SIGNAL RECONSTRUCTION FROM DISCRETE-TIME WIGNER DISTRIBUTION

by

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(ABSTRACT)

Wigner distribution is considered to be one of the most powerful tools for time-frequency analysis of non-stationary signals. Wigner distribution is a bilinear signal transformation which provides two dimensional time-frequency characterization of one dimensional signals. Although much work has been done recently in signal analysis and applications using Wigner distribution, not many synthesis methods for Wigner distribution have been reported in the literature.

This thesis is concerned with signal synthesis from discrete-time Wigner distribution and from discrete-time pseudo-Wigner distribution and their applications in noise filtering and signal separation. Various algorithms are developed to reconstruct signals from the modified or specified Wigner distribution and pseudo-Wigner distribution which generally do not have a valid Wigner distributions or valid pseudo-Wigner distribution structures. These algorithms are

successfully applied to the noise filtering and signal separation problems.

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TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

1.0 INTRODUCTION

In signal analysis, a stationary signal can be fully described either in the time domain f(t) or in the frequency domain $F(\omega)$. Because the stationary signal spectral contents do not change with time, f(t) and $F(\omega)$ each give complete, equivalent information of a signal.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t} dt \; . \tag{1.1}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega \; . \tag{1.2}$$

However, these two monodimensional representations (in time t or in frequency ω) do not always display sufficient information for signal processing and analysis, especially when the signal spectral contents vary with time (non-stationary signal). This problem has received considerable attention in recent years. Different approaches have been used to deal with this situation. One of them treats the non-stationary situation as a concatenation of quasi-stationary ones, such as short-time Fourier transform approach [1]. In other words, windowing is applied such that the signal is assumed to be stationary over the duration of the window. The Fourier transform of this windowed signal can be used to characterize the energy distribution of the signal at a time that is given by the center of the window. Sliding the window over the signal allows us to display the variations of this distribution with time. This yields the spectrogram of the signal. However, short-time analyses are known to suffer drawbacks, especially according to the quasi-stationary assumptions. The short-time Fourier transform then faces the well-known compromise of either widening the window for a better resolution, or shortening this duration to assume a better quasi-stationary but suffering then a poorer resolution. Another approach is to look for a true time-frequency representation. Various definitions of time-frequency representation have been proposed [2-5], each with its own merits and drawbacks. A recent series of papers by Claasen and Mecklenbrauker [6,7] has shown that the Wigner distribution and its evolutional version (the pseudo-Wigner distribution) were good candidates. The Wigner distribution is defined as [6]:

$$W_{f,g}(t,\omega) = \int_{-\infty}^{\infty} e^{-j\omega\tau} f(t+\tau/2)g^{*}(t-\tau/2)d\tau \ . \qquad (1.3)$$

Wigner distribution has many important properties which make it very attractive for time-frequency analysis of non-stationary signals and systems. Many other time-frequency representations, such as the ambiguity function used in radar and sonar, and the spectrogram used in speech enhancement and compression, can be derived from it via linear transformations [8]. There are various methods to evaluate the Wigner distribution. One efficient method is based on digital signal processing which is to convert an analog signal into a discrete-time signal and to determine a discrete-time version of the Wigner distribution. The discrete-time Wigner distribution is defined as [7]:

$$W_{f,g}(n,\theta) = 2 \sum_{k=-\infty}^{\infty} e^{-j2k\theta} f(n+k) g^{*}(n-k) \ . \tag{1.4}$$

The determination of this distribution function involves infinite summation which requires the signals to be known for all time. In practical implementations it is necessary to window the signals before the discrete-time Wigner distribution can be computed. This leads to the definition of the pseudo-Wigner distribution. Pseudo-Wigner distribution is a filtered version of the Wigner distribution of the signals, where filtering takes place in the frequency variable. The discrete-time pseudo-Wigner distribution is defined as [7]:

$$PWD_{f,g}(n,\theta) = 2 \sum_{k=-L}^{L} f(n+k) h(k) g^{*}(n-k) h^{*}(-k) e^{-j2k\theta} \ . \tag{1.5}$$

Utilizing Wigner distribution for signal analysis and applications has rapidly increased. This causes the demand of developing the synthesis technique for Wigner distribution. So far, few synthesis methods have been reported in the literature. In this thesis, the synthesis algorithms for discrete-time Wigner distribution and discrete-time pseudo-Wigner distribution are investigated. These synthesis procedures can be used for noise filtering and signal separation in applications.

Signal synthesis from Wigner distribution is a least squares approximation problem [23]. Three algorithms of signal synthesis from discrete-time Wigner distribution are investigated in this thesis. The first algorithm is a least squares procedure using basis functions expansion. Two sets of basis functions are investigated for signal synthesis applications. The second algorithm is first developed by Boudreaux-Bartels and Parks [9] using least squares procedures. Since the even-indexed and odd-indexed samples are not coupled, this synthesis procedure leads us to find the even-indexed and odd-indexed sequences separately. Wigner distribution gives a description of an one-dimensional signal f(n) by a two-dimensional pattern. Inverse Fourier transform of Wigner distribution gives a two-dimensional discrete-time function. This two-dimensional discrete-time sequence is recognized to be the outer product of two one-dimensional sequences. The third algorithm then involves the factorization procedure where the desired one-dimensional signal sequence can be recovered as the solution of this factorization problem.

Due to the fact that even-indexed and odd-indexed samples are not coupled, signal recovery from discrete-time Wigner distribution is possible up to a constant factor, but it requires different procedures for even-indexed and odd-indexed samples. In general, it is not possible to determine both even-indexed and odd-indexed samples of a signal up to the same constant factor. Combining the even-indexed and odd-indexed sequences is then not unique unless we assume that L samples of signal f(n) are known with $L \geq 2$. If the discrete signal f(n) is obtained by sampling a continuous signal with a sampling rate of at least twice the Nyquist rate, the even-indexed or the odd-indexed samples fully describe the signal. Two approaches to combine the even-indexed and odd-indexed samples are discussed in this thesis.

In a number of practical applications [10,11], it is desirable to modify the pseudo-Wigner distribution and then estimate the processed signal from the modified pseudo-Wigner distribution. In speech, for example, the original signals are often corrupted with noise. We use a smoothed pseudo-Wigner estimator to smooth the noise in a time-frequency plane and then the signal is reconstructed from the modified version of the pseudo-Wigner distribution. As another example, in the multicomponent signal case, the bilinear structure of the Wigner distribution creates cross-terms without any real physical significance. In order to separate the signal from the cross-terms, we use a smoothing process to reduce the cross-terms in time-frequency plane; then signals are estimated from this modified version of the pseudo-Wigner distribution. For application purposes, two synthesis algorithms for pseudo-Wigner distribution are developed in this thesis. Both algorithms are formulated in the similar way as the third synthesis method for Wigner distribution which involves the outer product approximation. The first method does not involve overlapped segments. The second synthesis method is an overlapping method where a section of unknown samples can be determined by using a portion of known samples determined in a previous section. This process continues as the reconstruction of a new section makes the reconstruction of the next overlapping section possible. This procedure is motivated by an overlapped method used in signal synthesis from Short Time Fourier Transform [12].

This thesis consists of eight chapters. Chapter 2 gives a brief review of Wigner distribution. Since most practical applications of the Wigner distribution will involve digital processing of sampled data, a numerical evaluation of the time-frequency distribution is required. The sampling techniques and aliasing effects are discussed in Chapter 3 [14,15]. Chapter 4 is devoted to signal synthesis from Wigner distribution. Three synthesis algorithms are presented there.

The concept of pseudo-Wigner distribution and its properties are discussed in Chapter 5. Two synthesis methods for pseudo-Wigner distribution are introduced in Chapter 6. Applications are discussed in Chapter 7. Chapter 8 concludes this thesis.

2.0 DISCRETE-TIME WIGNER DISTRIBUTION AND ITS PROPERTIES

Discrete-time Wigner distribution is a bilinear signal transformation which provides a time-frequency characterization of a signal. It has some important properties which make it attractive for analysis of non-stationary signals and systems. The Wigner distribution was proposed by Wigner for application in quantum mechanics by means of joint distributions of quasi-probabilities in 1932 [2]. The theory of Wigner distribution has been recently reviewed from a signal processing view point by Claasen and Mecklenbrauker [6,7,8,13].

2.1 DEFINITION OF WIGNER DISTRIBUTION

The Wigner distribution can be evaluated both from the discrete time signals and from the Fourier transform of the signals. The cross-Wigner distribution of two signals f(n) and g(n) is defined by [7]:

$$W_{f,g}(n,\theta) = 2 \sum_{k=-\infty}^{\infty} e^{-j2k\theta} f(n+k)\, g^{*}(n-k) \tag{2.1}$$

A similar expression of Wigner distribution for spectra $F(\theta)$ and $G(\theta)$ can be obtained by:

$$W_{F,G}(\theta,n) = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{j2n\omega}\, F(\theta+\omega)\, G^*(\theta-\omega)\, d\omega. \qquad (2.2)$$

By letting $g(n) = f(n)$ or $G(\theta) = F(\theta)$, we obtain the auto-Wigner distribution $W_f(n,\theta)$. The two distributions (2.1) and (2.2) have the relation:

$$W_{F,G}(\theta,n) = W_{f,g}(n,\theta). \qquad (2.3)$$

2.2 PROPERTIES OF WIGNER DISTRIBUTION

The properties of Wigner distribution reviewed here are based on the articles by Claasen and Mecklenbrauker [6,7] and a very detail description of the signal theoretical background of the Wigner distribution can be found in those references.

P1: The Wigner distribution is real for both real or complex discrete-time signal $f(n)$.

$$W_f(n,\theta) = W_f^*(n,\theta) \qquad (2.4)$$

P2: Wigner distribution is discrete in time domain n and continuous in frequency domain θ. With respect to θ, the function is periodic with period π.

$$W_{f,g}(n,\theta) = W_{f,g}(n,\theta+\pi) \tag{2.5}$$

P3: Wigner distribution has the same time support as signals, i.e.,

$$\text{if} \quad f(n) = g(n) = 0, \; n < n_a \text{ or } n > n_b \; .$$

$$\text{then} \quad W_{f,g}(n,\theta) = 0, \; n < n_a \text{ or } n > n_b. \tag{2.6}$$

P4: The band limited characteristics of discrete-time signal on the Wigner distribution is different from the continuous-time case due to the fact that the Wigner distribution for continuous-time case is not periodic, while the Wigner distribution for the discrete-time signal is periodic in θ with a period π, i.e.,

$$\text{if} \quad F(\theta) = G(\theta) = 0, \; \theta_a < \theta < \theta_b \text{ and } \theta_b - \theta_a > \pi$$

$$\text{then} \quad W_{f,g}(n,\theta) = 0, \; \theta_a < \theta < \theta_b - \pi. \tag{2.7}$$

P5: The time shift of the signal $f(n)$ yields the same time shift for the Wigner distribution. Similarly, modulation of a signal $f(n)$, which corresponds to a frequency shift of

$F(\theta)$, yields the same frequency shift for the Wigner distribution, i.e.,

$$\text{if} \quad f(n) = g(n-k),$$
$$\text{then} \quad W_f(n,\theta) = W_g(n-k,\theta). \tag{2.8}$$

Similarly,

$$\text{if} \quad f(n) = g(n)\, e^{jn\omega},$$
$$\text{then} \quad W_f(n,\theta) = W_g(n,\theta-\omega). \tag{2.9}$$

P6: The Wigner distribution of a modulated signal is a convolution in frequency domain, i.e.,

$$\text{if } f(n) = g(n)\, m(n),$$
$$\text{then} \quad W_f(n,\theta) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} W_g(n,\omega)\, W_m(n,\theta-\omega)\, d\omega. \tag{2.10}$$

P7: An inverse Fourier transform yields a two-dimensional time function $y(n,m)$,

$$y(n,m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\theta m}\, W_f(n,\theta/2) d\theta, \tag{2.11}$$

or this can be written in the form

$$y(n_1,n_2) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j(n_1-n_2)\theta} W_f\left(\frac{n_1+n_2}{2}, \theta\right) d\theta. \quad (2.12)$$

If $W_f(n,\theta)$ is a valid Wigner distribution, then $y(n_1,n_2) = f(n_1)f^*(n_2)$. That is, the necessary and sufficient condition of $W_f(n,\theta)$ that describes signal f(n) is that the matrix $[y(n_1,n_2)]$ is a rank 1 matrix. Then we can rewrite (2.12) as:

$$\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j(n_1-n_2)\theta} W_f\left(\frac{n_1+n_2}{2}, \theta\right) d\theta = f(n_1)f^*(n_2). \quad (2.13)$$

By letting $n_1 = 2n$, $n_2 = 0$, we obtain:

$$f(2n)\ f^*(0) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j2n\theta}\ W_f(n,\theta)\ d\theta. \quad (2.14)$$

Similarly if $n_1 = 2n-1$, $n_2 = 1$ gives

$$f(2n-1)\ f^*(1) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{j2(n-1)\theta}\ W_f(n,\theta)\ d\theta. \quad (2.15)$$

From the above two equations, we find the synthesis procedures of reconstructing the even and odd numbered samples of signal f(n) are different. This is due to the fact that in the Eq. (2.1) only the even or the odd samples occur simultaneously.

P8: Integral Wigner distribution over one period in frequency domain is equal to the instantaneous signal power,

$$\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} W_f(n,\theta)\, d\theta = |f(n)|^2. \tag{2.16}$$

P9: The summation of the Wigner distribution defined in Eq. (2.1) over the time variable at certain frequency θ does not yield the energy density spectrum of f(n) at this frequency. Instead we have,

$$\sum_{n=-\infty}^{\infty} W_f(n,\theta) = |F(\theta)|^2 + |F(\theta+\pi)|^2. \tag{2.17}$$

The aliasing term in (2.17) will cause some problems later in various equations. In next section, we will discuss the sampling methods to avoid aliasing if the signal f(n) is the sampled data.

3.0 SAMPLING TECHNIQUES AND ALIASING EFFECTS

Although the Wigner distribution for continuous-time signal $f_c(t)$ (we use $f_c(t)$ to denote a continuous time signal hereafter) has a number of attractive properties, the evaluation for continuous-time Wigner distribution is very difficult and may even be impossible. One solution is to sample the continuous-time signal. The discrete-time Wigner distribution can be determined efficiently by digital computer or by other digital hardwares. The determination of the discrete-time Wigner distribution involves infinite summation which requires the signal to be known for all time. In practical implementations it is necessary to window the signal before the discrete-time Wigner distribution can be computed. This leads to the pseudo-Wigner distribution, which will be discussed in Chapter 5.

If the continuous-time signal $f_c(t)$ is band-limited,

$$F(\omega) = 0, \quad |\omega| > \omega_c,$$

the Wigner distribution can be completely determined by the samples $\{f(nT)\}$, where $\{f(nT)\}$ is a sequence of samples of the continuous-time signal $f_c(t)$ taken periodically with sampling period T.

$$W_f(nT,\omega) = 2T \sum_{k=-\infty}^{\infty} e^{-j2\omega kT}[f(n+k)T][f^*(n-k)T],$$
$$\text{for } T \le \pi/2\omega_c. \qquad (3.1)$$

The continuous-time Wigner distribution can be recovered by using the interpolation:

$$W_f(t,\omega) = \sum_{n=-\infty}^{\infty} W_f(nT,\omega) \frac{\sin \pi(t/T - n)}{\pi(t/T - n)},$$
$$\text{for } T \le \pi/2\omega_c. \qquad (3.2)$$

When T = 1 of (3.1), we can obtain the Wigner distribution for a discrete-time signal f(n) with a unit sample period:

$$W_f(n,\theta) = 2 \sum_{k=-\infty}^{\infty} e^{-j2k\theta} f(n+k)\, f^*(n-k). \qquad (3.3)$$

The problem with discrete-time Wigner distribution is that it suffers from the effect of aliasing. The aliasing problem occurs because of the fact that $W_f(n,\theta)$ has a period π whereas $F(\theta)$ has a period of 2π.

In order to illustrate the effect of aliasing on Wigner distribution, we present two examples below. For convenience, we restrict ourselves to real signals only.

As a first example, Fig. 1 presents discrete-time Wigner distribution for sampled continuous-time FM signal $f_c(t)$:

$$f_c(t) = \cos\,[w_0 t + \beta \sin w_m t] \quad 100 \le t \le 200, \qquad (3.4)$$
$$= 0 \quad \text{otherwise.}$$

where $\omega_0 = \pi/30$, $\omega_m = \pi/150$, $\beta = 107$. We sample the signal $f_c(t)$ at a rate higher than the Nyquist rate, but lower than twice of the Nyquist rate. As a second example, Fig. 2 is the discrete-time Wigner distribution of a linear chirp signal which is given by

$$g_c(t) = \cos\,(\alpha t^2) \quad 0 \le t \le 240, \qquad (3.5)$$
$$= 0 \quad \text{otherwise.}$$

where $\alpha = \pi/500$. We again sample the chirp signal at a rate higher than the Nyquist rate but lower than twice of the Nyquist rate. In both examples, frequency and time increments are respectively $\Delta\omega = \pi/121$ and $\Delta n = 5$. It can be seen that severe aliasing contributions occur because Wigner distribution repeats itself around $\theta = \pi$. (Since Wigner distribution involves infinite duration, the Wigner distribution we computed above is actually the pseudo-Wigner distribution.)

There are several methods to avoid the aliasing:

OVERSAMPLING: The signal is sampled with at least twice the Nyquist rate; that is, the discrete signal has a Fourier transform $F(\theta)$ that is zero over an interval covering at least half of its period, i.e.,

$$F(\theta) = 0 \qquad \pi/2 < |\theta| < \pi. \tag{3.6}$$

Now we want to test the two signals in example 1 and 2 with a sampling rate higher than twice the Nyquist rate. Fig 3 and 4 shows that the Wigner distribution has no aliasing contributions for any signal that samples at least twice of the Nyquist rate, where Fig. 3 corresponds to Fig. 1, and Fig. 4 corresponds to Fig. 2.

INTERPOLATION: The second method involves interpolation. An analog waveform $f(n) = f_c(nT)$ is sampled at the Nyquist rate and then interpolated by a factor 2, i.e.,

$$g(n) = f_c(nT') = f_c(nT/2). \tag{3.7}$$

Clearly, $g(n) = f(n/2)$ for $n = 0, \pm 2, \pm 4, \ldots$ but we must fill in the unknown samples for all other values of n by an interpolation process. This can be done using a digital filter [1].

ANALYTIC SIGNAL: For an analytic signal, we can avoid the aliasing by using data sampled at the Nyquist rate. This is because the frequency spectrum of the analytic signal vanishes for negative frequencies. A discrete-time analytic signal is defined in [16] by

$$f_a(n) = f(n) + j\,\hat{f}(n) \tag{3.8}$$

where f(n) is real and $\hat{f}$ is the discrete Hilbert transform of f, defined by

$$\hat{f}(n) = \sum_{m \neq n} f(m)\, \frac{\sin^2(m-n)\pi/2}{(m-n)\,\pi/2} \tag{3.9}$$

The relation between the spectrum $F(\theta)$ of the original signal f(n) and the spectrum $F_a(\theta)$ is given by:

$$\begin{aligned} F_a(\theta) &= 2F(\theta) && 0 < \theta < \pi, \\ &= F(\theta) && \theta = 0, \\ &= 0 && -\pi < \theta < 0. \end{aligned} \tag{3.10}$$

By evaluating the Wigner distribution from the analytic signal rather than the signal itself, we avoid the interference between positive and negative frequency components.

