MATHEMATICAL MODELING OF ADHESIVE LAYER CRACKS UTILIZING INTEGRAL EQUATIONS

by

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(ABSTRACT)

Within recent years, crack analysis in adhesive layers has become a topic of interest for many researchers. A common model which is used incorporates a 3-region elasticity problem consisting of only 2 materials, the adhesive layer bounded by 2 layers of a stiffer elastic substrate. Cracks have been experimentally observed to propagate in straight paths as well as wavy paths within the adhesive layer and even at its boundaries.

A theoretical model based on work done by Fleck, Hutchinson, and Suo (1991) is used to study crack path selection. Complex stress potential functions are employed to develop a symbolic derivation. The method of distributed dislocations is utilized to represent the crack. A series of Chebyshev polynomials is used to approximate the unknown dislocations. The resulting integral equations are solved through the collocation method and the series coefficients are recovered. Several numerical packages, Mathcad 5.0+ and Mathematica 2.2.1, were used to study the computational aspects of the problem. The focus of the research was to develop efficient modular software packages to be run on a standard PC system. Several numerical techniques were utilized to reduce computational time and control the numerical accuracy of the problem. Some of these techniques included a "numerical freeze" algorithm, Fast Fourier Transform techniques, Gaussian inversion,
Gaussian quadrature and Romberg quadrature. The numerically sensitive regions were identified. Finally, recommendations for future work and possible solutions to handle the numerically sensitive regions were presented.
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ANALYSIS OF ADHESIVE JOINTS

1.0 Introduction

Fracture analysis of adhesively bonded joints has attracted considerable attention in recent years. These advanced materials are used in high performance composites, connections between dissimilar materials and bonded joints which help produce light, stiff and economic structures. There are many advantages to adhesive bonding over conventional assembly methods such as reduced cost, lightweight designs, and reduced residual stresses which are implicit in mechanical fasteners and riveted joints.

Crack initiation and propagation have been held responsible for the failure of many adhesive joints. Various crack propagation mechanisms have been tentatively identified. Cracks in adhesive layers have been experimentally observed to propagate both interfacially as well as cohesively in straight or wavy paths. These problems have many practical applications. If a crack were to grow perpendicular to a material interface, it could either penetrate or deflect and grow parallel into the interface. For instance, within a fiber-reinforced composite a crack propagating through a matrix which impinges upon a fiber can either penetrate the interface and break the fiber or it could deflect into the interface causing fiber slippage and debonding (Popejoy et al., 1992). Another example as suggested by Ho et al. (1993) involves cracks in multilayer capacitors. A crack within the capacitor would lead to conducting paths for electrical leaks.

Many of these problems have been addressed in the literature, but different approaches and models were used with various underlying assumptions to solve these problems. The
approaches range from simplistic models, where the adhesive layer is assumed to be small and negligible, to complex finite-element models. Several of the techniques utilized will be discussed in this chapter.

1.1 Various Methods

A classical model used assumes a homogeneous specimen which in essence ignores the adhesive layer and employs the far-field loading conditions to analyze the larger assembly with a more complex method. This approach might be used on a larger structural assembly like a car frame or an airplane fuselage to get rough measurements of strength.

In an approach to analyze an interfacial crack within a bimaterial system introduced by Suo and Hutchinson (1989), they developed a universal relation between the global loading conditions (far-field stress intensity factors) and the values of the local stress intensity factors in the near-field. In particular, they suggest that the stress intensity factors for a homogeneous body can approximate the actual interface stress intensity factors.

Other studies which analyze cracks impinging on a bimaterial interface were done by Popejoy et al. (1992). In their approach, they utilized a Consistent Shear-Lag (COSL) model to help answer the question as to whether the crack will penetrate the interface or be deflected along it. Rather than restricting the problem to bimaterials, this model allows the study of more complex material systems, such as fiber-reinforced composites.

Through the use of elegant integral equations, Erdogan and Gupta (1971) developed a general method to solve for the local stress intensity factors in multi-layer composites.
Cotterell and Rice (1980) examined the nonsingular $T$-stress in Westergard’s asymptotic expansion of stresses. Theoretically, they determined the effect of this constant stress on crack path stability.

1.2 Crack Propagation Methods

In analyzing crack propagation there are several techniques which are numerically suited to handle the localized regions of high stress and high strain field gradients: Finite Element Method, Boundary Element Method and Displacement Discontinuity Method. Each of these techniques offer advantages and disadvantages.

With the finite element approach, special singular elements can be used to handle the crack tip stress singularities (Wang et al., 1978). However, a major limitation of the method revolves around the mesh formulation. In order to model an advancing crack, the mesh must be redefined for each step which is computationally inefficient and thus very expensive.

In a multizone boundary element method, the domain is split into smaller regions which surround the crack. This results in a larger system of equations. The boundary element method also suffers from the same shortcomings as the finite element method with the continuous refinement of the boundaries as the crack advances (Ameen and Raghuprasad, 1994).

Finally, the displacement discontinuity method (Crouch, 1976) may be a better approach to use on a crack. Since the discontinuity elements have strong stress singularities at the ends, they are ideal to model cracks with relative displacements across their surfaces. On the other hand, these strong stress singularities make it difficult to model boundaries with finite and smoothly distributed applied loads.
TWO-DIMENSIONAL MODELING OF ADHESIVE JOINTS

2.0 Introduction

The effective toughness of a joint is determined by the nature of the resulting crack path. To help predict the nature of a path in a brittle adhesive layer, a mathematical model developed by Fleck, Hutchinson, and Suo (1991) based on the asymptotic elasticity problem has been reconstructed and programmed onto a PC platform. The numerical aspects of this method has been studied in detail to help optimize the method and identify its numerical sensitive weaknesses.

2.1 The Asymptotic Model

The asymptotic stress fields of a stationary crack in the immediate vicinity of the crack-tip are given conveniently by the Williams asymptotic expansion of stresses

\[
\begin{bmatrix}
    \sigma_x(r, \theta) \\
    \sigma_y(r, \theta) \\
    \sigma_{xy}(r, \theta)
\end{bmatrix} = \frac{K_I}{\sqrt{2\pi}} \frac{1}{\sqrt{r}} \cos\left(\frac{\theta}{2}\right) \begin{bmatrix}
    1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta_0}{2}\right) \\
    \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta_0}{2}\right) \\
    1 + \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta_0}{2}\right)
\end{bmatrix} + \frac{K_{II}}{\sqrt{2\pi}} \frac{1}{\sqrt{r}} \begin{bmatrix}
    -\sin\left(\frac{\theta}{2}\right) \left[ 2 + \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta_0}{2}\right) \right] \\
    \cos\left(\frac{\theta}{2}\right) \left[ 1 - \sin\left(\frac{\theta}{2}\right) \sin\left(\frac{\theta_0}{2}\right) \right] \\
    \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta_0}{2}\right)
\end{bmatrix} \begin{bmatrix}
    T \\
    0 \\
    0
\end{bmatrix} + \mathcal{O}(\sqrt{r})
\]

(2.1)

where \(K_I\) and \(K_{II}\) are the scalar amplitudes of the asymptotic stress intensity factors in mode I and mode II, respectively, \(T\) is a nonsingular stress acting parallel to the crack plane, and \(\mathcal{O}(\sqrt{r})\) is a nonsingular stress term which vanishes at the crack-tip. The
coordinate axes, geometrical parameters \((r, \theta)\) and stress components for an element located in the vicinity of the crack-tip are shown in Figure 2.1.

![Coordinate Axes and Stress Components](image)

**Figure 2.1-** The coordinate axes, geometrical parameters and stress components for an element located in the vicinity of the crack-tip.

In the asymptotic problem, an adhesive layer is bound by two semi-infinite substrates. The materials are assumed to be linear elastic, isotropic and homogeneous but with an elastic mismatch between the materials. Because of the relatively small plastic zone size permitted, the adhesive layer is assumed to be brittle. In the far-field, the joint is loaded remotely by the three applied loads \(K_{r}^\infty\), \(K_{\theta}^\infty\) and \(T^\infty\), which are determined for a purely homogeneous specimen which neglects the brittle adhesive layer (Figure 2.2).
Figure 2.2- Far-field stress intensity factors and $T$-stress for two adhesively bonded adherends (adhesive layer is neglected)

Figure 2.3- Geometry and loading of an adhesive layer crack between two dissimilar adherends.

In order to determine the local $K_I$, $K_{II}$, and $T$-stress, the following elasticity problem presented in Figure 2.3 is used.
In the near field, the adhesive layer is bounded by two semi-infinite half planes of the same material. The adhesive layer shear modulus and Poisson ratio are $\mu_2$ and $\nu_2$, respectively, while the substrate shear modulus and Poisson ratio are $\mu_1$ and $\nu_1$, respectively. A crack within the adhesive layer lies parallel to the interfaces, and its location is identified by the geometric ratio $c/H$, where $c$ is the distance the crack is located above the lower interface and $H$ is the thickness of the adhesive layer. If the crack is located on the centerline ($c/H = 0.5$), symmetry arguments dictate that a purely mode I loading condition will result in pure mode I stresses. If however a crack is located off the centerline ($c/H \neq 0.5$), a pure mode I loading condition would result in mixed mode near-field stresses.

2.2 The Elastic Mismatch Parameters

The other nondimensional parameters which describe this system are the elastic mismatch parameters of Dundurs (1969). For plane strain problems Dundurs’ parameters are expressed as:

$$\alpha = \frac{(1-\nu_2)/\mu_2 - (1-\nu_1)/\mu_1}{(1-\nu_2)/\mu_2 + (1-\nu_1)/\mu_1}$$

$$\beta = \frac{i}{2} \frac{(1-2\nu_2)/\mu_2 - (1-2\nu_1)/\mu_1}{(1-\nu_2)/\mu_2 + (1-\nu_1)/\mu_1}$$

The parameter $\alpha$ can be physically described as the relative difference between the plane strain modulii given by $\alpha = \frac{\overline{E}_1 - \overline{E}_2}{\overline{E}_1 + \overline{E}_2}$ where $\overline{E}_1 = \frac{2\mu_1}{1-\nu_1}$ and $\overline{E}_2 = \frac{2\mu_2}{1-\nu_2}$. The parameter $\beta$ does not have a physical interpretation, nor does it have as strong an influence on the problem as $\alpha$ (Suo and Hutchinson, 1989).

Figure 2.4 shows a plot of the elastic mismatch parameters $\alpha$ and $\beta$ for several common material systems used in adhesive joints.
Figure 2.4- Typical values of the Dundurs' parameters for polymeric and inorganic adhesives joining various substrates.

In the literature, many solutions are plotted using only $\alpha$ and assuming $\beta = \alpha/4$. Figure 2.4 shows that if $\nu_1 = \nu_2 = 1/3$, then $\beta = \alpha/4$. Some bimaterial systems are clustered near this line, which therefore justifies that $\beta = \alpha/4$. Throughout this paper, all the numerical tests were done using an aluminum epoxy system where $\nu_1 = 0.35$, and $\mu_1 = 26.3$ GPa for the adherend and $\nu_2 = 0.34$ and $\mu_2 = 1.5$ GPa for the adhesive. The elastic mismatch parameters for this bimaterial system are $\alpha = 0.893$ and $\beta = 0.223$.

As mentioned previously, the far-field quantities, $K_i^\infty$, $K_{II}^\infty$ and $T^\infty$, are different from the local quantities $K_i$, $K_{II}$, and $T$. This results from the elastic mismatch between the materials and from the residual stress, $\sigma'$, present within the adhesive layer. These far-field stress intensity factors $K_i^\infty$ and $K_{II}^\infty$ can be determined from handbooks such as Tada
\textit{et al.} (1985) for specific specimen geometries. In the present work, $K_I^\infty$, $K_{II}^\infty$ and $T^\infty$, are taken as “applied loads” for the local elasticity problem in the 2-material, 3-region model shown in Figure 2.3. In order to determine the local quantities for the stress intensity factors and the $T$-stress in terms of the far-field quantities, the solution to this elasticity problem must be obtained. The solution procedure follows directly from Fleck, Hutchinson and Suo (1991).

### 2.3 The T-Stress

The $T$-stress as presented in equation (2.1) remains in the asymptotic expansion because of stability arguments. There is experimental evidence and theoretical explanations (Cottrell and Rice, 1980) that support these stability relations.

![Diagram of crack growth and dependence on non-singular stress $T$ acting parallel to the initial crack at its tip.](image)

**Figure 2.5** - Path of crack growth and dependence on non-singular stress $T$ acting parallel to the initial crack at its tip.

Figure 2.5 shows a straight crack is advancing in an isotropic homogeneous brittle solid where the crack-tip is under pure Mode I conditions (i.e. $K_{II} = 0$). If the crack encounters an imperfection such as a dust particle or a void, the crack will veer from the straight line when $T > 0$, whereas a negative $T$-stress will pull the crack-tip back to straightness. Therefore, for Mode I loading, a positive $T$-stress will cause unstable crack growth, and a negative $T$-stress will result in a stable crack path.

To determine the local $T$-stress, the principle of superposition is enforced. The local $T$-stress linearly depends on the four loading parameters through the following expression
\[ T = \left( \frac{1 - \alpha}{1 + \alpha} \right) T^\infty + \sigma^o + c_t \frac{K^o_t}{\sqrt{H}} + c_{tt} \frac{K^o_{tt}}{\sqrt{H}} \]  

(2.4)

where \( \sigma^o \) is the thermal residual stress parallel to the crack plane present in the adhesive layer, and \( c_t \) and \( c_{tt} \) are nondimensional coefficients which depend on the geometric ratio \( c/H \) and the elastic mismatch parameters \( \alpha \) and \( \beta \). These coefficients were calculated and tabulated by Fleck, Hutchinson and Suo (1991). The tabulated results were converted into contour plots which are easier to utilize in conjunction with actual \( \alpha \) and \( \beta \) values for practical adhesive/adherend pairs. Such contour plots are shown in Figures 2.6 and 2.7.
Figure 2.6- Contour plots of the $c_l$ coefficient values computed by Fleck, Hutchinson, and Suo (1991). Typical values of $(\alpha, \beta)$ for several structural adhesives are shown as data points.

Figure 2.6 shows the values for the $c_l$ coefficient as a function of crack placement in the adhesive layer. As can be seen, for common material systems the values of $c_l$ change slightly or are fairly insensitive to crack placement in the adhesive layer. The values of $c_l$ are in the range from (-0.1,-0.17) for $c/H = 0.5$, and (-0.13,-0.2) for $c/H = 0.8$. The values of $c_{ll}$ are more sensitive to crack placement, as seen in Figure 2.7.
Figure 2.7- Contour plots of the $c_{ll}$ coefficient values computed by Fleck, Hutchinson, and Suo (1991). Typical values of $(\alpha, \beta)$ for several structural adhesives are shown as data points.

As the crack is displaced off the centerline, the values change drastically. At $c/H = 0.6$, common systems have values for $c_{ll}$ from $(0.025, 0.034)$. When the crack is closer to the upper interface ($c/H = 0.8$), the values change by an order of magnitude to $(0.1, 0.13)$. At $c/H = 0.5$, the crack on the centerline is symmetrical and under pure mode I locally so $c_{ll} = 0$. 

$c/H = 0.5$

$C_{ll} = 0$

for

$c/H = 0.5$

$c/H = 0.6$

$c/H = 0.7$

$c/H = 0.8$
2.4 Stress Intensity Factors

Through the conservation of the \( J \)-integral, the local stress intensity factors can be related to the remote stress intensity factors in the following manner

\[
G = J = \frac{1}{E_2} \left( K_i^2 + K_u^2 \right) = \frac{1}{E_1} \left[ \left( K_i^\infty \right)^2 + \left( K_u^\infty \right)^2 \right]
\]

or equivalently

\[
(K_i + iK_u) = \left( \frac{E_2}{E_1} \right)^{1/2} \left( K_i^{\infty} + iK_u^{\infty} \right) e^{i\phi}
\]

where \( \phi = \tan^{-1} \left( \frac{K_u}{K_i} \right) - \tan^{-1} \left( \frac{K_u^{\infty}}{K_i^{\infty}} \right) \). Like the \( c_i \) and \( c_u \) coefficients, the phase angle \( \phi \) is dependent only on the geometric ratio \( c/H \) and the elastic mismatch parameters \( \alpha \) and \( \beta \). Fleck, Hutchinson, and Suo (1991) computed values for \( \phi \). Figure 2.8 shows the approximation and numerical solution for \( \phi \) plotted as a function of \( c/H \) for several values of \( \alpha \). The values needed to compute the approximation are tabulated in Fleck, Hutchinson, and Suo (1991). The following approximation developed by Fleck et al. (1991) was used in Figure 2.8 to fit the numerical solution with a high degree of accuracy:

\[
\phi = \varepsilon \ln \left( \frac{H - c}{c} \right) + 2 \left( \frac{c}{H} - \frac{1}{2} \right) (\phi_{\text{H}}(\alpha, \beta) + \omega(\alpha, \beta))
\]

where \( \varepsilon = \frac{1}{2\pi} \ln \left( \frac{1 - \beta}{1 + \beta} \right) \), the function \( \phi_{\text{H}}(\alpha, \beta) \) is given in Hutchinson et al. (1987) and \( \omega(\alpha, \beta) \) is tabulated in Suo and Hutchinson (1989).
Figure 2.8- Plot of phase angle $\phi$ against values of $c/H$ for various values of $\alpha$ (where $\beta = \alpha/4$).

It can be seen that $\phi$ is antisymmetric about the centerline of the adhesive ($c/H = 0.5$). For compliant adhesives ($\alpha > 0$), the values of $\phi$ are in the range (-5° to +5°) as $c/H$ varies in the range (0.2, 0.8). However, as the crack moves close to the interface, the values of $\phi$ increase drastically.

2.5 Crack Path Selection in an Adhesive Layer

If the crack propagates along the centerline, a pure mode I loading is also present locally ($K_{II} = 0$). If a crack is propagating slightly above the centerline, the crack will veer towards the centerline if locally $K_{II} > 0$. In other words for a crack which is slightly displaced from the $K_{II} = 0$ path, the value of $\partial K_{II}/\partial \alpha$ must be greater than zero in order
for the crack to veer towards that path (Fleck, Hutchinson, and Suo, 1991). From equation (2.6), the link between \( K_\| \) and \( \partial K_\| / \partial \varepsilon \) can be shown as:

\[
K_\| = \left( \frac{E_2}{E_1} \right)^{1/2} \left( K_\|^{\infty} \sin \phi + K_\|^{\infty} \cos \phi \right)
\]  

(2.8)

From Figure 2.8, the values of \( \phi \) were shown to be small for a crack located in the adhesive layer and not along the interfaces. Therefore, for small \( \phi \) equation (2.8) reduces to

\[
K_\| \equiv \left( \frac{E_2}{E_1} \right)^{1/2} \left( K_\|^{\infty} \phi + K_\|^{\infty} \right)
\]  

(2.9)

Now, by taking the partial derivative with respect to \( \varepsilon \) of equation (2.9) an expression for \( \partial K_\| / \partial \varepsilon \) results

\[
\frac{\partial K_\|}{\partial \varepsilon} \equiv \left( \frac{E_2}{E_1} \right)^{1/2} \left( K_\|^{\infty} \frac{\partial \phi}{\partial \varepsilon} \right)
\]  

(2.10)

Recalling the stability criteria imposed by the \( T \)-stress and keeping in mind the contribution from \( \partial K_\| / \partial \varepsilon \) on crack path selection, four distinctly different crack patterns can be identified. Figure 2.9 shows the four possible cracking patterns identified by Fleck, Hutchinson, and Suo (1991).
Figure 2.9- Four possible cracking patterns identified by Fleck, Hutchinson, and Suo (1991).

For pattern A, a crack slightly above the centerline will kink towards the centerline because $\partial K / \partial c > 0$ and run stably through the adhesive layer because of the negative $T$-stress.

For pattern B, the crack runs cohesively following a wavy path through the adhesive layer. This pattern results when a crack slightly displaced off the centerline is forced back to the centerline by a positive $\partial K / \partial c$ value. However, the positive $T$-stress will cause an unstable cracking pattern. The path will be wavy if $T$ is not too large and $\partial K / \partial c$ can force the crack back to the centerline. However, if $T$ is very large, the crack will grow into the interface.
Pattern C represents propagation of a crack very near or into the interface. In this case a crack displaced off the centerline is forced away from the centerline by negative $\partial K_{II} / \partial \alpha$, while a negative $T$-stress causes the crack to stably approach the interface at a small angle.

On the other hand, pattern D is represented by a crack which will run unstably into the interface. The negative $\partial K_{II} / \partial \alpha$ causes the crack to diverge towards the interface, while the tensile $T$-stress moves the crack unstably into the interface at large angles.

These four cracking patterns are shown in Figure 2.10 on an $\alpha, \beta$ plot.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure2.10.png}
\caption{Four cracking patterns plotted on an $\alpha, \beta$ plot (Fleck et al., 1991). Note that the contour corresponding to $T = 0$ must be shifted if the thermal residual stress, $\sigma^r$, or the far-field $T$-stress, $T^c$, are significant (Fleck, Hutchinson, and Suo, 1991).}
\end{figure}
2.6 Case Study of a DCB Test

In a double cantilever beam (DCB) test, a pure mode I loading is applied (i.e. $K_{I}^{c} = 0$). One adhesive system was considered as an example and case study. The system consisted of a DOW Chemical Company rubber toughened model epoxy with 8% rubber and 1018 cold rolled steel from a study by Rakestraw et al. (1996). The material properties of the system are given in Table 2.1.

**Table 2.1- Material Properties of Adhesive System**

<table>
<thead>
<tr>
<th>MATERIAL</th>
<th>Thermal Expansion Coefficient ($10^{6}/\text{C}$)</th>
<th>Young's Modulus (GPa)</th>
<th>Poisson Ratio</th>
<th>Shear Modulus $\mu_s$ (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dow rubber epoxy</td>
<td>78</td>
<td>3</td>
<td>0.33</td>
<td>1.13</td>
</tr>
<tr>
<td>1018 cold rolled Steel</td>
<td>200</td>
<td></td>
<td>0.33</td>
<td>75.11</td>
</tr>
</tbody>
</table>

In these DCB tests, the strain energy release rate (SERR), $G$, was calculated from the applied loads utilizing the compliance method. According to Rakestraw et al. (1996), the $G_{IC}$ value for the aforementioned system was computed around 300 J/m$^2$. Now, the bulk adhesive layer fracture toughness can be calculated through the conservation of the J-integral (equation 2.5) as

$$K_{lc}^c = \sqrt{\frac{2\mu_s}{1-\nu_s}} G_{IC}.$$

The usual assumption ignores the thickness of the adhesive layer and just uses a homogeneous base specimen with a centerline crack. On the other hand, the present method (Fleck et al., 1991) offers corrections to account for crack placement within the adhesive layer.

