SOME PROBLEMS IN THE AXIALLY SYMMETRICAL BENDING
OF A THICK CIRCULAR PLATE
RESTING ON AN ELASTIC FOUNDATION

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\( \beta, b, \alpha \) Parameters
\[ \beta = \frac{(2-\nu)}{[10(1-\nu)]} \]
\[ b^2 = \frac{h^2 \beta}{L^2} \]
\[ \alpha = \cos^{-1} \left( -\frac{b^2}{2} \right) \]

\( U_0, U_1 \) Real parts of Bessel functions of the first kind

\( V_0, V_1 \) Imaginary parts of Bessel functions of the first kind

\( \bar{U}_0, \bar{U}_1 \) Real parts of Bessel functions of the second kind

\( \bar{V}_0, \bar{V}_1 \) Imaginary parts of Bessel functions of the second kind
IV. INTRODUCTION

If a thin plate is bent with small deflection, i.e., when the deflection of the middle plane is small compared with the thickness of the plate, the following assumptions can be made in the classical theory:

(1) the thickness of the plate is small in comparison with other dimensions.

(2) the displacements are small, so that the quantities of the second and higher order are neglected, hence the linearity of the resulting differential equations is established.

(3) the component of stress normal to the middle surface is small compared with other components of stress, hence it may be neglected in the stress-strain relations.

(4) plane cross-sections normal to the undeformed middle surface remain plane and normal to the deformed middle surface.

In dealing with a thick plate under bending, the effect of shear deformation and the component of stress normal to the middle surface must be taken into account. A theory including this effect was developed by E. Reissner in 1945 [1], [2]e.

The number listed in the parenthesis gives the reference which is listed by the same number in the bibliography.
In this paper, it is desirable to adapt Reissner's theory to the axially symmetrical bending of a thick circular plate with a circular hole at the center, resting on an elastic foundation. The solutions will be expressed in terms of Bessel functions of the first and second kind with a complex argument. The resulting moments and deflection will be compared with the results obtained from the formulas based on the classical theory.

The theory of circular plates on an elastic foundation has direct application to the analysis and design of footings and foundations of buildings where platelike elements are surrounded by soil such that their behavior under applied loads is approximated by the assumptions of the underlying analysis.
V. THE REVIEW OF LITERATURE

Bending of circular plates resting on an elastic foundation have been discussed by H. Hertz, A. Föppl, F. Schleicher, M. Hetenyi, S. Timoshenko and other authors (3). In their treatment, these authors have restricted their discussion to thin circular plates, neglecting the effect of transverse-shear deformation. A theory of bending of elastic plates accounting for this effect was given by Eric Reissner. The Reissner's theory was developed from Castigliano's theorem of least work which states that if an elastic system is acted on by given forces on the boundary, and the changes of stress components which satisfy the equations of equilibrium and the boundary conditions are to be considered, the true stress components are those that make the strain energy minimum.

The fundamental equations of Reissner's theory are obtained by an application of the above theorem combined with the Lagrangian multiplier method of the calculus of variations.

Reissner's theory has been applied to the solution of several problems in the static bending of plates. He solved the problem of the bending and twisting of an infinite plate with a circular hole. Naghdi and Rowley (4) extended the theory to include an elastic foundation which reacts with pressure proportional to the displacement at every point of
the foundation and solved two problems involving axially-symmetric bending of an infinite plate. Naghdi (5) also solved the problem of the bending and twisting of an infinite plate with an elliptic hole. D. Frederick (6) applied the theory to solve problems of (a) the axially symmetric bending of a finite circular plate on an elastic foundation under a partial uniform loading, (b) the non-symmetric bending of an infinite plate on an elastic foundation with a rigid circular inclusion and (c) the non-symmetric bending of a clamped circular plate with no foundation due to an eccentric concentrated load. A more detailed history on Reissner's theory is also found in reference (5).
VI. THE INVESTIGATION

A. METHOD OF PROCEDURE

The Basic Equations

Based on Reissner's theory, it is assumed that the bending stresses are distributed linearly over the thickness of the plate as in the standard theory of thin plates, i.e.,

\[ \sigma_r = \frac{M_r}{h^2/6 \times h/2} \]

\[ \sigma_\theta = \frac{M_\theta}{h^2/6 \times h/2} \]  \[ \text{[1]} \]

\[ \tau_{r\theta} = \frac{M_{r\theta}}{h^2/6 \times h/2} \]

In Equations [1] \( M_r \) and \( M_\theta \) are the bending couples, \( M_{r\theta} \) the twisting couple, \( h \) the thickness of the plate (which is assumed to be uniform). The coordinate system and the usual stress resultants in their positive directions are shown in Figure 1.

