

A MODIFIER-BASED PHILOSOPHY OF WHOLE NUMBER

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(ABSTRACT)

This paper offers an alternative philosophy of whole number in which number-words are treated as being of the semantic-type *modifier*. Other accounts of number in which number-words are treated as names, syncategoremata, determiners, and predicates are considered and rejected based on their failure to provide number-words with the necessary compositional semantics. This leaves only modifiers as plausible candidates to play number-words' role in natural language. After the semantic-type modifier is chosen, a decision between number-words' being adjectival or adverbial modifiers must then be made. I argue that due to a lack of entities to be ascribed adjectival numerical properties we must settle on an adverbial treatment. After developing this treatment, I close with an attempt to explain seemingly singular-term uses of number-words in arithmetical statements like ' $2 + 2 = 4$ ' in terms of these claims' stating the rules for substituting equivalent modifier-phrases in non-mathematical usages.

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I. INTRODUCTION AND OVERVIEW*

[W]hen one has been occupied with these questions for a long time one comes to suspect that our way of using language is misleading, that number-words are not proper names of objects at all.

—Gottlob Frege, *Posthumous Writings*, March 1925.

1. Frege initiated the modern inquiry into the concept of number with a pair of evidently loaded questions: The first sentence of the *Grundlagen* asks, What is the number one? and To what does the sign 1 refer?² Taken for granted here appears to be that there is such a *thing* as the number one.

Despite his early advocacy of this strong kind of platonist view, Frege ultimately seems to have changed his mind about the objectual status of numbers (as witnessed by the above quotation). And I think he was quite right to do so. However, once this naïve, or pretheoretical, notion of numbers as objects has been abandoned, it's not exactly clear how, and with what, it should be replaced. A variety of more or less interesting proposals for proceeding from this initial position have been made, but there is as of yet no consensus about which might be pursued most fruitfully. In the spirit of "letting a hundred flowers blossom" in such an environment, then, the present work will provide the beginnings of what I take to be an attractive and new non-objectual philosophical theory of number. It

* Thanks to all the members of my committee for raising many difficult objections to this material. Whether I was able to adequately respond to them or not is an open question, but the work has undoubtedly been improved by their careful reading and suggestions.

² (Frege 1884b: 11)

is my hope that the view will not be merely another option, but will be deemed worthy of further serious investigation.

2. Recent work in the philosophy of mathematics has tended to conclude that traditional ways of arguing for or against numbers as objects quickly lead to stalemate.³ It's not that the typical arguments advanced are particularly bad or unconvincing ones, but that each seems to be counterbalanced by an equipollent argument or based on premises that wouldn't be accepted by intended targets. Nominalists roll out Benacerraf's Non-Uniqueness and Access concerns⁴ or some type of thought experiment in which we discover that "all there is is concrete."⁵ Platonists respond by continuing to rely on the Quine/Putnam Indispensability Thesis⁶ or something like Lewis's Philosophical Humility Credo.⁷ But one's initial intuitions about abstract objects seem to be the most important determiners of one's final view on the matter.

I hope to avoid the trenches of this old debate by focusing on an idea recently put forward by Stephen Yablo. Yablo suggests that maybe the hope for an argument showing numbers and the like to be "*obnoxious*" is in fact "dead and gone," but that there may be another way. Perhaps it's not that mathematical objects are intolerable, but that when we examine our mathematical practices in a

³ E.g., (Burgess & Rosen 1997: I.A), (Balaguer 1998: Chapter 8), (Yablo 2001: §12).

⁴ Non-Uniqueness in (Benacerraf 1965), Epistemic-access concerns in (Benacerraf 1973).

⁵ E.g. Burgess and Rosen's Oracle of Philosophy in (1997: 3). See also (Azzouni 1994: 56).

⁶ See esp. (Putnam 1979).

⁷ I.e., "Do *you* want to tell the mathematicians that there really are no numbers?" See (Lewis 1991).

