INFLATED CONICAL MEMBRANE
SUBJECTED TO AXIAL COMPRESSIVE LOAD

by

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Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

MASTER OF SCIENCE

in

ENGINEERING MECHANICS

August 1962

Blacksburg , Virginia

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I. INTRODUCTION

In recent years, inflated fabric has drawn attention as a possible new structural material. It has the obvious advantages of being light in weight, easily adaptable to different architectual design, compact for easy transportation, and very economical.

Theoretical solutions to the problem of finding the critical buckling load for thin shells using large deflection theory has been advanced [1]* and subsequent investigations have followed. However, these solutions are based on series approximations which become extremely cumbersome for large displacements. A number of experimental investigations have been conducted by aircraft and fabric companies.

The purpose of this thesis is to present a theoretical method of determining the relation between the applied load and the internal pressure, stresses, and deflection of an inflated conical fabric membrane subjected to axial compressive loads. A conical shape is chosen since it is a common shape in structures of this type and can be reduced to a range of shapes between and including a flat membrane to a cylinder. The method of solution is an energy approach. The resulting equations are solved with the aid of the I. B. M. 650 Ditigal Computer. A program, designed to minimize the amount of work needed to solve any problem in the above range, is presented.

* Numbers in square brackets refer to references.

II. NOMENCLATURE

E Modulus of elasticity.

l Upward displacement in the axial direction.

U Total strain energy.

T Total potential energy.

W_1 Work done by applied compressive load.

W_2 Work done by internal pressure.

P Internal pressure.

q Compressive load per unit area.

t Thickness of the membrane.

u Component of displacement of a point on the middle surface in the x direction.

h Component of displacement of a point on the middle surface in the y direction.

w Component of displacement of a point on the middle surface in the z direction.

D Length of an element of the cone measured in the x direction.

r Radial distance measured to a point on the middle surface.

$\triangle$ Radial displacement of any point on the middle surface.

a Radius of the upper plate.

b Radius of the lower plate.

α Slope angle of cone element.

θ Slope of the middle surface at any point after loading.

ε_x Strain component in the x direction.

ε_y Strain component in the y direction.

ν Poisson's ratio.

δ Deflection of the upper plate.

V_o Initial volume.

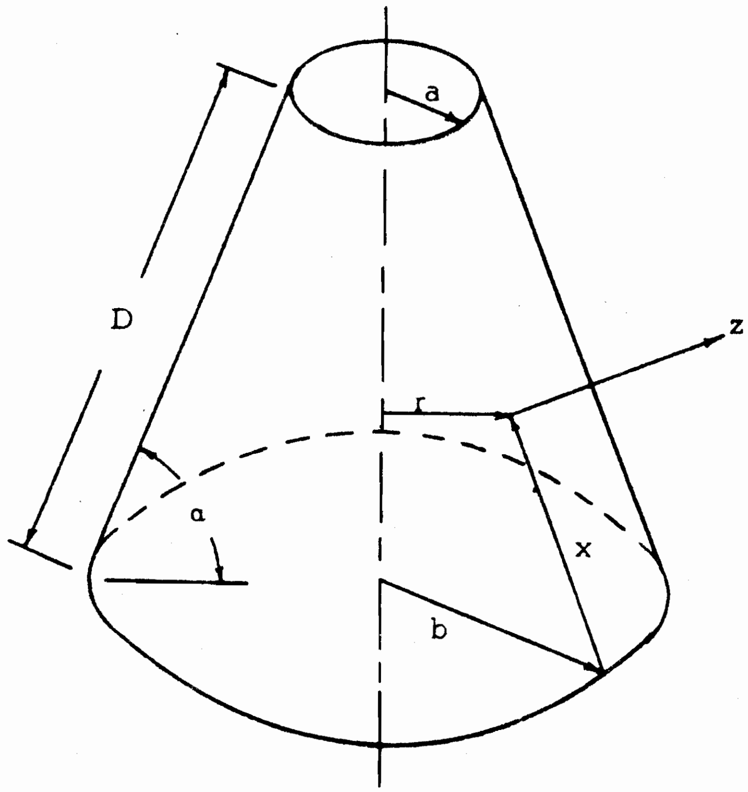
V Volume after deformation.

Other symbols defined where first used.

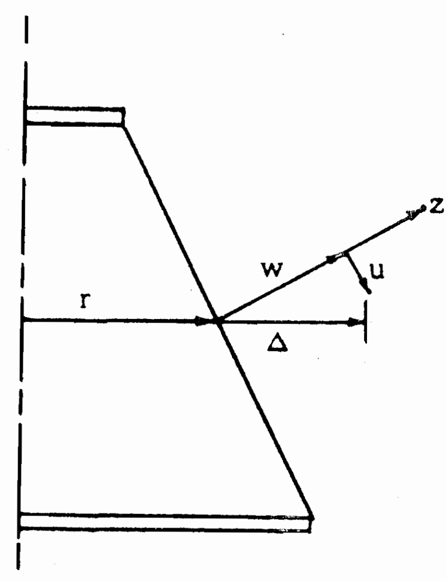
III. DIFFERENTIAL EQUATIONS OF EQUILIBRIUM

A. Description of the Structure

The structure used for this analysis has the shape of a frustum of a cone. At each end is a flat, circular plate which is assumed rigid. The lower plate is assumed fixed and a uniform load is applied to the upper plate. The walls are made of a material which is capable of undergoing large deflections and strains. The wall thickness is very small compared with the other dimensions of the structure.



a.



b.

Figure 1. Shape of the Membrane Before Loading

B. Assumptions

In order to simplify the analysis, the following assumptions are necessary.

1. The material of which the walls of the structure are made is perfectly elastic and obeys Hooke's law under the applied loads.

2. The wall thickness is constant during deformation.

3. A biaxial state of stress exists.

4. Strains in the membrane are small but the deflections are not, in general, limited.

5. Internal pressure is constant although a slight modification of the method allows taking into account variable pressures. See conclusions.

6. The geometry of the shell is entirely defined by specifying the form of the middle surface.

C. Strain - Displacement Relations

Let u, h, w be the displacement components of any point on the middle surface of the conical shell in the x, y and z directions after deformation. Because of symmetry, the displacement of all points lying in the same horizontal circle are equal in the x and z directions. Also, there is no displacement in the circumferential direction for any point on the middle surface. Therefore, h equals zero for any point. The datum for potential energy will be taken as the initial undeflected conical shape with an internal pressure. It can be seen that if an undeflected conical membrane is subjected to an internal pressure, the walls will bulge and the membrane will assume some other shape. However, in choosing a potential energy datum, we are not restricted to possible equilibrium configurations of the membrane.