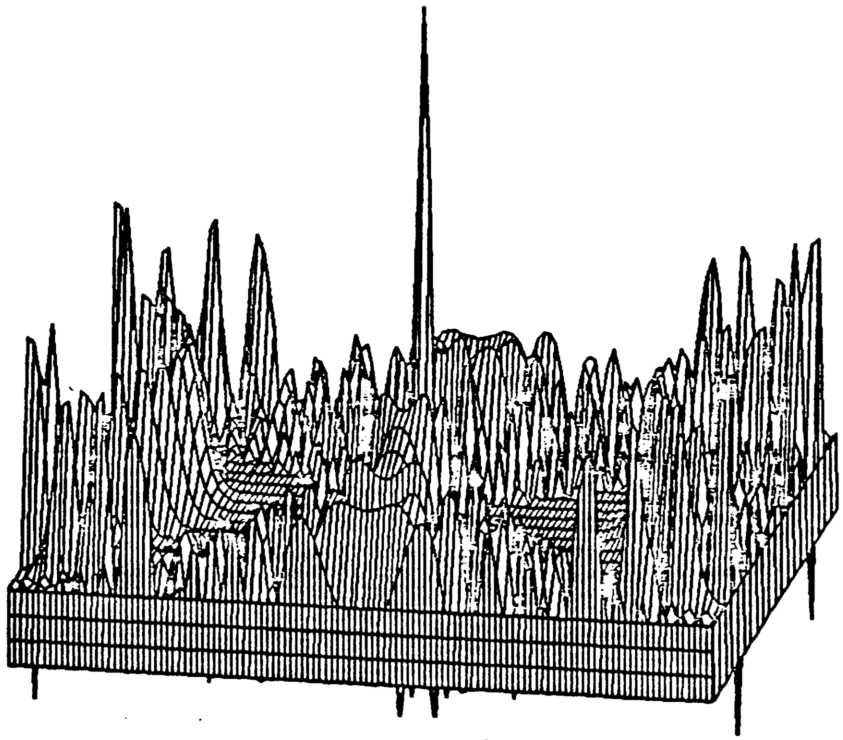

Figure 1. The WD of Eq.(3.4) with aliasing.

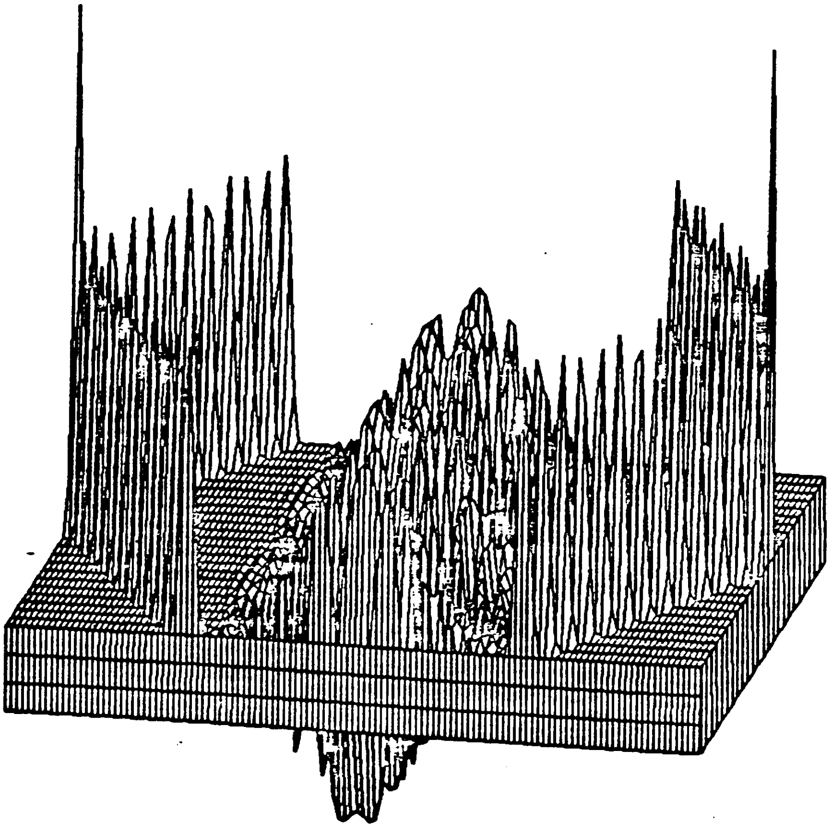

Figure 2. The WD of Eq.(3.5) with aliasing.

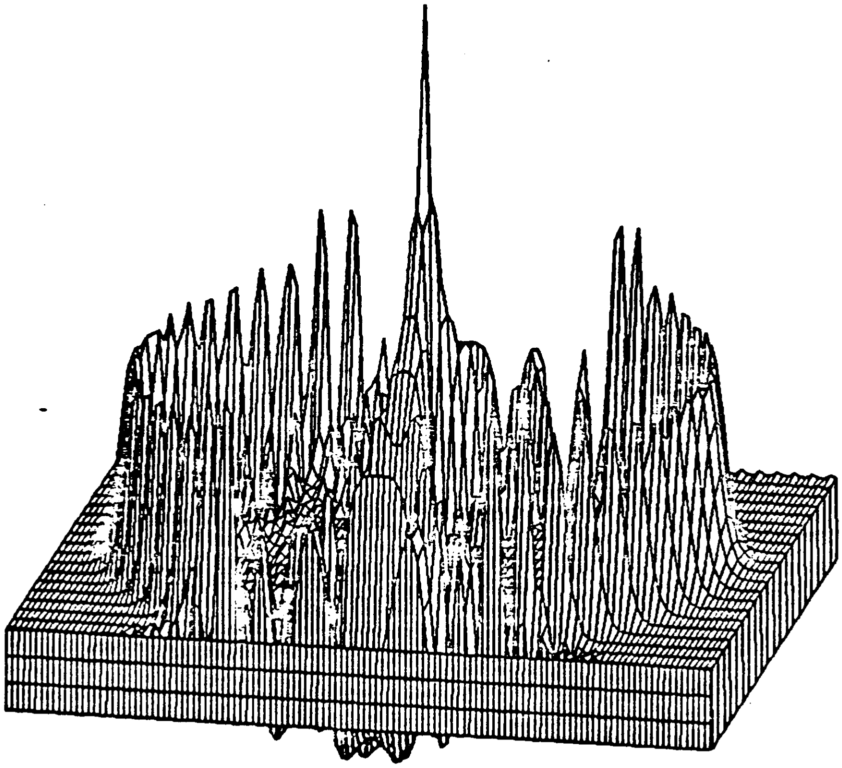

Figure 3. The WD of Eq.(3.4) with no aliasing.

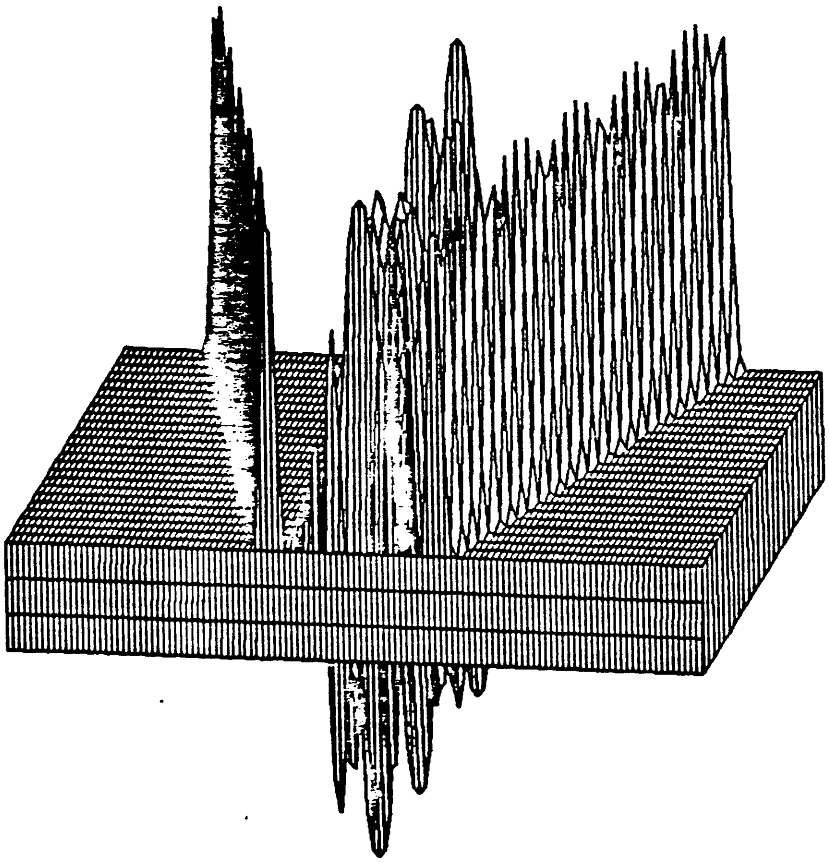

Figure 4. The WD of Eq.(3.5) with no aliasing.

4.0 SIGNAL SYNTHESIS FROM WIGNER DISTRIBUTION

This chapter is concerned with signal synthesis from Wigner distribution. Three algorithms are investigated for determining the discrete-time sequence of which the Wigner distribution best approximates a modified or specified Wigner distribution. In section 4.1 we derive a least-square synthesis procedure using basis functions expansion. In section 4.2 we discuss another least-square synthesis method in which the error minimization procedure can be broken up into two separable time-domain minimization procedures that depend upon only the even indexed samples or the odd indexed samples for the optimal solution. A third algorithm is given in section 4.3. This is an ad-hoc synthesis procedure which involves factorization by using outer product.

4.1 LEAST-SQUARE SYNTHESIS USING BASIS FUNCTIONS EXPANSION

In this section we present a least square fitting algorithm for signal reconstruction for modified Wigner distribution. The synthesis algorithm developed here follows the synthesis procedure of radar ambiguity function described in [17].

4.1.1 BASIS FUNCTIONS AND THEIR PROPERTIES

Wigner distribution can be expanded according to cross-Wigner distribution associated with a set of special orthonormal basis. Two sets of basis functions are investigated in this section. Related work can be found in [22]. We suppose that for a discrete-time signal f(n) there exists a set of basis functions $\{\phi_i(n)\}$ which satisfy:

$$f(n) = \sum_{i=-L}^{L} a_i \phi_i(n). \tag{4.1}$$

The basis functions are assumed to vanish outside an interval $-L \le n \le L$ and to be orthonormal:

$$\begin{aligned} &\phi_i(n) = 0 \qquad |n| > L, \\ &\sum_{n=-L}^{L} \phi_i(n)\ \phi_j^*(n) = \delta_{i,j}. \end{aligned} \tag{4.2}$$

Because the even-indexed and the odd-indexed values of a cross-Wigner distribution of the basis functions are not coupled, the sequence $\{\phi_i(n)\}$ is required to satisfy an additional constraint below in order to obtain an orthogonal set of the cross-Wigner distribution of the basis functions,

$$\sum_{n,m=-L}^{L} \phi_i(n+m)\phi_k^*(n+m)\phi_\ell(n-m)\phi_p^*(n-m) = \delta_{i,k}\delta_{\ell,p} \tag{4.3}$$

Let's illustrate the sequence of basis functions by two examples.

Example 1:

$$\phi_i(n) = \begin{cases} \delta_{i,n} & -L \le n \le L \\ 0 & \text{otherwise.} \end{cases} \tag{4.4}$$

$$\sum_{n=-L}^{L} \phi_i(n)\ \phi_j^*(n) = \sum_{n=-L}^{L} \delta_{i,n}\delta_{j,n} = \delta_{i,j}, \tag{4.5}$$

which satisfies (4.2), the basis functions are orthonormal. And

$$\sum_{n=-L}^{L}\sum_{m=-L}^{L} \phi_i(n+m)\ \phi_k^*(n+m)\ \phi_\ell(n-m)\ \phi_p^*(n-m)$$

$$= \sum_{n=-L}^{L}\sum_{m=-L}^{L} \delta_{i,n+m}\ \delta_{k,n+m}\ \delta_{\ell,n-m}\ \delta_{p,n-m}$$

$$= \sum_{n=-L}^{L} \delta_{k,i}\ \delta_{\ell,2n-i}\ \delta_{p,2n-i}$$

$$= \delta_{i,k}\ \delta_{\ell,p}\ , \tag{4.6}$$

satisfies the condition (4.3).

Example 2:

$$\phi_i(n) = \begin{cases} \exp(j\ \frac{2\pi}{2L+1}\ in)\ /\ \sqrt{2L+1} & -L \le n \le L \\ 0 & \text{otherwise.} \end{cases} \tag{4.7}$$

$$\sum_{n=-L}^{L} \phi_i(n)\ \phi_k^*(n) = \frac{1}{2L+1} \sum_{n=-L}^{L} \exp(j\frac{2\pi}{2L+1}in)\ \exp(-j\frac{2\pi}{2L+1}kn)$$

$$= \frac{1}{2L+1} \sum_{n=-L}^{L} \exp(j\frac{2\pi}{2L+1}n(i-k))$$

$$= \delta_{i,k}. \tag{4.8}$$

And

$$\sum_{n=-L}^{L} \sum_{m=-L}^{L} \phi_i(n+m)\ \phi_k^*(n+m)\ \phi_\ell(n-m)\ \phi_p^*(n-m)$$

$$= \frac{1}{(2L+1)^2} \sum_{n,m=-L}^{L} \exp[j\frac{2\pi}{2L+1}i(n+m)]\ \exp[-j\frac{2\pi}{2L+1}k(n+m)]$$

$$\bullet \exp[j\frac{2\pi}{2L+1}\ell(n-m)]\ \exp[-j\frac{2\pi}{2L+1}p(n-m)]$$

$$= \frac{1}{(2L+1)^2} \{ \sum_{n=-L}^{L} \exp[j\frac{2\pi}{2L+1}n(i-k+p-\ell)]$$

$$\bullet \sum_{m=-L}^{L} \exp[j\frac{2\pi}{2L+1}m(i-k-p+\ell)]\}$$

$$= \delta_{i-k+p-\ell,0}\ \delta_{i-k-p+\ell,0} \tag{4.9}$$

The last expression in the above equation equals 1 when

$$\left\{ \begin{array}{l} i-k+p-\ell = 0 \\ i-k-p+\ell = 0 \end{array} \right. \tag{4.10}$$

From (4.10) we have

$$\left\{ \begin{array}{l} i = k \\ p = \ell \end{array} \right. \tag{4.11}$$

Rewriting (4.9)

$$\sum_{n,m=-L}^{L} \phi_i(n+m)\phi_k^*(n+m)\phi_\ell(n-m)\phi_p^*(n-m) = \delta_{i,k}\delta_{\ell,p}. \tag{4.12}$$

The above two examples demonstrate that both sets of basis functions satisfy the requirements (4.2) and (4.3).

Substituting (4.1) into the auto-Wigner distribution (3.3) and interchanging the summation gives

$$W_f(n,\theta) = 2 \sum_{i,j=-L}^{L} a_i a_j^* \sum_{k=-\infty}^{\infty} e^{-j2k\theta} \phi_i(n+k)\phi_j^*(n-k) \tag{4.13}$$

$K_{i,j}(n,\theta)$ which is denoted as a cross-Wigner distribution between a pair of basis functions:

$$\begin{aligned} K_{i,j}(n,\theta) &= 2 \sum_{k=-\infty}^{\infty} e^{-j2k\theta} \phi_i(n+k)\ \phi_j^*(n-k) \\ &= 2 \sum_{k=-K}^{K} e^{-j2k\theta} \phi_i(n+k)\ \phi_j^*(n-k) \end{aligned} \tag{4.14}$$

where $|n| \le L$, and $K \ge L - |n|$.

Substituting (4.14) in (4.13) gives

$$W_f(n,\theta) = \sum_{i=-L}^{L} \sum_{j=-L}^{L} a_i a_j^*\ K_{i,j}(n,\theta) \tag{4.15}$$

It is easy to show that the set $\{K_{i,j}(n,\theta)\}$ is orthogonal,

$$(K_{i,j}, K_{p,q}) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} K_{i,j}(n,\theta)\, K^{*}_{p,q}(n,\theta)\, d\theta \tag{4.16}$$

Inserting (4.14) in (4.16) and interchanging the order of integration and summation give

$$(K_{i,j}, K_{p,q}) =$$

$$4 \sum_{n=-L}^{L} \sum_{k,\ell=-K}^{K} \phi_i(n+k)\phi^{*}_j(n-k)\phi^{*}_p(n+\ell)\phi_q(n-\ell)$$

$$\cdot \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{j2\theta(\ell-k)} d\theta \tag{4.17}$$

The integration of the above expression gives

$$\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} e^{j2\theta(\ell-k)} d\theta = \begin{cases} 1, & \ell = k, \\ 0, & \ell \neq k. \end{cases} \tag{4.18}$$

Return to (4.17) we have

$$(K_{i,j}, K_{p,q}) = 4 \sum_{n=-L}^{L} \sum_{k=-K}^{K} \phi_i(n+k)\phi^{*}_p(n+k)\phi_q(n-k)\phi^{*}_j(n-k) \tag{4.19}$$

By setting the summation limit K = L and employing (4.3) to above equation, we finally obtain

$$\sum_{n=-L}^{L} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} K_{i,j}(n,\theta)K^{*}_{p,q}(n,\theta)d\theta = 4\delta_{i,p}\delta_{j,q} \tag{4.20}$$

4.1.2 SIGNAL ESTIMATION FROM MODIFIED AUTO-WIGNER DISTRIBUTION

$Y(n,\theta)$ is a given modified auto-Wigner distribution. In general, $Y(n,\theta)$ is not a valid auto-Wigner distribution in the sense that there is not a sequence of which the auto-Wigner distribution is given by $Y(n,\theta)$. We develop the algorithm to estimate a sequence $f(n)$ whose auto-Wigner distribution $W_f(n,\theta)$ is closest to $Y(n,\theta)$ in the squared error sense. In other words, we use the least-square algorithm that minimizes

$$\varepsilon = \sum_{n=-L}^{L} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} |Y(n,\theta) - W_f(n,\theta)|^2 d\theta \qquad (4.21)$$

Inserting (4.15) into (4.21) gives

$$\varepsilon = \sum_{n=-L}^{L} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} |Y(n,\theta) - \sum_{i,j=-L}^{L} a_i a_j^* K_{i,j}(n,\theta)|^2 d\theta. \qquad (4.22)$$

This expression is to be minimized by choosing the coefficients $\{a_k\}$. Expanding the equation and utilizing the orthogonality of $K_{i,j}(n,\theta)$ gives

$$\varepsilon = \|Y\|^2 + 4 \sum_{i,j=-L}^{L} |a_i a_j^*|^2 - \sum_{i,j=-L}^{L} a_i a_j^* B_{ij}^* - \sum_{i,j=-L}^{L} a_i^* a_j B_{ij} \qquad (4.23)$$

where

$$\|Y\|^2 = \sum_n \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} |Y(n,\theta)|^2 d\theta, \tag{4.24}$$

$$B_{i,j} = \sum_n \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} Y(n,\theta)\; K^*_{i,j}(n,\theta)\; d\theta. \tag{4.25}$$

The minimization of ε with respect to the coefficients a_i will provide a least-square approximation to the desired function. We treat a_i and a_i^* as independent variables when minimizing the mean square error ε.

$$\frac{\partial \varepsilon}{\partial a_p^*} = 0 = 8\; a_p \sum_{i=-L}^{L} |a_i|^2 - \sum_{i=-L}^{L} a_i B^*_{ip} - \sum_{i=-L}^{L} a_i B_{pi}, \tag{4.26}$$

where $\sum_{i=-L}^{L} |a_i|^2$ is the signal energy E. The equation (4.26) can be formulated in matrix notation as

$$(\underline{B} + \tilde{\underline{B}})\; \underline{a} = 8E\; \underline{a} = \lambda\; \underline{a}, \tag{4.27}$$

where $\underline{B} = (B_{i,j})$ is a (2L+1) by (2L+1) matrix and $\tilde{\underline{B}}$ is the conjugate transpose of $\underline{B}$. $\underline{a}$ is the normalized eigenvector of signal coefficients that needed to be determined. By carrying out the eigenvalues and eigenvectors decomposition of the Hermitian matrix $(\underline{B} + \tilde{\underline{B}})$, we observe that there are two significant positive eigenvalues λ_1 and λ_2, i.e.,

$$(\underline{B} + \tilde{\underline{B}}) \simeq \lambda_1\; \underline{a}^{(1)}\underline{a}^{(1)} + \lambda_2\; \underline{a}^{(2)}\underline{a}^{(2)} \tag{4.28}$$

$$f_1(n) = \sum_{i=-L}^{L} \sqrt{\lambda_1/8}\; a_i^{(1)} \phi_i(n), \tag{4.29}$$

and

$$f_2(n) = \sum_{i=-L}^{L} \sqrt{\lambda_2/8}\; a_i^{(2)} \phi_i(n), \tag{4.30}$$

where $a_i^{(1)}$ is the ith element of eigenvector $\underline{a}^{(1)}$, and $a_i^{(2)}$ is the ith element of eigenvector $\underline{a}^{(2)}$. We observe that $f_1(n)$ is the even indexed sequence of f(n) since all the odd-indexed values of $f_1(n)$ are zero. Similarly, $f_2(n)$ is the odd-indexed sequence of f(n).

$$\{f(n)\}_{n=-L}^{L} = \alpha\{f_1(2n)\}_{n=-L/2}^{L/2} + \beta\{f_2(2n+1)\}_{n=-L/2}^{L/2-1}, \tag{4.31}$$

where L is an even integer. From the observation, we find the reconstructed signal f(n) is a separable sequence which divides into an even-indexed sequence and an odd-indexed sequence. This is due to the fact that in the sum (3.3) only even-indexed or odd-indexed samples occur simultaneously.