For DCB tests, the loading is pure mode I ($K_{II}^c=0$). Therefore, equation (2.4) reduces to
\[ T = \frac{1 - \alpha}{1 + \alpha} T^\infty + \sigma^\circ + c_l \frac{K_l^\circ}{\sqrt{H}} \]  

(2.11)

where \( c_l \) is the correction factor which is a function of \( c/H, \alpha \) and \( \beta \) as discussed in Section 2.3. Now the local stress intensity factors can be calculated from equation (2.6) as

\[ K_l = \left( \frac{1 - \alpha}{1 + \alpha} \right)^{1/2} (K_l^\circ \cos \phi) \]  

(2.12)

\[ K_H = \left( \frac{1 - \alpha}{1 + \alpha} \right)^{1/2} (K_l^\circ \sin \phi) \]  

(2.13)

where \( \phi \) is used as a further correction for crack placement and is also a function of \( c/H, \alpha \) and \( \beta \) like \( c_l (c/H, \alpha, \beta) \).

According to Fleck, Hutchinson, and Suo (1991) the first term in equation (2.8) is negligible for compliant adhesive layers. Therefore, for the case study, only the last two terms of equation (2.8) are considered. Since \( c_l \) is always negative for compliant adhesive layers, the local \( T \)-stress will always be negative if \( \sigma^\circ \) is negative as well. However, if \( \sigma^\circ \) is positive, then the sign of the local \( T \)-stress will depend on the comparable magnitudes of \( \sigma^\circ \) and the last term in equation (2.8).

For the rubber-toughened epoxy/steel system, the DCB specimens are cured at 155°C for 1½ hours. The difference between the glass transition temperature and room temperature is about 70°C, so that the thermal residual stress is approximately 16 MPa. The thickness of the adhesive layer \( H \) is 0.5 mm. Taking into account the applied critical load and the coefficient \( c_l(c/H, \alpha, \beta) \) for a preexisting crack located at the centerline of the adhesive layer \( (c/H = 0.5) \), the \( T \)-stress will be approximately -10 MPa. If the pre-crack in the DCB specimen is closer to the interface \( (c/H = 0.8) \), the \( T \)-stress will be about -16 MPa. This lies in the stable regime of the cracking patterns. However, to determine if the crack will move towards the centerline or into the interface, \( \partial \mathcal{K}_H / \partial \alpha \) must be considered. For this
material system, Figure 2.10 indicates that $\partial K_{II}/\partial c > 0$. This can also be confirmed using Figure 2.8 and equation (2.10). For compliant adhesive layers, $\phi$ is monotonically increasing through the thickness of the adhesive layer. Or in other words, $\partial \phi / \partial c > 0$. So therefore, $\partial K_{II}/\partial c$ must be greater than zero. From equations (2.12) and (2.13), the near-field stress intensity factors for $c/H=0.5$ were calculated as $K_I = 1.005 \text{ MPa} \cdot \sqrt{\text{m}}$ and $K_{II} = 0 \text{ MPa} \cdot \sqrt{\text{m}}$. For a crack closer to the interface ($c/H=0.8$), $K_I = 1.003 \text{ MPa} \cdot \sqrt{\text{m}}$ and $K_{II} = 0.058 \text{ MPa} \cdot \sqrt{\text{m}}$. Thus, the crack should run cohesively in the adhesive layer.

According to Rakestraw et al. (1996), the predominant mode of fracture for their test specimens was cohesive. This is what was predicted with the above method. However, some of the specimens showed cracks running interfacially. This might be explained because the specimens were precracked interfacially. They observed that the interface toughness is about 17.5% lower than the bulk adhesive toughness. Therefore, the crack might continue to propagate along the interface following the path of least resistance.
Solution to the Elasticity Problem

3.0 Introduction

This chapter focuses on Fleck, Hutchinson and Suo's (1991) method to solve the 3-region 2-material elasticity problem of an adhesive layer sandwiched between two half-planes of a stiffer substrate (Figure 2.1) using an integral equations method.

The general approach was introduced by Erdogan and Gupta (1971) with a general integral equation formulation for cracks in layered composites. In recent years, others have adopted this approach. Hutchinson, Mear and Rice (1987) used this method to solve the problem of a crack paralleling a bimaterial interface. Thouless, Evans, Ashby, and Hutchinson (1987) utilized the method to look at the edge cracking and spalling of brittle plates. Through this method Suo and Hutchinson (1989) developed a universal relation between the actual interface stress intensity factors at the crack-tip of an interface crack and the apparent Mode I and Mode II stress intensity factors associated with the corresponding problem for the crack in a homogeneous material. Dewynne, Hills and Nowell (1992) make improvements on the method to help solve problems with the calculation of the opening displacement of surface-breaking plane cracks. Kelly, Nowell, and Hills (1994) employ it to analyze an edge interface crack. Recently as well, Cheung and Chen (1994) utilized this method to develop a new boundary integral equation for a notch problem of antiplane elasticity. Finally, Fleck, Hutchinson and Suo (1991) used this approach to solve the present elasticity problem.

The method detailed in the following sections follows directly from Fleck, Hutchinson and Suo's (1991) work. Their symbolic derivations were reproduced and several discrepancies were identified, as shown in Appendices A and B. The work in these
sections concentrated on furthering the understanding of sensitive issues such as computational speed, convergence and accuracy of the solution procedure.

### 3.1 Solution Summary

Figure 3.1 shows an analysis flow chart of the integral equation method as applied to the elasticity problem of an adhesive crack. The approach consists of first modeling the crack with a distribution of dislocations. Then the resulting Cauchy-type integral equations are solved. Since the solution is unknown, the dislocations are represented by a series expansion in Chebyshev polynomials. The resulting integral equations are solved via the collocation method forming a linear system of algebraic equations. The Chebyshev coefficients are then recovered. Finally, the characteristics of the local stress field can be determined. In particular, the local stress intensity factors can be computed, crack path selection can be.
determined and crack propagation stability can be ascertained. The goal of this research has been to study the accuracy of the method, and to develop a modular package which has been optimized for speed on the PC. Standard numerical packages such as Mathcad 5+ and Mathematica 2.2.1 have been used to develop the modules.

3.2 Symbolic Derivation of the Integral Equations

Two problems are introduced and their solutions superposed to obtain the resulting stress field. The first problem introduces an isolated dislocation at the origin of the adhesive layer. The half planes, \( y > d \) and \( y < -c \), are transformed from the adhesive material to the substrate material, thus introducing a displacement mismatch \( \Delta u(x) \) at the interfaces while keeping the stresses fixed. The second problem has no dislocation in the adhesive layer. Instead a displacement mismatch of equal magnitude and opposite sign from problem 1 (i.e. \( -\Delta u(x) \)) is applied at the interfaces. This displacement mismatch gives rise to finite stresses everywhere which are represented by stress functions with unknown coefficients.

Each of these problems consists of two separate sub-problems: one for the parallel dislocation of Burger vector \( b_1 \) and the other for the perpendicular dislocation of Burger vector \( b_2 \). The boundary conditions are imposed symbolically in the Fourier domain: the displacement mismatch at each interface and stress continuity across these interfaces. The resulting system is

**for the \( b_1 \) case**

\[
\begin{pmatrix}
M_1 & M_2 & 0 \\
M_3 & 0 & M_4 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
D \\
E \\
C
\end{pmatrix}
= 
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix}
\]  \hfill (3.1a)

**Similarly, for the \( b_2 \) case**

\[
\begin{pmatrix}
M_1 & M_2 & 0 \\
M_3 & 0 & M_4 \\
0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
G \\
H \\
F
\end{pmatrix}
= 
\begin{pmatrix}
w_1 \\
w_2
\end{pmatrix}
\]  \hfill (3.1b)
where the vectors $D$, $E$ and $C$ correspond to the Fourier coefficients for the adhesive layer, the lower substrate layer and the upper substrate layer, respectively. The Fourier coefficients $G$, $H$ and $F$ correspond in a similar manner for the $b_2$ case. The vectors $D$ and $G$ contain elements $D_i(\lambda)$ and $G_i(\lambda)$ ($i=1,2,3,4$) which are functions of the Fourier variable, $\lambda$. The details on obtaining the matrices $M_1$, $M_2$, $M_3$ and $M_4$, as well as, the vectors $v_1$, $v_2$, $w_1$, and $w_2$ are given in Appendix B. Some differences were noted in our symbolic derivation and the work of Fleck, Hutchinson and Suo (1991). But one obvious mistake resulted in applying the displacement gradient mismatch boundary condition. This resulted in a distinct sign difference in the last terms of the $w_1$ and $w_2$ vectors. After applying this change, the antisymmetry properties of the problem were restored.

3.3 Partial Closed Form Solutions for Adhesive Layer Coefficients, $D_i(\lambda)$ and $G_i(\lambda)$

Ultimately, the problem is to determine the coefficients in the adhesive layer, $D_i(\lambda)$ for the $b_1$ case and $G_i(\lambda)$ for the $b_2$ case, within the Fourier plane. In Fleck, Hutchinson and Suo (1991), they solved equations (3.1a) and (3.1b) for $D_i(\lambda)$ and $G_i(\lambda)$ numerically through Gaussian elimination at each value of $\lambda$. In order to reduce computational time, a "partial" closed form solution was obtained using Gaussian inversion.

First, the larger $M$ matrices needed to be reduced to smaller $2 \times 2$ matrices. The $M_i$ and $M_j$ matrices are $4 \times 4$ matrices, and the $M_2$ and $M_4$ matrices are $4 \times 2$ matrices. Figure 3.2 shows the divided matrices in a super matrix form for the $b_1$ case.
A similar procedure is followed for the $b_2$ case. The vectors $v_1$ and $v_2$ are also reduced from 4-element vectors to two 2-element vectors each. For the $b_1$ case, to isolate $D$, the vectors $C$ and $E$ were eliminated. Likewise, for the $b_2$ dislocation solution, the vector $G$ was isolated by eliminating $H$ and $F$. As a result, non-singular matrices, i.e. $M_{41}$, $M_{21}$, $M_{12}$, and $M_{21}$, were used as pivots to develop the following symbolic solution:

**for the $b_1$ case**

\[
\begin{bmatrix}
D_1(\lambda) \\
D_2(\lambda)
\end{bmatrix} = \left( M_{22}' - M_{22}'M_{12}'^{-1}M_{11}' \right)^{-1} \left( v_{22}' - M_{22}'M_{12}'^{-1}v_{11}' \right)
\]

\[
\begin{bmatrix}
D_3(\lambda) \\
D_4(\lambda)
\end{bmatrix} = \left( M_{12}' - M_{11}'M_{21}'^{-1}M_{22}' \right)^{-1} \left( v_{11}' - M_{11}'M_{21}'^{-1}v_{22}' \right)
\]

**Similarly, for the $b_2$ case**

\[
\begin{bmatrix}
G_1(\lambda) \\
G_2(\lambda)
\end{bmatrix} = \left( M_{21}' - M_{22}'M_{12}'^{-1}M_{11}' \right)^{-1} \left( v_{22}' - M_{22}'M_{12}'^{-1}v_{11}' \right)
\]

\[
\begin{bmatrix}
G_3(\lambda) \\
G_4(\lambda)
\end{bmatrix} = \left( M_{12}' - M_{11}'M_{21}'^{-1}M_{22}' \right)^{-1} \left( v_{11}' - M_{11}'M_{21}'^{-1}v_{22}' \right)
\]

where

\[
M_{11}' = M_{321} - M_{42}M_{41}^{-1}M_{311} \quad M_{21} = M_{121} - M_{22}M_{21}^{-1}M_{111}
\]

\[
M_{12}' = M_{322} - M_{42}M_{41}^{-1}M_{312} \quad M_{22} = M_{122} - M_{22}M_{21}^{-1}M_{112}
\]
\[ v_{11}^* = v_{22} - M_{42} M_{41}^{-1} v_{21} \]
\[ w_{11}^* = w_{22} - M_{42} M_{41}^{-1} w_{21} \]
\[ v_{22}^* = v_{12} - M_{22} M_{21}^{-1} v_{11} \]
\[ w_{22}^* = w_{2} - M_{22} M_{21}^{-1} w_{11} \]

Now, for any values of \( \lambda \) the values of \( D_i(\lambda) \) and \( G_i(\lambda) \) can be obtained. The details of this method are shown in section B.5.

### 3.4 “Numerical Freeze” Method

Since the functions like \( D_i(\lambda) \) and \( G_i(\lambda) \) were used repetitively, several numerical techniques were utilized to reduce computational time. One of the main techniques developed and utilized was named the “numerical freeze” method.

A problem was encountered in dealing with the software package in use, Mathcad 5+. One of the nice features of this package allows function definitions such as equation (3.2). However, each time the function would be recalled, the individual matrices would be recalculated. Unfortunately, this was redundant and computationally time consuming. Therefore to reduce time, the functions, i.e. \( D_i(\lambda) \) and \( G_i(\lambda) \), were evaluated at discrete points.

Using built in algorithms, splines were applied to the “frozen” data. Throughout the work, cubic splines were utilized to fit smooth curves to the data which were continuous to the 2nd derivative. Now, the spline coefficients could be stored and recalled later to construct interpolated functions. These new interpolated functions could then be used for integration calculations with increased speed over calculations of the real functions and with extreme accuracy. Figure 3.3 shows an example of this technique.
Figure 3.3- Interpolated function superposed over the real function.

No apparent differences are identified between the values of the real function $G_r(\lambda)$ and its interpolated counterpart $G_{intrp}(\lambda)$. This “numerical freeze” technique was used repeatedly on many functions in the solution process.
3.5 The Kernel Functions, \( f_\theta(\zeta) \)

The displacement mismatch at the \( x_2 = d \) and \( x_2 = -c \) boundary gives rise to finite stresses. The Fourier coefficients of the adhesive layer, \( D_i(\lambda) \) and \( G_i(\lambda) \), are the components which describe these stress fields. The kernel functions \( f_\theta(\zeta) \) and \( g_i(\zeta) \) listed below inverse transform the Fourier coefficients describing the adhesive layer back into the real plane.

\[
\begin{align*}
    f_{11}(\zeta) &= \frac{4\pi(1 - \nu_2)}{\mu_2} \int_{0}^{\infty} (-D_1 + D_2 + D_3 + D_4) \sin \lambda \zeta \, d\lambda \\
    f_{21}(\zeta) &= \frac{4\pi(1 - \nu_2)}{\mu_2} \int_{0}^{\infty} (-D_1 - D_3) \cos \lambda \zeta \, d\lambda \\
    f_{12}(\zeta) &= \frac{4\pi(1 - \nu_2)}{\mu_2} \int_{0}^{\infty} (G_1 - G_2 - G_3 - G_4) \cos \lambda \zeta \, d\lambda \\
    f_{22}(\zeta) &= \frac{4\pi(1 - \nu_2)}{\mu_2} \int_{0}^{\infty} (-G_1 - G_3) \sin \lambda \zeta \, d\lambda \\
    g_1(\zeta) &= \frac{4\pi(1 - \nu_2)}{\mu_2} \int_{0}^{\infty} (D_1 - 2D_2 + D_3 + 2D_4) \cos \lambda \zeta \, d\lambda \\
    g_2(\zeta) &= \frac{4\pi(1 - \nu_2)}{\mu_2} \int_{0}^{\infty} (G_1 - 2G_2 + G_3 + 2G_4) \sin \lambda \zeta \, d\lambda
\end{align*}
\]

(3.3)

These inverse Fourier cosine and sine integral transforms are evaluated on the \( \lambda \) interval from 0 to \(+\infty\). To evaluate these integrals numerically, an appropriate truncated interval of \( \lambda \) was determined. To get an idea of this interval, the integrands behavior was explored. For example, Figure 3.3 showed \( G_1(\lambda) \) trailing off to zero around 10. Therefore, this was a good starting place. For \( \lambda \) greater than 10, all the \( D_i(\lambda) \) and \( G_i(\lambda) \) coefficients trailed off to zero and would have no contribution to the integrals they were associated with. To be certain, a convergence check was done. As an example, Table 3.1 shows for different \( \lambda_{\text{max}} \) values, the convergence of \( f_{22}(\zeta) \) at three different locations from the crack-tip.
Table 3.1: $f_{22}(\zeta)$ Convergence Check

<table>
<thead>
<tr>
<th>$\lambda_{\max}$</th>
<th>$f_{22}(-01)$</th>
<th>$f_{22}(-10)$</th>
<th>$f_{22}(-45)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>-0.21097</td>
<td>-2.9177</td>
<td>-0.73657</td>
</tr>
<tr>
<td>10</td>
<td>-0.24688</td>
<td>-2.9507</td>
<td>-0.73970</td>
</tr>
<tr>
<td>20</td>
<td>-0.24971</td>
<td>-2.9520</td>
<td>-0.73943</td>
</tr>
<tr>
<td>30</td>
<td>-0.24972</td>
<td>-2.9520</td>
<td>-0.73943</td>
</tr>
<tr>
<td>40</td>
<td>-0.24972</td>
<td>-2.9520</td>
<td>-0.73943</td>
</tr>
</tbody>
</table>

This table shows that good convergence has been reached for $\lambda_{\max} = 30$. Since later calculations depend on the accuracy of these integrals, we decided to use $\lambda_{\max} = 30$.

As mentioned previously, the kernel functions $f_6(\zeta)$ and $g_6(\zeta)$ inverse transform the Fourier coefficients from the Fourier plane back into the physical plane. Evaluating these cosine and sine Fourier integrals proved challenging. When plotting the $f_6(\zeta)$ kernel functions, periodic numerical “spikes” were present where decaying behavior was expected. Figure 3.4 shows an example of these numerical “spikes.”

![Graph showing $f_{22}(\zeta)$ with Spurious "Spikes".](image)

**Figure 3.4** - $f_{22}(\zeta)$ with Spurious "Spikes".
It was not known if these "spikes" were actually present or they were a result of the complicated numerics inherent within the problem. An alternate method to evaluate these integrals was employed.

A FFT algorithm was used to evaluate these Fourier cosine and sine integrals. The FFT was originally utilized to examine the numerical "spikes."

The Nyquist critical frequency principle (Paris, ****) was used to examine the numerical "spikes." As can be seen from Figure 3.5, when the FFT method was employed on the continuous kernel functions, the transform approached zero as the Nyquist critical frequency ($\zeta = \pm 50$ in this case) was reached. Also, no "spikes" were present in this critical range, so aliasing or "folding over" had not occurred.

![Figure 3.5- FFT of $f_{22}(\zeta)$ with no "Spurious Spikes"

Since the "spikes" did not fold over into the critical range, it was concluded that they were not actually present in the signal, but just numerical anomalies.
The greatest advantage to the FFT approach was the reduction of computational time. Unfortunately, the peak values using the FFT algorithm were approximately 10% greater than just utilizing numerical integration.

The FFT algorithm rearranges the original discrete real signal (a set of \(2^n\) numbers) into two sets, a real set (a series of \(2^{m-1}\) numbers) and an imaginary set (a series of \(2^{m-1}\) numbers) of values. The real components correspond to the cosine transform, and the imaginary components correspond to the sine transform. These transform components follow the mapping in Figure 3.6.

**Figure 3.6- Mapping Between Real Domain and Fourier Domain**

To transform from the Fourier plane back into the physical plane, the scale factor, \(\delta_\zeta\), needed to be used. The expression for \(\delta_\zeta\) is given below

\[
\delta_\zeta = \frac{\pi}{\lambda_{\text{max}}}.
\]  

(3.4)

In Press et al. (1992), they set certain guidelines for the FFT of a smooth function for an infinite interval. When the range of integration is infinite or semi-infinite, they suggest that the integral be split at a large enough value of \(t\) so that the remaining interval is small. For example,
\[ \int_{a}^{\infty} e^{i\alpha t} h(t) dt = \int_{a}^{b} e^{i\alpha t} h(t) dt + \int_{b}^{\infty} e^{i\alpha t} h(t) dt \]

\[ = \int_{a}^{b} e^{i\alpha t} h(t) dt - \frac{h(b)e^{i\alpha b}}{i\omega} + \frac{h'(b)e^{i\alpha b}}{(i\omega)^2} \ldots \]  

(3.5)

where \( h(t) \) are the Fourier coefficients, \( G_\ell(\lambda) \) and \( D_\ell(\lambda) \), in our case. The extra terms are the asymptotic expansion obtained from integrating by parts on the remaining interval \((b, \infty)\). Figure 3.7 shows the negligible contribution from just the 1st order asymptotic expansion.

**Figure 3.7 - \( f_{22}(\zeta) \) with the 1st-Order Asymptotic Expansion**

This plot is on a log-scale to help zoom in closer to the dislocation origin. The \( -\zeta \)-axis is plotted with a very small value away from the crack tip because the \( \log(0) \) is undefined. Even though the figure does not show it, the \( f_{22}(\zeta) \) function does indeed go to zero at \( \zeta = 0 \).

Also as can be seen in the figure, the 1st-order expansion values are small compared to the kernel function itself. The expansion terms will make no contribution to the overall value of the function. Therefore, it was concluded that the values obtained for the FFT were valid and no end-point correction factors were needed.
### 3.5.1 Other FFT Pitfalls

Even though the FFT has many computationally efficient advantages, the use of the FFT does present a problem. Figure 3.6 shows that the step, \( d\lambda \), corresponds to a maximum value, \( \zeta_{\text{max}} \), in the physical plane. The value for \( \pm \zeta_{\text{max}} \) is 157, so any data outside this range of the FFT is useless. Unfortunately, there is data which needs to be extracted from this region later during other integration steps.

On the other hand, there are some redeeming qualities about the \( f_{\xi}(\zeta) \) kernel functions themselves. As noted by Fleck, Hutchinson, and Suo (1991), the kernel functions, \( f_{\xi}(\zeta) \), are well behaved in the whole range \(-\infty < \zeta < +\infty\), with asymptotes

\[
f_{\xi}(\zeta) = O\left(\frac{1}{\zeta}\right) \quad \text{as} \quad |\zeta| \to \infty.
\]

This makes sense, since at \( \zeta = 0 \) lies the dislocation and the disturbance created by it is

---

**Figure 3.8** - Asymptotic behavior of \( f_{22}(\zeta) \) kernel function. centered here and decays to zero further out. This is displayed in Figure 3.8.
To handle the data outside the FFT usable range, the kernel functions were extrapolated to zero further out. This will ultimately give better results in the integration process. Section B.6 shows the plots for all the kernel functions for the aforementioned test case.