To determine the circumferential strain, consider an element cut from the cone by two radial planes and two planes transverse to the axis. The sides of the element have lengths dx and dy. Referring to figure 2a the strain ε_y is

$$\varepsilon_y = \frac{dy' - dy}{dy}$$

$$= \frac{\triangle}{r} \qquad (1)$$

From figure 1b,

$$\triangle = w \sin\alpha + u \cos\alpha .$$

From figure 1 ,

$$r = (D - x)\cos\alpha + a.$$

But $a + D\cos\alpha = b$, and equation (1) becomes

$$\varepsilon_y = \frac{w \sin\alpha + u \cos\alpha}{b - x\cos\alpha} \qquad (2)$$

To determine the strain in the x direction, first denote the end points of the infinitesimal length dx by A and B. (See Figure 2b). After deformation, point A moves to a new position, A', and point B to B'. Since h = 0, the length A'B' remains in the xz plane. The x and z components of the displacement from B to B' are u and w respectively. Thus, the components of the displacement from A to A' can be written as $u + \frac{dy}{dx}\,dx$ and $w + \frac{dw}{dx}\,dx$. By definition, strain is given by

$$\varepsilon \quad = \frac{A'B' - AB}{AB}$$

Referring to figure 2b,

$$A'B' = \sqrt{\left(1 - \frac{du}{dx}\right)^2 + \left(-\frac{dw}{dx}\right)^2}$$

$$AB = dx$$

therefore

$$\varepsilon_x = \sqrt{\left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} \quad -1 \qquad (3)$$

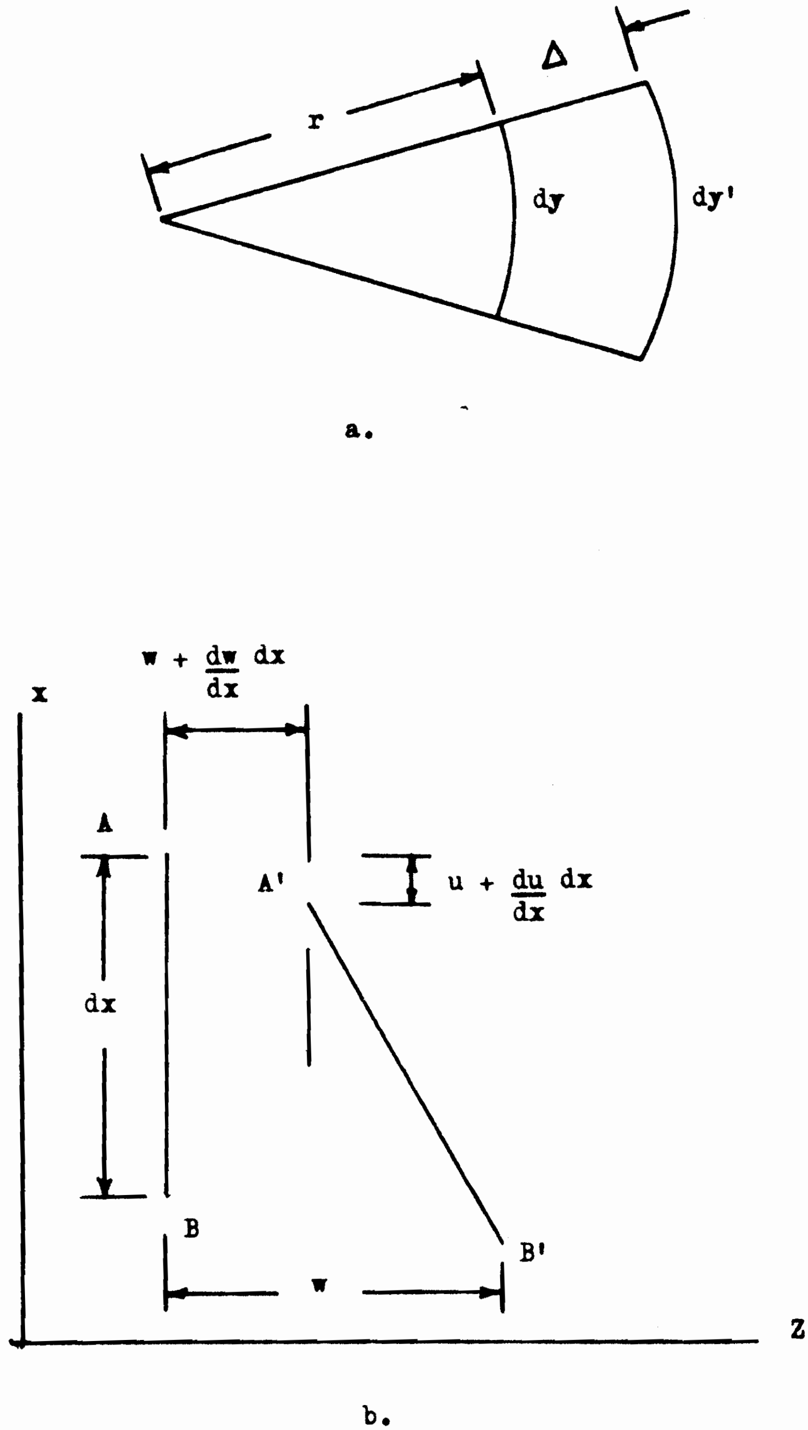


Figure 2. Shape of an Element of the Membrane Before and After Loading.

D. Total Potential Energy

1. Strain Energy due to Stretching of the Membrane

To determine the energy stored in the membrane, consider an infinitesimal element cut from the membrane wall as in Section C. During deformation, assume that the element is acted on by a force in the x direction only and the stress increases from zero to σ_x. Thus the net strain energy stored in the element equals the average stress times the strain in that direction times the volume. Therefore,

$$dU_1 = \tfrac{1}{2}\, \sigma_x \quad \varepsilon'_x \quad t\ dx\ dy$$

From Hooke's law

$$\varepsilon'_x = \frac{\sigma_x}{E}$$

thus

$$dU_1 = \tfrac{1}{2}\, \frac{\sigma_x^{\,2}}{E} \quad t\ dx\ dy \tag{4}$$

Now let the stress in the y direction increase from zero to σ_y. The additional strain energy stored in the element is

$$dU_2 = \tfrac{1}{2}\, \varepsilon'_y \quad \sigma_y \quad t\ dx\ dy - \nu\ \varepsilon'_y \quad \sigma_x \quad t\ dx\ dy \tag{5}$$

From Hooke's law

$$\varepsilon'_y = \frac{\sigma_y}{E}$$

thus

$$dU_2 = \tfrac{1}{2}\, \frac{\sigma_y^{\,2}}{E} \quad t\ dx\ dy - \nu \frac{\sigma_y}{E} \quad \sigma_x \quad t\ dx\ dy$$

Thus the total strain energy in the element is

$$dU = dU_1 + dU_2$$

adding equations (4) and (5)

$$dU = \left(\frac{t}{2} \frac{\sigma_x^2}{E} + \frac{t}{2} \frac{\sigma_y^2}{E} - \nu \frac{\sigma_x \sigma_y}{E} t \right) dx\, dy$$

