570	-23.840	-51.650	19.440	73.620	
62.890	84.970	21.840	-39.350	118.720	
37.480	93.240	84.390	26.750	38.750	
-19.370	54.380	14.590	62.850	57.380	
43.570	-0.5472	-1.1854	0.4462	1.6897	0.6377
62.890	119.3812	0.8074	-0.5647	0.1043	1.4045
37.480	113.7477	36.9764	2.0083	-0.9857	-1.4489
-19.370	43.7814	-43.7229	184.0189	0.2307	0.2307

Thus: $x_1 = 0.006377$

$x_2 = 0.014045$

$x_3 = -0.014489$

$x_4 = 0.002307$

Inversion of a matrix by Zurmuhl's method, as when using Dwyer's, consists of solving the n sets of n simultaneous equations resulting from the expansion of $\left[K\right]\left[K\right]^{-1} = \left[I\right]$.

Table 3 illustrates Zurmuhl's method as used in the solution of a single set of simultaneous equations for which the matrix of coefficients is non-symmetric - use of the method for inversion of a non-symmetric matrix is illustrated in a later section of the paper.

The author considers the most satisfactory combination for the inversion of the stiffness matrix, when many joints translate, to be the method of partitioning. Zurmuhl's method may be used to invert the (usually) non-symmetric submatrix $\left[\theta\right]$, and Dwyer's method, which is a special case of Zurmuhl's, to invert the symmetric submatrix $\left[K_{11}\right]$. If Zurmuhl's method is used on the original non-symmetric stiffness matrix, little saving of time accrues from the existence of a sizeable symmetric submatrix. Zurmuhl's method, applied to the original stiffness matrix, is recommended only when the number of joint rotations is small compared to the number of translations, a situation which will rarely be encountered in any frame which is to be completely analyzed.

D. Frames Analyzed

1. By Matrix Power Series Approximation

Since the purpose of the investigation of the matrix power series was primarily to determine rates of convergence, very few frames were completely analyzed by this method. The stiffness matrix was established for frames of varying number of joints and for widely varying stiffness ratios, first neglecting and then including sidesway. The power series approximation was used to invert the stiffness matrix (or auxiliary stiffness matrix).

The accuracy of the approximate inverse was checked by computing the product $\left[K\right]\left[K\right]^{-1}$, and noting its deviation from the unit matrix. In the few frames which were completely analyzed, the maximum error in any end moment was found to be less than four percent of the maximum end moment when the elements of $\left[K\right]\left[K\right]^{-1}$ agreed with the elements of $\left[I\right]$ to three decimals. Therefore, the investigation of the rate of convergence consisted of determining the number of terms of the power series and the time of computation required to obtain such agreement. If the number of terms required exceeded the order of the matrix, the computation was halted at that point. In the case of matrices for which the power series diverged, the computation was stopped as soon as the divergence was unmistakably evident. All computations were carried out using five computer decimals.

Figure 2 illustrates the frames investigated in this manner. The relative stiffness of each member is indicated by a circled number.