### 3.6 Antisymmetry Between the Components of the $b_1$ and $b_2$ Solutions

The stresses defined by the superposition of the isolated dislocation problem and the dislocation free strip problem are given as

\[
\sigma_{11}(\zeta) = \frac{\mu_2}{4\pi(1 - \nu_2)} \left( \frac{2b_2}{\zeta} + g_1(\zeta)b_1 + g_2(\zeta)b_2 \right) \tag{3.7a}
\]

\[
\sigma_{22}(\zeta) = \frac{\mu_2}{4\pi(1 - \nu_2)} \left( \frac{2b_2}{\zeta} + f_{22}(\zeta)b_2 + f_{21}(\zeta)b_1 \right) \tag{3.7b}
\]

\[
\sigma_{12}(\zeta) = \frac{\mu_2}{4\pi(1 - \nu_2)} \left( \frac{2b_1}{\zeta} + f_{11}(\zeta)b_1 + f_{12}(\zeta)b_2 \right) \tag{3.7c}
\]

As can be seen from the equation (3.7), $\sigma_{22}(\zeta)$ is primarily a $b_2$ induced stress, whereas, the $\sigma_{12}(\zeta)$ stress is primarily a $b_1$ induced stress. As a result, the $f_{21}(\zeta)$ term in $\sigma_{22}(\zeta)$ and the $f_{12}(\zeta)$ term in $\sigma_{12}(\zeta)$ are coupling terms. In changing from the $b_1$ problem to the $b_2$ problem, there is an antisymmetry present (Appendix B). Therefore, since the $f_{21}(\zeta)$ kernel function is derived from the $b_1$ solution, and the $f_{12}(\zeta)$ kernel function is derived from the $b_2$ solution, these functions should display antisymmetry to each other. Figure 3.9 indeed shows these antisymmetric properties.
Figure 3.9- Antisymmetry of the coupling terms.

These coupling terms were not antisymmetric when Fleck, Hutchinson and Suo’s (1991) vectors \( w_1 \) and \( w_2 \) were used as such. When the last element of each vector \( w_1 \) and \( w_2 \) changes sign, the antisymmetric properties of the problem are restored.

### 3.7 Integral Equation Formulation

The dislocation distribution semi-infinite interval is reduced to a finite interval with the following change of variables

\[
x = \frac{u - 1}{u + 1}, -1 < u < 1
\]

\[
\xi = \frac{t - 1}{t + 1}, -1 < t < 1
\]

which gives

\[
\zeta = x - \xi = \frac{2(u - t)}{(u + 1)(t + 1)},
\]

This allows for several numerical integration techniques.
A complete approximation for the dislocation distribution, \( b_i(\xi) + i b_\xi(\xi) \), in the literature is as a Chebyshev series approximation

\[
c_i(t) = \left( \frac{1+t}{1-t} \right)^{\frac{1}{2}} \sum_{k=0}^{\infty} a_{ik} T_k(t), \quad i = 1, 2
\]  

(3.10)

where \( T_k(t) \) is the Chebyshev polynomial of the 1st kind of degree \( k \), and \( a_{ik} \) is a double set of infinite real constants determined through the solution of the linear system of equations.

If the traction free boundary condition for \( x_i < 0 \) is imposed, the change of variable mentioned above is made, and the Chebyshev series approximation for \( c_i(t) \) substituted, the following linear system of integral equations results

\[
\int_{-1}^{1} \frac{(u+1)}{(u - t)(t + 1)} c_i(t) dt + \int_{-1}^{1} \frac{f_{ij}(\xi)c_i(t)}{(1 + t)z} dt = 0, \quad |u| < 1
\]  

(3.11)

where the first integral is the Cauchy Principal Value integral.

The closed form solution for the first integral is shown below (Fleck, Hutchinson, and Suo, 1991)

\[
\int_{-1}^{1} \frac{(u+1)}{(u - t)(t + 1)} c_i(t) dt = -\pi(1 + u) \sum_{k=1}^{\infty} a_{ik} U_{k-1}(u).
\]  

(3.12)

The pole, \( (t + 1) \) term, present in the second integral of equation (3.12) could present some problems, so Fleck, Hutchinson, and Suo (1991) proposed the introduction of a new function \( F_{ij}(\xi) \) to extract a factor of \( (t + 1) \) from \( f_{ij}(\xi) \):

\[
F_{ij}(\xi) = f_{ij}(\xi) \sqrt{\xi^2 + 4} = \frac{f_{ij}(\xi)}{(t + 1)} \frac{1}{p(t,u)}
\]  

(3.13)
where

\[
p(t,u) = \frac{u+1}{2} \left[ (u+1)^2 (t+1)^2 + (u-t)^2 \right]^{-1/2}
\]  

(3.14)

The new \( F_\zeta(\zeta) \) functions are calculated from the known kernel functions \( f_\zeta(\zeta) \). Plots of these \( F_\zeta(\zeta) \) functions are in section B.7.2. Then the representation

\[
f_\zeta(\zeta) = (t+1) p(t,u) F_\zeta(\zeta)
\]  

(3.15)

and the representation for \( c(t) \) (equation 3.10) are substituted into the second integral of equation (3.11) which gives

\[
\int_1^\infty f_\zeta(\zeta) c_j(t) \frac{dt}{(1+t)^3} = \sum_{k=0}^{\infty} a_{jk} I_{jk}(u), \quad |u| < 1
\]  

(3.16)

where

\[
I_{jk}(u) = \int_1^\infty \frac{p(t,u) F_\zeta(\zeta(u,t)) T_k(t) dt}{\sqrt{1-t^2}}
\]  

(3.17)

Originally, a Romberg algorithm was utilized to numerically integrate the integral in the above equation. Because the semi-infinite interval was mapped into the finite range from -1 to 1, Gaussian integration could also be used on the interval to shorten computer time. Now through the substitution of equations (3.12) and (3.16) into equation (3.11), the below system of linear integral equations results

\[
-\pi(1+u) \sum_{k=1}^{N} a_{ik} U_{k-1}(u) + \sum_{k=0}^{N} \sum_{j=1}^{2} a_{jk} I_{jk}(u) = 0, \quad -1 < u < 1, \quad i = 1,2
\]  

(3.18)
3.8 Convergence Studies on the $I_{ijk}(u)$ Integrals

As mentioned previously, a Romberg algorithm was utilized to numerically integrate $I_{ijk}(u)$. This scheme was the most convergent method used and thus considered an "exact" integration. Unfortunately, this method was computationally intensive, and thus not practical.

Since the finite range was from -1 to 1, several types of Gaussian integration could be utilized. The first type used was based on a Gaussian-Legendre quadrature scheme. Fleck et al. (1991) utilized this scheme for collocating on the linear system of equations.

Another type of numerical integration follows directly from Erdogan and Gupta (1972). They realized that Gauss-Chebyshev integration could fairly accurately integrate the square root singularity. The expression for Gauss-Chebyshev integration is given by

$$\int_{-1}^{1} \frac{\varphi(x)dx}{\sqrt{1-x^2}} = \frac{\pi}{n} \sum_{k=1}^{n} \varphi(x_k) \quad \text{where} \quad x_k = \cos\left(\frac{2k-1}{2n} \pi\right)$$

(3.19)

This can be directly applied to the integral in equation (3.16)

$$I_{ijk}(u) = \int_{-1}^{1} \frac{p(t,u)}{\sqrt{1-t^2}} F_{ij}[\zeta(u,t)]T_k(t)dt$$

(3.20)

$$= \frac{\pi}{n} \sum_{m=1}^{n} p(t_m,u) F_{ij}[\zeta(u,t_m)]T_k(t_m)$$

(3.21)

where \( t_m = \cos\left(\frac{2m-1}{2n} \pi\right), \quad m = 1, ..., n \)

As seen in Figure 3.10, the differences between the Gauss-Legendre integration and the Gauss-Chebyshev integration are negligible.
\[ I_{22k}(u) \text{ with } k=20 \]

**Figure 3.10** - Comparison of \( I_{22k}(u) \) with \( k=20 \) for different types and orders of Gaussian quadratures.

However, the Gauss-Chebyshev integration was used from here on because it better models the square root singularity.

As previously mentioned, to help eliminate the pole at \( t = -1 \) in equation (3.16), a new function \( F_y(\mathcal{G}) \) was introduced. From preliminary results this seems like it had worked. Figure 3.11 shows that the \( I_{22k}(u) \) with \( k = 10 \) converged to zero at \( u = -1 \). This figure also displays the convergence of the integration technique. It should be noted that the integrals are evaluated using the same number of integration points as collocation points. With as few as 20 Gauss-Chebyshev integration points, it appears that convergence has been reached. However, it does not appear that 20 collocation points is enough to accurately represent the integral. With 40 and 60 points, the graphs of the integral are superposed on the
Figure 3.11- Convergence of $I_{22k}(u)$ with $k = 10$ comparing Romberg method and various orders of Gauss-Chebyshev integration.

integral using the Romberg method. Therefore, 40 integration and collocation points would sufficiently model the $I_{11k}(u)$ and $I_{22k}(u)$ integrals.

The antisymmetry properties present in the $f_{12}(\zeta)$ and $f_{21}(\zeta)$ kernel functions are still present in the $I_{12k}(u)$ and $I_{21k}(u)$ integrals as shown in Figure 3.12.
Figure 3.12 - Antisymmetry check between the $I_{120}(u)$ and the $I_{210}(u)$ integrals.

Unlike the $I_{11k}(u)$ and $I_{22k}(u)$ integrals, the coupling integrals display convergence problems at $u = -1$. Figure 3.13 shows the Romberg algorithm integral and the Gauss-Chebyshev

Figure 3.13 - Convergence of $I_{120}(u)$ integral.
integration using different orders for the $I_{120}(u)$. The Romberg algorithm makes sense. It does not make sense for the Gauss-Chebyshev to display the singular behavior. Nevertheless, it can be seen that the Gaussian integration is starting to converge as the order of Gauss points is increased from 20 to 60. The spikes close to $u = -1$ for the Romberg algorithm result from the algorithm’s inability to converge at certain points. This shouldn’t be of any concern, since these points are tolerant to within 1%.

As the degree of the Chebyshev polynomial is increased (i.e. as $k$ is increased), the singular behavior at $u = -1$ is still present in the $I_{12k}(u)$ and $I_{21k}(u)$ integrals (Figure 3.14).

**Figure 3.14** - Convergence problems at $u = -1$ for coupling integral, $I_{12k}(u)$.

In both Figures 3.13 and 3.14, for 40 and 60 Gauss points the integrals begin to diverge from the “exact” integration around $u = -0.55$. This behavior is consistent for all degrees of $k$. 

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3.8.1 Endpoint Convergence on $I_{ijk}(u)$ Integrals

The areas of the integrals which raise the most concern for this system are the endpoints. But as can be seen in Figure 3.15 and 3.16, convergence is reached.

**Figure 3.15** - Convergence of the $I_{12k}(u)$ integral for $k = 0, \ldots, 20$ at the crack-tip ($u = 1$).

**Figure 3.16** - Convergence of the $I_{12k}(u)$ and $I_{21k}(u)$ integrals for $k = 0, \ldots, 20$ far from the crack-tip ($u = -1$).
The $I_{12k}(u)$ integral is plotted for many degrees of the Chebyshev polynomial, and it can be seen that at low orders of $k$, there are nonzero values at the crack-tip ($u = 1$). However, the integral decays to zero as the degree $k$ is increased. These plots were done utilizing the Romberg "exact" integration scheme with a specified tolerance of 1%. Therefore, it can be concluded that there is convergence at both endpoints $u = -1$ and $u = 1$, and that the singular behavior displayed in Figure 3.13 for $I_{12k}(u)$ at $u = -1$ is just a product of the Gauss-Chebyshev integration method. Plots for endpoint convergence are given for the other $I_{10k}(u)$ integrals in the Section B.7.3.

3.8.2 Identity Crisis

One other problem identified is when the degree of the Chebyshev polynomial $k$ is equal to the number of Gauss-Chebyshev integration points, $N_{int}$. Figure 3.17 displays this problem.

![Figure 3.17](image)

**Figure 3.17** - Plot of $I_{12k}(u)$ integral with $k = 20$ showing identity problem with Gaussian-quadrature.
As it can be seen for \( k = N_{int} \), the integral is identically zero which is a false result. Figure 3.17 also shows what the integral should look like. This plot was generated with 20 Gauss-Chebyshev quadrature points for integration plotted over the Romberg representation. Obviously the integral is not identically zero. What has happened in Figure 3.17 revolves around an identity conflict. The zeros for the Gauss-Chebyshev quadrature are given by

\[
t_m = \cos \left( \frac{2m - 1}{2N_{int}} \pi \right), \quad m = 1, \ldots, N_{int}
\]

and the Chebyshev polynomial \( T_k(t_m) \) can be represented by

\[
T_k(t) = \cos(k \cdot \text{acos}(t)).
\]  
(3.22)

So, by substituting \( t_m \) into equation (3.22), the following results:

\[
T_k(t_m) = \cos \left( k \cdot \text{acos} \left( \cos \left( \frac{2m - 1}{2N_{int}} \pi \right) \right) \right)
\]

\[
T_k(t_m) = \cos \left( k \left( \frac{2m - 1}{2N_{int}} \pi \right) \right).
\]

Now, by letting \( k = N_{int} \) and substituting into the above equation

\[
T_{k=N_{int}}(t_m) = \cos \left( N_{int} \left( \frac{2m - 1}{2N_{int}} \pi \right) \right)
\]

\[
T_{k=N_{int}}(t_m) = \cos \left( \frac{(2m - 1) \pi}{2} \right), \quad m = 1, \ldots, N_{int}.
\]

the identity crisis is identified. For any value of \( m \), \( T_{k=N_{int}}(t_m) \) is identically zero. This is a false answer as has been shown. Therefore, in order to get the highest degree \( k \) values, a larger integration order will be needed. So, \( N_{int} > k \). Alternatively, another integration method without this identity problem could be used like the Romberg algorithm.
3.9 Chebyshev Series Representation for $h_1(\zeta)$ and $h_2(\zeta)$

In a similar fashion to introducing the $F_i(\zeta)$ functions, the $h_i(\zeta)$ functions are introduced to help remove the pole at $(1 + i)$ through

$$g_i(\zeta) = h_i(\zeta)\left(\frac{1+i}{2}\right) = \frac{h_i(\zeta)}{1+\zeta}$$

(3.23)

A more detailed discussion is given in the section B.7.6.

A Chebyshev-series approximation is used to represent the $h_i(\zeta)$ functions:

$$h_i(\zeta(t)) = \sum_{k=1}^{M} d_{ik} T_{k-1}(t) - \frac{1}{2} d_{i1}, \quad -\zeta = \frac{t-1}{i+1}.$$  

(3.24)

where the coefficients are found from the known kernel functions, $h_i(\zeta)$, and $M$ is suggested as 40. As shown in Figures 3.18 and 3.19, the series are very convergent for $M = 40$.

![Figure 3.18](image)

**Figure 3.18-** Chebyshev coefficients for $h_1(\zeta)$. 

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Figure 3.19 - Chebyshev coefficients for $h_2(\zeta(t))$.

Figures 3.20 and 3.21 plot the real function and the Chebyshev series representation for $h_1(\zeta)$ and $h_2(\zeta)$.

Figure 3.20 - Plot of $h_1(\zeta(t))$. 
Figure 3.21- Plot of $h_2(\zeta(t))$.
Both functions are superposed so $M = 40$ is indeed enough to accurately represent the $h_2(\zeta)$ functions. The $d_{ik}$ coefficients are important later on in the evaluation of the $T$-stress.

3.10 Reduction of the Linear System of Equations
The linear system of equations in $a_y$ equation (3.18) provides $2N$ equations. The additional 2 equations are provided by the asymptotic representation for the loading conditions, $K_i^\infty$.

$$
\sum_{k=0}^{N} (-1)^k a_{ik} = \left(\frac{2}{\pi}\right)^{1/2} \left(1 - \nu_2\right) \frac{(1-\alpha)}{\mu_2} \frac{1}{1 + \alpha} K_i^\infty.
$$

Fleck, Hutchinson and Suo (1991) solved for the $2(N+1)$ unknowns $a_{ik}$ ($i=1,2; k=0,...,N$) by collocating eqn. (3.18) at $N$ Gauss-Legendre points for $u$ in the interval $-1 < u < 1$.

The resulting system to be solved will be of the form

$$
[A][x] = [b]
$$
where \([b]\) is a \(2(N + 1)\) vector containing the known loading conditions (\(i.e. K^e_i\)), \([A]\) is a \(2(N + 1) \times 2(N + 1)\) matrix containing the left hand sides of equations (3.18) and (3.24), and \([x]\) is the vector containing the \(2(N + 1)\) \(a_{ik}\) coefficients to be solved for.

Several problems identified with the convergence of \(I_{jk}(u)\) integrals at \(u = -1\) and for high orders of \(k\) give insight into the original results. In Figure 3.22, the series coefficients \(a_{ik}\) for mode II loading are plotted for \(N = 20\).

![Graph showing series coefficients for mode II loading](image)

**Figure 3.22** - Series coefficients \(a_{ik}\) for mode II loading. This shows no convergence properties for \(N = 20\). As \(N\) is increased, convergence was expected. But again, the results were not convergent.

From Figure 3.22, it appears that there could be “noise” or accumulated error in the system. To check this, a reduced system was employed. Figure 3.23, shows a representation for the \([A]\) matrix. The \([A]\) matrix was programmed with 4 quadrants.
Quadrants 1 and 4 will carry the diagonal terms $-\pi(1 + u_i)U_{i-1}(u_i) + I_{2i}(u_i)$ and quadrants 2 and 3 will contain the coupling integrals, $I_{12k}(u)$ and $I_{21k}(u)$. 

**Figure 3.23**—Reduced [A] matrix 

The figure also shows how $m$ and $k$ are configured for each quadrant. It should be noted that at $m = 0$, $u \equiv 1$ and at $m = N$, $u \equiv -1$.

The cross-hatched area indicates how the system has been reduced. In the reduction, the high order $k$ terms have been left out, and the $I_{yk}(u)$ terms closest to $u = -1$ have been eliminated. Figure 3.24 plots the $a_{ik}$ coefficients for the original $N = 20$ system which has been reduced to $N = 18$. 

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Figure 3.24- Convergence of the reduced series for $a_{1k}$ under Mode II loading conditions.

The result is a very convergent system. So, it has been identified that these terms introduce much accumulated error. Plots of the other series coefficients are given in section B.7.4.
Concluding Remarks

4.1 Highlights of the Problem Areas

One of the many important features in the solution procedure revolved around the use of the FFT to evaluate the Fourier sine and cosine transforms. A drawback to the method is the resulting usable range. In later procedures, data needed to be extracted outside this range (section 3.5.1 and B.6). However, keeping in mind the asymptotic behavior of the $f_0(\zeta)$ kernel functions, they were eventually extrapolated further out in a decaying nature to zero. Thus the useful data range was extended.

It has been determined through the reduction of the linear system where the main error enters the system through the numerical evaluation of the $I_{jk}(u)$ integrals. These are the key to the solution procedure. More specifically, the higher order $k$ terms and the terms closest to $u = -1$ give erroneous results causing error build up (Figure 3.21). When these terms were removed, a convergent system resulted (Figure 3.23).

4.2 Possible Remedies

Since it is now believed that a convergent solution is dependent on the accuracy of the integrals, the solutions will focus on this area.

It was shown that the Romberg integration gave the most convergent results (Figures 3.11, 3.13 and 3.14). So, one solution could be to evaluate the integrals utilizing the Romberg method. This is not very computationally efficient, but the error in the results would most likely be reduced. However, several things need to be kept in mind. Utilizing the Romberg method, the $I_{12k}(u)$ and $I_{21k}(u)$ integrals seem to behave fairly well. But as
seen in figure 3.13, there are some local convergence problems. Both the $I_{12k}(u)$ and $I_{21k}(u)$ integrals for a certain degree $k$ and collocation mesh on $u$ could be generated and stored. An outside program like Microsoft Excel could then be used to help smooth the stored curves. In the same manner, the $I_{11k}(u)$ and $I_{22k}(u)$ integrals could also be generated and stored. Unfortunately some of the points closest to $u = -1$ will not converge (Figure 3.11) for either integral, but they do converge at $u = -1$. So, these integrals could also be exported to another program and points could be added to complete the mesh on $u$. These integrals will not be as accurate as the $I_{12k}(u)$ and $I_{21k}(u)$ integrals.

On the other hand, a hybrid approach could be tried. The $I_{12k}(u)$ and $I_{21k}(u)$ integrals could still be evaluated by the Romberg method, but the $I_{11k}(u)$ and $I_{22k}(u)$ integrals could be evaluated utilizing the Gauss-Chebyshev integration. Figure 3.11 shows that the Romberg method superposes the Gauss-Chebyshev integration for 40 and 60 quadrature points at $k = 10$. This behavior is consistent for all degrees of $k$.

4.3 Future Work

The collocation method might not be the best approach. Several other possible methods which seem to yield experimentally consistent results are the boundary element method, finite difference method, and the finite element method. These will be discussed in the subsequent sections.

4.3.1 Boundary Element Method

Mear and Newman (1995) utilize a special boundary element technique to analyze cracks in two-dimensional isotropic and anisotropic solids. Their approach starts by developing the boundary integral equations for the problem which consist of the conventional boundary terms plus an additional term for the distribution of dislocations along the crack line. They use standard boundary element techniques to evaluate the solution, as well as,
novel approaches for the crack line integrals. With the graphical user interface, "FADD: Fracture Analysis by Distributed Dislocations," very accurate numerical results were obtained along with the ability to predict and simulate fatigue crack growth.

In studies done by Heo and Gerstle (1995), a boundary force method was used to determine the mode II fracture toughness of materials in a single edge-notched shear beam test specimen. Similar to procedures done in the work presented so far, to evaluate the singular integrals some of the unknowns were removed to arrive at convergent solutions. In this study, they also used the boundary element method to determine the mode I and mode II fracture toughness of an acrylic material utilizing an edge-notched three-point bend specimen and the single edge-notched shear beam, respectively. The numerical results were in good agreement with experimental results.

In recent work done by Ameen and Raghuprasad (1994), a hybrid technique is used to model both internal and edge cracks. A direct boundary element method is utilized to model the finite domain of the body; whereas, the displacement discontinuity elements represent the crack. One of the main advantages of this hybrid technique over other techniques like finite element approaches is the ability to model crack propagation without having to redefine the mesh during each step of crack advancement. In essence for each step of crack extension, only one displacement discontinuity element is added.

4.3.2 Finite Difference Method

The Finite Difference Method (FDM) has been utilized as an effective method for fracture analysis of some standard specimens. For example, Shmuely (1977) used FDM to perform a dynamic fracture analysis on a DCB specimen utilizing two-dimensional displacements and the dynamic relaxation method.
In another study done by Yuanhan, Zaihua and Jian (1993), a fracture analysis including static, dynamic and crack arrest in a stiffened panel with an edge crack is performed. The FDM was employed to calculate the stress intensity factors, crack propagation lengths and velocities. The advantages of the method were little preparation and calculation time for very accurate results.