For this state of stress, Hooke's law is

$$\varepsilon_x = \frac{1}{E} (\sigma_x - \nu \sigma_y); \qquad \varepsilon_y = \frac{1}{E} (\sigma_y - \nu \sigma_x)$$

therefore dU may be written as

$$dU = \frac{E\, t}{2(1 - \nu^2)} (\varepsilon_x^2 + \varepsilon_y^2 + 2 \varepsilon_x \varepsilon_y \nu) dx\, dy$$

The total strain energy can be found by integrating dU over the entire surface of the membrane.

$$U = \int_0^D \int_0^{2\pi (r + \triangle)} \frac{E\, t}{2(1 - \nu^2)} (\varepsilon_x^2 + \varepsilon_y^2 + 2 \nu \varepsilon_x \varepsilon_y) dx\, dy \qquad (6)$$

Substituting from equation (2) and equation (3) and writing $r + \triangle$ as

$$r + \triangle = D \operatorname{Cos}\alpha + a + w \operatorname{Sin}\alpha - x \operatorname{Cos}\alpha + u \operatorname{Cos}\alpha$$

$$= b + w \operatorname{Sin}\alpha - x \operatorname{Cos}\alpha + u \operatorname{Cos}\alpha$$

equation (6) becomes

$$U = \int_0^D \pi(b - x \operatorname{Cos}\alpha + w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha) \frac{E\, t}{1 - \nu^2} \left\{ \left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2 \right.$$

$$+ \left(\frac{w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha}{b - x \operatorname{Cos}\alpha} \right)^2 + 2 \nu \left(\frac{w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha}{b - x \operatorname{Cos}\alpha} \right)$$

$$\left. \left[\sqrt{\left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} - 1 \right] - 2 \sqrt{\left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} + 1 \right\} dx \qquad (7)$$

2. Potential Energy of the External Forces and Internal Pressure

Denote by q the uniform distributed load per unit area acting on the upper surface. The total compressive load is then given by $\pi a^2 q$. Thus, the work*done by the axial compressive force is

$$W_1 = \pi a^2 q \delta$$

where δ is the downward displacement of the upper plate of the cone. This displacement is given by

$$\delta = - \int_0^D \frac{dl}{dx} dx$$

Since $l = w \cos\alpha - u \sin\alpha$, the above expression becomes

$$\delta = - \int_0^D \left(\cos\alpha \frac{dw}{dx} - \sin\alpha \frac{du}{dx}\right) dx$$

Thus

$$W_1 = \pi a^2 q \int_0^D \left(\sin\alpha \frac{du}{dx} - \frac{dw}{dx} \cos\alpha\right) dx \qquad (8)$$

The work*done by the internal pressure P is

$$W_2 = \int_V P \, dV \qquad (9)$$

An elemental volume, of height $dx \sin\alpha$, cut by two adjacent horizontal planes before deformation, is

$$dV_o = -\pi r^2 dx \sin\alpha$$

After deformation, the element of volume will change to

$$dV = -\pi (r + \Delta)^2 \sqrt{\left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} \cos\left[\theta + \left(\frac{\pi}{2} - \alpha\right)\right] dx$$

* Virtual Work

$$= -\pi\,(r+\triangle)^2 \sqrt{\left(1-\frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2}\;(\mathrm{Cos}\,\theta\,\mathrm{Sin}\,\alpha - \mathrm{Sin}\,\theta\;\mathrm{Cos}\,\alpha)\,dx$$

From Figure (2b)

$$\mathrm{Cos}\,\alpha = \frac{1-\frac{du}{dx}}{\sqrt{\left(1-\frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2}} \tag{10}$$

$$\mathrm{Sin}\,\alpha = \frac{-\frac{dw}{dx}}{\sqrt{\left(1-\frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2}} \tag{11}$$

Substituting these into the expression for dv

$$dV = -\pi\,(r+\triangle)^2 \left[\left(1-\frac{du}{dx}\right)\mathrm{Sin}\,\alpha + \frac{dw}{dx}\,\mathrm{Cos}\,\alpha\right] dx$$

Thus, the work done by the internal pressure becomes, from equation (9)

$$W_2 = \int_0^D P\pi \left\{ (b + w\,\mathrm{Sin}\,\alpha + u\,\mathrm{Cos}\,\alpha - x\,\mathrm{Cos}\,\alpha)^2 \left[\left(1-\frac{du}{dx}\right)\mathrm{Sin}\,\alpha + \frac{dw}{dx}\,\mathrm{Cos}\,\alpha\right] - (b + x\,\mathrm{Cos}\,\alpha)^2\;\mathrm{Sin}\,\alpha \right\} dx \tag{12}$$

The total potential energy of the system is then

$$T = U - W_1 - W_2$$

From equations (7), (8) and (12)

$$T = \int_0^D \left\{ \frac{\pi E t_2}{1-\nu}\,(b - x\,\mathrm{Cos}\,\alpha + w\,\mathrm{Sin}\,\alpha + u\,\mathrm{Cos}\,\alpha) \right.$$

$$\left[\left(1-\frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2 - 2\sqrt{\left(1-\frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} + 1\right.$$

$$+ \left(\frac{w\,\mathrm{Sin}\,\alpha + u\,\mathrm{Cos}\,\alpha}{b - x\,\mathrm{Cos}\,\alpha}\right)^2 + 2\,\nu \left(\frac{w\,\mathrm{Sin}\,\alpha + u\,\mathrm{Cos}\,\alpha}{b - x\,\mathrm{Cos}\,\alpha}\right)$$

$$\left[\sqrt{\left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} - 1 \right] \quad \Bigg] \quad - \quad a^2 q \left(\frac{du}{dx} \operatorname{Sin} \alpha - \frac{dw}{dx} \operatorname{Cos} \alpha \right)$$

$$- \pi P \Bigg[\left(b + w \operatorname{Sin} \alpha + u \operatorname{Cos} \alpha - x \operatorname{Cos} \alpha \right)^2 \left(1 - \frac{du}{dx}\right) \operatorname{Sin} \alpha$$

$$+ \frac{dw}{dx} \operatorname{Cos} \alpha - \left(b - x \operatorname{Cos} \alpha \right)^2 \operatorname{Sin} \alpha \Bigg] \Bigg\} \; dx. \qquad (13)$$

E. Equation of Static Equilibrium

Consider a free body diagram of the upper portion of the cone with a plane parallel to the upper plate. (See Figure 3).