Inversion of the auxiliary stiffness matrix for the frame of

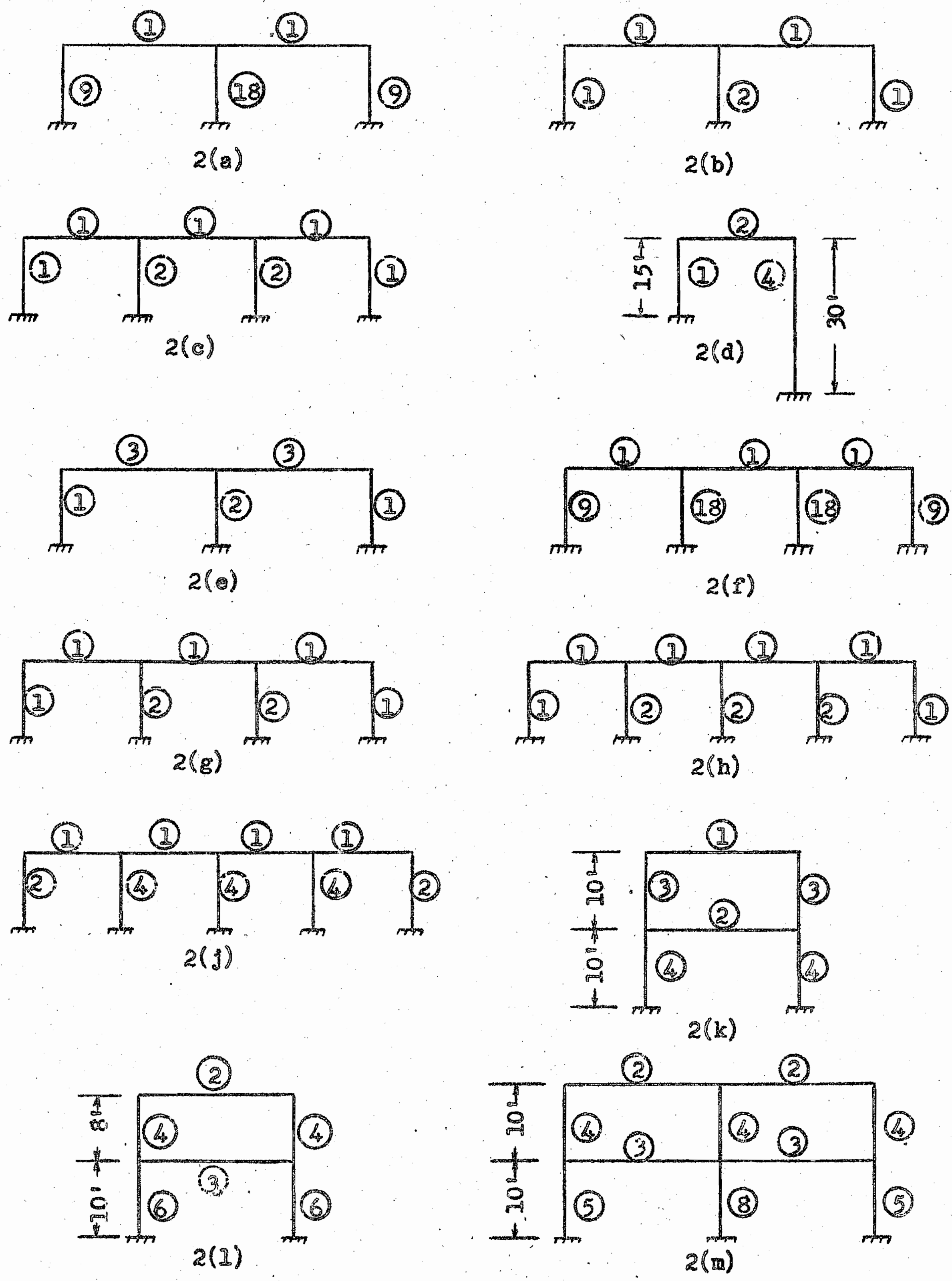


FIGURE 2

FRAMES INVESTIGATED BY MATRIX POWER SERIES

Figure 2(k) by the power series method is indicated below, sidesway being neglected. Only the diagonal and superdiagonal elements are written, since the matrix is symmetric.

$$[K] = \begin{bmatrix} 18 & 3 & 0 & 2 \\ & 8 & 1 & 0 \\ & & 8 & 3 \\ & & & 18 \end{bmatrix}$$

$$[K'] = \begin{bmatrix} 1.00000 & 0.37500 & 0.00000 & 0.11111 \\ & 1.00000 & 0.12500 & 0.00000 \\ & & 1.00000 & 0.16667 \\ & & & 1.00000 \end{bmatrix}$$

$$[K'] = [I] - [R']$$

$$[R'] = - \begin{bmatrix} 0.00000 & 0.37500 & 0.00000 & 0.11111 \\ & 0.00000 & 0.12500 & 0.00000 \\ & & 0.00000 & 0.16667 \\ & & & 0.00000 \end{bmatrix}$$

Norm $[R'] = 1.5989$. The series is not necessarily convergent.

$$[R']^2 = \begin{bmatrix} 0.07485 & 0.00000 & 0.08854 & 0.00000 \\ & 0.07639 & 0.00000 & 0.03935 \\ & & 0.07639 & 0.00000 \\ & & & 0.07485 \end{bmatrix}$$

$$\left[R'\right]^3 = -\begin{bmatrix} 0.00000 & 0.03848 & 0.00000 & 0.02307 \\ & 0.00000 & 0.02431 & 0.00000 \\ & & 0.00000 & 0.01739 \\ & & & 0.00000 \end{bmatrix}$$

$$\left[R'\right]^4 = \begin{bmatrix} 0.00908 & 0.00000 & 0.01339 & 0.00000 \\ & 0.00945 & 0.00000 & 0.00886 \\ & & 0.00945 & 0.00000 \\ & & & 0.00908 \end{bmatrix}$$

$$\left[R'\right]^5 = -\begin{bmatrix} 0.00000 & 0.00503 & 0.00000 & 0.00433 \\ & 0.00000 & 0.00341 & 0.00000 \\ & & 0.00000 & 0.00252 \\ & & & 0.00000 \end{bmatrix}$$

$$\left[R'\right]^6 = \begin{bmatrix} 0.00143 & 0.00000 & 0.00184 & 0.00000 \\ & 0.00126 & 0.00000 & 0.00104 \\ & & 0.00126 & 0.00000 \\ & & & 0.00143 \end{bmatrix}$$

$$\left[K'\right]^{-1} = \left[I\right] + \left[R'\right] + \left[R'\right]^2 + \left[R'\right]^3 + \left[R'\right]^4 + \left[R'\right]^5 + \left[R'\right]^6$$

$$\left[K'\right]^{-1} = \begin{bmatrix} 1.08536 & -0.41851 & 0.10377 & -0.13851 \\ & 1.08710 & -0.15272 & 0.04925 \\ & & 1.08710 & -0.18658 \\ & & & 1.08536 \end{bmatrix}$$

$$[K'][K']^{-1} = \begin{bmatrix} 0.91303 & -0.00538 & 0.02577 & 0.00055 \\ & 0.91110 & 0.02208 & -0.02601 \\ & & 1.03691 & 0.00047 \\ & & & 1.03887 \end{bmatrix}$$

To illustrate the value of the Frazer, Duncan, Collar correction method:

Let $[C_o] = [I] + [R'] + [R']^2$

$$[C_o] = \begin{bmatrix} 1.07485 & -0.37500 & 0.08854 & -0.11111 \\ & 1.07639 & -0.12500 & 0.03935 \\ & & 1.07639 & -0.16667 \\ & & & 1.07485 \end{bmatrix}$$

$$[2\ I] - [K'][C_o] = \begin{bmatrix} 1.00000 & -0.03848 & 0.00000 & -0.02308 \\ & 1.00174 & -0.02431 & 0.03537 \\ & & 1.00174 & -0.01739 \\ & & & 1.00000 \end{bmatrix}$$

$$[C_o]\{[2\ I] - [K'][C_o]\} = [C_1] = \begin{bmatrix} 1.08706 & -0.41917 & 0.10209 & -0.15072 \\ & 1.08771 & -0.15289 & 0.08342 \\ & & 1.08771 & -0.19072 \\ & & & 1.08706 \end{bmatrix}$$

$$[K'][C_1] = \begin{bmatrix} 0.91312 & -0.00211 & 0.02357 & 0.00135 \\ & 0.91141 & 0.02136 & 0.00312 \\ & & 1.03681 & 0.00089 \\ & & & 1.03853 \end{bmatrix}$$

Thus, it is evident that one cycle of this correction method, which theoretically should be the equivalent of considering the first six terms of the power series, actually gives better accuracy than the first seven terms, when a constant number of computer decimals is used.

2. By Dwyer's Method Plus Partitioning

Since Dwyer's method and the method of partitioning are direct, rather than approximate, methods of inversion, the accuracy of the inverse matrix depends only upon the number of significant figures used in computation. Inversion by these methods was performed for the purpose of determining the time required for solution, and the accuracy obtainable using a fixed number of computer decimals, rather than a fixed number of significant figures.

The frames of Figure 3 were completely analyzed, inversion of the stiffness matrix being performed by using the method of partitioning, with Dwyer's method used for inverting $\left[K_{11}\right]$. Final end moments in all members were computed for the loadings shown, and compared with published values. Comparative solutions for the frames of (3a) through (3d) were obtained from an article by Wilson[15], who analyzed the frames by using slope-deflection methods combined with superposition to reduce the number of simultaneous equations to be solved. The frame of (3c) was analyzed by Cornish [4] using moment balance, an approximate method. His results are admittedly inaccurate, both because he neglected sidesway and because he used only two cycles of balancing.

A complete analysis of the frame of (3c) follows:

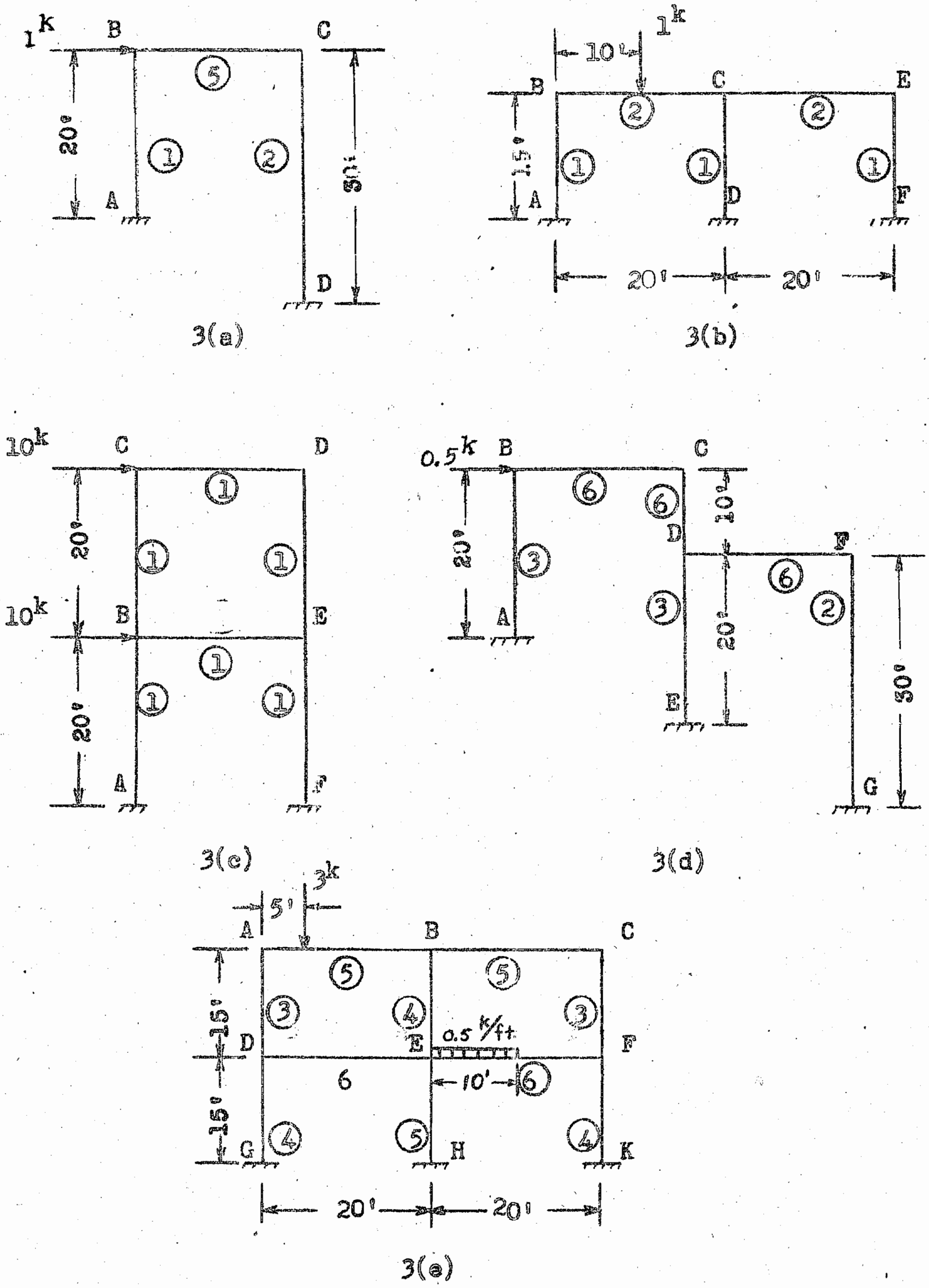


FIGURE 3

FRAMES ANALYZED BY DWYER'S METHOD AND PARTITIONING

$$[K] = \begin{bmatrix} 6 & 1 & 0 & 1 & 1 & 1 \\ 1 & 4 & 1 & 0 & 0 & 1 \\ 0 & 1 & 4 & 1 & 0 & 1 \\ 1 & 0 & 1 & 6 & 1 & 1 \\ 3 & 0 & 0 & 3 & 4 & 0 \\ 3 & 3 & 3 & 3 & 0 & 4 \end{bmatrix}$$

Inversion of $[K_{11}]$ by Dwyer's Method

i	1	2	3	4	5	6	7	8
a_{i1}	6	1	0	1	1	0	0	0
a_{i2}		4	1	0	0	1	0	0
a_{i3}			4	1	0	0	1	0
a_{i4}				6	0	0	0	1
a_{i1}	6.0000	1.0000	0.0000	1.0000	1.0000			
b_{i1}	1.0000	0.1667	0.0000	0.1667	0.1667			
a_{i2-1}		3.8333	1.0000	-0.1667	-0.1667	1.0000		
b_{i2-1}		1.0000	0.2609	-0.0435	-0.0435	0.2609		
a_{i3-2}			3.7391	1.0435	0.0435	-0.2609	1.0000	
b_{i3-2}			1.0000	0.2791	0.0116	-0.0698	0.2674	
a_{i4-3}				5.5348	-0.1861	0.1163	-0.2791	1.0000
b_{i4-3}				1.0000	-0.0336	0.0210	-0.0504	0.1807
					0.1807	-0.0504	0.0210	-0.0336
						0.2816	-0.0757	0.0210
							0.2815	-0.0504
								0.1807

$$[K][K]^{-1} = \begin{bmatrix} 1.0002 & 0.0002 & -0.0001 & 0.0001 \\ & 1.0003 & -0.0003 & 0.0000 \\ & & 0.9999 & 0.0001 \\ & & & 1.0002 \end{bmatrix}$$

Inversion of $[K]$ by partitioning:

		3.0000	0.0000	0.0000	3.0000	4.0000	0.0000
		3.0000	3.0000	3.0000	3.0000	0.0000	4.0000
0.1471	0.1177	0.1807	-0.0504	0.0210	-0.0336	1.0000	1.0000
-0.0294	0.1765	-0.0504	0.2816	-0.0757	0.0210	0.0000	1.0000
-0.0294	0.1765	0.0210	-0.0757	0.2815	-0.0504	0.0000	1.0000
0.1471	0.1177	-0.0336	0.0210	-0.0504	0.1807	1.0000	1.0000
0.3455	0.1092	0.4413	-0.0882	-0.0882	0.4413	3.1174	-0.7062
0.1092	0.4819	0.3531	0.5295	0.5292	0.3531	-0.7062	2.2351

$$[K]^{-1} = \begin{bmatrix} 0.2345 & -0.0175 & 0.0539 & 0.0202 & -0.0637 & -0.0728 \\ -0.0175 & 0.3241 & -0.0332 & 0.0539 & -0.0091 & -0.0818 \\ 0.0539 & -0.0332 & 0.3241 & -0.0175 & -0.0091 & -0.0818 \\ 0.0202 & 0.0539 & -0.0175 & 0.2345 & -0.0637 & -0.0728 \\ -0.1910 & -0.0273 & -0.0273 & -0.1910 & 0.3455 & 0.1092 \\ -0.2183 & -0.2455 & -0.2455 & -0.2183 & 0.1092 & 0.4819 \end{bmatrix}$$

$$[K][K]^{-1} = \begin{bmatrix} 1.0004 & 0.0002 & -0.0001 & 0.0003 & -0.0003 & -0.0003 \\ 0.0001 & 1.0002 & -0.0004 & 0.0000 & 0.0000 & 0.0001 \\ 0.0000 & -0.0003 & 0.9998 & 0.0001 & 0.0000 & 0.0001 \\ 0.0003 & -0.0001 & 0.0001 & 1.0004 & -0.0003 & -0.0003 \\ 0.0001 & 0.0000 & 0.0000 & 0.0001 & 0.9998 & 0.0000 \\ 0.0001 & -0.0001 & -0.0001 & 0.0001 & 0.0000 & 1.0000 \end{bmatrix}$$

