ON THE FREQUENCY ANALYSIS OF BEAMS WITH NON-UNIFORM CROSS SECTIONS

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Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

MASTER OF SCIENCE

in

STRUCTURAL ENGINEERING

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August 22, 1955

Blacksburg, Virginia

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I. Introduction

The determination of the natural frequencies of lateral vibrations of beams has been the subject of numerous papers and books. The method most frequently used in engineering applications is known as the energy method, or the method of Rayleigh and Ritz. However for beams with variable cross sections, the mass and the rigidity properties are unusually complicated functions making the integrals to be evaluated very complex. The method under investigation, first suggested by Dr. H. Marcus, deals solely with uncoupled bending vibrations and utilizes only elementary principles of engineering mechanics.

The beams may be of variable cross section and may have any number of spans. The basic shape-function equation is developed by using the principle of virtual displacements and the equations satisfying different boundary conditions have also been developed.

II. The Review of Literature

As described in the introduction, since a rigorous solution of the basic equation in the general case presents difficulties, various approximate methods of solving the problems have been developed. Much of that work was done in connection with the analysis of hulls of ships and aircraft flutter. Approximate solutions were obtained using step-by-step methods of numerical integration.

C. E. Inglis (1) approximated a beam of continuously varying moment of inertia by a "stepped" beam and derived an algebraic expression for the deflection curve. N. O. Myklestad (2) developed a very useful scheme of numerical integration for a stepped beam, which has the slight disadvantage that the quantities and coefficients appearing in the basic formulas are rather remotely related to those quantities with which engineers are familiar from the theory of structures. A. I. Bellin (3) developed a method which involves only moments in the basic equation. M. A. Prohl's (4) method is similar to Myklestad's method although different from it in several minor points.

III. Investigation

A. Derivations

i. Basic Equation

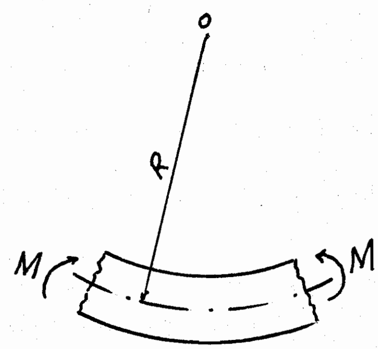

Fig. 1 - A section of a prismatic beam.

Consider, as shown in Fig. 1, a section of a prismatic beam in equilibrium under an applied bending moment M. The following facts are readily adduced and are derived in elementary texts (5) on structural theory. The radius of curvature R is given by the expression

$$\frac{1}{R} = \frac{M}{EI} \quad \ldots\ldots\ldots\ldots\ldots \text{(i-1)}$$

where I is the section moment of inertia (inches^4)

and E is Young's modulus (p.s.i.)

From this,

$$\frac{d^2w}{d\,x^2} = \frac{M}{EI} \quad \ldots\ldots\ldots\ldots\ldots \text{(i-2)}$$

which is the differential relation between the transverse beam deflection w and the bending moment M.

If q represents the load distribution on the beam, while V, M represent shear and moment distribution, respectively, we have the relations

$$\frac{dM}{dx} = V \quad \ldots\ldots\ldots\ldots\ldots \text{(i-3)}$$

$$\frac{dV}{dx} = q \quad \ldots\ldots\ldots\ldots\ldots \text{(i-4)}$$

whence

$$\frac{d^2M}{d\,x^2} = q \quad \text{...............} \quad (i\text{-}5)$$

Hence the differential relation between load distribution q and beam deflection w is

$$\frac{d^2}{dx^2} \quad EI\,\frac{d^2w}{dx^2} = q \quad \text{..........} \quad (i\text{-}6)$$

It should be noted that the above results are subject to the following limitations usually assumed in the elastic range-plane stress distribution during bending; homogeneous beam material obeying Hooke's law; no shear stresses contribute to deflection; beam is initially prismatic; elastic moduli in tension and compression; are equal.

If the beam is subjected to free vibrations, it is loaded only by inertia forces.

$$q = -\gamma\frac{A}{g}\frac{\partial^2 w}{\partial t^2} \quad \text{..........} \quad (i\text{-}7)$$

where

$\gamma =$ weight of material of the beam per unit volume (lb./in.^{3})

$A =$ cross sectional area of beam (in.^{2})

$g =$ gravitational acceleration (in./sec.^{2})

Substituting (i-7) into (i-6), we get

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 W}{\partial x^2}\right) = -\frac{\gamma A}{g}\frac{\partial^2 W}{\partial t^2} \quad \text{..........} \quad (i\text{-}8)$$

Here, I and A are functions of x.

Equation (i-8) is the basic equation for free vibration of beams with variable cross sectional area. However, since the cross section, the mass, and the rigidity properties may be unusually complicated functions of x, we do not attempt to solve the basic equation exactly but reduce it to a simpler algebraic form so as to be able to employ a step-by-step method of integration.

ii. Equations For Beams With Variable Cross Sections

a. General Equation

Consider the beam shown in Fig. (2)

We subdivide the beam in n segments having equal lengths. Let $F_o, F_1, \dots F_{n-1}$ be the forces acting parallel to the z-axis at the points o, 1, 2, ... n-1. They produce

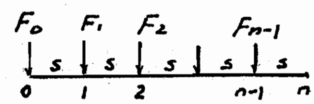

Fig. 2 - Beam with n segments.

in the first segment a shear force $V_1 = -F_o$

in the second segment a shear force $V_2 = -(F_o+F_1)$

...

in the n^{th} segment a shear force $V_r = -(F_o+F_1- \dots +F_{r-1})$

The bending moments for the section are

$1{:}M_1 = -F_oS = V_1S$

$2{:}M_2 = -S\,F_o + (F_o+F_1) = (V_1+V_2)\,S$

$n{:}M_n = (V_1+V_2+ \dots +V_n)\,S$

These equations yield

$$V_{m-1} - V_m = F_{m-1} \quad \dots\dots\dots\dots \text{(ii-1)}$$

$$M_m - M_{m-1} = SV_m \quad \dots\dots\dots\dots \text{(ii-2)}$$

Substituting (ii-2) into (ii-1), we get

$$V_{m\,1} - V_m = \frac{(M_{m\,1} - M_m) - (M_m - M_{m-1})}{S} = F_m \quad \text{(ii-3)}$$

Frmm Fig. 3, we obtain the moment at a section located at a distance x from (m-1) or a distance x^1 from (m)

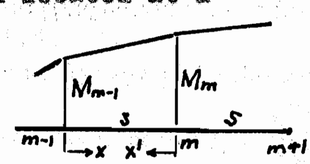

Fig. 3 Moment diagram.

$$M = M_{m-1} + (M_m - M_{m-1}) \frac{x}{s} = M_m - (M_m - M_{m-1}) \frac{x^1}{s} \quad \dots \text{(ii-4)}$$

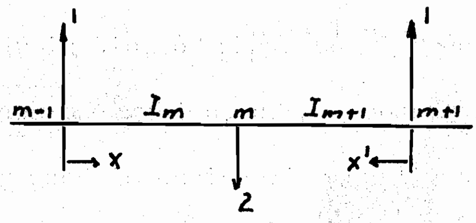

Fig. 4 - Segments m-1, m and m_1 m+1

In Fig. 4, we assume that the adjacent segments m-1, m and m, m+1

are subjected to fictitious forces.

$F_{m-1} = F_{m+1} = -1;\ F_m - +2,$

producing the bending moments

$M = x$ in the segment m-1, m

$M = x^1$ in the segment m, m+1

Applying the principle of virtual displacements, then

$$-W_{m-1} + 2W_m - W_{m+1} = \frac{1}{EI_m}\int_0^s \left\{M_{m-1} + (M_m - M_{m-1})\frac{x}{s}\right\} x\, dx$$

$$+ \frac{1}{EI_{m+1}}\int_0^s \left\{M_{m+1} - (M_{m+1} - M_m)\frac{x^1}{s}\right\} x^1 dx^1$$

or

$$\frac{6EI_c}{S^2}(2W_m - W_{m-1} - W_{m+1}) = (M_{m-1} + 2M_m)\, i_m + (2M_m + M_{m+1})\, i_{m+1} \quad \dots \text{(ii-5)}$$

where I_m is the moment of inertia for the segment m-1, m

and I_{m+1} is the moment of inertia for the segment m, m-1, $i_m = \frac{I_c}{I_m}$

With the abbreviation

$$U_m = \frac{6EI_c}{S^2}(W_{m-1} - 2W_m + W_{m+1}) \quad \dots \text{(ii-6)}$$

equation (ii-5) yields

$$M_{m-1}\, i_m + 2M_m(i_m + i_{m+1}) + M_{m+1}\, i_{m+1} = -U_m \quad \dots \text{(ii-7)}$$

Eliminating M_{m-1} and M_{m+1} from equations (ii-7) and (ii-3) leads to the correlation

$$M_{m-1}(i_{m+1}-i_m)-2M_m(i_m+2i_{m+1}) = -i_{m+1}\, SF_m+U_m \quad \dots (a)$$

$$M_{m+1}(i_m-i_{m+1})-2M_m(i_{m+1}-2i_m) = -i_m\, sF_m-U_m \quad \dots (b)$$

Equation (b) yields also

$$-2M_{m-1}(i_m+2i_{m-1}) - M_m(i_m-i_{m-1}) = -i_{m-1}\, sF_{m-1}+U_{m-1} \quad \dots (c)$$

From (a) and (c):

$$M_m = \frac{(i_{m-1}sF_{m-1}-U_{m-1})(i_{m+1}-i_m) + 2(i_{m-1}sF_m-U_m)(i_m+2\, i_{m-1})}{(i_{m+1}-i_m)(i_m-i_{m-1}) + 4(i_m+2i_{m-1})(i_m+2\, i_{m+1})}$$

$$M_{m-1} = \frac{-(i_{m+1}sF_m-U_m)(i_m-i_{m-1}) + 2(i_{m-1}\, sF_{m-1} - U_{m-1})(i_m+2i_{m+1})}{(i_{m+1}-i_m)(i_m- i_{m-1}) + 4(i_m+2i_{m-1})(i_m+2i_{m+1})}$$

Using the abreviations

$$(i_{m+1}-i_m)(i_m-i_{m-1}) + 4(i_m+2i_{m-1})(i_m+2i_{m+1}) = D_m$$

$$\dots \text{(ii-8)}$$

$$\frac{i_m-i_{m-1}}{D_m} = a_m$$

$$\frac{i_{m+1}-i_m}{D_m} = a_m$$

$$\frac{2(i_m+2i_{m+1})}{D_m} = b_m$$

$$\frac{2(i_m+2i_{m-1})}{D_m} = b_m$$

we write

$$M_{m-1} = -a_m(i_{m+1}sF_m-U_m) + b_m(i_{m-1}sF_{m-1}-U_{m-1}) \quad \dots (d)$$

$$M_m = a_m(i_{m-1}sF_{m-1}-U_{m-1}) + b_m(i_{m+1}sF_m-U_m) \quad \dots (e)$$

From (d):

$$M_m = -a'_{m+1}(i_{m+2}sF_{m+1} - U_{m+1}) + b''_{m+1}(i_m sF_m - U_m) \qquad \ldots (f)$$

Equating (e) and (f):

$$a''_m (i_{m-1}sF_{m-1} - U_{m-1}) + sF_m(i_{m-1}b'_m - i_m b''_{m+1})$$
$$- U_m(b'_m - b''_{m+1}) + a'_{m+1}(i_{m+2}sF_{m+1} - U_{m+1}) = 0$$

or

$$a''_m i_{m-1} sF_{m-1} + sF_m(i_{m+1}b'_m - i_m b''_{m+1}) + a'_{m+1} i_{m+2} sF_{m+1}$$
$$= a''_m U_{m-1} + U_m(b'_m - b''_{m+1}) + a'_{m+1}U_{m+1} \qquad \ldots (ii\text{-}9)$$

This recursive equation gives us a correlation between the forces F_{m-1}, F_m, F_{m+1} and the curvatures U_{m-1}, U_m, U_{m+1}

In order to apply the equation to the problem of lateral vibrations, the displacement of the point m may be taken as $W_m = x_m \sin pt$ and the inertia force due to the mass θ_m located at the point m can be written:

$$F_m = -\theta_m \frac{\partial^2 W_m}{\partial t^2} = p^2 \theta_m X_m \sin pt\ ,$$

Thus we have

$$U_m = \frac{6EI_c}{s^2}(X_{m-1} - 2X_m + X_{m+1}) \sin pt\ .$$

Substituting these values in equation (ii-9) gives

$$sp^2\{a''_m i_{m-1}\theta_{m-1}X_{m-1} + (i_{m+1}b'_m - i_m b''_{m+1})\theta_m X_m + a'_{m+1} i_{m+2}\theta_{m+1}X_{m+1}\}$$
$$= \frac{6EI_c}{s^2}\{a''_m(X_{m-2} - 2X_{m-1} + X_m) + (b'_m - b''_{m+1})(X_{m-1} - 2X_m + X_{m+1})$$
$$+ a'_{m+1}(X_m - 2X_{m+1} + X_{m+2})$$

or

$$X_{m-2}a''_m - X_{m-1}\left\{2a''_m + b''_{m+1} - b'_m + a''_m i_{m-1}\theta_{m-1}\frac{p^2s^3}{6EI_c}\right\}$$
$$+ X_m\left\{a''_m + a'_{m+1} + 2(b''_{m+1} - b'_m) - (i_{m+1}b'_m - i_m b''_{m+1})\theta_m \frac{p^2s^3}{6EI_c}\right\}$$
$$- X_{m+1}\left\{2a'_{m+1} + b''_{m+1} - b'_m + a'_{m+1} i_{m+2}\theta_{m+1}\frac{p^2s^3}{6EI}\right\} + X_{m+2}a'_{m+1} = 0 \qquad \ldots (ii\text{-}10)$$

This recurisve correlation between the five values X_{m-2}, X_{m-1}, X_m, X_{m+1}, X_{m+2} is the general equation for the beam.

b. Equation For Free End

Assuming that the end $m=0$ is free, we have $M = 0$; $M_2 = -F_o s$. See Fig. (5)

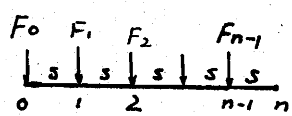

Fig. 5 - Beam with n segments.