From Statics,

$$\Sigma F_{\dot{z}} = 0$$

and this becomes

$$0 = P\,(r + \Delta)^2\,\pi - q\,\pi\,a^2 - 2\pi\,(r + \Delta)\,\sigma_x\,t\,\mathrm{Cos}\left[\theta + \left(\frac{\pi}{2} - \alpha\right)\right]$$

But

$$\sigma_x = \frac{E}{1 - \nu^2}\left[\sqrt{\left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} - 1 + \nu\left(\frac{w\,\mathrm{Sin}\,\alpha + u\,\mathrm{Cos}\,\alpha}{b - x\,\mathrm{Cos}\,\alpha}\right)\right]$$

$$r + \Delta = b + w\,\mathrm{Sin}\,\alpha + u\,\mathrm{Cos}\,\alpha - x\,\mathrm{Cos}\,\alpha$$

and using equations (10) and (11),

$$P\,(b + w\,\mathrm{Sin}\,\alpha + u\,\mathrm{Cos}\,\alpha - x\,\mathrm{Cos}\,\alpha)^2 - q\,a^2 - \frac{2\,E\,t}{1 - \nu^2}$$

$$(b + w\,\mathrm{Sin}\,\alpha + u\,\mathrm{Cos}\,\alpha - x\,\mathrm{Cos}\,\alpha)\left\{1 - \left[\left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right]^{-\frac{1}{2}}\right.$$

$$\left. + \nu\left(\frac{w\,\mathrm{Sin}\,\alpha + u\,\mathrm{Cos}\,\alpha}{b - x\,\mathrm{Cos}\,\alpha}\right)\left[\left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right]^{-\frac{1}{2}}\right\}$$

$$\left[\left(1 - \frac{du}{dx}\right)\mathrm{Sin}\,\alpha + \frac{dw}{dx}\,\mathrm{Cos}\,\alpha\right] = 0 \qquad (14)$$

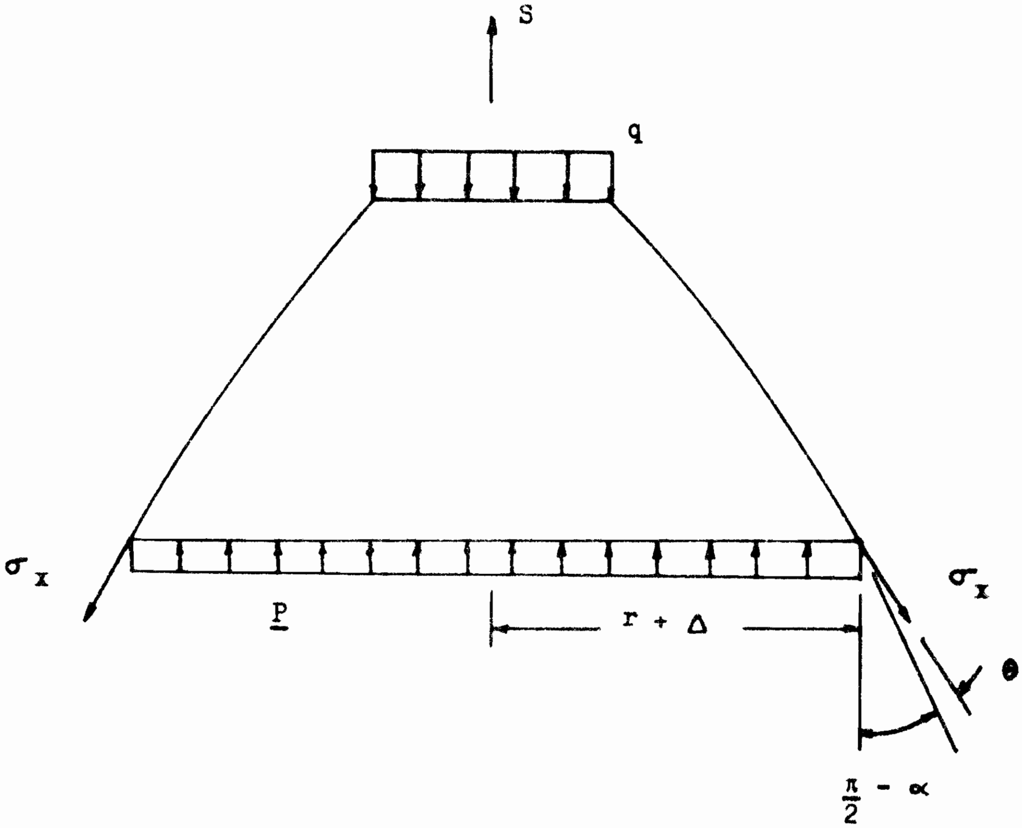


Figure 3. Free Body Diagram of the Upper Portion of the Membrane

F. Euler Equations

Consider an equation of the form

$$V = \int_a^b f\ (x, w, u, w', u')\ dx\ ,\ \text{where } (\ )' = \frac{d(\ )}{dx} .$$

We wish to find functional relationships between x, w, and x, u which render V a minimum. From the calculus of variations, it can be shown that the necessary conditions that such functional relationships extremize V is that f satisfy the Euler Equations

$$\frac{\partial f}{\partial u} - \frac{d}{dx}\left(\frac{\partial f}{\partial u'}\right) = 0 \qquad (15)$$

and

$$\frac{\partial f}{\partial w} - \frac{d}{dx}\left(\frac{\partial f}{\partial w'}\right) = 0 \qquad (16)$$

From equation (13)

$$\frac{\partial f}{\partial u} = \frac{\pi Et_2}{1-\nu^2} \operatorname{Cos}\alpha \left\{ \left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2 - 2\sqrt{\left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} + 1 \right.$$

$$+ \left(\frac{w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha}{b - x \operatorname{Cos}\alpha}\right) + 2\nu \left(\frac{w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha}{b - x \operatorname{Cos}\alpha}\right)\left.\left[\sqrt{\left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} - 1\right]\right\}$$

$$+ \frac{2\pi Et_2}{1-\nu^2} (b - x \operatorname{Cos}\alpha + w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha) \left\{ \left(\frac{w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha}{b - x \operatorname{Cos}\alpha}\right)^2 \operatorname{Cos}\alpha \right.$$

$$\left. + \frac{\nu \operatorname{Cos}\alpha}{b-x \operatorname{Cos}\alpha}\left[\sqrt{\left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} - 1\right]\right\} - \pi\, 2P\, (b + w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha$$

$$- x \operatorname{Cos}\alpha\ ) \left[\left(1 - \frac{du}{dx}\right) \operatorname{Sin}\alpha + \frac{dw}{dx} \operatorname{Cos}\alpha \right] \operatorname{Cos}\alpha\ .$$

This can be written as

$$\frac{\partial f}{\partial u} = A \operatorname{Cos}\alpha\ .$$

In a similar manner

$$\frac{\partial f}{\partial w} = A \operatorname{Sin}\alpha \quad ,$$

where

$$A = \frac{\pi Et_2}{1-\nu^2}\left\{\left(1-\frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2 - 2\sqrt{\left(1-\frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} + 1\right.$$

$$+ \left(\frac{w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha}{b - x \operatorname{Cos}\alpha}\right)^2 + 2\nu\left(\frac{w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha}{b - x \operatorname{Cos}\alpha}\right)$$

$$\left.\left[\sqrt{\left(1-\frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} - 1\right]\right\} + \frac{2\pi Et_2}{1-\nu^2}\left(b - x \operatorname{Cos}\alpha + w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha\right)$$

$$\left\{\left(\frac{w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha}{b - x \operatorname{Cos}\alpha}\right)_2 + \frac{\nu}{b-x \operatorname{Cos}\alpha}\left[\sqrt{\left(1-\frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2} - 1\right]\right\} - 2\pi P$$