From Equations [1] and by means of the differential equations of equilibrium (neglecting the body force):

\[ \frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \]

\[ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{z\theta}}{\partial z} + \frac{2 \tau_{r\theta}}{r} = 0 \]  \[ \text{[2]} \]

\[ \frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{r\theta}}{r} = 0 \]
there are obtained expressions for transverse shear stresses and transverse normal stress

\[ T_{rs} = \frac{V_r}{2h^3} \left[ 1 - \left( \frac{z}{h/2} \right)^2 \right] \]

\[ T_{\theta z} = \frac{V_\theta}{2h^3} \left[ 1 - \left( \frac{z}{h/2} \right)^2 \right] \]  

\[ \sigma_z = -\frac{1}{2}(q_1 + q_2) + (q_1 - q_2) \left( \frac{2z}{2h} - \frac{2z^3}{3h^3} \right) \]  

These satisfy the following boundary conditions automatically:

\[ T_{rs}(r, \theta, \pm h/2) = 0 \]

\[ T_{\theta z}(r, \theta, \pm h/2) = 0 \]

\[ \sigma_z (r, \theta, -h/2) = -q_1 \]

\[ \sigma_z (r, \theta, +h/2) = -q_2 \]

where \( q_1 \) is the pressure on the upper face and \( q_2 \) is the pressure on the lower face of the plate. For a plate resting on an elastic foundation, \( q_2 \) represents the reaction of the foundation and is equal to \(-kw\); \( k \) is called the modulus of the foundation, usually expressed in pounds per square inch per inch of deflection.

Taking \( u_r \), \( u_\theta \) and \( w \) as the displacements in \( r- \), \( \theta- \) and \( z- \) directions, it is consistent with the assumption of linear bending stress distribution to assume that the displacements \( u_r \) and \( u_\theta \) vary linearly over the thickness of the plate and that \( w \) does not vary over the thickness of the plate, i.e.,
\[ u_r = \gamma_r z \]
\[ u_\phi = \gamma_\phi z \]
\[ w = \bar{w} \]

where \( \gamma_r \) and \( \gamma_\phi \) are rotations of a line element originally perpendicular to the middle surface about a point in the middle surface and \( \bar{w} \) is the vertical displacement for all points on this line. These are defined as

\[ \gamma_r = \frac{12}{h^3} \int_{-h/2}^{h/2} u_r z dz \]
\[ \gamma_\phi = \frac{12}{h^3} \int_{-h/2}^{h/2} u_\phi z dz \]

\[ \bar{w} = \frac{3}{2h} \int_{-h/2}^{h/2} w \left[ 1 - \left( \frac{z}{h/2} \right)^2 \right] dz \]

**Fundamental Equations for the Bending of Circular Plates**

From Reissner's Variational theorem (1), the fundamental equations for the bending of a plate on an elastic foundation in polar coordinates are:

\[ D \nabla^4 \bar{w} - \beta h^2 \nabla^2 (q_1 + k \bar{w}) + (q_1 + k \bar{w}) = 0 \]

\[ \frac{h^2}{10} \nabla^2 V_r - V_r + \frac{h^2}{10} \left( \frac{2}{r} + \frac{j \nu_r}{3 r} + \frac{V_r}{r^2} \right) = \frac{\partial}{\partial r} \frac{3}{r^2} \nabla^2 \bar{w} \]

\[ - \frac{h^2}{10(1-\nu)} \frac{\partial}{\partial r} (q_1 + k \bar{w}) + \frac{h^2}{r} \frac{q_1 + k \bar{w}}{r} \]

[5]

[6]

[7]
\[
\frac{h^2}{10} \nabla^2 v_\theta - V_\theta + \frac{h^2}{10} \left( \frac{2}{r^2} \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r^2} \right) = D \frac{1}{r} \frac{\partial}{\partial r} \nabla^2 \bar{w}
\]

\[
\frac{h^2}{10(1-\nu)} \frac{1}{r} \frac{\partial}{\partial \theta} \left( q_1 + k \bar{w} \right)
\]

\[
M_r = - \frac{h^2}{5} \frac{\partial v_\theta}{\partial r} - \frac{v h^2}{10(1-\nu)} \left( q_1 + k \bar{w} \right) = -D \left( \frac{\partial \bar{w}}{\partial r^2} + \frac{v}{r} \frac{\partial \bar{w}}{\partial r} \right)
\]

\[
+ \frac{v}{r^2} \frac{\partial \bar{w}}{\partial \theta^2}
\]

\[
M_\theta = - \frac{h^2}{5} \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v}{r} \right) - \frac{v h^2}{10(1-\nu)} \left( q_1 + k \bar{w} \right) = -D \left( \frac{1}{r} \frac{\partial \bar{w}}{\partial r} \right) \frac{\partial}{\partial r} \left[ \frac{\bar{w}}{r} \right] [7]
\]

\[
+ \frac{1}{r^2} \frac{\partial \bar{w}}{\partial \theta^2} + \nu \frac{\partial \bar{w}}{\partial r^2}
\]

\[
M_{r\theta} = - \frac{h^2}{10} \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + r \frac{\partial}{\partial r} \left[ \frac{v_\theta}{r} \right] \right) = -D(1-\nu) \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \bar{w}}{\partial \theta} \right)
\]

\[
\gamma_r = \frac{\partial \bar{w}}{\partial r} = \frac{6}{50}\bar{w} \quad v_r
\]

\[
\gamma_\theta = \frac{1}{r} \frac{\partial \bar{w}}{\partial \theta} = \frac{6}{50}\bar{w} \quad v_\theta
\]

where \( \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta^2} \) and \( \beta = \frac{2 - \nu}{10(1-\nu)} \)