Equation (ii-3) for $m = 1$

$$M_o - 2M_1 + M_2 = -sF_1$$

and equation (ii-7) for $m=1$

$$M_o i_1 + 2M_1(i_1 + i_2) + M_2 i_2 = -U_1$$

then yield

$$i_2 M_2 = i_2 s(F_1 + 2F_o)$$

$$i_2 M_2 = -U_1 - 2M_1(i_1 + i_2) = -U_1 + 2sF_o(i_1 + i_2)$$

or

Then

$$U_1 = s\left\{i_2 F_1 + 2F_o(i_1 + 2i_2)\right\} = s\left\{2i_1 F_o + (F_1 + 4F_o)i_2\right\}$$

$$\frac{p^2 s^3}{6EI_c}\left\{i_2\theta_1 X_1 + 2\theta_o X_o(i_1 + 2i_2)\right\} = X_o - 2X_1 + X_2$$

or

$$X_o\left\{1 - 2\theta_o\frac{p^2 s^3}{6EI_c}(i_1 + 2i_2)\right\} - X_1\left(2 + \frac{p^2 s^3}{6EI_c}i_2\theta_1\right) + X_2 = 0 \quad \text{... (ii-11)}$$

In the same way we obtain from

$$M_1 - 2M_2 + M_3 = -sF_2$$

and

$$M_1 i_2 + 2M_2(i_2 + i_3) + i_3 M_3 = -U_2$$

$$M_1(i_3 - i_2) - 2M_2(i_2 + 2i_3) = -sF_2 i_3 + U_2$$

the expression

$$2M_2(i_2 + 2i_3) = -sF_o(i_3 - i_2) + sF_2 i_3 - U_2$$

Since

$$M_2 = - sF_1 + 2M_1 = -s (F_1 + 2F_o)$$

we have

$$- 2s (i_2 + 2i_3) (F_1 + 2F_o) + F_o s (i_3 - i_2) - F_2 s i_3 = -U_2$$

or

$$-sF_o (5i_2 + 7i_3) + 2sF_1 (i_2 + 2i_3) + sF_2 i_3 = \frac{6EI_c}{s^2}(X_1 - 2X_2 + X_3)$$

or

$$-\frac{p^2 s^3}{6EI_c} \theta_o X_o (5i_2 + 7i_3) + X_1\left\{1 - \frac{2p^2s^3}{6EI_c}\theta_1 (i_2 + 2i_3)\right\}$$
$$- X_2\left\{2 + \frac{p^2s^3}{6EI_c}\theta_2 i_3\right\} + X_3 = 0 \qquad \text{... (ii-12)}$$

The next equation between X_o, X_1, X_2, X_3, X_4 has the form given by equation (ii-10) for m=2.

c. Equation For Fixed End

Now let "n" be fixed. (Fig. 6)

It is readily shown by the area moment method that the moments M_n, M_{n-1} of the segment n-1, n must satisfy the equations

$$w_n - w_{n-1} = \frac{M_n s^2}{2EI_n} + \frac{M_{n-1} - M_n}{EI_n}\frac{s^2}{3}$$

and

$$M_{n-1} + 2M_n = \frac{6EI_n}{s^2}(w_n - w_{n-1})$$

and

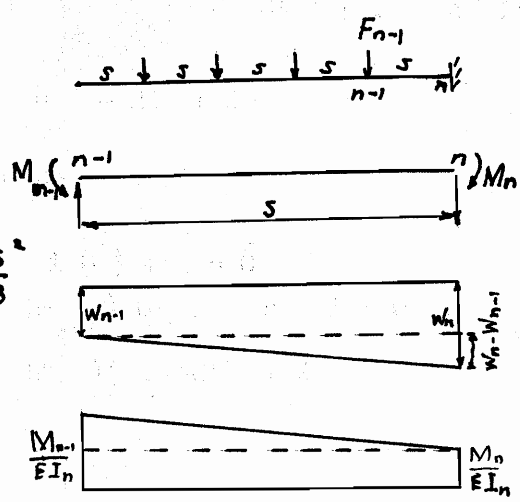

Fig. 6 - $\frac{M}{EI}$ diagram

$$M_{n-1} + 2M_n = \frac{6EI_c}{i_n s^2}(w_n - w_{n-1}) \qquad \text{... (ii-13)}$$

The basic equations

$$M_{n-2} - 2M_{n-1} + M_n = - s F_{n-1}$$

imply that

$$2M_{n-1}(2\,i_{n-1}+i_n)+M_n(i_n-i_{n-1})=i_{n-1}sF_{n-1}-U_{n-1} \qquad \ldots\ (ii\text{-}14)$$

From (ii-13) and (ii-14) we get

$$M_{n-1}=c'_{n-1}(i_{n-1}sF_{n-1}-U_{n-1})-c''_{n-1}\frac{6EI_c}{S^2}(w_n-w_{n-1}) \qquad \ldots\ (ii\text{-}15)$$

where

$$c'_{n-1}=\frac{2}{3(i_n+3i_{n-1})}\ ; \qquad c''_{n-1}=\frac{(i_n-i_{n-1})}{3(i_n+3i_{n-1})i_n}$$

Equation(e) requires that

$$M_{n-1}=a''_{n-1}(i_{n-2}sF_{n-2}-U_{n-2})+b'_{n-1}(i_n sF_{n-1}-U_{n-1}) \qquad \ldots\ (ii\text{-}16$$

Equation (ii-16) is compatible with (ii-16) provided

$$i_{n-2}a''_{n-1}sF_{n-2}+sF_{n-1}(i_n b'_{n-1}-c'_{n-1}i_{n-1})-a''_{n-1}U_{n-2}$$
$$-U_{n-1}(b'_{n-1}-c'_{n-1})+c''_{n-1}\frac{6EI_c}{s^2}(w_n-w_{n-1})=0$$

or

$$a''_{n-1}\frac{ps^3}{6EI_c}\theta_{n-2}X_{n-2}+\frac{ps^3}{6EI_c}\theta_{n-1}X_{n-1}(i_n b'_{n-1}-c''_{n-1}i_{n-1})$$
$$-a''_{n-1}(X_{n-3}-2X_{n-2}+X_{n-1})-(b'_{n-1}-c'_{n-1})(X_{n-2}-2X_{n-1}+X_n)$$
$$+c''_{n-1}(X_n-X_{n-1})=0$$

or finally

$$a''_{n-2}X_{n-3}$$
$$+X_{n-2}\left\{2a''_{n-1}+c'_{n-1}-b'_{n-1}+a''_{n-1}\frac{ps^3}{6EI_c}\theta_{n-1}\right\}$$
$$+X_{n-1}\left\{-a''_{n-1}+2(c'_{n-1}-b'_{n-1})-c''_{n-1}+\frac{ps^3}{6EI_c}\theta_{n-1}(i_n b'_{n-1}-c'_{n-1}i_{n-1})\right.$$
$$-X_n(b'_{n-1}-c'_{n-1}-c''_{n-1})=0 \qquad \ldots\ (ii\text{-}17)$$

The next equation between X_n, X_{n-1}, X_{n-2}, X_{n-3}, X_{n-4} has the form given by equation (ii-10).

d. Equation For Hinged End

Now let "m-1" be a hinged ended. (Fig. 7)

From Equation (d):

$$M_{m-1} = -a'_m \left(i_{m+1} sF_m - U_m \right) + b''_m \left(i_{m-1} sF_{m-1} - U_{m-1} \right)$$

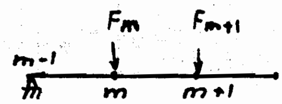

Fig. 7 Beam with a hinged end.

Since $M_{m-1} = 0$

$$a'_m \left(i_{m+1} sF_m - U_m \right) = b''_m \left(i_{m-1} sF_{m-1} - U_{m-1} \right) \qquad \ldots \text{(ii-18)}$$

or

$$i_{m-1} sF_{m-1} - U_{m-1} = \frac{a'_m}{b''_m} \left(i_{m+1} sF_m - U_m \right)$$

from equation (e) and equation (ii-18)

$$M_m = a''_m \left(i_{m-1} sF_{m-1} - U_{m-1} \right) + b'_m \left(i_{m+1} sF_m - U_m \right)$$

$$= \left(i_{m+1} sF_m - U_m \right) \left(\frac{a''_m a'_m}{b''_m} + b'_m \right) \qquad \ldots \text{(ii-19)}$$

From Equation (d), substituting m for m-1, we get

$$M_m = -a'_{m+1} \left(i_{m+2} sF_{m+1} - U_{m+1} \right) + b''_{m+1} \left(i_m sF_m - U_m \right) \qquad \ldots \text{(ii-20)}$$

And (ii-19) is compatible with (ii-20) provided

$$\left(i_{m+1} sF_m - U_m \right) \left(\frac{a''_m a'_m}{b''_m} + b'_m \right)$$
$$= -a'_{m+1} \left(i_{m+2} s F_{m+1} - U_{m+1} \right) + b''_{m+1} \left(i_m sF_m - U_m \right) \qquad \ldots \text{(ii-21)}$$

Using the same expressions as on section (ii-a) that

$$F_m = p^2 \theta_m X_m \sin pt; \quad U_m = \frac{6EI_c}{s^2} \left(X_{m-1} - 2X_m + X_{m+1} \right) \sin pt$$

equation (ii-21) yields

$$\left\{ i_{m+1} sp^2 \theta_m X_m - \frac{6EI_c}{s^2} \left(X_{m-1} - 2X_m + X_{m+1} \right) \right\} \left\{ \frac{a''_m a'_m + b'_m b''_m}{b''_m} \right\}$$
$$= -a'_{m+1} \left\{ i_{m+2} sp^2 \theta_{m+1} X_{m+1} - \frac{6EI_c}{s^2} \left(X_m - 2X_{m+1} + X_{m+2} \right) \right\}$$
$$+ b''_{m+1} \left\{ i_m sp^2 \theta_m X_m - \frac{6EI_c}{s^2} \left(X_{m-1} - 2X_m + X_{m+1} \right) \right\}$$

or

$$X_m\left\{sp^2\theta_m\left(i_{m+1}\phi_m - b''_{m+1}\, i_m\right) + \frac{6EI_c}{s^2}\left(2\phi_m - a'_{m+1} - 2\, b''_{m+1}\right)\right\}$$

$$+X_{m+1}\left\{sp^2\theta_{m+1}\left(a'_{m+1}\, i_{m+2}\right) - \frac{6EI_c}{s^2}\left(\phi_m + 2a'_{m+1} - b''_{m+1}\right)\right\}$$

$$+X_{m+2}\left\{-\frac{6EI_c}{s^2}a'_{m+1}\right\} = 0 \qquad \ldots \text{(ii-22)}$$

where $\phi_m = \dfrac{a''_m a'_m + b'_m b''_m}{b''_m}$

The next equation between X_{m-1}, X_m, X_{m+1}, X_{m+2}, X_{m+3}, has the form given by equation (ii-10).

e. Equation For Intermediate Support

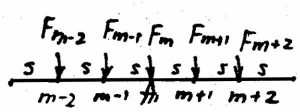

Fig. 8 - Beam with an intermediate support.

Now let "m" be an intermediate support. (Fig. 8)

From equation (ii-10):

$$X_{m-2}a''_m - X_{m-1}\left(2a''_m + b''_{m+1} - b'_m + a''_m i_{m-1}\theta_{m-1}\frac{ps}{6EI}\right)$$

$$+ X_m\left(a''_m + a'_{m+1} + 2\left(b''_{m+1} - b'_m\right) - \left(i_{m+1} b'_m - i_m b''_{m+1}\right)\theta_m \frac{p^2s^3}{6EI_c}\right)$$

$$- X_{m+1}\left(2a'_{m+1} + b''_{m+1} - b'_m + a'_{m+1} i_{m+2}\theta_{m+1}\frac{p^2s^3}{6EI_c}\right)$$

$$+ X_m\left(a'_{m+1}\right) = 0$$

Since $X_m = 0$, then the above equation yields

$$X_{m-2}a''_m - X_{m-1}\left(2a''_m + b''_{m+1} - b'_m + a''_m i_{m-1}\theta_{m-1}\frac{ps}{6EI}\right)$$

$$- X_{m+1}\left(2a'_{m+2} + b''_{m+1} - b'_m + a'_{m+1} i_{m+2}\theta_{m+1}\frac{ps}{6EI}\right)$$

$$+ X_{m+2}\, a'_{m+1} = 0$$

$$\ldots \text{(ii-23)}$$

iii. Equations For Beams With Constant Cross Sections

a. General Equation

From equation (ii-3) and equation (ii-7), we have

$$M_{m-1} - 2M_m + M_{m+1} = -sF_m$$

$$M_{m-1} i_m + 2M_m(i_m + i_{m-1}) + i_{m+1} M_{m+1} = -U_m$$

Since the moment of inertia is constant in this case, then $i = 1$, and the above two equations yield

$$M_{m-1}\ 2M_m + M_{m+1} = -sF_m \qquad \text{... (iii-1)}$$

$$M_{m-1} + 4M_m + M_{m+1} = -U_m \qquad \text{... (iii-2)}$$

Subtracting these two equations, we get

$$M_m = \frac{1}{6}(sF_m - U_m) \qquad \text{... (iii-3)}$$

or
$$M_{m-1} = \frac{1}{6}(sF_{m-1} - U_{m-1})$$

or
$$M_{m+1} = \frac{1}{6}(sF_{m+1} - U_{m+1})$$

Substitute back into equation (iii-1), we have

$$(sF_{m-1} - U_{m-1}) - 2(sF_m - U_m) + (sF_{m+1} - U_{m+1}) = -6sF_m$$

or
$$sF_{m-1} + 4sF_m + sF_{m+1} = (U_{m-1} - 2U_m + U_{m+1}) \qquad \text{... (iii-4)}$$