4.1.3 SIGNAL ESTIMATION FROM MODIFIED CROSS-WIGNER DISTRIBUTION

For signal synthesis from a given modified cross-Wigner distribution $Y(n,\theta)$, we write the total square error of the approximation as in section 4.1.2.

$$\varepsilon = \sum_{n=-L}^{L} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} |Y(n,\theta) - W_{f,g}(n,\theta)|^2 d\theta \tag{4.32}$$

We suppose the signals f(n) and g(n) satisfy:

$$f(n) = \sum_{i=-L}^{L} a_i \phi_i(n) \tag{4.33}$$

$$g(n) = \sum_{i=-L}^{L} b_i \phi_i(n) \tag{4.34}$$

Substituting (4.33) and (4.34) into (2.1), the cross-Wigner distribution can be written as

$$W_{f,g}(n,\theta) = \sum_{i=-L}^{L} \sum_{j=-L}^{L} a_i b_j^* K_{i,j}(n,\theta), \tag{4.35}$$

and the mean-square error is given by

$$\varepsilon = \sum_{n=-L}^{L} \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} |Y(n,\theta) - \sum_{i=-L}^{L} \sum_{j=-L}^{L} a_i b_j^* K_{i,j}(n,\theta)|^2 d\theta \tag{4.36}$$

Following the same procedure as in section 4.1.2, equation (4.36) becomes

$$\begin{aligned} \varepsilon = \|Y\|^2 &- \sum_{i=-L}^{L} \sum_{j=-L}^{L} a_i b_j^* B_{ij}^* \\ &- \sum_{i=-L}^{L} \sum_{j=-L}^{L} a_i^* b_j B_{ij} + 4 \sum_{i=-L}^{L} \sum_{j=-L}^{L} |a_i|^2 |b_j|^2 \end{aligned} \tag{4.37}$$

Now we treat a_i, a_i^*, b_j and b_j^* as four independent variables when minimizing the mean-square error ε.

$$\frac{\partial \varepsilon}{\partial a_p^*} = 0 = -\sum_{j=-L}^{L} b_j \, B_{pj} + 4 \, a_p \sum_{j=-L}^{L} |b_j|^2 \tag{4.38}$$

$$\frac{\partial \varepsilon}{\partial a_p^*} = 0 = -\sum_{i=-L}^{L} a_i \, B^*_{ip} + 4 \, b_p \sum_{i=-L}^{L} |a_i|^2 \tag{4.39}$$

Formulating (4.38) and (4.39) in matrix notation, we have

$$\underline{B} \, \underline{b} = 4E_b \, \underline{a} \tag{4.40}$$

$$\tilde{\underline{B}} \, \underline{a} = 4E_a \, \underline{b} \tag{4.41}$$

Substituting (4.40) into (4.41) and vice versa gives

$$\underline{B} \, \tilde{\underline{B}} \, \underline{a} = 16 \, E_a E_b \, \underline{a} \tag{4.42}$$

$$\tilde{\underline{B}} \, \underline{B} \, \underline{b} = 16 \, E_a E_b \, \underline{b} \tag{4.43}$$

By carrying out the eigenvalues and eigenvectors decomposition of matrices $\underline{B\tilde{B}}$ and $\underline{\tilde{B}B}$, we obtain the reconstructed signals f(n) and g(n) respectively up to a constant factor.

4.2 THE SECOND METHOD OF LEAST SQUARES ERROR ESTIMATION FROM MODIFIED WIGNER DISTRIBUTION

Let f(n) and $W_f(n,\theta)$ denote a sequence of signal and its auto-Wigner distribution. From the definition, the auto-Wigner distribution is

$$W_f(n,\theta) = 2 \sum_{k=-\infty}^{\infty} e^{-j2k\theta} f(n+k)\ f^*(n-k) \tag{4.44}$$

Given a modified auto-Wigner distribution $Y(n,\theta)$, the inverse transform of this distribution is a two dimensional time function

$$y(n,m) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} Y(n,\theta)\ e^{jm\theta} d\theta \tag{4.45}$$

In this section we discuss another algorithm to estimate a sequence $f(n)$ of which the modified Wigner distribution $W_f(n,\theta)$ is closest to $Y(n,\theta)$ in the total square error sense. This algorithm is first developed by Boudreaux-Bartels and Parks [9].

$$\varepsilon = \sum_n \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} |Y(n,\theta) - W_f(n,\theta)|^2 d\theta, \tag{4.46}$$

where $W_f(n,\theta)$ is a valid Wigner distribution while $Y(n,\theta)$ is not necessarily a valid Wigner distribution. By Parseval's theorem, equation (4.46) can be written as

$$\varepsilon = \sum_n \sum_m |y(n,2m) - 2\ f(n+m)f\ (n-m)|^2 \tag{4.47}$$

Recalling the Wigner distribution property (7) in section 2, or Eq. (2.14) and Eq. (2.15), we see that even-indexed or odd-indexed sample sequences can be reconstructed up to a constant factor with different procedure. To minimize the mean square error, Equation (4.47) can be broken up into two separate parts. That will minimize the mean square error of even indexed sequence and odd indexed sequence separately. Equation (4.47) can be rewritten as

$$\begin{aligned} \varepsilon &= \varepsilon_e + \varepsilon_o \\ &= \sum_n \sum_k |y(n,4k-2n) - 2f(2k)f^*(2n-2k)|^2 \\ &\quad + \sum_n \sum_k |y(n,4k-2n-2) - 2f(2k-1)f^*(2n-2k+1)|^2 \end{aligned} \tag{4.48}$$

To minimize the mean square error ε_e, viz.,

$$\varepsilon_e = \sum_n \sum_k |y(n,4k-2n) - 2f(2k)f^*(2n-2k)|^2, \tag{4.49}$$

can be obtained by treating f(n) and $f^*(n)$ as independent variables.

$$\begin{aligned} (\partial\varepsilon_e/\partial f^*(2p)) = 0 = 4f(2p) \sum_m |f(2m)|^2 \\ - \sum_m [y(m+p,2p-2m) + y^*(m+p,2m-2p)]\ x(2m) \end{aligned} \tag{4.50}$$

The above expression can be formulated in matrix notation as

$$\underline{Y}_e\ \underline{f}_e = 4\ \|\underline{f}_e\|^2 \underline{f}_e \tag{4.51}$$

where $\underline{f}_e$ is the normalized eigenvector and $\underline{Y}_e = (y_{m,p})$ with

$$y_{m,p} = y(m+p,2p-2m) + y^*(m+p,2m-2p) \tag{4.52}$$

Similarly, we have

$$\varepsilon_o = \sum_n \sum_k |y(n,4k-2n-2) - 2f(2k-1)f^*(2n-2k+1)|^2 \tag{4.53}$$

$$(\partial\varepsilon_o/\partial f^*(2p-1)) = 0 = 4f(2p-1) \sum_m |f(2m-1)|^2$$
$$- \sum_m [y(m+p-1,2p-2m) + y^*(m+p-1,2m-2p)]\, x(2m-1) \tag{4.54}$$

$$\underline{Y}_o\, \underline{f}_o = 4\, \|\underline{f}_o\|^2\, \underline{f}_o \tag{4.55}$$

where

$$y_{m,p} = y(m+p-1,2p-2m) + y^*(m+p-1,2m-2p). \tag{4.56}$$

It can be shown [9] that optimal solutions of even-indexed sequence $\underline{f}_e$ and odd-indexed sequence $\underline{f}_o$ are normalized eigenvectors of $\underline{Y}_e$ and $\underline{Y}_o$ associated with the largest eigenvalues λ_e and λ_o, respectively.

There are two approaches of combining the even-indexed sequence and odd-indexed sequence. One of them, which was introduced by Boudreaux-Bartels and Parks, is to find the real parameters α and β such that f(n) is closest to the original sequence s(n), i.e. minimize

$$\varepsilon_e(\alpha) = \sum_n |s(2n) - f_e(2n)e^{j\alpha}|^2 \quad . \tag{4.57}$$

and

$$\varepsilon_o(\beta) = \sum_m |s(2m-1) - f_o(2m-1)e^{j\beta}|^2 . \tag{4.58}$$

Clearly, the mean square error ε_e is a function of α and is minimum if

$$\partial\varepsilon_e(\alpha)/\partial\alpha = 0 = -2\mathrm{Im}[e^{-j\alpha} \sum_n s(2n)f^*(2n) - \|\underline{f}_e\|^2] . \tag{4.59}$$

The above equation yields

$$\alpha = \tan^{-1}\{\mathrm{Re}[\sum_n s(2n)f_e^*(2n)]/\mathrm{Im}[\sum_n s(2n)f_e^*(2n)]\} . \tag{4.60}$$

Similarly,

$$\beta=\tan^{-1}\mathrm{Re}[\sum_n s(2n-1)f_o^*(2n-1)]/\mathrm{Im}[\sum_n s(2n-1)f_o^*(2n-1)] . \tag{4.61}$$

The second approach is based on a smoothness criterion of which the phase is chosen in such a way that the resulting sequence will be as smooth as possible.

$$\varepsilon = \sum_n |f_e(2n)e^{j\alpha} - f_o(2n-1)e^{j\beta}|^2 \tag{4.62}$$

Error ε will be minimum if

$$\partial\varepsilon/\partial\alpha = 0,$$
$$\partial\varepsilon/\partial\beta = 0 \tag{4.63}$$

Then the reconstructed signal is given by

$$\underline{f} = \underline{f}_e e^{j\alpha} + \underline{f}_o e^{j\beta} \tag{4.64}$$

4.3 SIGNAL RECONSTRUCTION USING OUTER PRODUCT APPROXIMATION

The third algorithm is an ad-hoc procedure. Given a desired time-frequency function $Y(n,\theta)$, the inverse transform of $Y(n,\theta)$ gives

$$y(n,2m) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} Y(n,\theta)\, e^{j2m\theta} d\theta. \tag{4.65}$$

Measuring the error distance between $y(n,2m)$ and a valid auto-Wigner distribution $W_f(n,\theta)$ gives

$$\begin{aligned} |e(n,m)| &= |y(n,2m) - \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} W_f(n,\theta)\, e^{j2m\theta} d\theta| \\ &= |y(n,2m) - 2\, f(n+m)\, f^*(n-m)|. \end{aligned} \tag{4.66}$$

The total square error is

$$\varepsilon = \sum_n \sum_m e^2(n,m) = \sum_n \sum_m |y(n,2m) - 2f(n+m)f^*(n-m)|^2, \tag{4.67}$$

which is the same expression we obtained in previous section. If the given distribution $Y(n,\theta)$ is a valid Wigner distribution and if there is no noise in the measurement, we should have

$$y(n,2m) = 2\, f(n+m)\, f^*(n-m) \tag{4.68}$$

or

$$y\left(\frac{n+m}{2},\, n-m\right) = 2f(n)f^*(m) \tag{4.69}$$

Since $(n+m)/2$ must be an integer, n and m can only occur with both even or odd integer. This tells us that the even-indexed samples and the odd-indexed samples are not coupled, that is, the even-indexed samples and the odd-indexed samples cannot occur simultaneously. If $f(n)$ is a time-limited signal, i.e.,

$$f(n) = 0 \qquad n < n_a \text{ or } n > n_b \tag{4.70}$$

We can form an outer product expression for even-indexed samples or odd-indexed samples separately. For convenience we let both n_a and n_b be even integer numbers, the even part of the outer product expression can be written as

$$\begin{bmatrix} f(n_a) \\ f(n_a+2) \\ \cdot \\ f(m) \\ \cdot \\ f(n_b) \end{bmatrix} \begin{bmatrix} f^*(n_a) & f^*(n_a+2) & \ldots & f^*(n) & \ldots & f^*(n_b) \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} y_{11} & y_{12} & \cdots & \cdots & \cdots & y_{1N} \\ y_{21} & y_{22} & \cdots & \cdots & \cdots & y_{2N} \\ . & . & .. & & & . \\ . & . & & y_{ij}(m,n). & & \\ y_{N1} & y_{N2} & \cdots & . & \cdots & y_{NN} \end{bmatrix} \tag{4.71}$$

where

$$\begin{aligned} y_{i,j}(m,n) &= y(\tfrac{m+n}{2},\ m-n) && \text{for } i \geq j, \\ y_{i,j}(m,n) &= y^{*}_{j,i}(m,n) && \text{for } i < j. \end{aligned} \tag{4.72}$$

where i,j are the indices of the matrix, m,n are the corresponding coefficients of the signal, and both n and m are even integers. Equation (4.71) can be written in matrix notation as

$$\underline{Y}_e = 2\ \underline{f}_e\ \underline{f}_e^{+} \tag{4.73}$$

where $\underline{f}_e^{+}$ denotes the conjugate transpose of $\underline{f}_e$. The even-indexed samples of f(n) can be reconstructed by solving the eigenvalues and eigenvectors problem of matrix $\underline{Y}_e$,

$$\underline{Y}_e\ \underline{f}_e = 2\|\underline{f}_e\|^2\ \underline{f}_e = \lambda_e \underline{f}_e. \tag{4.74}$$

$\underline{Y}_e$ is a rank 1 Hermitian matrix if $Y(n,\theta)$ corresponds to a valid Wigner distribution. This can be used as a test for validity. A similar procedure can be carried out for the odd-indexed samples, viz.,

$$\underline{Y}_o \underline{f}_o = 2 \, \|\underline{f}_o\|^2 \, \underline{f}_o = \lambda_o \underline{f}_o \tag{4.75}$$

The expression of (4.72) can also be used to determine the matrix $\underline{Y}_o$, but now both n and m are odd integers.

If the given distribution $Y(n,\theta)$ is a modified Wigner distribution, the matrices $\underline{Y}_e$ and $\underline{Y}_o$ may have rank greater than 1. Then we have mean square errors as:

$$\varepsilon_e = \|\underline{Y}_e/2 - \underline{f}_e \underline{f}_e^+\|^2 \tag{4.76}$$

$$\varepsilon_o = \|\underline{Y}_o/2 - \underline{f}_o \underline{f}_o^+\|^2 \tag{4.77}$$

The eigenvectors that minimize the mean square error functions are the normalized eigenvectors of $\underline{Y}_e$ and $\underline{Y}_o$ associated with the largest eigenvalues λ_e and λ_o, respectively.

Three algorithms for signal synthesis from the discrete-time Wigner distribution have been discussed. Let us test those algorithms by an example. Fig. 5 and Fig. 6 show an FM chirp signal and its Wigner distribution.

$$\begin{aligned} f(t) &= 50 \cos(\pi t^2/1000) \qquad -40 \le t \le 40 \\ &= 0 \qquad\qquad\qquad\qquad\quad \text{otherwise} \end{aligned} \tag{4.78}$$

Fig. 7, Fig. 8 and Fig. 9 show the reconstructed signal from Fig. 6 using the synthesis methods discussed in Section 4.1, 4.2, and 4.3, respectively.

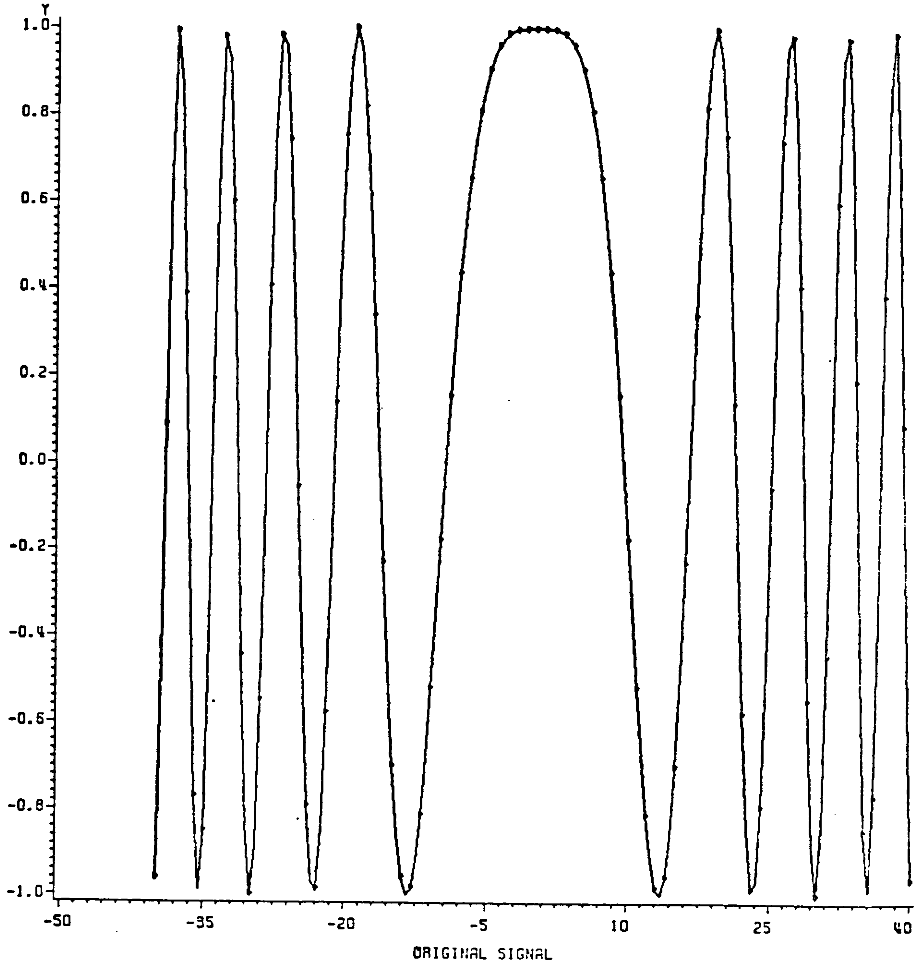

Figure 5. The FM chirp signal f(t).

