178
52

SIMULTANEOUS PROCESS CONTROL OF SEVERAL INDEPENDENT QUALITY VARIABLES

by

Marshall Alan Wise

Thesis submitted to the Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in

Industrial Engineering and Operations Research

APPROVED:

Joel A. Nachlas, Chairman

J. William Schmidt

Jeffrey D. Tew

May, 1990

Blacksburg, Virginia

SIMULTANEOUS PROCESS CONTROL OF SEVERAL INDEPENDENT QUALITY VARIABLES

by

Marshall Alan Wise

Joel A. Nachlas, Chairman

Industrial Engineering and Operations Research

(ABSTRACT)

A method for multivariate quality control with the dual objectives of providing a true level of sampling error probabilities for the joint control of several quality variables while also giving problem diagnoses for the quality variables individually. The method is comprised of an afine transformation of the multiple quality variables which creates a univariate test statistic used to monitor the quality and provide problem diagnoses. In practice, realized values of this statistic would be plotted as a time series on a control chart with multiple diagnosis intervals. For the analysis of the method's effectiveness, the quality variables are assumed to be independent and normally distributed.

The method is shown to be successful in achieving desired sampling error probabilities for any m quality variables in the case of positive shifts in the means of the variables. A second transformed variable is added for the diagnosis of shifts of unrestricted direction, and its effectiveness is analyzed. The sample size requirement of the afine transformation method is compared to the total sample size necessary when a separate Shewhart chart for the mean is maintained for each quality variable with the same overall sampling plan objectives. The power of the method to detect quality problems in general while disregarding specific diagnoses is compared to the power of Hotelling's T^2 test for multivariate quality control. A comprehensive evaluation of the relative worth of the two methods is not possible since the T^2 statistic does not consider diagnoses of the individual quality variables.

Acknowledgements

I would like to take this opportunity to express my gratitude to Dr. Joel Nachlas for allowing me to use as my thesis a research topic which he had initiated. I also want to offer my sincere thanks and appreciation for his support and assistance not only on this project but throughout my academic career.

Special thanks are also due to Dr. J. William Schmidt and Dr. Jeffrey Tew for serving on my thesis committee and offering their time and insight.

Table of Contents

List of Illustrations

List of Tables

Chapter 1

INTRODUCTION

Virtually all of the existing technical literature and textbook instruction concerning control charts is based on monitoring a univariate test statistic, such as a sample mean, and comparing this statistic to predetermined control limits. There exist, however, many manufactured and assembled items with two or more distinct criteria which are significant to the quality of the unit as a whole. Typical practice in these cases is to monitor each quality characteristic of the item separately by constructing a different control chart for each of the characteristics and treating them as independent univariate tests. Apart from the obvious administrative difficulties of maintaining multiple control charts for each type of product, this practice yields results which can be misleading both theoretically and statistically.

When separate control charts are employed to monitor multiple quality characteristics independently, the true sampling error probabilities for the joint control of all the characteristics of an item will not be the same as the stated levels of the individual charts. In the case of independent quality characteristics, the composite sampling error is greater than that of the error for the component characteristics (Montgomery 1980). For dependent or correlated characteristics, analysis of the ac-

cumulation of sampling errors becomes more complicated. In either case, the discrepancy becomes more pronounced as the number of quality characteristics increases.

The issue of appropriate sample size also can become a problem when separate control charts are used. The selection of a sample size for control chart plots of sample means is in practice often arbitrary or subjective, but a statistically appropriate sample size could be computed by defining a desired power of detection for an alternative hypothesis. This hypothesis would be based upon some critical value pertinent to the particular application. Using this approach for multiple characteristics would most likely result in a different sample size for each quality characteristic while use of the same sample size for each could result in underprotection for some characteristics and overprotection for others. Having several different sample sizes for a single product could conceivably be a source of confusion in the inspection process.

Multivariate quality control techniques have been developed to compensate for the shortcomings of the separate control chart practice. These techniques are few in number and are primarily based on the T^2 distribution of Hotelling (1947). The T^2 statistic does provide the stated level of Type I sampling error for the joint control of multiple quality variables. It is an exact method, and it is applicable for both independent and dependent normally distributed quality variables. Because it is a single statistic incorporating all of the individual quality characteristics, the sample values of T^2 can be plotted as a time series on a single control chart.

There are, however, some significant shortcomings of the T^2 method. The two disadvantages to be described here directly result from the fact that the sample T^2 statistic is a dimensionless quantity and therefore offers little concrete information about the nature of the physical process it represents. One major disadvantage is that it is practically impossible to define a meaningful alternative hypothesis based on the dimensionless T^2 distribution. Consequently there is no thorough and physically significant basis for judging the power of the T^2 control chart or to select a statistically meaningful sample size. A second and equally critical shortcoming of Hotelling's method in practical application is that, while it can provide a signal that product quality is out of control, it does

not support diagnosis with regard to which of the quality characteristics are causing the signal and require correction.

Nachlas and Barreau (1989) identify the need for a multivariate quality control chart technique with diagnostic capabilities. They propose a method in which a single, univariate test statistic is computed with an afine transformation of the individual quality variables. This statistic is then plotted on a control chart consisting of multiple decision intervals. These intervals correspond to the different possible diagnoses of the state of the multivariate process control situation. Their method also incorporates detection of an alternative hypothesis at a specified level of power for each of the quality characteristics while maintaining a single sample size for the unit as a whole.

Nachlas and Barreau (1989) demonstrate the effectiveness of their strategy in the somewhat restricted case of two independent and normally distributed quality variables with the added stipulation that only positive shifts in the means of the quality variables are considered. They acknowledge this limitation and propose methods for generalizing their strategy to less restrictive assumptions.

The research in this paper uses the afine transformation method of Nachlas and Barreau as its foundation. The main objective is to extend their method to incorporate as many quality characteristics as desired while eliminating the restriction concerning offsetting shifts. This paper considers the case of quality variables which are independent and normally distributed. To be considered effective, the resulting control chart for the transformed test statistic must not only be able to signal an out of control situation for the multiple characteristics, subject to the stated level of sampling error probability, but also provide a high probability of correct diagnosis as to which quality variables are causing the out of control reading.

In summary, a generalization of the afine transformation method which allows for multiple quality variables and offsetting shifts is developed in this paper, and an analysis of its effectiveness is presented. The sample size requirements are discussed and compared to the sample sizes associated

with other control chart methods. As a final analysis, a comparison of the effectiveness of the afine transformation in detecting various out-of-control situations to that of Hotelling's T^2 statistic is provided.

Chapter 2

LITERATURE REVIEW

The T^2 statistic of Hotelling (1947) has become the standard technique for multivariate quality control. In spite of the practical limitations described previously, Hotelling's statistic does provide an exact solution for any desired region of joint coverage for multiple normally distributed quality variables. Much subsequent research in multivariate quality and process control has its foundation in Hotelling's work, and the effectiveness of any newly developed method must be judged in relation to his T^2 technique.

Montgomery (1985) gives a textbook treatment of the T^2 statistic. He begins his presentation with a simple example demonstrating how the use of separate control charts of sample means to monitor multiple quality characteristics of an item does not result in the stated level of coverage for the joint control region. The example is as follows: consider m independent component quality variables each controlled separately with a Type I sampling error probability of α. The true probability of Type I error for the item as a whole is given by

$$\alpha' = 1 - (1 - \alpha)^m$$

which is strictly greater than α. He states that a similar but more complicated accumulation of sampling error occurs when the component variables are not independent but correlated.

As presented by Montgomery, Hotelling's T^2 statistic for m component quality variables is computed in the following manner.

$$T^2_{m,n-1} = n(\vec{\bar{x}} - \vec{\mu})' S^{-1} (\vec{\bar{x}} - \vec{\mu}) \qquad [2.1]$$

where

n = sample size (each of m characteristics)

$\vec{\bar{x}}$ = vector of m sample means

$\vec{\mu}$ = vector of m population means

S = m by m matrix of covariances

In practice, the population means and the covariance matrix would probably be estimated by some preliminary sampling.

Critical values of the T^2 distribution can be computed using its relationship to the F distribution.

$$T^2_{\alpha, m, n-1} = m \frac{n-1}{n-m} F_{\alpha, m, n-m} \qquad [2.2]$$

The hypothesis test using the T^2 statistic at a level of significance of α would be to reject that the multiple quality characteristics of an item are in control if the sample T^2 exceeds the critical value of the T^2 distribution and to accept that they are in control otherwise. A univariate control chart for the multiple quality variables may be constructed using the desired critical value of the T^2 distribution as the upper control limit. Because this statistic is a univariate quantification of the joint control problem, it can be plotted as a time series so that runs and other nonrandom phenomena may be observed.

When the vector of means has shifted from the target value used in the T^2 calculation, the distribution of the statistic becomes noncentral. The power of the T^2 test to detect deviations from the target means can be determined using its relationship to the noncentral F distribution. Tiku (1985) gives the noncentrality parameter for the nonnull ($\vec{\mu} \neq \vec{0}$) distribution of F resulting from the T^2 statistic as

$$n\vec{\mu}' S^{-1}\vec{\mu}. \qquad [2.3]$$

The probabilistic analysis of the T^2 test can thus be entirely performed using the more familiar F distribution.

Jackson (1956) studied the case of controlling two component quality characteristics of an item graphically. He states that monitoring the two variables with separate conventional control charts implicitly assumes that the region of joint control is rectangular, but using the rectangle results in areas of overcontrol and of undercontrol. Jackson then explains that the true geometry of the region of joint control at a given level of significance is elliptical. In the case of two correlated component variables, the major axis of the ellipse is tilted in the direction of the correlation. The joint control of two variables could be achieved by plotting sample observations of one variable versus the other on a two-dimensional grid. If the plotted point falls outside the control ellipse, then the quality of the item would be viewed as out of control.

Jackson demonstrates that his method of control ellipses is mathematically identical to Hotelling's T^2 control charts. He also argues that his method is superior in practical application for one significant reason. The control ellipses signal not only that the joint distribution of quality characteristics is out of control but also graphically suggest which of the characteristics is causing the signal. Jackson concedes, however, that this method is limited in that it can only incorporate two quality variables. The control ellipse plots also lack the time series representation achieved with the T^2 control charts.

Ghare and Torgersen (1968) demonstrate that the analysis of multivariate quality control problems is greatly simplified when the covariances are assumed to be known and not considered as estimates. For two component quality variables, they show that control ellipses can be constructed just as in Jackson's paper (1956). For more than two quality characteristics, the known covariance assumption simplifies the joint control problem by allowing the more familiar χ^2 distribution to be used in place of Hotelling's T^2 The sample χ^2 statistic is calculated in the same manner as the sample T^2 statistic. A χ^2 control chart with a level of significance alpha is then drawn with the critical value $\chi^2_{\alpha, m}$ as the upper control limit. The previously discussed advantages and disadvantages of using a control ellipse chart versus plotting the sample χ^2 statistics on a control chart as a time series are also relevant in this problem formulation.

Fuchs and Kenett (1987) approach the problem from a different perspective and propose the use of multivariate tolerance regions instead of the usual hypothesis test formulation of multivariate quality control. They demonstrate that a tolerance region with a prescribed coverage and level of confidence can be described by the same quadratic form equation used to compute the sample T^2 statistic. They also assert that critical values for the tolerance region can be approximated by the F distribution which has already been shown to have a direct relationship to Hotelling's T^2 distribution. The authors conclude that while the tolerance region formulation of the multivariate quality control problem may be more theoretically sound, it sees limited practical use because it is more difficult to interpret than traditional hypothesis tests.

Jackson and Bradley (1961) were confronted with one of the important deficiencies of Hotelling's T^2 method when they attempted to constructed a sequential T^2 test. The authors begin with the observation that Wald had not satisfactorily addressed the procedure for a multivariate sequential test. Wald (1947) suggests assigning weighting constants to the multiple hypotheses of the different quality characteristics but fails to provide a definite method for doing so. Jackson and Bradley design a sequential probability ratio test in the manner of Wald but use Hotelling's T^2 distribution to incorporate the multiple characteristics into a single statistic. In such a procedure, a ratio of the likelihood that the observed data represent some undesirable alternative hypothesis to the likelihood

that they represent the target or null hypothesis is constructed. The two hypothesis are presented as follows.

$$H_0 : (\vec{\mu} - \vec{\mu}_0)' S^{-1} (\vec{\mu} - \vec{\mu}_0) = \lambda_0$$

vs.

$$H_1 : (\vec{\mu} - \vec{\mu}_0)' S^{-1} (\vec{\mu} - \vec{\mu}_0) = \lambda_1$$

where

$\vec{\mu}$ = vector of population means of the components

$\vec{\mu}_0$ = vector of target means for the components

S = covariance matrix

λ_0 = value of T^2 representing null hypothesis

λ_1 = value of T^2 representing alternate hypothesis

The difficulty in designing the sequential test lies with the selection of meaningful critical values of the T^2 distribution corresponding to the two hypotheses. The authors suggest λ_0 be chosen as zero since this implies that the individual characteristics are all at their target mean values. They concede that it is virtually impossible to prescribe a theoretically correct procedure for choosing the value of λ_1. This obstacle is a direct consequence of one of the major weaknesses of the T^2 method. Because this statistic is a dimensionless quantity, it cannot provide a concrete or intuitively meaningful analogy of the physical process which it is monitoring. The individual quality characteristics may be thought of as losing their identities in the T^2 statistic. In the words of the authors, a suitable value of the alternative hypothesis depends on the visualization of an m-dimensional ellipsoid. Other related consequences of this problem are that neither the power of the T^2 test to detect out of control situations nor a rule for selecting a statistically appropriate sample size can be obtained from the T^2 distribution. The authors are eventually successful in designing a sequential T^2 test, but this success is qualified with an assumed value for the alternative hypothesis which they readily admit is arbitrary and even tenuous.