**Governing Equations for the Axially Symmetric Bending of Circular Plates Resting on an Elastic Foundation.**

For the axially symmetric bending of circular plates resting on an elastic foundation, \( \bar{w} \) is a function of \( r \) only, and the twisting moment \( M_{r\theta} \) and shear force \( V_\theta \) vanish identically. Hence Equations [7] become

\[
D \nabla^2 \bar{w} - \beta h^2 \nabla^2 \left( q_1 + k \bar{w} \right) + \left( q_1 + k \bar{w} \right) = 0
\]

\[
M_r = - \frac{h^2}{5} \frac{dV_r}{dr} - \frac{\nu h^2}{10(1-\nu)} \left( q_1 + k \bar{w} \right) = -D \left( \frac{\partial \bar{w}}{\partial r} + \frac{\nu}{r} \frac{\partial \bar{w}}{\partial r} \right) [8]
\]
\[ \begin{align*}
\kappa_0 &= \frac{h^2}{3} \frac{V_r}{r} - \frac{\nu}{10(1-\nu)} \left( q_1 + k \bar{w} \right) = -\delta \left( \frac{1}{r^2} \frac{d^2}{dr^2} + \nu \frac{d^2}{dr^2} \right) \\
V_r &= \frac{h^2}{10} \frac{2\nu}{1-\nu} \frac{d}{dr} \left( q_1 + k \bar{w} \right) - \delta \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d^2}{dr^2} - \frac{1}{r^2} \frac{d^2}{dr^2} \right)
\end{align*} \]

where \( \nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \)

In case of a uniformly distributed load over the upper surface of the plate, \( \frac{d q_1}{dr} = 0 \).

By introducing the notations:
\[
\begin{align*}
\rho &= \bar{w} + \frac{q_1}{k} \\
b &= D/k_a \\
r &= l x_a \\
b_0 &= \frac{h^2 k_a}{k_l}
\end{align*}
\]
(since \( k \) is measured in pounds per cubic inch and \( D \) in pound inches, the quantity \( l \) has the dimension of a length; therefore, \( x \) is a dimensionless quantity), Equations [8] can be written as
\[
\begin{align*}
\nabla^4 \rho - b^2 \nabla^2 \rho + \rho &= 0 \\
\frac{\kappa_0}{k_l} &= -\left( \frac{d^2 \rho}{dx^2} + \frac{\nu}{x} \frac{d \rho}{dx} \right) - \frac{1}{5} \frac{h^2}{k_l} \frac{d^2 \rho}{dx^2} \left( \nabla^2 \rho \right) + \frac{h^2}{5} \frac{h^2}{k_l} \frac{d^2 \rho}{dx^2} \\
&+ \frac{h^2}{k_l} \frac{\nu}{10(1-\nu)} \rho \\
\end{align*}
\]

\[
\begin{align*}
\frac{\kappa_0}{k_l} &= -\left( \frac{d^2 \rho}{dx^2} + \frac{1}{x} \frac{d \rho}{dx} \right) - \frac{1}{5} \frac{h^2}{k_l} \frac{d}{dx} \left( \nabla^2 \rho \right) + \frac{h^2}{5} \frac{h^2}{k_l} \frac{d \rho}{dx} \\
&+ \frac{h^2}{k_l} \frac{\nu}{10(1-\nu)} \rho \\
\end{align*}
\]

\[ V_r/k_l = b^2 \frac{de}{dx} - \frac{d}{dx} (\nabla^2 \rho) \]

where \( \nabla^2 = \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} \)
The first one of Equations [9] can be factored as

\[(\nabla^2 + e^{ia})(\nabla^2 + e^{-ia})\rho = 0\]  \[10\]

where \[\cos a = -\frac{b^2}{2}\]