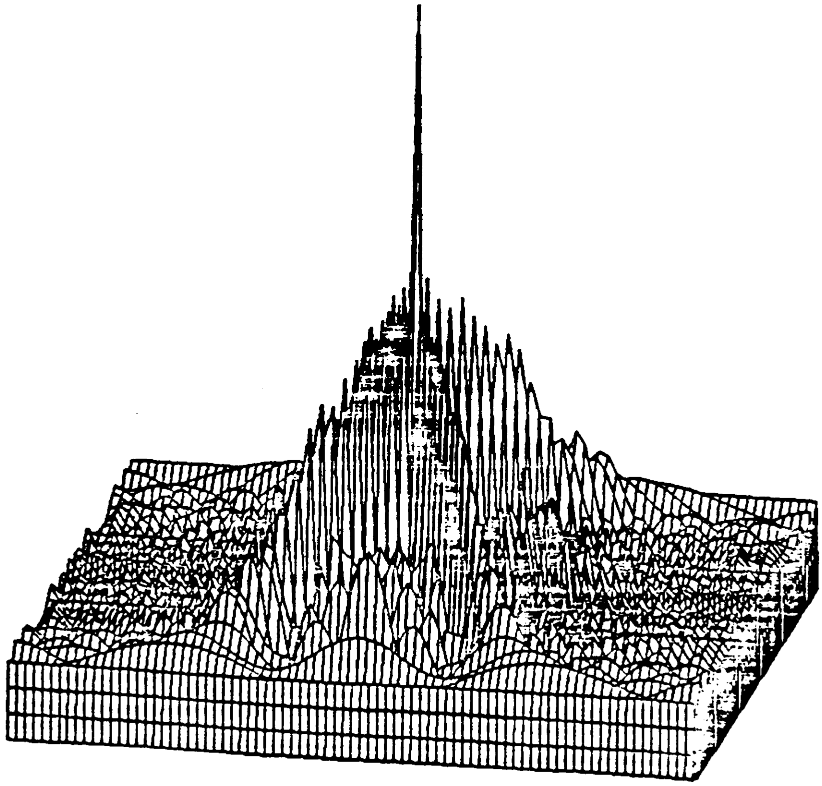

Figure 6. The WD of Fig. 5.

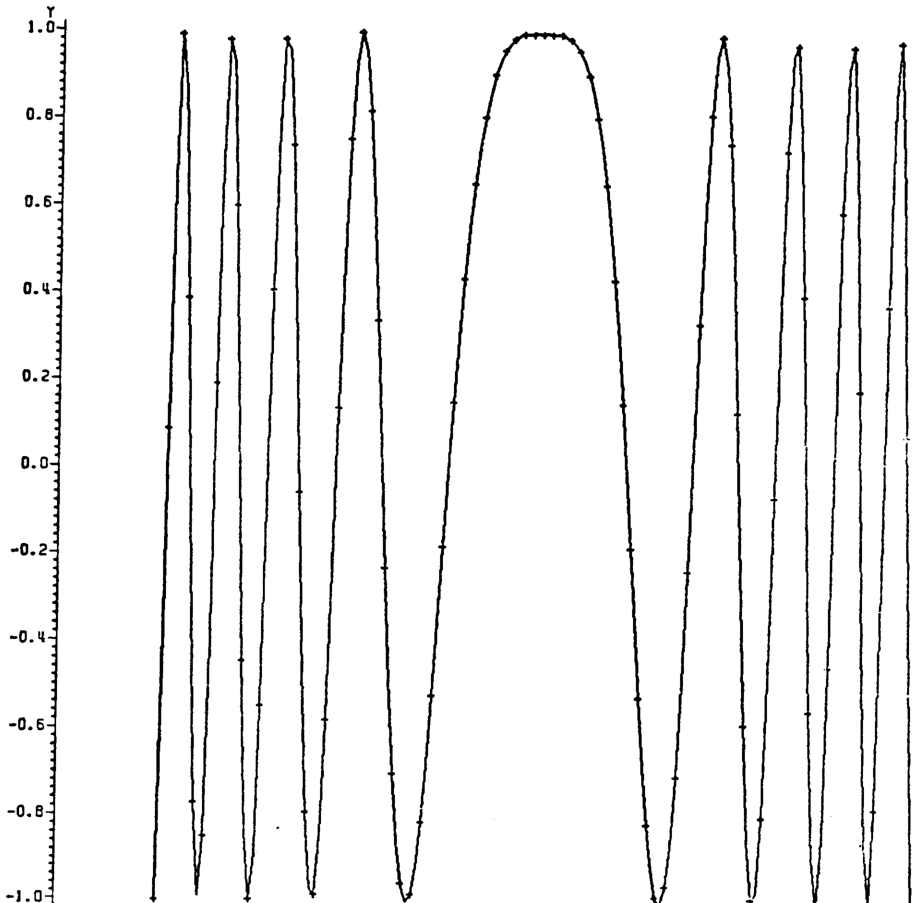

Figure 7. f(t) recovered by using the method in Sec.4.1.

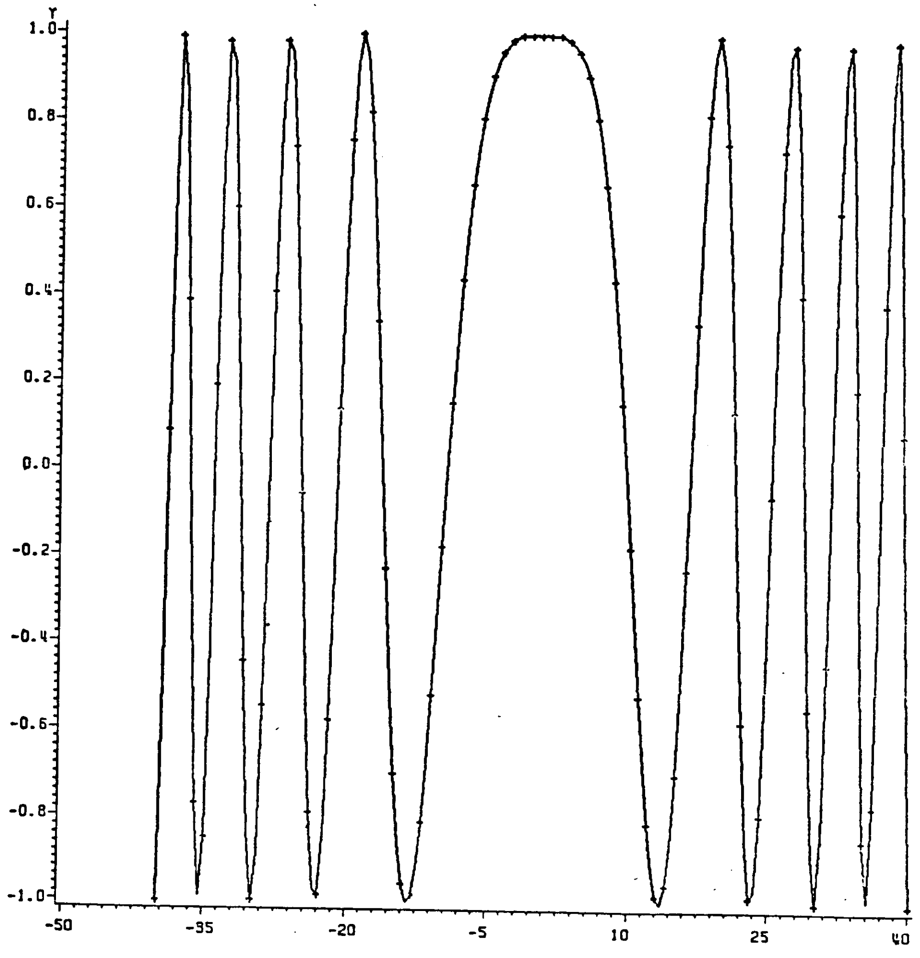

Figure 8. f(t) recovered by using the method in Sec.4.2.

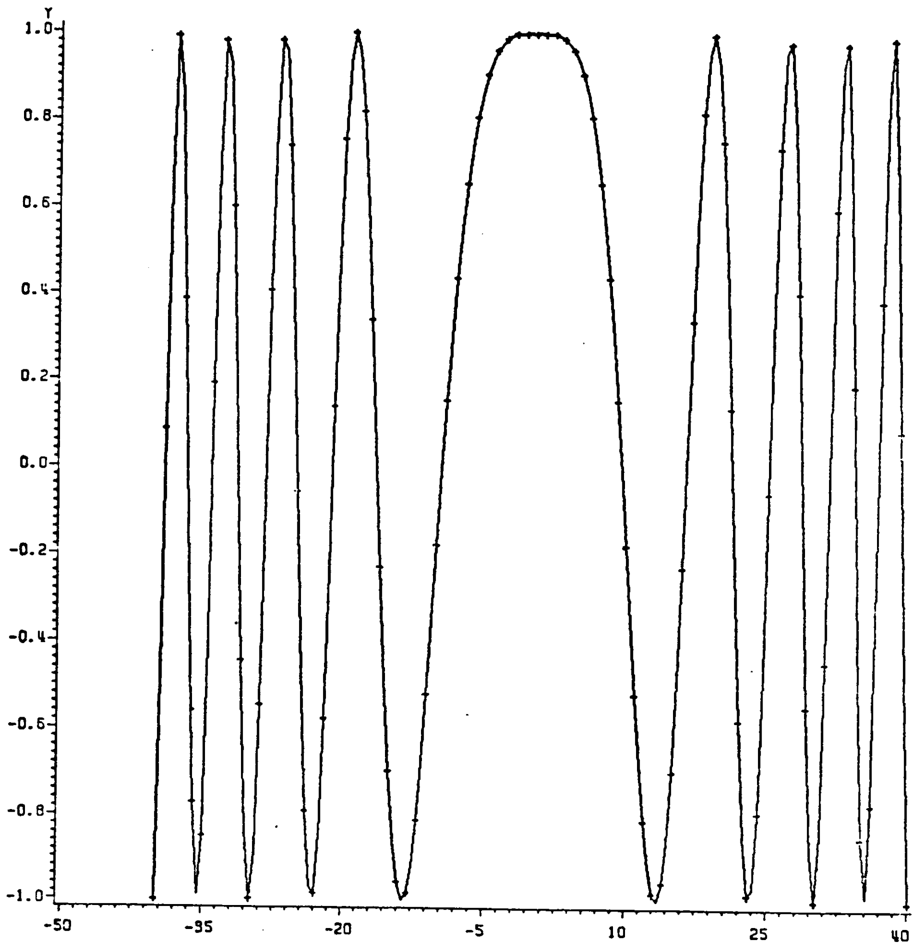

Figure 9. f(t) recovered by using the method in Sec.4.3.

5.0 DISCRETE-TIME PSEUDO-WIGNER DISTRIBUTION AND ITS PROPERTIES

For practical implementations it is necessary to weight the signals f and g by functions h_f and h_g, respectively, before the Wigner distribution can be computed. These weighting functions are called windows and will slide along the time axis with the instant k when the Wigner distribution has to be evaluated. Denoting the windowed signals be $f_k(n)$ and $g_k(n)$. Hence

$$\begin{aligned} f_k(n) &= f(n)\ h_f(n-k) \\ g_k(n) &= g(n)\ h_g(n-k) \end{aligned} \tag{5.1}$$

With Eq.(2.10) we can evaluate the Wigner distribution of the windowed signals with each fixed k:

$$W_{f_k,g_k}(n,\theta) = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} W_{f,g}(n,\omega) W_{h_f,h_g}(n-k,\theta-\omega)d\omega \tag{5.2}$$

For each window position one gets another Wigner distribution. We consider only the case when n=k. In this case, the window is symmetrically located around n; and we obtain

$$W_{f_k,g_k}(n,\theta)\Big|_{n=k} = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} W_{f,g}(n,\omega) W_{h_f,h_g}(0,\theta-\omega)d\omega. \tag{5.3}$$

The new function of n and θ is the so called pseudo-Wigner distribution proposed by Claasen and Mecklenbrauker [7].

$$PWD_{f,g}(n,\theta) = W_{f_k,g_k}(n,\theta)\ \Big|_{\,n=k} \tag{5.4}$$

Using the Wigner distribution definition (2.1), the above equation can be rewritten as

$$PWD_{f,g}(n,\theta) = 2 \sum_{k=-\infty}^{\infty} e^{-j2k\theta} f(n+k)h(k)g^*(n-k)h^*(-k) \tag{5.5}$$

If the windows have a duration 2L-1, i.e., if

$$h(k) = 0, \qquad |k| \geq L \tag{5.6}$$

then

$$PWD_{f,g}(n,\theta) = 2 \sum_{k=-L+1}^{L-1} e^{-j2k\theta} f(n+k)h(k)g^*(n-k)h^*(-k) \tag{5.7}$$

The explicit derivation of (5.3) shows that the pseudo-Wigner distribution is nothing but a frequency smoothed Wigner distribution [13,18]. This smoothing is equivalent to a convolution in the frequency direction of the Wigner distribution of the non-windowed signals with the Wigner distribution of the window functions. An important point is that the pseudo-Wigner distribution does not give a spread in the time direction. Consequently, time properties which are held in the Wigner distribution are still held in the pseudo-Wigner distribution. In particular, properties P1, P3, P5, and P8 of Wigner distribution in Section 2 hold for the pseudo-Wigner distribution.

We shall illustrate the concept of the pseudo-Wigner distribution by some examples. We use a normed Hamming window in all the examples below, i.e.,

$$h(n) = [0.54+0.46\cos(2\pi n/2L)]/(2L+1), \quad -L \le n \le L$$
$$= 0 \qquad \text{otherwise.} \qquad (5.8)$$

As a first example, we consider a cosine wave form:

$$f(n) = \cos\ (\pi n/8) \qquad (5.9)$$

with $L = 24$, $\Delta\omega = \pi/49$, and $\Delta n = 2$. Fig. 10 shows the pseudo-Wigner distribution result. Fig. 11 is the pseudo-Wigner distribution of the same signal with $L = 60$, $\Delta\omega = \pi/121$, and $\Delta n = 5$. From these two plots, we observe that the wider the window duration, the smaller the spread version occurs in frequency direction. Furthermore it can be seen that the spectral contents do not change with time.

As a second example, we consider a chirp signal:

$$f(n) = \cos(\pi n^2/1000) \tag{5.10}$$

Fig. 12 shows the pseudo-Wigner distribution of the signal $f(n)$ with window duration $L = 60$, $\Delta\omega = \pi/121$, and $\Delta n = 5$. This pseudo-Wigner distribution shows that the frequency linearly increases with time.

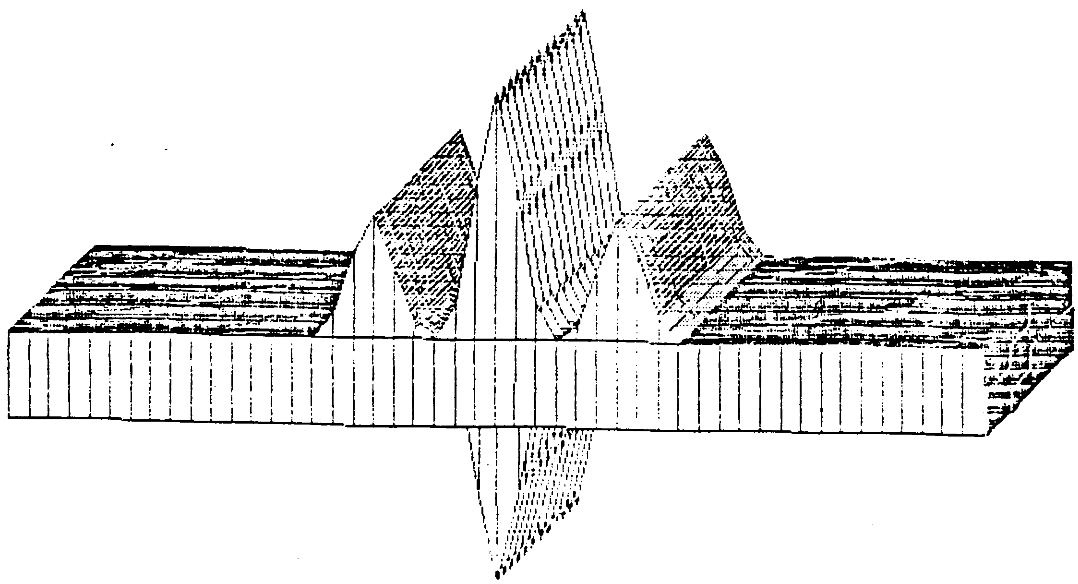

Figure 10. The PWD of Eq.(5.9) with L=24.

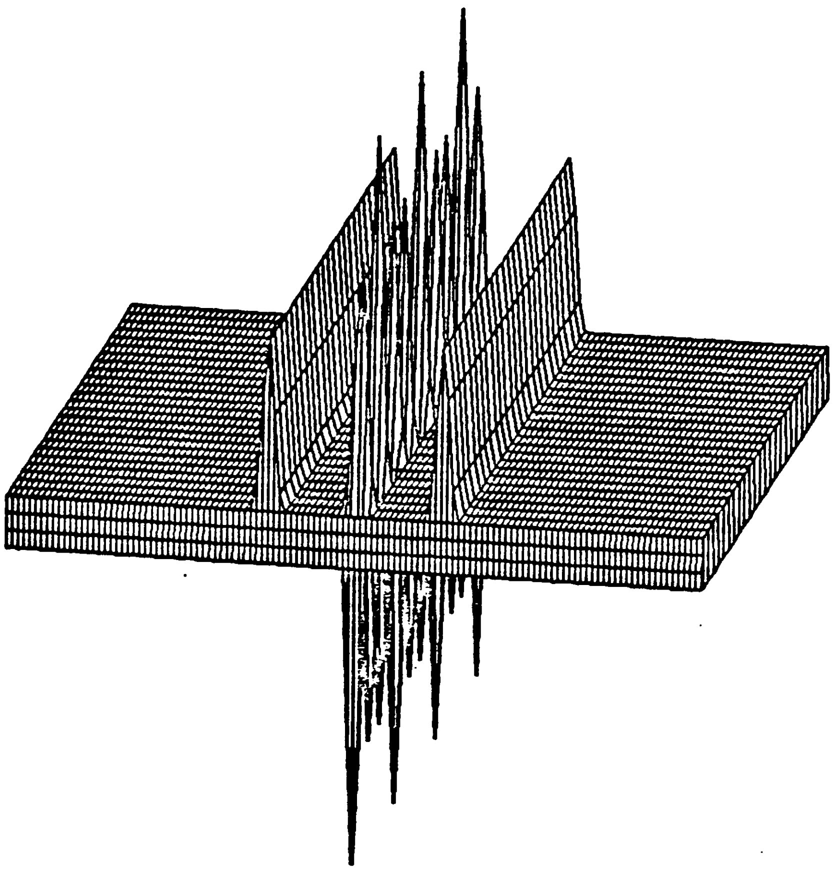

Figure 11. The PWD of Eq.(5.9) with L=60.