In another paper, Jackson (1959) provides an extension of his earlier work on the two quality characteristic control ellipses to allow for multiple characteristics. His goal is to maintain a ge-

ometric interpretation, as he did with the ellipses, so that the plotted sample statistics offer a diagnosis as to which of the quality characteristics are causing an out of control signal. He constructs a two step computational procedure based in part on the eigenvectors of the covariance matrix. The first step has a geometrical interpretation in that it corresponds to a rotation of principal axes. The second step results in a sample T^2 statistic which may be interpreted in the usual manner. The difference between this method and the direct T^2 computation is that Jackson uses the additional first step to infer a diagnosis about the state of the component quality characteristics.

Jackson's first step results in an m-dimensional vector of which the components are m linear combinations consisting of weighted sums and differences of the m quality variables. Jackson describes the proposed procedure in the following manner. A T^2 control chart is maintained as before except for the two-step method of computing the sample test statistic. In addition to this chart, a separate supplementary control chart is plotted for each of the m transform variables computed in the first step. A total of $m + 1$ control charts are therefore required. When the T^2 chart signals an out of control situation, the user refers to the supplementary control charts to try to infer which quality characteristics are the source of the signal.

The interpretation of the supplementary control charts as explained by Jackson seems somewhat arbitrary. Jackson prescribes no general method for using these charts for problem diagnosis. The rules appear to be entirely dependent on the situation and largely based on judgment and assumptions. There is also no way to analyze the sampling error probabilities associated with the various diagnoses. Jackson acknowledges that these supplementary control charts are prone to the same problems as conventional charts for controlling means of multicharacteristic quality situations separately. If this is indeed the case, then Jackson's proposed method seems inferior to simply maintaining a T^2 chart for the joint problem while plotting m different Shewhart control charts for the sample means for use in signal diagnoses.

The research of Jackson and Bradley (1961) on the sequential T^2 test and of Jackson (1959) on multivariate tests with diagnostic capability both directly address two salient issues examined by

Nachlas and Barreau (1989). They propose a method of an afine transformation of the quality variables in order to obtain a univariate test statistic. The objective of Nachlas' and Barreau's work is to construct a univariate control chart which, in addition to providing any desired level of significance for the joint control hypothesis test, also gives the user a diagnosis when the quality is out of control. This diagnostic power is not to be based on judgment but will be in terms of user-defined sampling error probabilities. Their method incorporates a distinct, physically meaningful alternative hypothesis for each of the component quality characteristics and prescribes the computation of one statistically appropriate sample size for the joint region of control and diagnosis.

Nachlas and Barreau, however, come to the conclusion that the initial formulation of their method is insensitive to the possibility of offsetting shifts in the means of the quality variables. They then restrict their analysis to two independent quality variables with only positive shifts in order to get an assessment of the potential effectiveness of the afine transformation technique. With this restriction, their method does successfully achieve the desired sampling error probabilities at the in-control case and at the critical shifts. The authors do not extend their analysis beyond the two-variable positive-shift case, but they do propose methods for generalizing the technique which they consider promising. A detailed presentation of their afine transformation method, especially as it applies to this paper, is included at the beginning of the following chapter.

Chapter 3

SOLUTION METHODOLOGY

The objective of this research is to extend the afine transformation method for multivariate quality control of Nachlas and Barreau (1989) to allow for some more general cases. Specifically, their restriction stating that only positive shifts in the quality variable means are possible is be relaxed; and a formulation of the method which incorporates more than two quality variables is developed. Their assumptions that the quality variables are independent and normally distributed are maintained here. The following presentation of the basic model of the transformation is based on their work.

Consider the case of an item or unit of production having m distinct quality characteristics, each critical to the quality of the item as a whole. Also assume that measurements of these characteristics can be represented as the components of an m-dimensional random vector, $\vec{X}$, which has a multivariate normal distribution. The following definitions are critical to the development of the solution.

defn 1 $\vec{X} = (X_1, X_2, \ldots, X_m)$ is a normal random vector $\sim N(\vec{\mu}, V)$.

defn 2 $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_m)$ with $\mu_i = \mathrm{E}[X_i]$.

defn 3 V = covariance matrix comprised of elements σ_{ij}^2 .

The afine transformation on the quality variables X_i is computed as follows.

$$y = \sum_{i=1}^{m} a_i x_i \qquad [3.1]$$

where

y = transformed quality variable.

a_i = parameter or transform coefficient of the i^{th} quality variable.

The transform variable y has a univariate normal distribution with mean

$$\mu_y = \sum_{i=1}^{m} a_i \mu_i \qquad [3.2]$$

and variance (for independent variables with zero covariances)

$$\sigma_y^2 = \sum_{i=1}^{m} a_i^2 \sigma_{x_i}^2. \qquad [3.3]$$

The parameters a_i are determined from the sampling plan objectives. Nachlas and Barreau (1989) propose a single parameter model with $a_i = a^{i-1}$.

The following definitions and notational conventions facilitate the presentation of the solution and are also taken freely from the Nachlas and Barreau formulation of the two independent quality variable model. Without loss of generality, the quality variables are assumed to be standard (or at

least standardized) normal random variables with means $\mu_i = 0$ and variances $\sigma_i^2 = 1$. Possible shifts in the means of the quality variables are represented as μ_i^r where

$$\mu_i^r = \mu_i + \delta_i \sigma_i. \qquad [3.4]$$

For the standard normal quality variables, the shift in $\overline{X}_i$ can be completely described by δ_i. Let δ'_i represent the critical shift in the mean of quality variable X_i against which protection is desired at a defined sampling error probability of θ The probability of a false signal when all quality variables are in control will also be set at θ. The numbering convention for the quality variables is such that $\delta'_i \leq \delta'_{i+1}$.

The two quality variable case leads to nine possible hypotheses concerning the quality variable means which must be simultaneously evaluated. These hypotheses consist of all resulting combinations of none, either, or both of these means having negative shifts, no shifts, or positive shifts. The nine hypotheses imply that there are nine different critical values of $E[Y]$ and consequently of $E[\overline{Y}]$. With all δ'_i defined as positive, these critical values are $-\delta'_1 - a\delta'_2, -a\delta'_2, \delta'_1 - a\delta'_2, -\delta'_1, 0, \delta'_1, -\delta'_1 + a\delta'_2, a\delta'_2, \delta'_1 + a\delta'_2$. In constructing the control chart for the sample statistic, $\bar{y}$, the hypotheses are manifested as bands or decision intervals (w_{jl}, w_{ju}), each with a critical value of $\bar{y}$ at the midpoint. In practice, observed values of $\bar{y}$ are plotted on the chart, and a diagnosis about the control status would be implied by the interval within which this sample value falls. In order to make a diagnosis in this manner, the assumption that a sample represents a homogeneous subgroup must be applicable.

The intervals (w_{jl}, w_{ju}) are defined by two sets of constraints. The first constraint is simply that the upper endpoint of one interval must be the lower endpoint of the next highest interval. The second constraint is, given that the true mean of the transform variable $\bar{y}$ is located at the midpoint of an interval, the sample statistic $\bar{y}$ must fall in that interval with probability $1 - \theta$, or

$$P[w_{jl} \leq \bar{y} \leq w_{ju} \mid \mu_y = \mu_{y_j}] = 1 - \theta. \qquad [3.5]$$

This formulation results in

$$(w_{jl}, w_{ju}) = \mu_{y_j} \pm Z_{1-\frac{\theta}{2}} \frac{\sigma_y}{\sqrt{n}}. \qquad [3.6]$$

For the two variable case, the authors give the following solution for the transform parameter and the sample size.

$$a = 2\frac{\delta'_1}{\delta'_2} \qquad [3.7]$$

$$n = \left(2\frac{Z_{1-\frac{\theta}{2}}}{\delta'_1}\right)^2 \sigma_y^2 \qquad [3.8]$$

Substituting the sample size equation into the expression for the decision intervals yields

$$(w_{jl}, w_{ju}) = \mu_{y_j} \pm \frac{\delta'_1}{2}. \qquad [3.9]$$

The variance of the sample statistic $\bar{y}$ is given by

$$\sigma_{\bar{y}}^2 = \left(\frac{\delta'_1}{2Z_{1-\frac{\theta}{2}}}\right)^2. \qquad [3.10]$$

Nachlas and Barreau acknowledge that this initial formulation of their method offers virtually no power to detect offsetting shifts in the means of the two quality variables. In order to provide some analysis of the potential effectiveness of their method, the authors explore its performance for a hypothetical situation in which only positive shifts in the means of the quality variables are possible. With this stipulation, they demonstrate that their afine transformation method is successful in achieving the desired sampling error probabilities when the means are in control and when either or both have shifted to their predetermined critical values.

The afine transformation method for multivariate quality control of Nachlas and Barreau has been presented in some detail here because it serves as the foundation of the research presented in the remainder of this paper.

Extension to More than Two Variables, Positive Shifts Only

The case in which only positive shifts in the quality variable means are possible is an important step in the development of a more general model. While it may indeed find practical applications in a manufacturing situation, its primary importance here is that its development is instructive and provides insight into the formulation of a more general solution.

Nachlas and Barreau suggest that the afine transformation be performed with a single parameter, a, where $a_i = a^{i-1}$. In their solution, the value of a is a function of the critical shifts of the first two quality variables. With three quality variables, for example, the coefficient of the third quality variable would be $a_3 = a^2 = 4(\frac{\delta'_1}{\delta'_2})^2$. Consider a hypothetical case in which the third quality variable has shifted to its critical mean, δ'_3. The expected value of the transform variable, y, using their solution is then $4(\frac{\delta'_1}{\delta'_2})^2\delta'_3$ which depends on the relative magnitudes of the δ'_i s. Compare this expected value to that which results from a second case in which the first and second quality variables have critical shifts. In this case the expected value of y is $3\delta'_1$. The ability of the transformed statistic to distinguish between these two cases depends on the proximity of the two expected values which is a function of the relative magnitudes of the δ'_i s. Although this solution may provide acceptable results for plausible critical shifts in a three-variable model, the method can not guarantee exact sampling error probabilities for shifts in its mean. This deficiency becomes more pronounced as the number of quality variables is increased further.

The two variable solution for positive shifts of Nachlas and Barreau results in the following critical values of E[Y]

$$0, \delta'_1, a\delta'_1, \delta'_1 + a\delta'_1.$$

or equivalently by substituting for a

$$0, \delta'_1, 2\delta'_1, 3\delta'_1.$$

The effect of the transform parameter a is to place these critical values at equal spacings in units of δ'_1, and the corresponding decision intervals are therefore of equal width. This result is expected because Y is a normal random variable with constant variance independent of the mean. Equal-width decision intervals are thus required to achieve equal sampling error probabilities at each of the critical values of E[Y]. Because this single parameter model does not guarantee this equal spacing when the number of quality variables exceeds two, it appears necessary to increase the number of parameters to achieve equal sampling error probabilities for the general multivariate case.

To solve the three quality variable case, an additional transform parameter b is introduced into the transform model. The equation for the transform variable thus becomes

$$y = x_1 + ax_2 + bx_3. \qquad [3.11]$$

With the same solution procedure used by Nachlas and Barreau, the following interval constraint equations can be solved for b.

$$3\delta'_1 + Z_{1-\frac{\theta}{2}} \frac{\sigma_y}{\sqrt{n}} = b\delta'_3 - Z_{1-\frac{\theta}{2}} \frac{\sigma_y}{\sqrt{n}} \qquad [3.12a]$$

$$b\delta'_3 + Z_{1-\frac{\theta}{2}} \frac{\sigma_y}{\sqrt{n}} = b\delta'_3 + \delta'_1 - Z_{1-\frac{\theta}{2}} \frac{\sigma_y}{\sqrt{n}} \qquad [3.12b]$$

Subtracting [3.12 a] from [3.12 b] gives

$$b = 4\frac{\delta'_1}{\delta'_3}. \qquad [3.13]$$

Note that the sample size solution does not change with the addition of the parameter.