In order to apply it to the problem of lateral vibrations, we make the same assumption that

$$W_m = X_m \sin pt$$

and

$$F_m = -\theta_m \frac{\partial^2 W_m}{\partial t^2} = p^2 \theta_m X_m \sin pt$$

then

$$U_m = \frac{6EI_c}{s^2}\left(X_{m-1} - 2X_m + X_{m+1}\right) \sin pt$$

Substituting those expressions into the equation (iii-4), we get

$$\frac{p^2 s^3}{6EI_c}\left(\theta_{m-1} X_{m-1} + 4\theta_m X_m + \theta_{m+1} X_{m+1}\right)$$

$$= (X_{m-2} - 4X_{m-1} + 6X_m - 4X_{m+1} + X_{m+2})$$

or

$$X_{m-2} - X_{m-1}\left(4 + \frac{p^2 s^3}{6EI_c}\theta_{m-1}\right)$$
$$+ X_m\left(6 - \frac{4p^2 s^3}{6EI_c}\theta_m\right)$$
$$- X_{m+1}\left(4 + \frac{p^2 s^3}{6EI_c}\theta_{m+1}\right)$$
$$+ X_{m+2}$$
$$= 0$$

... (iii-5)

If all masses θ_m have the same value θ_c, then equation (iii-5) yields

$$(X_{m-2} + X_{m+2}) - (X_{m-1} + X_{m+1})(4 + \phi) + X_m(6 - 4\phi) = 0 \qquad \text{... (iii-6)}$$

where $\phi = \dfrac{p^2 s^3 \theta_c}{6EI_c}$

b. Equations For Free End

From equation (iii-1) and equation (iii-2), for $m=1$, we have

$$M_o - 2M_1 + M_2 = -sF_1$$

$$M_o + 4M_1 + M_2 = -U_1$$

for $M_o = 0$ the values

$$M_1 = \frac{sF_1 - U_1}{6} = -F_o s$$

$$M_2 = -\frac{1}{3}(U_1 + 2sF_1) = \frac{1}{6}(sF_2 - U_2)$$

from which

$$s(6F_o + F_1) = U_1 = \frac{6EI_c}{s^2}(X_o - 2X_1 + X_2)\sin pt$$

$$s(4F_1 + F_2) = -2U_1 + U_2 = -\frac{6EI_c}{s^2}(2X_o - 5X_1 + 4X_2 - X_3)\sin pt$$

or

$$X_o(1-\phi) - X_1(2+\phi) + X_2 = 0 \qquad \text{... (iii-7)}$$

$$-2X_o + X_1(5-4\phi) + X_2(4-\phi) - X_3 = 0 \qquad \text{... (iii-8)}$$

c. Equation For Fixed End

The boundary conditions for the rigidly supported end section (n) are easily formulated by considering the beam as prolonged symmetrically with respect to the point n. See Fig. 9

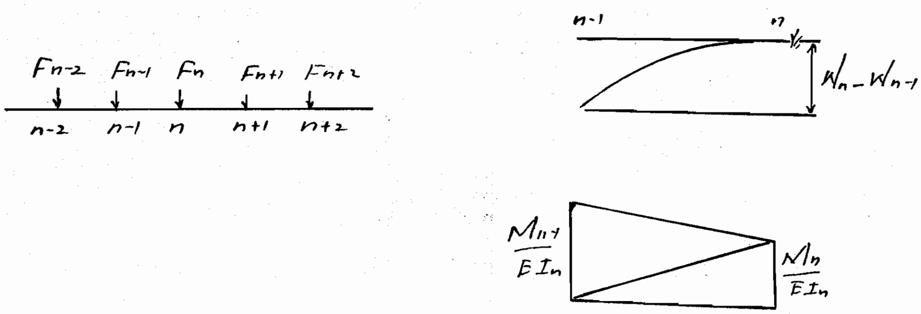

Fig. 9 - $\frac{M}{EI}$ diagram

The symmetry requires that

$$X_{n+1} = X_{n-1}, \quad X_{n+2} = X_{n-2}$$

$$F_{n+1} = F_{n-1}, \quad F_{n+2} = F_{n-2}$$

and

$$sF_n = 2(M_n - M_{n-1})$$

The total restraint at n implies that

$$W_n = 0$$

and by using Area Moment Method, we can show that

$$W_n - W_{n-1} = (M_{n-1} + 2M_n)$$

from which

$$M_n = -\frac{M_{n-1}}{2} - \frac{3EI_c}{s^2} W_{n-1}$$

$$F_n = -\frac{6EI_c W_n}{s^3} - \frac{3M_{n-1}}{s}$$

or $F_n = -\frac{6EI_c}{s^3}W_{n-1} + \frac{1}{2s}(sF_{n-1} - U_{n-1})$

or

$$F_n - \frac{p^2}{2}\theta_c W_{n-1} - 3\frac{EI_c}{s}W_{n-2}$$

Substituting these values in the equation (iii-4)

$$X_{n-3} - 4X_{n-2} + 6X_{n-1} - 4X_n + X_{n+1}$$

$$= \frac{p^2 s^3}{6EI_c}\theta_c \left(X_{n-2} + \frac{7}{2} X_{n-1} \right) - \frac{X_{n-2}}{2}$$

we get finally

$$X_{n-3} - X_{n-2}\left(\frac{7}{2} + \phi\right) + X_{n-1}\left(7 - \frac{9}{2}\phi\right) = 0 \qquad \ldots \text{(iii-9)}$$

d. Equation For Hinged End

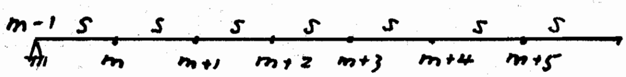

Fig. 10 - Beam with a hinged end.

For point m, the equation (iii-1) is

$$M_{m-1} - 2M_m + M_{m+1} = -sF_m$$

Since $M_{m-1} = 0$ in this case, then the above equation yields

$$- 2M_m + M_{m+1} = -sF_m \qquad \ldots \text{(iii-10)}$$

From equation (iii-2), for $M_{m-1} = 0$

$$4M_m + M_{m+1} = -U_m \qquad \ldots \text{(iii-11)}$$

Subtracting the above two equations, we have

$$6M_m = -(U_m - sF_m)$$

or

$$M_m = \frac{1}{6}(sF_m - U_m) \qquad \ldots \text{(iii-12)}$$

or

$$M_{m+1} - \frac{1}{6}\,(sF_{m+1} - U_{m+1})$$

Substituting these values into equation (iii-10), we have

$$-\frac{2}{6}\,(sF_m - U_m) + \frac{1}{6}\,(sF_{m+1} - U_{m+1}) = -sF_m$$

or

$$4sF_m + sF_{m+1} = -2U_m + U_{m+1}$$

or

$$\frac{s^3}{6EI_c}\left(4\,\theta_m\,p^2X_m + \theta_{m+1}\,p^2X_{m+1}\right) = -2(X_{m-1} - 2X_m + X_{m+1})$$
$$-X_m - 2X_{m+1} + X_{m+2}$$

and $X_{m-1} = 0$, the above equation yields

$$X_m\left(\frac{2s^3p^2\theta_m}{3\,EI_c} - 5\right) + X_{m+1}\left(\frac{s^3p^2\,\theta_{m+1}}{6\,EI_c} + 4\right) - X_{m+2} = 0 \quad \ldots \text{(iii-13)}$$

The next equation between X_m , X_{m+1}, X_{m+2}, X_{m+3} has the form given by equation (iii-5)

e. Equations For Intermediate Support

The general equation (iii-5) can be applied except that $X_m = 0$. See Fig. 11

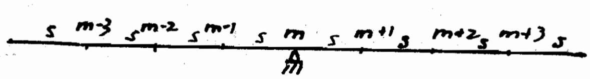

Fig. 11 - Beam with an intermediate support.

B. Results

i. Applications

a. Beams With Variable Cross Sections

1. Cantilever Beam

In order to be able to compare the results found by using other methods, the author chooses the example of a cantilever beam (6) which has already been solved by using the Stodola-Viannello method. In order to conveniently formulate our equations, the order of the segments of the original problem is as shown in Fig. 12. All the necessary calculations of coefficients are listed in the tabular scheme.

The method calculating the roots of the determinant has been described in detail in the Appendix on page

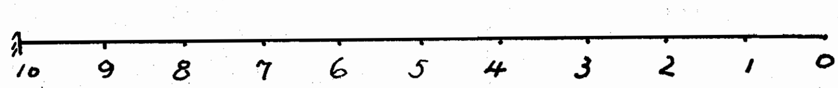

Fig. 12 - Cantilever beam with variable cross sections.

1-2 Formulation of Equations

Since m=0 is a free end, then from equation (ii-11):

$$X_o\left\{1 - 2\,\theta_o \frac{p^2 s^3}{6\,EI_c}(i_1+2i_2)\right\} - X_1\left\{2 + \frac{p^2 s^3}{6\,EI_c} i_2 \theta_1\right\} + X_2 = 0$$

Substituting all the necessary coefficients from Table 1 into above equation, we get

$$X_o(1 - 0.315 \times 10^{-6} p^2) - X_1(2 + 0.144 \times 10^{-6} p^2) + X_2 = 0 \qquad \ldots (1 - 1)$$

Similarly for m=1, from equation (ii-12):

$$-\frac{p^2 s^3}{6\,EI_c}\theta_o X_o\{5i_2 + 7i_3\} + X_1\left\{1 - \frac{2p^2 s^3}{6\,EI_c}\theta_1(i_2 + 2i_3)\right\} - X_2\left\{2 + \frac{p^2 s^3}{6\,EI_c}\theta_2 i_3\right\} + X_3 = 0$$

Substituting all the coefficients from Table 1 into the above equation, we get

$$-0.358 \times 10^{-6} p^2 X_o + (1 - 0.66 \times 10^{-6} p^2)\ X_1 - (2 + 0.131 \times 10^{-6} p^2)\ X_2 + X_3 = 0 \qquad (1 - 2)$$

For m-2, we can use the general equation (ii-10) that

$$\begin{aligned} X_o a_2'' &- X_1\left\{2a_2'' + b_3'' - b_2' + a_2''\ i_1\theta_1 \frac{p^2 s^3}{6\,EI_c}\right\} \\ &+ X_2\left\{a_2'' + 2(b_3'' - b_2') - (i_3 b_2' - i_2 b_3'')\theta_2 \frac{p^2 s^3}{6\,EI_c} + a_3''\right\} \\ &- X_3\left\{2a_3' + b_3'' - b_2' + a_3'\ i_4\theta_3 \frac{p^2 s^3}{6\,EI_c}\right\} \\ &+ X_4 a'_3 = 0 \end{aligned}$$

Substituting all coefficients into the above equation, then

$$-0.00068\ X_0 + (0.01236 + 0.000198 \times 10^{-6} p^2)\ X_1$$
$$- (0.00728 - 0.00102 \times 10^{-6} p^2)\ X_2$$
$$+ (0.0060 + 0.0000108 \times 10^{-6} p^2)\ X_3$$
$$- 0.0018\ X_4 \qquad = 0$$

$$\qquad \ldots (1\text{-}3)$$

For m=3, using equation (ii-10), then

$$X_1\, a''_2 - X_2 \left\{ 2a''_3 + b''_4 - b'_3 + a''_3\, i_2\, \theta_2\, \frac{p^2 s^3}{6\,EI_c} \right\}$$
$$+ X_3 \left\{ a''_3 + a'_4 + 2(b''_4 - b'_3) - (i_4\, b'_3 - i_3\, b''_4)\,\theta_3\, \frac{p^2 s^3}{6\,EI_c} \right\}$$
$$- X_4 \left\{ 2a'_4 + b''_4 - b'_3 + a'_4\, i_5\, \theta_4\, \frac{p^2 s^3}{6\,EI_c} \right\}$$
$$+ X_5\, a'_4 \quad = 0$$

and substituting all coefficients, we get

$$-0.00092\ X_1 + (0.01234 + 0.000208 \times 10^{-6} p^2)\ X_2$$
$$- (0.00722 - 0.000875 \times 10^{-6} p^2)\ X_3$$
$$+ (0.0058 + 0.00000952 \times 10^{-6} p^2)\ X_4$$
$$- 0.0017 \qquad X_5$$
$$= 0 \qquad \ldots (1\text{-}4)$$

For m=4, using equation (ii-10), then

$$X_2a''_4 - X_3\left\{2a''_4 + b''_5 - a''_4\, i_3\, \theta_3 \frac{p^2s^3}{6\,EI_c} - b'_4\right\}$$

$$+ X_4\left\{a''_4 + a'_5 + 2(b''_5 - b'_4) - (i_5 b'_4 - i_4 b''_5)\theta_4 \frac{p^2s^3}{6\,EI_c}\right\}$$

$$- X_5\left\{2a'_5 + b''_5 - b'_4 + a'_5\, i_6\, \theta_5 \frac{p^2s^3}{6\,EI_c}\right\}$$

$$- X_6a'_5 = 0$$

and substituting all coefficients into the equation, we get

$$- 0.001\ X_2$$

$$+ (0.0125 + 0.000179 \times 10^{-6} p^2)\ X_3$$

$$- (0.00818 - 0.000775 \times 10^{-6} p^2)\ X_4$$

$$+ (0.00596 + 0.0000107 \times 10^{-6} p^2)\ X_5$$

$$- 0.00158 \qquad X_6 = 0$$

... (1-5)

