APPLICATION OF THE HODOGRAPH METHOD TO SEVERAL PROBLEMS IN FLUID MECHANICS

by

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II. LIST OF SYMBOLS

Symbol	Definition
ϕ	potential function
ψ	stream function
x, y	Cartesian coordinates in physical plane
z	complex variable in physical plane
ξ	complex variable in auxiliary plane
σ	complex variable in distorted hodograph plane
F	complex potential function
u, v	velocity components in x and y directions respectively
q	magnitude of the velocity vector
U	velocity in undisturbed stream
θ	direction of the velocity vector in physical plane
w	modified velocity in distorted hodograph plane
p	pressure
ρ	density
γ	ratio of specific heats (specific heat at constant pressure to specific heat at constant volume)
a	speed of sound
M	Mach number
K, C, L	constants
$\alpha, \beta, \varepsilon$	angles
a, b, c	points on the real axis of the ξ-plane

Symbol	Definition
A, B, C, D, E	points in the physical plane
l	a characteristic dimension in physical plane
n	contraction coefficient
o	as subscript, refers to stagnation conditions
∞	as subscript, refers to conditions in undisturbed stream

III: INTRODUCTION

The general problem in fluid mechanics involves finding an expression which is a solution to certain mathematical equations and which satisfies certain boundary conditions. These mathematical equations are the equations expressing the physical laws of conservation of mass and conservation of momentum. The equations expressing conservation of momentum, generally referred to as the equations of motion are, however, non-linear, and as yet no method of obtaining solutions to these equations is known. For this reason certain simplifying assumptions are generally made, namely, that the fluid is perfect and incompressible and that the fluid motion is irrotational. By making these assumptions and by introducing the velocity potential, ϕ, the problem is reduced to that of finding a solution to Laplace's equation which satisfies the boundary condition for the particular problem.

The first of these assumptions, that of a perfect fluid, is in effect an assumption that the fluid is nonviscous. It was Prandtl who in 1904 first advanced the theory that the effects of viscosity, in fluids of small viscosity, are limited to a small layer of fluid next to the boundary, and therefore the fluid outside of this boundary layer may be treated as a perfect fluid. This theory is borne out by experiment; thus reasonably accurate solutions may be obtained for many problems by considering them as nonviscous. The second of these assumptions, that of an incompressible fluid renders accurate solutions for liquids and for gases at low velocities where the effects of compressibility are negligible.

The solutions to fluid mechanics problems in which the fluid is assumed to be perfect (nonviscous), incompressible and irrotational comprise the Classical Hydro and Aerodynamic Theory.

In problems in which gases attain high velocities, classical theory fails to predict the physical flow accurately since the effects of compressibility are no longer negligible. If the fluid is not considered incompressible, however, the equations of motion are again non-linear and again no method of obtaining solutions to these equations is known. Molenbroeck in 1908 and Chaplygin (1) in 1902 discovered that these exact non-linear equations of motion could be transformed into exact linear equations by a mathematical transformation. This transformation is known as the hodograph transformation. Chaplygin (1) made an important simplifying assumption in the hodograph methods and applied these methods to problems of gaseous jets in a paper which appeared in Russia in 1902.

Several investigators applied Chaplygin's methods to problems in gas dynamics in the early Thirties. The hodograph methods in general use today, however, contain an improvement in the method of approximation which was suggested by Theodore von Karman (9, 10) and worked out by H. S. Tsien (8) in 1939. The method of Karman-Tsien has been applied to several problems of fluid mechanics, the most notable being the flow about certain airfoils.

It is important to note that although the hodograph transformation transforms the non-linear equations of motion into linear equations, there is a distinct difficulty which prohibits its use in solving more than a few compressible flow problems. This difficulty arises from the fact that the physical boundaries are difficult to satisfy. This difficulty is basically due to the fact that the boundary conditions are given in terms of the physical coordinates, x and y, while the equations of motion in the hodograph plane involve not the physical coordinates but the magnitude of the velocity, q, and the velocity direction, θ, as independent variables.

It is the purpose of this thesis to investigate several problems in the flow of a compressible fluid by the hodograph method. These problems will be of the general class in which the "free" streamlines are involved.

IV. THE HODOGRAPH METHOD

A. The Equations of Motion

A steady, two-dimensional, irrotational motion of a compressible fluid is governed by the equations expressing conservation of mass, conservation of momentum and irrotationality. The equation of conservation of mass, or the continuity equation, may be written

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (1)$$

where ρ is the mass density of the fluid and u and v are the velocity components in the x and y directions respectively. The equation of conservation of momentum or the equation of motion of the fluid in the form commonly known as the Bernoulli equation is

$$q\,dq + \frac{dp}{\rho} = 0 \qquad (2)$$

Here q is the magnitude of the velocity vector and p is the static pressure in the fluid. The assumption that the fluid motion is irrotational gives rise to an equation relating the partial derivatives of the velocity components, namely

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \qquad (3)$$

If the flow is also assumed to be isentropic, there exists a fourth equation

$$p = \text{constant}\ \rho^{\gamma} \qquad (4)$$

which relates the pressure, p, and the density, ρ, of the fluid by the ratio of specific heats, γ. The velocity of sound, a, in a gas undergoing an isentropic process is defined by

$$a^2 = \frac{dp}{d\rho} \qquad (5)$$

Equations (1) through (4) comprise a system of four equations in four unknowns, u, v, ρ, and p. The general approach is to eliminate three of the unknowns and to obtain one equation in one unknown which is to be solved. To this end an additional variable and two additional equations are added to the four above. This additional variable, the velocity potential ϕ, is defined by the equations

$$\frac{\partial \phi}{\partial x} = u \, ; \quad \frac{\partial \phi}{\partial y} = v \qquad (6)$$

It is obvious that the velocity potential satisfies the equation of irrotational motion (3).

Introducing the velocity potential as the dependent variable into equations (1) through (3) and reducing equations (1) through (4) to a single equation, the expression

$$\left[a^2 - \frac{\partial \phi^2}{\partial x}\right] \frac{\partial^2 \phi}{\partial x^2} - 2 \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} \frac{\partial^2 \phi}{\partial x \partial y} +$$

$$\left[a^2 - \frac{\partial \phi^2}{\partial y}\right] \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (7)$$

is obtained. This is the potential equation for a steady, two-dimensional, irrotational motion of a compressible fluid and is obviously a non-linear partial differential equation.

At present there are two main methods of obtaining solutions to this equation:

1. Approximate solutions to the potential equation are

obtained by making certain assumptions by which the potential equation is linearized. Perhaps the most notable of these methods of linearizing the potential equation is the method of small perturbations.

2. The non-linear potential equation is transformed into a set of linear partial differential equations by a mathematical transformation. This is the hodograph transformation. It must be noted, however, that not all problems involving the potential equation can be solved by hodograph methods. This is due to the fact that the physical boundaries are difficult to satisfy in the hodograph plane.

The following equations, obtained from equations (1), (2), (4) and (5) are inserted here for future reference,

$$a^2 = a_o^2 - \frac{\gamma - 1}{2} q^2 \tag{8}$$

$$p = p_o \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{\gamma}{\gamma - 1}} \tag{9}$$

$$\rho = \rho_o \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{1}{\gamma - 1}} \tag{10}$$

$$M^2 = \frac{q^2}{a_o^2 - \frac{\gamma - 1}{2} q^2} \tag{11}$$

In these equations M is the Mach number or the ratio of the local velocity to the local velocity of sound, and the subscript zero refers to stagnation conditions.

B. The Hodograph Transformation

In addition to equations (1) through (4) which represent two-dimensional, potential motion and equations (6) which define the velocity potential, a second function, the stream function, Ψ , may be introduced. The stream function is defined by the equations

$$u = \frac{\rho_o}{\rho} \frac{\partial \Psi}{\partial y} ; \quad v = -\frac{\rho_o}{\rho} \frac{\partial \Psi}{\partial x} \tag{12}$$

and obviously satisfies the continuity equation.

Introducing the magnitude of the velocity vector, q, and the angle of inclination of the velocity vector with the x axis, θ, as new variables, the equations for the velocity potential and stream function may be written

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = q \cos \theta \, dx + q \sin \theta \, dy$$

$$d\Psi = \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial y} dy = -\frac{\rho}{\rho_o} q \sin \theta \, dx + \frac{\rho}{\rho_o} q \cos \theta \, dy \tag{13}$$

Solving these equations for dx and dy yields

$$dx = \frac{\cos \theta}{q} d\phi - \frac{\sin \theta}{q} \frac{\rho_o}{\rho} d\Psi$$

$$dy = \frac{\sin \theta}{q} d\phi + \frac{\cos \theta}{q} \frac{\rho_o}{\rho} d\Psi \tag{14}$$

As long as the correspondence between the physical and hodograph planes is one to one, x and y can be expressed as functions of q and θ, and ϕ and Ψ as functions of q and θ. Thus

$$d\phi = \frac{\partial \phi}{\partial q} dq + \frac{\partial \phi}{\partial \theta} d\theta$$

$$d\Psi = \frac{\partial \Psi}{\partial q} dq + \frac{\partial \Psi}{\partial \theta} d\theta \tag{15}$$

Introducing equations (15) into equations (14) yields

$$dx = \left(\frac{\cos\theta}{q}\frac{\partial\phi}{\partial q} - \frac{\sin\theta}{q}\frac{\rho_o}{\rho}\frac{\partial\psi}{\partial q}\right)dq$$

$$+ \left(\frac{\cos\theta}{q}\frac{\partial\phi}{\partial\theta} - \frac{\sin\theta}{q}\frac{\rho_o}{\rho}\frac{\partial\psi}{\partial\theta}\right)d\theta$$

$$dy = \left(\frac{\sin\theta}{q}\frac{\partial\phi}{\partial q} + \frac{\cos\theta}{q}\frac{\rho_o}{\rho}\frac{\partial\psi}{\partial q}\right)dq$$

$$+ \left(\frac{\sin\theta}{q}\frac{\partial\phi}{\partial\theta} + \frac{\cos\theta}{q}\frac{\rho_o}{\rho}\frac{\partial\psi}{\partial\theta}\right)d\theta \qquad (16)$$