For positive shifts, this three variable model results in the following critical values of E[Y]

$$0, \delta'_1, 2\delta'_1, 3\delta'_1, b\delta'_3, \delta'_1 + b\delta'_3, 2\delta'_1 + b\delta'_3, 3\delta'_1 + b\delta'_3$$

or equivalently by substituting for b

$$0, \delta'_1, 2\delta'_1, 3\delta'_1, 4\delta'_1, 5\delta'_1, 6\delta'_1, 7\delta'_1.$$

The same solution procedure could be repeated for any $m > 3$ quality variables, but a much simpler solution may be obtained by recognizing the general pattern.

To facilitate the presentation, define $\gamma_i = \frac{\delta'_i}{\delta'_1}$ and $\lambda_i = \frac{\delta_i}{\delta'_i}$ where as before δ_i represents the actual shift in standardized units of the mean of quality variable x_i and δ'_i is its predetermined critical shift. The general formulation of the transform parameters a_i giving equal sampling error probabilities is

$$a_i = \frac{2^{i-1}}{\gamma_i} \quad , i = 1, m. \qquad [3.14]$$

The general form of the transform variable y is then

$$y = \sum_{i=1}^{m} \frac{2^{i-1}}{\gamma_i} x_i. \qquad [3.15]$$

The sample size equation is the same as given previously as [3.8] except that the variance of y will be different here. With the assumption of standard normal quality variables, the general sample size equation is

$$n = \left(\frac{2Z_{1-\frac{\theta}{2}}}{\delta'_1} \right)^2 \sum_{i=1}^{m} \frac{2^{2i-2}}{\gamma_i^2}. \qquad [3.16]$$

The variance of the sample statistic $\bar{y}$ does not change in this formulation and is again independent of the transform parameters.

$$\sigma_{\bar{y}}^2 = \left(\frac{\delta'_1}{2Z_{1-\frac{\theta}{2}}} \right)^2 \qquad [3.10]$$

This general formulation is now shown mathematically to provide the desired level of sampling error probability for any m quality variables regardless of the relative δ'_i s. The actual shifts can be described completely in terms of the λ_i s since $\delta_i = \lambda_i \delta'_i$. Let μ_y represent the true mean of the transform variable. Then the general equation for the true mean of the transform variable can be given as

$$\mu_y = a_1\lambda_1\delta'_1 + a_2\lambda_2\delta'_2 + a_3\lambda_3\delta'_3 + \cdots + a_m\lambda_m\delta'_m$$

$$\mu_y = \delta'_1 \sum_{i=1}^{m} \lambda_i 2^{i-1}. \qquad [3.17]$$

To determine in which decision interval on the control chart the sample statistic $\bar{y}$ should fall in order for the correct diagnosis to be made, define λ_i' as the shift indicator variable such that

$$\lambda_i' = \begin{cases} 1 & \text{if } \lambda_i > 0 \\ 0 & \text{if } \lambda_i = 0. \end{cases}$$

Also let μ_y' represent the midpoint of the decision interval providing the correct inference or diagnosis. This value is computed as

$$\mu_y' = \delta'_1 \sum_{i=1}^{m} \lambda_i' 2^{i-1}. \qquad [3.18]$$

The correct diagnosis interval limits are then given by

$$\delta'_1 \left(\sum_{i=1}^{m} \lambda_i' 2^{i-1} \pm \frac{1}{2} \right). \qquad [3.19]$$

The probability of a correct diagnosis is the probability that the sample statistic $\bar{y}$ is in the correct decision interval given the true mean μ_y. This probability is given by

$$P\left[\delta'_1 \left(\sum_{i=1}^{m} \lambda_i' 2^{i-1} - \frac{1}{2} \right) < \bar{y} < \delta'_1 \left(\sum_{i=1}^{m} \lambda_i' 2^{i-1} + \frac{1}{2} \right) \right] \qquad [3.20]$$

which reduces to

$$P\left[\left(\sum_{i=1}^{m} \lambda_i' 2^i - 1 - \sum_{i=1}^{m} \lambda_i 2^i \right) Z_{1-\frac{\theta}{2}} < Z < \left(\sum_{i=1}^{m} \lambda_i' 2^i + 1 - \sum_{i=1}^{m} \lambda_i 2^i \right) Z_{1-\frac{\theta}{2}} \right]. \qquad [3.21]$$

Note that this probability is independent of the δ'_i s. The intermediate solution steps are listed in Appendix A of this text.

From equation [3.21], the probability of a correct diagnosis for any m quality variables at any combination of critical or zero shifts by substituting λ_i' for λ_i may be obtained. This yields

$$P[-Z_{1-\frac{\theta}{2}} < Z < Z_{1-\frac{\theta}{2}}] = 1-\theta \qquad [3.22]$$

which is the desired level of sampling error probability at the critical shifts.

For the two quality variable case, this general formulation is equivalent to the solution of Nachlas and Barreau (1989). In their paper, they present a table showing the probability of correct diagnoses for various combinations of shifts in the means of the two variables. A set of similarly constructed tables for the two and three variable cases is provided here in Chapter 4.

Multiple Variables with No Restrictions on Shift Direction

Nachlas and Barreau suggest computing a second transform variable, y_2, to provide diagnostic power capable of detecting offsetting shifts in the means of the quality variables. This second transform variable is computed from the same sample as the first, so it requires no additional inspection.

The second transform variable is used in conjunction with the first to provide a diagnosis about the quality variables. The form of this second transform should complement the form of the first in such a way that, when the two are used together, the method is not disrupted by the presence of offsetting shifts in the means of the component quality variables.

For the two quality variable case, the following strategy is suggested both for its simplicity and its ability to detect offsetting shifts with the same effectiveness as it does one-directional shifts. The two afine transform variables are defined as

$$y_1 = a_1x_1 + a_2x_2 \qquad [3.23a]$$

$$y_2 = a_1x_1 - a_2x_2. \qquad [3.23b]$$

The transform parameters a_i are defined as before, so the first transform variable y_1 is identical to the single transform variable y used in the positive shifts model. The critical values and decision intervals for both y_1 and y_2 are also identical to those of the positive shifts model except that the corresponding critical values and intervals on the negative side are included here.

The first transform variable y_1 represents a sum of the two quality variables while y_2 is a difference between them. In the most pathological case of offsetting shifts, the mean μ_{y_1} would be zero and the statistic $\bar{y}_1$ would not be able to distinguish this case from an in-control or zero-shift case. The mean μ_{y_2} would, however, be either extremely positive or negative in this offsetting shift situation. Only in an in-control case would the means of both the transform variables be zero. The combined use y_1 and y_2 thus provides a method of quality diagnosis which is not adversely affected by the direction of the shifts in the quality variable means.

This logic is similar to that offered by Jackson (1959) for diagnosis in a multivariate quality control problem. Jackson does not, however, give definite rules for diagnosis nor an analysis of the sampling error probabilities for the possible diagnoses.

The diagnosis for the two component quality variables is given by the pair of decision intervals of y_1 and y_2 in which the sample statistics $\bar{y}_1$ and $\bar{y}_2$ fall. Each pair of decision intervals corresponds to a particular quality diagnosis. Because the objective here is to detect only the existence and di-

rection of shifts and not to provide an inference about the magnitude of shifts, more than one pair of intervals may correspond to the same diagnosis. The assignment of diagnoses to the interval pairs is a consequence of the analysis of the joint probabilities of $\bar{y}_1$ and $\bar{y}_2$ which is presented first.

Let $[w_{1jl}, w_{1ju}]$ represent the upper and lower endpoints of the j^{th} interval of y_1 and $[w_{2kl}, w_{2ku}]$ be the k^{th} interval of y_2. As is described below, the two quality variable case results in seven intervals each for y_1 and y_2 and therefore forty-nine pairs of intervals. Because the quality variables are assumed to be standard normal random variables, it is appropriate and algebraically convenient to perform the probability analysis on $\bar{\lambda}_i = \frac{\bar{x}_i}{\delta'_1}$ rather than on the $\bar{x}_i$. The mean and variance of the $\bar{\lambda}_i$ are

$$\mu_{\bar{\lambda}_i} = \lambda_i \qquad [3.24]$$

$$\sigma^2_{\bar{\lambda}_i} = \left(\frac{1}{\delta'_i}\right)^2 \sigma^2_{\bar{x}_i} = \frac{1}{n}\left(\frac{1}{\delta'_i}\right)^2 \qquad [3.25]$$

where λ_i is defined as before ($\lambda_i = \frac{\delta_i}{\delta'_i}$).

The probability that the combination of $\bar{y}_1$ and $\bar{y}_2$ will fall in a particular pair of intervals j and k is given by

$$P[w_{1jl} < \bar{y}_1 < w_{1ju} \text{ and } w_{2kl} < \bar{y}_2 < w_{2ku}] \qquad [3.26]$$

which algebraically reduces to

$$P\left[\frac{w_{1jl}}{\delta'_1} < \bar{\lambda}_1 + 2\bar{\lambda}_2 < \frac{w_{1ju}}{\delta'_1} \text{ and } \frac{w_{2kl}}{\delta'_1} < \bar{\lambda}_1 - 2\bar{\lambda}_2 < \frac{w_{2ku}}{\delta'_1}\right]. \qquad [3.27]$$

This expression results in four inequalities which must be satisfied for $\bar{y}_1$ and $\bar{y}_2$ to fall in this interval pair.

$$\bar{\lambda}_1 + 2\bar{\lambda}_2 > \frac{w_{1jl}}{\delta'_1} \qquad [3.28a]$$

$$\bar{\lambda}_1 + 2\bar{\lambda}_2 < \frac{w_{1ju}}{\delta'_1} \qquad [3.28b]$$

$$\bar{\lambda}_1 - 2\bar{\lambda}_2 > \frac{w_{2kl}}{\delta'_1} \qquad [3.28c]$$

$$\bar{\lambda}_1 - 2\bar{\lambda}_2 < \frac{w_{2ku}}{\delta'_1} \qquad [3.28d]$$

These inequalities define a two-dimensional "diamond" region shown in Figure 1 on page 25 when plotted with $\bar{\lambda}_1$ and $\bar{\lambda}_2$ as the $x\,y$ axes. The forty-nine pairs of intervals of $\bar{y}_1$ and $\bar{y}_2$ result in forty-nine adjacent and equal-area diamond regions on the $\bar{\lambda}_1\,\bar{\lambda}_2$ plane with the in-control or zero shift diagnosis diamond centered about the origin. The joint probability distribution of the independent normal random variables, $\bar{\lambda}_1$ and $\bar{\lambda}_2$, is on the axis coming out of the page and perpendicular to the plane of the diamond. The probability of $\bar{y}_1$ and $\bar{y}_2$ being in a particular pair of intervals is equal to the area of the joint distribution surface of $\bar{\lambda}_1$ and $\bar{\lambda}_2$ over the corresponding diamond region.

The probability is given by the double integral of the joint distribution function of $\bar{\lambda}_1$ and $\bar{\lambda}_2$ over the region. Because of the diamond shape, the region is divided into equal left and right halves before the integration is performed. The probability of being in the region is then the sum of the integrals of the two halves. All of the limits of integration are determined using simple algebra with the four inequalities of equation [3.28]. The formal definition of the probability that $\bar{y}_1$ and $\bar{y}_2$ are in a given pair of intervals j and k is