For m=5, using equation (ii-10), then

$$X_3a''_5 - X_4\left\{2a''_5 + b''_6 - b'_5 + a''_5\, i_4\, \theta_4 \frac{p^2S^3}{6EI_c}\right\}$$

$$+ X_5\left\{a''_5 + a'_6 + 2(b''_6 - b'_5) - (i_5b'_5 - i_5b''_6)\theta_5 \frac{p^2S^3}{6EI_c}\right\}$$

$$- X_6\left\{2a'_6 + b''_6 - b'_5 + a'_6\, i_7\, \theta_6 \frac{p^2S^3}{6EI_c}\right\}$$

$$- X_7\, a'_6 = 0$$

and substituting all coefficients, we get

$$- 0.00107\, X_3 + (0.01284 + 0.000179 \times 10^{-6}\, p^2)\, X_4$$

$$- (0.00803 - 0.000945 \times 10^{-6}\, p^2)\, X_5$$

$$+ (0.00582 + 0.0000197 \times 10^{-6}\, p^2)\, X_6$$

$$- 0.00156 \qquad X_7 = 0$$

$$\ldots (1\text{-}6)$$

For m=6, using equation (ii-10), then

$$X_4\, a''_6$$

$$- X_5 \left\{ 2a''_6 + b''_7 - b'_6 + a''_6\, i_5 \theta_5 \frac{p^2 s^3}{6EI_c} \right\}$$

$$+ X_6 \left\{ a''_6 + a'_7 + 2(b''_7 - b'_6) - (i_7 b'_6 - i_6 b''_7)\theta_6 \frac{p^2 s^3}{6EI_c} \right\}$$

$$- X_7 \left\{ 2a'_7 + b''_7 - b'_6 + a'_7 i_8\, \theta_7 \frac{p^2 s^3}{6EI_c} \right\}$$

$$+ X_8\;\; a'_7$$

$$= 0$$

and substituting all coefficients, we get

$$- 0.0012\, X_4$$

$$+ (0.01324 + 0.000211 \times 10^{-6}\, p^2)\, X_5$$

$$- (0.00766 - 0.000945 \times 10^{-6}\, p^2)\, X_6$$

$$+ (0.00558 - 0.0000272 \times 10^{-6}\, p^2)\, X_7$$

$$- (0.00154) \qquad X_8$$

$$= 0 \qquad \ldots (1\text{-}7)$$

For m=7, using equation (ii-10), then

$$X_5 a''_7$$

$$- X_6 \left\{ 2a''_7 + b''_8 - b'_7 + a''_7\, i_6\, \Theta_6 \frac{p^2 s^3}{6EI_c} \right\}$$

$$+ X_7 \left\{ a''_7 + a'_8 + 2\,(b''_8 - b'_7) - (i_8\, b'_7 - i_7\, b''_8)\Theta_7 \frac{p^2 s^3}{6EI_c} \right\}$$

$$- X_8 \left\{ 2a'_8 + b''_8 - b'_7 + a'_8\, i_9\, \Theta_8 \frac{p^2 s^3}{6EI_c} \right\}$$

$$- X_9 \quad a'_8 \quad = 0$$

and substituting all the coefficients, we get

$$- X_5\ (0.00115)$$

$$+ X_6\ (0.01330 + 0.000406 \times 10^{-6} p^2)$$

$$- X_7\ (0.00942 - 0.000173 \times 10^{-6} p^2)$$

$$+ X_8\ (0.00634 + 0.0000273 \times 10^{-6} p^2)$$

$$- X_9\ (0.00147)$$

$$= 0 \qquad \ldots (1\text{-}8)$$

For m=8, using equation (ii-10), then

$$X_6\ (a''_8)$$

$$- X_7 \left\{ 2a''_8 + b''_9 - b'_8 - a''_8\, i_7\, \Theta_7 \frac{p^2 s^3}{6EI_c} \right\}$$

$$+ X_8 \left\{ a''_8 + a'_9 + 2(b''_9 - b'_8) - (i_9 b'_8 - i_8 b''_9)\, \Theta_8 \frac{p^2 s^3}{6EI_c} \right\}$$

$$- X_9 \left\{ 2a'_9 + b''_9 - b'_8 + a'_9\, i_{10}\, \Theta_9 \frac{p^2 s^3}{6EI_c} \right\}$$

$$+ X_{10} \quad a'_9 \quad = 0$$

and substituting all the coefficients and note that

$X_{10} = 0$, we get

$$
\begin{aligned}
&- X_6\ (0.00116) \\
&+ X_7\ (0.01222 + 0.000561 \times 10^{-6} p^2) \\
&- X_8\ (0.00525 - 0.00196 \times 10^{-6} p^2) \\
&+ X_9\ (0.00428 + 0.0000268 \times 10^{-6} p^2) \\
&= 0 \qquad \ldots (1\text{-}9)
\end{aligned}
$$

For m-10, since it is a fixed end, we must use equation (ii-17), then

$$
\begin{aligned}
&X_7 a''_9 \\
&+ X_8 \left\{ 2a''_9 + c'_9 - b'_9 + a''_9\, \theta_8 \frac{p^2 s^3}{6EI_c} \right\} \\
&+ X_9 \left\{ -a''_9 + 2(c'_9 - b'_9) - c''_9 + \frac{p^2 s^3}{6EI_c}\, \theta_9 (i_{10}\, b'_9 - c'_9\, i_9) \right\} \\
&- X_{10} \left\{ b'_9 - c'_9 - c''_9 \right\} = 0
\end{aligned}
$$

and substituting all the coefficients and note that $X=0$, we get

$$
\begin{aligned}
&- X_7\ (0.00114) \\
&+ X_8\ (0.00528 + 0.00022 \times 10^{-6} p^2) \\
&- X_9\ (0.00876 + 0.00185 \times 10^{-6} p^2) \\
&= 0 \qquad \ldots (1\text{-}10)
\end{aligned}
$$

1-3 Calculation of Roots of Determinants

The method has been described in the Appendix. The p^2 term is small compared with that constant term if p is less than one hundred radians per second. For this case the ten equations yield

$$X_o - 2X_1 + X_2 = o \qquad (1\text{-}1')$$

$$-0.358\times10^{-6}\ X_o + X_1 - 2X_2 + X_3 = o \qquad (1\text{-}2')$$

$$-6.8\ X_o + 123.6\ X_1 - 72.8\ X_2 + 60X_3 - 18\ X_4 = o \qquad (1\text{-}3')$$

$$-9.2\ X_1 + 123.4\ X_2 - 72.2\ X_3 + 58\ X_4 - 17\ X_5 = o \qquad (1\text{-}4')$$

$$-10\ X_2 - 125\ X_3 - 81.8\ X_9 - 59.6\ X_5 - 15.8\ X_6 = o \qquad (1\text{-}5')$$

$$-10.7\ X_3 + 128.4\ X_4 - 80.3\ X_5 + 58.2\ X_6 - 15.6\ X_7 = o \qquad (1\text{-}6')$$

$$-11.2\ X_4 + 132.4\ X5 - 76.6\ X_6 + 55.8\ X_7 - 15.4\ X_8 = o \qquad (1\text{-}7')$$

$$-11.5\ X_5 + 133\ X_6 - 74.2\ X_7 + 63.4\ X_8 - 14.7\ X_9 = o \qquad (1\text{-}8')$$

$$-11.6\ X_6 + 122.2\ X_7 - 52.5\ X_8 + 42.8\ X_9 = o \qquad (1\text{-}9')$$

$$-11.4\ X_7 + 52.8\ X_8 - 87.5\ X_9 = o \qquad (1\text{-}10')$$

1. Assume $X'_9 = o$ and $X'_8 = A$

from (1-10')

$$X'_7 = 4.632A$$

(1-9')

$$X'_6 = 44.270A$$

(1-8')

$$X'_5 = 479.56A$$

(1-7')

$$X'_4 = 5,388.01A$$

(1-6')

$$X'_3 = 61,291.2A$$

(1-5')

$$X'_2 = 724,854A$$

(1-4')

$$X'_1 = 9274,577A$$

(1-3')

$$X'_o = 160,081,880A$$

2. Assume $X''_8 = o$ and $X''_9 = B$

$$X''_7 = -7.67B$$

$$X''_6 = -77.1B$$

$$X''_5 = -830B$$

$$X''_4 = -9323B$$

$$X''_3 = -106070B$$

$$X''_2 = -1,262,500B$$

$$X''_1 = -16,163,000B$$

$$X''_o = -281,590,000B$$

3. Let $X_i = X'_i + X''_i$ e.g. $X_9 = o + B = B$

Substitute the new values into the first two equations

(1-1') and (1-2'), we get from equation (1-1')

$$A = 1.7423B$$

from equation (1-2'), we get

$$+ X\ 0.358 \times 10^{-6} p^2 = 1332$$

$$p^2 = \frac{1332}{0.358} \times 10^6$$

$$= 3721$$

$$p = 61.0 \text{ radion/sec.}$$

The above answer is 3 percent higher than the answer found by the Stodola method which is given in section (ii-a-1)

B. Results

i. Applications

a. Beam With Variable Cross Sections

2. Hinged End Beam

Use four segments, see Fig. 13

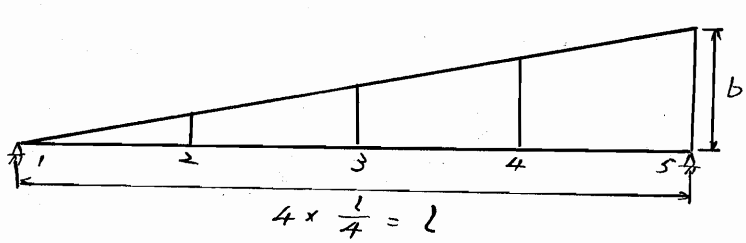

Fig. 13 - Beam with four segments.

2-2 Formulation of Equations

For m=2, equation (ii-22) yields

$$X_2\left\{s\,p^2\theta_2\left(i_3\phi_2-b''_3\,i_2\right)+\frac{6EI_c}{s^2}\left(2\phi_2-a'_3-2b''_3\right)\right\}$$

$$+X_3\left\{s\,p^2\theta_3\,i_4\,a'_3-\frac{6EI_c}{s^2}\left(\phi_2+2a'_3-b''_3\right)\right\}$$

$$+X_4\left\{-\frac{6EI_c\,a'_3}{s^2}\right\}=0$$

Substituting all the coefficients from Table 2 into the above equation, we get

$$X_2\left\{0.002935\,p^2\frac{b\gamma l^2}{g}-0.004846\,\frac{E\,b^3}{l^2}\right\}$$

$$+X_3\left\{0.0022603\,p^2\frac{b\gamma l^2}{g}+0.001586\,\frac{E\,b^3}{l^2}\right\}$$

$$+X_4\left\{-0.00037029\,\frac{E\,b^3}{l^2}\right\}$$

$$=0 \qquad \ldots (2\text{-}1)$$

For m=3, since $X_1 = X_5 = 0$, then equation (ii-10) yields

$$-X_2\left\{s p^2\theta_2\,i_2\,a''_3+\frac{6EI_c}{s^2}\left(2a''_3-b'_3+b''_4\right)\right\}$$

$$+X_3\left\{-s p^2\theta_2\left(i_4 b'_3-i_3 b'_4\right)+\frac{6EI_c}{s^2}\left(a''_3-2b'_3+a'_4+2b''_4\right)\right\}$$

$$-X_4\left\{s p^2\theta_4\,i_5\,a'_4+\frac{6EI_c}{s^2}\left(2a'_4-b'_3+b''_4\right)\right\}$$

$$=0$$

Substituting all the coefficients into the above equation, we get

$$- X_2 \left\{ 0.00011361 \ \frac{p^2 b \gamma l^2}{g} + 0.002282 \ \frac{E b^3}{l^2} \right\}$$

$$+ X_3 \left\{ -0.003615 \ \frac{p^2 b \gamma l^2}{g} + 0.001925 \ \frac{E b^3}{l^2} \right\}$$

$$- X_4 \left\{ 0.002091 \ \frac{p^2 b \gamma l^2}{g} + 0.0006382 \ \frac{E b^3}{l^2} \right\}$$

$$= 0$$

... (2-2)

For m=4, it is the same as m=2 and $X_5 = 0$, then equation (ii-22) yields

$$X_4 \left\{ s p^2 \theta_4 (i_3 \ \phi_4 - b''_3 \ i_4) + 6 \frac{EI_s}{s^2} (2\phi_4 - a'_3 - 2b''_3) \right\}$$

$$+ X_3 \left\{ s p^2 \theta_3 a'_3 \ i_2 - \frac{6 EI_s}{s^2} (\phi_4 + 2a'_3 - b''_3) \right\}$$

$$+ X_2 \left\{ - 6 \frac{EI_s}{s^2} a'_3 \right\} = 0$$

Substituting all the coefficients into the above equation, we get

$$X_4 \left\{ -0.013177 \ \frac{p^2 b \gamma l^2}{g} - 0.01139 \frac{Eb^3}{l^2} \right\}$$

$$+ X_3 \left\{ 0.00008371 \ \frac{p^2 b \gamma l^2}{g} + 0.011794 \ \frac{Eb^3}{l^2} \right\}$$

$$+ X_2 \left\{ -0.00037029 \ \frac{E b^3}{l^2} \right\}$$

... (2-3)

$$= 0$$

2-3 Calculation of Roots of Determinant

First try: $p = 2.8$

then equation (2-1) yields

$$X_2\ (0.003372) + X_3\ (0.0079148) + X_4\ (-0.0010368) = 0$$

and equation (2-2) yields

$$X_2\ (-0.002600) + X_3\ (-0.008197) + X_4\ (-0.006494) = 0$$

and equation (2-3) yields

$$X_2\ (-0.0010368) + X_3\ (0.0120284) + X_4\ (-0.048286) = 0$$

In order that the three equations be independent of one another, their determinant must vanish. Now we try different values of p until we find the value that makes the determinant vanish. In the first try, the determinant is

$$\Delta = \begin{vmatrix} 0.003372 & 0.0079148 & -0.00037029 \\ -0.002600 & -0.008197 & -0.006494 \\ -0.00037029 & 0.0120284 & -0.048286 \end{vmatrix}$$

or

$$\Delta \times 100 = \begin{vmatrix} 0.3372 & 0.7915 & -0.0370 \\ -0.2600 & -0.8197 & -0.6494 \\ -0.0370 & 1.2028 & -4.8286 \end{vmatrix} = 0.7000$$

Second try:

$p^2 = 2.5$, following the same procedures as the first try, the determinant is

$$\Delta \times 100 = \begin{vmatrix} 0.2492 & 0.7236 & -0.0370 \\ -0.2566 & -0.7112 & -0.5805 \\ -0.0370 & 1.2000 & -4.4330 \end{vmatrix}$$

$$= 0.1800$$

Third try: the determinant of $p^2 = 2$ is

$$\Delta \times 100 = \begin{vmatrix} 0.1024 & 0.6106 & -0.0370 \\ -0.2509 & -0.5305 & -0.4820 \\ -0.0370 & 1.196 & -3.774 \end{vmatrix}$$

$$= -0.2691$$

From the above three tries, we can plot a curve of the determinant vs the values of p^2 as in Fig. 14. The intercept at the p^2 axis gives the value as 2.35.