"calm and unprejudiced way," we find that the most plausible account simply doesn't involve them.⁸ This (hopefully) unprejudiced examination of our practices will be the main project of Chapter II.

Instead of jumping off from the use of number-words within *mathematical* practice, however, I'll begin by trying to get a better understanding of how number-words function within natural language, especially as they occur within phrases such as, 'thirty-two books' or 'two hundred dollars.' It will be seen that, in order to explain our capacity to understand infinitely many number-words, these number-words must somehow have a compositional semantics, i.e., 'two' must make the same contribution to 'thirty-two' and 'two hundred' (and maybe even to 'twenty'). After settling on this desideratum of semantic compositionality, it will become clear that the eligible semantic-types for number-words in numerical phrases is extremely limited: Specifically, I'll argue that common accounts that take number-words to be the names of abstract objects, syncategorematic, determiners, or predicates all fail to meet the requirement. The failure of these other theories of number will leave only the semantic-type modifier as a plausible candidate for number-words' role in natural language.

The reason this result is significant in regard to the question whether or not numbers are objects is that, while modifiers do have a semantic value (i.e., they're not syncategorematic), they're non-referential. Thus, if number-words can only be modifiers and modifiers are non-referring, we have good reason to believe

⁸ (Yablo 2001: 87)

that numbers are not objects at all. This is a conclusion, then, that favors a nominalistic treatment of mathematics and makes a positive contribution to the deadlocked nominalism/platonism debate. Furthermore, this point in favor of number irrealists seems to be made in a way less prone to provoke platonist worries about metaphysical axe grinding than most of the aforementioned traditional methods. So, from this initial investigation into the functioning of number-words in natural language, some deeper philosophical implications seem to immediately suggest themselves.⁹

After settling on modifiers as the appropriate semantic-type for number-words in simple numerical phrases, we'll need to give an account of three further aspects of statements involving these number-words. As Peter Simons suggests,

An adequate philosophical theory of whole numbers has to be able to both [(i)] give an account of what we accomplish when we make empirical ascriptions of number.... and also to [(ii)] provide an account of the content and validity of the propositions of arithmetic...An adequate account must further [(iii)] provide some explanation of the link between the two. (1982a: 160)

An account of the content of empirical ascriptions of number will be the project of Chapter III, while Chapter IV will be reserved for consideration of the content and validity of arithmetical claims. A discussion of the link between the two types of statements will serve to link the respective chapters as well.

3. Before moving on to the examination of number-words within natural language, I want to pause to make note of, and quickly argue for, a

⁹ Cf. (Hofweber 2005a: §6).

methodological assumption that will be in force throughout: Whatever our account of number ultimately turns out to be, it should be what Thomas Hofweber has termed *uniform*.¹⁰ The notion of a uniform account of number can be clarified by considering the following statements exemplifying what we'll call the attributive, the singular-term, and the symbolic uses of a number-word.¹¹

(1_{att}) Jupiter has four moons.

(1_{sing}) The number of Jupiter's moons is four.

(1_{sym}) Jupiter has 4 moons / The number of Jupiter's moons is 4.

A uniform account of number allows every occurrence of a number-word to make *the same contribution* to the underlying logical form of any statement involving it. Uniformity is not an aspect of the most well-known theory of number, so it will be useful to say a little about why uniformity seems so desirable and how this relatively familiar account fails to meet its demands.

The most commonsense account of number-words in natural language takes 'four' as it appears in (1_{att}) and (1_{sing}) to represent a different word in each. According to this account, in (1_{att}), 'four' disappears at the level of logical form; in other words, it's *syncategorematic*.¹² In (1_{sing}) on the other hand, 'four' names the mathematical object four. Also on this account, the pair of sentences in (1_{sym}) simply abbreviate (1_{att}) and (1_{sing}) respectively. The account is therefore clearly

¹⁰ See (Hofweber 2005a: 180-181).

¹¹ Unless one or the other seems more appropriate in a particular context I'll use 'number-words' in an expanded sense to include numerals from this point forward.