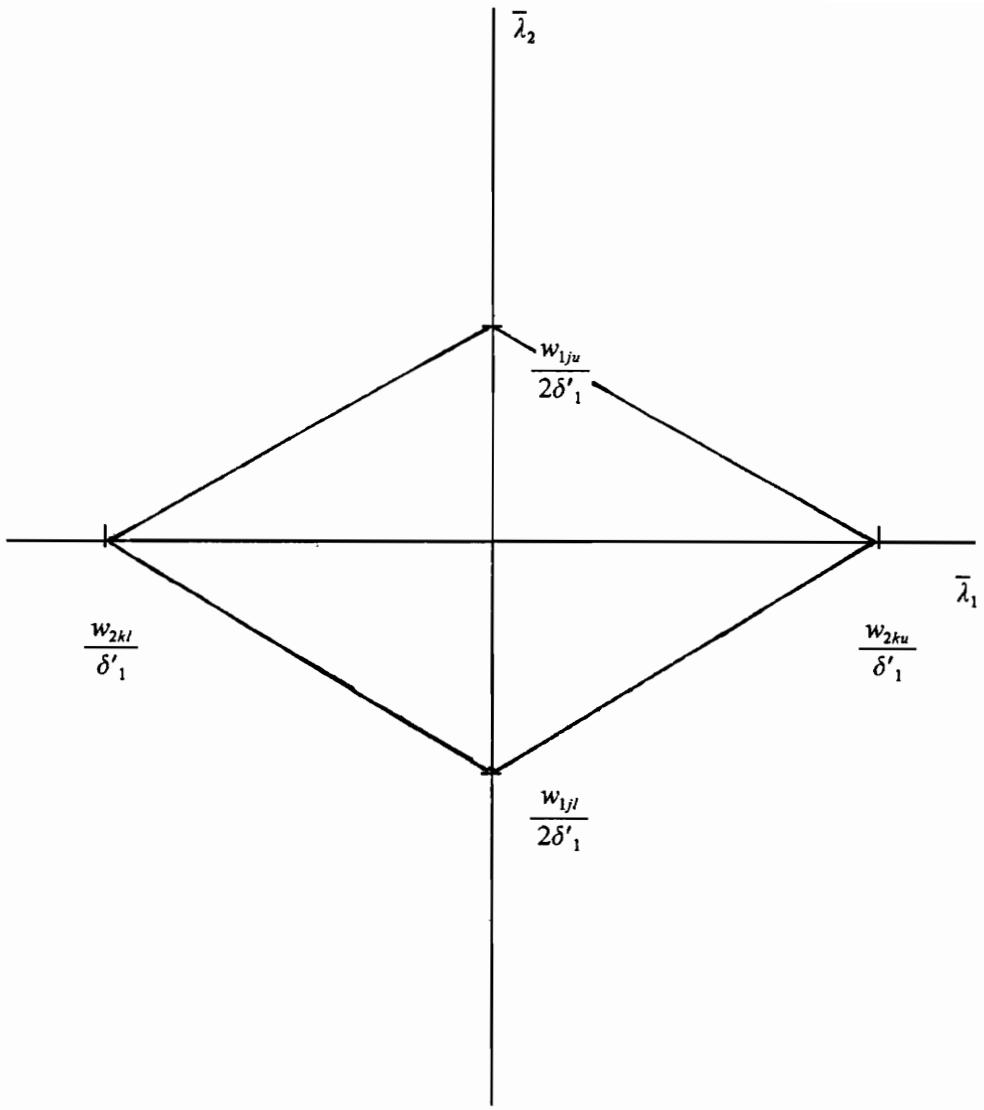

Figure 1. Region of Quality Variable Values Satisfying a Decision Interval Pair: There are 7 rows of 7 adjacent and equivalent regions each. The in-control diagnosis region is at the center of all the regions.

$$\int_{\frac{w_{1jl}+w_{2kl}}{2\delta'_1}}^{\frac{w_{1ju}+w_{2kl}}{2\delta'_1}} \int_{\frac{w_{1jl}}{2\delta'_1}-\frac{\bar{\lambda}_1}{2}}^{-\frac{w_{2kl}}{2\delta'_1}+\frac{\bar{\lambda}_1}{2}} f(\bar{\lambda}_1)f(\bar{\lambda}_2)d\bar{\lambda}_2 d\bar{\lambda}_1 + \int_{\frac{w_{1ju}+w_{2kl}}{2\delta'_1}}^{\frac{w_{1ju}+w_{2ku}}{2\delta'_1}} \int_{-\frac{w_{2ku}}{2\delta'_1}+\frac{\bar{\lambda}_1}{2}}^{\frac{w_{1ju}}{2\delta'_1}-\frac{\bar{\lambda}_1}{2}} f(\bar{\lambda}_1)f(\bar{\lambda}_2)d\bar{\lambda}_2 d\bar{\lambda}_1. \quad [3.29]$$

Let $\Phi_A^B(\bar{\lambda}_2)$ represent the cumulative normal distribution of $\bar{\lambda}_2$ from A to B. The probability equation then becomes

$$\int_{\frac{w_{1jl}+w_{2kl}}{2\delta'_1}}^{\frac{w_{1ju}+w_{2kl}}{2\delta'_1}} \Phi_{\frac{w_{1jl}}{2\delta'_1}-\frac{\bar{\lambda}_1}{2}}^{-\frac{w_{2kl}}{2\delta'_1}+\frac{\bar{\lambda}_1}{2}} (\bar{\lambda}_2)f(\bar{\lambda}_1)d\bar{\lambda}_1 + \int_{\frac{w_{1jl}+w_{2kl}}{2\delta'_1}}^{\frac{w_{1ju}+w_{2ku}}{2\delta'_1}} \Phi_{-\frac{w_{2ku}}{2\delta'_1}+\frac{\bar{\lambda}_1}{2}}^{\frac{w_{1ju}}{2\delta'_1}-\frac{\bar{\lambda}_1}{2}} (\bar{\lambda}_2)f(\bar{\lambda}_1)d\bar{\lambda}_1 \quad [3.30]$$

where

$$f(\bar{\lambda}_1) = \frac{1}{\sqrt{2\pi}\,\sigma_{\bar{\lambda}_1}} e^{-\frac{1}{2}\left(\frac{\bar{\lambda}_1-\mu_{\bar{\lambda}_1}}{\sigma_{\bar{\lambda}_1}}\right)^2}.$$

Note that in practice there will probably be no upper or lower limits above or below the extreme critical values of y_1 and y_2, so the region under the joint probability surface will be without either one or two of the constraints in these cases.

One could conceivably assign diagnoses to several of the interval pairs merely by reasoning out what must happen for an interval pair to occur, but there is a more definite way to mathematically determine the most logical diagnoses. For each diamond region on the $\bar{\lambda}_1\,\bar{\lambda}_2$ plane, the coordinates of the centroid of the region can be determined algebraically from the four inequalities in [3.28] as

$$\text{mid}\,\bar{\lambda}_1 = \frac{w_{1ju}+w_{2kl}}{2\delta'_1} \quad [3.31a]$$

$$\text{mid}\ \bar{\lambda}_2 \;=\; \frac{w_{1jl} - w_{2kl}}{4\delta'_1}. \qquad [3.31b]$$

In its strict probabilistic definition, a sample point falling in a particular region has a greater likelihood of coming from the joint distribution with means at the centroid of that region than from a joint distribution with means at the centroids of any of the other regions. These centroids are thus analogous to the critical values of E[Y] in the positive shifts model.

The centroid coordinate expressions give exact numerical values of $\bar{\lambda}_1$ and $\bar{\lambda}_2$, but the objective here is to detect only the existence and direction of shifts. Therefore, several of the regions' centroids give the same quality diagnosis since they differ only in the magnitude of the shifts. The diagnoses for each of the forty-nine pairs of y_1 and y_2 intervals are given in Table 1 on page 28.

As expected, the diagnoses form a regular pattern with all pairs giving the same diagnosis forming distinct subgroups. To determine the probability of a certain diagnosis given a certain pair of shifts λ_1 and λ_2, the sum of the probabilities of being in each of the pairs of intervals giving that diagnosis is computed. Keep in mind that, to use the two transform variable method, all that is required of the user is to compare the sample values of $\bar{y}_1$ and $\bar{y}_2$ to this table. In practice, the table would be referred to only when one of these two sample statistics falls outside its middle decision interval. The translation into $\bar{\lambda}_1$ and $\bar{\lambda}_2$ and the integration is only necessary to analyze the sampling error probabilities.

Also note that since $\sigma^2_{\bar{\lambda}_2}$ is a function of δ'_2 and the sample size n which in turn is a function of δ'_1, the probabilistic behavior is not independent of the magnitude of the δ'_is as it was in the positive shifts case. For reasonable values of δ'_1 and δ'_2, however, the results do seem to be insensitive to the changes in the δ'_is. Sampling error probabilities for various shifts and combinations of δ'_1 and δ'_2 are calculated using a combination of Simpson's rule and a numerical approximation of the

Table 1. Quality Diagnosis for Each Pair of Diagnosis Intervals of the 2 Transform Variables

$y_1 \backslash y_2$	$-3\delta'_1$	$-2\delta'_1$	$-\delta'_1$	0	δ'_1	$2\delta'_1$	$3\delta'_1$
$-3\delta'_1$	$\delta_1 < 0$ $\delta_2 = 0$	$\delta_1 < 0$ $\delta_2 < 0$	$\delta_1 < 0$ $\delta_2 < 0$	$\delta_1 < 0$ $\delta_2 < 0$	$\delta_1 < 0$ $\delta_2 < 0$	$\delta_1 < 0$ $\delta_2 < 0$	$\delta_1 = 0$ $\delta_2 < 0$
$-2\delta'_1$	$\delta_1 < 0$ $\delta_2 > 0$	$\delta_1 < 0$ $\delta_2 = 0$	$\delta_1 < 0$ $\delta_2 < 0$	$\delta_1 < 0$ $\delta_2 < 0$	$\delta_1 < 0$ $\delta_2 < 0$	$\delta_1 = 0$ $\delta_2 < 0$	$\delta_1 > 0$ $\delta_2 < 0$
$-\delta'_1$	$\delta_1 < 0$ $\delta_2 > 0$	$\delta_1 < 0$ $\delta_2 > 0$	$\delta_1 < 0$ $\delta_2 = 0$	$\delta_1 < 0$ $\delta_2 < 0$	$\delta_1 = 0$ $\delta_2 < 0$	$\delta_1 > 0$ $\delta_2 < 0$	$\delta_1 > 0$ $\delta_2 < 0$
0	$\delta_1 < 0$ $\delta_2 > 0$	$\delta_1 < 0$ $\delta_2 > 0$	$\delta_1 < 0$ $\delta_2 > 0$	$\delta_1 = 0$ $\delta_2 = 0$	$\delta_1 > 0$ $\delta_2 < 0$	$\delta_1 > 0$ $\delta_2 < 0$	$\delta_1 > 0$ $\delta_2 < 0$
δ'_1	$\delta_1 < 0$ $\delta_2 > 0$	$\delta_1 < 0$ $\delta_2 > 0$	$\delta_1 = 0$ $\delta_2 > 0$	$\delta_1 > 0$ $\delta_2 > 0$	$\delta_1 > 0$ $\delta_2 = 0$	$\delta_1 > 0$ $\delta_2 < 0$	$\delta_1 > 0$ $\delta_2 < 0$
$2\delta'_1$	$\delta_1 < 0$ $\delta_2 > 0$	$\delta_1 = 0$ $\delta_2 > 0$	$\delta_1 > 0$ $\delta_2 > 0$	$\delta_1 > 0$ $\delta_2 > 0$	$\delta_1 > 0$ $\delta_2 > 0$	$\delta_1 > 0$ $\delta_2 = 0$	$\delta_1 > 0$ $\delta_2 < 0$
$3\delta'_1$	$\delta_1 = 0$ $\delta_2 > 0$	$\delta_1 > 0$ $\delta_2 > 0$	$\delta_1 > 0$ $\delta_2 > 0$	$\delta_1 > 0$ $\delta_2 > 0$	$\delta_1 > 0$ $\delta_2 > 0$	$\delta_1 > 0$ $\delta_2 > 0$	$\delta_1 > 0$ $\delta_2 = 0$

cumulative normal distribution adapted freely from Abramowitz and Stegun (1965) and are given in Chapter 4.

This geometric determination of the quality diagnoses for the decision interval pairs of y_1 and y_2 fails when the number of quality variables exceeds two. Consider the case of two transform variables with three quality variables. Each decision interval pair results in a set of two parallel planes in the three-dimensional space spanned by the three quality variables. In two dimensions, the decision intervals formed a closed region which is the diamond shown in Figure 1. In three dimensions, however, a closed region cannot be formed by two pairs of parallel planes. The region that results can be visualized as a slanted box without a top or bottom. A third transform variable would be necessary to provide the two additional parallel planes which could then "close" the box region.

The effect of having a diagnosis region which is open at two opposite ends is that the method again becomes incapable of detecting offsetting shifts in at least two of the three quality variables. For example, consider a diagnosis region formed by two pairs of parallel planes situated about the origin in three dimensions. This region would contain the in-control diagnosis with all variables at their target mean values. However, since the region is open at two opposite ends, two of the quality variables can fool the diagnostic test by shifting in opposite directions indefinitely. The assignment of a diagnosis to this and all other regions would be somewhat arbitrary in the sense that neither the in-control nor the offsetting shifts diagnoses can be thought the more likely diagnosis in the strict statistical definition.

At least in the formulation presented here, the two transform variable method does not appear capable of providing statistically adequate diagnoses for more than two quality variables. It does appear that the strategy developed in this section could be extended to any m quality variables by using m transform variables, but the practical implications of such a strategy preclude any possible benefits.

Discussion of the Required Sample Size

The purpose of the analysis given in this section is to arrive at a general idea about the magnitude of the required sample size of the afine transformation method relative to the sample size necessary when maintaining a separate Shewhart control chart for each of the multiple quality variables. It is not straightforward to objectively assess the advantage gained from using a single chart instead of several charts for one item, and the separate Shewhart charts scheme is still susceptible to the inadequacies described in the previous chapter. Therefore, the sample size comparison between the two strategies is not a comprehensive test of their relative effectiveness and efficiency.