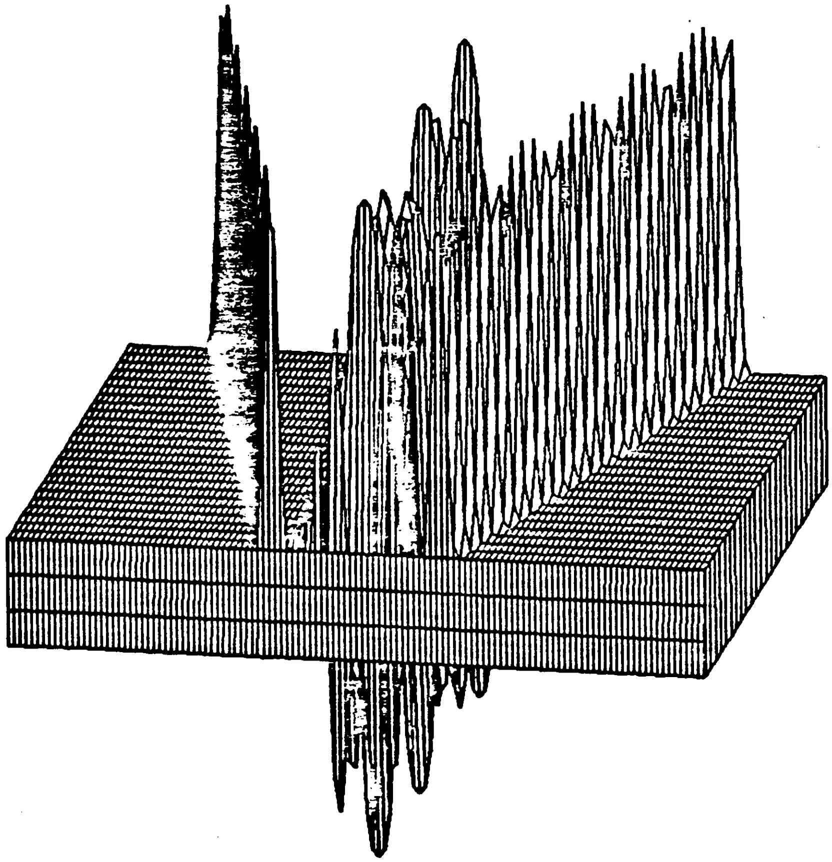

Figure 12. The PWD of Eq.(5.10).

6.0 SIGNAL SYNTHESIS FROM PSEUDO-WIGNER DISTRIBUTION

In a number of applications, it is desirable to modify the pseudo-Wigner distribution and then estimate the processed signal from it. In this section two synthesis algorithms are developed for estimating a signal from the time-frequency function.

6.1 OUTER PRODUCT APPROXIMATION FOR SIGNAL SYNTHESIS FROM PSEUDO-WIGNER DISTRIBUTION

The synthesis algorithm developed here is an ad-hoc procedure which follows the same algorithm we introduce in section 4.3. For a given pseudo-Wigner distribution

$$Y(n,\theta) = 2 \sum_{k=-L+1}^{L-1} f(n+k)f^*(n-k)h(k)h^*(-k)e^{-j2k\theta}, \qquad (6.1)$$

with

$$h(k) = 0, \qquad |k| \geq L. \qquad (6.2)$$

The inverse transform of $Y(n,\theta)$ gives

$$\begin{aligned} y(n,2k) &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} Y(n,\theta)\, e^{j2k\theta} d\theta \\ &= 2\, f(n+k)f^*(n-k)h(k)h^*(-k) \end{aligned} \qquad (6.3)$$

Hence

$$f(n+k)f^*(n-k) = \frac{y(n,2k)}{2h(k)h^*(-k)} \tag{6.4}$$

or

$$\begin{aligned} f(n)f^*(m) &= c(n,m) \\ &= y(\tfrac{n+m}{2}, n-m) \;/\; 2h(\tfrac{n-m}{2})h^*(\tfrac{m-n}{2}) \end{aligned} \tag{6.5}$$

This is in the outer product form again and thus we can synthesize the desired even-indexed and odd-indexed sequences using an outer product approximation procedure discussed in section 4.3. We assume the window h(k) is known within the interval -L < K < L and no zero within this interval, i.e.,

$$\begin{aligned} h(k) &\neq 0 \qquad -L < k < L \\ h(k) &= 0 \qquad |k| \geq L \end{aligned} \tag{6.6}$$

The desired even indexed sequence can again be recovered by solving the eigenvalues and eigenvectors problem:

$$\underline{C}_e\ \underline{f}_e = \|\underline{f}_e\|^2\ \underline{f}_e = \lambda_e\ \underline{f}_e\ , \tag{6.7}$$

where

$$C_e(n,m) = y(\tfrac{n+m}{2},\ n-m) \;/\; 2h(\tfrac{n-m}{2})h^*(\tfrac{m-n}{2}) \tag{6.8}$$

with both n and m are even integers. Similarly, for the odd-indexed sequence we have,

$$\underline{C}_o \; \underline{f}_o = \|\underline{f}_o\|^2 \; \underline{f}_o = \lambda_o \; \underline{f}_o \; . \tag{6.9}$$

The expression for $C_o(n,m)$ is exactly the same as Eq.(6.8) except that both n and m are odd integers now. Both $\underline{C}_e$ and $\underline{C}_o$ are rank 1 Hermitian matrix if $Y(n,\theta)$ corresponds to a valid pseudo-Wigner distribution. However, for a modified pseudo-Wigner distribution, the matrix $\underline{C}_e$ or $\underline{C}_o$ may have rank greater than 1. Then we have mean square error functions as:

$$\begin{aligned} \varepsilon_e &= \|\underline{C}_e - \underline{f}_e \; \underline{f}_e^+\|^2 \\ \varepsilon_o &= \|\underline{C}_o - \underline{f}_o \; \underline{f}_o^+\|^2 \end{aligned} \tag{6.10}$$

The eigenvectors that minimize the mean square error functions are vectors associated with the largest positive eigenvalues. Thus, the synthesized signals are given by,

$$\begin{aligned} \underline{f}_e &= \sqrt{\lambda_e} \; \underline{u}_e, \\ \underline{f}_o &= \sqrt{\lambda_o} \; \underline{v}_o, \end{aligned} \tag{6.11}$$

and $\underline{f} = \alpha \; \underline{f}_e + \beta \; \underline{f}_o$

where λ_e and λ_o are the largest eigenvalues corresponding to $\underline{C}_e$ and $\underline{C}_o$, and $\underline{u}_e$ and $\underline{v}_o$ are the normalized eigenvectors associated with λ_e and λ_o, respectively. α and β are constant factors. The approximation errors are given by

$$\varepsilon_e = \| \underline{C}_e - \lambda_e \, \underline{u}_e \underline{u}_e^+ \|^2 = \sum_{i \neq e} \lambda_i$$

$$\varepsilon_o = \| \underline{C}_o - \lambda_o \, \underline{v}_o \underline{v}_o^+ \|^2 = \sum_{i \neq o} \lambda_i \qquad (6.12)$$

The outer product procedure can also be used for signal synthesis from cross-pseudo-Wigner distribution. Inverse transform of cross-pseudo-Wigner distribution of $Y(n,\theta)$ gives

$$y(n,2k) = 2 \; f(n+k)g^*(n-k)h(k)h^*(-k), \qquad (6.13)$$

or

$$c(n,k)^* = f(n)g^*(k)$$

$$= y(\tfrac{n+k}{2}, \; n-k) \; / \; 2h(\tfrac{n-k}{2})h^*(\tfrac{k-n}{2}) \qquad (6.14)$$

In general, matrix $\underline{C}$ is a rectangular matrix and the desired signal f(n) and g(n) can be recovered by carrying out the singular value decomposition procedure [21]. Thus

$$\underline{C}_e = [\underline{u}] \; [\Lambda]^{\frac{1}{2}} [\underline{v}]^+ \; . \qquad (6.15)$$

where $[\Lambda]$ is a diagonal matrix, and matrices [u] and [v] are unitary matrices, i.e.,

$$[u] \; [u^+] = [I] \qquad (6.16)$$

$$[v] \; [v^+] = [I] \qquad (6.17)$$

and

$$[u] = [\underline{u}_1 \ \underline{u}_2 \ \dots \ \underline{u}_p] \tag{6.18}$$

$$[v] = [\underline{v}_1 \ \underline{v}_2 \ \dots \ \underline{v}_p] \tag{6.19}$$

We can reconstruct even-indexed sequence of f(n) by finding the largest eigenvalue and eigenvector of matrix $\underline{C}_e\underline{C}^+_e$; viz.,

$$\underline{C}_e \ \underline{C}_e^+ = \lambda_1\underline{u}_1\underline{u}_1^+ + \lambda_2\underline{u}_2\underline{u}_2^+ + \dots + \lambda_p\underline{u}_p\underline{u}_p^+ \tag{6.20}$$

$$\underline{f}_e = \underline{u}_1 , \tag{6.21}$$

where we assume λ_1 is the largest eigenvalue which is the product of the energy of signals f and g, and u_1 is the corresponding eigenvector. Similarly, we can reconstruct even-indexed sequence of g(n) by finding the largest eigenvalue and eigenvector of matrix $\underline{C}_e^+\underline{C}_e$;

$$\underline{C}_e^+ \ \underline{C}_e = \lambda_1\underline{v}_1\underline{v}_1^+ + \lambda_2\underline{v}_2\underline{v}_2^+ + \dots + \lambda_p\underline{v}_p\underline{v}_p^+ \tag{6.22}$$

$$\underline{g}_e = \underline{v}_1 , \tag{6.23}$$

A similar procedure can be carried out for reconstructing odd-indexed samples of f(n) and g(n).

6.2 OVERLAPPING METHOD FOR SIGNAL SYNTHESIS FROM PSEUDO-WIGNER DISTRIBUTION

We assume the window function h(n) is a known sequence with no zero samples over its finite length and 2P samples of f(n) for $1 \leq n \leq 2P$ are known. Given a desired pseudo-Wigner distribution $Y(n,\theta)$, we can write the outer product expansion as in section 6.1.

$$\underline{f}_e \, \underline{f}_e^+ = \underline{C}_e \tag{6.24}$$

where

$$\underline{f}_e = \begin{bmatrix} f(2) \\ f(4) \\ . \\ f(2p) \\ . \\ f(2L) \end{bmatrix} \tag{6.25}$$

$$C_e(n,m) = y(\tfrac{n+m}{2},\ n-m) \ / \ 2h(\tfrac{n-m}{2})h^*(\tfrac{m-n}{2}) \tag{6.26}$$

with

$$h(k) = 0 \qquad |k| \geq L. \tag{6.27}$$

Now let

$$\underline{f}_e = \begin{bmatrix} \underline{X}_e \\ \\ \underline{A}_e \end{bmatrix}, \tag{6.28}$$

with vector $\underline{X}_e$ contains all the known even samples, and vector $\underline{A}_e$ is the unknown samples needed to be determined, i.e.,

$$\underline{X}_e = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_p \end{bmatrix} = \begin{bmatrix} f(2) \\ f(4) \\ . \\ . \\ f(2p) \end{bmatrix} \tag{6.29}$$

$$\underline{A}_e = \begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_{L-p} \end{bmatrix} = \begin{bmatrix} f(2p+2) \\ f(2p+4) \\ . \\ . \\ f(2L) \end{bmatrix} \tag{6.30}$$

Let

$$\underline{C}_e = \left[\begin{array}{ccc|cccc} C_{1,2} & \cdots & C_{1,p} & C_{1,p+1} & \cdots & \cdots & C_{1,\ell} \\ . & & . & . & & & . \\ C_{p,1} & \cdots & C_{p,p} & C_{p,p+1} & \cdots & \cdots & C_{p,\ell} \\ \hline C_{p+1,1} & & C_{p+1,p} & C_{p+1,p+1} & & \cdots & C_{p+1,\ell} \\ . & & & . & & & \\ . & & . & . & & & . \\ C_{\ell,1} & \cdots & C_{\ell,p} & C_{\ell,p+1} & \cdots & \cdots & C_{\ell,\ell} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \underline{U} & \underline{V}^{+} \\ \hline \underline{V} & \underline{W} \end{array}\right] \tag{6.31}$$

Inserting (6.28), and (6.31) in (6.24), we have

$$\begin{bmatrix} \underline{X}_e \\ \underline{A}_e \end{bmatrix} \begin{bmatrix} \underline{X}_e^{+} & \underline{A}_e^{+} \end{bmatrix}$$

$$= \left[\begin{array}{c|c} \underline{U} & \underline{V}^{+} \\ \underline{V} & \underline{W} \end{array}\right] \tag{6.32}$$

where

$$\underline{X}_e \, \underline{X}_e^{+} = \underline{U} \tag{6.33}$$

$$\underline{A}_e \, \underline{X}_e^{+} = \underline{V} \tag{6.34}$$

$$\underline{A}_e \, \underline{A}_e^{+} = \underline{W} \tag{6.35}$$

Multiply vector $\underline{X}_e$ to both sides of (6.34) we have

$$\underline{A}_e \, \|\underline{X}_e\|^2 = \underline{V} \, \underline{X}_e, \tag{6.36}$$

The unknown vector $\underline{A}_e$ can be determined by

$$\underline{A}_e = \underline{V}\ \underline{X}_e \ / \ \|\underline{X}_e\|^2 \tag{6.37}$$

Once $\underline{A}_e$ is determined, we can use $\underline{A}_e$ as known samples to determine the next unknown section. This process continues until all the unknown samples are determined. It turns out that this synthesis algorithm is the most straight forward one. It does not involve eigenvalue and eigenvector decomposition. It can compute much faster than any of the algorithms that were introduced before. The approximation error is given by

$$\varepsilon_e = \|\underline{W} - \underline{A}_e\underline{A}_e^+\|^2 \tag{6.38}$$

A similar procedure can also be carried out to determine the odd-indexed sequence.

7.0 APPLICATION

This chapter presents two experiments to test the reconstruction algorithms discussed in Chapter 6. The first experiment is an application for noise filtering. It is, in essence, to reconstruct a signal from the modified version of the pseudo-Wigner distribution after a smoothing process in the time-frequency plane. In the second experiment, separation of multicomponent signals is considered. In the case of multicomponent signals, the bilinear structure of Wigner distribution is known to create cross-terms without any physical significance. In order to reduce these unwanted cross-terms, a smoothing process is applied to the time-frequency distribution. In our example, two components of the signal overlap in both time and frequency domains. However, they are discernible and separable in the time-frequency plane. The smoothed pseudo-Wigner distribution is modified, and a component of the signal is separated and reconstructed from modified smoothed Wigner distribution.

7.1 NOISE FILTERING

In this experiment, we reconstruct a signal from the smoothed version of the pseudo-Wigner distribution. A smoothed pseudo-Wigner estimator is defined as [10],

$$SPW(n,\theta) = \sum_{k=-N+1}^{N-1} |h(k)|^2 \cdot \sum_{\ell=-M+1}^{M-1} g(\ell)x(n+\ell+k)x^*(n+\ell-k)e^{-j2k\theta} \qquad (7.1)$$

where $g(\ell)$ is a symmetric, normed data window, this smoothing process only smooths the distribution in the time direction independently from the frequency smoothing performed by h(k). It is then much more versatile than the double smoothing used in spectrogram.

As a first example, Fig. 13 shows the pseudo-Wigner distribution for a signal+noise case, i.e.,

$$x(t) = 50 \cos(\pi t/8) + n(t), \quad t=0,1,...,300 \qquad (7.2)$$

where n(t) is a zero-mean Gaussian white noise with variance $\sigma^2 = 25^2$, h(k) is a normed Hamming window, i.e.,

$$h(k) = \frac{0.54 + 0.46\cos(2k\pi/2L)}{2L+1} \qquad |k| < L$$

$$= 0 \qquad |k| \geq L \qquad (7.3)$$

We set L=25, $\Delta\omega=\pi/49$ and sampling period T=1. Time origin t_0=48. Fig. 14 shows x(t) in time domain. Fig. 15 and Fig. 16 show the uncorrupted signal and its pseudo-Wigner distribution. Smoothed versions of the pseudo-Wigner distribution are shown in Figs. 17-20 by using the smoothed pseudo-Wigner estimator (7.1). The data window $g(\ell)$ is a rectangular window and the degree of smoothing is specified on each figure as M. Increasing M from 6 to 50 clearly shows the noise reduction. In order to compare the synthesis results with the input, we define the input signal-to-noise ratio as:

$$SNR_{in} = \sum_k f^2(k) \,/\, \sum_k n^2(k) \tag{7.4}$$

And the output signal-to-noise ratio is defined as:

$$SNR_{out} = \sum_k f^2(k) \,/\, \sum_k (f(k) - c\hat{f}(k))^2 \tag{7.5}$$

where

$$c = \sqrt{\sum f^2(k) \,/\, \sum \hat{f}^2(k)}$$

$\hat{f}(k)$ is the reconstructed signal. The reconstructions from Fig. 17 and Fig. 20 are given in Fig. 21 and Fig. 22, respectively, using the synthesis method developed in Section 6.1. Input signal-to-noise ratio is 2.5 dB. Signal-to-noise ratios corresponding to Fig. 21 and Fig. 22 are -3 dB and 24.3 dB, respectively. Fig. 21 and Fig. 22 show that the phase of the reconstructed signal is different from the original signal (Fig. 15). Consequently, the signal can be recovered up to a phase factor only. Fig. 23 and Fig. 24 show the reconstructed signal corresponding to Fig. 17, and Fig. 20 using the overlapping method developed in Section 6.2 with 10 unknown samples being reconstructed in each window section and the length of each section is 49. Input signal-to-noise ratio is 3.2 dB. Signal-to-noise ratios corresponding to Fig. 23 and Fig. 24 are 3.9 dB and 17.7 dB. Fig. 23 and Fig. 24 show that the phase of the reconstructed signal is same as the original signal. Indeed, the overlapping method reserves the phase information.

In the second example, we consider the cases:

$$f(t) = 50 \cos(\pi t^2/1000), \qquad t=0,1,\ldots,300 \tag{7.6}$$

$$x(t) = f(t) + n(t) \qquad t=0,1,\ldots,300 \tag{7.7}$$

Fig. 25 and Fig. 26 show the signals $f(t)$ and $x(t)$ in time domain, and Fig. 27 and Fig. 28 are the pseudo-Wigner distribution of the signals. We use the same white noise function $n(t)$ and same window function $h(k)$ as in the first example with $L=61$, $\Delta\omega=\pi/121$, and sampling period $T=1$. Time origin $t_0=60$. The smoothed versions of pseudo-Wigner distribution of signal $x(t)$ are given in Fig. 29 and Fig. 30. The reconstructed signals from Fig. 30 are given in Fig. 31 and Fig. 32 using the synthesis methods developed in Section 6.1 and Section 6.2, respectively.

The above two examples show the ability of the smoothed pseudo-Wigner estimator to discriminate the signal from the noise, and the signals are successfully reconstructed from these modified versions of pseudo-Wigner distribution using the synthesis methods developed in Chapter 6.