Bazant and Beissel (1994) utilized a finite difference solution scheme to adapt an integral equation formulation with an asymptotic series solution for cohesive fracture. This is directly applicable to the current research in this thesis. Like this thesis, an asymptotic series expansion was employed to solve the integral equation resulting from a crack analysis. Bazant and Beissel went further by developing a method for solving general integro-differential equations using finite differences. From this method, the crack path is subdivided into small intervals which ultimately reduces the problem to a system of nonlinear algebraic equations to be solved by a nonlinear optimization algorithm.

4.3.3 Finite Element Method

Finite element approaches are also good in modeling cracks. However, it is difficult to model the singularity around the crack-tip. A very fine mesh is utilized to model the crack-tip singularity but this results in large amounts of computer memory, round off errors, and in general converges too slowly.

Leung and Su (1994) eliminate the singularity from the domain by the fractal two level finite element method. In this method, an infinitesimal mesh is generated around the crack-tip. The resulting large number of degrees of freedom are transformed to a small set of generalized coordinates. The advantages to this method are increased computational efficiency and highly accurate results of the stress intensity factors and the stresses.
In a study by Yong-Li (1994), FEM was used for a crack normal to the bimaterial interface. He introduces a special quadratic and cubic zero width singular interface finite element to model the singular stress conditions present at the crack-tip. The numerical results obtained show that the special elements adequately model the perpendicular interface crack.

Finally, Yoshibash and Schiff (1993) introduce a superelement to deal with two-dimensional singular boundary value problems in linear elasticity. Instead of using a very fine mesh over the singular region, which results in a large number of degrees of freedom, or even incorporating difficult to use singular elements, they develop a singular-superelement (SSE) which can be used along with linear and quadratic elements to model a whole domain. The SSE incorporates the analytic form of the solution near the singularity and smoothing functions to blend the functions near the internal region and those used over the remainder of the domain. As a result, very accurate and rapidly converging solutions are obtained.
Appendix

Problem 1 (Isolated Edge Dislocation)

A.0 Introduction

A detailed overview of the solution procedure as already discussed in previous chapters will be given in the Appendices A and B. The solution follows directly from Fleck, Hutchinson, and Suo (1991) and Ionita (1994).

An edge dislocation (Figure A.1) with strengths $b_1$ and $b_2$ is introduced to the origin of the adhesive layer of the elasticity problem previously discussed.

![Figure A.1 - Edge dislocation in an adhesive layer.](image)

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The dislocation is located by the geometric parameter $c/H$ where $c$ is the distance above the lower interface, and $H$ is the thickness of the adhesive layer. Two problems are posed independently and then their linear solutions superposed. The first problem, which will be discussed in Appendix A, consists of an isolated dislocation in a full plane made of the adhesive material. The two half planes, $y > d$ and $y < -c$, are then transformed from material 2 into material 1 which will induce a displacement mismatch, $\Delta u(x)$, at the upper interface ($y = d$) and the lower interface ($y = -c$). The stresses will remain fixed everywhere. The second problem, which will be discussed in Appendix B, will contain a dislocation free adhesive layer of height $H$ sandwiched between two half-planes of a stiffer substrate. At each interface, there will be a displacement mismatch, $-\Delta u(x)$, from problem 1.

A.1 Complex Potential Functions

The expressions for the Muskhelishvili potentials and the Burger vectors are given below.

$$\varphi(z) = A \ln(z), \quad \Omega(z) = \overline{A} \ln(z)$$

$$\varphi'(z) = \frac{A}{z}, \quad \Omega'(z) = \frac{\overline{A}}{z} \quad \text{and} \quad \varphi''(z) = \frac{-A}{z^2}$$

where

$$z = x + i x_2 \quad \text{and} \quad A = \frac{\mu_2}{4\pi(1 - \nu_2)} \left( b_2 - ib_1 \right) \quad (A.1)$$

The stresses are given by

$$\sigma_{11} + \sigma_{22} = 2 \left( \varphi'(z) + \overline{\varphi'(z)} \right) \quad (A.2a)$$

$$\sigma_{22} - \sigma_{11} + i\sigma_{12} = 2 \left[ (\overline{z} - z)\varphi''(z) + \Omega'(z) - \varphi'(z) \right]. \quad (A.2b)$$

In material 2, the complex displacement, $u(z) = u_1 + iu_2$, has the general expression

$$u_{\text{material #2}}(z) = \frac{1}{2\mu_2} \left[ (3 - 4\nu_2)\varphi(z) + (\overline{z} - z)\varphi'(z) + \Omega(z) \right] \quad (A.3)$$

and for material 1 the same potentials are used, but the only quantities to change are $\mu_2$ and $\nu_2$ into $\mu_1$ and $\nu_1$. 

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\[ u_{\text{material} \#1}(z) = \frac{1}{2 \mu_1} \left[ (3 - 4 \nu_1) \varphi(z) + (\bar{z} - z)\bar{\varphi}'(z) - \bar{\Omega}(z) \right] \] (A.4)

Hence, by letting the material 2 transform into material 1, the displacement jump at the interfaces is defined as

\[ \Delta u(z) = u_{\text{material} \#1} - u_{\text{material} \#2} \] (A.5)

By substituting equations (A.3) and (A.4) into (A.5), the general expression for the displacement jump is given as

\[ \Delta u(z) = \left( \frac{3 - 4 \nu_1}{2 \mu_1} - \frac{3 - 4 \nu_2}{2 \mu_2} \right) \varphi(z) + \left( \frac{1}{2 \mu_1} - \frac{1}{2 \mu_2} \right) [(\bar{z} - z)\bar{\varphi}'(z) - \bar{\Omega}(z)] \] (A.6)

The constants in equation (A.6) can be related to the elastic mismatch parameters \( \alpha \) and \( \beta \).

\[ \left( \frac{3 - 4 \nu_1}{2 \mu_1} - \frac{3 - 4 \nu_2}{2 \mu_2} \right) = \frac{4(1 - \nu_2)}{2 \mu_2} \left[ -\left( \frac{\alpha + \beta}{1 + \alpha} \right) \right] \] (A.7)

and

\[ \left( \frac{1}{2 \mu_1} - \frac{1}{2 \mu_2} \right) = -\left( \frac{\mu_1}{2 \mu_1 \mu_2} - \frac{\mu_2}{2 \mu_1 \mu_2} \right) = -\frac{4}{2 \mu_2} \left[ \frac{1}{4 \mu_1} (\mu_1 - \mu_2) \right] = \frac{4(1 - \nu_2)}{2 \mu_2} \left[ -\left( \frac{\alpha + \beta}{1 + \alpha} \right) \right] \] (A.8)

Substitute these relations (A.7) and (A.8) into (A.6)

\[ \Delta u(z) = \frac{-4(1 - \nu_2)}{2 \mu_2} \frac{1}{1 + \alpha} \left[ (\alpha + \beta)\varphi(z) + (\alpha - \beta)((\bar{z} - z)\bar{\varphi}'(z) - \bar{\Omega}(z)) \right] \] (A.9)

To simplify the work, a new constant is introduced into equation (A.9) as such

\[ c_{\text{on}} = \frac{-4(1 - \nu_2)}{2 \mu_2} \frac{1}{1 + \alpha} \] (A.10)

and the resulting simplification is

\[ \Delta u(z) = c_{\text{on}} \left[ (\alpha + \beta)\varphi(z) + (\alpha - \beta)((\bar{z} - z)\bar{\varphi}'(z) - \bar{\Omega}(z)) \right] . \] (A.11)

From equation (A.5), the displacement gradient mismatch is defined as
\[
\Delta' u(z) = \frac{\partial \Delta u_1(z)}{\partial x_1} + i \frac{\partial \Delta u_2(z)}{\partial x_1} \tag{A.12}
\]

A general expression for \(\Delta' u(z)\) is shown below

\[
\Delta' u(z) = \text{con}_1 \left[ (\alpha + \beta) \varphi'(z) + (\alpha - \beta) \left( \frac{\varphi''(z)}{\varphi(z)} - \Omega(z) \right) \right] \tag{A.13}
\]

Upon substitution of the Muskhelishvili potentials (A.1) into equation (A.13), the following expression is obtained:

\[
\Delta' u(z) = \text{con}_1 \left[ (\alpha + \beta) \frac{A}{x_1 + ix_2} + (\alpha - \beta) \left( 2ix_2 \frac{A}{(x_1 + ix_2)^2} - \frac{A}{x_1 + ix_2} \right) \right] \tag{A.14}
\]

The substitution of the expression for the burger vector A into equation (A.14) yields

\[
\Delta' u(z) = \text{con}_1 \left[ (\alpha + \beta) \frac{b_2 - ib_1}{x_1 + ix_2} + (\alpha - \beta) \left( 2ix_2 (b_2 - ib_1) \frac{x_2}{(x_1^2 - x_2^2)^2 + 4x_1^2 x_2^2} \right) \right.
+ \left. 2ix_1 \frac{x_2}{(x_1^2 - x_2^2)^2 + 4x_1^2 x_2^2} \right] \tag{A.15}
\]

To help further reduce the equations, a new constant, \(\text{con}_2\), can be defined as:

\[
\text{con}_2 = \text{con}_1 \frac{\mu_2}{4\pi(1 - \nu_2)} = \frac{-4(1 - \nu_2)}{2\mu_2} \frac{1}{1 + \alpha} \frac{\mu_2}{4\pi(1 - \nu_2)} = \frac{-1}{2\pi(1 + \alpha)} \tag{A.16}
\]

After equation (A.16) is substituted into equation (A.15), the expression for \(\Delta' u(z)\) is reduced and can be separated into the \(b_1\) and \(b_2\) components.

**The \(b_1\) component**

\[
\frac{-1}{2\pi(1 + \alpha)} \left[ \frac{-i}{x_1 + ix_2} \alpha + \beta \left( \frac{x_2}{(x_1^2 + x_2^2)^2} + 2ix_1 \frac{x_2}{(x_1^2 + x_2^2)^2} + \frac{i}{x_1 - ix_2} \right) \right] \tag{A.17}
\]
The $b_2$ component

\[
-\frac{1}{2\pi(1+\alpha)} \left[ \frac{\alpha + \beta}{x_1 + i x_2} + (\alpha - \beta) \left[ 2ix_2 \left( \frac{(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^2} + 2ix_1 \frac{x_2}{(x_1^2 + x_2^2)^2} \right) - \frac{1}{x_1 - ix_2} \right] \right]
\]

(A.18)

Now, separate equations (A.17) and (A.18) into their real and imaginary parts to get $\partial \Delta u_1 / \partial x_1$ and $\partial \Delta u_2 / \partial x_1$.

**REAL PARTS**

$b_2$ component

\[
-\frac{1}{2\pi(1+\alpha)} \text{Re} \left[ -i \frac{\alpha + \beta}{x_1 + i x_2} + (\alpha - \beta) \left[ -2x_2 \left( \frac{(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^2} + 2ix_1 \frac{x_2}{(x_1^2 + x_2^2)^2} \right) + i \frac{1}{x_1 - ix_2} \right] \right]
\]

yields

\[
-\frac{1}{2\pi(1+\alpha)} \left[ -(\alpha + \beta) \frac{x_2}{(x_1^2 + x_2^2)} + (\alpha - \beta) \left[ -2x_2 \left( \frac{(x_1^2 - x_2^2)}{(x_1^2 + x_2^2)^2} - \frac{x_2}{(x_1^2 + x_2^2)^2} \right) \right] \right]
\]

which simplifies to

\[
\frac{x_2}{\pi(1+\alpha)} \frac{2\alpha x_1^2 + \beta x_1^2 - \beta x_2^2}{(x_1^2 + x_2^2)^2}
\]

(A.19)

For comparison, the corresponding expression from Fleck et al.'s (1991) work is

\[
\frac{1}{\pi(1+\alpha)} x_2 \frac{\left( (2\alpha - \beta) x_1^2 + 2(\alpha - \beta) x_2^2 \right)}{(x_1^2 + x_2^2)^2}
\]

(A.20)

which is identical to equation (A.19) when the numerator is expanded. Therefore, the real part of the $b_1$ component is verified.
\[-\frac{1}{2\pi(1+\alpha)} \text{Re} \left[ \frac{\alpha + \beta}{x_1 + ix_2} + (\alpha - \beta) \left[ 2ix_2 \left( \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2} + 2ix_1 \frac{x_2}{(x_1^2 + x_2^2)^2} \right) - \frac{1}{x_1 - ix_2} \right] \right] \]
yields
\[-\frac{1}{2\pi(1+\alpha)} \left[ (\alpha + \beta) \frac{x_1}{(x_1^2 + x_2^2)} + (\alpha - \beta) \left[ -4x_1^2 \frac{x_1}{(x_1^2 + x_2^2)^2} - \frac{x_1}{(x_1^2 + x_2^2)} \right] \right] \]
which simplifies to
\[\frac{x_1}{\pi(1+\alpha)} \frac{(2\alpha^2 - \beta x_1^2 - 3\beta x_1^2)}{(x_1^2 + x_2^2)^2} \tag{A.21} \]

For comparison, the corresponding expression from Fleck, Hutchinson and Suo’s (1991) work is
\[\frac{x_1}{\pi(1+\alpha)} \frac{(-\beta(x_1^2 + x_2^2) + 2(\alpha - \beta)x_2^2)}{(x_1^2 + x_2^2)^2} \tag{A.22} \]
which is identical to equation (A.21) when the numerator is expanded. Therefore, the real part of the \(b_2\) component is verified.

**IMAGINARY PARTS**

**\(b_2\) component**

\[-\frac{1}{2\pi(1+\alpha)} \text{Im} \left[ -i \frac{\alpha + \beta}{x_1 + ix_2} + (\alpha - \beta) \left[ -2x_2 \left( \frac{x_1^2 - x_2^2}{(x_1^2 + x_2^2)^2} + 2ix_1 \frac{x_2}{(x_1^2 + x_2^2)^2} \right) + \frac{i}{x_1 - ix_2} \right] \right] \]
yields
\[-\frac{1}{2\pi(1+\alpha)} \left[ (\alpha + \beta) \frac{x_1}{(x_1^2 + x_2^2)} + (\alpha - \beta) \left[ -4x_1^2 \frac{x_1}{(x_1^2 + x_2^2)^2} + \frac{x_1}{(x_1^2 + x_2^2)^2} \right] \right] \]
which simplifies to
\[
\frac{x_i}{\pi(1 + \alpha)} \left[ \frac{(2\alpha x_i^2 - \beta x_i^2 + \beta x_i^2)}{(x_i^2 + x_i^2)^2} \right]
\]

(A.23)

For comparison, the corresponding expression from Fleck, Hutchinson and Suo's (1991) work is

\[
\frac{x_i}{\pi(1 + \alpha)} \left[ \frac{\beta(x_i^2 + x_i^2) + 2(\alpha - \beta)x_i^2}{(x_i^2 + x_i^2)^2} \right]
\]

(A.24)

which is identical to equation (A.23) when the numerator is expanded. Therefore, the imaginary part of the \(b_1\) component is verified.

**b_2 component**

\[-\frac{1}{2\pi(1 + \alpha)} \text{Im} \left[ \frac{\alpha + \beta}{x_i + ix_i^2} + (\alpha - \beta) \left[ 2ix_i \left( \frac{x_i^2 - x_2^2}{(x_i^2 + x_i^2)^2} + 2ix_i \frac{x_i}{(x_i^2 + x_i^2)^2} \right) - \frac{1}{x_i - ix_i^2} \right] \right] \]

where evaluation over the complex plane yields

\[-\frac{1}{2\pi(1 + \alpha)} \left[ (\alpha + \beta) \frac{x_i^2}{(x_i^2 + x_i^2)^2} + (\alpha - \beta) \left[ 2x_i \left( \frac{x_i^2 - x_2^2}{(x_i^2 + x_i^2)^2} - \frac{x_2}{(x_i^2 + x_i^2)^2} \right) \right] \right] \]

which simplifies to

\[
\frac{x_2}{\pi(1 + \alpha)} \left[ \frac{2\alpha x_2^2 - \beta x_2^2 + \beta x_2^2}{(x_2^2 + x_2^2)^2} \right]
\]

(A.25)

For comparison, the corresponding expression from Fleck, Hutchinson and Suo's (1991) work is

\[
\frac{x_2}{\pi(1 + \alpha)} \left[ \frac{\beta(x_i^2 + x_i^2) + 2(\alpha - \beta)x_2^2}{(x_i^2 + x_i^2)^2} \right]
\]

(A.26)

which is identical to equation (A.25) when the numerator is expanded. Therefore, the imaginary part of the \(b_2\) component is verified.

The final expressions for \(\partial u_i / \partial x_i\) and \(\partial u_2 / \partial x_i\) are
\[
\frac{\partial \Delta u_1}{\partial x_1} = \frac{b_1}{\pi(1 + \alpha)} \left[ (2 \alpha - \beta) \frac{x_2}{r^2} - 2(\alpha - \beta) \frac{x_1^3}{r^4} \right] + \frac{b_2}{\pi(1 + \alpha)} \left[ -\beta \frac{x_1}{r^2} + 2(\alpha - \beta) \frac{x_1 x_2^2}{r^4} \right]
\]

(A.27a)

\[
\frac{\partial \Delta u_2}{\partial x_1} = \frac{b_1}{\pi(1 + \alpha)} \left[ \beta \frac{x_1}{r^2} + 2(\alpha - \beta) \frac{x_1 x_2^2}{r^4} \right] + \frac{b_2}{\pi(1 + \alpha)} \left[ \beta \frac{x_2}{r^2} + 2(\alpha - \beta) \frac{x_2^3}{r^4} \right]
\]

(A.27b)

where \( r^2 = x_1^2 + x_2^2 \).

By superposing the solution from problem 2, this displacement gradient mismatch at the interfaces will be canceled.
Problem 2 (Adhesive Layer without Edge Inclusion)

B.0 Introduction

An adhesive layer of thickness $H$ is bounded by two half-planes of a stiffer substrate. There are no edge dislocations in the adhesive layer. However, a displacement gradient mismatch of equal magnitude but opposite sign in equation (A.27) from problem 1 is imposed at the interfaces. The subsequent derivation follows Ionita’s (1994) work quoting Fleck, Hutchinson, and Suo (1991).

B.1 $b_1$ Solution ($b_2 = 0$)

The complex stress potentials satisfy

$$\nabla^4 U = 0 \quad \nabla^2 X = 0 \quad \frac{\partial^2 X}{\partial x_1 \partial x_2} = \frac{1}{4} \nabla^2 U$$

(B.1)

where the stresses and displacements are given by

$$\sigma_{11} = \frac{\partial^2 U}{\partial x_2^2} \quad \sigma_{22} = \frac{\partial^2 U}{\partial x_1^2} \quad \sigma_{12} = -\frac{\partial^2 U}{\partial x_1 \partial x_2}$$

(B.2)

$$2\mu u_1 = -\frac{\partial U}{\partial x_1} + 4(1 - v) \frac{\partial X}{\partial x_2} \quad 2\mu u_2 = -\frac{\partial U}{\partial x_2} + 4(1 - v) \frac{\partial X}{\partial x_1}$$

(B.3)

This notation became confusing in Fleck, Hutchinson, and Suo (1991), so we use $x_1 = x$ and $x_2 = y$ for the remaining parts of the solution.

B.1.1 Derivation of the Complex Potentials for $b_1$

Each layer of the problem can be separated into a symmetric and an antisymmetric part. For the $b_1$ solution, the potential $U(x, y)$ is symmetric and $X(x, y)$ is antisymmetric. When switching to the $b_2$ solution, the antisymmetry properties are retained. However, $U(x, y)$ is now antisymmetric and $X(x, y)$ is symmetric.
The symbolic derivation is done in the Fourier plane. Keeping this in mind, the appropriate Fourier transforms $\tilde{U}^s$ and $\tilde{X}^a$ are given by

$$\mathcal{F}(U^s) = \tilde{U}^s = \frac{2}{\pi} \int_0^\infty U^s \cos \lambda x dx \quad \text{and} \quad \mathcal{F}(X^a) = \tilde{X}^a = \frac{2}{\pi} \int_0^\infty X^a \sin \lambda x dx. \quad (B.4)$$

To obtain the characteristic equation for $\tilde{U}^s$, substitute the biharmonic equation (B.1) into the appropriate transform in (B.4).

$$\mathcal{F}((\nabla^4 U^s) = \lambda^4 \tilde{U}^s - 2\lambda^2 \frac{\partial^2 \tilde{U}^s}{\partial y^2} + \frac{\partial^4 \tilde{U}^s}{\partial y^4} = 0$$

The characteristic equation is:

$$\lambda^4 - 2\lambda^2 r^2 + r^4 = 0$$

$$(r - \lambda)^2 (r + \lambda)^2 = 0$$

where the roots are $r_{1,2} = -\lambda$ and $r_{3,4} = \lambda$.

Therefore, the solution for $\tilde{U}^s$ is of the form

$$\tilde{U}^s = (A_1 + A_2 y)e^{-\lambda y} + (A_3 + A_4 y)e^{\lambda y} \quad (B.5)$$

where $A_i$ are functions of $\lambda$. These Fourier coefficients are now chosen to be of the form

$$A_1 \to \frac{A_1}{\lambda^2}, \quad A_2 \to \frac{A_2}{\lambda}, \quad A_3 \to \frac{A_3}{\lambda^2}, \quad A_4 \to \frac{A_4}{\lambda}$$

and substituted into equation (B.5) to get the following expression for $\tilde{U}^s$:

$$\tilde{U}^s = \left(\frac{A_1}{\lambda^2} + \frac{A_2}{\lambda} y\right)e^{-\lambda y} + \left(\frac{A_3}{\lambda^2} + \frac{A_4}{\lambda} y\right)e^{\lambda y}. \quad (B.6)$$

This expression can be inverse transformed to get the corresponding real function $U^s$

$$U^s(x, y) = \Re \left[ \int_0^\infty \left(\frac{A_1}{\lambda^2} + \frac{A_2}{\lambda} y\right)e^{-\lambda y} + \left(\frac{A_3}{\lambda^2} + \frac{A_4}{\lambda} y\right)e^{\lambda y} \right] \cos \lambda x dx$$

where the Fourier coefficients $A_i$ are designated as $C_i$ in top layer, $D_i$ in the middle adhesive layer and $E_i$ in the lower substrate layer. Since the stresses derived from $U^s(x, y)$
and $X'(x, y)$ remain finite everywhere, as $y \to \pm \infty$ the coefficients $C_3$, $C_4$, $E_1$, and $E_2$ will all equal zero.