To provide a succinct comparison, a ratio of the sample size required by the two strategies is constructed. First, the sample size of the afine transformation method has been previously stated as equation [3.9]. By substituting for the variance term, the sample size equation becomes

$$n = \frac{4Z^2_{1-\frac{\theta}{2}}}{\delta'^2_1} \sum_{i=1}^{m} \frac{2^{2i-2}}{\gamma_i^2} . \qquad [3.32]$$

To calculate the sample size required by the separate control charts for the mean, first consider n_i which is defined as the sample size requirement for the i^{th} quality variable. Maintaining consistency with the formulation of the afine transform model, assume that both the Type I and Type II sampling error for the i^{th} control chart are set at θ with the alternate hypothesis set for a shift of δ'_i. This critical shift of δ'_i can also be expressed as $\delta'_i = \gamma_i \delta'_1$. The assumption of standard normal random quality variables is also applied here. The sample size required to meet the objectives of the i^{th} control chart is

$$n_i = \frac{4Z^2_{1-\theta}}{\gamma_i^2 \delta'^2_1} . \qquad [3.33]$$

The cumulative sample size required to maintain a separate control chart for each of m quality variables is the sum of the n_i or

$$n = \frac{4Z_{1-\theta}^2}{\delta_1'^2} \sum_{i=1}^{m} \frac{1}{\gamma_i^2} . \qquad [3.34]$$

The ratio of sample sizes required for the afine transformation method versus using separate Shewhart charts is

$$\frac{Z_{1-\frac{\theta}{2}}^2 \sum_{i=1}^{m} \frac{2^{2i-2}}{\gamma_i^2}}{Z_{1-\theta}^2 \sum_{i=1}^{m} \frac{1}{\gamma_i^2}} . \qquad [3.35]$$

This ratio is greater than one for any $m > 0$ and approaches infinity in the limiting case. To summarize, the sample size requirement of the afine transformation method does not compare favorably with that required by using separate Shewhart control charts.

Comparison to Hotelling's Statistic

Because Hotelling's method does not consider the power to detect shifts in the quality variable means, it will not be possible to give a direct comparison of the performance of the afine transformation method to Hotelling's for a general case. However, some examples or plausible case studies can be constructed for which the performance of these two methods can be evaluated.

The power of the positive shifts model is used to represent the afine transformation method for the comparison. This model is chosen because it is the potentially most effective case for the afine transformation method. The T^2 statistic is obviously indifferent to the direction of mean shifts because of its squared terms.

To put it on equal terms with the T^2 method, the specific quality diagnoses will not be considered with the afine transformation variables. All that will be measured is the probability of detecting an out-of-control situation. The same Type I sampling error probability for the in-control case, or θ, will be used in both methods. The sample size required to achieve θ for the afine transformation method will also be used for the T^2 test. The performance of the afine transformation method for positive shifts has been shown to be independent of the δ'_i s. The power of the T^2 test will not, however, be insensitive to the values of δ'_i because the statistic is not defined in terms Type II error at an alternate hypothesis. Therefore, more than one set of δ'_i s must be considered to evaluate the power of the T^2 test versus the afine transformation method.

The power of the afine transformation method is simply one minus the probability that the transformed sample statistic is in the in-control or zero shift decision interval which is formally stated as

$$1 - P\left[-\frac{\delta'_1}{2} < \bar{y} < \frac{\delta'_1}{2} \mid \mu = \mu_y \right] \qquad [3.36]$$

which reduces to

$$1 - P\left[\left(-1 - \sum_{i=1}^{m} \lambda_i 2^i \right) Z_{1-\frac{\theta}{2}} < Z < \left(1 - \sum_{i=1}^{m} \lambda_i 2^i \right) Z_{1-\frac{\theta}{2}} \right]. \qquad [3.37]$$

The solution is similar to that given in Appendix A."

The power of the T^2 test is computed using its relationship to the F distribution given previously as [2.2]. When the vector of quality variable means has shifted from zero, the distribution becomes a noncentral F with noncentrality parameter given as [2.3] and repeated here.

$$n\vec{\mu}' S^{-1}\vec{\mu} \qquad [2.3]$$

This calculation can also be simplified in terms of the λ_i s. With the assumption that the quality variables remain independent with unit variances when their means have shifted, the noncentrality parameter calculation becomes

$$n\sum_{i=1}^{m}(\lambda_i\delta'_i)^2. \qquad [3.38]$$

The entire analysis of the T^2 test can then be performed in terms of the F distribution. First, a critical value of F with m and $n-m$ degrees of freedom and level of significance θ is computed or found from a statistical table. The power of the T^2 test is the probability than an F random variable with the computed noncentrality parameter exceeds the predetermined critical value of F.

Tables displaying the power of the afine transformation method and the T^2 test for various values of δ'_i and for two and three quality variable cases are presented in Chapter 4.

Chapter 4

NUMERICAL RESULTS AND ANALYSIS

The various computations used in evaluating the performance of the afine transformation method are performed here with FORTRAN and SAS programs which are included here as appendices. A numerical approximation to the standard normal density function, adapted freely from Abramowitz and Stegun (1965), is coded as a FORTRAN subprogram and is also included as an appendix.

The numerical results are arranged into tables of probability calculations for various combinations of shifts in the component quality variable population means. These shifts are indexed according to the λ_i s (where $\lambda_i = \frac{\delta_i}{\delta'_i}$ as before) instead of the usual δ_i s to emphasize the fact that the sampling error probabilities for the afine transform method can be fixed at any set of critical shifts. In other words, the sampling error can be set for any combination of alternate hypotheses for the quality variable means.

Correct Diagnosis Probabilities for the Positive Shifts Model

Numerical results for the single transform variable method for two quality variables are given in Table 2 on page 36 with the level of sampling error set arbitrarily at $\theta = 0.05$. The values in this table represent the probabilities of correctly diagnosing the existence of shifts in either or both of the quality variables. This table does not show the power of the method to detect in general an out-of-control situation for the two variable set. The power probabilities (without considering diagnoses) are generally much higher than these and are shown in Table 7 on page 50.

As noted in the previous chapter, the general form of the transform developed in this paper for the positive shifts model is identical to the model given by Nachlas and Barreau (1989) when there are only two quality variables. Nachlas and Barreau also provide a table of correct diagnosis probabilities similar to Table 2. At the higher shifts, however, the correct diagnosis probabilities shown in Table 2 here are higher than those given by Nachlas and Barreau. This discrepancy is not due to any real difference in effectiveness but results from the assumption made here that in practice there would be no upper limit for the decision interval which corresponds to the diagnosis of all of the variables having shifted. It appears that the authors did not make this assumption and therefore show results which are not as favorable as they might have shown.

The assumption that there is no practical lower limit for the no-shifts decision interval is not made here. It is believed by this author that making such an assumption would be using the restrictive assumption of positive shifts to unfairly exaggerate the effectiveness of the method.

As expected, Table 2 indicates that the desired level of sampling error is achieved at all combinations of zero and critical shifts. The only exception is for the case of both variable means reaching their critical values simultaneously. In this case the level of error is cut by one-half because of the assumption of no upper limit at the extreme end of the decision interval chart.

Table 2. Correct Diagnosis Probabilities for 2 Variables, Positive Shifts

$\lambda_1 \backslash \lambda_2$	0.00	0.25	0.50	0.75	1.00	1.25	1.50
0.00	.9500	.0000	.0250	.5001	.9500	.4998	.0250
0.25	.1636	.0000	.0000	.0016	.1636	.8364	.9984
0.50	.5001	.0000	.0000	.0250	.5001	.9750	1.000
0.75	.8348	.0000	.0016	.1636	.8364	.9984	1.000
1.00	.9500	.0000	.0250	.5001	.9750	1.000	1.000
1.25	.8348	.0016	.1636	.8364	.9984	1.000	1.000
1.50	.4998	.0250	.5001	.9750	1.000	1.000	1.000

Table 3. Correct Diagnosis Probabilities for 3 Variables, Positive Shifts

$\lambda_1 = 0.00$

$\lambda_2 \backslash \lambda_3$	0.00	0.25	0.50	0.75	1.00	1.25	1.50
0.00	.9500	.0000	.0000	.0250	.9500	.0250	.0000
0.25	.0000	.0000	.0000	.0000	.0000	.5001	.4998
0.50	.0250	.0000	.0000	.0000	.0250	.9500	.0250
0.75	.5001	.0000	.0000	.0000	.5001	.4998	.0000
1.00	.9500	.0000	.0000	.0250	.9500	.0250	.0000
1.25	.4998	.0000	.0000	.5001	.4998	.0000	.0000
1.50	.0250	.0000	.0250	.9500	.0250	.0000	.0000

$\lambda_1 = 0.25$

$\lambda_2 \backslash \lambda_3$	0.00	0.25	0.50	0.75	1.00	1.25	1.50
0.00	.1636	.0000	.0000	.0000	.1636	.8348	.0016
0.25	.0000	.0000	.0000	.0000	.0000	.0016	.8364
0.50	.0000	.0000	.0000	.0000	.0000	.1636	.9984
0.75	.0016	.0000	.0000	.0000	.0016	.8364	1.000
1.00	.1636	.0000	.0000	.0000	.1636	.9984	1.000
1.25	.8348	.0000	.0000	.0016	.8364	1.000	1.000
1.50	.8348	.0000	.0000	.1636	.9984	1.000	1.000

$\lambda_1 = 0.50$

$\lambda_2 \backslash \lambda_3$	0.00	0.25	0.50	0.75	1.00	1.25	1.50
0.00	.5001	.0000	.0000	.0000	.5001	.4998	.0000
0.25	.0000	.0000	.0000	.0000	.0000	.0250	.9750
0.50	.0000	.0000	.0000	.0000	.0000	.5001	1.000
0.75	.0250	.0000	.0000	.0000	.0250	.9750	1.000
1.00	.5001	.0000	.0000	.0000	.5001	1.000	1.000
1.25	.9500	.0000	.0000	.0250	.9750	1.000	1.000
1.50	.4998	.0000	.0000	.5001	1.000	1.000	1.000

$\lambda_1 = 0.75$

$\lambda_2 \backslash \lambda_3$	0.00	0.25	0.50	0.75	1.00	1.25	1.50
0.00	.8348	.0000	.0000	.0016	.8348	.1636	.0000
0.25	.0000	.0000	.0000	.0000	.0000	.1636	.9984
0.50	.0016	.0000	.0000	.0000	.0016	.8364	1.000
0.75	.1636	.0000	.0000	.0000	.1636	.9984	1.000
1.00	.8348	.0000	.0000	.0016	.8364	1.000	1.000
1.25	.8348	.0000	.0000	.1636	.9984	1.000	1.000
1.50	.1636	.0000	.0016	.8364	1.000	1.000	1.000

$\lambda_1 = 1.00$

$\lambda_2 \backslash \lambda_3$	0.00	0.25	0.50	0.75	1.00	1.25	1.50
0.00	.9500	.0000	.0000	.0250	.9500	.0250	.0000
0.25	.0000	.0000	.0000	.0000	.0000	.5001	1.000
0.50	.0250	.0000	.0000	.0000	.0250	.9750	1.000
0.75	.5001	.0000	.0000	.0000	.5001	1.000	1.000
1.00	.9500	.0000	.0000	.0250	.9750	1.000	1.000
1.25	.4998	.0000	.0000	.5001	1.000	1.000	1.000
1.50	.0250	.0000	.0250	.9750	1.000	1.000	1.000

$\lambda_1 = 1.25$

$\lambda_2 \backslash \lambda_3$	0.00	0.25	0.50	0.75	1.00	1.25	1.50
0.00	.8348	.0000	.0000	.1636	.8348	.0016	.0000
0.25	.0016	.0000	.0000	.0000	.0016	.8364	1.000
0.50	.1636	.0000	.0000	.0000	.1636	.9984	1.000
0.75	.8348	.0000	.0000	.0016	.8364	1.000	1.000
1.00	.8348	.0000	.0000	.1636	.9984	1.000	1.000
1.25	.1636	.0000	.0016	.8364	1.000	1.000	1.000
1.50	.0016	.0000	.1636	.9984	1.000	1.000	1.000

$\lambda_1 = 1.50$

$\lambda_2 \backslash \lambda_3$	0.00	0.25	0.50	0.75	1.00	1.25	1.50
0.00	.4998	.0000	.0000	.5001	.4998	.0000	.0000
0.25	.0250	.0000	.0000	.0000	.0250	.9750	1.000
0.50	.5001	.0000	.0000	.0000	.5001	1.000	1.000
0.75	.9500	.0000	.0000	.0250	.9750	1.000	1.000
1.00	.4998	.0000	.0000	.5001	1.000	1.000	1.000
1.25	.0250	.0000	.0250	.9750	1.000	1.000	1.000
1.50	.0000	.0000	.5001	1.000	1.000	1.000	1.000

The method appears to be more effective when one of the quality variables has shifted alone and when both quality variables have shifted substantially. The method appears ineffective when both quality variables have shifted but to values below their respective critical values. This characteristic is, however, also typical of conventional Shewhart control charts.

The same general comments about the method's performance apply in the case of three quality variables. Tables displaying the correct diagnosis probabilities for the three-variable case begin with Table 3 on page 37. The desired level of sampling error (also set at $\theta = 0.05$) is achieved at all combinations of zero and critical shifts. The effectiveness when only one of the variables has shifted has diminished somewhat compared to the two-variable case. The ineffectiveness of the method when more than one variable has shifted to a point below the critical values is much more evident here, and in several of these cases the probability of making the correct diagnosis is essentially zero.