7.2 SIGNAL SEPARATION

In the second experiment, we reconstruct monocomponent signals from the smoothing version of multicomponent pseudo-Wigner distribution. If a signal x(n) happens to be multicomponent, it could be written as combination of monocomponent ones:

$$x(n) = \sum_{i=1}^{N} x_i(n) \tag{7.8}$$

The Wigner distribution of this signal x(n) is given by

$$W_x(n,\theta) = \sum_{i=1}^{N} W_{x_i}(n,\theta) + 2\,\mathrm{Re} \sum_{i=1}^{N} \sum_{j=1}^{i} W_{x_i x_j}(n,\theta), \tag{7.9}$$

and a similar relation holds for pseudo-Wigner distribution. Hence, the bilinear structure of Wigner distribution adds cross-terms without any real physical significance to the sum of the Wigner distribution of each component. Fig. 33, Fig. 34 and Fig. 35 show two windowed FM chirp signals and the sum of them, i.e.,

$$f(t) = 50 \cos (\pi t^2/1450)\, e^{-(t-140)^2/1000} \tag{7.10}$$

$$g(t) = 50 \cos (\pi t^2/900)\, e^{-(t-140)^2/1000} \tag{7.11}$$

$$x(t) = f(t) + g(t) \tag{7.12}$$

$f(t)$ and $g(t)$ overlap in both time and frequency domains. The pseudo-Wigner distribution of $x(t)$ is given in Fig. 36 using the same Hamming window Eq.(7.3), where $L=61, \Delta\omega=\pi/121$ and sampling period $T=1$. Fig. 36 shows that $f(t)$ and $g(t)$ are clearly discernible and separable in the time-frequency plane, and the pseudo-Wigner distribution in Fig. 36 is modified to produce the modified pseudo-Wigner distribution of $f(t)$ in Fig. 37. Fig. 38 and Fig. 39 show the contour plots of Fig. 36 and Fig. 37. Fig. 40 shows the smoothed pseudo-Wigner distribution of Fig. 36 with the degree of smoothing $M=7$, and Fig. 40 is modified to produce the smoothed time-frequency function of $f(t)$ in Fig. 41. Fig. 42 and Fig. 43 show the contour plots of Fig. 40 and Fig. 41. The reconstructed signal from Fig. 37 and Fig. 41 are shown in Fig. 44 and Fig. 45, respectively, using the outer product synthesis method. This example shows that the pseudo-Wigner distribution synthesis algorithm can be used for signal separation.

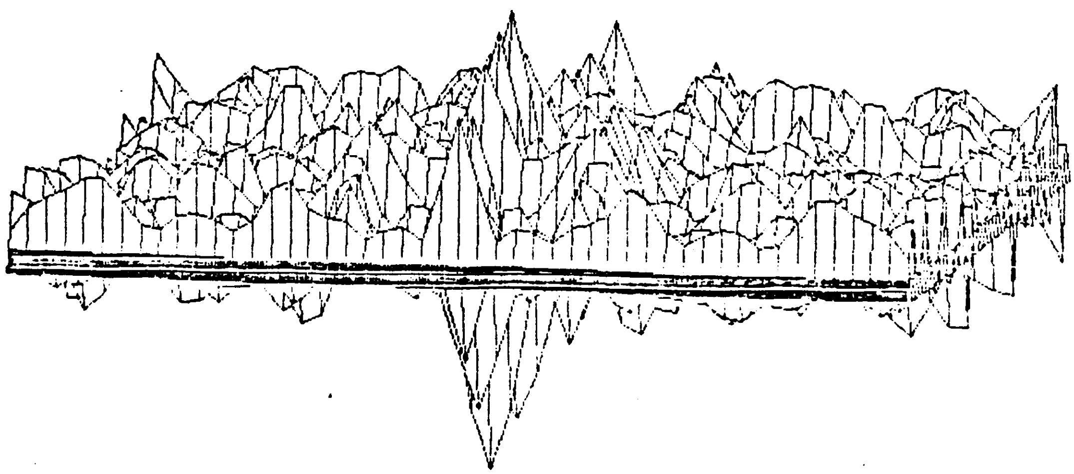

Figure 13. The PWD of Eq.(7.2).

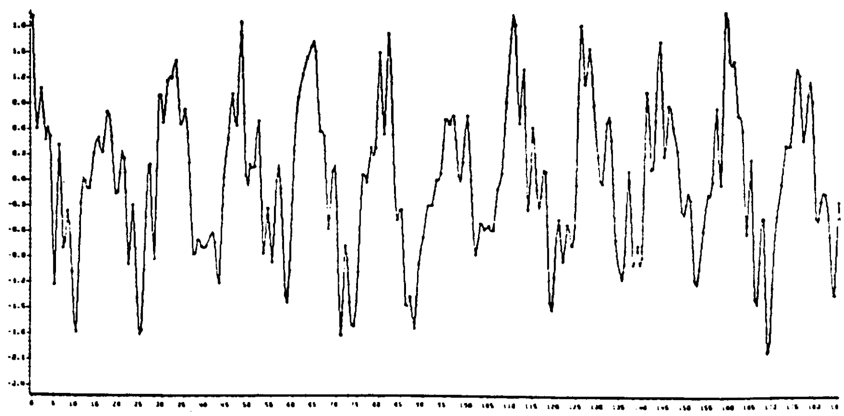

Figure 14. The signal x(t) in time domain.

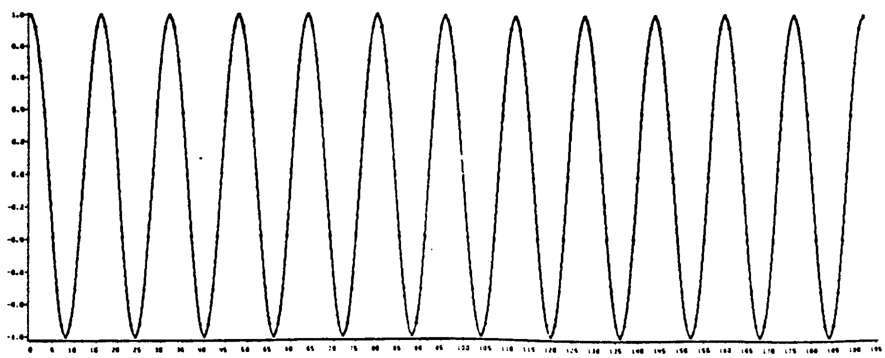

Figure 15. The signal $f(t)=\cos(\pi t/8)$.

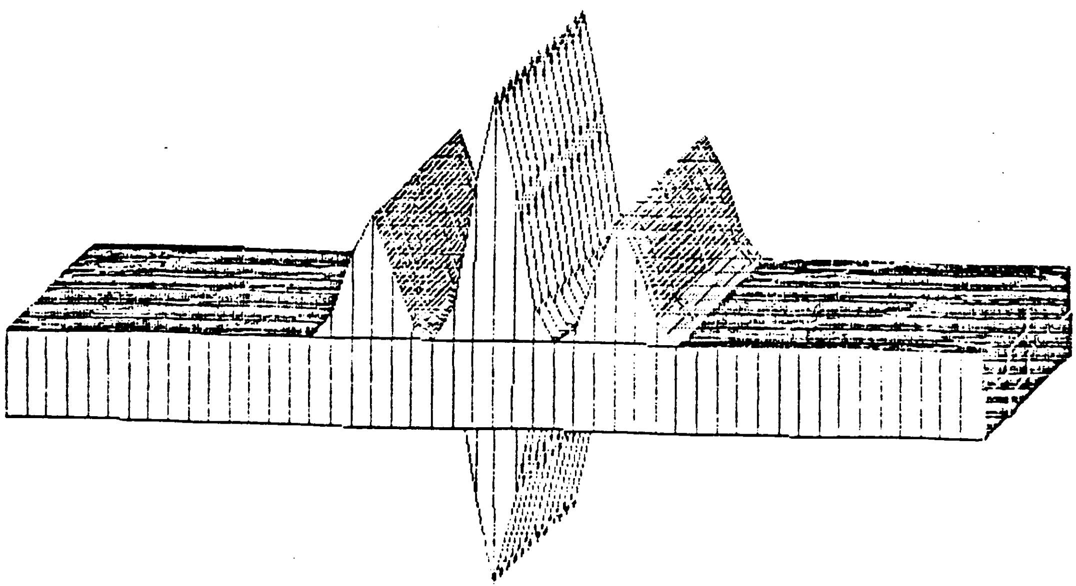

Figure 16. The PWD of Fig. 15.

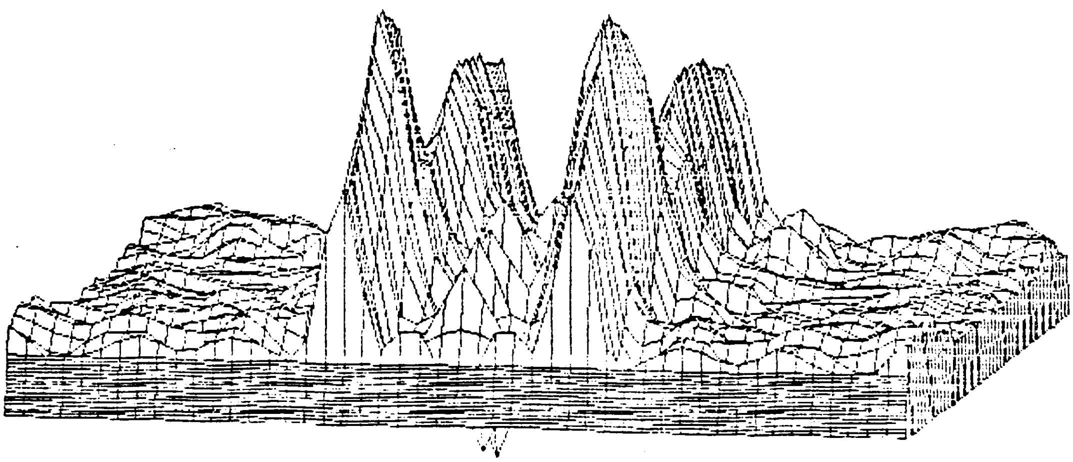

Figure 17. The smoothed PWD with M=6.

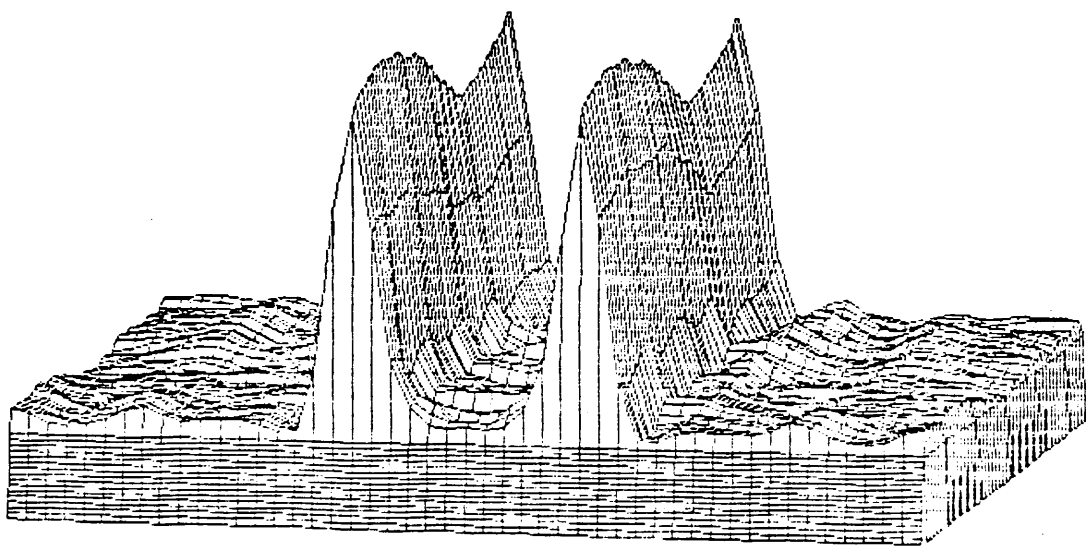

Figure 18. The smoothed PWD with M=24.

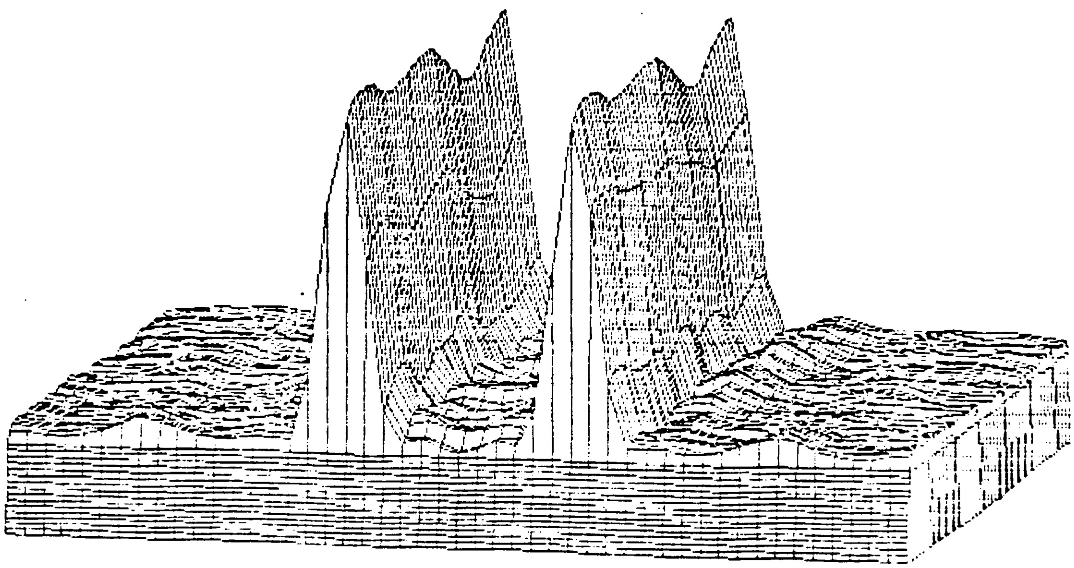

Figure 19. The smoothed PWD with M=36.

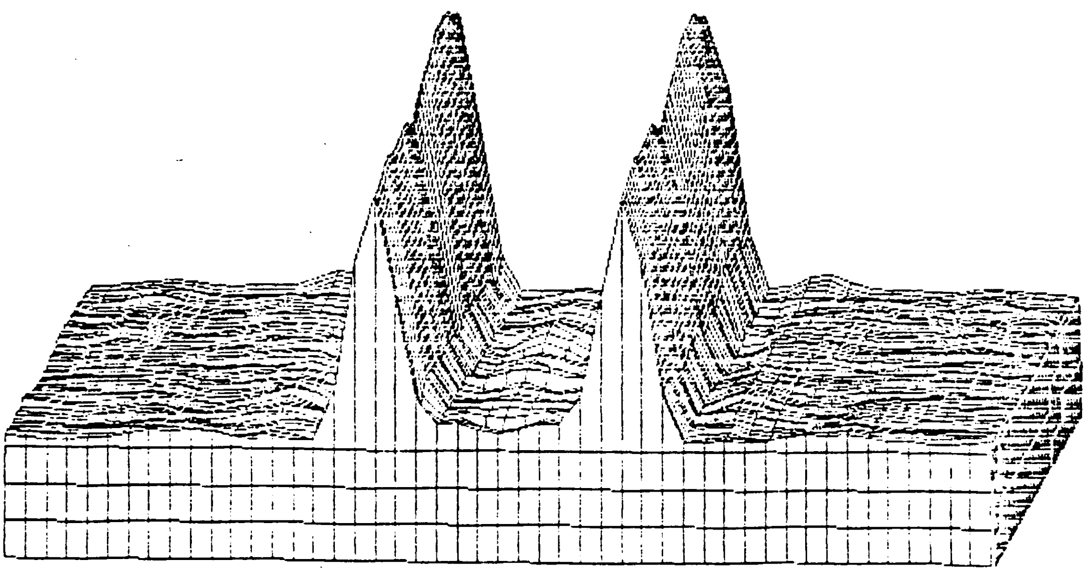

Figure 20. The smoothed PWD with M=50.

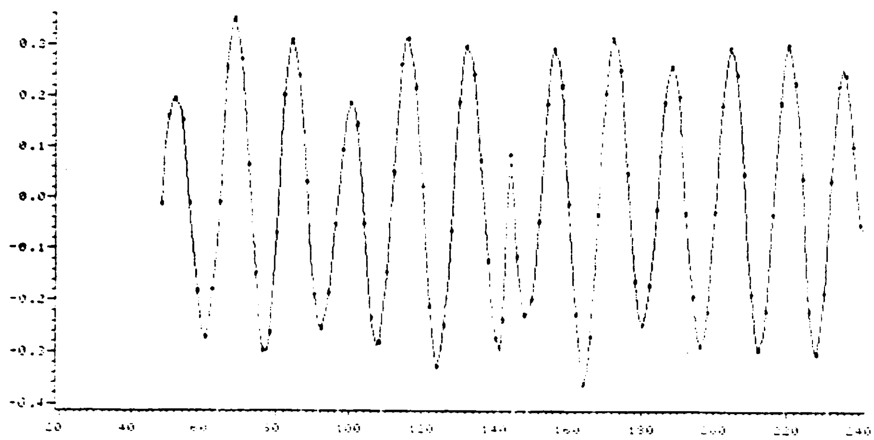

Figure 21. The f(t) recovered from Fig.17.

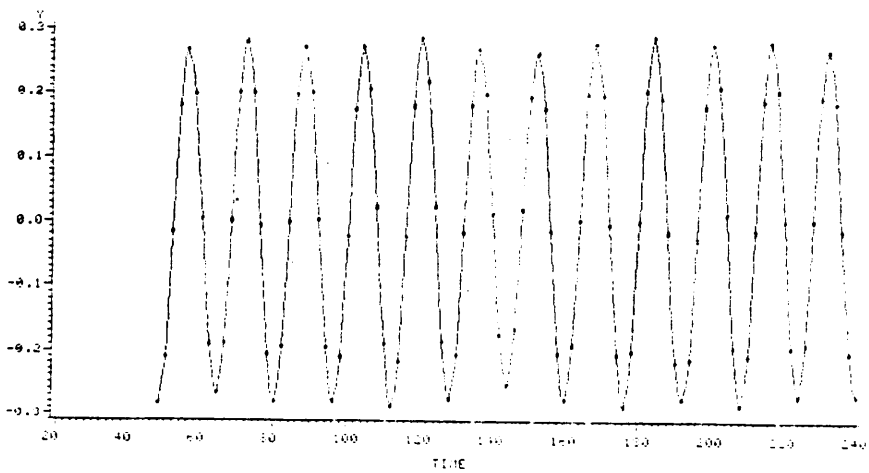

Figure 22. The f(t) recovered from Fig.20.

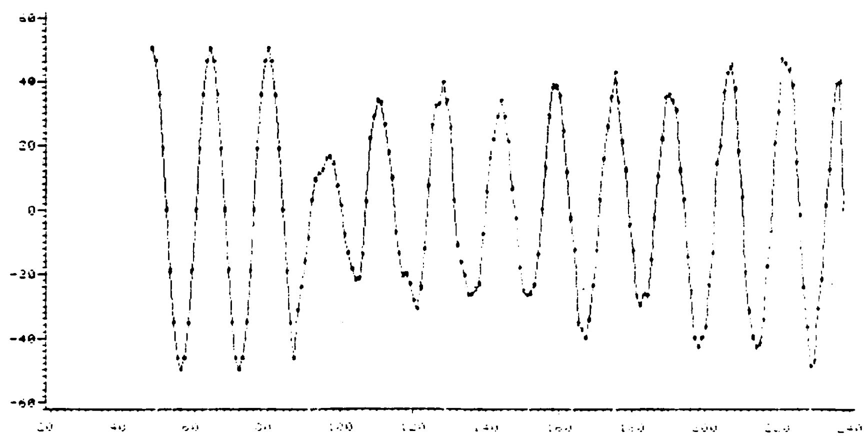

Figure 23. The f(t) recovered from Fig.17.