Since the left-hand side of equations (16) are exact differentials, the reciprocity relation can be applied. Therefore

$$\frac{\partial}{\partial\theta}\left(\frac{\cos\theta}{q}\frac{\partial\phi}{\partial q} - \frac{\sin\theta}{q}\frac{\rho_o}{\rho}\frac{\partial\psi}{\partial q}\right) = \frac{\partial}{\partial q}\left(\frac{\cos\theta}{q}\frac{\partial\phi}{\partial\theta} - \frac{\sin\theta}{q}\frac{\rho_o}{\rho}\frac{\partial\psi}{\partial\theta}\right)$$

$$\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{q}\frac{\partial\phi}{\partial q} + \frac{\cos\theta}{q}\frac{\rho_o}{\rho}\frac{\partial\psi}{\partial q}\right) = \frac{\partial}{\partial q}\left(\frac{\sin\theta}{q}\frac{\partial\phi}{\partial\theta} + \frac{\cos\theta}{q}\frac{\rho_o}{\rho}\frac{\partial\psi}{\partial\theta}\right) \qquad (17)$$

Carrying out the indicated differentiation, equations (17) reduce to

$$-\sin\theta\frac{\partial\phi}{\partial q} - \cos\theta\frac{\rho_o}{\rho}\frac{\partial\psi}{\partial q} = -\frac{\cos\theta}{q}\frac{\partial\phi}{\partial\theta} + \frac{\sin\theta}{q}\frac{\rho_o}{\rho}\left(1 + \frac{q}{\rho}\frac{d\rho}{dq}\right)\frac{\partial\psi}{\partial\theta}$$

$$\cos\theta\frac{\partial\phi}{\partial q} - \sin\theta\frac{\rho_o}{\rho}\frac{\partial\psi}{\partial q} = -\frac{\sin\theta}{q}\frac{\partial\phi}{\partial\theta} - \frac{\cos\theta}{q}\frac{\rho_o}{\rho}\left(1 + \frac{q}{\rho}\frac{d\rho}{dq}\right)\frac{\partial\psi}{\partial\theta} \qquad (18)$$

Multiplying the first of equations (18) by cos θ and the second by sin θ and adding yields

$$\frac{\partial\phi}{\partial\theta} = q\frac{\rho_o}{\rho}\frac{\partial\psi}{\partial q} \qquad (19)$$

Multiplying the first of equations (18) by sin θ and the second by cos θ and subtracting yields

$$\frac{\partial \phi}{\partial q} = -\frac{1}{q}\frac{\rho_o}{\rho}\left(1 + \frac{q}{\rho}\frac{d\rho}{dq}\right)\frac{\partial \psi}{\partial \theta} \qquad (20)$$

From equations (2) and (5), however

$$\frac{q}{\rho}\frac{d\rho}{dq} = -\frac{q^2}{a^2} = -M^2 \qquad (21)$$

Equation (21) defines M the Mach number of the fluid. Introducing equation (21) into equation (20) renders

$$\frac{\partial \phi}{\partial q} = -\frac{\rho_o}{\rho q}\left(1 - M^2\right)\frac{\partial \psi}{\partial \theta} \qquad (22)$$

C. The Transformed Equations

Equations (19) and (22) are the equations of motion in the hodograph plane written in terms of the velocity magnitude, q, and velocity direction, θ. These equations are linear since the coefficients of the derivatives are functions of the independent variables q and θ alone.

It is convenient to introduce at this point a new variable in order to put these equations in symmetric form. Introducing

$$dw = \sqrt{1-M^2}\;\frac{dq}{q} \qquad (23)$$

into the hodograph equations yields

$$\frac{\partial \phi}{\partial \theta} = \frac{\rho_o}{\rho}\sqrt{1-M^2}\;\frac{\partial \psi}{\partial w} \qquad (24)$$

$$\frac{\partial \phi}{\partial w} = -\frac{\rho_o}{\rho}\sqrt{1 - M^2}\,\frac{\partial \psi}{\partial \theta} \tag{25}$$

These are the equations of motion in the so-called "distorted" hodograph plane where the independent variables are w and θ. Integration of equation (23) gives the relation between the variable w and the velocity magnitude, q, to be

$$w = \log \frac{Cq}{a_o + \sqrt{a_o^{\,2} + q^2}} \tag{26}$$

Here C is the constant of integration.

D. Chaplygin's Approximation

Chaplygin (1) noticed that the factor $\frac{\rho_o}{\rho}\sqrt{1 - M^2}$ in equations (24) and (25) differs little from unity for values of M not too close to one. Noticing that

$$\frac{\rho_o}{\rho}\sqrt{1 - M^2} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma - 1}} \left(1 - M^2\right)^{\frac{1}{2}} \tag{27}$$

by virtue of equation (10), and developing this in powers of M^2 yields

$$\begin{aligned}\frac{\rho_o}{\rho}\sqrt{1 - M^2} &= \left(1 + \frac{M^2}{2} + (2 - \gamma)\frac{M^4}{8} + \cdots\right)\left(1 - \frac{M^2}{2} - \frac{M^4}{8} + \cdots\right)\\ &= \left(1 - \frac{\gamma - 1}{8} M^4 + \cdots\cdots\right)\end{aligned} \tag{28}$$

Thus $\frac{\rho_o}{\rho}\sqrt{1 - M^2}$ differs from unity by terms of M^4 and higher.

With this approximation, $\frac{\rho_o}{\rho}\sqrt{1 - M^2} \simeq 1$, the hodograph equations reduce to

$$\frac{\partial \phi}{\partial \theta} = \frac{\partial \psi}{\partial w}; \quad \frac{\partial \phi}{\partial w} = -\frac{\partial \psi}{\partial \theta} \tag{29}$$

Thus the hodograph equations are reduced to the Cauchy-Riemann relations by the Chaplygin condition

$$\frac{\rho_o}{\rho} \sqrt{1 - M^2} \cong 1 \tag{30}$$

This condition may be interpreted in a different way. In equation (27) the Chaplygin condition, (30), is satisfied if γ is taken to be $\gamma = -1$ for M less than one. Naturally no gas with this property, $\gamma = -1$, exists; however, the true meaning of the assumption $\gamma = -1$ can be found by considering the isentropic relation for an "imaginary" gas with this property. For a perfect gas, the isentropic relation is given by

$$\frac{p}{\rho^{\gamma}} = \text{constant} \tag{31}$$

For the "imaginary" gas, this relation becomes

$$p\rho = \text{constant} \tag{32}$$

Hence the condition $\gamma = -1$ defines a straight line isentrope in the p versus $\frac{1}{\rho}$ plane. In other words, by making the Chaplygin approximation the true isentrope of a perfect gas has been approximated by a straight line tangent to the true isentrope. Chaplygin (1), in his original work, used as the point of tangency the point corresponding to stagnation conditions, p_o and ρ_o. An essential refinement to the method of Chaplygin was made in 1939 by Theodore von Karman (9) (10) and H. S. Tsien (8) who used a straight line isentrope which was tangent to the true isentrope at a point corresponding to the undisturbed stream conditions p_∞ and ρ_∞ .

E. The Relation Between the Hodograph and the Physical Planes

Since equations (29) are equations of the Cauchy-Riemann type they are satisfied by the real and imaginary parts of any complex analytic function $F(\theta + iw) = \phi + i\psi$. Now the problem is to find the analytic function, F, which satisfies the given boundary conditions in the physical plane. It is, therefore, necessary to consider the transformation from the physical plane to the hodograph plane. In the physical plane the relation between the coordinates is given by

$$dz = dx + idy \tag{33}$$

Introduction of equations (14), which express dx and dy in terms of q, θ, ϕ and ψ, into equation (33) yields the expression

$$dz = \frac{e^{i\theta}}{q} (d\phi + id\psi) \tag{34}$$

This equation may be written in terms of the analytic function $F = \phi + i\psi$ which is a solution to equations (29). Therefore

$$dz = \frac{e^{i\theta}}{q} \left[dF + \overline{dF} + \frac{\rho_o}{\rho} (dF - \overline{dF}) \right] \tag{35}$$

since $F = \phi + i\psi$ and $dF = d\phi + id\psi$. Here the bar refers to the conjugate function. Equation (26) may now be solved for $\frac{1}{q}$ to give

$$\frac{1}{q} = \frac{1}{2a_o} \left[Ce^{-w} - \frac{1}{C} e^{w} \right] \tag{36}$$

Also by considering equations (10), (11) and (36), the following expressions are obtained

$$1 + \frac{\rho_o}{\rho} = \frac{2\,Ce^{-w}}{Ce^{-w} - \frac{1}{C}e^{w}}$$

$$1 - \frac{\rho_o}{\rho} = - \frac{\frac{2}{C}e^{w}}{Ce^{-w} - \frac{1}{C}e^{w}} \tag{37}$$

Introducing equations (36) and (37) into equation (35) yields the relation between the physical plane, $z = x + iy$, and the "distorted" hodograph plane, $\sigma = \theta + iw$. This relation is

$$dz = \frac{1}{2a_o}\left[Ce^{i\sigma}\, dF(\sigma) - \overline{\frac{1}{C}e^{-i\sigma}\, dF(\sigma)}\right] \tag{38}$$

V. DISCUSSION OF THE PROBLEM

It has been shown that the equations of motion in the "distorted" hodograph plane are reduced to the Cauchy-Riemann relations by the Chaplygin approximation. It is well known that the Cauchy-Riemann relations are satisfied by any analytic function, $F = \phi + i\psi$, and therefore the problem becomes one of finding an analytic function which satisfies the given boundary conditions.

In the several problems considered in this thesis, the analytic function, $F(\sigma)$, which satisfies the boundary conditions, is found by relating the "distorted" hodograph plane to the potential, F, plane through one or more intermediate planes, a procedure which is well known in the treatment of classical theory. This simple procedure is possible since the problems considered herein are problems in which the bounding streamlines are made up partially of straight boundaries and partially of "free" streamlines. Therefore, in these problems the velocity direction is known along the straight boundaries, and the velocity magnitude is known along the free streamlines. The velocity magnitude along the straight boundary and the velocity direction along the free streamline are, however, not known until the problem is solved. Since the velocity direction along the straight boundary and the velocity magnitude along the free streamlines are known, it is possible to transform the bounding streamlines from the physical plane into a boundary of definite slope in the distorted hodograph plane. It is therefore possible to formulate the problems considered as direct boundary value problems.