¹² I.e., "Jupiter has four moons" $\Rightarrow \exists w \exists x \exists y \exists z [\text{MoonOfJupiter}(w) \wedge \text{MoonOfJupiter}(x) \wedge \text{MoonOfJupiter}(y) \wedge \text{MoonOfJupiter}(z) \wedge \neg \exists u \{ \text{MoonOfJupiter}(u) \wedge u \neq (w \text{ or } x \text{ or } y \text{ or } z) \}]$.

not uniform in above sense. 'Four' in (1_{att}) makes no single contribution to the statement's logical form, while 'four' in (1_{sing}) contributes a reference to four.

To see why this lack of uniformity is problematic, we'll need to have a slightly closer look at this account's characterization of the logical forms of (1_{att}) and (1_{sing}) . Let's take '#' to be a function mapping all concepts (or monadic predicates) whose extensions can be put into one-one correspondence to a unique object: their number.¹³ Then if we let 'J' stand for the predicate *__ is a moon of Jupiter*, this account symbolizes the logical form of the attributive and singular-term uses of 'four' in our examples as (2_{att}) and (2_{sing}) .¹⁴

$$(2_{\text{att}}) \quad \exists_4 x Jx$$

$$(2_{\text{sing}}) \quad \#J = 4$$

These logical forms are quite different, yet (1_{att}) and (1_{sing}) seem to be obviously truth-functionally equivalent. This equivalence is hard to explain if the logical forms of the two expressions differ as drastically as (2_{att}) and (2_{sing}) . If (1_{att}) is a claim about only Jupiter's moons while (1_{sing}) is about Jupiter's moons *and* the number four, what guarantees that the truth-conditions never vary between them?¹⁵

To get a little clearer on why the differing logical forms of (2_{att}) and (2_{sing}) is objectionable, consider the following. If (1_{att}) and (1_{sing}) are equivalent claims

¹³ $\#(\text{__ is a blind mouse}) = 3$ and $\#(\text{__ is a day of the week}) = 7$, for example. Cf. (Boolos 1998).

¹⁴ ' $\exists_n x Fx$ ' is defined recursively: $\exists_0 x Fx \Rightarrow \neg \exists x Fx$ and $\exists_{n+1} x Fx \Rightarrow \exists x [Fx \wedge \exists_n y (Fy \wedge x \neq y)]$.

¹⁵ See (Hofweber 2005b: 262-264).

(and this seems to be beyond question), then the truth-value of each must be the same in all possible worlds. However, if there is a world in which numbers don't exist, then, granting that (2_{sing}) is the logical form of (1_{sing}) , (1_{sing}) will be false in this numberless world, even if Jupiter has four moons there: The identity claim of (2_{sing}) will be undefined in a numberless world because '4' doesn't refer in such a world. Someone might object here that numbers necessarily exist so they exist in all possible worlds; that is, this supposed example could never possibly be realized. Even if we accept this counterproposal, however, there is still trouble with treating (2_{att}) and (2_{sing}) as the logical forms of (1_{att}) and (1_{sing}) .

Suppose someone (a nominalist perhaps) states, (i) "Jupiter has four moons, but there are no numbers." This supposed individual may even phrase her claim, (ii) "The number of Jupiter's moons is four, but there are no numbers." Now, if (2_{sing}) is the logical form of (1_{sing}) , then (ii) is an outright contradiction. It is tantamount to saying that something is identical to something that doesn't exist. However, again relying on the apparent logical equivalence between (1_{att}) and (1_{sing}) , the pair must be intersubstitutable *salva veritate*. This turns out not to be the case though since (i) is not contradictory: Whatever you think about the claims of the nominalist, surely they're not blatantly inconsistent.¹⁶ So, as previously stated, it seems as if it's preferable if 'four' can be made to make the same contribution to the form of each of (1_{att}) and (1_{sing}) since this would make the above equivalence more explainable. Without 'four' in the singular-term

¹⁶ Cf. (Field 1980: 94)

statements being supposed to contribute this additional entity, the abstract object four, the objections raised no longer can find a foothold.