$$\{\phi\} = [K]^{-1} \times \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -400 \\ -200 \end{bmatrix} = \begin{bmatrix} 40.04 \\ 20.00 \\ 20.00 \\ 40.04 \\ -160.04 \\ -140.06 \end{bmatrix} \quad ----(8)$$

$$\begin{bmatrix} M_{BE} \\ M_{CD} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 40.04 \\ 20.00 \\ 20.00 \\ 40.04 \\ -160.04 \\ -140.06 \end{bmatrix} = \begin{bmatrix} 120.12 \\ 60.00 \end{bmatrix} \quad --(9)$$

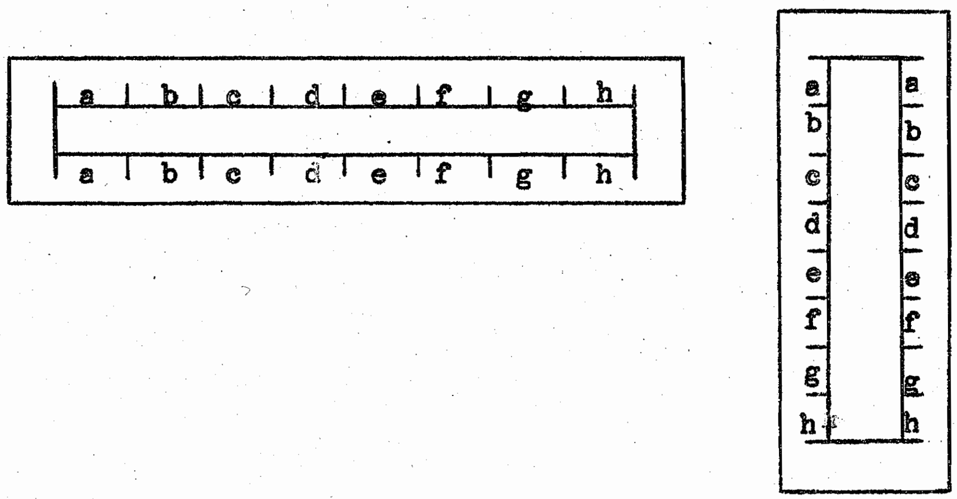
$$\begin{bmatrix} M_{AB} \\ M_{BC} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \{\phi\} = \begin{bmatrix} -120.00 \\ -39.98 \end{bmatrix} \quad -----(10)$$

$$\begin{bmatrix} M_{BA} \\ M_{CB} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \{\phi\} = \begin{bmatrix} -79.96 \\ -60.02 \end{bmatrix} \quad - (12)$$

Moments are in foot-kips. Because of the symmetry of the structure, the above moments are the only ones it is necessary to compute. Note that, in practice, it is not necessary to rewrite any matrix in order to perform the various multiplications necessary. If the two matrices to be multiplied have been written once, the multiplication may be performed without confusion by use of a simple aid which is rapidly constructed. This aid may consist simply of two strips of cardboard, one containing a horizontal slit, the other a vertical slit, as indicated in Figure 4(a). The horizontal slit is used to separate the row of the premultiplier, and the vertical slit, the column of the postmultiplier. Figure 4(b) illustrates a refinement of the multiplication aid which the author found convenient in multiplying matrices of varying orders. Made of wood, the device consists of a horizontal and vertical bar, each equipped with a slide to facilitate setting the order of the matrix. The "matrix multiplier", as used by the author, provides a space of three-quarters by one-quarter inch for each matrix element. In use, the element (a) of the horizontal row is multiplied by element (a) of the vertical row, etc.

3. By Zurmuhl's Method

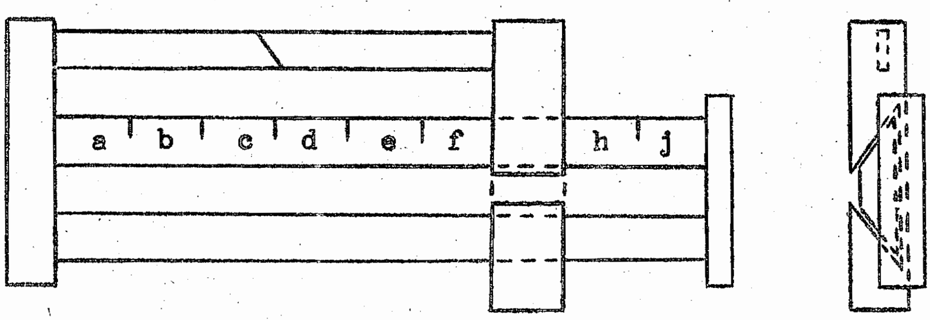
The number of operations and, consequently, the time required for inversion using Dwyer's method, the method of partitioning, or Zurmuhl's method, varies as the square of the order of the matrix. To the author's mind, the relative merit of Zurmuhl's method, as compared to the combined method of the preceding section, could be determined by



4(a)

Cardboard Strips to Identify

Row and Column Being Multiplied



4(b)

Wooden Matrix Multiplier

Order of Matrix Adjustable

FIGURE 4

MATRIX MULTIPLICATION AIDS

inverting a very few matrices by both methods, and extrapolating the results. The stiffness matrices of the frames of Figure 3(b) and 3(c) were, therefore, inverted by both methods, the accuracy of the two methods being investigated by a comparison of the products $[K][K]^{-1}$.