If, however, this were not the case, that is if the velocity direction or the velocity magnitude were not known on some portion of the bounding streamline, the problem would be considerably more difficult.

It will be shown that the boundaries of the flow problems considered here form polygons in both the "distorted" hodograph plane and in the potential plane ($F = \phi + i\Psi$). Since this is true both planes can be transformed into an intermediate plane by the theorem of Schwarz-Christoffel.*

The theorem of Schwarz-Christoffel states that the boundary of a regular polygon, in say the z-plane, is transformed into the real axis of some new plane, say the ξ-plane, by the transformation

$$\frac{dz}{d\xi} = K(\xi - a)^{\frac{a}{\pi} - 1} (\xi - b)^{\frac{\beta}{\pi} - 1} (\xi - c)^{\frac{\varepsilon}{\pi} - 1} \ldots \quad (39)$$

Moreover, if the polygon is simple the interior of the polygon is mapped onto the upper half of the ξ-plane by this transformation. In the transformation, K is a constant which may be complex; a, b, c,....are the locations on the real axis of the ξ-plane representing the vertices of the polygon, and α, β, ε.... are the interior angles of the polygon corresponding to the vertices, a, b, c,....

* For a comprehensive treatment of the Schwarz-Christoffel Transformation, see THEORETICAL HYDRODYNAMICS by L. M. Milne-Thomson

The Schwarz-Christoffel transformation has an important property which will be used in this thesis: Vertices which lie at infinity do not enter into the transformation equation. The proof of this property is not shown here, however, this proof can be found in several texts which treat the Schwarz-Christoffel Transformation.

VI. APPLICATIONS OF THE HODOGRAPH METHOD

A. Free Streamlines

When a fluid is required to turn a sharp corner, radius of curvature zero, the acceleration of the fluid particle becomes infinite. This requires an infinite force. In ideal fluid flow this infinite force is obtained by assuming an infinite velocity at the corner. Since this type of flow is not physically possible, the assumption is generally made that the fluid separates from the body and does not negotiate the corner. This leads to a class of problems involving "free" streamlines, a free streamline being one which emanates from a sharp corner. In the problems considered in this thesis the fluid is assumed to separate at points on the body where the body form makes a sudden turn with the exception of stagnation points.

The fluid which is in contact with the body downstream from the body and which is separated from the main fluid motion by the free streamlines is known as the wake. In steady flow the fluid in the wake is assumed to be at rest. This assumption is considerably in error for actual fluids. However, if the wake contains fluid of much less density than the fluid in the main flow, the theory should give results which compare favorably with experiment.

Neglecting the effect of gravity, the condition that the wake be at rest indicates that the pressure intensity in the wake is constant, and therefore the pressure intensity along the free streamline is constant. By virtue of this fact and Bernoulli's equation, the velocity magnitude must be constant along a free streamline.

B. Two-Dimensional Orifice

In the flow from a two-dimensional orifice, Figure 1(a), a large tank is assumed to have a rectangular slot of great length out of which the fluid flows. The points B, B' and I' are assumed to be a great distance from the slot, and therefore have zero velocity. The flow along the wall BA has a direction $\theta = 0$ and the flow along the wall B'A' has the direction $\theta = -\pi$. To avoid an infinite velocity at A and A', the flow is assumed to separate and leaves in a tangential direction. If the velocity at I_∞ is taken to be U, then the velocity along the free streamlines AI_∞ and $A'I_\infty$ is also U. The velocity direction along the free streamline AI_∞ varies from $\theta = 0$ at A to $\theta = -\frac{\pi}{2}$ at I_∞, and the velocity direction along $A'I_\infty$ varies from $-\pi$ at A' to $-\frac{\pi}{2}$ at I_∞. The velocity along the streamline $I'I_\infty$ varies from zero at I' to U at I_∞ and has the constant direction $\theta = -\frac{\pi}{2}$.

Since the velocity direction is known along the boundaries BA and B'A' and since the velocity magnitude is known along the free streamlines AI_∞ and $A'I_\infty$, these bounding streamlines may be transformed into the distorted hodograph plane. If the constant C in equation (26) is taken to be

$$C = \frac{a_o + \sqrt{a_o^2 + U^2}}{U}$$

then at A, A' and I_∞, where the velocity magnitude is $q = U$,

$$w = \log \frac{a_o + \sqrt{a_o^2 + U^2}}{U} \quad \frac{U}{a_o + \sqrt{a_o^2 + U^2}} = 0$$

Also at B, B' and I' where the velocity magnitude is zero

$$w = \log \frac{a_o + \sqrt{a_o^2 + U^2}}{U} \quad \frac{0}{2a_o} = -\infty$$

Thus the σ-plane may be plotted as shown in Figure 1(b).

The bounding streamlines BAI_∞ and $B'A'I_\infty$ form a rectangle in the σ-plane and therefore may be transformed into the real axis of the ξ-plane, Figure 1(c), by the Schwarz-Christoffel transformation. Taking A at $\xi = +1$ and A' at $\xi = -1$, and noting that the interior angles at A and A' are $\alpha = \beta = \frac{\pi}{2}$, the transformation becomes

$$\frac{d\sigma}{d\xi} = K(\xi - 1)^{-\frac{1}{2}}(\xi + 1)^{-\frac{1}{2}}$$
$$= \frac{K}{\sqrt{\xi^2 - 1}}$$

Integration of this expression yields

$$\sigma = K \cosh^{-1} \xi + L$$

where L is the constant of integration.

The constants K and L may be evaluated by considering the following conditions: At A $\sigma = 0$, $\xi = 1$; at A' $\sigma = -\pi$, $\xi = -1$.

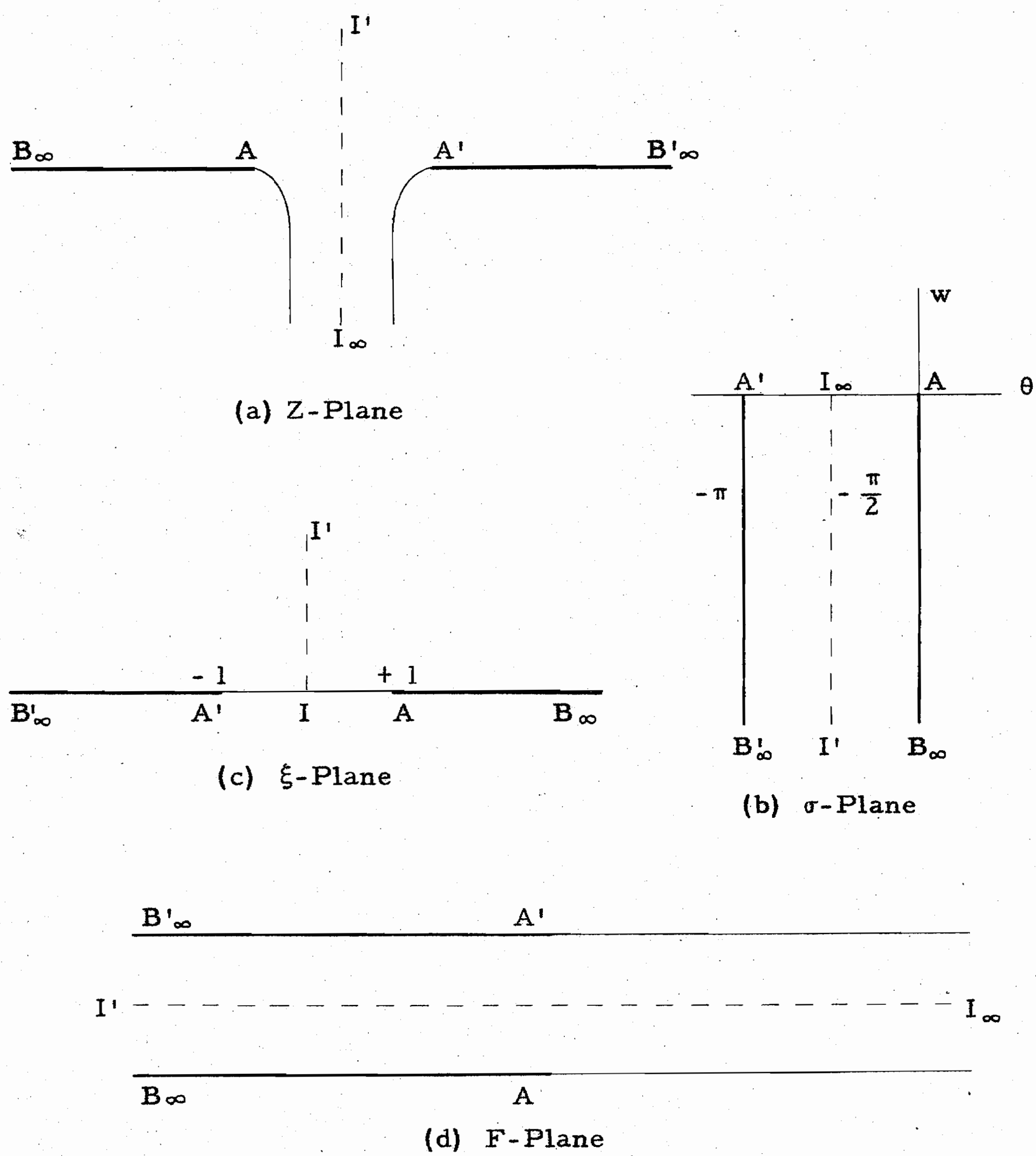

Figure 1. Planes Used in the Solution of Flow From A Two-Dimensional Orifice

Therefore

$$0 = K \cosh^{-1}(1) + L$$

$$-\pi = K \cosh^{-1}(-1) + L$$

for which

$$K = i$$
$$L = 0$$

The transformation then becomes

$$\sigma = i \cosh^{-1} \xi$$

which, when solved for ξ, yields

$$\xi = \cos \sigma \tag{40}$$

Now if the point A in the real plane is taken at the point of zero potential, then at point A' the potential is also zero, and if the bounding streamline BAI_∞ is taken as the zero streamline, the potential plane, $F = \phi + i\psi$, is shown by Figure 1(d). Again the bounding streamlines form a polygon in the potential plane and therefore can be transformed into the ξ-plane, Figure 1(c), by the Schwarz-Christoffel transformation. Again taking A at $\xi = +1$, A' at $\xi = -1$ and I_∞ at $\xi = 0$ and noting that the interior angles at A and A' in the F-plane are $\alpha = \beta = \pi$ and the angle at I_∞ is $\varepsilon = 0$, the transformation becomes

$$\frac{dF}{d\xi} = K(\xi - 0)^{-1}$$

Integrating this expression yields

$$F = K \log \xi + L \tag{41}$$

where L is again the constant of integration.