This same procedure is followed to obtain the representation for $X^a$. To acquire the characteristic equation for $\tilde{X}^a$, substitute the harmonic equation (B.1) into the appropriate transform in (B.4).

$$\mathcal{F}_s (\nabla^2 X^a) = -\lambda^2 \tilde{X}^a + \frac{\partial^2 \tilde{X}^a}{\partial y^2} = 0$$

The characteristic equation is:

$$r^2 - \lambda^2 = 0$$

$$(r - \lambda)(r + \lambda) = 0$$

where the roots are $r_1 = -\lambda$ and $r_3 = \lambda$.

Therefore, the solution for $\tilde{X}^a$ is chosen to be

$$\tilde{X}^a = b_1\left(R e^{-\lambda y} + P_2 e^{\lambda y}\right)$$  \hspace{1cm} (B.7)

where $P_i$ are functions of $\lambda$.

Now, express $\tilde{X}^a$ in the real plane

$$X^a(x, y) = b_1 \int_0^\infty \left(R e^{-\lambda y} + P_2 e^{\lambda y}\right) \sin \lambda x d\lambda$$  \hspace{1cm} (B.8)

This still has to satisfy the last equation in (B.1) to find the relation between $P_i$ and $A_i$.

$$\frac{\partial^2 X^a}{\partial x \partial y} = b_1 \int_0^\infty \lambda^2 \left(-R e^{-\lambda y} + P_2 e^{\lambda y}\right) \cos \lambda x d\lambda$$  \hspace{1cm} (B.9)

and $\nabla^2 U^s = \frac{\partial^2 U^s}{\partial x^2} + \frac{\partial^2 U^s}{\partial y^2}$

where

$$\frac{\partial^2 U^s}{\partial x^2} = -b_1 \int_0^\infty \left[(A_1 + A_2 \lambda y)e^{-\lambda y} + (A_3 + A_4 \lambda y)e^{\lambda y}\right] \cos \lambda x d\lambda$$

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\[
\frac{\partial^2 U^s}{\partial y^2} = b \int_0^\infty \left[ (A_1 + A_2(\lambda y - 2))e^{-\lambda y} + (A_3 + A_4(\lambda y + 2))e^{i\lambda y} \right] \cos \lambda x d\lambda
\]

so that

\[
\nabla^2 U^s = b \int_0^\infty \left( -2A_2 e^{-\lambda y} + 2A_4 e^{i\lambda y} \right) \cos \lambda x d\lambda . \tag{B.10}
\]

Now recalling the relation from equation (B.1)

\[
\frac{\partial^2 X}{\partial x_1 \partial x_2} = \frac{1}{4} \nabla^2 U
\]

and substituting into this equation with equations (B.9) and (B.10) will yield the following equation:

\[
\int_0^\infty \left[ \left( -\lambda^2 P_1 + \frac{A_2}{2} \right) e^{-\lambda y} + \left( \lambda^2 P_2 - \frac{A_4}{2} \right) e^{i\lambda y} \right] \cos \lambda x d\lambda = 0 \tag{B.11}
\]

The following relations can be made from equation (B.11):

\[
y \to -\infty \Rightarrow -\lambda^2 P_1 + \frac{A_2}{2} = 0 \Rightarrow P_1 = \frac{A_2}{2\lambda^2} \]

\[
y \to +\infty \Rightarrow \lambda^2 P_2 - \frac{A_4}{2} = 0 \Rightarrow P_2 = \frac{A_4}{2\lambda^2} \]

Finally, by substituting these relations into equations (B.7) and (B.8), the expressions for \(\tilde{X}^a\) and \(X^a\) are obtained

\[
\tilde{X}^a = \frac{1}{2\lambda^2} \left[ A_2 e^{-\lambda y} + A_4 e^{i\lambda y} \right] \tag{B.12}
\]

\[
X^a(x, y) = b \int_0^\infty \frac{1}{2\lambda^2} \left( A_2 e^{-\lambda y} + A_4 e^{i\lambda y} \right) \sin \lambda x d\lambda . \tag{B.13}
\]

### B.1.2 Functions \(U^s\) and \(X^a\) Expressed in Terms of Individual Layers

In layer 1 of material 1 (\(y \geq a\)):

\[
\tilde{U}^s = b \left( \frac{C_1}{\lambda^2} + \frac{C_2}{\lambda} \right) e^{-\lambda y}
\]
\[ \tilde{X}^a = b_1 \frac{C_2}{2 \lambda^2} e^{-\lambda y} \]

\[ U^s(x, y) = b_1 \int_0^\infty \left( C_1 \frac{1}{\lambda^2} + C_2 \frac{y}{\lambda} \right) e^{-\lambda y} \cos \lambda x d\lambda \]

\[ X^a(x, y) = b_1 \int_0^\infty \frac{C_2}{2 \lambda^2} e^{-\lambda y} \sin \lambda x d\lambda \]  

(B.14)

In layer 2 of material 2 (\(-c \leq y \leq d\)):

\[ \tilde{U}^s = b_1 \left[ \left( D_1 \frac{1}{\lambda^2} + D_2 \frac{y}{\lambda} \right) e^{-\lambda y} + \left( \frac{D_3}{\lambda^2} + \frac{D_4}{\lambda} y \right) e^{\lambda y} \right] \]

\[ \tilde{X}^a = b_1 \frac{1}{2 \lambda^2} \left( D_2 e^{-\lambda y} + D_4 e^{\lambda y} \right) \]

\[ U^s(x, y) = b_1 \int_0^\infty \left[ \left( \frac{D_1}{\lambda^2} + \frac{D_2}{\lambda} y \right) e^{-\lambda y} + \left( \frac{D_3}{\lambda^2} + \frac{D_4}{\lambda} y \right) e^{\lambda y} \right] \cos \lambda x d\lambda \]

\[ X^a(x, y) = b_1 \int_0^\infty \frac{1}{2 \lambda^2} \left( D_2 e^{-\lambda y} + D_4 e^{\lambda y} \right) \sin \lambda x d\lambda \]

(B.15)

In layer 3 of material 1 (\(y \leq -c\)):

\[ \tilde{U}^s = b_1 \left( E_3 \frac{1}{\lambda^2} + E_4 \frac{y}{\lambda} \right) e^{\lambda y} \]

\[ \tilde{X}^a = b_1 \frac{E_4}{2 \lambda^2} e^{\lambda y} \]

\[ U^s(x, y) = b_1 \int_0^\infty \left( E_3 \frac{1}{\lambda^2} + E_4 \frac{y}{\lambda} \right) e^{\lambda y} \cos \lambda x d\lambda \]

\[ X^a(x, y) = b_1 \int_0^\infty \frac{E_4}{2 \lambda^2} e^{\lambda y} \sin \lambda x d\lambda \]

(B.16)
B.1.3 Fourier Transform of the Tractions and the Displacement Gradients

The following two sections will give general expressions for the tractions and displacement gradients in the Fourier domain for the \( b_1 \) solution. It is assumed that the stress potentials are continuous across the boundaries \( y = d \) and \( y = -c \).

**B.1.3.1 Tractions**

The general expression for the transform of the tractions in any layer is given by equations (B.17) and (B.18)

\[
\tilde{\mathcal{F}}(\sigma_{22}) = \tilde{\sigma}_{22} = -\lambda^2 \tilde{U}^x \\
\tilde{\mathcal{F}}(\sigma_{12}) = \tilde{\sigma}_{12} = \lambda \frac{\tilde{U}^x}{\partial y}
\]

\[
\tilde{\sigma}_{22} = b_1 \left[ (-A_1 - A_2 \lambda y) e^{-\lambda y} + (-A_3 - A_4 \lambda y) e^{\lambda y} \right] \\
\tilde{\sigma}_{12} = b_1 \left[ (-A_1 + (1 - \lambda y) A_2) e^{-\lambda y} + \left[ A_3 + (1 + \lambda y) A_4 \right] e^{\lambda y} \right].
\]  

**B.1.3.2 Displacement Gradient Mismatch**

The following derivation will result in the general expressions for the displacement gradient mismatch.

\[
2 \mu \mu_1 = -\frac{\partial U}{\partial x} + 4(1 - \nu) \frac{\partial X}{\partial y} \\
2 \mu \mu_2 = -\frac{\partial U}{\partial y} + 4(1 - \nu) \frac{\partial X}{\partial x}
\]

\[
\frac{\partial u_1}{\partial x} = -\frac{1}{2 \mu} \frac{\partial^2 U}{\partial x^2} + \frac{2(1 - \nu)}{\mu} \frac{\partial^2 X}{\partial x \partial y} \\
\frac{\partial u_2}{\partial x} = -\frac{1}{2 \mu} \frac{\partial^2 U}{\partial x \partial y} + \frac{2(1 - \nu)}{\mu} \frac{\partial^2 X}{\partial x^2}
\]

Take the appropriate Fourier transform

\[
\tilde{\mathcal{F}} \left( \frac{\partial u_1}{\partial x} \right) = \tilde{u}_{1,1} = \frac{1}{2 \mu} \lambda^2 \tilde{U}^x + \frac{2(1 - \nu)}{\mu} \lambda \frac{\tilde{X}^x}{\partial y} \\
\tilde{\mathcal{F}} \left( \frac{\partial u_2}{\partial x} \right) = \tilde{u}_{2,1} = \frac{1}{2 \mu} \lambda \frac{\tilde{U}^x}{\partial y} - \frac{2(1 - \nu)}{\mu} \lambda^2 \tilde{X}^x
\]
\[
\ddot{u}_{1,1} = \frac{b_1}{2\mu_1} \left[ (A_1 + A_2 \lambda y) e^{-\lambda y} + (A_3 + A_4 \lambda y) e^{\lambda y} \right] + \frac{b_1(1 - \nu)}{\mu} \left[ -A_2 e^{-\lambda y} + A_4 e^{\lambda y} \right]
\]

\[
\ddot{u}_{2,1} = \frac{b_1}{2\mu} \left\{ -A_1 + (1 - \lambda y) A_2 \right\} e^{-\lambda y} + \left[ A_3 + (1 + \lambda y) A_4 \right] e^{\lambda y} \} - \frac{b_1}{\mu} \left[ A_2 e^{-\lambda y} + A_4 e^{\lambda y} \right]
\]

(B.20)

The displacement gradient mismatch is given by

\[
\frac{\partial \Delta u}{\partial \hat{x}} = \left( \frac{\partial u_1}{\partial \hat{x}} \right)_{\text{material } 1} - \left( \frac{\partial u_1}{\partial \hat{x}} \right)_{\text{material } 2}
\]

The Fourier transform of the displacement gradient mismatch follows below

\[
\Delta \ddot{u}_{1,1} = b_1 \left\{ \left( \frac{1}{2\mu_1} - \frac{1}{2\mu_2} \right) \left[ (A_1 + A_2 \lambda y) e^{-\lambda y} + (A_3 + A_4 \lambda y) e^{\lambda y} \right] \right\} + \ldots
\]

\[
\left( \frac{2(1 - \nu_1)}{\mu_1} - \frac{2(1 - \nu_2)}{\mu_1} \right) \frac{1}{2} \left[ -A_2 e^{-\lambda y} + A_4 e^{\lambda y} \right]
\]

\[
\Delta \ddot{u}_{2,1} = b_1 \left\{ \left( \frac{1}{2\mu_1} - \frac{1}{2\mu_2} \right) \left[ -A_1 + (1 - \lambda y) A_2 \lambda y e^{-\lambda y} + (A_3 + (1 + \lambda y) A_4 \lambda y e^{\lambda y} \right] \right\} + \ldots
\]

\[
\left( \frac{2(1 - \nu_1)}{\mu_1} - \frac{2(1 - \nu_2)}{\mu_1} \right) \frac{1}{2} \left[ A_2 e^{-\lambda y} + A_4 e^{\lambda y} \right]
\]

factoring out \( \frac{2(1 - \nu_1)}{\mu_1} \) yields:

\[
\Delta \ddot{u}_{1,1} = \frac{2(1 - \nu_1)}{\mu_1} b_1 \left\{ \left( \frac{1}{2\mu_1} - \frac{1}{2\mu_1} \right) \right\} \left\{ \ldots \right\} + \left( \frac{2(1 - \nu_2)}{\mu_2} \right) \left\{ \ldots \right\}
\]

\[
= \frac{2(1 - \nu_1)}{\mu_1} b_1 \left\{ -\frac{\alpha - \beta}{1 - \alpha} \right\} \left\{ \ldots \right\} + \left( 1 - \Sigma \right) \left\{ \ldots \right\}
\]

\[
= \frac{2(1 - \nu_1)}{\mu_1} b_1 \left\{ -\frac{\alpha - \beta}{1 - \alpha} \right\} \left\{ \ldots \right\} + \left( 1 - \Sigma \right) \left\{ \ldots \right\}
\]
\[
= \frac{b_1 2(1 - \nu_2)}{\Sigma} \left\{ \frac{-\alpha - \beta}{1 - \alpha} \left\{ (A_1 + A_2 \lambda y) e^{-i\lambda y} + (A_3 + A_4 \lambda y) e^{i\lambda y} \right\} \right\} \\
\] + \left\{ 1 - \Sigma \right\} \frac{1}{2} \left\{ -A_2 e^{-i\lambda y} + A_4 e^{i\lambda y} \right\} \\
\] (B.21)

\[
\Delta \tilde{u}_{2,1} = \frac{b_1 2(1 - \nu_2)}{\Sigma} \left\{ \frac{-\alpha - \beta}{1 - \alpha} \left\{ (-A_1 + (1 - \lambda y)A_2 \lambda y) e^{-i\lambda y} + (A_3 + (1 + \lambda y)A_4 \lambda y) e^{i\lambda y} \right\} \right\} \\
\] - \left\{ 1 - \Sigma \right\} \frac{1}{2} \left\{ A_2 e^{-i\lambda y} + A_4 e^{i\lambda y} \right\} \\
\] (B.22)

### B.1.4 Boundary Conditions

The boundary conditions for the tractions are given by the equilibrium equation as

\[
\left( \sigma_{11} n_1 + \sigma_{21} n_2 \right)^{\text{layer #1 or #3}} + \left( \sigma_{11} n_1 + \sigma_{21} n_2 \right)^{\text{layer #2}} = 0 \\
\left( \sigma_{12} n_1 + \sigma_{22} n_2 \right)^{\text{layer #1 or #3}} + \left( \sigma_{12} n_1 + \sigma_{22} n_2 \right)^{\text{layer #2}} = 0 \
\] (B.23)

where \( n_1 \) and \( n_2 \) represent the normals to the interfaces.

By matching the gradient displacement mismatch from problem 1 and problem 2, the second set of boundary conditions are obtained:

\[
\Delta u_{1,1} + (\Delta u_{1,1})_{\text{dislocation}} = 0 \\
\Delta u_{2,1} + (\Delta u_{2,1})_{\text{dislocation}} = 0 \
\] (B.24)

where \( (\Delta u_{1,1})_{\text{dislocation}} \) are the expressions obtained from the first problem.
B.1.4.1 The Interface Between Layer 1 and Layer 2 \((y = d)\)

Figure B.1 shows the normal vectors at the upper interface \((y = d)\).

\[
n_{\text{layer1}} = 0\hat{r} - 1\hat{j} \\
n_{\text{layer2}} = 0\hat{r} + 1\hat{j}
\]

**Figure B.1** - Normal vectors at the upper interface \((y = d)\).

Now, the expressions in (B.23) can be written as:

\[
\sigma_{11}(0)^{\text{layer1}} + \sigma_{21}(1)^{\text{layer1}} + \sigma_{11}(0)^{\text{layer2}} + \sigma_{21}(1)^{\text{layer2}} = 0
\]

which reduces to

\[
\sigma_{21}^{\text{layer2}} - \sigma_{21}^{\text{layer1}} = 0 \quad (B.25)
\]

and

\[
\sigma_{12}(0)^{\text{layer1}} + \sigma_{22}(1)^{\text{layer1}} + \sigma_{12}(0)^{\text{layer2}} + \sigma_{22}(1)^{\text{layer2}} = 0
\]

which when simplified yield

\[
\sigma_{22}^{\text{layer2}} - \sigma_{22}^{\text{layer1}} = 0 \quad (B.26)
\]

Substitute equations (B.17) and (B.18) into (B.25) and (B.26) in the Fourier transform domain:

\[
b_1\left((-D_1 - D_2\lambda d)e^{-\lambda d} + (-D_3 - D_4\lambda d)e^{\lambda d}\right) - b_1\left((-C_1 - C_2\lambda d)e^{-\lambda d}\right) = 0
\]

\[
b_1\left([-D_1 + (1 - \lambda d)D_2]e^{-\lambda d} + [D_3 + (1 + \lambda d)D_4]e^{\lambda d}\right) - b_1\left([-C_1 + (1 - \lambda d)C_2]e^{-\lambda d}\right) = 0
\]

(B.27)

Substitute equations (B.21) and (B.22) into (B.24) in the Fourier transform domain:
\[-\frac{(\alpha - \beta)}{1 - \alpha} D_1 e^{-\lambda d} + \left[ -\frac{(\alpha - \beta)}{1 - \alpha} \lambda d + \frac{\Sigma}{2} \right] D_2 e^{-\lambda d} - \frac{(\alpha - \beta)}{1 - \alpha} D_3 e^{\lambda d} \ldots \]

\[+ \left[ -\frac{(\alpha - \beta)}{1 - \alpha} \lambda d - \frac{\Sigma}{2} \right] D_4 e^{\lambda d} + \left( -\frac{1}{2} D_2 e^{-\lambda d} + \frac{1}{2} D_4 e^{\lambda d} \right) = -\frac{\Sigma}{b_1} \frac{\mu_2}{2(1 - \nu_2)} \Delta \tilde{u}_{1,1}(\lambda, d)_{\text{dislocation}} \]

(B.28)

\[\frac{(\alpha - \beta)}{1 - \alpha} D_1 e^{-\lambda d} + \left[ -\frac{(\alpha - \beta)}{1 - \alpha} (\lambda d) + \frac{\Sigma}{2} \right] D_2 e^{-\lambda d} - \frac{(\alpha - \beta)}{1 - \alpha} D_3 e^{\lambda d} \ldots \]

\[+ \left[ -\frac{(\alpha - \beta)}{1 - \alpha} (1 + \lambda d) + \frac{\Sigma}{2} - 1 \right] D_4 e^{\lambda d} + \left( \frac{1}{2} D_2 e^{-\lambda d} - \frac{1}{2} D_4 e^{\lambda d} \right) = -\frac{\Sigma}{b_1} \frac{\mu_2}{2(1 - \nu_2)} \Delta \tilde{u}_{2,1}(\lambda, d)_{\text{dislocation}} \]

In relation to (B.28), the first stress invariant is given as

\[\sigma_{11} + \sigma_{22} = \text{constant} \]

and substituting in the complex potential expressions for the traction (B.2) yields

\[\sigma_{11} + \sigma_{22} = \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial x^2} = \nabla^2 U = 4 \frac{\partial^2 \tilde{X}^a}{\partial x \partial y}. \]

(B.29)

The Fourier transform of (B.29) is below

\[\bar{\sigma}_{11} + \bar{\sigma}_{22} = 4 \lambda \frac{\partial \tilde{X}^a}{\partial y} = \text{constant} \]

Therefore at the upper interface where \(y = d\):

\[-\frac{C_2}{2} e^{-\lambda d} = \frac{1}{2} \left[ -D_2 e^{-\lambda d} + D_4 e^{\lambda d} \right] \]

(B.30)

Substitute equation (B.30) into equation (B.28) to produce the following:

\[-\frac{(\alpha - \beta)}{1 - \alpha} D_1 e^{-\lambda d} + \left[ -\frac{(\alpha - \beta)}{1 - \alpha} \lambda d + \frac{\Sigma}{2} \right] D_2 e^{-\lambda d} - \frac{(\alpha - \beta)}{1 - \alpha} D_3 e^{\lambda d} \ldots \]

\[+ \left[ -\frac{(\alpha - \beta)}{1 - \alpha} \lambda d - \frac{\Sigma}{2} \right] D_4 e^{\lambda d} - \frac{C_2}{2} e^{-\lambda d} = -\frac{\Sigma}{b_1} \frac{\mu_2}{2(1 - \nu_2)} \Delta \tilde{u}_{1,1}(\lambda, d)_{\text{dislocation}} \]

(B.31)
\[
\left(\frac{\alpha - \beta}{1 - \alpha}\right) D_1 e^{-i\beta d} + \left[-\left(\frac{\alpha - \beta}{1 - \alpha}\right) (1 - \beta d) + \frac{\Sigma}{2}\right] D_2 e^{-i\beta d} - \left(\frac{\alpha - \beta}{1 - \alpha}\right) D_3 e^{i\beta d} - \\
\left[-\left(\frac{\alpha - \beta}{1 - \alpha}\right) (1 + \beta d) + \frac{\Sigma}{2} - 1\right] D_4 e^{i\beta d} - \frac{1}{2} C_2 e^{-i\beta d} = -\frac{\Sigma}{b_1 2(1 - \nu_2)} \Delta \tilde{u}_{2,1}(\lambda, d)_{\text{dislocation}}
\]

Now relating equations (B.27) and (B.31) with Fleck, Hutchinson, and Suo's (1991) matrices:

\[
[m_{311}] [D_{12}] + [m_{312}] [D_{34}] + [m_{41}] [C] = -\frac{\mu_2}{2(1 - \nu_2)} \frac{\Sigma}{b_1} \{v_{21}\}
\]

\[
[m_{321}] [D_{12}] + [m_{322}] [D_{34}] + [m_{42}] [C] = -\frac{\mu_2}{2(1 - \nu_2)} \frac{\Sigma}{b_1} \{v_{22}\}
\]

(B.32)

where

\[
[m_{41}] = e^{-i\beta d} \begin{bmatrix} 1 & \lambda d \\ i & \lambda d - 1 \end{bmatrix} \quad \quad \quad [m_{42}] = e^{-i\beta d} \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}
\]

\[
[m_{311}] = e^{i\beta d} \begin{bmatrix} -e^{-i\beta d} & -\lambda d e^{-i\beta d} \\ -e^{-i\beta d} & (1 - \lambda d) e^{-i\beta d} \end{bmatrix} \quad \quad \quad [m_{312}] = e^{i\beta d} \begin{bmatrix} -1 & -\lambda d \\ 1 & (1 + \lambda d) \end{bmatrix}
\]

\[
[m_{321}] = e^{i\beta d} \begin{bmatrix} \left(\frac{\alpha - \beta}{1 - \alpha}\right) e^{-i\beta d} & \left(\frac{\alpha - \beta}{1 - \alpha}\right) (\lambda d + \frac{\Sigma}{2}) e^{-i\beta d} \\ \left(\frac{\alpha - \beta}{1 - \alpha}\right) e^{-i\beta d} & \left(\frac{\alpha - \beta}{1 - \alpha}\right) (\lambda d + 1 + \frac{\Sigma}{2}) e^{-i\beta d} \end{bmatrix}
\]

\[
[m_{322}] = e^{i\beta d} \begin{bmatrix} \left(\frac{\alpha - \beta}{1 - \alpha}\right) e^{-i\beta d} & \left(\frac{\alpha - \beta}{1 - \alpha}\right) \lambda d - \frac{\Sigma}{2} \\ \left(\frac{\alpha - \beta}{1 - \alpha}\right) e^{-i\beta d} & \left(\frac{\alpha - \beta}{1 - \alpha}\right) (\lambda d + 1 + \frac{\Sigma}{2}) e^{-i\beta d} \end{bmatrix}
\]

\[
\Sigma = \frac{1 + \alpha}{1 - \alpha} \quad \{D_{12}\} = \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} \quad \{D_{34}\} = \begin{bmatrix} D_3 \\ D_4 \end{bmatrix} \quad \{C\} = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}
\]

\[
\{v_{21}\} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \quad \quad \{v_{22}\} = \begin{bmatrix} \Delta \tilde{u}_{1,1}(\lambda, d)_{\text{dislocation}} \\ \Delta \tilde{u}_{2,1}(\lambda, d)_{\text{dislocation}} \end{bmatrix}
\]
There are some discrepancies between the matrices obtained here and those matrices in Fleck, Hutchinson, and Suo’s (1991) work. The circled component is an extra term so instead of $\frac{\alpha - \beta}{1 - \alpha}(\lambda d + 1) + \frac{\gamma}{2} - 1$, Fleck et al. (1991) obtained $\frac{\alpha - \beta}{1 - \alpha}(\lambda d + 1) + \frac{\gamma}{2}$.