For multivariate situations beyond these two cases, the desired level of sampling error has been shown to be achievable at the zero and critical shifts. However, it appears that the method is ineffective at combinations of shifts other than that which is defined as the critical set for the alternate hypothesis. The performance of the method when some or all of the quality variable means have shifted to values below critical appears to deteriorate as the number of quality variables is increased.

Correct Diagnosis Probabilities for Unrestricted Shift Direction Model

Correct diagnosis probabilities for the two transform variable, two quality variable afine transformation model are shown in the tables which begin with Table 4 on page 46. The probabilities in this case are not independent of the critical shifts, so the results from three different pairs of values for δ'_1 and δ'_2 are given here. The results do appear insensitive to the choice of δ'_1 and δ'_2. As before, the level of sampling error is set at $\theta = 0.05$. This sampling error is set as for the single

transform variable model, and it does not come out directly in the results of the two transform variable model.

The strategy presented in the previous chapter for selecting the form of the second transform variable is successful in making the method sensitive to offsetting shifts in the quality variable means. In fact, the performance of the method is completely independent of the direction of the shifts: it yields equal correct diagnosis probabilities for shifts of the same magnitude regardless of the direction of the shifts. The column and row headings of the sampling error tables reflect this result with the plus-or-minus signs in front of each entry.

The correct diagnosis probabilities are highest when both quality variables have shifted. In most of these cases, the sampling error is even less than the level stated in the design of the plan. For two quality variables, the two transform variable model is much more successful than the single transform model in making the correct diagnosis when both quality variable means shift.

There are some irregularities in the performance, however, when only one of the variables has a shifted mean. Referring to the first row of Table 4 on page 46 which corresponds to shifts in the second variable only, the correct diagnosis probability exhibits an up-and-down pattern. In other words, there are situations in which a larger deviation of the mean from its target actually has a lower probability of being correctly diagnosed. The same pattern also exists for the first quality variable as its mean shifts down the first column of the table. This phenomenon is a consequence of the diamond shapes of the diagnosis regions on the plane of the two quality variables as shown in the previous chapter. In the case of one variable shifting alone, at some values the coordinates of the quality variable means are at the centroid of a diamond corresponding to the correct diagnosis. As the mean of the shifted variable is increased beyond these points, the coordinates of the pair of means moves away from the centroid as the probability of a sample point falling within that region decreases. The diagnosis in these situations will more likely be that both the quality variables have shifted in a given direction.

Table 4. Correct Diagnosis Probabilities for 2 Variables, No Shift Restrictions

$\delta'_1 = 1.5\,,\ \delta'_2 = 1.75$

$\lambda_1 \backslash \lambda_2$	0.00	±0.25	±0.50	±0.75	±1.00	±1.25	±1.50
0.00	.9011	.3299	.9095	.6598	.9139	.6558	.9470
±0.25	.0044	.6769	.3190	.6785	.3166	.6725	.1593
±0.50	.1680	.9260	.8238	.9506	.8076	.9149	.4798
±0.75	.6759	.8382	.9919	.9943	.9860	.9889	.8333
±1.00	.9051	.6640	.9871	.9757	.9604	.9798	.9745
±1.25	.6803	.8386	.9938	.9904	.9910	.9969	.9983
±1.50	.3360	.9461	.9756	.9604	.9798	.9995	.9999

Table 5. Correct Diagnosis Probabilities for 2 Variables, No Shift Restrictions (Ex. 2)

$\delta'_1 = 2.00$, $\delta'_2 = 2.20$

$\lambda_1 \backslash \lambda_2$	0.00	±0.25	±0.50	±0.75	±1.00	±1.25	±1.50
0.00	.9107	.3378	.9113	.6757	.9163	.6715	.9477
±0.25	.0033	.6761	.3201	.6777	.3180	.6723	.1596
±0.50	.1600	.9269	.8314	.9514	.8146	.9138	.4789
±0.75	.6750	.8375	.9932	.9951	.9883	.9901	.8337
±1.00	.9062	.6557	.9861	.9746	.9587	.9789	.9744
±1.25	.6783	.8378	.9944	.9916	.9921	.9973	.9983
±1.50	.3199	.9464	.9745	.9587	.9789	.9995	.9999

Table 6. Correct Diagnosis Probabilities for 2 Variables, No Shift Restrictions (Ex. 3)

$\delta'_1 = 2.50$, $\delta'_2 = 2.85$

$\lambda_1 \backslash \lambda_2$	0.00	±0.25	±0.50	±0.75	±1.00	±1.25	±1.50
0.00	.9015	.3330	.9102	.6661	.9148	.6620	.9473
±0.25	.0039	.6766	.3195	.6781	.3172	.6724	.1594
±0.50	.1648	.9264	.8268	.9509	.8104	.9145	.4794
±0.75	.6755	.8379	.9924	.9946	.9869	.9894	.8334
±1.00	.9055	.6607	.9867	.9753	.9598	.9794	.9745
±1.25	.6794	.8382	.9940	.9908	.9914	.9970	.9983
±1.50	.3296	.9462	.9752	.9598	.9795	.9995	.9999

The two transform variable method does not appear to perform adequately when both quality variables are at their target values. The resulting level of sampling error is significantly higher than the level set in the design. This error could be decreased by using a smaller designed sampling error, but doing so would greatly increase the already high sample size requirements. The ineffectiveness of the method in the in-control situation is unfortunate since in practical use one would hope that the quality variables were at there target means the majority of the time. In effect, the method performs least effectively in the situation which should occur the most frequently.

Comparison of Power to Hotelling's Statistic

The purpose of the analysis presented in this section is not to offer a general evaluation of the power of Hotelling's T^2 statistic. Instead, the intent here is merely to make some comparisons between the power of the afine transformation method to that of Hotelling's statistic in various out-of-control situations. To put the two methods on equal terms for the comparison, the two tests are both set with a level of sampling error at $\theta = 0.05$ and with the sample size computed for the afine transformation method. This sample size is what is required by the afine transform to achieve $\theta = 0.05$. The Type I error of the T^2 test can be set for any sample size, but its power to detect various out-of-control situations is certainly affected by the sample size used. The power calculations for the T^2 have been performed using the noncentral F distribution in a SAS program.

The power of the afine transform method in various situations is presented in Table 7 on page 50. The power of the T^2 test is shown in tables which begin with Table 8 on page 51. As has been explained, the power of the T^2 is not independent of the δ'_i s, so three different pairs of δ'_i s are presented here to avoid the chance of making the comparison on an idiosyncratic case.

For the two quality variable case, neither the afine transform nor the T^2 appear to be more dominant in terms of power. The T^2 statistic does have in general a greater probability of detecting a

Table 7. Power of Afine Transformation for 2 Variables

$\lambda_1 \backslash \lambda_2$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	.0500	.3478	.8802	.9970	1.000	1.000
0.20	.1229	.6524	.9750	.9998	1.000	1.000
0.40	.3478	.8802	.9970	1.000	1.000	1.000
0.60	.6524	.9750	.9998	1.000	1.000	1.000
0.80	.8802	.9970	1.000	1.000	1.000	1.000
1.00	.9750	.9998	1.000	1.000	1.000	1.000

Table 8. Power of Hotelling's Statistic for 2 Variables

$\delta'_1 = 1.20$, $\delta'_2 = 1.50$

$\lambda_1 \backslash \lambda_2$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	.0500	.3356	.8979	.9987	1.000	1.000
0.20	.2270	.5171	.9377	.9993	1.000	1.000
0.40	.7204	.8588	.9875	.9999	1.000	1.000
0.60	.9750	.9895	.9994	1.000	1.000	1.000
0.80	.9996	.9998	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

Table 9. Power of Hotelling's Statistic for 2 Variables (Ex. 2)

$\delta'_1 = 1.75\,,\ \delta'_2 = 2.00$

$\lambda_1 \backslash \lambda_2$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	.0500	.3092	.8662	.9970	1.000	1.000
0.20	.2447	.5088	.9204	.9984	1.000	1.000
0.40	.7586	.8716	.9856	.9998	1.000	1.000
0.60	.9835	.9926	.9994	1.000	1.000	1.000
0.80	.9998	.9999	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

control problem when the first quality variable shifts alone, and it appears to perform equally well for shifts in either variable. The power of the afine transform method, however, shows a marked decrease in power for shifts in the first variable in comparison to the second. This difference is due to the higher coefficient or transform parameter of the second quality variable in the transform equation.

The power of the afine transform method for three quality variables is given in tables which begin with Table 11 on page 55. For three variables, the power to detect control problems is extremely high in almost all cases. This power is probably not so much a reflection of any intrinsic effectiveness of the method as it is a result of the high sample sizes required by the afine transformation method for fixing the level of sampling error at the various critical shifts.

The power of the T^2 statistic for three quality variables is given in tables that start with Table 12 on page 61. The high sample size requirement for the afine transform method causes the power of the T^2 test to almost immediately go to one in the three variable situation. Tables with other sets of δ'_1, δ'_2, and δ'_3 are not included here because the power reaches one so quickly in all cases. The general trend shows that Hotelling's T^2 statistic is more powerful than the afine transform method at the same sample size. It must be re-emphasized though that this comparison can not be considered the ultimate evaluation of the performance of the two methods since the T^2 statistic is incapable of giving diagnoses at the component quality variable level.

Table 10. Power of Hotelling's Statistic for 2 Variables (Ex. 3)

$$\delta'_1 = 2.50\,,\ \delta'_2 = 2.75$$

$\lambda_1 \backslash \lambda_2$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	.0500	.2634	.7879	.9876	.9999	1.000
0.20	.2239	.4461	.8609	.9926	.9999	1.000
0.40	.7041	.8180	.9653	.9985	1.000	1.000
0.60	.9679	.9824	.9974	.9999	1.000	1.000
0.80	.9992	.9996	.9999	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

Table 11. Power of Afine Transformation for 3 Variables

$\lambda_1 = 0.00$

$\lambda_2 \backslash \lambda_3$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	.0500	.8802	1.000	1.000	1.000	1.000
0.20	.3478	.9970	1.000	1.000	1.000	1.000
0.40	.8802	1.000	1.000	1.000	1.000	1.000
0.60	.9970	1.000	1.000	1.000	1.000	1.000
0.80	1.000	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

$\lambda_1 = 0.20$

$\lambda_2 \backslash \lambda_3$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	.1229	.9750	1.000	1.000	1.000	1.000
0.20	.6524	.9998	1.000	1.000	1.000	1.000
0.40	.9750	1.000	1.000	1.000	1.000	1.000
0.60	.9998	1.000	1.000	1.000	1.000	1.000
0.80	1.000	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

$\lambda_1 = 0.40$

$\lambda_2 \backslash \lambda_3$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	.3478	.9970	1.000	1.000	1.000	1.000
0.20	.8802	1.000	1.000	1.000	1.000	1.000
0.40	.9970	1.000	1.000	1.000	1.000	1.000
0.60	1.000	1.000	1.000	1.000	1.000	1.000
0.80	1.000	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

$\lambda_1 = 0.60$

$\lambda_2 \backslash \lambda_3$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	.6524	.9998	1.000	1.000	1.000	1.000
0.20	.9750	1.000	1.000	1.000	1.000	1.000
0.40	.9998	1.000	1.000	1.000	1.000	1.000
0.60	1.000	1.000	1.000	1.000	1.000	1.000
0.80	1.000	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

$\lambda_1 = 0.80$

$\lambda_2 \backslash \lambda_3$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	.8802	1.000	1.000	1.000	1.000	1.000
0.20	.9970	1.000	1.000	1.000	1.000	1.000
0.40	1.000	1.000	1.000	1.000	1.000	1.000
0.60	1.000	1.000	1.000	1.000	1.000	1.000
0.80	1.000	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

$\lambda_1 = 1.00$

$\lambda_2 \backslash \lambda_3$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	.9750	1.000	1.000	1.000	1.000	1.000
0.20	.9998	1.000	1.000	1.000	1.000	1.000
0.40	1.000	1.000	1.000	1.000	1.000	1.000
0.60	1.000	1.000	1.000	1.000	1.000	1.000
0.80	1.000	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

Table 12. Power of Hotelling's Statistic for 3 Variables

$$\delta'_1 = 1.50, \delta'_2 = 1.75, \delta'_3 = 2.00$$

$$\lambda_1 = 0.00$$

$\lambda_2 \backslash \lambda_3$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	.0500	.8871	1.000	1.000	1.000	1.000
0.20	.7802	.9910	1.000	1.000	1.000	1.000
0.40	1.000	1.000	1.000	1.000	1.000	1.000
0.60	1.000	1.000	1.000	1.000	1.000	1.000
0.80	1.000	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