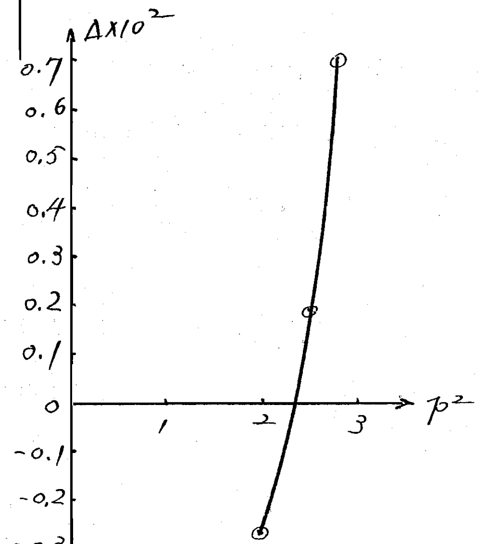

Fig. 14 - $\Delta \times 10^2$-p^2 curve

Now we can check it by the last try.

Let $p^2 = 2.35$, then the determinant is

$$\Delta \times 100 = \begin{vmatrix} 0.2051 & 0.6898 & -0.0370 \\ -0.2549 & -0.6570 & -0.5552 \\ -0.0375 & 1.1990 & -4.225 \end{vmatrix}$$

$$= 0.01$$

It is quite small so that we can say that the p^2 - 2.35. The difference of the frequency between this method and the Rayleigh method which is given in section (B-ii-a-2) is 9.5%.

B. Results

i. Applications

a. Beams With Variable Cross Sections

2. Hinged End Beams

Use 6 segments, see Fig. 15.

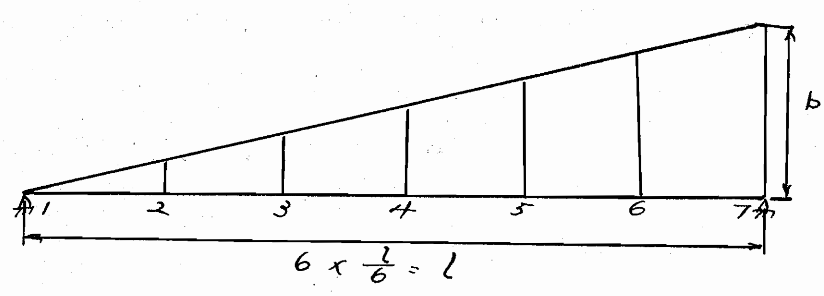

Fig. 15 - Beam with six segments.

2-2' Formulation of Equations

For m=2, we use the equation (ii-22) and note that $X_1 = 0$, then

$$X_2 \left\{ sp^2 \theta_2 (i_3 \phi_2 - b''_3 i_2) + \frac{6\,EI_c}{s^2}(2\phi_2 - a'_3 - 2b''_3) \right\}$$

$$+ X_3 \left\{ sp^2 \theta_3 i_4 a'_3 - \frac{6\,EI_c}{s^2}(\phi_2 + 2a'_3 - b''_3) \right\}$$

$$+ X_4 \left\{ - \frac{6\,EI_c}{s^2} a'_3 \right\} = 0$$

Substituting all the coefficients from Table 3 into the above equation, we get

$$X_2 \left\{ 0.000869 \frac{p^2 b \gamma l^2}{g} - 0.003230 \frac{E\,b^3}{l^2} \right\}$$

$$+ X_3 \left\{ 0.0005491 \frac{p^2 b \gamma l^2}{g} + 0.001057 \frac{E\,b^3}{l^2} \right\}$$

$$+ X_4 \left\{ - 0.0002469 \frac{E\,b^3}{l} \right\} = 0 \qquad \text{... (2-1')}$$

For m=3, use equation (ii-10) and note $X_1 = 0$, then

$$- X_2 \left\{ sp^2 \theta_2 i_2 a''_3 + \frac{6\,EI_c}{s^2}(2a''_3 - b'_3 + b''_4) \right\}$$

$$+ X_3 \left\{ sp^2 \theta_3 (i_3 b''_4 - i_4 b'_3) + \frac{6\,EI_c}{s^2}(a''_3 - 2b'_3 + a'_4 + 2b''_4) \right\}$$

$$- X_4 \left\{ sp^2 \theta_4 a'_4 i_5 + \frac{6EI_c}{s^2} (2a'_4 - b'_3 + b''_4) \right\}$$

$$+ X_5 \left\{ 6 \frac{EI_c}{s^2} a'_4 \right\} = 0$$

Substituting all the coefficients into the equation, we get

$$
\begin{aligned}
&- X_2 \left\{ \frac{p^2 b \gamma l^2}{g} (0.00003366) + 0.001521 \frac{E b^3}{l^2} \right\} \\
&+ X_3 \left\{ \frac{p^2 b \gamma l^2}{g} (-0.001071) + 0.001283 \frac{E b^3}{l^2} \right\} \\
&- X_4 \left\{ \frac{p^2 b \gamma l^2}{g} (0.0006194) + 0.0004255 \frac{E b^3}{l^2} \right\} \\
&+ X_5 \left\{ 0.000058 \frac{E b^3}{l^2} \right\} = 0
\end{aligned}
\qquad \ldots (2\text{-}2')
$$

For m=4, same as above equation, we have

$$
\begin{aligned}
&X_2 \left\{ \frac{6 E I_c}{s^2} a''_4 \right\} \\
&- X_3 \left\{ s p^2 \theta_3 i_3 a''_4 + \frac{6 E I_c}{s^2} (2a''_4 - b'_4 + b''_5) \right\} \\
&+ X_4 \left\{ s p^2 \theta_4 (i_4 b''_5 - i_5 b'_4) + \frac{6 E I_c}{s^2} (a'_4 + a'_5 + 2(b''_5 - b'_4) \right\} \\
&- X_5 \left\{ s p^2 \theta_5 i_6 a'_5 + \frac{6 E I_c}{s^2} (2a'_5 + b''_5 - b'_4) \right\} \\
&+ X_6 \left\{ \frac{6 E I_c}{s^2} a'_5 \right\} = 0
\end{aligned}
$$

and substituting all the coefficients into above equation, we get

$$
\begin{aligned}
&X_2 \left\{ 0.000113 \frac{E b^3}{l^2} \right\} \\
&- X_3 \left\{ 0.00003769 \frac{p^2 b \gamma l^2}{g} + 0.000412 \frac{E b^3}{l^2} \right\} \\
&+ X_4 \left\{ - 0.0007834 \frac{p^2 b \gamma l^2}{g} + 0.0005119 \frac{E b^3}{l^2} \right\} \\
&- X_5 \left\{ 0.0007347 \frac{p^2 b \gamma l^2}{g} + 0.000247 \frac{E b^3}{l^2} \right\} \\
&+ X_6 \left\{ 0.00002645 \frac{E b^3}{l^2} \right\} = 0
\end{aligned}
\qquad \ldots (2\text{-}3')
$$

For m=5, using equation (ii-10), we have

$$X_3 \left\{ \frac{6\,EI_c}{s^2}\; a''_5 \right\}$$

$$- X_4 \left\{ sp^2 \theta_4\, i_4\, a''_5 + \frac{6\,EI_c}{s^2} \left(2a''_5 - b'_5 + b''_6 \right) \right\}$$

$$+ X_5 \left\{ sp^2 \theta_5 \left(i_5 b''_6 - i_6 b'_5 \right) + \frac{6EI_c}{s^2} \left(a''_5 + a'_6 + 2b''_6 - 2b'_5 \right) \right\}$$

$$- X_6 \left\{ sp\, \theta_6\, i_7\, a'_6 + \frac{6\,EI_c}{s^2} \left(2a'_6 - b'_5 + b''_6 \right) \right\}$$

$$= 0$$

and substituting all the coefficients into the equation, we get

$$X_3 \left\{ 0.00004362\ \frac{E\,b^3}{l^2} \right\}$$

$$- X_4 \left\{ \frac{p^2 b \gamma l^2}{g} (0.0001963) + 0.0001361\ \frac{Eb^3}{l^2} \right\}$$

$$+ X_5 \left\{ \frac{p^2 b \gamma l^2}{g} (-0.001593) + 0.0001532\ \frac{Eb^3}{l^2} \right\}$$

$$- X_6 \left\{ \frac{p^2 b \gamma l^2}{g} (0.000715) + 0.0000726\ \frac{Eb^3}{l^2} \right\}$$

$$= 0 \qquad \ldots (2\text{-}4')$$

For m-6, we must use equation (ii-22), we have

$$X_4 \left\{ - \frac{6\,EI_c}{s^2}\; a'_5 \right\}$$

$$+ X_5 \left\{ sp^2 \theta_5\, a'_5\, i_4 - \frac{6\,EI_c}{s^2} \left(\phi_6 + 2a'_5 - b''_5 \right) \right\}$$

$$+ X_6 \left\{ sp^2 \theta_6 \left(i_5 \phi_6 - b''_5\, i_6 \right) + \frac{6\,EI_c}{s^2} \left(2\phi_6 - a'_5 - 2b''_5 \right) \right\}$$

$$= 0$$

and substituting all the coefficients into the equation, we get

$$- X_4 \left\{ 0.00002645 \ \frac{E\,b^3}{l^2} \right\}$$

$$+ X_5 \left\{ \frac{p^2 b \gamma l^2}{g}(0.0001587) + 0.0002963 \ \frac{Eb^3}{l^2} \right\}$$

$$+ X_6 \left\{ \frac{p^2 b \gamma l^2}{g}(-0.013816) - 0.0007249 \ \frac{Eb^3}{l^2} \right\}$$

$$= 0 \qquad \ldots (2\text{-}5')$$

2-3' Calculation of Roots of Determinant

First try: let $p^2 = 2.5$, then the equations (2-1') will yield (2-2'), equation (2-3'), equation (2-4') and equation (2-5') will yield

X_2	X_3	X_4	X_5	X_6	
-0.0010575	0.002430	-0.0002469			= 0
-0.001605	-0.001395	-0.001974	0.000058		= 0
0.0001130	-0.0005062	-0.0014466	-0.0020838	0.00002645	= 0
	0.00004362	-0.0006268	-0.0038288	-0.001860	= 0
		-0.00002645	0.00069305	-0.035265	= 0

Using the method shown in the Appendices, let

$X_2 = o \qquad X_3 = A$

then from

equ. (2-1') $X_4 = 9.842A$

equ. (2-2') $X_5 = 359.0A$

equ. (2-3') $X_6 = 28840A$

let

$X_2 = B \qquad X_3 = o$

then from

equ. (2-1') $X_4 = -4.283B$

equ. (2-2') $X_5 = -118.1B$

equ. (2-3') $X_6 = -9534B$

Substituting the value of X in terms of A and B, into

equation (2-4'), we get

$3.055A - B = o$

equation (2-5'), we get

$3.024A - B = o$

The determinant of the above two equations is

$$\Delta = \begin{vmatrix} 3.055 & -1 \\ 3.024 & -1 \end{vmatrix} = -0.031$$

Second try : let $p^2 = 2.8$, then the five equations will yield

X_2	X_3	X_4	X_5	X_6	
-0.0007968	0.002694	-0.0002469			= 0
-0.001615	-0.001716	-0.002160	0.000058		= 0
0.000113	-0.0005175	-0.001682	-0.002304	0.00002645	= 0
	0.00004362	-0.0006857	-0.004307	-0.002075	= 0
		-0.00002645	0.0012407	-0.03941	= 0

Let

$X_2 = 0 \quad X_3 = A$

then

$X_4 = 10.91A$

$X_5 = 435.9A$

$X_6 = 46569A$

$X_2 = B \quad X_3 = 0$

$X_4 = -3.227B$

$X_5 = -92.33B$

$X_6 = -8243.6B$

Substitute into equation (2-4'), we get

$$-98.5149A + 17.5031B = 0$$

or $\quad 5.6284A - B = 0$

Substitute into equation (2-5'), we get

$$-1830.29A + 323.74B = 0$$

$$5.6536A - B = 0$$

From the above equations in terms of A and B, we find the determinant is

$$\Delta = \begin{vmatrix} 5.6284 & -1 \\ 5.6536 & -1 \end{vmatrix} = +0.0252$$

From the two trials of values of p^2, we can plot the curve of determinants vs p^2, and the value of p^2 for intercept $\Delta = 0$ is $p^2 = 2.67$

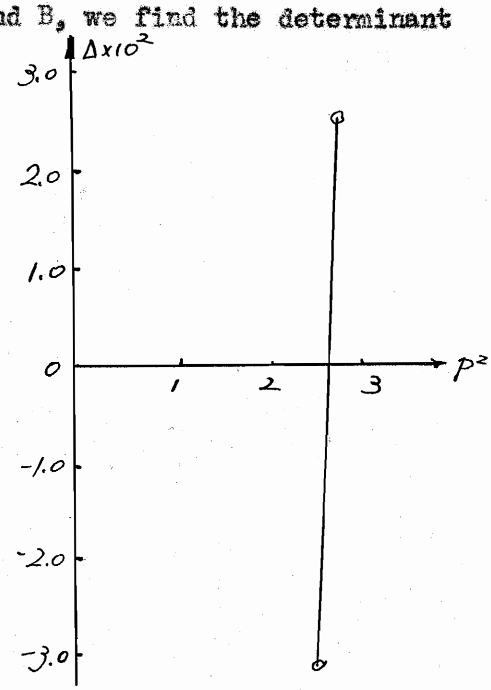

Fig. 16 - $\Delta \times 10^2$ - p^2 curve.