The solutions of Equation [10] are \[J_0(x e^{ia/2}), J_0(x e^{-ia/2})\], \[Y_0(x e^{ia/2})\] and \[Y_0(x e^{-ia/2})\]; where \[J_0(x e^{ia/2})\] and \[J_0(x e^{-ia/2})\] are Bessel functions of the first kind, \[Y_0(x e^{ia/2})\] and \[Y_0(x e^{-ia/2})\] are Bessel functions of the second kind. These can be expressed as

\[J_0(x e^{ia/2}) = U_0 + iV_0\]
\[J_0(x e^{-ia/2}) = U_0 - iV_0\]  \[11\]
\[Y_0(x e^{ia/2}) = U_0 + iV_0\]
\[Y_0(x e^{-ia/2}) = U_0 - iV_0\]

or

\[U_0 = \frac{1}{2} \left[ J_0(x e^{ia/2}) + J_0(x e^{-ia/2}) \right]\]
\[V_0 = \frac{1}{2i} \left[ J_0(x e^{ia/2}) - J_0(x e^{-ia/2}) \right]\]

\[U_0 = \frac{1}{2} \left[ Y_0(x e^{ia/2}) + Y_0(x e^{-ia/2}) \right]\]
\[V_0 = \frac{1}{2i} \left[ Y_0(x e^{ia/2}) - Y_0(x e^{-ia/2}) \right]\]  \[12\]

hence the solution of Equation [10] can be written as

\[\rho = U + \frac{q_1}{k}\]  \[13\]

\[= A U_0 + B V_0 + C U_0 + D V_0\]

or

\[\bar{\nu} = A U_0 + B V_0 + C U_0 + D V_0 - \frac{q_1}{k}\]  \[14\]

where \(A, B, C,\) and \(D\) are integral constants to be determined by the boundary conditions.

Substitute Equation [13] into the last three of Equations [9], the expressions for \(M_r, M_0\) and \(V_r\) can be obtained in terms of \(U_0, V_0, U_1, V_1, U_0, V_0, U_1\) and \(V_1\) through recurrence
relations which are listed in the appendix. The final expressions of \( N_r \), \( M_\theta \) and \( V_r \) are presented below:

\[
N_r = A k l^2 \left\{ \frac{\nu - 1}{x} - \frac{h^2}{l^2} \frac{1}{5x} \right\} u_1 \cos \alpha/2 - \left[ \frac{\nu - 1}{x} + \frac{h^2}{l^2} \frac{1}{5x} \right] v_1 \sin \alpha/2
+ \left[ \frac{h^2}{l^2} \frac{\nu}{10(1-\nu)} + \frac{1}{5} \frac{h^2}{l^2} \cos \alpha \right] u_0 - \left[ \sin \alpha \right] v_0
\]

\[
+ B k l^2 \left[ \frac{\nu - 1}{x} + \frac{h^2}{l^2} \frac{1}{5x} \right] u_1 \sin \alpha/2 + \left[ \frac{\nu - 1}{x} - \frac{h^2}{l^2} \frac{1}{5x} \right] v_1 \cos \alpha/2
+ \left[ \frac{h^2}{l^2} \frac{\nu}{10(1-\nu)} + \frac{1}{5} \frac{h^2}{l^2} \cos \alpha \right] u_0 + \left[ \sin \alpha \right] u_0
\]

\[
+ C k l^2 \left[ \frac{\nu - 1}{x} - \frac{h^2}{l^2} \frac{1}{5x} \right] u_1 \cos \alpha/2 - \left[ \frac{\nu - 1}{x} + \frac{h^2}{l^2} \frac{1}{5x} \right] v_1 \sin \alpha/2
+ \left[ \frac{h^2}{l^2} \frac{\nu}{10(1-\nu)} + \frac{1}{5} \frac{h^2}{l^2} \cos \alpha \right] u_0 - \left[ \sin \alpha \right] v_0
\]

\[
+ D k l^2 \left[ \frac{\nu - 1}{x} + \frac{h^2}{l^2} \frac{1}{5x} \right] u_1 \sin \alpha/2 + \left[ \frac{\nu - 1}{x} - \frac{h^2}{l^2} \frac{1}{5x} \right] v_1 \cos \alpha/2
+ \left[ \frac{h^2}{l^2} \frac{\nu}{10(1-\nu)} + \frac{1}{5} \frac{h^2}{l^2} \cos \alpha \right] u_0 + \left[ \sin \alpha \right] u_0
\]

\[
M_\theta = A k l^2 \left\{ \frac{1-\nu}{x} + \frac{h^2}{l^2} \frac{1}{5x} \right\} u_1 \cos \alpha/2 - \left[ \frac{1-\nu}{x} - \frac{h^2}{l^2} \frac{1}{5x} \right] v_1 \sin \alpha/2
+ \left[ \frac{h^2}{l^2} \frac{\nu}{10(1-\nu)} + \nu \cos \alpha \right] u_0 - \left[ \nu \sin \alpha \right] v_0
\]

\[
+ B k l^2 \left[ \frac{1-\nu}{x} - \frac{h^2}{l^2} \frac{1}{5x} \right] u_1 \sin \alpha/2 + \left[ \frac{1-\nu}{x} + \frac{h^2}{l^2} \frac{1}{5x} \right] v_1 \cos \alpha/2
\]