The constant of integration, L, can be evaluated by considering point A. At A, $\phi = 0$, $\psi = 0$ and $\xi = +1$, therefore

$$F = \phi + i\,\psi = 0 = K \log(1) + L$$

for which $L = 0$.

The constant K may be evaluated by considering the values of ψ and ϕ at point A' in the F-plane and the value of ξ at point A' in the ξ-plane. In the ξ-plane A' has previously been taken at $\xi = -1$, and in the F-plane A' has previously been taken at $\phi = 0$. It remains then to find the value of ψ corresponding to A'.

It can be shown, from the definition of stream function, that the mass rate of flow between two streamlines is given by

$$\text{Mass rate of flow} = \rho_o\,(\psi_1 - \psi_2)$$

where ρ_o is the stagnation density and ψ_1 and ψ_2 are the values of the stream function along the two streamlines. Since the value of the stream function on BAI_∞ has arbitrarily been taken as zero, the value of the stream function along $B'A'I_\infty$ may be correlated with the mass flow between BAI_∞ and $B'A'I_\infty$. If l is taken as the width of the slot and n as the coefficient of contraction, then the width of the stream at I_∞ is nl and the mass flow at I_∞ is ρnlU. Therefore

$$\psi_{B'A'I_\infty} = \frac{\rho nlU}{\rho_o}$$

Thus at A'

$$F = \phi + i\,\psi = i\,\frac{\rho nlU}{\rho_o} = K \log(-1)$$

for which $K = -\dfrac{\rho nlU}{\rho_o \pi}$.

The solution to the problem consists of the combination of equations (40) and (41) to give

$$F(\sigma) = \phi + i\Psi = -\frac{\rho n l U}{\rho_o \pi} \log \cos \sigma \tag{42}$$

It is now important to find the equation of the free streamline and the value of the contraction coefficient, n. If the point A is taken as the origin in the real plane, then the equation of the free streamline AI_∞ is found by integrating the expression for dz,

$$dz = \frac{1}{2a_o} \left[Ce^{i\sigma} dF(\sigma) - \overline{\frac{1}{C} e^{-i\sigma} dF(\sigma)} \right] \tag{38}$$

between the origin and some point z. The corresponding integration in the σ-plane is carried out along the line AI_∞ from A, the origin, to some point $\sigma = \theta$ since $w = 0$ along the free streamline.

Introducing the value of dF into equation (38) and integrating yields

$$\int_o^z dz = z = x + iy = \frac{\rho n l U}{2a_o \rho_o \pi} \int_o^\theta \left[Ce^{i\sigma} \frac{\sin\sigma}{\cos\sigma} d\sigma \right.$$

$$\left. - \overline{\frac{1}{C} e^{-i\sigma} \frac{\sin\sigma}{\cos\sigma} d\sigma} \right]$$

carrying out the integration on the right-hand side and equating real and imaginary parts yields

$$x = \frac{\rho n l}{\rho_o \pi} (1 - \cos\theta) = \frac{2\rho n l}{\rho_o \pi} \sin^2 \frac{\theta}{2}$$

$$y = \frac{\rho n l}{\rho_o \pi} \left(\log \tan \left(\frac{\theta}{2} + \frac{\pi}{4} \right) - \sin\theta \right) \tag{43}$$

These are the parametric equations for the free streamline AI_∞ . The asymptotic value for x is $\frac{\rho n l}{\rho_o \pi}$ (for $\theta = -\frac{\pi}{2}$); hence, the total width of the stream at I_∞ is $l - 2\frac{\rho n l}{\rho_o \pi}$. The coefficient of contraction then is

$$n = \frac{\pi}{\pi + 2\frac{\rho}{\rho_o}}$$

From equation (10) $\frac{\rho}{\rho_o}$ can be expressed in terms of the Mach number M as

$$\frac{\rho}{\rho_o} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{-1}{\gamma - 1}}$$

Within the present approximation ($\gamma = -1$) this gives

$$\frac{\rho}{\rho_o} = (1 - M^2)^{\frac{1}{2}}$$

and the coefficient becomes

$$n = \frac{\pi}{\pi + 2\sqrt{1 - M^2}} \tag{44}$$

C. Flow From A Tank With 45° Walls

The problem of the flow out of a tank with 45° walls is similar to the preceding problem except that the tank walls are now inclined at 45° with the horizontal as shown in Figure 2(a). Again the velocity at B, B' and I' is assumed to be zero and the velocity at I_∞ is assumed to be U. To avoid infinite velocities at A and A', the flow is assumed to separate at these points, and therefore the streamlines AI_∞ and $A'I_\infty$ are free streamlines along which the velocity is U. The flow direction along BA is $-\frac{\pi}{4}$ and along B'A' is $-\frac{3\pi}{4}$. Along the free streamline AI_∞ the flow direction varies from $-\frac{\pi}{4}$ to $-\frac{\pi}{2}$, and along the free streamline $A'I_\infty$ the flow direction varies from $-\frac{3\pi}{4}$ to $-\frac{\pi}{2}$. Again taking the constant C as

$$C = \frac{a_o + \sqrt{a_o^{\,2} + U^2}}{U}$$

consideration of the flow magnitude and direction along the bounding streamlines gives the distorted hodograph plane shown in Figure 2(b).

The transformation from the distorted hodograph plane to the ξ-plane is made by the Schwarz-Christoffel transformation, the general form being

$$\sigma = K \cosh^{-1} \xi + L$$

The conditions to be substituted for evaluation of K and L are as follows: At A, $\sigma = -\frac{\pi}{4}$, $\xi = +1$; at A', $\sigma = -\frac{3\pi}{4}$, $\xi = -1$; thus

$$-\frac{\pi}{4} = K \cosh^{-1}(1) + L$$

$$-\frac{3\pi}{4} = K \cosh^{-1}(-1) + L$$

for which

$$L = -\frac{\pi}{4}$$

$$K = \frac{i}{2}$$

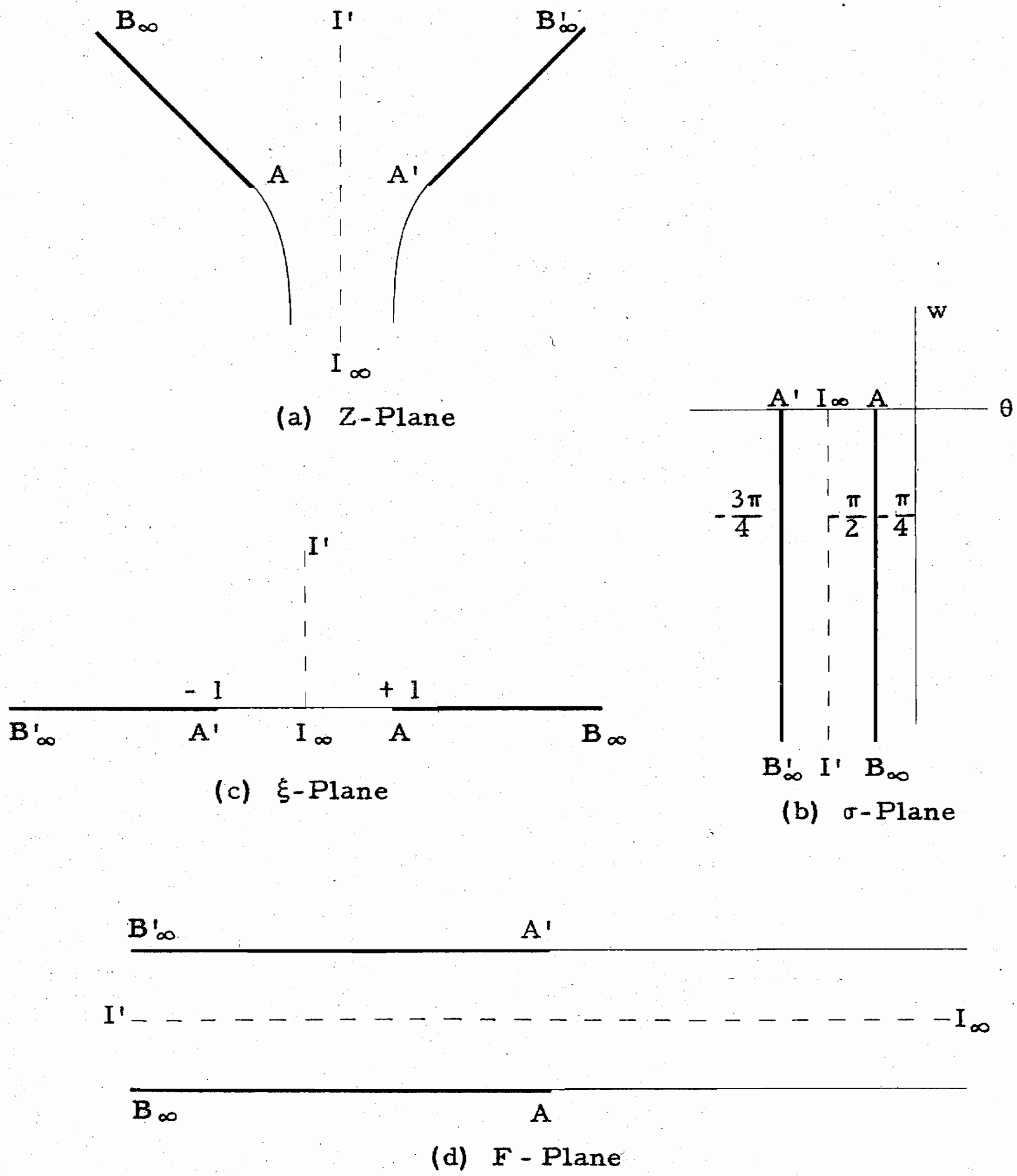

Figure 2. Planes Used in the Solution of Flow From A Tank With 45^o Walls

Therefore

$$\sigma = \frac{i}{2} \cosh^{-1} \xi - \frac{\pi}{4}$$

which, when solved for ξ, yields

$$\xi = - \sin 2\sigma \tag{45}$$

If the point A is taken as the point of zero potential, then A' will also have zero potential, and if the bounding streamline BAI $_\infty$ is taken as the zero streamline, the potential, F, plane is represented by Figure 2(d). Transforming the potential plane into the ξ -plane by the Schwarz-Christoffel transformation, yields the general transformation