Now, let us try

$p^2 = 2.67$ The five equations yield

X_2	X_3	X_4	X_5	X_6	
-0.00091	0.0025231	-0.0002469			= 0
-0.0016109	-0.001577	-0.002080	0.000058		= 0
0.000113	-0.0005126	-0.001500	-0.002209	0.00002645	= 0
	0.00004362	-0.0006601	-0.004100	-0.001982	= 0
		-0.00002645	0.0007200	-0.0376136	= 0

Let

$X_2 = 0$

$X_3 = A$

follow the same method and procedure, we get

$X_4 = 10.22$

$X_5 = 393.0$

$X_6 = 33500$

Let

$X_2 = B \quad X_3 = 0$

$X_4 = \quad -3.69B$

and follow the same method and procedure to get

$X_5 = -104\ B$

$X_6 = -8940\ B$

Substitute equation (2-4') and equation (2-5'), we get

$-68.01A + 18.176B = 0$

$-1260.8A + 336.73B = 0$

or

$3.75A - B = 0$

$3.75A - B = 0$

The determinant is

$\Delta = 0$

And for $p^2 = 2.67\ (\text{radian/sec})^2$ is 3.0% lower than the answer found by using Rayleigh's method which is given in section B-ii-a-2.

B. Results

i. Applications

b. Beams With Constant Cross Sections

1. Hinged Beam With Variable Masses

This problem had already been solved in the paper by Bleich (7). The variable masses have been shown in the Figure 17 according to the original problem.

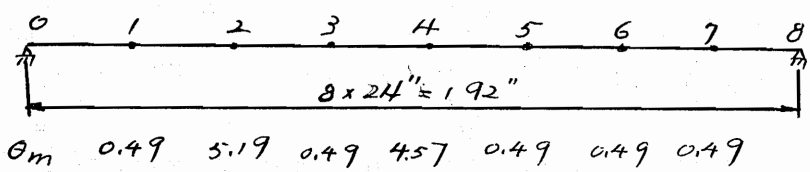

Fig. 17 - Hinged beam with variable masses.

1-1 Formulation of Equations:

For m=1, using equation (iii-13)

$$X_1\left(\frac{2\,s^3 p^2 \theta_1}{3\,E\,I_c} - 5\right) + X_2\left(\frac{s^3 p^2 \theta_2}{6\,EI_c} + 4\right) - X_3 = 0$$

then

$$X_1\,(12.30\mathrm{x}10^{-6}p^2 - 5) + X_2\,(32.58\mathrm{x}10^{-6}p^2 + 4) - X_3 = 0$$

... (1-1)

For m=2, m=3, m=4, m=5, m=6, using equation (iii-5) and note that X = 0, and $X_8 = 0$, then

m=2:

$$-X_1\left(4 + \frac{p^2 s^3}{6\,EI_c}\,\theta_1\right) + X_2\left(6 - \frac{4\,p^2 s^3}{6\,EI_c}\,\theta_2\right) - X_3\left(4 + \frac{p^2 s^3}{6\,EI_c}\,\theta_3\right) + X_4 = 0$$

and

$$-X_1\,(4+3.076\mathrm{x}10^{-6}p^2) + X_2\,(6-130.3\mathrm{x}10^{-6}p^2) \qquad \ldots (1\text{-}2)$$

$$-X_3\,(4-3.076\mathrm{x}10^{-6}p^2) + X_4 = 0$$

m=3:

$$X_1 - X_2\left(4+\frac{p^2s^3}{6\,EI_c}\theta_2\right) + X_3\left(6-\frac{4\,p^2s^3}{6\,E\,I_c}\theta_3\right) - X_4\left(4+\frac{p^2s^3}{6EI_c}\theta_4\right) + X_5 = 0$$

and

$$X_1 - X_2\,(4+32.58\text{x}10^{-6}p^2) + X_3\,(6-12.30\text{x}10^{-6}p^2)$$

$$-X_4\,(4+28.70\text{x}10^{-6}p^2) + X_5 = 0 \qquad \ldots\ (1\text{-}3)$$

m=4:

$$X_2 - X_3\left(4+\frac{p^2s^3}{6\,EI_c}\theta_3\right) + X_4\left(6-\frac{4p^2s^3}{6\,EI_c}\theta_4\right) - X_5\left(4+\frac{p^2s^3}{6\,EI_c}\theta_5\right) + X_6 = 0$$

and

$$X_2 - X_3\,(4+3.076\text{x}10^{-6}p^2) + X_4\,(6-114.8\text{x}10^{-6}p^2)$$

$$-X_5\,(4+3.076\text{x}10^{-6}p^2) + X_6 = 0 \qquad \ldots\ (1\text{-}4)$$

m=5:

$$X_3 - X_4\left(4+\frac{p^2s^3}{6\,EI_c}\theta_4\right) + X_5\left(6-\frac{4\,p^2s^3}{6\,EI_c}\theta_5\right) - X_6\left(4+\frac{p^2s^3}{6\,EI_c}\theta_6\right) + X_7 = 0$$

and

$$X_3 - X_4\,(4+28.70\text{x}10^{-6}p^2) + X_5\,(6-12.30\text{x}10^{-6}p^2)$$

$$-X_6\,(4+3.07\text{x}10^{-6}p^2) + X_7 = 0 \qquad \ldots\ (1\text{-}5)$$

m=6:

$$X_4 - X_5\left(4+\frac{p^2s^3}{6\,EI_c}\theta_5\right) + X_6\left(6-\frac{4\,p^2s^3}{6\,EI_c}\theta_6\right) - X_7\left(4+\frac{p^2s^3}{6\,EI_c}\theta_7\right) = 0$$

and

$$X_4 - X_5\,(4+3.076\text{x}10^{-6}p^2) + X\ \ (6-12.30\text{x}10^{-6}p^2)$$

$$-X_7\,(4+3.076\text{x}10^{-6}p^2) = 0 \qquad \ldots\ (1\text{-}6)$$

but for m=7, we must use equation (iii-13), then

$$-X_5 + X_6\left(\frac{p^2s^3}{6\,EI_c}\theta_6 + 4\right) + X_7\left(\frac{2\,p^2s^3}{3\,EI_c}\theta_7 - 5\right) = 0$$

and

$$-X_5 + X_6\,(3.076\text{x}10^{-6}p^2+4) + X_7\,(12.30\text{x}10^{-6}p^2-5) = 0 \qquad \ldots\ (1\text{-}7)$$

1-2 Calculation of Roots of Determinant

From the above seven equations, we can see that the p term is much smaller than the constant term if p is less than 100

For illustration, we try first

$$p^2 = 3000 \text{ (radion/sec)}^2$$

then the above seven equations yield

$$X_1 (-4.963) + X_2 (4.097) - X_3 = 0 \qquad \ldots (1\text{-}1')$$

$$-X_1 (4.01) + X_2 (5.61) - X_3 (4.01) + X_4 = 0 \qquad \ldots (1\text{-}2')$$

$$X_1 - X_2 (4.097) + X_3 (5.960) - X_4 (4.086) + X_5 = 0 \qquad \ldots (1\text{-}3')$$

$$X_2 - X_3 (4.01) + X_4 (5.66) - X_5 (4.01) + X_6 = 0 \qquad \ldots (1\text{-}4')$$

$$X_3 - X_4 (4.086) + X_5 (5.96) - X_6 (4.01) + X_7 = 0 \qquad \ldots (1\text{-}5')$$

$$X_4 - X_5 (4.01) + X_6 (5.96) - X_7 (4.01) = 0 \qquad \ldots (1\text{-}6')$$

$$X_5 + X_6 (4.01) + X_7 (-4.963) = 0 \qquad \ldots (1\text{-}7')$$

Let $X_1 = 0$, and $X_2 = A$

from equation (1-1')

$$X_3 = 4.097\ A$$

from equation (1-2')

$$X_4 = 10.82\ A$$

from equation (1-3')

$$X_5 = 23.9\ A$$

from equation (1-4')

$$X_6 = 50\ A$$

from equation (1-5')

$$X_7 = 98.17\ A$$

Similarly let $X_2 = 0$ and $X_1 = B$

from equation (1-1')

$X_3 = -4.963\ B$

from equation (1-2')

$X_4 = -15.91\ B$

from equation (1-3')

$X_5 = -36.42\ B$

from equation (1-4')

$X_6 = -75.9\ B$

from equation (1-5')

$X_7 = -148.34\ B$

Substituting the values of X in terms of A and B into equation (1-6')

$-180.48A + 272.49B = 0 \quad$ or $\quad A - 1.5098B = 0$

and into equation (1-7')

$-310.6A + 468.26B \quad$ or $\quad A - 1.5075B = 0$

The determinant of the above two equations is

$$\Delta = \begin{vmatrix} 1 & -1.5098 \\ 1 & -1.5075 \end{vmatrix} = +\ 0.0023$$

Second try for $p^2 = 2900\ (\text{rad/sec})^2$ and follow the same procedure, we get seven equations are

$X_1\ (-4.964) + X_2\ (4.095) - X_3 = 0$

$-X_1\ (4.009) + X_2\ (5.63) - X_3\ (4.009) + X_4 = 0$

$X_1 - X_2\ (4.094) + X_3\ (5.97) - X_4\ (4.083) + X_5 = 0$

$X_2 - X_3\ (4.009) + X_4\ (5.67) - X_5(4.009) + X_6 = 0$

$$X_3 - X_4\,(4.083) + X_5\,(5.97) - X_6\,(4.009) - X_7 = 0$$

$$X_4 - X_5\,(4.009) + X_6\,(5.97) - X_7\,(4.009) = 0$$

$$-X_5 - X_6\,(4.009) + X_7\,(-4.97) = 0$$

and for $X_1 = 0$ and $X_2 = A$

$X_3 = 4.095$

$X_4 = 10.78\ A$

$X_5 = 23.65\ A$

$X_6 = 49.10\ A$

$X_7 = 95.6\ A$

For $X_2 = 0$ and $X_1 = B$, we have

$X_3 = -4.964\ B$

$X_4 = -15.00\ B$

$X_5 = -32.61\ B$

$X_6 = -64.68\ B$

$X_7 = -120.9$

Substituting into the last two equations and we find the determinants

$$\Delta = \begin{vmatrix} +2.038 & -1 \\ +1.232 & -1 \end{vmatrix} = +\,0.796$$

By these two values, we can plot the curve of Δ vs p^2.

From the intercept at the p^2 axis, the root of p^2 will be

$p^2 = 3001$ (rad./sec)^2.

It is 0.3% higher from the results found by Stodola's method in

Bleich's paper.

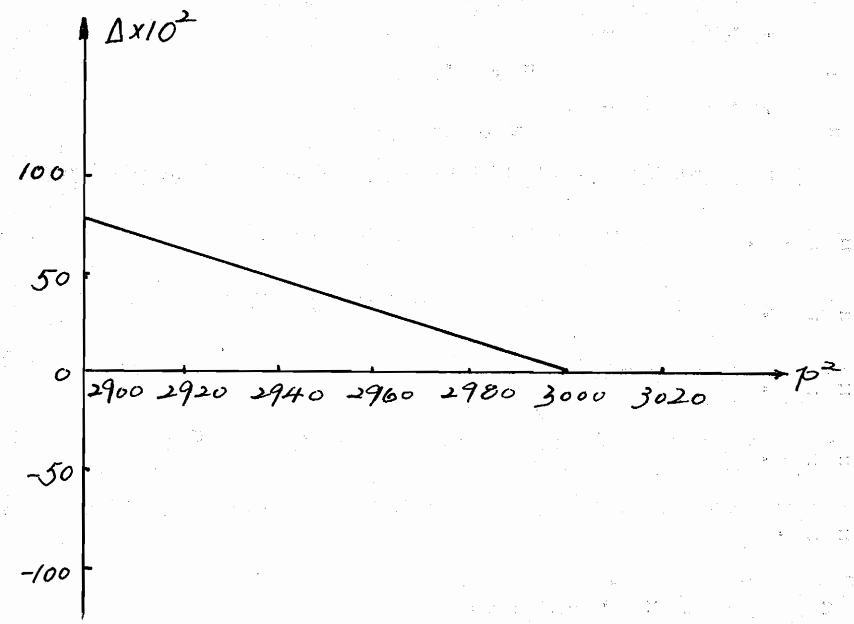

Fig. 18 - $\Delta\times10^2$ - p^2 curve.