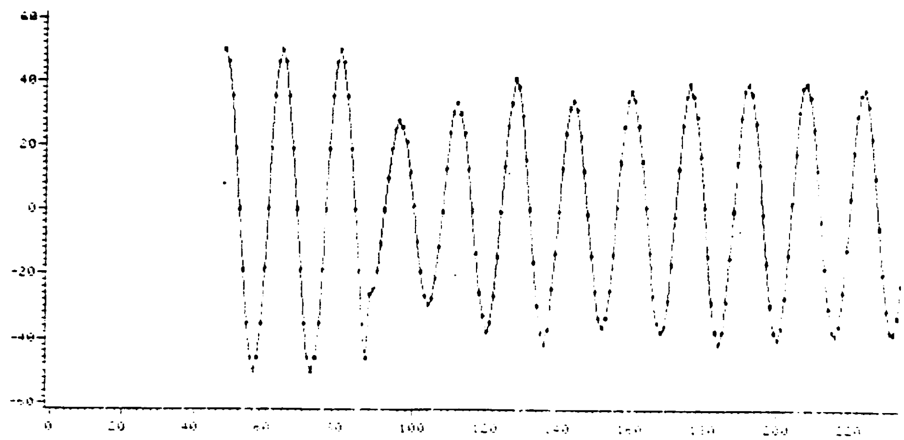

Figure 24. The f(t) recovered from Fig.20.

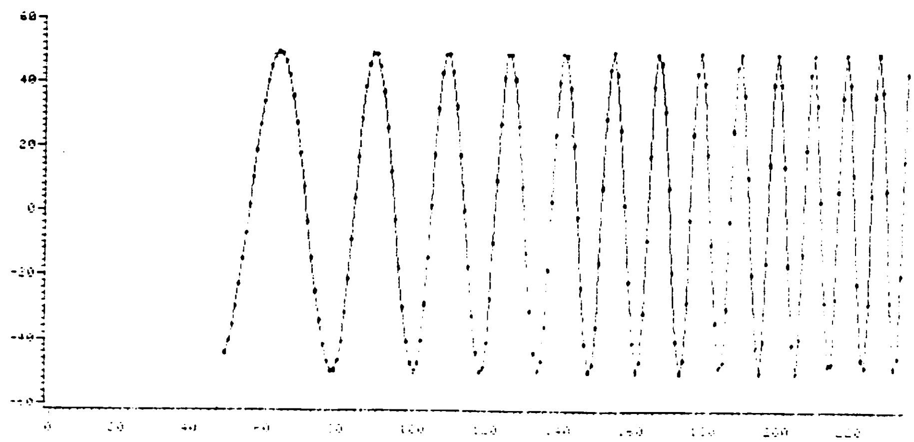

Figure 25. The signal f(t) of Eq.(7.6).

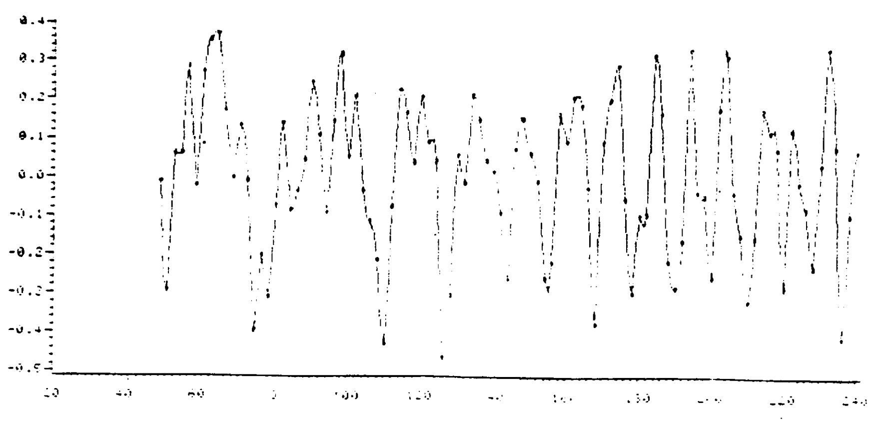

Figure 26. The signal x(t) of Eq.(7.7).

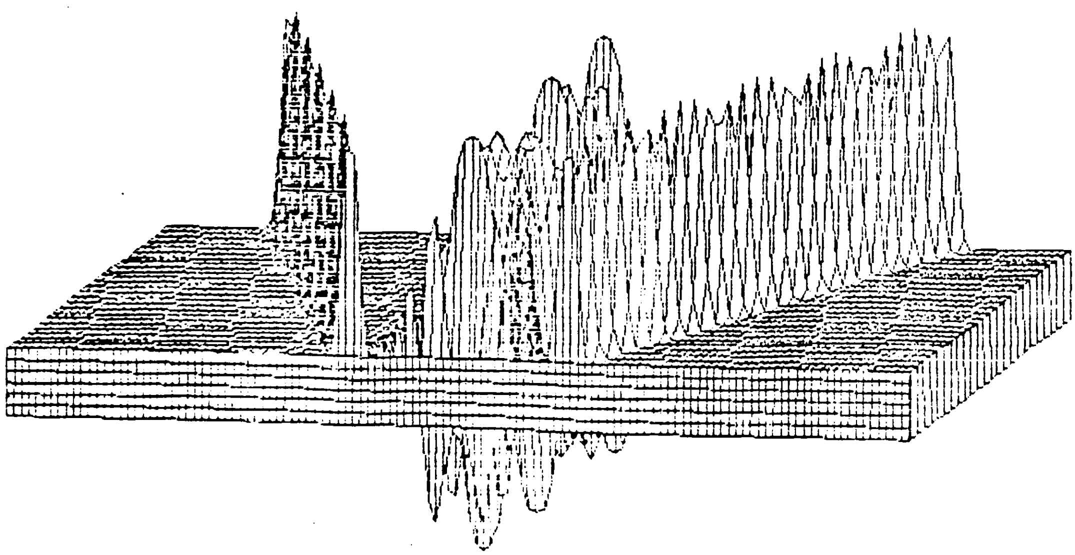

Figure 27. The PWD of Fig.25.

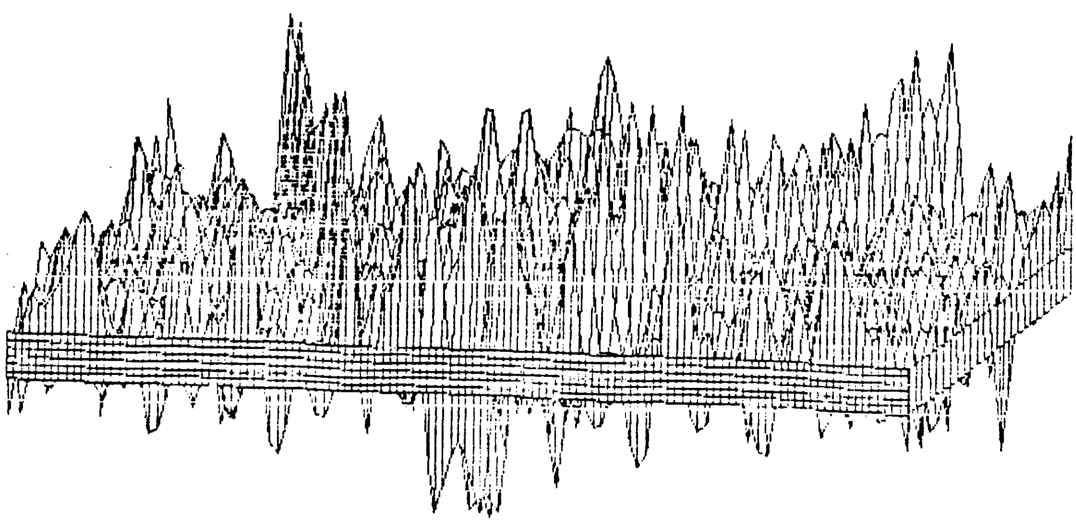

Figure 28. The PWD of Fig.26

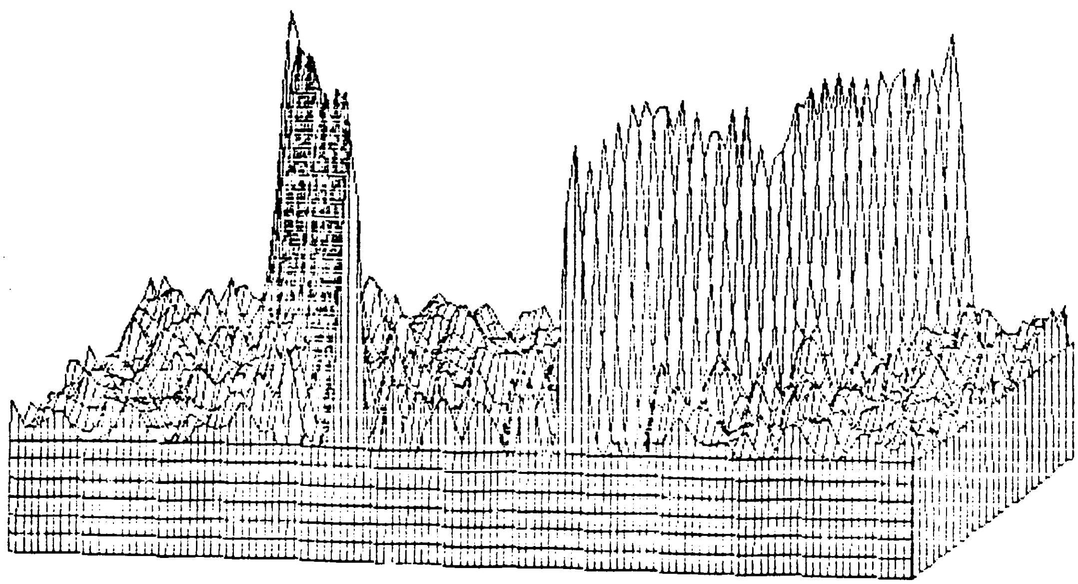

Figure 29. The smoothed PWD of Fig.28 with M=6.

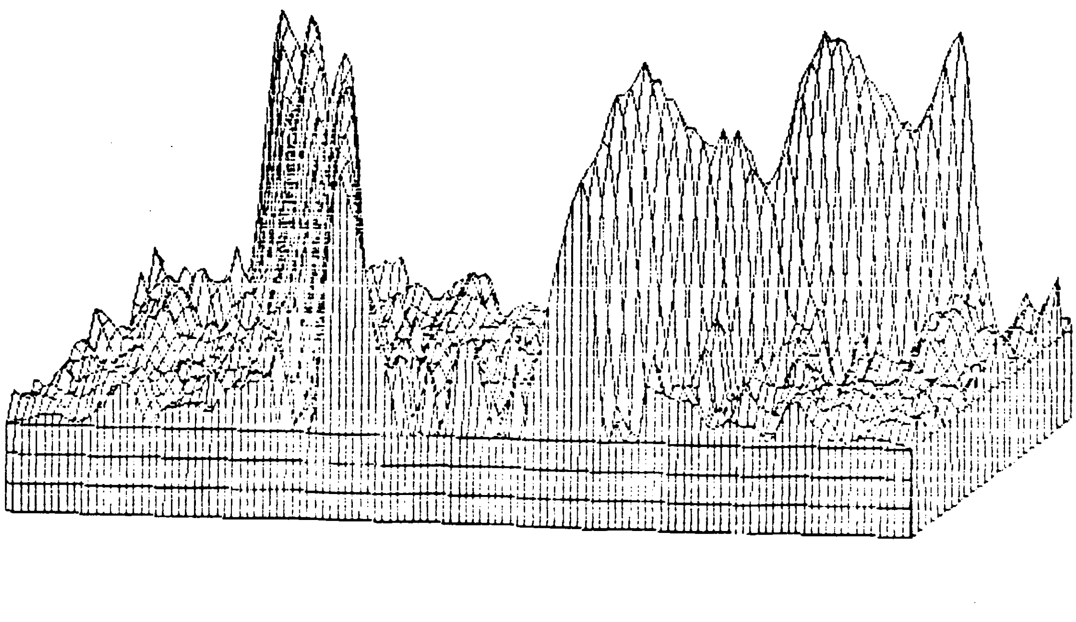

Figure 30. The smoothed PWD of Fig.28 with M=12.

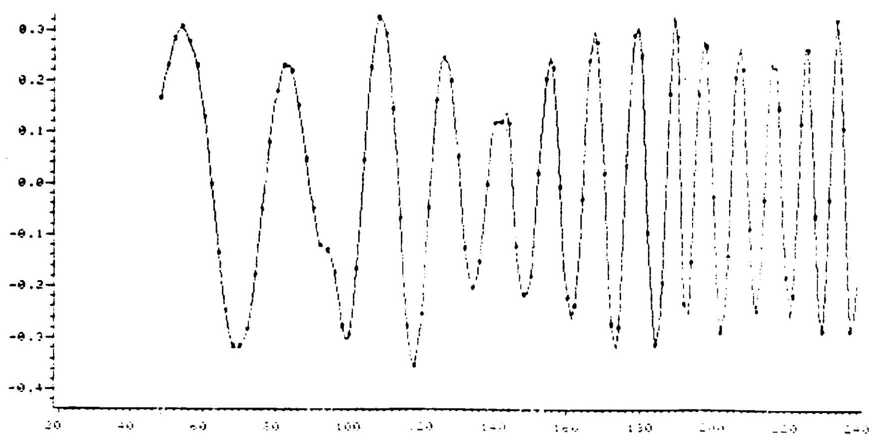

Figure 31. The f(t) recovered from Fig.30.

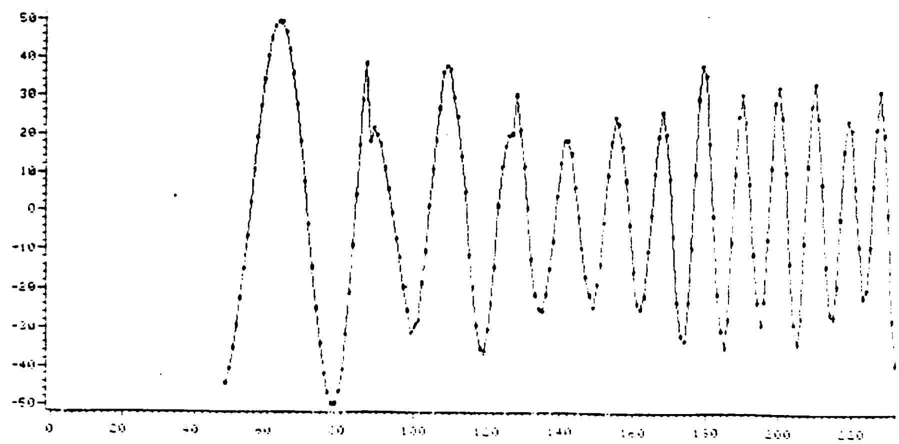

Figure 32. The f(t) recovered from Fig.30.

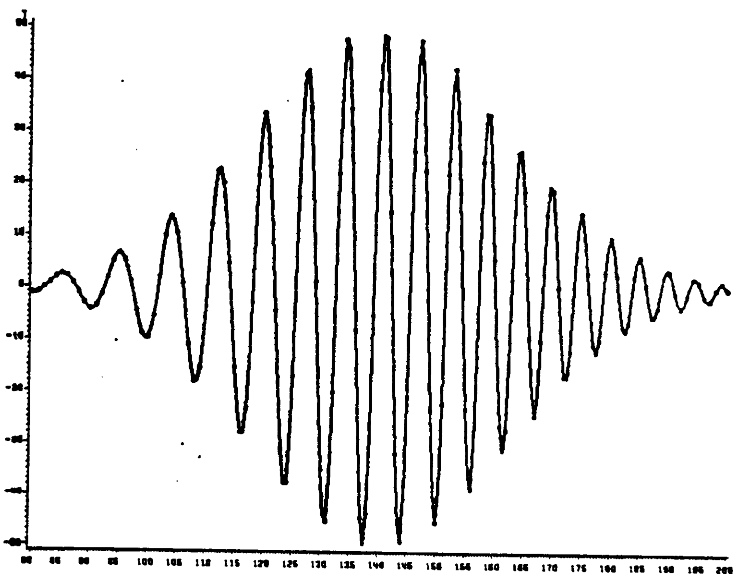

Figure 33. The signal f(t).

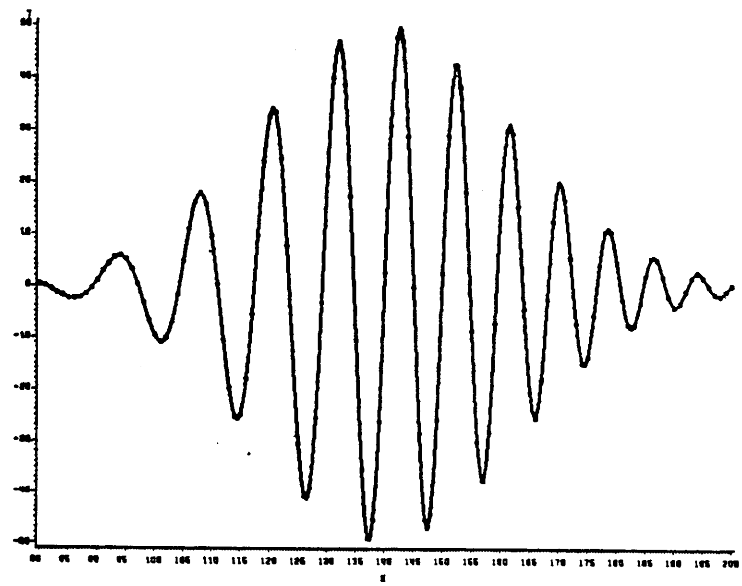

Figure 34. The signal g(t).

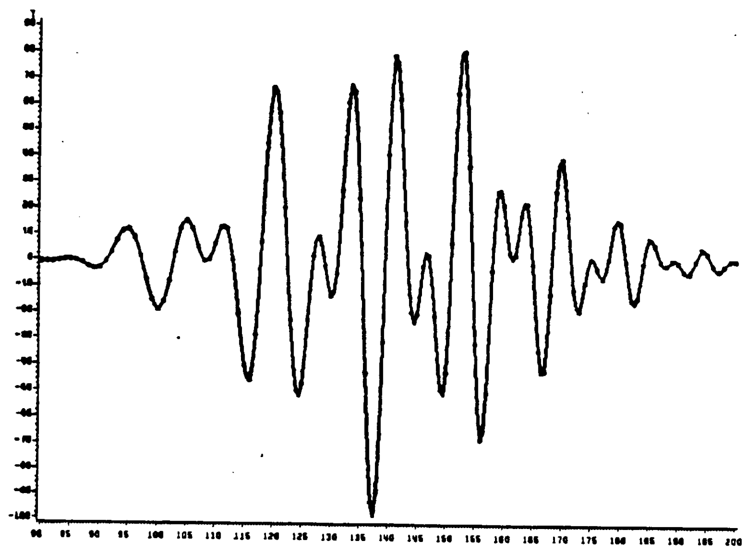

Figure 35. The signal x(t)=f(t)+g(t).