$$F = K \log \xi + L$$

The conditions necessary for the evaluation of K and L are: At A, $\phi = 0$, $\psi = 0$, $\xi = +1$; at A', $\phi = 0$, $\psi = \frac{\rho n l U}{\rho_o}$, $\xi = -1$; thus

$$0 = K \log (1) + L$$

$$i \frac{\rho n l U}{\rho_o} = K \log (-1) + L$$

for which

$$L = 0$$

$$K = - \frac{\rho n l U}{\rho_o \pi}$$

Therefore

$$F = - \frac{\rho n l U}{\rho_o \pi} \log \xi \tag{46}$$

Combining equations (45) and (46) renders the general solution to this problem

$$F = -\frac{\rho n l U}{\rho_o \pi} \log (- \sin 2\sigma) \qquad (47)$$

Following the procedure outlined in the preceding problem, the parametric equations for the streamline AI are outlined as

$$x = -\frac{\rho n l}{\rho_o \pi}\left[2\cos\theta + \log\tan\frac{|\theta|}{2} - \sqrt{2} - \log\left(\frac{2-\sqrt{2}}{\sqrt{2}}\right)\right]$$

$$y = -\frac{\rho n l}{\rho_o \pi}\left[2\sin\theta - \log\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) + \sqrt{2} + \log\left(\frac{2-\sqrt{2}}{\sqrt{2}}\right)\right]$$

$$(48)$$

The asymptotic value for x is $\frac{\rho n l}{\rho_o \pi}\left[\sqrt{2} + \log\left(\frac{2-\sqrt{2}}{\sqrt{2}}\right)\right]$ (for $\theta = -\frac{\pi}{2}$).

The total width of the stream at I_∞ is $l - 2\frac{\rho n l}{\rho_o \pi}\left[\sqrt{2} + \log\left(\frac{2-\sqrt{2}}{\sqrt{2}}\right)\right]$

and the coefficient of contraction is $n = \dfrac{\pi}{\pi + 2\frac{\rho}{\rho_o}\left[\sqrt{2} + \log\left(\frac{2-\sqrt{2}}{\sqrt{2}}\right)\right]}$.

Within the present approximation ($\gamma = -1$), this reduces to

$$n = \frac{\pi}{\pi + 2\left[\sqrt{2} + \log\left(\frac{2-\sqrt{2}}{\sqrt{2}}\right)\right]\sqrt{1 - M^2}} \qquad (49)$$

D. Borda's Mouthpiece

Borda's mouthpiece, in two dimensions, is a re-entrant slot in a large container as shown in Figure 3(a). The points B, B' and I' are assumed to be far removed from the entrance so that the velocity at these points is zero. The velocity at I_∞ in the stream is assumed to be at U and therefore the velocity along the free streamlines AI $_\infty$ and A'I $_\infty$ is U. The velocity direction along BA is $\theta = 0$ and along B'A' is $\theta = 2\pi$. The velocity direction varies along the streamline AI $_\infty$ from 0 to π and along A'I$_\infty$ from 2π to π. I'I is a streamline.

If the constant C is again taken as

$$C = \frac{a_o + \sqrt{a_o^{\,2} + U^2}}{U}$$

the bounding streamlines BAI $_\infty$ and B'A'I $_\infty$ and the streamline I'I $_\infty$ are represented in the σ-plane by Figure 3(b).

Again the transformation equation for transforming the rectangle BAIA'B' in the σ-plane into the upper-half of the ξ -plane, Figure 3(c), is

$$\sigma = K \cosh^{-1} \xi + L$$

The constants K and L are evaluated from the conditions at A and A': At A, $\sigma = 0$, $\xi = +1$; at A', $\sigma = 2\pi$, $\xi = -1$. Making these substitutions yields

$$0 = K \cosh^{-1} (1) + L$$

$$2\pi = K \cosh^{-1} (-1) + L$$

for which

$$L = 0$$

$$K = 2i$$

Therefore the transformation from the σ-plane is given by

$$\sigma = 2i \cosh^{-1} \xi$$

or

$$\xi = \cos \frac{\sigma}{2} \tag{50}$$

If the points A and A' are taken as points of zero potential and if the bounding streamline BAI $_\infty$ is taken as the zero streamline, the potential plane for Borda's mouthpiece is shown by Figure 3(d). The transformation from the potential plane to the intermediate ξ -plane is again made by the Schwarz-Christoffel transformation, the general form for this problem being

$$F = K \log \xi + L$$

The constants K and L are again evaluated from the conditions at A and A' which are: At A, $\phi = 0$, $\Psi = 0$, $\xi = +1$; at A', $\phi = 0$, $\Psi = \frac{\rho n l U}{\rho_o}$, $\xi = -1$. Making these substitutions yields

$$0 = K \log (1) + L$$

$$i\frac{\rho n l U}{\rho_o} = K \log (-1) + L$$

for which

$$L = 0$$

$$K = -\frac{\rho n l U}{\rho_o \pi}$$

The transformation from the F-plane to the ξ -plane is then

$$F = -\frac{\rho n l U}{\rho_o \pi} \log \xi \tag{51}$$

The solution to the problem is found by combining equations (50) and (51) to obtain

$$F = -\frac{\rho n l U}{\rho_o \pi} \log \cos \frac{\sigma}{2} \tag{52}$$

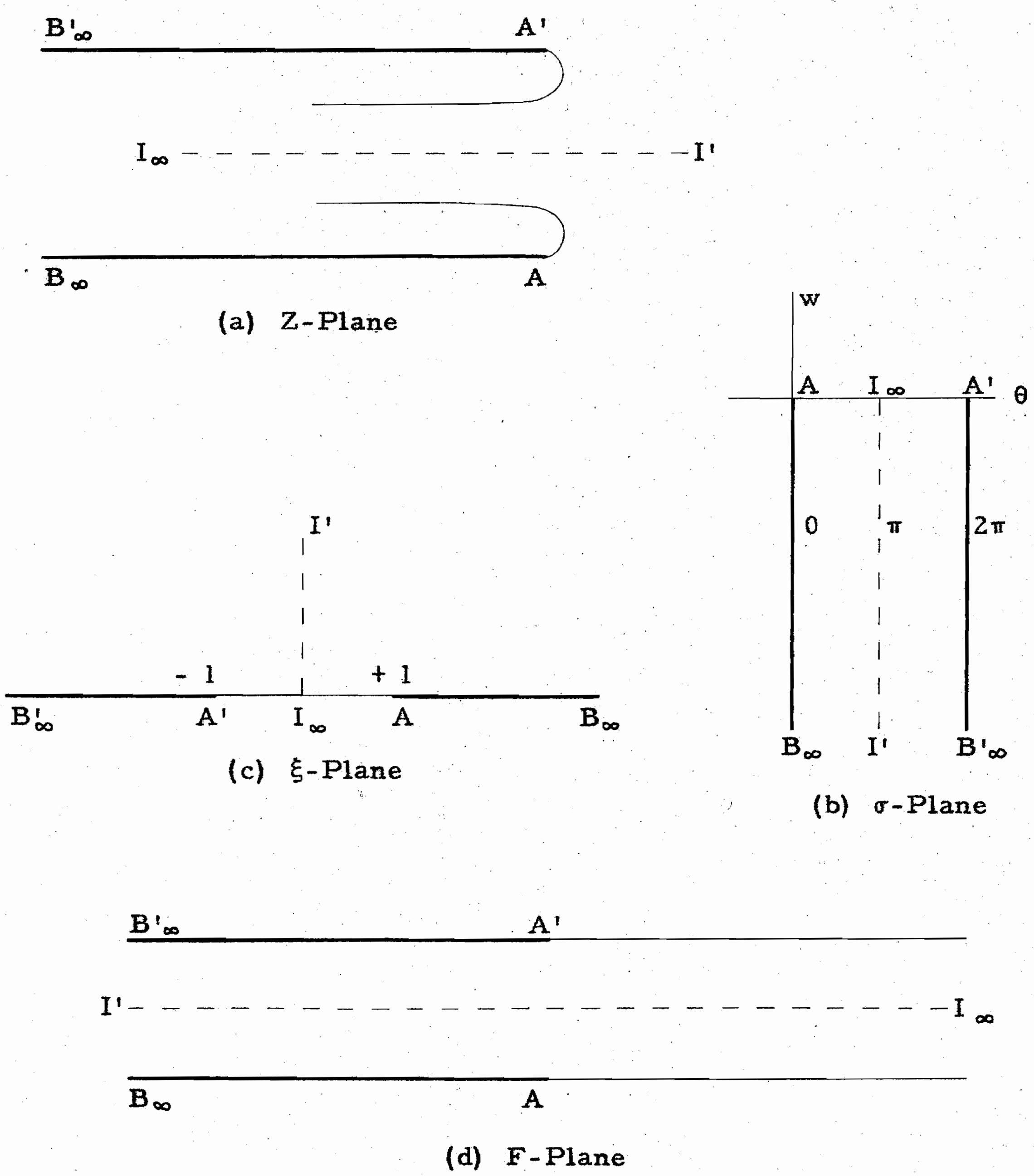

Figure 3. Planes Used in the Solution of Borda's Mouthpiece

The parametric equations for the free streamline AI_∞ can be obtained as previously outlined and are found to be

$$x = \frac{\rho n l}{\rho_o \pi}\left(\sin^2 \frac{\theta}{2} - \log \sec \frac{\theta}{2} \right)$$

$$y = \frac{\rho n l}{2\rho_o \pi}\left(\theta - \sin \theta \right) \qquad (53)$$

Here again the point A has been taken at the origin in the z - plane.