### B.1.4.2 The Interface Between Layer 2 and Layer 3 ($y = -c$)

Figure B.2 shows the normal vectors at the lower interface ($y = -c$).

**Figure B.2** - Normal vectors at the lower interface ($y = -c$).

Now, the expressions in (B.23) can be written as:

$$\sigma_{11}^{layer\#1} + \sigma_{21}(-1)^{layer\#1} + \sigma_{11}^{layer\#2} + \sigma_{21}(1)^{layer\#2} = 0$$

which reduces to

$$\sigma_{21}^{layer\#2} - \sigma_{21}^{layer\#1} = 0 \quad (B.33)$$

and

$$\sigma_{12}^{layer\#1} + \sigma_{22}(-1)^{layer\#1} + \sigma_{12}^{layer\#2} + \sigma_{22}(1)^{layer\#2} = 0$$

which when simplified yield

$$\sigma_{22}^{layer\#2} - \sigma_{22}^{layer\#1} = 0. \quad (B.34)$$

Substitute equations (B.17) and (B.18) into (B.33) and (B.34) in the Fourier transform domain:

$$b_1\left[(-D_1 - D_2\lambda d)e^{\lambda c} + (-D_3 + D_4\lambda c)e^{-\lambda c}\right] - b_1\left((-E_3 + E_4\lambda c)e^{-\lambda c}\right) = 0$$

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\[ b_i \left\{ -D_1 + (1 + \lambda c) D_2 \right\} e^{i\kappa c} + \left\{ D_3 + (1 + \lambda c) D_4 \right\} e^{-i\kappa c} \right\} - b_i \left\{ -E_3 + (1 + \lambda c) E_4 \right\} e^{-i\kappa c} \] = 0

(B.35)

Substitute equations (B.21) and (B.22) into (B.24) in the Fourier transform domain:

\[ -\left( \frac{\alpha - \beta}{1 - \alpha} \right) E_3 e^{-i\kappa c} + \left[ \left( \frac{\alpha - \beta}{1 - \alpha} \right) \lambda c + \frac{1}{2} - \frac{\Sigma}{2} \right] E_4 e^{-i\kappa c} = -\frac{\Sigma}{b_i} \frac{\mu_2}{2(1 - \nu_2)} \Delta \tilde{u}_{1,1}(\lambda, -c)_{\text{dislocation}} \]

(B.36)

\[ -\left( \frac{\alpha - \beta}{1 - \alpha} \right) E_3 e^{i\kappa c} + \left[ \left( \frac{\alpha - \beta}{1 - \alpha} \right)(1 + \lambda c) - \frac{1}{2} + \frac{\Sigma}{2} \right] E_4 e^{+i\kappa c} = -\frac{\Sigma}{b_i} \frac{\mu_2}{2(1 - \nu_2)} \Delta \tilde{u}_{1,1}(\lambda, -c)_{\text{dislocation}} \]

Use the 1st stress invariant condition (B.35) at \( y = -c \):

\[ \frac{E_4}{2} e^{-i\kappa c} = \frac{1}{2} \left[ -D_2 e^{i\kappa c} + D_4 e^{-i\kappa c} \right] \]

(B.37)

Introduce equation (B.37) into equation (B.36) to produce the following:

\[ -\left( \frac{\alpha - \beta}{1 - \alpha} \right) E_3 e^{i\kappa c} + \left[ \left( \frac{\alpha - \beta}{1 - \alpha} \right) \lambda c + \frac{1}{2} \right] E_4 e^{+i\kappa c} + \frac{\Sigma}{2} D_2 e^{+i\kappa c} - \frac{\Sigma}{2} D_4 e^{-i\kappa c} \ldots \]

\[ = -\frac{\Sigma}{b_i} \frac{\mu_2}{2(1 - \nu_2)} \Delta \tilde{u}_{1,1}(\lambda, -c)_{\text{dislocation}} \]

(B.38)

Now relating equations (B.35) and (B.38) with Fleck, Hutchinson, and Suo's (1991) matrices:

\[ [m_{11}] \{D_{12}\} + [m_{12}] \{D_{34}\} + [m_{31}] \{E\} = -\frac{\mu_2}{2(1 - \nu_2)} \frac{\Sigma}{b_i} \{v_{11}\} \]

\[ [m_{12}] \{D_{12}\} + [m_{12}] \{D_{34}\} + [m_{32}] \{E\} = -\frac{\mu_2}{2(1 - \nu_2)} \frac{\Sigma}{b_i} \{v_{12}\} \]
where

\[
\begin{align*}
    [m_{21}] &= e^{-\lambda c} \begin{bmatrix} 1 & -\lambda c \\ -1 & \lambda c - 1 \end{bmatrix} \\
    [m_{22}] &= e^{-\lambda c} \begin{bmatrix} \frac{a - \beta}{1 - a} & \frac{a - \beta}{\lambda c - 1} \\ \frac{a - \beta}{1 - a} & -\frac{a - \beta}{1 - a} \end{bmatrix} \\
    [m_{11}] &= e^{\lambda c} \begin{bmatrix} -1 & \lambda c \\ -1 & 1 + \lambda c \end{bmatrix} \\
    [m_{12}] &= e^{\lambda c} \begin{bmatrix} 0 & \frac{\Sigma}{2} \\ 0 & -\frac{\Sigma}{2} \end{bmatrix} \\
    [m_{12}] &= e^{\lambda c} \begin{bmatrix} 0 & -\frac{\Sigma}{2} e^{-2\lambda c} \\ 0 & \frac{\Sigma}{2} e^{-2\lambda c} \end{bmatrix}
\end{align*}
\]

\[
\{ E \} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} \quad \{ v_{11} \} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \{ v_{12} \} = \begin{bmatrix} \Delta u_{1,1}(\lambda, -c)_{\text{dislocation}} \\ \Delta u_{2,1}(\lambda, -c)_{\text{dislocation}} \end{bmatrix}
\]

There are some discrepancies between the matrices obtained here and those matrices in Fleck, Hutchinson, and Suo's (1991) work. The circled component is where the discrepancy lies. Instead of \(-\frac{\Sigma}{2}\), Fleck et al. (1991) had \(\frac{\Sigma}{2}\).

The matrices systems in (B.32) and (B.39) can now be combined to get a full system of equations for the \(b_1\) solution in the following manner:

\[
e^{\lambda c} \begin{bmatrix} [m_{11}] & [m_{12}] \\ [m_{21}] & [m_{22}] \end{bmatrix} \begin{bmatrix} D_{12} \\ D_{34} \end{bmatrix} + e^{-\lambda c} \begin{bmatrix} [m_{21}] \\ [m_{22}] \end{bmatrix} \{ E \} = -\frac{\mu_2 \Sigma}{2(1 - \nu_2)} b_1 \begin{bmatrix} 0 \\ e^{\lambda c} \{ v_{12} \} \end{bmatrix}
\]

\[
e^{i\delta} \begin{bmatrix} [m_{311}] & [m_{312}] \\ [m_{321}] & [m_{322}] \end{bmatrix} \begin{bmatrix} D_{12} \\ D_{34} \end{bmatrix} + e^{-i\delta} \begin{bmatrix} [m_{41}] \\ [m_{42}] \end{bmatrix} \{ C \} = -\frac{\mu_2 \Sigma}{2(1 - \nu_2)} b_1 \begin{bmatrix} 0 \\ e^{i\delta} \{ v_{12} \} \end{bmatrix}
\]

Let

\[
\{ D \} = \begin{bmatrix} D_{12} \\ D_{34} \end{bmatrix} \quad \{ v_1 \} = \begin{bmatrix} 0 \\ e^{\lambda c} \{ v_{12} \} \end{bmatrix} \quad \{ v_2 \} = \begin{bmatrix} 0 \\ e^{i\delta} \{ v_{12} \} \end{bmatrix}
\]

Multiply both sides by \(e^{-\lambda H}\) and regroup:
\[
\begin{align*}
[M_1][D] + [M_2][E] &= \{v_1\} \\
[M_3][D] + [M_4][C] &= \{v_2\}
\end{align*}
\]

In matrix form

\[
\begin{pmatrix}
[M_1] & [M_2] & 0 \\
[M_3] & 0 & [M_4]
\end{pmatrix}
\begin{pmatrix}
\{D\} \\
\{E\} \\
\{C\}
\end{pmatrix}
= 
\begin{pmatrix}
\{v_1\} \\
\{v_2\}
\end{pmatrix}
\] (B40)

where

\[
[M_1] = e^{-\lambda d} \begin{pmatrix}
m_{111} & m_{112} \\
m_{121} & m_{122}
\end{pmatrix}
\]

\[
[M_2] = e^{-\lambda (H+d)} \begin{pmatrix}
m_{211} & m_{212} \\
m_{221} & m_{222}
\end{pmatrix}
\]

\[
[M_3] = e^{-\lambda c} \begin{pmatrix}
m_{311} & m_{312} \\
m_{321} & m_{322}
\end{pmatrix}
\]

\[
[M_4] = e^{-\lambda (H+c)} \begin{pmatrix}
m_{411} & m_{412} \\
m_{421} & m_{422}
\end{pmatrix}
\]

\[
\{v_1\} = -\frac{\mu_2 \Sigma}{2(1 - \nu_2)} \frac{e^{-\lambda (H+c)}}{b_1} \begin{pmatrix}
0 \\
\frac{b_1 \left( a(1-\lambda c) + \beta c \right)}{\pi} \\
\frac{b_2 \left( a(1-\lambda c) + a \lambda c \right)}{\pi}
\end{pmatrix}
\]

\[
\{v_2\} = -\frac{\mu_2 \Sigma}{2(1 - \nu_2)} \frac{e^{-\lambda (H+d)}}{b_1} \begin{pmatrix}
0 \\
\frac{b_1 \left( a(1-\lambda d) + \beta d \right)}{\pi} \\
\frac{b_2 \left( a(1-\lambda d) + a \lambda d \right)}{\pi}
\end{pmatrix}
\]

B.1.4.3 The Fourier Transform of the Displacement Gradient Mismatch

\[\Delta u_{i,1}(\lambda, y)_{\text{dislocation}}\]

The appropriate Fourier transforms for the displacement gradient mismatches is given by

\[\Delta \tilde{u}_{i,1}(\lambda, y) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\partial \Delta u_{i,1}}{\partial \lambda x_1} \cos \lambda x_1 dx\]

\[\Delta \tilde{u}_{i,1}(\lambda, y) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\partial \Delta u_{i,1}}{\partial \lambda x_1} \sin \lambda x_1 dx\]
Now, the corresponding $b_1$ expressions from equation (A.27) are substituted into the above Fourier transforms. These integrals were evaluated in MATHEMATICA 2.2.1. The following closed form solutions for the gradient mismatches at each interface were obtained.

\[
\Delta \tilde{u}_{1,1}(\lambda, d) = \frac{b_1}{\pi} \left( \frac{\alpha(1 - \lambda d) + \beta \lambda d}{(1 + \alpha)} \right) e^{-\lambda d}
\]

\[
\Delta \tilde{u}_{2,1}(\lambda, d) = \frac{b_1}{\pi} \left( \frac{\beta(1 - \lambda d) + \alpha \lambda d}{(1 + \alpha)} \right) e^{-\lambda d}
\]

\[
\Delta \tilde{u}_{1,1}(\lambda, -c) = -\frac{b_1}{\pi} \left( \frac{\alpha(1 - \lambda c) + \beta \lambda c}{(1 + \alpha)} \right) e^{-\lambda c}
\]

\[
\Delta \tilde{u}_{2,1}(\lambda, -c) = \frac{b_1}{\pi} \left( \frac{\beta(1 - \lambda c) + \alpha \lambda c}{(1 + \alpha)} \right) e^{-\lambda c}
\]

which are consistent with the expressions in Fleck et al. (1991).

**B.2 $b_2$ Solution ($b_1 = 0$)**

Similar to the procedure followed for solution 1, the potentials can be obtained:

\[
U^a(x, y) = b_2 \int_0^\infty \left[ \left( \frac{B_3}{\lambda^2} + \frac{B_4}{\lambda} \right) e^{-\lambda y} + \left( \frac{B_3}{\lambda^2} + \frac{B_4}{\lambda} \right) e^{\lambda y} \right] \sin \lambda x \, d\lambda \quad (B.41)
\]

\[
X^a(x, y) = -b_2 \int_0^\infty \frac{1}{2\lambda^2} \left( B_2 e^{-\lambda y} + B_4 e^{\lambda y} \right) \cos \lambda x \, d\lambda \quad (B.42)
\]

As can be seen from equations (B.41) and (B.42), the antisymmetry properties of the stress potentials have reversed from the $b_1$ solution to the $b_2$ solution. Now, $U^a(x, y)$ is an antisymmetric function and $X^a(x, y)$ is a symmetric function. Therefore, the appropriate Fourier transforms for these potentials are

\[
\mathcal{F}(U^a) = \tilde{U}^a = \frac{2}{\pi} \int_0^\infty U^a \sin \lambda x \, d\lambda \quad \mathcal{F}(X^a) = \tilde{X}^a = \frac{2}{\pi} \int_0^\infty X^a \cos \lambda x \, d\lambda
\]
B.2.1 Functions $U^a$ and $X^s$ Expressed in Terms of Individual Layers

In layer 1 of material 1 ($y \geq d$):

$$
\tilde{U}^a = b_2 \left( \frac{F_1}{\lambda^2} + \frac{F_2}{\lambda} y \right) e^{-\lambda y}
$$

$$
\tilde{X}^s = -b_2 \frac{F_2}{2\lambda^2} e^{-\lambda y}
$$

$$
U^a(x, y) = b_2 \int_0^\infty \left( \frac{F_1}{\lambda^2} + \frac{F_2}{\lambda} y \right) e^{-\lambda y} \sin \lambda x d\lambda
$$

$$
X^s(x, y) = -b_2 \int_0^\infty \frac{F_2}{2\lambda^2} e^{-\lambda y} \cos \lambda x d\lambda
$$

(B.43)

In layer 2 of material 2 ($-c \leq y \leq d$):

$$
\tilde{U}^a = b_2 \left[ \left( \frac{G_1}{\lambda^2} + \frac{G_2}{\lambda} y \right) e^{-\lambda y} + \left( \frac{G_3}{\lambda^2} + \frac{G_4}{\lambda} y \right) e^{\lambda y} \right]
$$

$$
\tilde{X}^s = -b_2 \frac{1}{2\lambda^2} \left( G_2 e^{-\lambda y} + G_4 e^{\lambda y} \right)
$$

$$
U^a(x, y) = b_2 \int_0^\infty \left[ \left( \frac{G_1}{\lambda^2} + \frac{G_2}{\lambda} y \right) e^{-\lambda y} + \left( \frac{G_3}{\lambda^2} + \frac{G_4}{\lambda} y \right) e^{\lambda y} \right] \sin \lambda x d\lambda
$$

$$
X^s(x, y) = -b_2 \int_0^\infty \frac{1}{2\lambda^2} \left( G_2 e^{-\lambda y} + G_4 e^{\lambda y} \right) \sin \lambda x d\lambda
$$

(B.44)

In layer 3 of material 1 ($y \leq -c$):

$$
\tilde{U}^a = b_2 \left( \frac{H_3}{\lambda^2} + \frac{H_4}{\lambda} y \right) e^{\lambda y}
$$

$$
\tilde{X}^s = -b_2 \frac{H_4}{2\lambda^2} e^{\lambda y}
$$

$$
U^a(x, y) = b_2 \int_0^\infty \left( \frac{H_3}{\lambda^2} + \frac{H_4}{\lambda} y \right) e^{\lambda y} \sin \lambda x d\lambda
$$
\[ \lambda^a(x, y) = -b_2 \int_0^\infty \frac{H_4}{2\lambda^2} e^{i\lambda y} \cos \lambda x d\lambda \]

(B.45)

### B.2.2 Fourier Transform of the Tractions and the Displacement Gradients

The following two sections will give general expressions for the tractions and displacement gradients in the Fourier domain for the \( b_2 \) solution.

#### B.2.2.1 Tractions

The general expression for the transform of the tractions in any layer is given by equations (B.46) and (B.47)

\[ \mathcal{F}(\sigma_{22}) = \tilde{\sigma}_{22} = -\lambda^2 \tilde{U}^a \quad \mathcal{F}(\sigma_{12}) = \tilde{\sigma}_{12} = -\lambda \frac{\partial \tilde{U}^a}{\partial y} \]

\[ \tilde{\sigma}_{22} = b_2 \left[ (-B_1 - B_2 \lambda y)e^{-i\lambda y} + (-B_3 - B_4 \lambda y)e^{i\lambda y} \right] \quad \text{B.46} \]

\[ \tilde{\sigma}_{12} = -b_2 \left[ -B_1 + (1 - \lambda y)B_2 \right] e^{-i\lambda y} + \left[ B_3 + (1 + \lambda y)B_4 \right] e^{i\lambda y} \]  

#### B.2.2.2 Displacement Gradient Mismatch

Using equation (B.19) and substituting the appropriate potential functions for the \( b_2 \) solution (B.41) and (B.42), take the appropriate Fourier transform

\[ \mathcal{F}_c \left( \frac{\partial \tilde{u}_1}{\partial x} \right) = \tilde{u}_{1,1} = \frac{1}{2\mu} \lambda^2 \tilde{U}^a - \frac{2(1 - \nu)}{\mu} \lambda \frac{\partial \tilde{X}^a}{\partial y} \]

\[ \mathcal{F}_c \left( \frac{\partial \tilde{u}_2}{\partial x} \right) = \tilde{u}_{2,1} = -\frac{1}{2\mu} \lambda \frac{\partial \tilde{U}^a}{\partial y} - \frac{2(1 - \nu)}{\mu} \lambda^2 \tilde{X}^a \]

\[ \tilde{u}_{1,1} = \frac{b_2}{2\mu} \left[ (B_1 + B_2 \lambda y)e^{-i\lambda y} + (B_3 + B_4 \lambda y)e^{i\lambda y} \right] + \frac{b_2}{2\mu} \frac{2(1 - \nu)}{\mu} \left[ -B_2 e^{-i\lambda y} + B_4 e^{i\lambda y} \right] \]
\[ \tilde{u}_{z,1} = -\frac{b_2}{2\mu} \left\{ -B_1 + (1 - \lambda y)B_2 \right\} e^{-\lambda y} + \left\{ B_3 + (1 + \lambda y)B_4 \right\} e^{\lambda y} \] - \frac{b_2}{2\mu} \frac{2(1 - \nu)}{\mu} \left[ B_2 e^{-\lambda y} + B_4 e^{\lambda y} \right] \]

The Fourier transform of the displacement gradient mismatch follows below:

\[ \Delta\tilde{u}_{z,1} = b_2 \left\{ \left( \frac{1}{2\mu_1} - \frac{1}{2\mu_2} \right) \left[ (B_1 + B_2 \lambda y) e^{-\lambda y} + (B_3 + B_4 \lambda y) e^{\lambda y} \right] + \cdots \right\} \]

\[ \left( \frac{2(1 - \nu_1)}{\mu_1} - \frac{2(1 - \nu_2)}{\mu_1} \right) \frac{1}{2} \left[ B_2 e^{-\lambda y} + B_4 e^{\lambda y} \right] \]

\[ \Delta\tilde{u}_{z,1} = -b_2 \left\{ \left( \frac{1}{2\mu_1} - \frac{1}{2\mu_2} \right) \left[ -B_1 + (1 - \lambda y)B_2 \lambda y e^{-\lambda y} + (B_3 + (1 + \lambda y)B_4 \lambda y) e^{\lambda y} \right] + \cdots \right\} \]

\[ \left( \frac{2(1 - \nu_1)}{\mu_1} - \frac{2(1 - \nu_2)}{\mu_1} \right) \frac{1}{2} \left[ B_2 e^{-\lambda y} + B_4 e^{\lambda y} \right] \]

factoring out \( \frac{2(1 - \nu_1)}{\mu_1} \) yields:

\[ \Delta\tilde{u}_{z,1} = \frac{2(1 - \nu_1)}{\mu_1} b_2 \left\{ \left( \frac{1}{2\mu_1} \frac{1}{2\mu_2} \right) \left[ \cdots \right] \right\} + \left( \frac{2(1 - \nu_2)}{\mu_1} \frac{\mu_2}{\mu_1} \right) \left[ \cdots \right] \]

\[ = \frac{2(1 - \nu_1)}{\mu_1} b_2 \left\{ -\frac{\alpha - \beta}{1 - \alpha} \{ \cdots \} + (1 - \Sigma) \{ \cdots \} \right\} \]

\[ = \frac{2(1 - \nu_1)}{\mu_1} \frac{2(1 - \nu_2)}{\mu_2} b_2 \left\{ -\frac{\alpha - \beta}{1 - \alpha} \{ \cdots \} + (1 - \Sigma) \{ \cdots \} \right\} \]

\[ = \frac{b_2}{\Sigma} \frac{2(1 - \nu_2)}{\mu_2} \left\{ -\frac{\alpha - \beta}{1 - \alpha} \{ \cdots \} + (1 - \Sigma) \{ \cdots \} \right\} \]

A similar procedure is followed to obtain \( \Delta\tilde{u}_{z,2} \). Since the boundary conditions have not been imposed, intermediate expressions for \( \Delta\tilde{u}_{1,1} \) and \( \Delta\tilde{u}_{z,1} \) are given below.
\[
\Delta \tilde{u}_{1,1} = \frac{b_2}{\Sigma} \frac{2(1 - v_2)}{\mu_2} \left\{ -\frac{\alpha - \beta}{1 - \alpha} \left[ (B_1 + B_2 \lambda_y) e^{-\lambda y} + (B_3 + B_4 \lambda_y) e^{\lambda y} \right] \right. \\
\left. + [1 - \Sigma] \frac{1}{2} \left[ -B_2 e^{-\lambda y} + B_4 e^{\lambda y} \right] \right\} 
\] 
(B.48)

\[
\Delta \tilde{u}_{2,1} = -\frac{b_2}{\Sigma} \frac{2(1 - v_2)}{\mu_2} \left\{ -\frac{\alpha - \beta}{1 - \alpha} \left[ -(B_1 + (1 - \lambda y)B_2 \lambda_y) e^{-\lambda y} + (B_3 + (1 + \lambda y)B_4 \lambda_y) e^{\lambda y} \right] \right. \\
\left. - [1 - \Sigma] \frac{1}{2} \left[ B_2 e^{-\lambda y} + B_4 e^{\lambda y} \right] \right\} 
\] 
(B.49)

Notice the difference in equations (B.22) and (B.49). There is a sign difference which is indicative of the antisymmetry of switching from the \( b_1 \) solution to the \( b_2 \) solution. Fleck, Hutchinson, and Suo (1991) did not have this sign change in any of part of their solution. This difference is very important because without it the antisymmetry of the problem is lost. It will be shown later were the sign change is applied in the final matrix representation.