The apparent truth-condition equivalence of (1_{att}) and (1_{sing}) raises a further worry for an account on which the word four is treated as being ambiguous, standing for something different in each. Consider Groucho Marx's famous use of the ambiguous 'flies': "Time flies like an arrow. Fruit flies like a banana." Attempting to mean 'flies' in the second sentence as it's meant in the first and vice versa results in nonsense: there are no time flies, and fruit doesn't fly at all. But trying to use the 'four' in (1_{sing}) as it's meant in (1_{att}) and *vice versa* at least doesn't obviously change the meaning of either statement. If 'four' actually represents two different words in (1_{att}) and (1_{sing}) though, we should expect that the substitution of one for the other in these sentences to result in a loss of sense.¹⁷

Byeong-uk Yi offers a final interesting argument that further multiplies the concerns about treating 'four' as an ambiguous term. Yi asks us to consider the following brief argument:

Bill and Hilary are two Americans. Chelsea is another American. Altogether they're three Americans; for two plus one is three (2002: 80).

If 'two' really means different things in the first and last sentence though, then the above inference is unwarranted because of an equivocation on 'two'. The argument would be analogous to,

¹⁷ Cf. (Wittgenstein 1953: 176): "If I say "Mr. Scot is not a Scot", I mean the first "Scot" as a proper name, the second one as a common name....Try to mean the first "Scot" as a common name and the second one as a proper name...When I say the sentence with this exchange of meanings I feel that its sense disintegrates."

Wachovia and First National are both banks. All banks are adjacent to bodies of water. Therefore, Wachovia and First National are adjacent to bodies of water; for they're both banks.

This second "argument" is clearly quite confused. However, the number-word-involving argument about Bill, Hillary, and Chelsea appears to be unquestionable. A uniform treatment of number-words can make sense of this validity and so, again, looks considerably more appealing.

Based on these three general reasons, then, I'll take the uniformity assumption as established as at least highly desirable in what follows. If our theory of number can comply with its demands, it will undeniably be explanatorily neater than an account that does not.

II. NUMBER-WORDS AND COMPOSITIONALITY.

...the multiplication of Words would have perplexed their Use...
—John Locke, *Essay*: III.i.3.

1. The present chapter will begin the formulation of a modifier (non-objectual) theory of number by arguing that modifiers are the only plausible semantic-type available for number-words. The argument will run basically as follows: (i) The semantics of number-words must be compositional; (ii) Number-words can only be given a compositional semantics if they're modifiers;¹⁸ Therefore, (iii) That's what they must be.¹⁹

I'll begin filling in this sketch by first briefly reviewing the considerations that make compositionality seem to be such a non-negotiable feature of the semantic content of language in general, and then go on to suggest that the need for compositionality is particularly evident in the case of our understanding of number-words. Next, I'll show that prominent theories that take number-words to be (i) the names of abstract objects, (ii) syncategorematic, (iii) determiners, or (iv) predicates all fail to meet the semantic compositionality requirement. These failures will leave only the semantic-type modifier as a plausible candidate for a number-word's function in language. Finally, since modifiers can either be

¹⁸ *Cf.* (Ionin & Matushansky *MSa*, *MSb*).

¹⁹ Jerry Fodor has forcefully employed similarly structured arguments from the need for compositionality to attack prototype theories of concepts (1998: Chapter 5), epistemically based semantics (2001: 7-10), and the thesis that the content of language is explanatorily prior to the content of thought (2001: 10-15).

adjectival or adverbial, a decision between the two will have to be made before a fuller theory of number can subsequently be developed from this starting point. A brief overview of some of the factors that will go into making this decision in Chapter III will close out the chapter.