The asymptotic value for y is $\frac{\rho n l}{2\rho_o}$ (for $\theta = \pi$); hence the total width of the stream at I_∞ is $l - \frac{\rho n l}{\rho_o}$ and the coefficient of contraction is

$$n = \frac{1}{1 + \frac{\rho}{\rho_o}}$$

Within the present approximation ($\gamma = -1$) this reduces to

$$n = \frac{1}{1 + \sqrt{1 - M^2}} \qquad (54)$$

E. Bobyleff's Problem

An interesting variation of the preceding problems is the problem known as Bobyleff's Problem in which a stream is supposed to impinge symmetrically on a bent plate whose section consists of two equal straight lines forming an angle, Figure 4(a). The problem considered here will be one in which the included half-angle is 45° since consideration of a general angle yields a transformation for which the constant cannot be determined.

In this problem, the velocity at A in the undisturbed stream is assumed to be U. The velocity at a considerable distance downstream, at points D and F, is also assumed to be U and since the streamlines CD and EF are free streamlines the velocity is U along these streamlines. At point B, the velocity is zero since this is a stagnation point.

If the constant C is the same as previously used, consideration of the flow directions and velocities in the real plane renders a plot of the bounding streamlines in the σ-plane as shown in Figure 4(b).

Again the polygon made up of bounding streamlines in the σ-plane is transformed into the real axis of the ξ-plane by the Schwarz-Christoffel transformation, the general form for this particular case being

$$\sigma = K \cosh^{-1} \xi + L$$

The constants K and L are evaluated from the conditions: At C, $\sigma = +\frac{\pi}{4}$, $\xi = +1$; at E, $\sigma = -\frac{\pi}{4}$, $\xi = -1$. These conditions yield

$$\frac{\pi}{4} = K \cosh^{-1}(1) + L$$

$$-\frac{\pi}{4} = K \cosh^{-1}(-1) + L$$

for which

$$L = \frac{\pi}{4}$$

$$K = -\frac{i}{2}$$

The transformation equation becomes

$$\sigma = -\frac{i}{2} \cosh^{-1} \xi + \frac{\pi}{4}$$

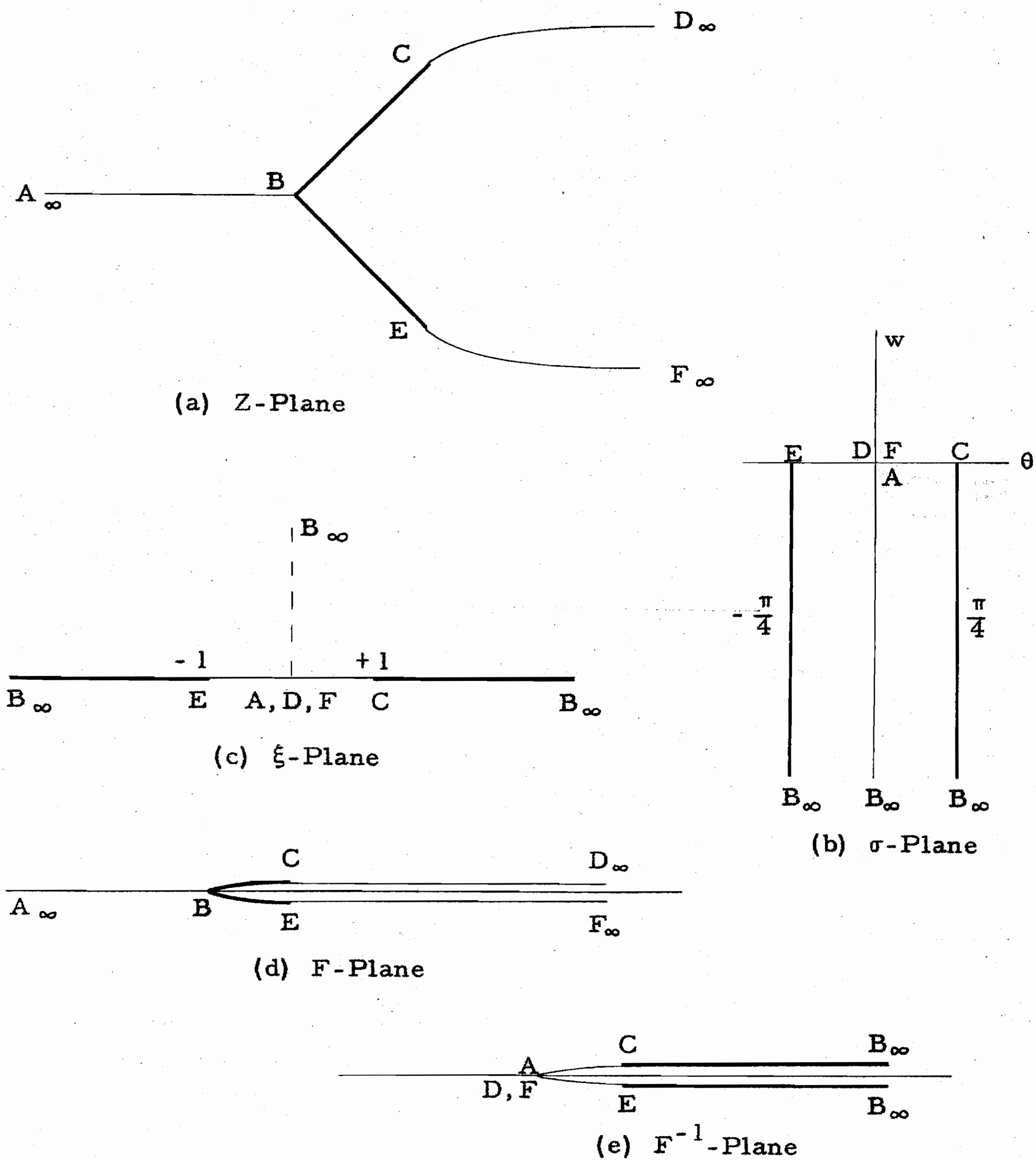

Figure 4. Planes Used in the Solution of Bobyleff's Problem

which, when solved for ξ yields

$$\xi = \sin 2\sigma \tag{55}$$

Now in the real plane, the zero streamline is taken as ABCD and ABEF. This streamline consists of a single streamline from A to B and two branches, BCD and BEF, extending to the right of point B. If point B is also chosen as the point of zero potential, then the potential plane will be represented by Figure 4(d). Here the branches BCD and BEF are shown slightly removed from the real ($\Psi = 0$) axis. Actually, however, these branches lie on the real axis. It becomes necessary now to introduce another intermediate plane, the F^{-1} plane, in order to relate the potential, F, plane to the ξ-plane. The bounding streamlines in the F^{-1}-plane are shown in Figure 4(e). The transformation from the F^{-1}-plane to the ξ-plane is now made by the Schwarz-Christoffel transformation. In the ξ-plane the points C and E have previously been located at $\xi = +1$ and $\xi = -1$ respectively. In the F^{-1}-plane the area between the streamlines BCD and BEF is now the interior of the polygon, hence the interior angles at C and E are $\alpha = \beta = \pi$ and the interior angle at A is $\epsilon = 2\pi$. Substitution of these values into the differential form of the Schwarz-Christoffel transformation yields

$$\frac{dF^{-1}}{d\xi} = K\xi$$

Upon integration, this equation becomes

$$F^{-1} = \frac{K}{2}\xi^2 + L$$

The constant L may be evaluated by considering the conditions at A which are: $\sigma = 0$, $F^{-1} = 0$. Thus

$$0 = \frac{K}{2}(0) + L$$

for which

$$L = 0$$

The relation between the F^{-1}- plane and the ξ-plane is therefore

$$F^{-1} = \frac{K}{2}\,\xi^2$$

and the relation between the potential, (F), plane and the ξ-plane is

$$F = \frac{2}{K\,\xi^2} \tag{56}$$

Introducing the relation between ξ and σ obtained above, equation (55), this becomes

$$F = \frac{2}{K\sin^2 2\sigma} \tag{57}$$

The constant K remains to be determined for a complete solution. This constant may be evaluated by consideration of the relation between the z-plane and the analytic function $F(\sigma)$. Integration of this expression from E to C gives

$$\int_E^C dz = \frac{1}{2a_0}\int_E^C \left[Ce^{i\sigma}\, dF(\sigma) - \overline{\frac{1}{C}\, e^{-i\sigma}\, dF(\sigma)} \right]$$

From equation (57) $dF(\sigma)$ is found to be

$$dF(\sigma) = \frac{8\cos 2\sigma}{K\sin^3 2\sigma}\, d\sigma$$

Making this substitution and carrying out the integration allows the evaluation of the constant K, which is

$$K = \frac{2}{1.5791U}$$

The final solution then, that is the analytic function, $F(\sigma)$, becomes

$$F = \frac{1.5791U}{\sin^2 2\sigma} \tag{58}$$

An expression for the force on this body may be obtained by summing up the pressure difference between the front and rear faces of the body over the entire body. Thus the drag force on the body is

$$D = 2 \int_B^C (p - p_1)\, dz$$

where p is the pressure on the front face of the body and p_1 is the pressure in the wake. Since the pressure intensity in the wake is the same as the pressure intensity along the free streamline where the velocity is U, p_1 may be written, by virtue of equations (9) and (11) as

$$p_1 = p_o \left[1 + \frac{\gamma - 1}{2} \left(\frac{U^2}{a_o^2 - \frac{\gamma - 1}{2} U^2} \right) \right]^{-\frac{\gamma}{\gamma - 1}}$$

Within the present approximation this becomes

$$p_1 = \frac{p_o}{a_o} \sqrt{a_o^2 - U^2}$$

Similarly, the local pressure on the face of the body, p, may be written

$$p = \frac{p_o}{a_o} \sqrt{a_o^2 - q^2}$$

As a result of the above, the force on the body may be expressed as:

$$D = \frac{2\,p_o}{a_o} \int_B^C \left[\left(\sqrt{a_o^{\,2} - q^2} - \sqrt{a_o^{\,2} - U^2} \right) \frac{dz}{d\sigma}\, d\sigma \right]$$

The quantity $\frac{dz}{d\sigma}$ may be determined from the analytic function, $F(\sigma)$, obtained previously and from the equation relating the z-plane and the analytic function, $F(\sigma)$, equation (38).

Unfortunately, it appears that when $\frac{dz}{d\sigma}$ is evaluated and substituted in this expression, the integration cannot be carried out; therefore, it is apparently impossible to determine the drag force on the body.

VII. DISCUSSION

In the first three problems discussed, The Two-Dimensional Orifice, Flow From A Tank With 45° Walls and Borda's Mouthpiece, the solution obtained by the hodograph method may be compared with the existing incompressible solution. A comparison of the solutions obtained by the hodograph method as presented herein and the classical incompressible solutions is shown in Table I.