**B.2.3 The Upper Interface Between Layer 1 and Layer 2 \( (y = d) \)**

Substitute equations (B.46) and (B.47) into (B.23) in the Fourier transform domain:

\[
b_2 \left\{ (-G_1 - G_2 \lambda d) e^{-\lambda d} + (-G_3 - G_4 \lambda d) e^{\lambda d} \right\} - b_2 \left\{ (-F_1 - F_2 \lambda d) e^{-\lambda d} \right\} = 0
\]

\[
b_2 \left\{ (-G_1 + (1 - \lambda d)G_2) e^{-\lambda d} + [G_3 + (1 + \lambda d)G_4] e^{\lambda d} \right\} - b_2 \left\{ (-F_1 + (1 - \lambda d)F_2) e^{-\lambda d} \right\} = 0
\]

(B.50)

Substitute equations (B.48) and (B.49) into (B.24) in the Fourier transform domain:

\[
-\left( \frac{\alpha - \beta}{1 - \alpha} \right) G_1 e^{-\lambda d} + \left[ \left( \frac{\alpha - \beta}{1 - \alpha} \right) \lambda d + \frac{\Sigma}{2} \right] G_2 e^{-\lambda d} - \left( \frac{\alpha - \beta}{1 - \alpha} \right) G_3 e^{\lambda d} \ldots
\]

\[
+ \left[ \left( \frac{\alpha - \beta}{1 - \alpha} \right) \lambda d - \frac{\Sigma}{2} \right] G_4 e^{\lambda d} - \left( \frac{1}{2} G_2 e^{-\lambda d} - \frac{1}{2} G_4 e^{\lambda d} \right) = -\frac{\Sigma}{b_2} \frac{\mu_2}{2(1 - v_2)} \Delta \tilde{u}_{1,1}(\lambda, d)_{\text{dislocation}}
\]

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\[
-\left(\frac{\alpha - \beta}{1 - \alpha}\right)G_1 e^{-\lambda d} + \left[\left(\frac{\alpha - \beta}{1 - \alpha}\right)(1 - \lambda d) - \frac{\Sigma}{2}\right]G_2 e^{-\lambda d} + \left(\frac{\alpha - \beta}{1 - \alpha}\right)G_3 e^{\lambda d} \ldots
\]
\[
+ \left[\left(\frac{\alpha - \beta}{1 - \alpha}\right)(1 + \lambda d) - \frac{\Sigma}{2} + 1\right]G_4 e^{\lambda d} + \left(\frac{1}{2} G_2 e^{-\lambda d} - \frac{1}{2} G_4 e^{\lambda d}\right) = -\frac{\Sigma}{b_2} \frac{\mu_2}{2(1 - v_2)} \Delta \tilde{u}_{2,1}(\lambda, d)_{\text{dislocation}}
\]

Therefore at the upper interface where \( y = d \) the first stress invariant is given as:
\[
\frac{F_z}{2} e^{-\lambda d} = \frac{1}{2} \left[ G_2 e^{-\lambda d} - G_4 e^{\lambda d} \right]
\]

(B.52)

Substitute equation (B.52) into (B.51) and yield the following:
\[
-\left(\frac{\alpha - \beta}{1 - \alpha}\right)G_1 e^{-\lambda d} + \left[\left(\frac{\alpha - \beta}{1 - \alpha}\right)(1 - \lambda d) + \frac{\Sigma}{2}\right]G_2 e^{-\lambda d} - \left(\frac{\alpha - \beta}{1 - \alpha}\right)G_3 e^{\lambda d} \ldots
\]
\[
+ \left[\left(\frac{\alpha - \beta}{1 - \alpha}\right)\lambda d - \frac{\Sigma}{2} \right]G_4 e^{\lambda d} - \frac{1}{2} F_z e^{-\lambda d} = -\frac{\Sigma}{b_2} \frac{\mu_2}{2(1 - v_2)} \Delta \tilde{u}_{2,1}(\lambda, d)_{\text{dislocation}}
\]

(B.53)

As noted previously, there is a sign difference in the \( \Delta \tilde{u}_{2,1} \) line when comparing the \( b_1 \) solution (B.31) to the \( b_2 \) solution (B.53). Therefore, in order to keep the \( M \) matrices from the \( b_2 \) solution consistent with those in the \( b_1 \) solution, the \( \Delta \tilde{u}_{2,1}(\lambda, d)_{\text{dislocation}} \) term must be multiplied by a minus sign.

\[
\{w_2\} = -\frac{\mu_2 \Sigma}{2(1 - v_2)} \frac{e^{-\lambda d}}{b_2} \begin{cases}
0 \\
0 \\
e^{\lambda d} \Delta \tilde{u}_{1,1}(\lambda, d)_{\text{dislocation}} \\
-e^{\lambda d} \Delta \tilde{u}_{2,1}(\lambda, d)_{\text{dislocation}}
\end{cases}
\]

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Now relating equations (B.50) and (B.53) with Fleck, Hutchinson, and Suo's (1991) matrices:

\[
[m_{311}][G_{12}] + [m_{312}][G_{34}] + [m_{411}][F] = -\frac{\mu_2}{2(1 - \nu_2)} \sum b_2 \begin{bmatrix} w_{21} \end{bmatrix}
\]

\[
[m_{321}][G_{12}] + [m_{322}][G_{34}] + [m_{421}][F] = -\frac{\mu_2}{2(1 - \nu_2)} \sum b_2 \begin{bmatrix} w_{22} \end{bmatrix}
\]

(B.54)

where

\[
\begin{align*}
\{G_{12}\} &= \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} & \{G_{34}\} &= \begin{bmatrix} G_3 \\ G_4 \end{bmatrix} & \{F\} &= \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \\
\{w_{21}\} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \{w_{22}\} &= \begin{bmatrix} \Delta \tilde{u}_{1,1}(\lambda, d)_{\text{dislocation}} \\ -\Delta \tilde{u}_{2,1}(\lambda, d)_{\text{dislocation}} \end{bmatrix}
\end{align*}
\]

There are the same discrepancies as those from the \(b_1\) solution between the matrices obtained here and those matrices in Fleck, Hutchinson, and Suo's (1991) work.

**B.2.4 The Lower Interface Between Layer 2 and Layer 3 (\(\nu = -c\))**

Substitute equations (B.46) and (B.47) into (B.23) in the Fourier transform domain:

\[
b_2 \left[ (-G_1 - G_2 \lambda d)e^{\lambda c} + (-G_3 + G_4 \lambda c)e^{-\lambda c} \right] - b_2 \left[ (-H_3 + H_4 \lambda c)e^{-\lambda c} \right] = 0
\]

\[
b_2 \left[ (-G_1 + (1 + \lambda c)G_2)e^{\lambda c} + [G_3 + (1 - \lambda c)G_4]e^{-\lambda c} \right] - b_2 \left[ (-H_3 + (1 - \lambda c)H_4)e^{-\lambda c} \right] = 0
\]

(B.55)

Substitute equations (B.48) and (B.49) into (B.24) in the Fourier transform domain:

\[
-\frac{\alpha - \beta}{1 - \alpha} H_3 e^{-\lambda c} + \left[ \frac{\alpha - \beta}{1 - \alpha} \lambda c + \frac{1}{2} \right] H_4 e^{-\lambda c} - \Sigma \left( \frac{1}{2} \right) H_4 e^{-\lambda c} = -\frac{\Sigma}{b_2} \frac{\mu_2}{2(1 - \nu_2)} \Delta \tilde{u}_{1,1}(\lambda, -c)_{\text{dislocation}}
\]

(B.56)

\[
\left( \frac{\alpha - \beta}{1 - \alpha} \right) H_3 e^{-\lambda c} + \left[ \frac{\alpha - \beta}{1 - \alpha} (1 - \lambda c) + \frac{1}{2} \right] H_4 e^{-\lambda c} - \Sigma \left( \frac{1}{2} \right) H_4 e^{-\lambda c} = -\frac{\Sigma}{b_2} \frac{\mu_2}{2(1 - \nu_2)} \Delta \tilde{u}_{2,1}(\lambda, -c)_{\text{dislocation}}
\]
Use the 1\textsuperscript{st} stress invariant condition (B.29) at \(y = -c\):

\[-\frac{H_4}{2} e^{-\lambda c} = \frac{1}{2} \left[ G_2 e^{\lambda c} - G_4 e^{-\lambda c} \right] \tag{B.57}\]

Introducing equation (B.57) into equation (B.56) yields:

\[-\left(\frac{\alpha - \beta}{1 - \alpha}\right) H_4 e^{-\lambda c} + \left(\frac{\alpha - \beta}{1 - \alpha}\right) \lambda c + \frac{1}{2} \right] H_4 e^{-\lambda c} + \frac{\Sigma}{2} G_2 e^{\lambda c} - \frac{\Sigma}{2} G_4 e^{-\lambda c} \ldots

= -\frac{\Sigma}{b_2} \frac{\mu_2}{2(1 - \nu_2)} \Delta \tilde{u}_{1,1}(\lambda, -c)_{\text{dislocation}} \tag{B.58}\]

\[\left(\frac{\alpha - \beta}{1 - \alpha}\right) H_4 e^{-\lambda c} + \left(\frac{\alpha - \beta}{1 - \alpha}\right) (1 - \lambda c) + \frac{1}{2} \right] H_4 e^{-\lambda c} + \frac{\Sigma}{2} G_2 e^{\lambda c} - \frac{\Sigma}{2} G_4 e^{-\lambda c} \ldots

= -\frac{\Sigma}{b_2} \frac{\mu_2}{2(1 - \nu_2)} \Delta \tilde{u}_{2,1}(\lambda, -c)_{\text{dislocation}} \tag{B.58}\]

As noted previously, there is a sign difference in the \(\Delta \tilde{u}_{2,1}\) line when comparing the \(b_1\) solution (B.36) to the \(b_2\) solution (B.58). Therefore, in order to keep the \(M\) matrices from the \(b_2\) solution consistent with those in the \(b_1\) solution, the \(\Delta \tilde{u}_{2,1}(\lambda, -c)_{\text{dislocation}}\) term must be multiplied by a minus sign.

\[
\begin{bmatrix}
\{w_1\} = -\frac{\mu_2 \Sigma}{2(1 - \nu_2)} \frac{e^{-\lambda c}}{b_2} \begin{bmatrix}
0 \\
0 \\
\Delta \tilde{u}_{1,1}(\lambda, -c)_{\text{dislocation}} \\
-\Delta \tilde{u}_{2,1}(\lambda, -c)_{\text{dislocation}}
\end{bmatrix}
\end{bmatrix}
\]

Now relating equations (B.55) and (B.58) with Fleck, Hutchinson, and Suo’s (1991) matrices:

\[
\begin{bmatrix}
m_{111} \right\{G_{12} \right\} + m_{112} \right\{G_{34} \right\} + m_{211} \right\{H \right\} = -\frac{\mu_2}{2(1 - \nu_2)} \frac{\Sigma}{b_2} \{w_{11}\}
\end{bmatrix}
\]

\[
\begin{bmatrix}
m_{121} \right\{G_{12} \right\} + m_{122} \right\{G_{34} \right\} + m_{221} \right\{H \right\} = -\frac{\mu_2}{2(1 - \nu_2)} \frac{\Sigma}{b_2} \{w_{12}\}
\end{bmatrix}
\]

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where
\[
\{H\} = \begin{pmatrix} H_3 \\ H_4 \end{pmatrix}, \quad \{w_{11}\} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \{w_{12}\} = \begin{pmatrix} \Delta \tilde{u}_{1,1}(\lambda, -c)_{\text{dislocation}} \\ -\Delta \tilde{u}_{2,1}(\lambda, -c)_{\text{dislocation}} \end{pmatrix}
\]

There are the same discrepancies from the \(b_1\) solution between the matrices obtained here and those matrices in Fleck, Hutchinson, and Suo's (1991) work.

The matrices systems in (B.54) and (B.59) can now be combined to get a full system of equations for the \(b_2\) solution in the following manner:
\[
e^{jc} \begin{bmatrix} m_{111} & m_{112} \\ m_{121} & m_{122} \end{bmatrix} \begin{bmatrix} G_{12} \\ G_{34} \end{bmatrix} + e^{-jc} \begin{bmatrix} m_{211} \\ m_{222} \end{bmatrix} \{H\} = \frac{\mu_2 \Sigma}{2(1-\nu_2)} \frac{e^{-jc}}{b_2} \begin{bmatrix} 0 \\ e^{jc} \{w_{12}\} \end{bmatrix}
\]
\[
e^{jd} \begin{bmatrix} m_{311} & m_{312} \\ m_{321} & m_{322} \end{bmatrix} \begin{bmatrix} G_{12} \\ G_{34} \end{bmatrix} + e^{-jd} \begin{bmatrix} m_{411} \\ m_{422} \end{bmatrix} \{F\} = \frac{\mu_2 \Sigma}{2(1-\nu_2)} \frac{e^{-jd}}{b_2} \begin{bmatrix} 0 \\ e^{jd} \{w_{22}\} \end{bmatrix}
\]

Let
\[
\{G\} = \begin{pmatrix} G_{12} \\ G_{34} \end{pmatrix}, \quad \{w_1\} = \begin{pmatrix} 0 \\ e^{jc} \{w_{12}\} \end{pmatrix}, \quad \{w_2\} = \begin{pmatrix} 0 \\ e^{jd} \{w_{22}\} \end{pmatrix}
\]

Multiply both sides by \(e^{-jH}\) and regroup:
\[
[M_1] \{G\} + [M_2] \{H\} = \{w_1\}
\]
\[
[M_3] \{G\} + [M_4] \{F\} = \{w_2\}
\]

In matrix form
\[
\begin{bmatrix} [M_1] & [M_2] \\ [M_3] & 0 \end{bmatrix} \begin{pmatrix} \{G\} \\ \{H\} \end{pmatrix} = \begin{pmatrix} \{w_1\} \\ \{w_2\} \end{pmatrix}
\]

(B.60)

where
\[
\{ w_1 \} = -\frac{\mu_2 \Sigma}{2(1 - \nu_2)} \frac{e^{-\lambda(H+c)}}{b_2} \begin{pmatrix}
0 \\
0 \\
b_2 \left( \begin{array}{c}
-\beta(1+\lambda c) + \alpha \lambda c \\
1+\alpha \\
b_2 \frac{-\beta(1+\lambda c) + \alpha \lambda c}{1+\alpha} \\
\end{array} \right) 
\end{pmatrix}
\]

\[
\{ w_2 \} = -\frac{\mu_2 \Sigma}{2(1 - \nu_2)} \frac{e^{-\lambda(H+d)}}{b_2} \begin{pmatrix}
0 \\
0 \\
b_2 \left( \begin{array}{c}
-\beta(1+\lambda d) + \alpha \lambda d \\
1+\alpha \\
b_2 \frac{-\beta(1+\lambda d) + \alpha \lambda d}{1+\alpha} \\
\end{array} \right) 
\end{pmatrix}
\]

B.2.5 The Fourier Transform of the Displacement Gradient Mismatch

\[\Delta u_{i,1}(\lambda, y)_{\text{dislocation}}\]

The appropriate Fourier transforms for the displacement gradient mismatches in the \( b_2 \) solution are given by

\[
\Delta \tilde{u}_{i,1}(\lambda, y) = \frac{2}{\pi} \int_0^\infty \frac{\partial \Delta u_i}{\partial x} \sin \lambda x dx
\]

\[
\Delta \tilde{u}_{2,1}(\lambda, y) = \frac{2}{\pi} \int_0^\infty \frac{\partial \Delta u_2}{\partial x} \cos \lambda x dx
\]

The corresponding \( b_2 \) components from equation (A.27) are substituted into the above Fourier transforms. These integrals were evaluated using MATHEMATICA 2.2.1 and the following closed form solutions resulted for the displacement gradient mismatches

\[
\Delta \tilde{u}_{i,1}(\lambda, d) = \frac{b_2}{\pi} \left( \frac{-\beta(1+\lambda d) + \alpha \lambda d}{1+\alpha} \right) e^{-\lambda d}
\]

\[
\Delta \tilde{u}_{2,1}(\lambda, d) = \frac{b_2}{\pi} \left( \frac{\alpha(1+\lambda d) - \beta \lambda d}{1+\alpha} \right) e^{-\lambda d}
\]

\[
\Delta \tilde{u}_{i,1}(\lambda, -c) = \frac{b_2}{\pi} \left( \frac{-\beta(1+\lambda c) + \alpha \lambda c}{1+\alpha} \right) e^{-\lambda c}
\]
\[
\Delta \tilde{u}_{2,1}(\lambda, -c) = -\frac{b_2}{\pi} \left( \frac{\alpha(1 + \lambda c) - \beta \lambda c}{1 + \alpha} \right) e^{-\lambda c}
\]
which are consistent with the expressions in Fleck et al. (1991).

### B.3 Summary Comparison of Fleck, Hutchinson and Suo’s (1991) Work

There are some differences between this derivation following Ionia's (1994) work and Fleck, Hutchinson and Suo’s (1991) original derivation. These are the following \( M \) matrices derived from this procedure.

\[
M_1 = e^{-\lambda c}
\begin{bmatrix}
-1 & \lambda c & -e^{-2\lambda c} & \lambda c e^{-2\lambda c} \\
-1 & (1 + \lambda c) e^{-2\lambda c} & (1 - \lambda c) e^{-2\lambda c} \\
0 & \Sigma/2 & 0 & -\frac{\Sigma}{2} e^{-2\lambda c} \\
0 & -\Sigma/2 & 0 & \frac{\Sigma}{2} e^{-2\lambda c}
\end{bmatrix}
\]

\[
M_2 = e^{\lambda (H+c)}
\begin{bmatrix}
1 & -\lambda c \\
-1 & (\lambda c - 1) \\
-\left(\frac{\alpha - \beta}{1 - \alpha}\right) & \left(\frac{\alpha - \beta}{1 - \alpha}\right) (\lambda c + 1/2) \\
-\left(\frac{\alpha - \beta}{1 - \alpha}\right) (1 - \lambda c) - 1/2
\end{bmatrix}
\]

\[
M_3 = e^{-\lambda c}
\begin{bmatrix}
-\lambda e^{-2\lambda d} & -\frac{\lambda}{\alpha} e^{-2\lambda d} & 1 & -\frac{\lambda}{(1 - \lambda d)} \\
-\frac{\lambda}{\alpha} e^{-2\lambda d} & \left(\frac{1 - \lambda d}{\alpha(1 - \alpha)} + \frac{\Sigma}{2}\right) e^{-2\lambda d} & 1 & \left(\frac{1}{\lambda d - 1}\right) e^{-2\lambda d} \\
\frac{\lambda}{\alpha} e^{-2\lambda d} & \left(\frac{1 - \lambda d}{\alpha(1 - \alpha)} + \frac{\Sigma}{2}\right) e^{-2\lambda d} & \left(\frac{1}{\lambda d - 1}\right) e^{-2\lambda d} & \frac{\lambda}{(1 - \lambda d)}
\end{bmatrix}
\]

\[
M_4 = e^{\lambda (H+d)}
\begin{bmatrix}
1 & \lambda d \\
1 & \lambda d - 1 \\
0 & \frac{1}{2} \\
0 & \frac{1}{2}
\end{bmatrix}
\]

The two differences between these matrices and those derived in Fleck, Hutchinson, and Suo (1991) are found in \( M_1 \) and \( M_3 \). In column 2, row 4 of \( M_1 \), there is a sign change from
\( \Sigma/2 \) to \(-\Sigma/2\). Also, the component in the 4th column, 4th row of \( M_3 \) has an extra term. So, instead of \(-\frac{\alpha - \beta}{1 - \alpha}(\lambda d + 1) + \frac{\Sigma}{2}\) there is \(-\frac{\alpha - \beta}{1 - \alpha}(\lambda d' + 1) + \frac{\Sigma}{2} - 1\). These are not simple misprints. They are fundamentally caused by imposing the stress invariant condition. The other difference was noted previously as the sign change in the \( w_1 \) and \( w_2 \) vector misprints.

**B.4 Introduction of Ionita’s (1994) Matrices**

The matrices derived following Ionita’s (1994) work were not ultimately used for several reasons. Even though the original matrices could not be obtained, the new matrices presented problems. The biggest problem was the antisymmetry properties between \( f_{12}(\zeta) \) and \( f_{21}(\zeta) \). Using Ionita’s (1994) matrices, a plot (B.3) of \( f_{12}(\zeta) \) and \( f_{21}(\zeta) \) was generated below.

![Graph of \( f_{12}(\zeta) \) and \( f_{21}(\zeta) \)](image)

**Figure B.3** Non-antisymmetry of \( f_{12}(\zeta) \) and \( f_{21}(\zeta) \) utilizing Ionita’s (1994) matrices.

It can be seen from the figure that \( f_{12}(\zeta) \neq -f_{21}(\zeta) \). This is important because it shows that with just the minor changes in the matrices, the antisymmetry properties which should be
present in the problem are lost. Therefore, Fleck, Hutchinson, and Suo’s (1991) original matrices were utilized making only the obvious sign change in \( w_1 \) and \( w_2 \).