Inversion of the stiffness matrix of the frame of Figure 3(b) is illustrated below:

$$[K] = \begin{bmatrix} 6 & 2 & 0 & 1 \\ 2 & 10 & 2 & 1 \\ 0 & 2 & 6 & 1 \\ 3 & 3 & 3 & 6 \end{bmatrix}$$

6.0000	2.0000	0.0000	1.0000	1.0000	0.0000	0.0000	0.0000
2.0000	10.0000	2.0000	1.0000	0.0000	1.0000	0.0000	0.0000
0.0000	2.0000	6.0000	1.0000	0.0000	0.0000	1.0000	0.0000
3.0000	3.0000	3.0000	6.0000	0.0000	0.0000	0.0000	1.0000
6.0000	0.3333	0.0000	0.1667	0.1667			
2.0000	9.3334	0.2143	0.0714	-0.0357	0.1071		
0.0000	2.0000	5.5714	0.1539	0.0128	-0.0384	0.1795	
3.0000	2.5000	2.3214	4.9641	-0.0887	-0.0360	-0.0839	0.2014

$$[K]^{-1} = \begin{bmatrix} 0.1932 & -0.0329 & 0.0257 & -0.0310 \\ -0.0350 & 0.1167 & -0.0352 & -0.0077 \\ 0.0265 & -0.0329 & 0.1924 & -0.0310 \\ -0.0887 & -0.360 & -0.0839 & 0.2014 \end{bmatrix}$$

$$[K][K]^{-1} = \begin{bmatrix} 1.0005 & 0.0000 & -0.0001 & 0.0000 \\ 0.0007 & 0.9994 & 0.0003 & 0.0004 \\ 0.0003 & 0.0000 & 1.0001 & 0.0000 \\ 0.0219 & -0.0633 & 0.0453 & 0.9993 \end{bmatrix}$$

Using Dwyer's method and partitioning,

$$[K]^{-1} = \begin{bmatrix} 0.1938 & -0.0348 & 0.0271 & -0.0310 \\ -0.0348 & 0.1162 & -0.0348 & -0.0078 \\ 0.0271 & -0.0348 & 0.1938 & -0.0310 \\ -0.0931 & -0.0233 & -0.0931 & 0.2016 \end{bmatrix}$$

$$[K][K]^{-1} = \begin{bmatrix} 1.0001 & 0.0003 & -0.0001 & 0.0000 \\ 0.0007 & 0.9995 & 0.0007 & 0.0004 \\ -0.0001 & 0.0003 & 1.0001 & 0.0000 \\ -0.0003 & 0.0000 & -0.0003 & 1.0002 \end{bmatrix}$$

E. Results

Since essentially different problems were involved in using the three methods of inversion, a uniform method of presentation of results is not feasible. Hence, results of the three separate investigations are presented below in varying forms.

1. Matrix Power Series Approximation

It is not enough to know, for a given frame, that the matrix power series will converge. Some indication of the time required to obtain acceptable accuracy is essential, if this method is to be compared with other analytical methods. The time required for inversion, using a desk calculator and a constant number of computer decimals, is almost directly proportional to the number of scalar products computed. Hence, the time of inversion is indicated approximately by the order of the stiffness matrix and the number of terms of the power series required for reasonable accuracy.

Table 4, below, presents the number of terms required and the time of inversion for each of the frames of Figure 2, together with the order of the matrix and the norm of matrix $\left[Q\right]$, which appears in the power series. These results might be more readily understandable if presented in the form of curves, for each matrix order, of the number of terms required for a given accuracy plotted against norm $\left[Q\right]$. The number of frames of each order investigated is too small to permit such curves to be drawn. That more frames were not investigated is due to the largely negative results of these early stages of the investigation.

TABLE 4

RESULTS OF MATRIX POWER SERIES INVESTIGATION

Frame*	Order of stiffness matrix	Norm [Q]	Terms of power series computed	Time of inversion (minutes)	Error**
2(a)	3	0.0791	3	36	0.00006
2(b)	3	0.7115	8	80	0.00170
2(c)	4	0.4330	5	95	0.00090
2(d)	3	1.1995	17	160	0.86899
2(e)	4	0.9312	10	175	0.00415
2(f)	5	1.3236	6	175	0.05447
2(g)	5	1.0004	6	175	0.02911
2(h)	6	1.0680	5	210	0.02186
2(j)	6	1.2381	5	210	0.04630
2(k)	6	1.5989	7	270	0.08890
2(l)	6	1.9744	6	250	Divergent
2(m)	8	0.94589	10	660	0.00172

* Numbers identify frames of Figure 2.