B. Results

i. Applications

b. Beams With Constant Cross Sections

2. With Constant Masses

Following same procedure as for the previous example, we have the seven equations

$m=1$

$$X_1\ (4\phi-5) + X_2\ (\phi-4) - X_3 = 0$$

$m=2$

$$-X_1(4-\phi) + X_2\ (6-4\phi) - X_3\ (4+\phi) + X_4 = 0$$

$m=3$

$$X_1 - X_2\ (4-\phi) + X_3\ (6-4\phi) - X_4\ (4-\phi) + X_5 = 0$$

$m=4$

$$X_2 - X_3\ (4+\phi) + X_4\ (6-4\phi) - X_5\ (4+\phi) + X_6 = 0$$

$m=5$

$$X_3 - X_4\ (4+\phi) + X_5\ (6-4\phi) - X_6\ (4-\phi) + X_7 = 0$$

$m=6$

$$X_4 - X_5\ (4+\phi) + X_6\ (6-4\phi) - X_7\ (4+\phi) = 0$$

$m=7$

$$-X_5 + X_6\ (4+\phi) + X_7\ (4\phi-5) = 0$$

First try for $p^2 = 400$ rad/sec , the seven equations yield

$$X_1 (-4.9882) + X_2 (4.0029) - X_3 = 0$$

$$-X_1 (4.0029) + X_2 (5.9882) - X_3 (4.0029) + X_4 = 0$$

$$X_1 - X_2 (4.0029) + X_3 (5.9882) - X_4 (4.0029) + X_5 = 0$$

$$X_2 - X_3 (4.0029) + X_4 (5.9882) - X_5 (4.0029) + X_6 = 0$$

$$X_3 - X_4 (4.0029) + X_5 (5.9882) - X_6 (4.0029) + X_7 = 0$$

$$X_4 - X_5 (4.0029) + X_6 (5.9882) - X_7 (4.0029) = 0$$

$$-X_5 + X_6 (4.0029) + X_7 (-4.9882) = 0$$

and let $X_1 = 0$ and $X_2 = A$, then let $X_2 = 0$ and $X_1 = B$, we find the determinant, using the last two equations, is

$$\Delta = \begin{vmatrix} -1 & 1.85992 \\ -1 & 1.85608 \end{vmatrix} = 0.00384$$

Second try set $p^2 = 600$ rad^2/sec^2, after the same routine work, we get the determinant

$$\Delta = \begin{vmatrix} -1 & 1.8412 \\ -1 & 1.8418 \end{vmatrix} = -0.0006$$

and plot the curve of the determinant vs p^2. We find the $p^2 = 575$ for $\Delta = 0$. This frequency is 3.4% higher than the frequency found by the Classic Method.

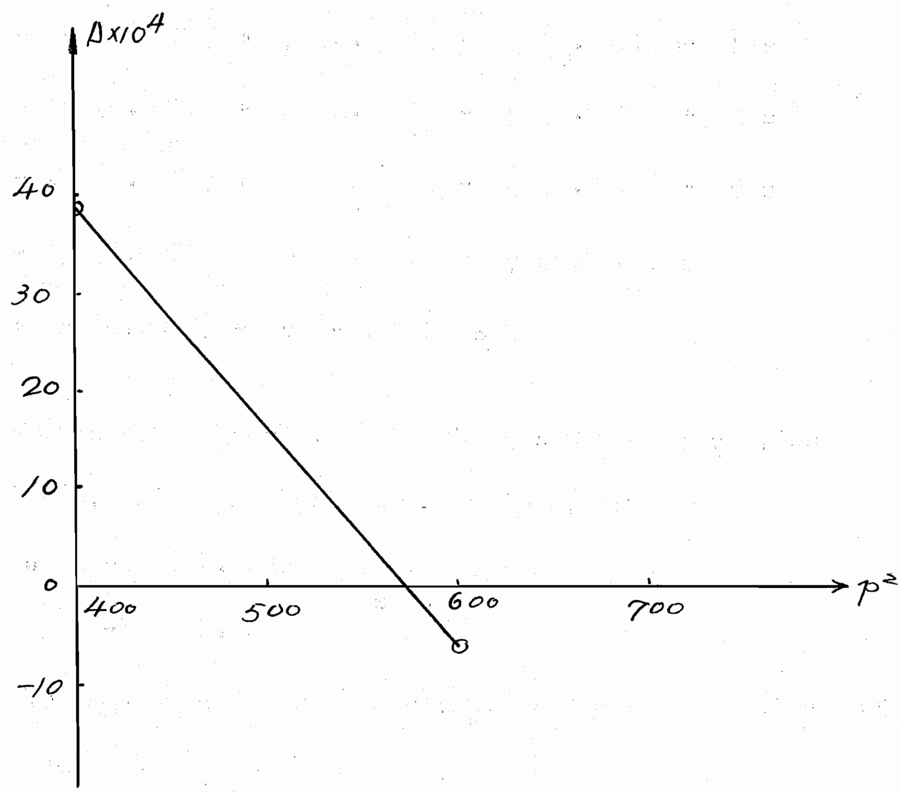

Fig. 19 - $\Delta \times 10^4$ - p^2 curve.

B. Results

ii. Applications By Other Methods

a. Beams With Variable Cross Sections

1. Cantilever Beam ... by Stodola Method

This example had already been solved in the book on "Aircraft Vibration and Flutter" by Scalan and Rosenbaum. (6) The cantilever beam is divided into 10 segments shown in Fig. 20.

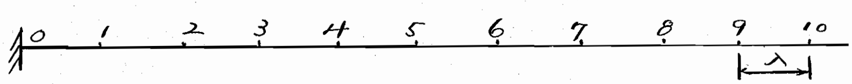

Fig. 20 - Cantilever beam.

All the calculations are in the Table 4. And from Table 4, finally

$$\frac{y_2}{y_3} = \frac{E}{(24)^4} \times 113.70 = \frac{10.3 \times 10^6 \times 113.70}{331776}$$

$$p^2 = \frac{y_2}{y_3} = \frac{1171.11}{331776} \times 10^6 = 3530$$

p = 59.41 radions/second.

= 567 cycles per minutes.

B. Results

ii. Applications By Other Methods

a. Beams With Variable Cross Sections

2. Hinged End Beam By Rayleigh Method

From equation (i-8), we have

$$\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 W}{\partial x^2} \right) = - \frac{\gamma A}{g} \frac{\partial^2 W}{\partial t^2}$$

Taking deflections of the beam, while vibrating, in the form

$$W = X \cos pt$$

then

$$V = \frac{1}{2}\int_0^l EI \left(\frac{d^2 X}{d x^2} \right)^2 dx$$

where V is the maximum potential energy

and

$$T = \frac{p^2}{2g}\int_0^l A \gamma X^2 \, dx$$

where T is the maximum kinetic energy.

Using Rayleigh's theory, then

$$V = T$$

therefore

$$p^2 = \frac{Eg}{\gamma} \frac{\int_0^l I \left(\frac{d X}{dx} \right) dx}{\int_0^l A X^2 \, dx}$$

For our problem, Fig. 13.

$$A = \frac{b}{l} x$$

$$I = \frac{1}{12}\left(\frac{b}{l} x \right)^2$$

End conditions

(1) $(EIX)_{x=0} = 0$

(2) $\left(EI \frac{d^2 X}{dx^2} \right)_{x=0} = 0$

(3) $(X)_{x=l} = 0$

(4) $\left(\frac{d^2 X}{dx^2} \right)_{x=l} = 0$

The first two end conditions will be always satisfied because $I = 0$ at $X = 0$.

In order to satisfy the condition at $X = 1$, take

$$X = a \sin \frac{x}{l}\pi$$

then

$$\frac{dX}{dx} = \frac{a\pi}{l} \cos \frac{x}{l}\pi$$

and

$$\frac{d^2X}{dx^2} = -\frac{a\pi^2}{l} \sin \frac{x}{l}\pi$$

Since $p^2 = \dfrac{Eg}{\gamma} \dfrac{\int_0^l I \left(\frac{d^2X}{dx^2}\right)^2 dx}{AX^2 dx}$

then

$$\int_0^l I \left(\frac{d^2X}{dx^2}\right)^2 dx = \int_0^l \frac{1}{12} \frac{b^3\pi^4}{l^4} a^2 \frac{x^3}{l^3} \sin^2 \frac{x}{l}\pi \, dx$$

$$= \frac{1}{12} \frac{b^3\pi^4 a^2}{l^7} \left(\frac{7}{80}\right) l^4$$

and

$$\int_0^l A X^2 dx = \frac{b}{l} a^2 \int_0^l x \sin^2 \frac{x}{l}\pi \, dx$$

$$= \frac{a^2 b l}{4}$$

$$p^2 = \frac{Eg}{\gamma} \frac{\frac{1}{12}\frac{b^3\pi^4}{l^7}a^2\frac{7}{80}l^4}{\frac{a^2bl}{4}}$$

$$= 2.87 \frac{E b^2 g}{\gamma l^4}$$

b. Beams With Constant Cross Sections

1. Hinged Beam With Variable Masses (7) ... By Stodola Method

This beam is divided into 8 segments shown in Fig. 21.

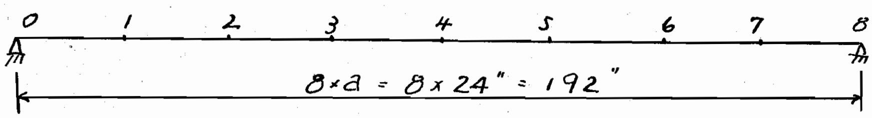

Fig. 21 - Hinged beam with variable masses.

All the calculations are in the Table 5. From the equation in Bleich's paper

$$p^4 = \frac{\sum M_i [y^{n-1}]^2}{\sum M_i [y^n]^2} = \frac{8.46 \times E^2 I^2}{66493 \times a^6}$$

$$p^2 = \sqrt{\frac{8.46}{66493}} \times \frac{EI}{a^3} = 0.01126 \cdot \frac{30 \times 10^6 \times 122.3}{(24)^3} = 2990$$

$$p = 54.7 \text{ radians/sec.}$$

2. Hinged Ends Beam With Constant Masses ... by Classic Method

From equation (i-8), for constant cross section and free vibration.

$$EI \frac{\partial^4 W}{\partial x^4} = -\frac{\gamma A}{g} \frac{\partial^2 W}{\partial t^2}$$

Let $W = X\,(A \cos pt + B \sin pt)$

$$\frac{\partial^4 X}{\partial x^4} = \frac{p^2}{a^2} X \quad , \quad \text{where } a^2 = \frac{EIg}{A\gamma} \text{ and let } k^4 = \frac{p^2}{a^2}$$

$$X = C_1 \sin kx + C_2 \cos kx + C_3 \sinh kx + C_4 \cosh kx$$

In our case, the ends are hinged, their boundary conditions are:

(1) $(X)_{x=0} = 0$ (2) $\left(\frac{d^2X}{dx^2}\right)_{x=0} = 0$

(3) $(X)_{x=l} = 0$ (4) $\left(\frac{d^2X}{dx^2}\right)_{x=l} = 0$

For convenience, we can put

$$X = C_1 (\cos kx + \cosh kx) + C_2 (\cos kx - \cosh kx)$$
$$+ C_3 (\sin kx + \sinh kx) + C_3 (\sin kx - \sinh kx)$$

From condition (3) and (4),

$C_3 = C_4$

and

$\sin kl = 0$

$kl = \pi,\ 2\pi,\ \ldots$

$$p = a\,k_1 = \frac{a\pi^2}{l^2}$$

where $a = \frac{EI\,g}{A}$

In our case, $I = 122.3$ in 4

$$E = 30x10^{-6} psi$$

$$\theta = 0.49 \ \frac{lbs.-sec.}{in.}$$

$$p_1^2 = \frac{a^2\pi^4}{l^4} = \frac{\pi^4 x\, 30x10^6 x\ 122.3}{0.49\ x\ (192)}$$

$$p_1 = 23.2\ \frac{radian}{sec.}$$

V. Discussion of Results and Conclusions

From the equations and the examples illustrated above, we find that the Marcus method can be applied to beams of both variable cross section and constant cross section. It can be applied for different boundary conditions: free supported, hinged supported, rigid supported, and intermediately supported. The beam may be of uniform section with different masses acting on it. The advantages of this method is its simplicity; and that the principle of virtual displacement is known to every student in engineering. All the equations are easy to formulate and the method of determining the roots can be very easily applied. For the natural frequency, three trials are enough to give you a good answer. The shape functions can be obtained immediately after you get the corresponding frequency. The more accurate the results you want, the more segments you should take.

In the case of forced vibrations, this method can also be applied. The only difference from the free vibration is that we add the effect of the external transient loads. We let

$$F_m' = T_m \operatorname{Sin} qt$$

and

$$W_m = X'_m \operatorname{Sin} qt$$

then the total force acting at the point m is

$$F_m = X'_m (q^2 \Theta_m + T_m)$$

Then we can derive all the equations for the forced vibration in the similar way as done for free vibration. The effect of damping will not be discussed here.

There is one disadvantage of this method in that it is very delicate in the convergence to the fundamental frequency. But this disadvantage disappears in finding the second mode frequency and the other higher frequencies although the accuracy will be reduced. It must be noted here that this is not a good method in solving problems with constant cross section.

In brief, being easy to follow and to understand makes it especially attractive to civil engineers. Further studies of ways of improving the rate of convergence of the results are recommended.

V. SUMMARY

This thesis presents an evaluation of a new method first suggested by Dr. H. Marcus for determining the natural frequencies of lateral vibrations for elastic beams of variable cross sections. The basic theory is the principle of virtual displacements. Using this theory and the relations between inertia force and displacements, one can derive all the equations necessary to solve problems with various end conditions.

VI. ACKNOWLEDGEMENT

The author wishes to express his appreciation to Dr. G. L. Rogers of the Civil Engineering Department for suggesting the Marcus method of solution and for his valuable advice during the preparation of this thesis.