$$\left(b + w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha - x \operatorname{Cos}\alpha\right)\left[\left(1-\frac{du}{dx}\right)\operatorname{Sin}\alpha + \frac{dw}{dx}\operatorname{Cos}\alpha\right]$$

then

$$\frac{d}{dx}\left[\frac{\partial f}{\partial u'}\right] = \frac{2\pi E t_2}{1-\nu^2}\left[\left(1-\frac{du}{dx}\right)\left(\frac{dw}{dx}\operatorname{Sin}\alpha + \frac{du}{dx}\operatorname{Cos}\alpha - \operatorname{Cos}\alpha\right)\right.$$

$$\left. - \left(b + w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha - x \operatorname{Cos}\alpha\right)\frac{d^2u}{dx^2}\right]\left\{-1 + \left[\left(1-\frac{du}{dx}\right)^2\right.\right.$$

$$\left.\left. + \left(\frac{dw}{dx}\right)^2\right]^{-\frac{1}{2}} - \nu\left(\frac{w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha}{b - x \operatorname{Cos}\alpha}\right)\left[\left(1-\frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right]^{-\frac{1}{2}}\right\}$$

$$+ \frac{2 Et \pi}{1-\nu^2}\left(b + w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha - x \operatorname{Cos}\alpha\right)\left(1-\frac{du}{dx}\right)$$

$$\left\{\left[\left(1-\frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right]^{-3/2}\left\{\left(1-\frac{du}{dx}\right)\left(-\frac{d^2u}{dx^2}\right) + \frac{dw}{dx}\frac{d^2w}{dx^2}\right\}\right.$$

$$\left[\nu\left(\frac{w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha}{b - x \operatorname{Cos}\alpha}\right) - 1\right] - \nu\left[\left(1-\frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right]^{-\frac{1}{2}}$$

$$\left.\frac{\left(b - x \operatorname{Cos}\alpha\right)\left(dw/dx \operatorname{Sin}\alpha + du/dx \operatorname{Cos}\alpha\right) + \left(w \operatorname{Sin}\alpha + u \operatorname{Cos}\alpha\right)\operatorname{Cos}\alpha}{\left(b - x \operatorname{Cos}\alpha\right)^2}\right\}$$

$$+ 2\pi P \sin \left(\frac{dw}{dx} \sin\alpha + \frac{du}{dx}\cos\alpha - \cos\alpha\right)(b + w\sin\alpha + u\cos\alpha - x\cos\alpha),$$

$$\frac{d}{dx}\left(\frac{\partial f}{\partial w'}\right) = \frac{2\pi E t_2}{1 - \lambda^2}\left[\left(\frac{dw}{dx}\sin\alpha + \frac{du}{dx}\cos\alpha - \cos\alpha\right)\frac{dw}{dx} + (b + w\sin\alpha\right.$$

$$\left. + u\cos\alpha - x\cos\alpha)\frac{d^2w}{dx^2}\right]\left\{1 - \left[\left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right]^{-\frac{1}{2}} +\right.$$

$$\left.\nu\left(\frac{w\sin\alpha + u\cos\alpha}{b - x\cos\alpha}\right)\left[\left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right]^{-\frac{1}{2}}\right\} + \frac{2\pi E t_2}{1 - \nu^2}\,(b + w$$

$$\sin\alpha + u\cos\alpha - x\cos\alpha)\ \frac{dw}{dx}\left\{\left[\left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right]^{-3/2}\right.$$

$$\left\{\left(1 - \frac{du}{dx}\right)\left(-\frac{d^2u}{dx^2}\right) + \frac{dw}{dx}\frac{d^2w}{dx^2}\right\}\left[1 - \nu\left(\frac{w\sin\alpha + u\cos\alpha}{b - x\cos\alpha}\right)\right]$$

$$+ \nu\left[\left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right]^{-\frac{1}{2}}\ \frac{(b - x\cos\alpha)(dw/dx\sin\alpha + du/dx\cos\alpha) +}{(b - x}$$

$$\left.\frac{(w\sin\alpha + u\cos\alpha)\cos\alpha}{\cos\alpha)^2}\right\}\ -2\pi P\cos\alpha\,(b + w\sin\alpha +$$

$$u\cos\alpha - x\cos\alpha)\left(\frac{dw}{dx}\sin\alpha + \frac{du}{dx}\cos\alpha - \cos\alpha\right)$$

Substitute these expressions into equations (15) and (16).

Let

$$R = b + w\sin\alpha + u\cos\alpha - x\cos\alpha$$

$$S = \frac{w\sin\alpha + u\cos\alpha}{b - x\cos\alpha}$$

$$D = \left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2$$

Solve equations (15) and (16) for $\frac{d^2w}{dx^2}$ and $\frac{d^2u}{dx^2}$

$$\frac{d^2w}{dx^2} = \frac{2\ E\ t}{1 - \nu^2} R \left[1 + (\nu S - 1)\ D^{-\frac{1}{2}} \right] \left\{ \left[A \text{ Sin } \alpha + 2P\ R \text{ Cos } \alpha \right.\right.$$

$$(\text{dw Sin } \alpha + \text{du Cos } \alpha - \text{Cos } \alpha) - \frac{2\ E\ t}{1 - \nu^2} \left(\frac{dw}{dx} \text{ Sin } \alpha + \frac{du}{dx} \text{ Cos } \alpha - \right.$$

$$\left. \text{Cos } \alpha \right) \left[1 + (\nu S - 1)\ D^{-\frac{1}{2}} \right] \frac{dw}{dx} - \frac{2\ E\ t}{1 - \nu^2} R \frac{dw}{dx}$$

$$\frac{(b - x \text{ Cos } \alpha)(dw/dx \text{ Sin } \alpha + du/dx \text{ Cos } \alpha) + (w \text{ Sin } \alpha + u \text{ Cos } \alpha) \text{ Cos } \alpha}{(b - x \text{ Cos } \alpha)^2}$$

$$\left. D^{-\frac{1}{2}} \right] - \frac{2\ E\ t\ R}{1 - \nu^2} \left(1 - \frac{du}{dx}\right) (\nu S - 1)\ D^{-3/2} \left[\left(A \text{ Cos } \alpha \frac{dw}{dx} + \right.\right.$$

$$\left(1 - \frac{du}{dx}\right) A \text{ Sin } \alpha - 2 \frac{dw}{dx} P \text{ Sin } \alpha\ R \left(\frac{dw}{dx} \text{ Sin } \alpha + \frac{du}{dx} \text{ Cos } \alpha - \text{Cos } \alpha\right)$$

$$\left.\left. + 2P \text{ Cos } \alpha\ R \left(1 - \frac{du}{dx}\right) \left(\frac{dw}{dx} \text{ Cos } \alpha + \frac{du}{dx} \text{ Cos } \alpha - \text{Cos } \alpha\right) \right] \right\}$$