* This equation is derived by the author under Professor D. Frederick's guidance. All other fundamental equations used in this chapter are based on Reference (6).
\[-19\] 
\[
\begin{align*}
+ & \left[ \frac{h^2}{l^2} \frac{\nu}{10(1-\nu)} + \nu \cos \alpha \right] V_o + \left[ \nu \sin \alpha \right] U_o \\
+ & Ck l^2 \left\{ \left[ \frac{1-\nu}{x} + \frac{h^2}{l^2} \frac{1}{5x} \right] U_1 \cos \alpha/2 - \left[ \frac{1-\nu}{x} - \frac{h^2}{l^2} \frac{1}{5x} \right] V_1 \sin \alpha/2 \\
+ & \left[ \frac{h^2}{l^2} \frac{\nu}{10(1-\nu)} + \nu \cos \alpha \right] U_o - \left[ \nu \sin \alpha \right] V_o \right\} \\
+ & Dk l^2 \left\{ \left[ \frac{1-\nu}{x} - \frac{h^2}{l^2} \frac{1}{5x} \right] U_1 \sin \alpha/2 + \left[ \frac{1-\nu}{x} + \frac{h^2}{l^2} \frac{1}{5x} \right] V_1 \cos \alpha/2 \\
+ & \left[ \frac{h^2}{l^2} \frac{\nu}{10(1-\nu)} + \nu \cos \alpha \right] U_o + \left[ \nu \sin \alpha \right] V_o \right\} \\
\end{align*}
\]
\[15\] 
\[V_r = Ak l \left[ U_1 \cos \alpha/2 + V_1 \sin \alpha/2 \right] + Bk l \left[ -U_1 \sin \alpha/2 + V_1 \cos \alpha/2 \right] \]
\[+ Ck l \left[ U_1 \cos \alpha/2 + V_1 \sin \alpha/2 \right] + Dk l \left[ -U_1 \sin \alpha/2 + V_1 \cos \alpha/2 \right] \]
B. APPLICATION AND RESULTS

A circular plate with a circular hole at the center, resting on an elastic foundation, is considered. The plate is acted upon by a uniformly distributed load \( q_1 \) on the upper surface and a uniformly distributed moment \( M_0 \) along the outer boundary (Figure 2). The boundary conditions are:

- at the outer boundary, \( (N_r)_{r=R_0} = N_0 \) and \( (V_r)_{r=R_0} = 0 \)
- at the inner boundary, \( (N_r)_{r=R_1} = 0 \) and \( (V_r)_{r=R_1} = 0 \)

To calculate the constants \( A, B, C \) and \( D \) in Equations \([14]\) and \([15]\) for this problem, the above boundary conditions must be used.

In order to use the tables for Bessel functions of the first and second kind for a complex argument \((6) (7)\), which are tabulated in 5° increments, \( \alpha/2 \) will be chosen as 45°, 50°, and 55°. The case where \( \alpha/2 = 45° \) corresponds to the classical solution, since the value of \( b \) is zero and the terms containing \( h^2/\ell^2 \) in Equations \([9]\) are omitted.

Calculations were made for \( \nu = 1/3 \) and \( R_0/R_1 = 2, 5 \) and 10. For these cases, the values of \( A, B, C \) and \( D \) are tabulated below:
Table 1. The values of constants A, B, C and D.

<table>
<thead>
<tr>
<th>( \frac{a}{2} )</th>
<th>( \frac{r_0}{r_1} )</th>
<th>2</th>
<th>5</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>45°</td>
<td>A</td>
<td>(2.66338) ( \frac{M_o}{k L^2} )</td>
<td>(0.31273) ( \frac{M_o}{k L^2} )</td>
<td>(-0.00439) ( \frac{M_o}{k L^2} )</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>(-0.50885) ( \frac{M_o}{k L^2} )</td>
<td>(-0.05165) ( \frac{M_o}{k L^2} )</td>
<td>(0.00591) ( \frac{M_o}{k L^2} )</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>(-2.43167) ( \frac{M_o}{k L^2} )</td>
<td>(0.13751) ( \frac{M_o}{k L^2} )</td>
<td>(-0.00873) ( \frac{M_o}{k L^2} )</td>
</tr>
<tr>
<td></td>
<td>D</td>
<td>(-1.86267) ( \frac{M_o}{k L^2} )</td>
<td>(-0.14529) ( \frac{M_o}{k L^2} )</td>
<td>(0.00086) ( \frac{M_o}{k L^2} )</td>
</tr>
<tr>
<td>50°</td>
<td>A</td>
<td>(1.74625) ( \frac{M_o}{k L^2} )</td>
<td>(0.34405) ( \frac{M_o}{k L^2} )</td>
<td>(-0.00973) ( \frac{M_o}{k L^2} )</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>(-0.44575) ( \frac{M_o}{k L^2} )</td>
<td>(-0.04112) ( \frac{M_o}{k L^2} )</td>
<td>(0.00112) ( \frac{M_o}{k L^2} )</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>(-1.98194) ( \frac{M_o}{k L^2} )</td>
<td>(0.08470) ( \frac{M_o}{k L^2} )</td>
<td>(-0.00278) ( \frac{M_o}{k L^2} )</td>
</tr>
<tr>
<td></td>
<td>D</td>
<td>(-1.06813) ( \frac{M_o}{k L^2} )</td>
<td>(-0.17908) ( \frac{M_o}{k L^2} )</td>
<td>(0.00504) ( \frac{M_o}{k L^2} )</td>
</tr>
<tr>
<td>55°</td>
<td>A</td>
<td>(1.37397) ( \frac{M_o}{k L^2} )</td>
<td>(0.36971) ( \frac{M_o}{k L^2} )</td>
<td>(-0.00900) ( \frac{M_o}{k L^2} )</td>
</tr>
<tr>
<td></td>
<td>B</td>
<td>(-2.00465) ( \frac{M_o}{k L^2} )</td>
<td>(-0.31253) ( \frac{M_o}{k L^2} )</td>
<td>(0.00813) ( \frac{M_o}{k L^2} )</td>
</tr>
<tr>
<td></td>
<td>C</td>
<td>(-3.79551) ( \frac{M_o}{k L^2} )</td>
<td>(-0.23788) ( \frac{M_o}{k L^2} )</td>
<td>(0.00761) ( \frac{M_o}{k L^2} )</td>
</tr>
<tr>
<td></td>
<td>D</td>
<td>(-0.69582) ( \frac{M_o}{k L^2} )</td>
<td>(-0.20734) ( \frac{M_o}{k L^2} )</td>
<td>(0.00500) ( \frac{M_o}{k L^2} )</td>
</tr>
</tbody>
</table>