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VIII. VITA

The writer was born on October 9, 1931, at Shanghai, China. He was awarded the B.Sc. degree in civil engineering from the National Taiwan University on July, 1954, after one year of service in the army. He matriculated at the Virginia Polytechnic Institute on September, 1954, as a graduate student in Civil Engineering Department. He is a member of Chi Epsilon Fraternity.

Fang An Lee

XI. APPENDIX

Calculation of the Roots of the Determinant

The computation of the determinant and the determination of the roots are a laborious operation, which can be considerably shortened by the following procedure.

Suppose the scheme for the group of equation for a beam with 8 segments is as follows:

$$
\text{(a)}\quad \left.\begin{array}{ccccccccc}
X_0 & X_1 & X_2 & & & & & & \\
X_0 & X_1 & X_2 & X_3 & & & & & \\
X_0 & X_1 & X_2 & X_3 & X_4 & & & & \\
 & X_1 & X_2 & X_3 & X_4 & X_5 & & & \\
 & & X_2 & X_3 & X_4 & X_5 & X_6 & & \\
 & & & X_3 & X_4 & X_5 & X_6 & X_7 & \\
 & & & & X_4 & X_5 & X_6 & X_7 & X_8 \\
 & & & & & X_5 & X_6 & X_7 & X_8
\end{array}\right\} \quad \text{...(A)}
$$

Let us first assume that $X_o = A$ and $X_1 = 0$ Solving equations we get a set of values

$$X_2 = X'_2\,A;\ X_3 = X'_3\,A;\ \ldots\ldots\ldots\ X_7 = X'_7\,A$$

A similar operation gives for $X_o = 0$ and $X_1 = B$ the values

$$X_2 = X''_2\,B;\ X_3 - X''_3\,B;\ \ldots\ldots\ldots\ X_7 = X''_7\,B$$

thus the total values are

$$X_o = A,\ X_1 = B,\ X_2 = X'_2\,A + X''_2\,B\ \ldots\ldots\ldots\ldots\ X_7 = X'_7\,A + X''_7 B$$

Substituting these values in the two last equations of (A) and taking account that $X_8 = 0$, we obtain two equations having the form

$$Aa_1 + Bb_1 = 0$$

$$Aa_2 + Bb_2 = 0$$

which are compatible one with the other only if

$$\Delta = a_1 b_2 - a_2 b_1 = 0$$

We start with an arbitrary values $\phi = \frac{p^2 S^3}{6EI_c}$, and determine the corresponding values X'_1, X''_1, a_1, b_1, a_2, b_2. If $\Delta \gtrless 0$, we repeat the same operation with a second, a third, a fourth ... value of ϕ and plot for each one of these values the corresponding values of Δ as shown below

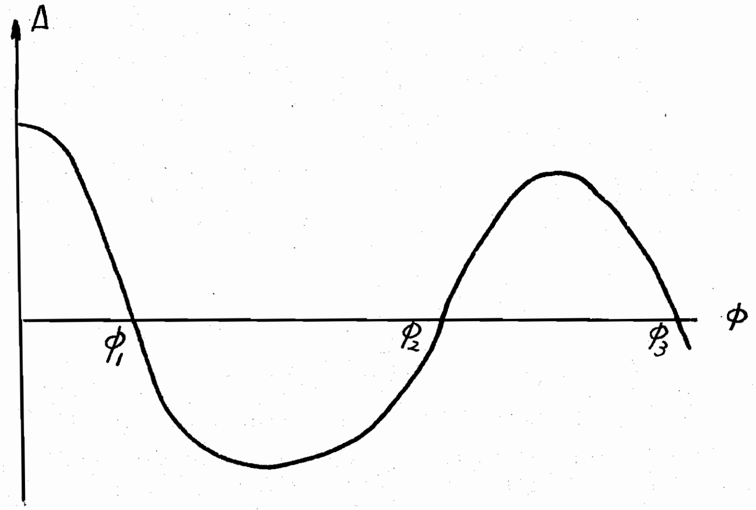

Fig. 22 Δ-ϕ curve

The successive intersection of the Δ-ϕ curve with the ϕ - axis determine

$P_1 = \sqrt{\frac{6EI_c\phi_1}{S^3}}$ $P_2 = \sqrt{\frac{6EI_c\phi_2}{S^3}}$ $P_3 = \sqrt{\frac{6EI_c\phi_3}{S^3}}$ ...

corresponding respectively to the frequencies of the first, second, third ... mode.

Table 1 Calculation of Coefficients for example 1

	1	2	3	4	5	6	7	8	9	10	11	12	13	
Station	I_m	$i_m = \frac{I_c}{I_m}$	$-(i_m - i_{m-1})$	$i_m + 2i_{m-1}$	$i_m + 2i_{m+1}$	$4\times(4)_m\times(5)_m$	$(3)_m \times (3)_{m+1}$	$(7)_m + (6)_m$	$a'_m = -\frac{(3)_m}{(8)_m}$	$a''_m = -\frac{(3)_{m+1}}{(8)_m}$	$b'_m = \frac{2(4)_m}{(8)_m}$	$b''_m = \frac{2(5)_m}{(8)_m}$	θ_m	Station
0	0												0.0156	0
1	48	20.90			4.17								0.0625	1
2	96	10.40	10.50	52.20	24.30	5050	34.500	5084.50	-0.00210	-0.00068	0.0206	0.0096	0.0975	2
3	144	6.95	3.45	27.35	17.35	1900	6.040	1906.04	-0.00180	-0.00092	0.0287	0.0182	0.1150	3
4	192	5.20	1.75	19.10	13.54	1030	1.800	1031.80	-0.00170	-0.00100	0.0368	0.0263	0.1440	4
5	246	4.17	1.03	14.57	11.13	650	0.720	650.72	0.00158	-0.00107	0.0447	0.0340	0.2110	5
6	288	3.47	0.70	11.81	9.41	447	0.350	447.35	0.00156	-0.00112	0.0530	0.0420	0.4540	6
7	336	2.97	0.50	9.91	8.15	323	0.185	323.19	0.00154	-0.00115	0.615	0.0505	0.7290	7
8	384	2.66	0.37	8.54	7.31	250	0.107	250.11	0.00148	-0.00116	0.0680	0.0581	0.8620	8
9	432	2.31	0.29	7.51	6.49	194	0.064	194.06	0.00150	-0.00114	0.0770	0.0667	0.9200	9
10	480	2.09	0.22	6.70	2.09	56							0.4680	10

Note - $I_c = 1000$; $c'_9 = 0.074$ $c''_9 = 0.00389$

Table 2 Calculations of coefficients for example 2 with four segments

Station	Depth	I	I_c	i_m	D_m	a'_m	a''_m	b''_m	b'_m	e_m	ϕ_m
Factor	b	$\frac{1}{12} \times \frac{1}{64} b^3$	$\frac{1}{12} \times \frac{1}{64} b^3$	$\frac{I}{I_c}$						$\frac{b \gamma l}{g}$	
1											
2	$\frac{1}{4}$	1	1	1	$\frac{1}{75}$	$\frac{1}{75}$	$\frac{7}{75}$	$\frac{34}{75}$	$\frac{2}{75}$	$\frac{1}{16}$	$\frac{1}{34}$
3	$\frac{1}{2}$	8	1	8	$\frac{1}{2613}$	$\frac{7}{2613}$	$\frac{19}{2613}$	$\frac{124}{2613}$	$\frac{20}{2613}$	$\frac{1}{8}$	
4	$\frac{3}{4}$	27	1	27	$\frac{1}{27263}$	$\frac{19}{27263}$	$\frac{37}{27263}$	$\frac{310}{27263}$	$\frac{86}{37263}$	$\frac{3}{16}$	$\frac{1}{309}$
5	1	64	1	64							

Table 3 Calculations of coefficients for example 2 with six segments

Station	Depth	I	I_c	i	D_m	a'_m	a''_m	b''_m	b'_m	θ_m	ϕ_m
Factor	b	$\frac{1}{12}\times\frac{1}{216}b^3$	$\frac{1}{12}\times\frac{1}{216}b^3$							$\frac{b\gamma l}{g}$	
1	0	0	1	0							
2	$\frac{1}{6}$	1	1	1	$\frac{1}{75}$	$\frac{1}{75}$	$\frac{7}{75}$	$\frac{34}{75}$	$\frac{2}{75}$	$\frac{1}{36}$	$\frac{1}{34}$
3	$\frac{1}{3}$	8	1	8	$\frac{1}{2613}$	$\frac{7}{2613}$	$\frac{19}{2613}$	$\frac{124}{2613}$	$\frac{20}{2613}$	$\frac{1}{18}$	
4	$\frac{1}{2}$	27	1	27	$\frac{1}{27263}$	$\frac{19}{27263}$	$\frac{37}{27263}$	$\frac{310}{27263}$	$\frac{86}{27263}$	$\frac{1}{12}$	
5	$\frac{2}{3}$	64	1	64	$\frac{1}{116553}$	$\frac{37}{116553}$	$\frac{61}{116553}$	$\frac{628}{116553}$	$\frac{236}{116553}$	$\frac{2}{18}$	
6	$\frac{5}{6}$	125	1	125	$\frac{6}{426643}$	$\frac{61}{426643}$	$\frac{91}{426643}$	$\frac{1114}{426643}$	$\frac{506}{426643}$	$\frac{5}{36}$	$\frac{1}{835}$
7	1	216	1	216							

Table 4. Calculation of coefficients for example ii-a-1

1	2	3	4	5	6	7	8	9
Station	I	m	y_1	$m y_1$	S	M	M/I	Slope
Factor →					Δx	$(\Delta x)^2$	$(\Delta x)^2$	$\frac{(\Delta x)^3}{E}$
10	0	0.0013	1.0	0.0013		0	0	
					0.0008			0.001272
9	48	0.0026	0.8	0.0021		0.0008	0.000017	
					0.0029			0.001255
8	96	0.0041	0.6	0.0025		0.0037	0.000039	
					0.0054			0.001216
7	144	0.0048	0.5	0.0024		0.0091	0.000063	
					0.0078			0.001153
6	192	0.0060	0.4	0.0024		0.0169	0.000088	
					0.0102			0.001065
5	240	0.0088	0.3	0.0050		0.0271	0.000113	
					0.0152			0.000752
4	288	0.0189	0.2	0.0038		0.0423	0.000147	
					0.0190			0.000805
3	336	0.0304	0.15	0.0046		0.0613	0.000182	
					0.0236			0.000623
2	384	0.0360	0.1	0.0036		0.0849	0.000221	
					0.0272			0.000402
1	432	0.0384	0.05	0.0019		0.1121	0.000259	
					0.0291			0.010143
0	480	0.0390	0.00	0		0.1421	0.000294	

	10	11	12	13	14	15	16	17	18
Sta.	y_2	y_2 (norm.)	$m y_2$	S	M	M/I	Slope	y_3	y_2/y_3
Factor →	$\frac{\Delta x^4}{E}$								
10	0.008886	1.000	0.0013		0	0		0.008954	111.68
				0.0008			0.001246		
9	0.007614	0.857	0.0022		0.0008	0.000017		0.007658	111.91
				0.0030			0.001279		
8	0.006359	0.716	0.0029		0.0038	0.000040		0.006379	112.29
				0.0059			0.001239		
7	0.005143	0.579	0.0028		0.0097	0.000067		0.005140	112.65
				0.0087			0.001172		
6	0.003990	0.449	0.0027		0.0184	0.000096		0.003968	113.16
				0.0114			0.001076		
5	0.002925	0.329	0.0029		0.0298	0.000124		0.002892	113.76
				0.0143			0.000952		
4	0.001973	0.222	0.0042		0.0441	0.000153		0.001940	114.43
				0.0185			0.000799		
3	0.001168	0.131	0.0040		0.0626	0.000186		0.001141	114.81
				0.0225			0.000613		
2	0.000545	0.061	0.0022		0.0851	0.000222		0.000528	115.58
				0.0247			0.000391		
1	0.000143	0.016	0.0006		0.1098	0.000254		0.000137	116.79
				0.0253			0.000151		
0	0	0	0		0.1351	0.000281		0	

Ave. 113.%

Table 5. Calculation of coefficients for example b-1

Pt.	$y^{(1)}$	Mass M	$M \times [y^{(1)}]^2$	$M y^{(1)}$	Inertia Loading: Shear V	Inertia Loading: Moment $\frac{\bar{M}}{d}$	Elastic Weight: $\frac{EI}{d^2} p$	Elastic Weight: Shear $\frac{EI}{d^2} V$	Elastic Weight: Moment $\frac{EI}{d^3} y^{(2)}$	$\frac{M(EI)^2}{d^6} \times [y^{(2)}]^2$	$p^2 y^{(2)}$	Second Step	Third Step
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)		
0	0	0.25	0	0		0	1.00		0	0	0	0	0
					5.998			35.412					
1	0.45	0.49	0.10	0.22		5.998	5.96		35.412	615	0.40	0.396	0.396
					5.778			29.452					
2	0.80	5.19	3.32	4.15		11.770	11.08		64.864	23012	0.732	0.727	0.727
					1.628			18.372					
3	1.00	0.49	0.49	0.49		13.404	13.32		83.236	3396	0.93	0.934	0.934
					1.138			5.052					
4	0.95	4.57	4.12	4.34		14.542	13.82		88.288	34145	0.994	0.993	0.993
					-3.202			8.768					
5	0.75	0.49	0.28	0.37		11.340	11.28		79.520	3100	0.89	0.896	0.896
					-3.572			-20.048					
6	0.50	0.49	0.12	0.25		7.768	7.73		59.472	1733	0.67	0.672	0.672
					-3.822			-27.778					
7	0.25	0.49	0.03	0.12		3.946	3.92		31.694	492	0.36	0.358	0.358
					-3.942			-31.698					
8	0	0.25	0	0		0.004	0.66		-0.006	0	0	0	0
9			8.46	9.94						66493			
											54.7	54.6	54.6