** Error is the maximum difference between the elements of $[K][K]^{-1}$ and the corresponding elements of $[I]$.

2. Dwyer's Method Plus Partitioning

As indicated in an above section, the time required for inversion of a symmetric matrix by Dwyer's method, for a given constant number of computer decimals, varies as the square of the order of the matrix. Using four decimals, the author found the time required for inversion to be approximately $n^2/2$ minutes, n being the order of the matrix. This time is affected only slightly by the number of zero elements in the matrix, but depends, of course, upon the analyst's proficiency in the use of a calculator.

The time required for completing the inversion of a non-symmetric matrix by partitioning, after the symmetric submatrix has been inverted, depends both upon the order of the original non-symmetric matrix and upon the order of the symmetric submatrix. In the present investigation, the author has limited the use of this method of inversion to stiffness matrices having only two rows and columns due to sidesway. For such matrices, the overall time of inversion is approximately twice the time required to invert the symmetric submatrix. Thus, the time of inversion in minutes, for the author, is given approximately by the square of the order of the matrix. It is estimated that frames with stiffness matrices of intermediate order, say ten to twenty, can be solved by a competent analyst in slightly less than half this time, on the average, using moment distribution and successive corrections.

Table 5 presents the results of the investigations using this combined method of inversion. The frame numbers identify frames of Figure 3. The "check value" column represents end moments obtained, for frames 3(a) through 3(d), by Wilson[15], using a theoretically exact solution, and for frame 3(e), by Cornish[4], using an approximate method and neglecting sidesway. All computations were performed using four computer decimals.

TABLE 5

RESULTS USING DWYER'S METHOD WITH PARTITIONING

Frame	Order of stiffness matrix	Time of inversion (minutes)	Overall time of solution (minutes)	Maximum error in [K] [K]$^{-1}$	Member	Final end moment (ft-lb)	Check value	Maximum deviation (%)
3(a)	3	5	12	.0010	AB	-5721	-5720	
					BA	-5412	-5410	
					BC	5415	5410	
					CB	6195	6180	0.24
					CD	-6180	-6180	
					DC	-7110	-7110	
3(b)	4	8	15	.0007	AB	397	390	1.79
					BA	969	963	
					BC	-969	-963	
					CB	2133	2135	
					CD	-930	-933	
					CE	-1200	-1202	
					DC	-552	-557	
					EC	-136	-137	
					EF	135	137	
					FE	-19.7	-	

TABLE 5 (CONTINUED)

Frame	Order of stiffness matrix	Time of inversion (minutes)	Overall time of solution (minutes)	Maximum error in $[K][K]^{-1}$	Member	Final end moment (ft-lb)	Check value	Maximum deviation (%)
3(c)	5	10	16	.0004	AB	-1200	-1200	
					BA	-800	-800	
					BE	1201	1200	0.08
					BC	-400	-400	
					CB	-600	-600	
					CD	600	600	
3(d)	6	20	35	.0003	AB	-2832	-2838	
					BA	-2326	-2345	
					BC	2320	-2345	1.06
					CB	1608	1612	
					CD	-1614	-1612	
					DC	-810	-799	
					DE	-1404	-1407	
					DF	2211	2203	
					FD	1112	1109	
					FG	-1112	-1109	
					GF	-1113	-1111	
					ED	-1956	-1954	
3(e)	6 (Sidesway neglected)	20	40	.0018	GD	-636	-	
					DG	-1272	-1300	

TABLE 5 (CONTINUED)

Frame	Order of stiffness matrix	Time of inversion (minutes)	Overall time of solution (minutes)	Maximum error in $[K][K]^{-1}$	Member	Final end moment (ft-lb)	Check value	Maximum deviation (%)
3(e)	6	20	40	.0018	DE	299	300	
(Cont'd)					DA	964	1000	
	(Sidesway				AD	3358	3500	
	neglected)				AB	-3531	-3500	
					BA	3039	3000	
					BC	-2224	-2200	
					BE	-904	-800	
					EB	1756	1800	
					ED	3461	3400	
					EF	-8856	-8800	
					EH	3679	-	
					FE	3790	3800	
					FK	-2417	-2400	
					FC	-1366	-1400	
					CF	-13	-100	
					CB	4	100	96
					KF	-1208	-	
					HE	-1840	-	

TABLE 5 (CONTINUED)

Frame	Order of stiffness matrix	Time of inversion (minutes)	Overall time of solution (minutes)	Maximum error in $[K][K]^{-1}$	Member	Final end moment (ft-lb)	Check value	Maximum deviation (%)
3(e)	8 (Sidesway included)	40	60	.0021	GD	-674	-	-
					DG	-1196	-	-
					DE	763	-	-
					DA	433	-	-
					AD	2896	-	-
					AB	-2897	-	-
					BA	3536	-	-
					BC	-1776	-	-
					BE	-1768	-	-
					EB	878	-	-
					ED	3878	-	-
					EF	-8432	-	-
					EH	3696	-	-
					FE	4268	-	-
					FK	-2332	-	-
					FC	-1919	-	-
					CF	-530	-	-
					CB	540	-	-
					KF	-1241	-	-
					HE	-2131	-	-

3. Zurmuhl's Method

No frames were completely analyzed by Zurmuhl's method. The stiffness matrices of the frames of Figure 3(b) and 3(c) were inverted by Zurmuhl's method to compare the accuracy and time required with those of Dwyer's method, using the same number of computer decimals. Referring to the inversion of frame 3(b), indicated above under "Frames Analyzed", it is seen that, using the same number of decimals throughout, Dwyer's method with partitioning gives a slightly more accurate inverse than Zurmuhl's method. Frazer, Duncan, and Collar[8] note that the method of partitioning does improve the accuracy of an inverse.