The correspondence between the analytic function obtained by the hodograph method and the analytic function obtained in classical theory is not easily seen since the analytic function, F, obtained by the hodograph method is expressed in terms of σ ($\sigma = \theta + iw$) while the analytic function, F', obtained by the classical theory is expressed in terms of ξ' ($\xi' = \log \frac{1}{q} + i\theta$). The correspondence between the hodograph solution and the classical solution is, however, readily seen by comparing the equations for the free streamline as obtained by each of these methods and by comparing the equations for the contraction coefficient as obtained by each of these methods. In each case the hodograph solution is modified by the factor $\frac{\rho_o}{\rho}$ which, within the present approximation, reduces to $\sqrt{1 - M^2}$. This factor ($\sqrt{1 - M^2}$) is well known in the field of fluid mechanics as a correction for compressibility. It should also be noted that in each case if the Mach number M is taken as 0 corresponding to an incompressible fluid the hodograph solution is identical to the incompressible solution.

The presence of the factor $\sqrt{1 - M^2}$ indicates that the free streamline and the contraction coefficient are functions of the Mach number, or the velocity, U, in the undisturbed fluid. This is not the case in the incompressible solution where the free streamlines and the contraction coefficients are the same for all values of undisturbed velocity, U. A comparison of the free streamlines for the incompressible case and for a Mach number of 0.5 is given in Figures 5, 6 and 7. Figure 8 is a comparison of the compressible and incompressible values of the contraction coefficient as a function of Mach number. It is to be noted that the contraction coefficient increases slightly with increasing Mach number.

Unfortunately, no counterpart was found in classical theory for the analytic function obtained here as a solution to Bobyleff's problem.

Table 1

Comparison of Hodograph Solutions and Classical Solutions

for Three Problems in Fluid Mechanics

A. Two-Dimensional Orifice

Classical Solution	Hodograph Solution
Analytic Function	
$F' = \frac{nlU}{\pi} \log(-\cosh \xi') - inlU$	$F = -\sqrt{1 - M^2}\ \frac{nlU}{\pi} \log \cos \sigma$
Parametric Equations of a Streamline	
$x = \frac{2nl}{\pi} \sin^2 \frac{\theta}{2}$	$x = \sqrt{1-M^2}\ \frac{2nl}{\pi} \sin^2 \frac{\theta}{2}$
$y = \frac{nl}{\pi}\left[\log \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) - \sin\theta\right]$	$y = \sqrt{1 - M^2}\ \frac{nl}{\pi}\left[\log \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) - \sin\theta\right]$
Contraction Coefficient	
$n = \frac{\pi}{\pi + 2}$	$n = \frac{\pi}{\pi + 2\sqrt{1 - M^2}}$

Table I (cont'd)

B. Flow From A Tank With 45° Walls

Classical Solution	Hodograph Solution
Analytic Function	
$F' = \frac{nlU}{\pi} \log \sinh 2\xi' - inlU$	$F = -\sqrt{1 - M^2}\,\frac{nlU}{\pi} \log(-\sin 2\sigma)$
Parametric Equations of a Streamline	
$x = -\frac{nl}{\pi}\left[2\cos\theta + \log\tan\frac{\lvert\theta\rvert}{2} - \sqrt{2} - \log\left(\frac{2-\sqrt{2}}{\sqrt{2}}\right)\right]$	$x = -\sqrt{1 - M^2}\,\frac{nl}{\pi}\left[2\cos\theta + \log\tan\frac{\lvert\theta\rvert}{2} - \sqrt{2} - \log\left(\frac{2-\sqrt{2}}{\sqrt{2}}\right)\right]$
$y = -\frac{nl}{\pi}\left[2\sin\theta - \log\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) + \sqrt{2} + \log\left(\frac{2-\sqrt{2}}{\sqrt{2}}\right)\right]$	$y = -\sqrt{1 - M^2}\,\frac{nl}{\pi}\left[2\sin\theta - \log\tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) + \sqrt{2} + \log\left(\frac{2-\sqrt{2}}{\sqrt{2}}\right)\right]$
Contraction Coefficient	
$n = \frac{\pi}{\pi + 2\left[\sqrt{2} + \log\left(\frac{2-\sqrt{2}}{\sqrt{2}}\right)\right]}$	$n = \frac{\pi}{\pi + 2\left[\sqrt{2} + \log\left(\frac{2-\sqrt{2}}{\sqrt{2}}\right)\right]\sqrt{1 - M^2}}$

Table 1 (cont'd)

C. Borda's Mouthpiece

Classical Solution	Hodograph Solution
Analytic Function	
$F' = \frac{nlU}{\pi} \log \cosh \frac{\xi'}{2} - inlU$	$F = -\sqrt{1 - M^2}\ \frac{nlU}{\pi} \log \cos \frac{\sigma}{2}$
Parametric Equations of a Streamline	
$x = \frac{nl}{\pi}\left[\sin^2 \frac{\theta}{2} - \log \sec \frac{\theta}{2}\right]$	$x = \sqrt{1 - M^2}\ \frac{nl}{\pi}\left[\sin^2 \frac{\theta}{2} - \log \sec \frac{\theta}{2}\right]$
$y = \frac{nl}{2\pi}\left[\theta - \sin \theta\right]$	$y = \sqrt{1 - M^2}\ \frac{nl}{2\pi}\left[\theta - \sin \theta\right]$
Contraction Coefficient	
$n = \frac{1}{2}$	$n = \frac{1}{1 + \sqrt{1 - M^2}}$

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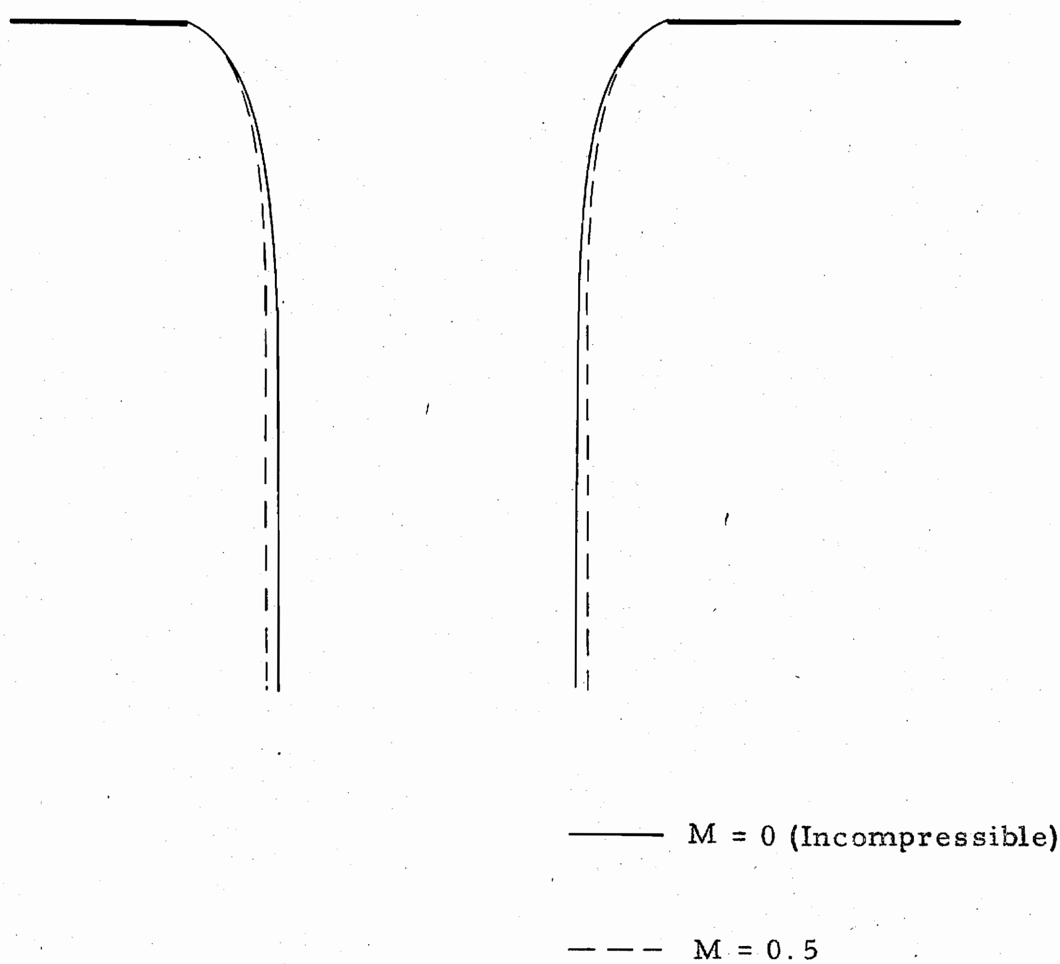