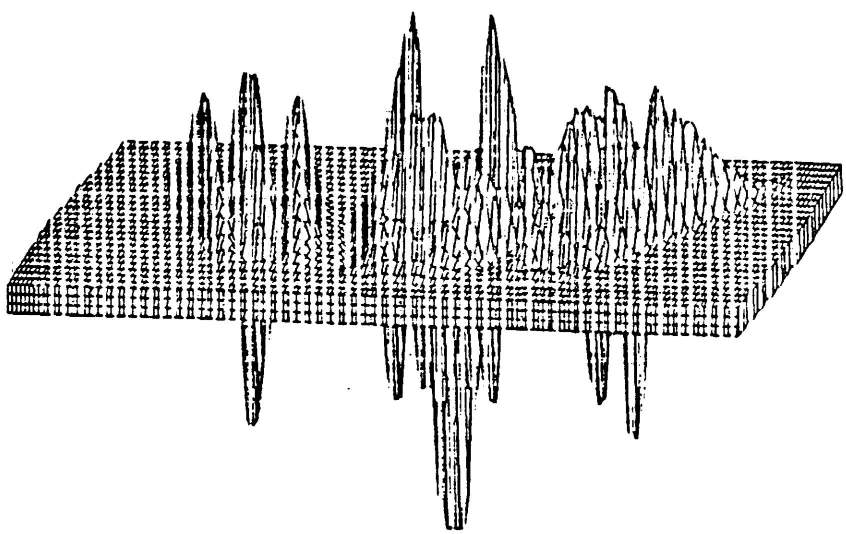

Figure 36. The PWD of x(t).

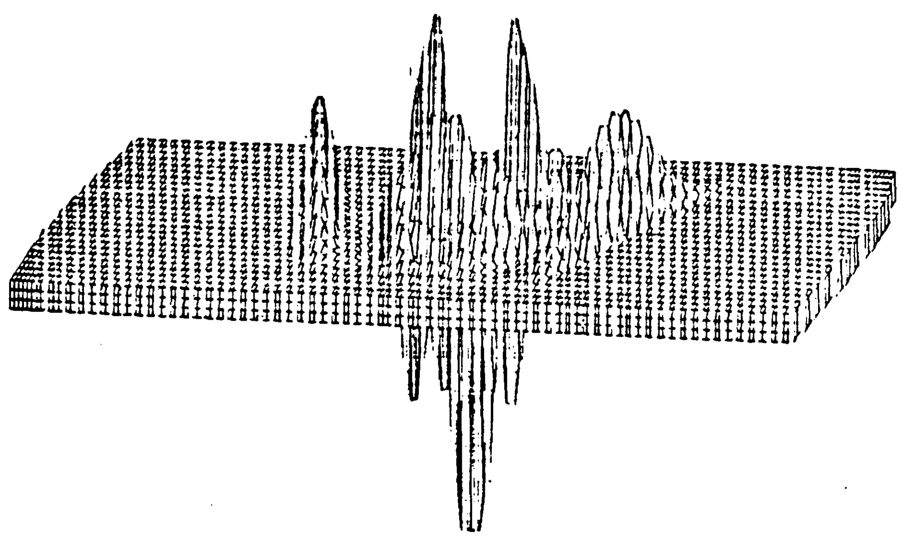

Figure 37. The modified PWD of Fig.36.

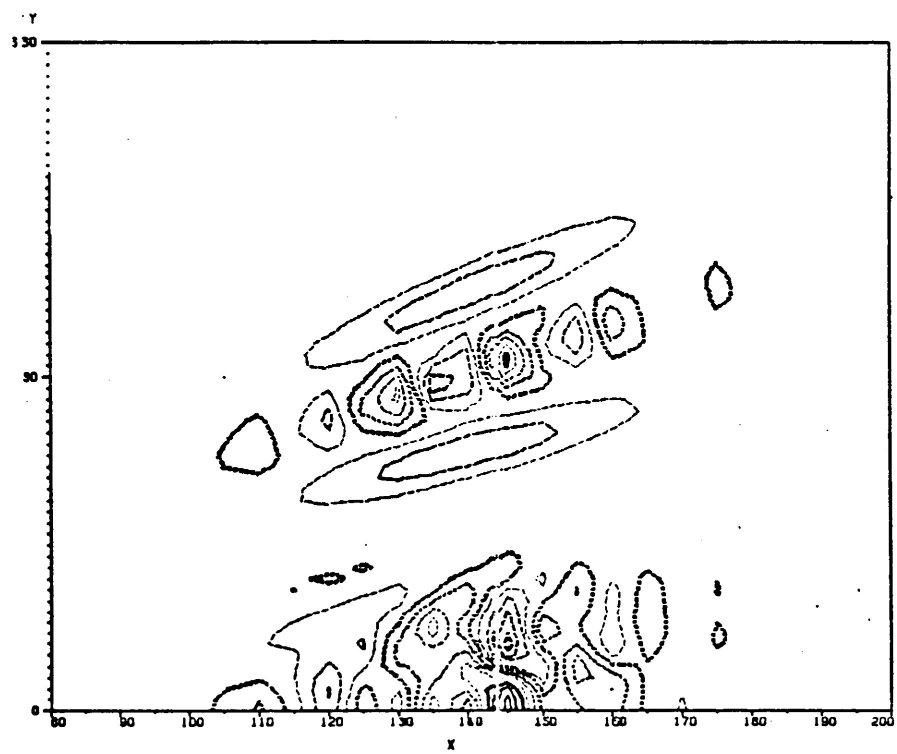

Figure 38. The contour plot of Fig.36.

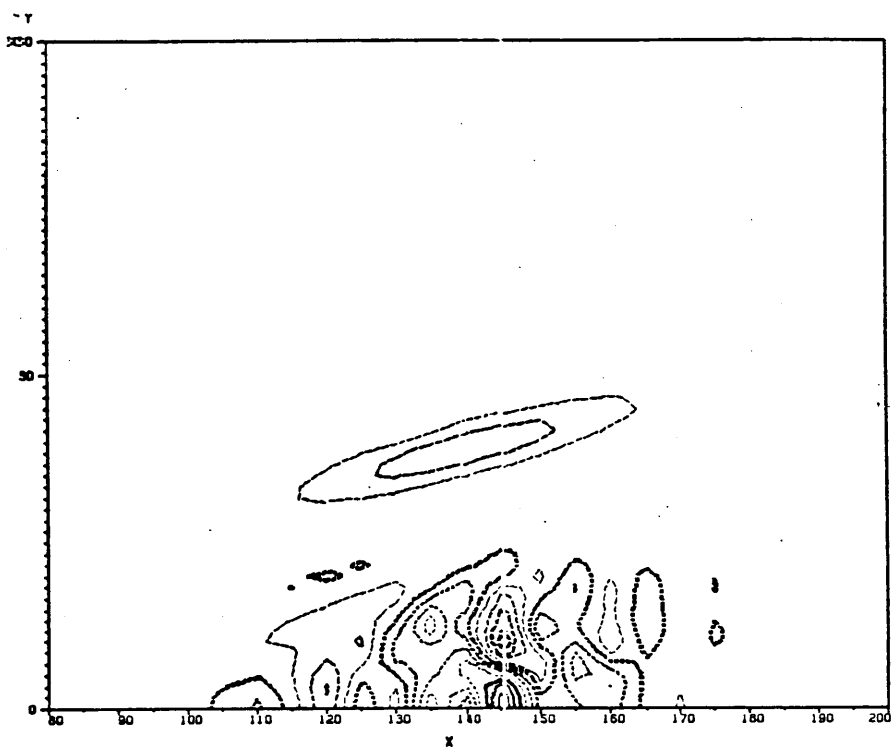

Figure 39. The contour plot of Fig.37.

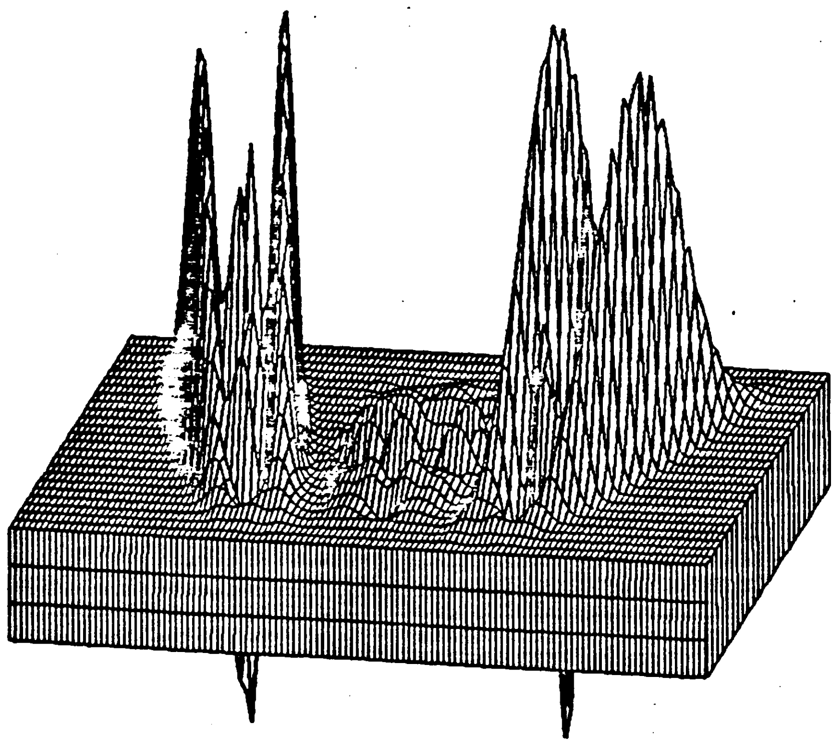

Figure 40. The smoothed PWD of Fig.36.

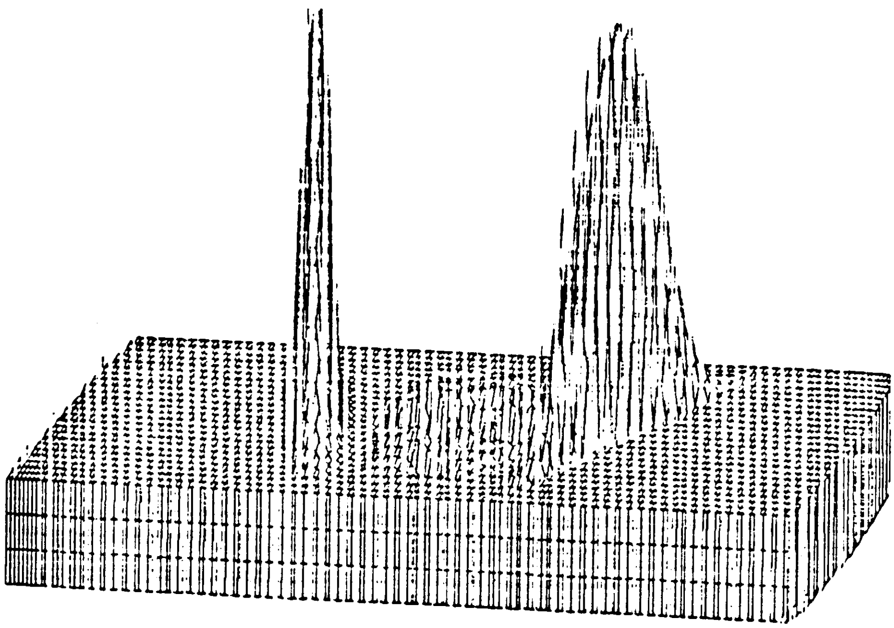

Figure 41. The modified PWD of Fig.40.

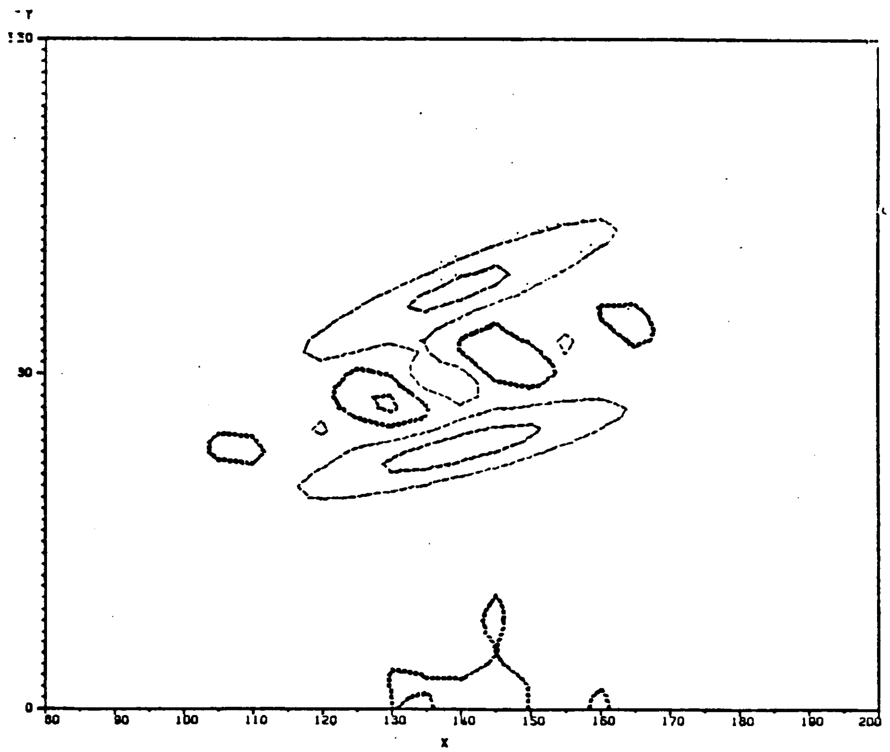

Figure 42. The contour plot of Fig.40.

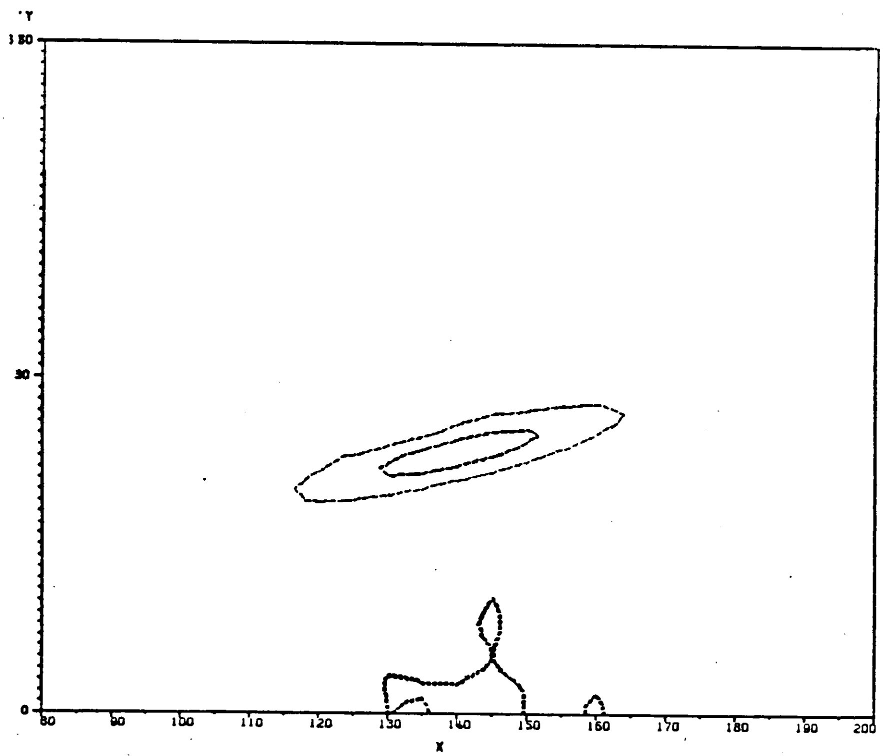

Figure 43. The contour plot of Fig.41.

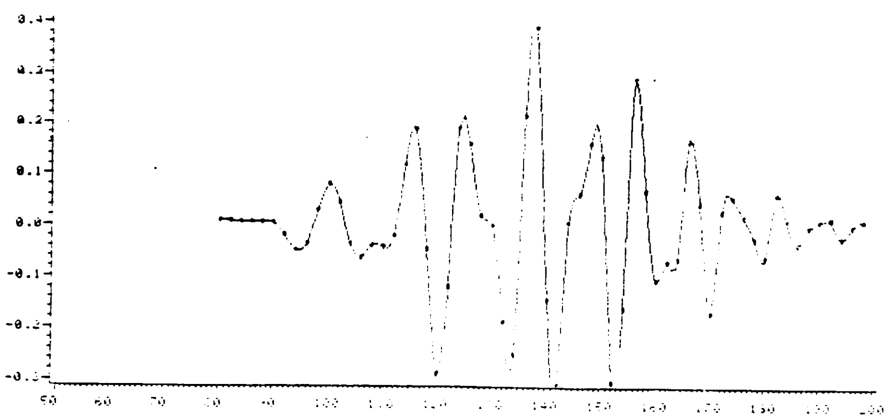

Figure 44. The f(t) recovered from Fig.37.

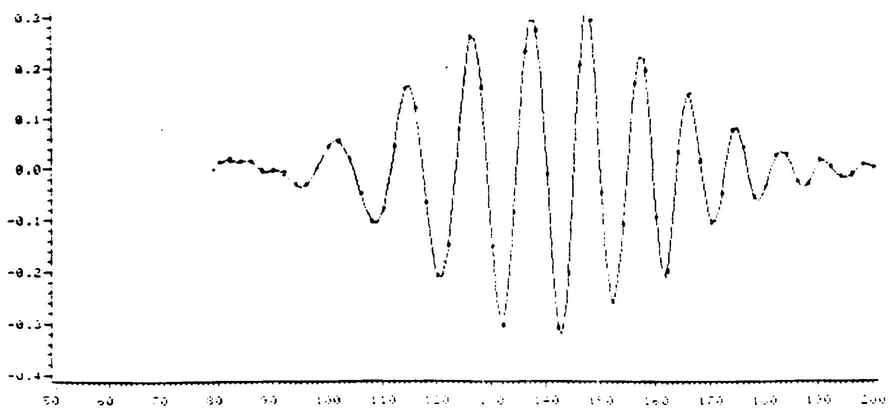

Figure 45. The f(t) recovered from Fig.41.

8.0 CONCLUSION

Wigner distribution is a mapping of a one dimensional signal into a two dimensional signal on the time-frequency plane. It possesses many important properties which make it very attractive for non-stationary signal analysis. Some important theoretical properties of discrete-time Wigner distribution have been reviewed in the first part of the thesis. Aliasing behavior of discrete-time Wigner distribution has been discussed with three methods being given to avoid the aliasing effect. The particular attention of this thesis has been given to develop signal synthesis algorithms. It has been shown that the inverse Fourier Transform of discrete-time Wigner distribution was a two dimensional time function which can be viewed as an outer product of two one dimensional functions.

The contribution of this work is developing the synthesis algorithms to reconstruct the signal from discrete-time Wigner distribution or from discrete-time pseudo-Wigner distribution. If the given distribution is a valid Wigner distribution or valid pseudo-Wigner distribution, signal recovery is possible up to a constant factor, and it requires a different procedure for even-indexed samples and odd-indexed samples because even-indexed and odd-indexed samples are not coupled. If the given distribution is not a valid Wigner distribution or valid pseudo-Wigner distribution, a least-square approximation procedure is formulated to recover the signal which minimizes the total square error. A previous work in synthesis of discrete-time Wigner distribution has been done by Boudreaux-Bartels and Parks [9] using least-square procedures. A new synthesis algorithm has been developed in the thesis which involves a factorization procedure where the desired one-dimensional signal sequence can be recovered as the solution of this factorization problem. In addition, two synthesis methods for discrete-time pseudo-Wigner distribution have been developed in the thesis and successfully applied to noise filtering and signal separation problems. Furthermore, these synthesis methods can be applied to test whether a given time-frequency function is a valid Wigner distribution.

An open question arises from these results and serves as a point of departure for further research. Since the synthesis methods developed in the thesis can only be applied to deterministic signals, to develop a synthesis algorithm for stochastic signal processing is one of the interesting topic for further research. In addition, the comparison of smoothing in time-frequency plane with smoothing in time domain, the window sizes and types, and the smoothing effects on different signals should receive further investigations.

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