Using Zurmuhl's method for inverting the non-symmetric stiffness matrix, no economy of time results from the fact that the matrix includes a symmetric submatrix. Thus, using Zurmuhl's method, more time is spent on the elements of the symmetric submatrix than when using Dwyer's method. A part of this lost time is regained in the computations involving the non-symmetric elements, since Zurmuhl's method requires less time than the method of partitioning. Extrapolating from admittedly slight evidence, it appears that Zurmuhl's method requires more time than Dwyer's method and partitioning, if no more than two rows and columns of the stiffness matrix are due to sidesway. The measured difference in time for the two frames investigated by both methods was slight, leading the author to believe that Zurmuhl's method would be slightly more rapid if more than two or three columns and rows are due to sidesway.

VI. DISCUSSION OF RESULTS

Results of the investigation of the matrix power series method of inversion are largely negative. Table 4 indicates an unreasonable length of time required for inverting matrices, even when the norm test indicates fairly rapid convergence. The only conceivable situation in which this method might prove advantageous is one in which the difference in pay scales between engineers and calculator operators is extremely great. In such a case, it might prove advantageous to have an engineer set up the matrix equations, using the inexpensive labor to perform the inversion. The author does not consider this situation extremely probable. If it should arise, however, then curves indicating the rate of convergence of the series for any value of norm $[Q]$ and any order matrix might be worth developing.

The method of partitioning, with Dwyer's method used to invert the symmetric submatrix, is reasonably rapid if the number of rows and columns of the stiffness matrix due to sidesway is small. Even though the matrix method requires more time than moment distribution for the analysis of a single loading condition, the matrix method may be more economical of time when many loading conditions are to be analyzed. Such is the case with a great number of modern structures, which are analyzed separately for dead load, snow and ice loads, and varying wind loads and live loads, in order to determine the worst condition for each member. In such cases, it frequently is economical of time to compute moment coefficients, giving the end moments in each member due to a unit unbalanced fixed-end moment at each joint. When such moment coefficients are determined by moment distribution, a separate analysis is necessary for each joint. Using the matrix method, joint rotation coefficients appear directly in the inverse stiffness matrix, giving the rotation of each joint for a unit unbalanced fixed-end moment at any joint. Determination of these joint rotation coefficients requires only one analysis by the matrix method. Consequently, the value of such coefficients in a given frame provides a good index to the value of the matrix methods.

Zurmuhl's method appears inferior to Dwyer's method used with partitioning when the number of rows and columns due to side-sway is small as compared to the order of the stiffness matrix. When the number of joint translations approaches the number of rotations, Zurmuhl's method is advantageous. For matrices of high order, including many joint translations, Zurmuhl's method may be used to advantage in inverting the submatrix $\left[\Theta\right]$, Dwyer's method being used to invert the symmetric submatrix, and these two inverse matrices being used with the method of partitioning to invert the original stiffness matrix.

VII. CONCLUSIONS

The matrix power series method of inversion is impractical, since the series diverges, or converges extremely slowly, except in the analysis of very special and, usually, impractical frames.

Dwyer's method, Zurmuhl's method, and the method of partitioning, used with discrimination, suffice to invert any stiffness matrix with satisfactory rapidity. These methods are rapid enough to compete with moment distribution when a frame is to be analyzed for many loading conditions. This condition is met frequently enough to make a study of matrix algebra worth while for the structural engineer.

Both Dwyer's method and Zurmuhl's method yield reasonable accuracy when four significant figures are used in the original stiffness matrix and four decimals are used throughout the computations, provided the stiffnesses of the frame members do not differ extremely.

VIII. ACKNOWLEDGMENTS

The author is indebted to Professor D. H. Pletta for the suggestion which led to the adoption of the thesis subject, and for his interest and guidance during the course of the investigation. He is grateful to Professor V. G. Szebehely whose enthusiasm for matrix methods has proved contagious, to Mr. E. E. Van Biervliet, for his able translation from the German of Zurmuhl's method, and to Professor F. J. Maher for his constructive criticism of the completed thesis.

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X. VITA

William P. Murden, Jr., was born in Newport News, Virginia, in 1924. He attended grammar school and high school in that city. In 1941, he entered the College of William and Mary, majoring in Chemistry until early 1943. At that time he left school to enter the Army Air Force, where he served as a chemical warfare instructor.

Upon discharge from the service in 1946, he entered Virginia Polytechnic Institute, graduating in 1949 with the degree of Bachelor of Science in Civil Engineering. In June, 1949, he began study for the degree of Master of Science in Applied Mechanics, at Virginia Polytechnic Institute.

From September 1948 to December 1949, he was a teaching assistant and teaching fellow of Applied Mechanics at Virginia Polytechnic Institute. He then became a full time instructor in that department.

The author is a member of Chi Epsilon, Tau Beta Pi, Phi Kappa Phi, Omicron Delta Kappa, and the Virginia Academy of Science. He is registered in Virginia as a professional engineer.