Deflections along the radius of the plates are listed in Tables 2, 3, and 4. The curves of deflection are given in Figures 3, 4 and 5.
Table 2. The values of $\rho/(M_0/k_i^2)$ for a plate with $r_0/r_1 = 2$.

<table>
<thead>
<tr>
<th>$a/2$</th>
<th>$x$</th>
<th>1</th>
<th>1.2</th>
<th>1.4</th>
<th>1.6</th>
<th>1.8</th>
<th>2.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>45°</td>
<td>1</td>
<td>1.33914</td>
<td>0.85197</td>
<td>0.37749</td>
<td>-0.09858</td>
<td>-0.58616</td>
<td>-1.09213</td>
</tr>
<tr>
<td>50°</td>
<td>1.00588</td>
<td>0.64915</td>
<td>0.29711</td>
<td>-0.06479</td>
<td>-0.44775</td>
<td>-0.86084</td>
<td></td>
</tr>
<tr>
<td>55°</td>
<td>1.28017</td>
<td>0.80210</td>
<td>0.35494</td>
<td>-0.08906</td>
<td>-0.55118</td>
<td>-1.04923</td>
<td></td>
</tr>
</tbody>
</table>

* Classical theory

Table 3. The values of $\rho/(M_0/k_i^2)$ for a plate with $r_0/r_1 = 5$.

<table>
<thead>
<tr>
<th>$a/2$</th>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>45°</td>
<td>0.23272</td>
<td>0.33216</td>
<td>0.34005</td>
<td>0.00743</td>
<td>-1.02125</td>
<td></td>
</tr>
<tr>
<td>50°</td>
<td>0.23389</td>
<td>0.30552</td>
<td>0.31738</td>
<td>0.02418</td>
<td>-1.01566</td>
<td></td>
</tr>
<tr>
<td>55°</td>
<td>0.29397</td>
<td>0.28902</td>
<td>0.29410</td>
<td>0.03319</td>
<td>-1.01336</td>
<td></td>
</tr>
</tbody>
</table>

Table 4. The values of $\rho/(M_0/k_i^2)$ for a plate with $r_0/r_1 = 10$.

<table>
<thead>
<tr>
<th>$a/2$</th>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>45°</td>
<td>-0.00448</td>
<td>-0.01205</td>
<td>-0.01252</td>
<td>0.09604</td>
<td>0.25232</td>
<td>-1.01540</td>
<td></td>
</tr>
<tr>
<td>50°</td>
<td>-0.00664</td>
<td>-0.00902</td>
<td>-0.00183</td>
<td>0.09431</td>
<td>0.23344</td>
<td>-1.01723</td>
<td></td>
</tr>
<tr>
<td>55°</td>
<td>-0.00727</td>
<td>-0.00571</td>
<td>0.00577</td>
<td>0.09010</td>
<td>0.21786</td>
<td>-1.01723</td>
<td></td>
</tr>
</tbody>
</table>

The values of $M_x$ and $M_y$ along the radius of a plate with $r_0/r_1 = 5$ are listed in Tables 5 and 6, the curves of $M_x$ and $M_y$ are given in Figures 6 and 7.
Table 5. The values of $M_\alpha / M_\theta$ for a plate with $r_0 / r_1 = 5$.