$$\frac{(1 - \nu^2)^2}{(2\ E\ t)^2\ R^2\ (1 + (\nu S - 1)\ D^{-\frac{1}{2}})} \qquad (19)$$

$$\frac{d^2u}{dx^2} = \left\{ A \text{ Cos } \alpha - 2P \text{ Sin } \alpha\ R \left(\frac{dw}{dx} \text{ Sin } \alpha + \frac{du}{dx} \text{ Cos } \alpha - \text{Cos } \alpha\right) \right.$$

$$+ \frac{2\ E\ t}{1 - \nu^2} \left(\frac{dw}{dx} \text{ Sin } \alpha + \frac{du}{dx} \text{ Cos } \alpha - \text{Cos } \alpha\right) \left[1 + (\nu S - 1)\ D^{-\frac{1}{2}} \right]$$

$$\left(1 - \frac{du}{dx}\right) + \frac{2\,E\,t_2}{1 - \nu^2}\, R\, \nu \left(1 - \frac{du}{dx}\right) \quad \frac{(b - x\,\mathrm{Cos}\,\alpha\ )\ (dw/dx\ \mathrm{Sin}\,\alpha\ +}{(b - x\,\mathrm{Cos}\,\alpha\ )^2}$$

$$\underline{du/dx\ \mathrm{Cos}\,\alpha\ \ ) + (w\ \mathrm{Sin}\,\alpha\ + u\ \mathrm{Cos}\,\alpha\ )\ \ \mathrm{Cos}\,\alpha} \quad D^{-\frac{1}{2}}$$

$$- \frac{2\,E\,t_2}{1 - \nu^2}\, R \quad \left(1 - \frac{du}{dx}\right) \quad (\nu S - 1) \quad D^{-3/2} \quad \frac{dw}{dx} \quad \left.\frac{d^2 w_2}{dx^2}\right\}$$

$$\frac{1 - \nu^2}{2\,E\,t\,R\left[\left(1 + (\nu S - 1)\,D^{-\frac{1}{2}}\right) - \left(1 - \frac{du}{dx}\right)^2 (\nu S - 1)\,D^{-3/2}\right]} \qquad (20)$$

IV. SOLUTION OF THE EULER EQUATIONS

Since an exact analytical solution of equations (19) and (20) appears impossible, consider the following substitutions:

Let $H = \frac{du}{dx}$ $G = \frac{dw}{dx}$

then $\frac{dH}{dx} = \frac{d^2u}{dx^2}$ and $\frac{dG}{dx} = \frac{d^2w}{dx^2}$

Substitution of the above into equations (19) and (20) along with

$$H = \frac{du}{dx} \tag{21}$$

$$G = \frac{dw}{dx} \tag{22}$$

gives a system of four simultaneous first order ordinary differential equations, which can be integrated numerically using an existing sub-routine [2] designed for the I. B. M. 650 digital computer.

Boundary Conditions

A solution of equations (19), (20), (21) and (22) requires four boundary conditions.

At $x = o$, $u = w = o$

From equation (14), for $x = o$, we have $u = w$, $Pb^2 - q\ a^2 -$

$$\frac{2\ E\ t\ b^2}{1 - \nu^2}\left\{1 - \left[\left(1 - \frac{du}{dx}\right)^2 + \left(\frac{dw}{dx}\right)^2\right]^{-\frac{1}{2}}\right\}$$

$$\left[\left(1 - \frac{du}{dx}\right) \operatorname{Sin} \alpha + \frac{dw}{dx} \operatorname{Cos} \alpha\right] = 0 \tag{23}$$

At $x = D$, the strain $\varepsilon_y = o$. Therefore

$$w \operatorname{Sin} \alpha + u \operatorname{Cos} \alpha = 0 \tag{24}$$

A solution to these equations can be obtained with the aid of the I. B. M. 650 digital computer. The equations were incorporated into a subroutine by Franz Edelman for the solution of systems of first order ordinary differential equations using the Bell Interpretive System. However, since all of the boundary conditions are not specified at one point, the final results must be obtained by trial and error.

The procedure for a computer solution for finding the deflection of the upper plate is as follows:

1. Enter in cards 1 and 2 of the program deck and the values of a, b, ν, E, t, α, P and q for each particular problem.

2. Assume a value for either H or G at x=0.

3. Calculate from equation (23) the value of the slope not assumed in 2.

4. Enter the values from 2 and 3 into the proper positions in card 3 of the program deck.

5. Select the desired interval and enter in card 4 of the program deck. Solution time is about 55 seconds for each interval.

6. The program is designed so that the value of $w \sin\alpha + u \cos\alpha$ at x = D is calculated for each run. If this value is zero, then equation (24) is satisfied and the assumed values in steps 2 and 3 are correct. If the value is not zero, repeat the procedure.

7. When equation (24) is satisfied by the proper assumption in step 2, the deflection of the upper plate can be obtained using the equation

$\delta = -w \cos\alpha + u \sin\alpha$ where w and u are the values at x = D.

8. The program is designed so that the values of w, u, $\frac{dw}{dx}$ and $\frac{du}{dx}$ are printed for each interval point. Thus the value for the strain and therefore the stress can be determined for each point.

V. NUMERICAL EXAMPLE

To illustrate the method of solution, a numerical example will be discussed. The values of E t and ν are chosen to approximate those values for a rubber coated fabric. The other dimensions are chosen so that the deflections are of the order of one foot and can be easily visualized.

Consider:

E t = 9600 lb per ft. ν = .4

α = 60 degrees a = 1 ft. b = 6 ft.

P = 100 lb per square ft.

Four intervals were considered:

.1 ft. 1/4 ft. 1/2 ft. 1 ft.

The difference in solutions for 1 foot intervals as compared to .1 foot intervals was less than 1%. The following solutions were determined using a 1 foot interval.

Four cases were considered.

Case I P = 100 lb per square foot

q = 0

Case II P = 100 lb per square ft.

q = 318.31 lb per square ft.

Case III P = 100 lb per square ft.

q = 636.63 lb per square ft.

Case IV P = 100 lb per square ft.

q = 3600 lb per square ft.

The results for Cases I - IV are tabulated below.