$\delta'_1 = 1.50, \delta'_2 = 1.75, \delta'_3 = 2.00$

$\lambda_1 = 0.20$

$\lambda_2 \backslash \lambda_3$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	.6330	.9822	1.000	1.000	1.000	1.000
0.20	.9600	.9991	1.000	1.000	1.000	1.000
0.40	1.000	1.000	1.000	1.000	1.000	1.000
0.60	1.000	1.000	1.000	1.000	1.000	1.000
0.80	1.000	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

$\delta'_1 = 1.50, \delta'_2 = 1.75, \delta'_3 = 2.00$

$\lambda_1 = 0.40$

$\lambda_2 \backslash \lambda_3$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	.9990	1.000	1.000	1.000	1.000	1.000
0.20	1.000	1.000	1.000	1.000	1.000	1.000
0.40	1.000	1.000	1.000	1.000	1.000	1.000
0.60	1.000	1.000	1.000	1.000	1.000	1.000
0.80	1.000	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

$$\delta'_1 = 1.50, \delta'_2 = 1.75, \delta'_3 = 2.00$$

$$\lambda_1 = 0.60$$

$\lambda_2 \backslash \lambda_3$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	1.000	1.000	1.000	1.000	1.000	1.000
0.20	1.000	1.000	1.000	1.000	1.000	1.000
0.40	1.000	1.000	1.000	1.000	1.000	1.000
0.60	1.000	1.000	1.000	1.000	1.000	1.000
0.80	1.000	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

$\delta'_1 = 1.50, \delta'_2 = 1.75, \delta'_3 = 2.00$

$\lambda_1 = 0.80$

$\lambda_2 \backslash \lambda_3$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	1.000	1.000	1.000	1.000	1.000	1.000
0.20	1.000	1.000	1.000	1.000	1.000	1.000
0.40	1.000	1.000	1.000	1.000	1.000	1.000
0.60	1.000	1.000	1.000	1.000	1.000	1.000
0.80	1.000	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

$\delta'_1 = 1.50, \delta'_2 = 1.75, \delta'_3 = 2.00$

$\lambda_1 = 1.00$

$\lambda_2 \backslash \lambda_3$	0.00	0.20	0.40	0.60	0.80	1.00
0.00	1.000	1.000	1.000	1.000	1.000	1.000
0.20	1.000	1.000	1.000	1.000	1.000	1.000
0.40	1.000	1.000	1.000	1.000	1.000	1.000
0.60	1.000	1.000	1.000	1.000	1.000	1.000
0.80	1.000	1.000	1.000	1.000	1.000	1.000
1.00	1.000	1.000	1.000	1.000	1.000	1.000

Chapter 5

CONCLUSIONS

A method for multivariate quality control is developed with the objective of providing a diagnosis of control problems for the individual quality variables in addition to giving a true level of sampling error probability for the joint control of the multiple quality variables. The method consists of an afine transformation of the quality variables to achieve a univariate test statistic and has its foundation with the work of Nachlas and Barreau (1989). The analysis here is restricted to independent and normally distributed quality variables.

For the case in which only positive shifts in the quality variable means can occur, a general form of the afine transformation for any m quality variables is developed and analyzed. This general formulation is shown both mathematically and by example to be effective in providing the correct diagnosis at any combination of zero and critical shifts in the quality variable means at any desired level of sampling error probability. The effectiveness is diminished, however, at shifts combinations other than those defined in the sampling plan objectives as critical. This ineffectiveness becomes more pronounced as the number of quality variables is increased. Whether or not this shortcoming is significant in practical use depends upon the specific needs of a given quality control situation. The values of shifts defined as critical could be decreased to make the method more sensitive, but

doing so would involve a trade-off of increasing the sample size. If the needs of the situation are such that any shifts below the critical values are insignificant, then the method may be quite effective.

One possible reason for the large sampling error probabilities at shifts less than the critical amounts has to do with the assignment of the diagnosis to each of the decision or diagnosis intervals of the transform variable. As explained earlier, the diagnosis assigned to each interval in the single transform variable model is determined by the combination of critical and zero shifts which results in the mean value of the transform variable which lies at the midpoint of the interval. The problem is that this critical shift situation is not the only situation which will cause the mean of the transform variable to reside at that midpoint. As an example, consider the decision interval with with δ'_1 at its midpoint. The diagnosis associated with this interval is that the first quality variable has shifted. Now consider the mean of the transform variable when the second quality variable undergoes a shift to half of its critical value. In this case, the mean of the transform variable is also δ'_1. Similar results also occur for other quality variables and in other decision intervals. The critical shift therefore does not represent a unique most likely parameter set for the underlying distribution and subsequently the underlying quality situation. The assignment of the diagnosis to the combination of critical shifts is then somewhat arbitrary. The sampling error probabilities will be fixed at these critical shifts, but other combinations of shifts will have almost zero probability of being correctly detected.

For the case of quality variables whose means can undergo either positive or negative shifts, a second transform variable is added to provide complementary diagnosis capability. A method of assigning a specific diagnosis to each pair of decision intervals of the first and second transformed variables is developed. For two quality variables, this strategy provides correct diagnoses with sampling error probabilities less than the designed level when both quality variables have shifted means. The two transform variable strategy presented here is successful in making the method capable of detecting shifts in opposite directions. The method results in sampling error probabilities higher than the designed levels when both quality variables are at their target means. The ineffec-

tiveness at this in-control case is a significant shortcoming since this case should be expected to occur more frequently than any of the out-of-control cases in practice. For more than two quality variables, the method of assigning quality diagnoses to pairs of decision intervals fails. It appears that m transform variables would be necessary to adequately monitor m quality variables with the strategy developed in this paper.

The sample size requirements of the afine transformation method compare unfavorably with the total sample size required when a separate Shewhart chart is used for each variable. The power of the afine transformation method to detect general control problems without specific diagnoses is analyzed and appears to be quite high. This power is probably not the result of any intrinsic effectiveness of the method but instead due to the high sample sizes required to provide the diagnoses for the individual quality variables. Its power is compared to the power of Hotelling's T^2 statistic at the same sample size. The analysis of the T^2 is performed using its relationship to the noncentral F distribution. For the two and three quality variable cases, the power of the two methods is similar, but the T^2 test appears to be more powerful than the afine transformation method under the assumptions of this paper. The T^2 statistic cannot provide diagnoses about the individual quality variables, so a complete objective comparison between the two methods is not possible.

REFERENCES CITED

Abramowitz, M. and Stegun, I. A., Handbook of Mathematical Functions, (New York, Dover Publications, Inc.) 1965.

Fuchs, Camil and Ron Kenett, "Multivariate Tolerance Regions and *F*-Tests", Journal of Quality Technology , Vol. 19, No. 3, (July 1987), pp. 122-131.

Ghare, P. M. and Paul E. Torgersen, "The Multicharacteristic Control Chart", The Journal of Industrial Engineering, Vol. 19, No. 6, (June 1968), pp. 269-272.

Hotelling, H., "Multivariate Quality Control", in Techniques of Statistical Analysis, Eisenhart, Hastay, and Wallis, eds., (New York, McGraw-Hill), 1947.

Jackson, J. Edward, "Quality Control Methods for Several Related Variables", Technometrics, Vol. 1, No. 4, (Nov. 1959), pp. 359-377.

Jackson, J. Edward, "Quality Control Methods for Two Related Variables", Industrial Quality Control, Vol. 12, No. 7, (Jan. 1956), pp. 2-6.

Jackson, J. Edward and Ralph A. Bradley, "Sequential χ^2 and T^2 Tests", Annals of Mathematical Statistics , Vol. 32, No. 4, (Dec. 1961), pp. 1063-1077.

Montgomery, Douglas C., Introduction to Statistical Quality Control, (New York, John Wiley & Sons), 1985.

Nachlas, Joel A. and Alain Barreau, "A Single Control Chart for Multivariate Quality Control", Proceedings of the Twenty-sixth Annual Meeting of the Southeast Chapter of The Institute of Management Sciences, (Oct. 1989), pp.379-381.

Tiku, M., "Noncentral F-Distribution", Encyclopedia of Statistical Sciences, Samuel Kotz and Norman L. Johnson, eds., (New York, John Wiley & Sons), 1985.

Wald, Abraham, Sequential Analysis, (New York, John Wiley & Sons), 1947.

Appendix A

Probability Solution for Positive Shifts Model

P(correct diagnosis)

$$= P(\bar{y} \text{ in correct interval} \mid \mu = \mu_y)$$

$$= P\left[\delta'_1\left(\sum_{i=1}^{m}\lambda_i' 2^{i-1} - \frac{1}{2}\right) < \bar{y} < \delta'_1\left(\sum_{i=1}^{m}\lambda_i' 2^{i-1} + \frac{1}{2}\right) \mid \mu = \mu_y\right]$$

$$= P\left[\frac{\delta'_1\left(\sum_{i=1}^{m}\lambda_i' 2^{i-1} - \frac{1}{2}\right) - \mu_y}{\sigma_{\bar{y}}} < Z < \frac{\delta'_1\left(\sum_{i=1}^{m}\lambda_i' 2^{i-1} + \frac{1}{2}\right) - \mu_y}{\sigma_{\bar{y}}}\right]$$

$$= P\left[\frac{\delta'_1\left(\sum_{i=1}^{m}\lambda_i' 2^{i-1} - \frac{1}{2}\right) - \delta'_1\sum_{i=1}^{m}\lambda_i 2^{i-1}}{\frac{\delta'_1}{2Z_{1-\frac{\theta}{2}}}} < Z < \frac{\delta'_1\left(\sum_{i=1}^{m}\lambda_i' 2^{i-1} + \frac{1}{2}\right) - \delta'_1\sum_{i=1}^{m}\lambda_i 2^{i-1}}{\frac{\delta'_1}{2Z_{1-\frac{\theta}{2}}}}\right]$$

$$= P\left[\frac{\sum_{i=1}^{m}\lambda_i' 2^{i-1} - \frac{1}{2} - \sum_{i=1}^{m}\lambda_i 2^{i-1}}{\frac{1}{2Z_{1-\frac{\theta}{2}}}} < Z < \frac{\sum_{i=1}^{m}\lambda_i' 2^{i-1} + \frac{1}{2} - \sum_{i=1}^{m}\lambda_i 2^{i-1}}{\frac{1}{2Z_{1-\frac{\theta}{2}}}}\right]$$

$$= P\left[\left(\sum_{i=1}^{m}\lambda_i' 2^{i} - 1 - \sum_{i=1}^{m}\lambda_i 2^{i}\right)Z_{1-\frac{\theta}{2}} < Z < \left(\sum_{i=1}^{m}\lambda_i' 2^{i} + 1 - \sum_{i=1}^{m}\lambda_i 2^{i}\right)Z_{1-\frac{\theta}{2}}\right]$$

Appendix B

Numerical Approximation Subroutine for Normal Probabilities

```
C  SUBROUTINE TO RETURN P[Z<z] FOR STANDARD NORMAL
C Taken from an approximation
C  given by Abramowitz and Stegun (1965)
      FUNCTION CUMNRM(Z)
      DATA P0,A1,A2,A3/.33267,.4361836,-.1201676,.937298/
      BX = (Z**2)/2.
      ZX = (EXP(-1.*BX))/SQRT(6.28)
      TT = 1./(1.+P0*ABS(Z))
      PZ = 1.-ZX*(A1*TT+A2*TT**2+A3*TT**3)
      IF(Z.LT.0.) PZ = 1.-PZ
      CUMNRM = PZ
      RETURN
      END
```

Appendix C

Program for Positive Shifts Model Probabilities

```
C   PROGRAM TO CALCULATE PROB OF CORRECT DIAGNOSIS FOR POSITIVE
C     SHIFTS MODEL WITH M=3
      REAL PC(7,7,7),TABLE(7),ROW(7),COLUMN(7)
      OPEN(1,FILE='M3DPOS.DAT')
      ZTH = 1.96
      DO 10 I = 1,7
      XI = I - 1
      XLAM1 = XI*0.25
C  Set indicator variables
      IF(XLAM1.EQ.0.)THEN
         XLAMP1 = 0.
      ELSE
         XLAMP1 = 1.
      END IF
      TABLE(I) = XLAM1
      DO 10 J = 1,7
      XJ = J - 1
      XLAM2 = XJ*0.25
      IF(XLAM2.EQ.0.)THEN
         XLAMP2 = 0.
      ELSE
         XLAMP2 = 1.
      END IF
      ROW(J) = XLAM2
      DO 10 K = 1,7
      XK = K - 1
      XLAM3 = XK*0.25
      IF(XLAM3.EQ.0.)THEN
         XLAMP3 = 0.
      ELSE
         XLAMP3 = 1.
      END IF
      COLUMN(K) = XLAM3
C  Standardize the normal random quality variables
      ZL=(2.*XLAMP1+4.*XLAMP2+8.*XLAMP3-1.-2.*XLAM1-4.*XLAM2-8.*XLAM3)
     &*ZTH
      ZU=(2.*XLAMP1+4.*XLAMP2+8.*XLAMP3+1.-2.*XLAM1-4.*XLAM2-8.*XLAM3)
     &*ZTH
      PZU = CUMNRM(ZU)
      PZL = CUMNRM(ZL)
      IF((XLAMP1+XLAMP2+XLAMP3).EQ.3.)PZU=1
      IF((XLAMP1+XLAMP2+XLAMP3).EQ.0.)PZL=0.
C  Calculate probability of being in the right decision interval
      PC(I,J,K) = PZU - PZL
   10 CONTINUE
      WRITE(1,*)'                    PROBABILITY OF CORRECT DIAGNOSIS'
      WRITE(*,*)'                    PROBABILITY OF CORRECT DIAGNOSIS'
      WRITE(1,*)
      WRITE(*,*)
      DO 20 I = 1,7
```

```
      WRITE(1,150)TABLE(I)
      WRITE(*,150)TABLE(I)
      WRITE(1,100) (COLUMN(K),K=1,7)
      WRITE(*,100) (COLUMN(K),K=1,7)
      DO 20 J = 1,7
      R=ROW(J)
      WRITE(1,110)R,(PC(I,J,K),K=1,7)
      WRITE(1,*)'.SK 1'
      WRITE(*,110)R,(PC(I,J,K),K=1,7)
   20 CONTINUE
  100 FORMAT(T7,'LAM3',T13,7(1H#,F5.2),/,T2,'LAM2')
  110 FORMAT(1H  ,F5.2,1X,7(1H#,F6.4))
  150 FORMAT(//,T26,'LAM1 = ',F8.4,/)
      STOP
      END
```