<table>
<thead>
<tr>
<th>$\alpha/2$</th>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^\circ 45$</td>
<td>0</td>
<td>0.06109</td>
<td>0.34087</td>
<td>0.76146</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$50^\circ$</td>
<td>0</td>
<td>0.11055</td>
<td>0.36865</td>
<td>0.76280</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$55^\circ$</td>
<td>0</td>
<td>0.11833</td>
<td>0.38167</td>
<td>0.76214</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

Table 6. The values of $M_\alpha / M_\theta$ for a plate with $r_0 / r_1 = 5$.

<table>
<thead>
<tr>
<th>$\alpha/2$</th>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^\circ 45$</td>
<td>-0.09943</td>
<td>-0.01515</td>
<td>0.14524</td>
<td>0.39220</td>
<td>0.59448</td>
<td></td>
</tr>
<tr>
<td>$50^\circ$</td>
<td>-0.05313</td>
<td>0.03146</td>
<td>0.16287</td>
<td>0.42687</td>
<td>0.54267</td>
<td></td>
</tr>
<tr>
<td>$55^\circ$</td>
<td>0.1139</td>
<td>0.07514</td>
<td>0.21518</td>
<td>0.45616</td>
<td>0.72623</td>
<td></td>
</tr>
</tbody>
</table>

The ratios of $M_\theta$ at the inner boundary for $\alpha/2 = 50^\circ$ and $55^\circ$ to $\alpha/2 = 45^\circ$ are presented in Table 7, their curves are given in Figure 8.

Table 7. Ratio of $M_\theta$ at $r=r_1$ for $\alpha/2=50^\circ$ and $55^\circ$ to $\alpha/2=45^\circ$

<table>
<thead>
<tr>
<th>$r_0/r_1$</th>
<th>$2$</th>
<th>$5$</th>
<th>$10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$</td>
<td>M_\theta</td>
<td>_{50^\circ}$ / $</td>
<td>M_\theta</td>
</tr>
<tr>
<td>$</td>
<td>M_\theta</td>
<td>_{55^\circ}$ / $</td>
<td>M_\theta</td>
</tr>
</tbody>
</table>
Fig. 3 Deflection along the radius for $r_0/r_1 = 2$.

\begin{align*}
\frac{p}{(E/I)} & \quad a/2 = 55^\circ \\
-2.0 & \\
-1.0 & \\
0.0 & \\
1.0 & \\
2.0 & \\
1.0 & 1.2 3.4 5.6 7.8 9.2.0 x
\end{align*}

\begin{align*}
-2.0 & \\
-1.0 & \\
0.0 & \\
1.0 & \\
2.0 & \\
1.0 & 1.2 3.4 5.6 7.8 9.2.0 x
\end{align*}

\begin{align*}
-2.0 & \\
-1.0 & \\
0.0 & \\
1.0 & \\
2.0 & \\
1.0 & 1.2 3.4 5.6 7.8 9.2.0 x
\end{align*}

\begin{align*}
-2.0 & \\
-1.0 & \\
0.0 & \\
1.0 & \\
2.0 & \\
1.0 & 1.2 3.4 5.6 7.8 9.2.0 x
\end{align*}
Fig. 4  Deflection along the radius for $r_0/r_1 = 5$.

- $\alpha/2 = 55^\circ$
- $\alpha/2 = 50^\circ$
- $\alpha/2 = 45^\circ$
Fig. 5 - a Deflection along the radius for \( \gamma_0/\gamma_1 = 19 \).

\[ \rho/(2d^2) \]

\( \alpha/2 = 45^\circ \)
Fig. 5-b  Deflection along the radius for $r_0/r_1 = 10$.

$a/2 = 50^\circ$

$\rho/(M_0/k^2)$
Fig. 5 - c  Deflection along the radius for $r_0/r_1 = 10$.  

$\rho/\left(\mu/k^2\right)$

$a/2 = 55^\circ$
Fig. 6 $M_r/M_0$ along the radius for $r_c/r_1 = 5$. 

- $a/2 = 55^\circ$
- $a/2 = 50^\circ$
- $a/2 = 45^\circ$
Fig. 7  $M_2/M_0$ along the radius for $r_2/r_1 = 5$

- $\alpha/2 = 55^\circ$
- $\alpha/2 = 50^\circ$
- $\alpha/2 = 45^\circ$
Fig. 8 \( \frac{M_{\text{elec}}}{M_{\text{elec},0}} \) vs. \( \alpha / \beta \).
VII. CONCLUSION

From the results of the cases discussed, it can be concluded that the neglect of transverse-shear deformation and normal stress results in expressions for bending moments and deflection which may be seriously in error. Since the results are affected by the factor \( h/\ell \) which is proportional to \( h \) and \( k^{1/4} \) \( (\ell = h\sqrt{E/k}) \), for a plate with considerably large value of "h" or a plate resting on an elastic foundation of which the modulus \( k \) is large, Reissner's theory should be applied.
VIII. APPENDIX