### B.5 Partial Closed Form Solution for \( D_i(\lambda) \) and \( G_i(\lambda) \) Coefficients

Fleck, Hutchinson, and Suo (1991) used Gaussian elimination to numerically evaluate the adhesive layer coefficients, \( D_i(\lambda) \) and \( G_i(\lambda) \). This is to time consuming. The method developed below will result in partial closed form solutions which can easily and quickly be evaluated for any value of \( \lambda \). Now utilizing the equations and matrices in (B.32) and (B.39), the following solutions resulted:

\[
M_{111} D_{12} + M_{112} D_{34} + M_{21} E = \nu_{11} \tag{B.62a}
\]

\[
M_{121} D_{12} + M_{122} D_{34} + M_{22} E = \nu_{12} \tag{B.62b}
\]

\[
M_{311} D_{12} + M_{312} D_{34} + M_{41} C = \nu_{21} \tag{B.62a}
\]

\[
M_{321} D_{12} + M_{322} D_{34} + M_{42} C = \nu_{22} \tag{B.62b}
\]

Premultiply equation (B.62a) by \( M_{41}^{-1} \):

\[
M_{41}^{-1} M_{311} D_{12} + M_{41}^{-1} M_{312} D_{34} + C = M_{41}^{-1} \nu_{21} \tag{B.63}
\]

Now, premultiply equation (B.63) by \( M_{42} \) and subtract from (B.62b):

\[
M_{11}^{\ast} D_{12} + M_{12}^{\ast} D_{34} = \nu_{11}^{\ast} \tag{B.64}
\]

where

\[
M_{11}^{\ast} = M_{321} - M_{42} M_{41}^{-1} M_{311}
\]

\[
M_{12}^{\ast} = M_{322} - M_{42} M_{41}^{-1} M_{312}
\]

\[
\nu_{11}^{\ast} = \nu_{22} - M_{42} M_{41}^{-1} \nu_{21}
\]

Premultiply eqn (B.62a) by \( M_{21}^{-1} \):

\[
M_{21}^{-1} M_{111} D_{12} + M_{21}^{-1} M_{112} D_{34} + E = M_{21}^{-1} \nu_{11} \tag{B.65}
\]

Now, premultiply equation (B.65) by \( M_{22} \) and subtract from (B.62b):

\[
M_{21}^{\ast} D_{12} + M_{22}^{\ast} D_{34} = \nu_{22}^{\ast} \tag{B.66}
\]
where
\[ M'_{21} = M_{121} - M_{122} M'_{21} M_{111} \]
\[ M'_{22} = M_{122} - M_{222} M'_{21} M_{112} \]
\[ v^*_{22} = v_{22} - M_{222} M^{-1}_{21} v_{11} \]

Premultiply equation (B.64) by \( M'^{-1}_{12} \):
\[ M'^{-1}_{12} M'_{11} D_{12} + D_{34} = M'^{-1}_{12} v^*_{11} \]  \( \text{(B.67)} \)

Now, premultiply equation (B.66) by \( M'^*_{22} \) and subtract from (B.67):
\[ (M'_{21} - M'^*_{22} M'^{-1}_{12} M'^*_{11}) D_{12} = (v^*_{22} - M'^*_{22} M'^{-1}_{12} v^*_{11}) \]
\[ D_{12} = (M'_{21} - M'^*_{22} M'^{-1}_{12} M'^*_{11})^{-1} (v^*_{22} - M'^*_{22} M'^{-1}_{12} v^*_{11}) \]  \( \text{(B.68)} \)

Premultiply equation (B.66) by \( M'^{-1}_{21} \):
\[ D_{12} + M'^{-1}_{21} M'^*_{22} D_{34} = M'^{-1}_{21} v^*_{22} \]  \( \text{(B.69)} \)

Now, premultiply equation (B.64) by \( M'^*_{11} \) and subtract from (B.69):
\[ (M'^*_{12} - M'^*_{11} M'^{-1}_{21} M'^*_{22}) D_{34} = (v^*_{11} - M'^*_{11} M'^{-1}_{21} v^*_{22}) \]
\[ D_{34} = (M'^*_{12} - M'^*_{11} M'^{-1}_{21} M'^*_{22})^{-1} (v^*_{11} - M'^*_{11} M'^{-1}_{21} v^*_{22}) \]  \( \text{(B.70)} \)

Following the exact same procedure as the \( b_1 \) solution, the coefficients for the adhesive layer in the \( b_2 \) solution are given below as:
\[ G_{12} = (M'^*_{21} - M'^*_{22} M'^{-1}_{12} M'^*_{11})^{-1} (v^*_{22} - M'^*_{22} M'^{-1}_{12} v^*_{11}) \]  \( \text{(B.71)} \)
\[ G_{34} = (M'^*_{12} - M'^*_{11} M'^{-1}_{21} M'^*_{22})^{-1} (v^*_{11} - M'^*_{11} M'^{-1}_{21} v^*_{22}) \]  \( \text{(B.72)} \)

These coefficients are functions of \( \lambda \). So, for each value of \( \lambda \) the coefficients for the adhesive layer are evaluated. One of the computer modules is programmed with these matrices and equations. The values for the coefficients are computed, then stored for later use. This procedure is very quick and efficient.
B.6 Superposition of Problem 1 and Problem 2

Since the problems are assumed linear, their solutions are able to be superposed. As a result, the stress fields for the combined solutions are given by

\[
\sigma_{11}(\zeta) = \frac{\mu_2}{4\pi(1 - \nu_2)} \left( \frac{2b_1}{\zeta} + g_1(\zeta)b_1 + g_2(\zeta)b_2 \right)
\]

\[
\sigma_{22}(\zeta) = \frac{\mu_2}{4\pi(1 - \nu_2)} \left( \frac{2b_2}{\zeta} + f_{22}(\zeta)b_1 + f_{12}(\zeta)b_2 \right)
\]

\[
\sigma_{12}(\zeta) = \frac{\mu_2}{4\pi(1 - \nu_2)} \left( \frac{2b_1}{\zeta} + f_{11}(\zeta)b_1 + f_{12}(\zeta)b_2 \right)
\]

where

\[
f_{11}(\zeta) = \frac{4\pi(1 - \nu_2)}{\mu_2} \int_0^\infty (-D_1 + D_2 + D_3 + D_4) \sin \lambda \zeta d\lambda
\]

\[
f_{21}(\zeta) = \frac{4\pi(1 - \nu_2)}{\mu_2} \int_0^\infty (-D_1 - D_3) \cos \lambda \zeta d\lambda
\]

\[
f_{12}(\zeta) = \frac{4\pi(1 - \nu_2)}{\mu_2} \int_0^\infty (G_1 - G_2 - G_3 - G_4) \cos \lambda \zeta d\lambda
\]

\[
f_{22}(\zeta) = \frac{4\pi(1 - \nu_2)}{\mu_2} \int_0^\infty (-G_1 - G_3) \sin \lambda \zeta d\lambda
\]

\[
g_1(\zeta) = \frac{4\pi(1 - \nu_2)}{\mu_2} \int_0^\infty (D_1 - 2D_2 + D_3 + 2D_4) \cos \lambda \zeta d\lambda
\]

\[
g_2(\zeta) = \frac{4\pi(1 - \nu_2)}{\mu_2} \int_0^\infty (G_1 - 2G_2 + G_3 + 2G_4) \sin \lambda \zeta d\lambda
\]

As explained earlier, these integrals were evaluated using a FFT algorithm. The following figures contain each kernel function based on the test case where \(\alpha = 0.893\) and \(\beta = 0.223\).
Figure B.4- Plot of the $f_{11}(\zeta)$ kernel function.

Figure B.5- Plot of the $f_{22}(\zeta)$ kernel function.
Figure B.6 - Plot of the $f_{12}(\zeta)$ kernel function.

Figure B.7 - Plot of $f_{21}(\zeta)$ kernel function.
Figure B.8 - Plot of $g_1(\zeta)$ kernel function.

Figure B.9 - Plot of $g_2(\zeta)$ kernel function.
B.7 Detailed Look at Integral Equation Formulation and Solution Procedure

Again this formulation and solution procedure follows directly from Fleck, Hutchinson, and Suo (1991).

B.7.1 Integral Equation Formulation

The edge dislocations, \( b_i(\xi) \), located at \( x_i = x, x_2 = 0 \) induce stresses on the crack line \( (x_1 = x, x_2 = 0) \) given by equation (3.7)

\[
\sigma_{12}(x) = \frac{\mu_2}{4\pi(1-\nu_2)} \left( \frac{2}{x-\xi} b_j(\xi) \right) + \sum_{j=1}^{2} f_j(x-\xi) b_j(\xi).
\]

Since mathematically the dislocations introduce a singularity, the crack can be modeled with a distribution of dislocations over the negative \( x_i \) axis. By imposing the traction free boundary condition on the crack face, the integral equation results

\[
\int_{0}^{\infty} \frac{2b_i(\xi)}{x-\xi} + \sum_{j=1}^{2} f_j(x-\xi) b_j(\xi) d\xi = 0, \quad x < 0, \quad i = 1, 2 \tag{B.73}
\]

where the first integral is the Cauchy Principal Value integral.

The range of integration is reduced to a finite interval through the following change of variable

\[
x = \frac{u - 1}{u + 1}, \quad -1 < u < 1
\]

\[
\xi = \frac{t - 1}{t + 1}, \quad -1 < t < 1
\]

out of which

\[
\zeta = x - \xi = \frac{2(u - t)}{(u + 1)(t + 1)} \tag{B.74}
\]
Ultimately, this will allow for several integration techniques to be utilized which would not otherwise be possible.

Now, by allowing \( c_i(t) = b_i(\xi) \), equation (B.73) can be written as

\[
\int_{-1}^{1} \frac{(u + 1)}{(u - t)(t + 1)} c_i(t) dt + \int_{-1}^{1} f'_i(\zeta(u,t)) c_j(t) \frac{dt}{(1 + t)^2} = 0, \quad |u| < 1
\]

(B.75)

where the repeated index \( j \) implies a summation of over \( j = 1,2 \).

Since the dislocation solution is unknown, a Chebyshev series of orthogonal polynomials is used to represent \( c_i(t) \)

\[
c_i(t) = \left( \frac{1 + t}{1 - t} \right)^{1/2} \sum_{k=0}^{\infty} a_{ik} T_k(t), \quad i = 1,2
\]

(B.76)

where \( T_k(t) \) is the Chebyshev polynomial of the first kind of degree \( k \) and \( a_{ik} \) are the series coefficients recovered from the solution. Once the coefficients are known, the dislocations can be reconstructed.

The far-field stress intensity factors \( K_i^\infty \) and the local stress intensity factors \( K_i \) are given in terms of the series coefficients \( a_{ik} \) as

\[
K_i^\infty = \left( \frac{\pi}{2} \right)^{1/2} \frac{\mu_2}{(1 - \nu_2)} \left( \frac{1 + \alpha}{1 - \alpha} \right) \sum_{k=0}^{\infty} (-1)^k a_{ik}
\]

(B.77)

\[
K_i = \left( \frac{\pi}{2} \right)^{1/2} \frac{\mu_2}{(1 - \nu_2)} \sum_{k=0}^{\infty} a_{ik}
\]

(B.78)

where \( K_i \) is the mode II stress intensity factor \( K_{II} \) and \( K_2 \) is the mode I stress intensity factor \( K_I \).
The series representation for \( c_i(t) \) will be truncated at a value \( N \), so that the integral equation in (B.75) will provide \( 2N \) equations. The remaining two equations will be obtained from equation (B.78). Therefore, there will be \( 2(N + 1) \) equations for \( 2(N + 1) \) unknowns, \( a_k \).

### B.7.2 Integral Equation Solution

When the series representation for \( c_i(t) \) (B.76) is substituted into (B.75), the first integral is exactly obtained as

\[
\int_{-1}^{1} \frac{(u + 1)}{(u - t)(t + 1)} c_i(t) dt = -\pi(1 + u) \sum_{k=1}^{\infty} a_k U_{k-1}(u) \tag{B.79}
\]

where \( U_k(u) \) is the Chebyshev polynomial of the second kind of degree \( k \).

Recall that in section 3.7, a new function \( F_0(\zeta) \) was introduced to help eliminate the \((1 + t)\) pole in the second integral of equation (B.75)

\[
F_0(\zeta) = f_0(\zeta) \sqrt{\zeta^2 + 4} = \frac{f_0(\zeta)}{(t + 1) p(t, u)}.
\]

Originally, the \( f_0(\zeta) \) were mistakenly not defined in the program outside the usable range of the FFT. If data was extracted outside this usable range, Mathcad would extrapolate on the last two points available. As a result, because of the \( \sqrt{\zeta^2 + 4} \) term in the \( F_0(\zeta) \) function, gross erroneous results would eventuate when evaluating the \( F_0(\zeta) \) functions for large \( \zeta \). This was corrected by better modeling the decaying behavior of the \( f_0(\zeta) \) kernel functions. The following plots show the \( F_0(\zeta) \) functions with and without this correction.
Figure B.10- Plot of $F_{11}(\zeta)$ with and without the correction.

Figure B.11- Plot of $F_{22}(\zeta)$ with and without the correction.
Figure B.12- Plot of $F_{12}(\zeta)$ with and without the correction.

Figure B.13- Plot of $F_{21}(\zeta)$ with and without the correction.
Now substitute equations (3.13) and (B.76) into the second integral of (B.75):

$$
\int_{-1}^{1} f_{0}(\zeta(u,t)) c_{j}(t) \frac{dt}{(1+t)^{2}} = \sum_{k=0}^{\infty} I_{jk}(u)a_{j}, \quad |u| < 1 \tag{B.80}
$$

where

$$
I_{jk}(u) = \int_{-1}^{1} \frac{p(t,u)}{\sqrt{1-t^2}} F_{0}(\zeta(u,t)) I_{k}(t) dt \tag{B.81}
$$

This integral needs to be evaluated accurately. The results of several different methods and the effects on the solution coefficients have been discussed in section (3.7).

Through equations (B.79) and (B.80), equation (B.75) can be written as the linear system of two integral equations in $a_{j}$:

$$
-\pi(1+u) \sum_{k=1}^{N} a_{j} U_{k-1}(u) + \sum_{k=0}^{N} I_{jk}(u)a_{j} = 0, \quad |u| < 1, \quad i = 1, 2 \tag{B.82}
$$

The infinite sums are replaced by a truncated finite series of $N + 1$ terms.

As mentioned previously, two more equations are provided by equation (B.77) in the form

$$
\sum_{k=0}^{N} (-1)^{k} a_{k} = \left( \frac{2}{\pi} \right)^{1/2} \frac{(1-v_{2})}{\mu_{2}} \left( \frac{1-\alpha}{1+\alpha} \right) K_{1}^{\infty} \tag{B.83}
$$

These $2(N+1)$ equations provided by (B.82) and (B.83) are used to solve for the $2(N+1)$ unknowns $a_{k}$. The method of collocation on the interval $-1 < u < 1$ was used on eqn (B.82) to solve for the unknowns. With as few as 20 collocation points, Fleck, Hutchinson, and Suo (1991) reported convergence. In the studies done here, it seems feasible for 20 collocation points to achieve convergence; however, several problems identified in section 3.7 concerning the integrals $I_{jk}(u)$ have not allowed convergence with any order of collocation points. It has been identified that accurate evaluation of the $I_{jk}(u)$ integrals is key to the convergence of the solution coefficients $a_{k}$. Until the problems
associated with the evaluation of the integrals can be worked out, no convergence can be meaningfully discussed.

Nevertheless, the solution coefficients are dependent on the loading conditions $K_i^\infty$, the geometric parameter $c/H$, and the elastic mismatch parameters $a$ and $b$. Once the solution coefficients are determined for any combination of these aforementioned conditions, the local stress intensity factors $K_i$ can be determined through equation (B.78).

**B.7.3 Plots of the Endpoint Convergence on the $I_{ijk}(u)$ Integrals**

As discussed in section 3.8.1, the endpoints of the integrals raise the most concern, but the following plots along with Figures 3.15 and 3.16 show that these $I_{ijk}(u)$ integrals all converge at $u = \pm 1$.

![Figure B.14- Convergence of the $l_{21k}(u)$ integral for $k = 0, ..., 20$ at the crack-tip $(u = 1)$.](image)

*Convergence at the crack-tip $(u = 1)$ for all degrees $k$ of $T_k(t)$*
Figure B.15 - Convergence of the $I_{11k}(u)$ integral for $k = 0, \ldots, 20$ at the crack-tip ($u = 1$).

Figure B.16 - Convergence of the $I_{22k}(u)$ integral for $k = 0, \ldots, 20$ at the crack-tip ($u = 1$).
**Figure B.17** - Convergence of the $I_{11k}(u)$ and $I_{22k}(u)$ integrals for $k = 0, \ldots, 2$ far from the crack-tip ($u = -1$).

**B.7.4 Plots of the $a_{ik}$ Coefficients for Mode I and Mode II Loading Conditions**

The following plots show the $a_{ik}$ coefficients ($i = 1,2$) under mode I and mode II loading conditions with the full series for $N = 20$ and a reduced series with $N = 18$. 


Figure B.18 - Convergence of $a_{1k}$ under mode I loading conditions for both the full series and the reduced series.

Figure B.19 - Convergence of $a_{2k}$ under mode I loading conditions for both the full series and the reduced series.
Figure B.20- Convergence of $a_{1k}$ under mode II loading conditions for both the full series and the reduced series.

Figure B.21- Convergence of $a_{2k}$ under mode II loading conditions for both full and the reduced series.
B.7.5 Determination of Error Through the Conservation of the J-Integral

The conservation of the J-integral provides an error check on the solution procedure. Recall from equation (2.5) that the local stress intensity factors can be related to the remote stress intensity factors in the following manner

\[ G = J = \frac{1}{E_2} (K_1^2 + K_\|^2) = \frac{1}{E_1} \left[ \left( K_1^\infty \right)^2 + \left( K_\|^\infty \right)^2 \right]. \]

From this principle, the error is given as

\[ Error = 100 \left( \frac{e_{local} - e_{far-field}}{e_{far-field}} \right) \quad (B.84) \]

where

\[ e_{local} = K_1^2 + K_\|^2 \]
\[ e_{far-field} = \frac{E_2}{E_1} \left[ \left( K_1^\infty \right)^2 + \left( K_\|^\infty \right)^2 \right]. \]

The far-field conditions are known as through the applied loads. On the other hand, the local conditions are determined the solution procedure. Thus a consistency check is provided on the solution through equation (B.84).

B.7.6 Determination of the T-stress

This evaluation of the T-stress follows directly from Fleck, Hutchinson, and Suo (1991).

The T-stress is the non-singular and non-vanishing stress parallel to the crack plane. For a distribution of dislocations, this stress is first given as the \( \sigma_{11}(x) \) in equation (3.7a):

\[ \sigma_{11}(x) = \frac{\mu_2}{4\pi(1 - \nu_2)} \int_{-\infty}^{0} \frac{2b_2(\xi)}{x - \xi} + g_1(x - \xi)b_1(\xi) d\xi \]

The local T-stress is determined as the limit \( x \to 0^+ \).
By letting \( c_i(t) = b_i(\xi) \), making the change of variable in equation (B.74), and substituting the representation for \( c_i(t) \) (B.76) into the \( \sigma_{11}(x) \) stress above, the closed form solution of the first integral can be obtained as

\[
\int_{-\infty}^{0} \frac{2b_i(\xi)}{x - \xi} d\xi = -2\pi \sum_{k=0}^{\infty} a_{2k}(1+u)U_{k-1}(u), \quad -1 < u < 1
\]  
(B.85)

which in the limit \( x \to 0^- \), \( u \to 1 \) and equation (B.85) reduces to

\[
\int_{-\infty}^{0} \frac{2b_i(\xi)}{x - \xi} d\xi = -4\pi \sum_{k=0}^{\infty} ka_{2k}.
\]  
(B.86)

Now, by performing the same change of variable and substitutions, the second integral present in the \( \sigma_{11}(x) \) stress can be given as

\[
I(x) = \int_{-\infty}^{0} g_i(x - \xi)b_i(\xi)d\xi = \int_{-1}^{1} g_i(\zeta(t))c_i(t) \frac{dt}{(1+t)^2}.
\]  
(B.87)

In the limit as \( x \to 0^- \), the variable \( \zeta \to -\xi \) (see equation B.74) and \( (1+t) = 2/(1+\zeta) \). In the same fashion as introducing a new function \( F_{\eta}(\zeta) \), another function is introduced to help eliminate the \( (1+t)^2 \) factor in equation (B.87). By separating out a factor of \( (1+t) \) from \( g_i(\zeta) \)

\[
g_i(\zeta) = h_i(\zeta) \left( \frac{1+t}{2} \right) = \frac{h_i(\zeta)}{1+\zeta}
\]  
(B.88)

the integral in equation (B.87) can be evaluated without problems. To increase the computational efficiency, the new functions \( h_i(\zeta) \) can be represented by a Chebyshev-series as such

\[
h_i(\zeta(t)) = \sum_{k=1}^{M} d_{2k} T_{k-1}(t) - \frac{1}{2} d_{11}, \quad \zeta = \frac{1-t}{1+t}
\]  
(B.89)

where \( d_{2k} \) are computed from the known kernel functions \( g_i(\zeta) \) and the relation (B.88) and \( M \) is taken as 40.
By substituting the expression for \( c_i (t) \) (B.76) and the representation for \( h_i (\zeta) \) (B.89) into equation (B.87) and utilizing the orthogonality relations associated with \( T_k (t) \), the integral \( I (\theta^+) \) can be simply evaluated as

\[
I (\theta^+) = \frac{\pi}{2} \sum_{k=0}^{N,M} \sum_{i=1}^{2} d_{ik} a_{ik} \tag{B.90}
\]

where the upper limit is taken to be the smaller of \( N \), the number of collocation points, and \( M \), the number of series coefficients in the representation of \( h_i (\zeta) \).

The local \( T \)-stress is given by substituting eqns (B.86) and (B.90) into the expression for \( \sigma_{11} (x) \):

\[
T = \sigma_{11} (x = 0^+) = \frac{\mu_2}{(1 - \nu_2)} \left( -\sum_{k=0}^{N} k a_{2k} + \frac{1}{8} \sum_{k=0}^{N,M} \sum_{i=1}^{2} a_{ik} d_{ik} \right) \tag{B.91}
\]

Recall the expression for the \( T \)-stress from section 2.3

\[
T = \left( \frac{1 - \alpha}{1 + \alpha} \right) T^* + \sigma^* + c_i \frac{K_i^o}{\sqrt{H}} + c_H \frac{K_H^o}{\sqrt{H}}.
\]

Keeping this in mind, in the solution procedure by letting \( K_i^o = 1 \) and \( K_H^o = 0 \), the \( T \)-stress value obtained in equation (B.91) will now correspond to \( c_i \). Likewise, by letting \( K_H^o = 1 \) and \( K_i^o = 0 \), the \( c_H \) value will be obtained through the \( T \)-stress. Since the \( H \) dependence is included in the solution procedure, it can be assumed as unity.
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