Appendix D

Program for Two Transform Variable Model Probabilities

```
C     PROBABILITIES OF BEING IN CORRECT DIAGNOSIS INTERVALS
C        FOR TWO TRANSFORM VARIABLE MODEL
C     Computation uses combination of Simpson's rule for numerical
C       integration and an approximation for the cumulative normal
      DIMENSION X1SH(6),X2SH(6),PC(6,6),Y1U(9),Y1L(9),Y2U(9),Y2L(9)
      DATA PI/3.141593/
C  Define normal distribution density function
      F(X,XMU,VAR)=(1./SQRT(2.*PI*VAR))*EXP(-1.*(X-XMU)**2/(2.*VAR))
      N=64
      XN=N
      WRITE(*,*)'INPUT DELTA1 AND DELTA2'
      READ(*,*)DELT1,DELT2
      GAM2=DELT2/DELT1
      WRITE(*,*)'INPUT Z OF THETA/2'
      READ(*,*)ZTHET
C  Compute sample size requirements
      SAMP=(2.*ZTHET/DELT1)**2*(1+4./GAM2**2)
      VAR1=(1./DELT1)**2/SAMP
      VAR2=(1./DELT2)**2/SAMP
      SD2=SQRT(VAR2)
      OPEN(1,FILE='MIDP.DAT')
      OPEN(2,FILE='X1NX2P.DAT')
      DO 2 IK=1,9
      READ(1,*)Y1L(IK),Y1U(IK),Y2L(IK),Y2U(IK)
    2 CONTINUE
      N1=-1
      N2=1
      DO 1 JO=1,6
      X1SH(JO) = FLOAT(N1*JO)/4.
    1 CONTINUE
      DO 4 LO=1,6
      X2SH(LO) = FLOAT(N2*LO)/4.
    4 CONTINUE
      DO 55 JO=1,6
      DO 55 LO=1,6
      PCSUM=0.
      XLAM1=X1SH(JO)
      XLAM2=X2SH(LO)
      DO 66 I4=1,9
CCC Perform Simpson's rule on left half of diamond
      A=0.5*(Y1L(I4)+Y2L(I4))
      B=A + 0.5
      IF(Y2L(I4).LT.-3.)A=A-2.0
      DELX=(B-A)/XN
      Y2UP = -0.5*Y2L(I4)
      Y2DO = .5*Y1L(I4)
      Y2UPP=0.5*Y1U(I4)
      ZU=(Y2UP+.5*(A) - XLAM2)/SD2
      IF(Y2L(I4).LT.-3.0)ZU=(Y2UPP-.5*(A) - XLAM2)/SD2
      ZL=(Y2DO-.5*(A) - XLAM2)/SD2
```

```
      PHI=CUMNRM(ZU) - CUMNRM(ZL)
      FA = F(A,XLAM1,VAR1)*PHI
      SUMODD=0.
C  Begin summation of odd terms for Simpson's rule
      DO 10 I=1,N-1,2
      XI=I
      ZU=(Y2UP+.5*(A+XI*DELX) - XLAM2)/SD2
      IF(Y2L(I4).LT.-3.0)ZU=(Y2UPP-.5*(A+XI*DELX) - XLAM2)/SD2
      ZL=(Y2DO-.5*(A+XI*DELX) - XLAM2)/SD2
      PHI=CUMNRM(ZU) - CUMNRM(ZL)
      SUMODD=SUMODD+ F((A+XI*DELX),XLAM1,VAR1)*PHI
   10 CONTINUE
      SUMEVE=0.
C  Begin summation of even terms for Simpson's rule
      DO 20 I=2,N-2,2
      XI=I
      ZU=(Y2UP+.5*(A+XI*DELX) - XLAM2)/SD2
      IF(Y2L(I4).LT.-3.0)ZU=(Y2UPP-.5*(A+XI*DELX) - XLAM2)/SD2
      ZL=(Y2DO-.5*(A+XI*DELX) - XLAM2)/SD2
      PHI=CUMNRM(ZU) - CUMNRM(ZL)
      SUMEVE=SUMEVE+ F((A+XI*DELX),XLAM1,VAR1)*PHI
   20 CONTINUE
      FB=0.
      PROB1=(1./3.)*DELX*(FA+4.*SUMODD+2.*SUMEVE +FB)
CCC  Perform same computations on second half of diamond
      A=B
      B=A+.5
CC  MOVE OUT UPPER LIMIT TO ARBITRARILY HIGH NUMBER SINCE IN
CC    PRACTICE THERE IS NO UPPER DECISION LIMIT FOR Y1
      DELX=(B-A)/XN
C     YOLDUP=Y2UP
      Y2UP = .5*Y1U(I4)
      Y2DO = -.5*Y2U(I4)
      ZU=(Y2UP-.5*(A) - XLAM2)/SD2
C     IF(Y1U(I4).GT.3.0)ZU=(YOLDUP+.5*(A) - XLAM2)/SD2
      ZL=(Y2DO+.5*(A) - XLAM2)/SD2
      PHI=CUMNRM(ZU) - CUMNRM(ZL)
      FA = F(A,XLAM1,VAR1)*PHI
      SUMODD=0.
      DO 30 I=1,N-1,2
      XI=I
      ZU=(Y2UP-.5*(A+XI*DELX) - XLAM2)/SD2
C     IF(Y1U(I4).GT.3.0)ZU=(YOLDUP+.5*(A+XI*DELX) - XLAM2)/SD2
      ZL=(Y2DO+.5*(A+XI*DELX) - XLAM2)/SD2
      PHI=CUMNRM(ZU) - CUMNRM(ZL)
      SUMODD=SUMODD+ F((A+XI*DELX),XLAM1,VAR1)*PHI
   30 CONTINUE
      SUMEVE=0.
      DO 40 I=2,N-2,2
      XI=I
```

```
      ZU=(Y2UP-.5*(A+XI*DELX) - XLAM2)/SD2
C     IF(Y1U(I4).GT.3.0)ZU=(YOLDUP+.5*(A+XI*DELX) - XLAM2)/SD2
      ZL=(Y2DO+.5*(A+XI*DELX) - XLAM2)/SD2
      PHI=CUMNRM(ZU) - CUMNRM(ZL)
      SUMEVE=SUMEVE+ F((A+XI*DELX),XLAM1,VAR1)*PHI
   40 CONTINUE
      FB=0.
C  Simpson's rule
      PROB2=(1./3.)*DELX*(FA+4.*SUMODD+2.*SUMEVE +FB)
      PROB = PROB1+PROB2
C  Total probability is sum of the two halves of diamond
      PCSUM = PCSUM+PROB
   66 CONTINUE
      PC(JO,LO)=PCSUM
      WRITE(*,*)JO,LO,PCSUM
   55 CONTINUE
      WRITE(2,*)'          PROB CORRECT DIAGNOSIS TWO Y S'
      WRITE(2,*)
      WRITE(2,100)(X2SH(LO), LO=1,6)
      WRITE(2,105)
      DO 33 JO=1,6
      WRITE(2,110)X1SH(JO),(PC(JO,LO), LO=1,6)
   33 CONTINUE
  105 FORMAT(1X,70('_'))
  100 FORMAT(1X,'L1\L2|',6F10.2)
  110 FORMAT(1X,F5.2,'|',6F10.4)
      STOP
      END
```

Appendix E

Program to Calculate Power for Afine Transformation

```
C   PROGRAM TO CALCULATE PROB OF CONTROL SIGNAL
C     FOR M=3  AFINE TRANSFORMATION MODEL
      REAL PC(6,6,6),TABLE(6),ROW(6),COLUMN(6)
      OPEN(1,FILE='M3DPOW.DAT')
      ZTH = 1.96
      DO 10 I = 1,6
      XI = I - 1
      XLAM1 = XI*0.20
      TABLE(I) = XLAM1
      DO 10 J = 1,6
      XJ = J - 1
      XLAM2 = XJ*0.20
      ROW(J) = XLAM2
      DO 10 K = 1,6
      XK = K - 1
      XLAM3 = XK*0.20
      COLUMN(K) = XLAM3
      ZL = (-1. - 2.*XLAM1 - 4.*XLAM2 - 8.*XLAM3)*ZTH
      ZU = (1. - 2.*XLAM1 - 4.*XLAM2 - 8.*XLAM3)*ZTH
      PZU = CUMNRM(ZU)
      PZL = CUMNRM(ZL)
      PC(I,J,K) = 1. - (PZU - PZL)
   10 CONTINUE
      WRITE(*,*)'  PROBABILITY OF CONTROL PROBLEM SIGNAL'
      WRITE(1,*)
      WRITE(*,*)
      DO 20 I = 1,6
      WRITE(1,150)TABLE(I)
      WRITE(*,150)TABLE(I)
      WRITE(1,100) (COLUMN(K),K=1,6)
      WRITE(*,100) (COLUMN(K),K=1,6)
      DO 20 J = 1,6
      R=ROW(J)
      WRITE(1,110)R,(PC(I,J,K),K=1,6)
      WRITE(1,222)
  222 FORMAT(5H.SK 1)
      WRITE(*,110)R,(PC(I,J,K),K=1,6)
   20 CONTINUE
  100 FORMAT(T7,'LAM3',T13,6(1H#,F5.2),/,T2,'LAM2')
  110 FORMAT(1H  ,F5.2,1X,6(1H#,F6.4))
  150 FORMAT(//,T26,'LAM1 = ',F8.4,/)
      STOP
      END
```

Appendix F

SAS Program to Calculate Power of Hotelling's Statistic

```
* SAS PROGRAM TO CALCULATE POWER OF T-SQUARED USING THE NONCENTRAL F
*    FOR DELTA1=1.2   DELTA2=1.5;
OPTIONS LINESIZE=72;
TITLE 'D1=1.2 D2=1.5';
DATA;
INPUT FCRIT DF1 DF2 NC;
POW = 1 - FPROB ( FCRIT , DF1 , DF2 , NC );
DROP FCRIT DF1 DF2 NC;
CARDS;
    3.259   2  36     .000
    3.259   2  36    3.420
    3.259   2  36   13.680
    3.259   2  36   30.780
    3.259   2  36   54.720
    3.259   2  36   85.500
    3.259   2  36    2.189
    3.259   2  36    5.609
    3.259   2  36   15.869
    3.259   2  36   32.969
    3.259   2  36   56.909
    3.259   2  36   87.689
    3.259   2  36    8.755
    3.259   2  36   12.175
    3.259   2  36   22.435
    3.259   2  36   39.535
    3.259   2  36   63.475
    3.259   2  36   94.255
    3.259   2  36   19.699
    3.259   2  36   23.119
    3.259   2  36   33.379
    3.259   2  36   50.479
    3.259   2  36   74.419
    3.259   2  36  105.199
    3.259   2  36   35.021
    3.259   2  36   38.441
    3.259   2  36   48.701
    3.259   2  36   65.801
    3.259   2  36   89.741
    3.259   2  36  120.521
    3.259   2  36   54.720
    3.259   2  36   58.140
    3.259   2  36   68.400
    3.259   2  36   85.500
    3.259   2  36  109.440
    3.259   2  36  140.220
     ;
PROC PRINT;
```

Vita

Marshall Alan Wise was born to Marshall and Susan Wise in Lorain, Ohio on May 6, 1966. He moved with his family to Roanoke, Virginia in 1977 and graduated from Cave Spring High School of Roanoke in 1984. He earned a Bachelor of Science degree in Industrial Engineering and Operations Research (IEOR) from the Virginia Polytechnic Institute and State University in May of 1988 and is seeking a Master of Science in IEOR with a concentration in Operations Research from that institution.