Following is a list of identities on the functions $U_n$, $V_n$, $\bar{U}_n$, $\bar{V}_n$, which were used in the solutions of the problem.

\[
\frac{2nl}{r} \left[ U_n \cos \alpha/2 + V_n \sin \alpha/2 \right] = U_{n+1} + U_{n-1}
\]

\[
\frac{2nl}{r} \left[ V_n \cos \alpha/2 - U_n \sin \alpha/2 \right] = V_{n+1} + V_{n-1}
\]

\[
\frac{2nl}{r} \left[ \bar{U}_n \cos \alpha/2 + \bar{V}_n \sin \alpha/2 \right] = \bar{U}_{n+1} + \bar{U}_{n-1}
\]

\[
\frac{2nl}{r} \left[ \bar{V}_n \cos \alpha/2 - \bar{U}_n \sin \alpha/2 \right] = \bar{V}_{n+1} + \bar{V}_{n-1}
\]

\[
\frac{dU_n}{dr} = \left\{ \frac{nU_n}{r} - \frac{1}{\ell} \left[ U_{n+1} \cos \alpha/2 - V_{n+1} \sin \alpha/2 \right] \right\}
\]

\[
\frac{dV_n}{dr} = \left\{ \frac{nV_n}{r} - \frac{1}{\ell} \left[ V_{n+1} \cos \alpha/2 + U_{n+1} \sin \alpha/2 \right] \right\}
\]

\[
\frac{\bar{d}U_n}{dr} = \left\{ \frac{n\bar{U}_n}{r} - \frac{1}{\ell} \left[ \bar{U}_{n+1} \cos \alpha/2 - \bar{V}_{n+1} \sin \alpha/2 \right] \right\}
\]

\[
\frac{\bar{d}V_n}{dr} = \left\{ \frac{n\bar{V}_n}{r} - \frac{1}{\ell} \left[ \bar{V}_{n+1} \cos \alpha/2 + \bar{U}_{n+1} \sin \alpha/2 \right] \right\}
\]

\[
\frac{dU_{n+1}}{dr} = \left\{ - \frac{(n+1)}{r} U_{n+1} + \frac{1}{\ell} \left[ U_n \cos \alpha/2 - V_n \sin \alpha/2 \right] \right\}
\]

\[
\frac{dV_{n+1}}{dr} = \left\{ - \frac{(n+1)}{r} V_{n+1} + \frac{1}{\ell} \left[ V_n \cos \alpha/2 + U_n \sin \alpha/2 \right] \right\}
\]

\[
\frac{\bar{d}U_{n+1}}{dr} = \left\{ - \frac{(n+1)}{r} \bar{U}_{n+1} + \frac{1}{\ell} \left[ \bar{U}_n \cos \alpha/2 - \bar{V}_n \sin \alpha/2 \right] \right\}
\]

\[
\frac{\bar{d}V_{n+1}}{dr} = \left\{ - \frac{(n+1)}{r} \bar{V}_{n+1} + \frac{1}{\ell} \left[ \bar{V}_n \cos \alpha/2 + \bar{U}_n \sin \alpha/2 \right] \right\}
\]
\[
\frac{d^2u_n}{dr^2} = \left\{ \left[ \frac{n(n-1)}{r^2} - \frac{\cos \alpha}{\ell^2} \right] u_n + \frac{\sin \alpha}{\ell^2} v_n + \frac{\cos \alpha/2}{\ell r} u_{n+1} - \frac{\sin \alpha/2}{\ell r} v_{n+1} \right\}
\]

\[
\frac{d^2v_n}{dr^2} = \left\{ \left[ \frac{n(n-1)}{r^2} - \frac{\cos \alpha}{\ell^2} \right] v_n - \frac{\sin \alpha}{\ell^2} u_n + \frac{\cos \alpha/2}{\ell r} v_{n+1} + \frac{\sin \alpha/2}{\ell r} u_{n+1} \right\}
\]

\[
\frac{d^2\bar{u}}{dr^2} = \left\{ \left[ \frac{n(n-1)}{r^2} - \frac{\cos \alpha}{\ell^2} \right] \bar{u} + \frac{\sin \alpha}{\ell^2} \bar{v} + \frac{\cos \alpha/2}{\ell r} \bar{u}_{n+1} - \frac{\sin \alpha/2}{\ell r} \bar{v}_{n+1} \right\}
\]

\[
\frac{d^2\bar{v}}{dr^2} = \left\{ \left[ \frac{n(n-1)}{r^2} - \frac{\cos \alpha}{\ell^2} \right] \bar{v} - \frac{\sin \alpha}{\ell^2} \bar{u} + \frac{\cos \alpha/2}{\ell r} \bar{v}_{n+1} + \frac{\sin \alpha/2}{\ell r} \bar{u}_{n+1} \right\}
\]
IX. ACKNOWLEDGEMENTS

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The author also wishes to express his heartfelt gratitude to Professor J. B. Jones for giving him the opportunity to undertake this investigation.
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