Case	G_o	H_o	w_{10}	u_{10}	δ - Ft.
I	.0755	-.02635	.552	-.764	-.938
II	.0776	-.02359	.302	-.618	-.686
III	.0810	-.0207	-.344	.548	.646
IV	-.537	.690	-4.90	8.49	9.80

where

$$G_o = \left.\frac{dw}{dx}\right|_{x = 0}$$

$$w_{10} = \left. w \right|_{x = 10}$$

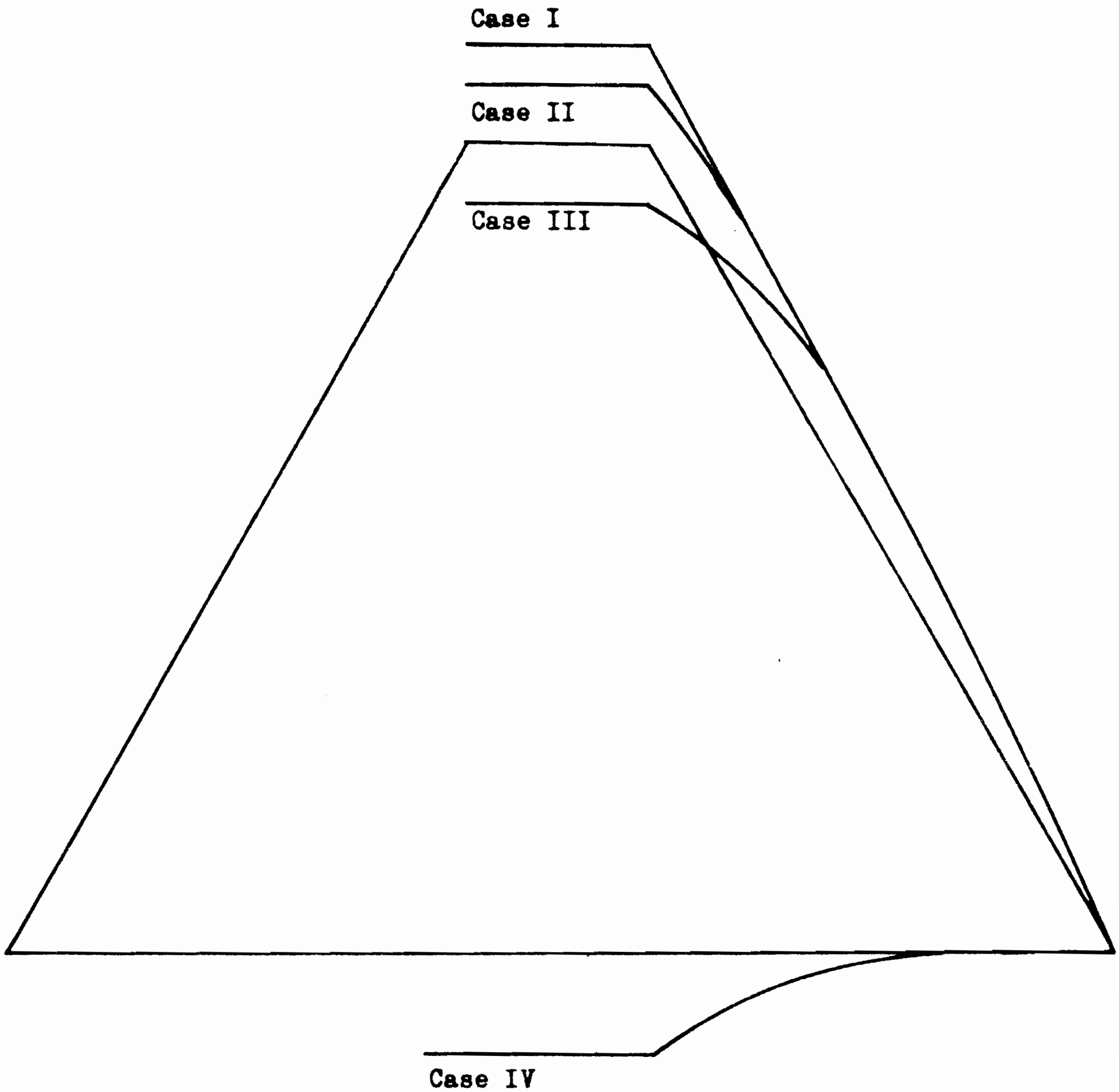


Figure 4. Deflection Profiles

BELL SYSTEM PROGRAM SHEET

PROBLEM: ____________________ WRITTEN BY: ____________

Card No. 1 - 4	Deck No. 5 6	Loc. 1st wd. 7 - 9	No wd 10	+ −	O_1	O_2 or A	B	C	Prob. No. 77-79	Tr. Inf 80	Remarks
0001	08	010	6	+	0	000	000	000			
				+	2	000	000	050			
				+	1	000	000	050			
				+	1	000	000	050			a
				+	6	000	000	050			b
				+	1	047	164	250	010	3	α
0002	08	016	5	+	9	600	000	053			E
				+	1	000	000	050			t
				+	4	000	000	049			y
				+	1	000	000	051			P
				+	1	000	000	051			q
									010	3	
0003	08	500	5	+	0	000	000	000			x.
				+	3	351	000	050			G.
				+	1	000	000	000			H.
				+	0	000	000	000			w.
				+	0	000	000	000			u.
									010	3	
0004	08	997	3	−	4	000	000	050			
				+	1	000	000	050			h
				+	4	000	000	050			
									010	3	
0005	08	035	6	+	0	303	015	021			
				+	0	304	015	022			
				+	3	016	017	023			
				+	3	018	018	000			
				+	2	012	000	000			
				+	4	023	000	024	010	3	
0006	08	041	3	+	3	024	011	025			
				+	9	000	000	050			D
				+	0	203	000	200			
									010	3	
0007	08	200	6	+	0	204	201	600			
				+	3	503	021	050			
				+	3	504	022	051			
				+	3	500	022	052			
				+	1	014	050	000			
				+	1	000	051	000	010	3	

BELL SYSTEM PROGRAM SHEET

PROBLEM: ______________________ WRITTEN BY: ______________

Card No. 1 - 4	Deck No. 5 6	Loc. 1st wd. 7 - 9	No wd 10	+ -	O_1	O_2 or A	B	C	Prob. No. 77-79	Tr. Inf 80	Remarks
0008	08	206	6	+	2	000	052	053			
				+	1	050	051	054			
				+	2	014	052	055			
				+	4	054	055	056			
				+	2	012	502	057			
				+	3	057	057	058	010	3	
0009	08	212	6	+	3	501	501	000			
				+	1	000	058	059			
				+	0	300	059	060			
				+	2	060	012	061			
				+	3	061	018	000			
				+	1	000	056	000	010	3	
0010	08	218	6	+	4	000	055	000			
				+	3	000	053	000			
				+	3	000	025	062			
				+	3	061	018	000			
				+	3	000	011	000			
				+	1	000	056	000	010	3	
0011	08	224	6	+	3	000	056	063			
				+	3	060	011	000			
				+	2	063	000	000			
				+	1	000	012	000			
				+	1	000	059	000			
				+	3	000	024	063	010	3	
0012	08	230	6	+	3	057	021	064			
				+	3	501	022	000			
				+	1	064	000	000			
				+	3	000	053	000			
				+	3	000	019	000			
				+	3	000	011	064	010	3	
0013	08	236	6	+	1	063	062	000			
				+	2	000	064	064			
				+	3	501	021	065			
				+	3	502	022	000			
				+	1	000	065	066			
				+	4	012	060	067	010	3	
0014	08	024	6	+	3	018	056	000			
				+	2	000	012	068			
				+	3	067	068	069			
				+	1	069	012	069			
				+	2	066	022	070			
				+	3	067	067	000	010	3	