Figure 5. Free Streamlines for the Flow Issuing From A Two-Dimensional Orifice

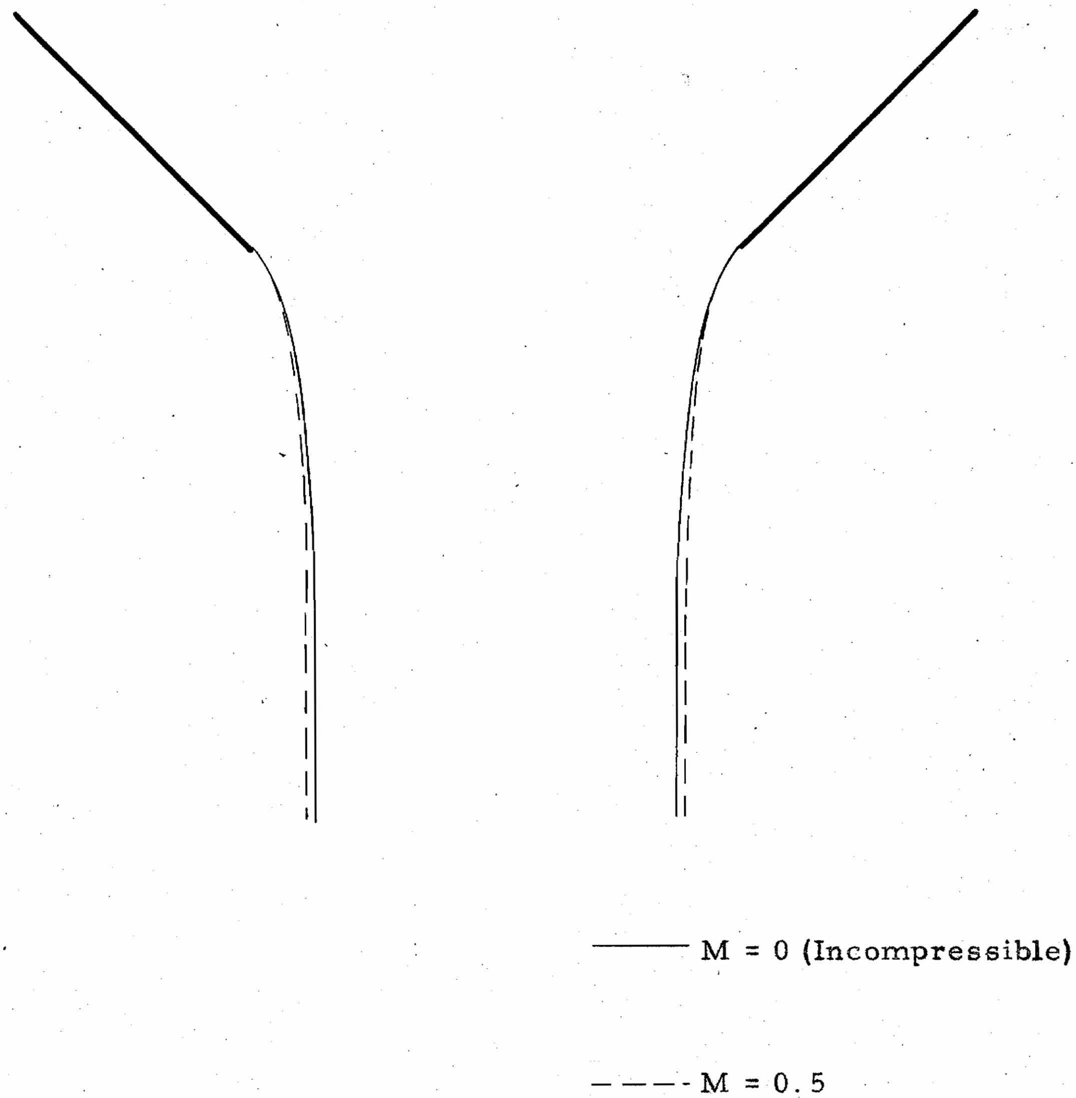

Figure 6. Free Streamlines for the Flow Issuing From A Tank With 45° Walls

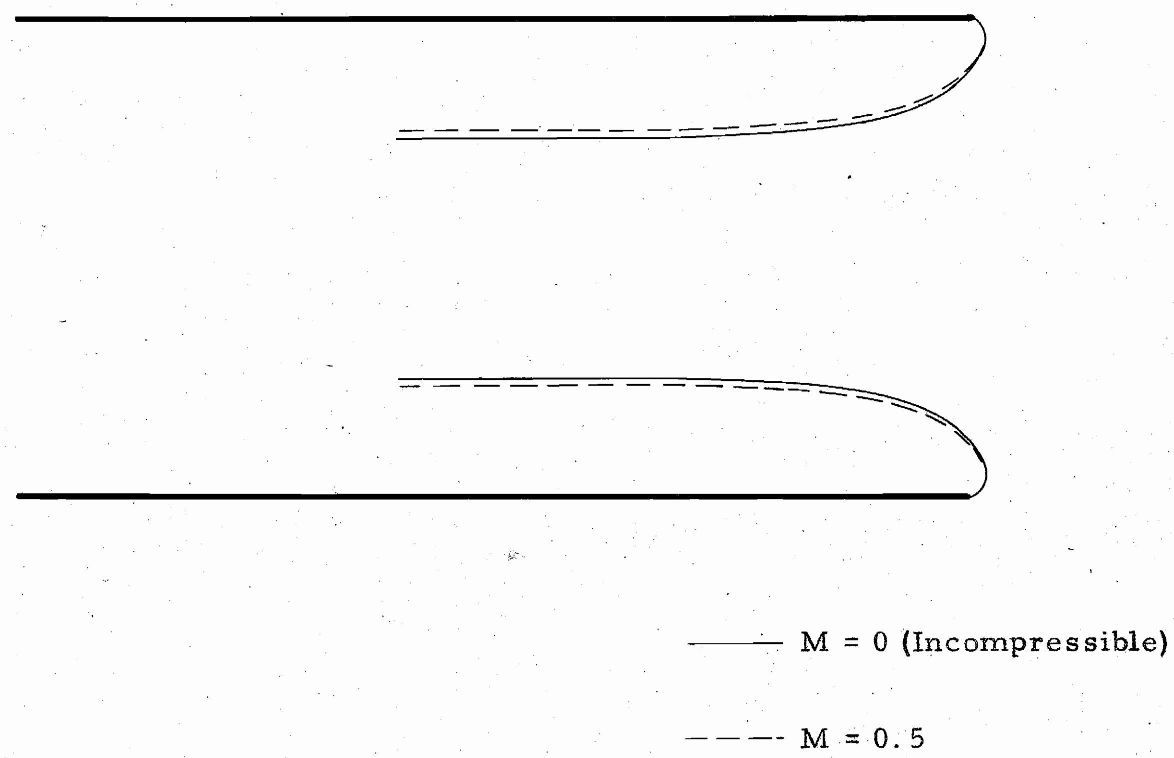

Figure 7. Free Streamlines for the Flow Issuing From Borda's Mouthpiece

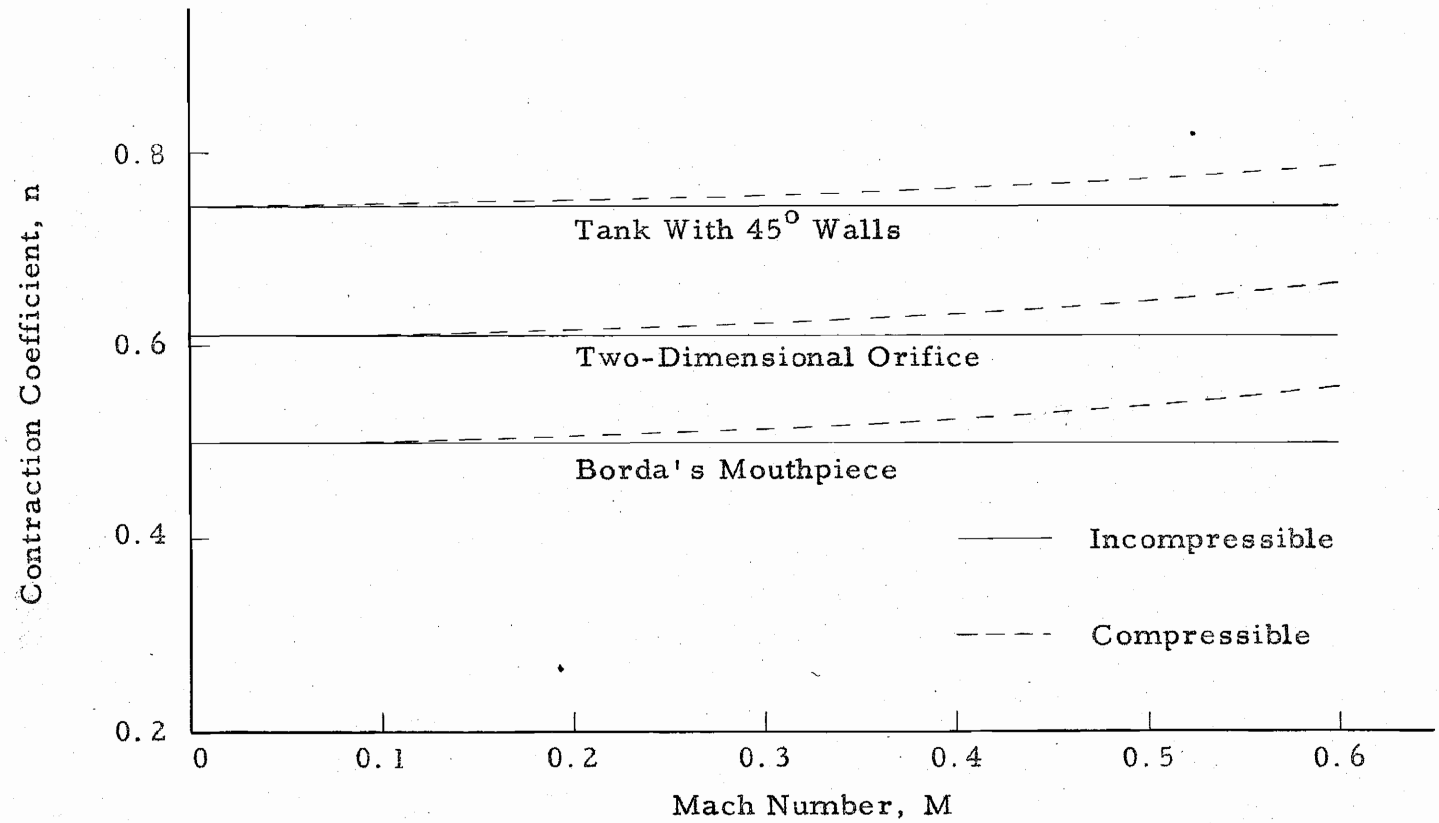

Figure 8. Variation of the Contraction Coefficient With Mach Number for Three Problems in Fluid Mechanics

VIII. CONCLUSIONS

It has been found that the hodograph method can be applied with considerable ease to problems of steady, two-dimensional, irrotational, compressible fluid motion in which the bounding streamlines are made up partially of straight boundaries and partially of free streamlines. As in classical theory, these solutions can be used to determine important physical characteristics of the problem such as the equations of the free streamlines and the contraction coefficient provided it is possible to carry out the mathematical operations.

It must be realized that the hodograph method gives only approximate solutions. However, the hodograph solution gives a more accurate description of a fluid flow than does the classical solution since the hodograph method takes into account the effects of compressibility.

It is hoped that the methods used and the solutions obtained in this thesis will aid in giving further insight into the problems of compressible fluid flow.

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XI. VITA

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