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BELL SYSTEM PROGRAM SHEET

PROBLEM: ______________________ WRITTEN BY: ______________

Card No. 1 - 4	Deck No. 5 6	Loc. 1st wd. 7 - 9	No wd 10	+ -	O_1	O_2 or A	B	C	Prob. No. 77-79	Tr. Inf 80	Remarks
0015	08	248	6	+	3	000	067	071			
				+	3	066	055	072			
				+	3	054	022	000			
				+	1	000	072	072			
				+	4	072	055	000			
				+	4	000	055	000	010	3	
0016	08	254	6	+	3	000	067	000			
				+	3	000	053	000			
				+	3	000	018	000			
				+	3	000	025	046			
				+	3	046	501	072			
				+	3	069	070	000	010	3	
0017	08	260	6	+	3	000	025	047			
				+	3	047	501	073			
				+	3	070	053	000			
				+	3	000	022	000			
				+	3	000	011	000			
				+	3	000	019	074	010	3	
0018	08	266	6	+	3	064	021	000			
				+	1	000	074	000			
				+	2	000	072	000			
				+	2	000	073	000			
				+	3	000	069	000			
				+	3	000	053	000	010	3	
0019	08	272	6	+	3	000	025	074			
				+	3	057	064	000			
				+	3	000	021	075			
				+	3	070	053	000			
				+	3	000	021	000			
				+	3	000	019	000	010	3	
0020	08	278	6	+	3	000	011	045			
				+	3	045	501	076			
				+	3	064	501	000			
				+	3	000	022	077			
				+	3	070	053	000			
				+	3	000	057	000	010	3	
0021	08	284	6	+	3	000	022	000			
				+	3	000	011	000			
				+	3	000	019	000			
				+	1	000	075	000			
				+	1	000	077	000			
				+	2	000	076	000	010	3	

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PROBLEM: ______________________ WRITTEN BY: ______________

Card No. 1 - 4	Deck No. 5 6	Loc. 1st wd. 7 - 9	No wd 10	+ -	O_1	O_2 or A	B	C	Prob. No. 77-79	Tr. Inf 80	Remarks
0022	08	290	6	+	3	000	071	000			
				+	3	000	068	000			
				+	3	000	057	000			
				+	3	000	053	000			
				+	3	000	025	077			
				+	3	071	071	000	010	3	
0023	08	296	6	+	3	000	068	000			
				+	3	000	068	000			
				+	3	000	501	000			
				+	3	000	501	000			
				+	3	000	058	000			
				+	3	000	010	000	010	3	
0024	08	302	6	+	1	000	069	000			
				+	3	000	053	000			
				+	3	000	053	000			
				+	3	000	025	000			
				+	3	000	025	078			
				+	2	074	077	079	010	3	
0025	08	308	6	+	4	079	078	079			
				+	3	046	057	080			
				+	3	047	057	081			
				+	3	064	022	082			
				+	3	079	501	000			
				+	3	000	071	000	010	3	
0026	08	314	6	+	3	000	068	000			
				+	3	000	057	000			
				+	3	000	053	000			
				+	3	000	025	083			
				+	2	082	045	000			
				+	1	000	080	000	010	3	
0027	08	320	6	+	1	000	081	000			
				+	2	000	083	083			
				+	3	058	068	000			
				+	3	000	071	000			
				+	2	069	000	000			
				+	3	000	053	000	010	3	
0028	08	326	6	+	3	000	025	000			
				+	4	083	000	085			
				+	9	001	085	551			
				+	9	001	079	552			
				+	9	002	501	553			
				+	0	204	332	620	010	3	

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PROBLEM: ______________________ WRITTEN BY: ______________

Card No. 1 - 4	Deck No. 5 6	Loc. 1st wd. 7 - 9	No wd 10	+ −	O_1	O_2 or A	B	C	Prob. No. 77-79	Tr. Inf 80	Remarks
0029	08	332	6	+	9	005	500	350			
				+	9	002	551	355			
				+	0	410	350	356			
				+	2	042	500	000			
				+	0	201	640	337			
				+	1	050	051	357	010	3	
0030	08	338	3	+	0	410	357	357			
				+	0	000	340	000			
				+	9	999	999	999			
									010	3	

VI. CONCLUSIONS

In the assumptions, it was indicated that a modification in the method of analysis would allow taking into account the variation in pressure. This could be done by assuming that the internal pressure P used in the analysis was the pressure after deformation. During deformation, the mass of the enclosed gas remains constant. Thus, knowing the initial and final volumes and the final pressure, the initial internal pressure could be determined using the appropriate gas law.

Consider the family of curves generated by plotting the load versus deflection for different values of internal pressure. Then, the value of the mass of the enclosed gas can be determined for different points along each curve of constant P. Thus, the curves of load versus deflection for variable pressure and constant mass can be determined by joining points of equal mass on the above curves.

A solution of the resulting equations on the 650 requires approximately ten trial runs. Thus, total machine time is less than two hours per solution. The accuracy of the solution did not depend greatly upon the length of the intervals tried but was very sensitive to the assumed values of H_o and G_o.

It can be shown from the data for Cases I - IV that all points on the middle surface of the membrane deflect approximately vertically. This result is in agreement with that of Shen [3] for a spherical membrane. For Cases I - III, the sign of the left hand side of the left hand side of equation (3) is positive so that tensile stresses exist in the membrane. However, for Case IV, the sign of the left hand side of equation (3) is negative so that compressive stresses exist in the membrane. In this case, wrinkling of the membrane would occur. Thus, the load would generally be considered excessive under these conditions and the solution found is not applicable.

Other investigations suggested by the results of this thesis are:

1. A comparison with the results of Ho [4]. Letting $\alpha = 90°$, the shape of the membrane becomes cylindrical.

2. An experimental investigation using a cylinder, which would be an easy shape to deal with.

3. Very important would be a simplified analysis assuming only vertical movement of the middle surface.

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VIII. ACKNOWLEDGEMENTS

The author wishes to express his appreciation to Dr. R. Chicurel, Major Professor of this thesis, for his help, guidance and advice.

Also, appreciation is extended to Professor J. H. Sword for his assistance in the computer phases of this thesis and to Professor D. H. Pletta, who originally suggested this topic to the author.

IX. VITA

The author was born in Walterboro, South Carolina, on August 21, 1937. He graduated from Hempstead High School, Hempstead, New York, on June 27, 1955. In September 1955, he entered Virginia Polytechnic Institute and received his B. S. degree in Engineering Mechanics in June 1959. Since September 1959, the author has been a one-half time Instructor in Engineering Mechanics at Virginia Polytechnic Institute and has been pursuing courses leading to the degree of Master of Science in Engineering Mechanics.

Charles D. Eskridge III

Charles D. Eskridge, III

INFLATED CONICAL MEMBRANE

SUBJECTED TO AXIAL COMPRESSIVE LOAD

by

Charles D. Eskridge, III

ABSTRACT

A theoretical analysis for finding the relationship between load, internal pressure and deflections is derived using energy methods. The total energy is minimized and the resulting equations solved with the aid of the I. B. M. 650 digital computer.

A numerical example is given to illustrate this method.