2. Wilhelm von Humboldt famously characterized language as "the infinite use of finite means." In attempting to explain how this infinite usage is possible for creatures with finite intellects, compositionality has appeared to many as an indispensable assumption—even, "the only game in town."²⁰ The basic argument leading to compositionality seems to run,

1. There are infinitely many expressions that a person can understand.
2. Every person's representational capacities are finite.
- ∴ 3. This infinity of expressions must be finitely representable.²¹

This finite representability then gets explained by an appeal to every word's making the same contribution to statements that employ it; in other words, to the language's semantics being compositional. With this assumption in force, a finite number of words can combine in an infinite number of ways, while at the same time ensuring that the meaning of any whole is determinable in terms of the meanings of its simpler constituent parts. As Jerry Fodor puts it, compositionality is whatever makes it the case "that the things that the expression (mutatis

²⁰ (Kaye 1993)

²¹ (Fodor 1998: 94-95)

mutandis the concept) 'brown cow' applies to are exactly the things to which the expressions 'brown' and 'cow' apply" (Fodor 2001: 6).

There is, perhaps, no clearer example of the infinite use of finite means in language than the case of number-words. Because the need for any useful language's semantics to exhibit semantic compositionality seems have become so universally accepted, taking a compositional semantics as a desideratum of an account of number-words (merely a small portion of any natural language) should *a fortiori* be accepted as well—and I will so accept it.

Thus, in giving an account of how number-words function in our language, any theory that can't provide them with a compositional semantics will have to be abandoned. This means that it will be required of any acceptable account that wherever, say, '2' appears, it must contribute what it always would to the meaning of the complete numerical expression. As it turns out, this compositionality requirement does quite a lot of work and rules out many of the currently accepted theories of number. Each of the following sections will examine a different theory with respect to its ability to provide the necessary compositionality for number-words and will show that only a modifier-based account will ultimately work.

3. The most familiar story about number-words and numerals takes each to function as a name for an abstract mathematical object. Elementary logic books and introductory mathematics books nearly all contain some passage giving a

brief explication of this type of theory: E.g., "Remember, '2' is not the number 2 itself, but is simply a name that refers to this object, whatever it may be." I'll suggest, however, that if we accept this kind of treatment, compositionality is sacrificed; therefore, the account will have to be deemed unacceptable.

If number-words really function as names, we lose compositionality almost by definition. Syntactically complex names (and names in general, of course) work as names precisely because they *don't* compose.²² What allows 'The Scarlet Pimpernel' to refer to "[t]hat demmed, elusive Pimpernel" rather than some contextually relevant small red flower is the fact that in a complex name the constituent words don't make their normal contributions; that is, they don't compose—The Scarlet Pimpernel is neither red nor a flower.

Now, suppose that a compositional semantics for a number-word like '25' can be given. Then '25' is syntactically complex: The meaning of '25' is only understood compositionally if '2' and '5' each make their own particular contributions. But if '2' and '5' are both names (for 2 and 5 respectively), then it seems like '25' shouldn't name the mathematical object 25, but the pair of objects, 2 and 5. If, on the other hand, we extend the analogy of the functioning of syntactically complex names, like 'The Scarlet Pimpernel', then '25' names 25 only on the assumption that '2' and '5' *don't* contribute to a compositionally determined whole. Here we can see, then, why an account of number-words as names is unable to provide for the necessary compositional semantics.

²² Cf. (Fodor 1998: 100): "Names...succeed in their job because they *aren't* compositional."

To pronounce the end of any names theory at this point would certainly be a bit hasty, however. It may be objected that the '25' being discussed is really just an abbreviation for 'twenty-five', and so it's wrong to look for the contribution from '2' to be what it would be when '2' appears in the units place, for example. This point may be correct, but it still ultimately leaves us with the same basic problem. If '25' really is short for 'twenty-five' then, supposing compositionality, we ought to get a name for 20 and 5 rather than for 25.

A further, related complaint also suggests itself immediately: Couldn't we say that this name just mentioned is not a name for 20 and 5, but for 20-and-5? The reply, though, remains essentially the same. If '20' contributes the mathematical object 20 and '5' 5, then the placement of an 'and' between them shouldn't form a name of an essentially distinct object. Names joined by a conjunction nevertheless name nothing more than the individuals so conjoined, consider e.g., 'Samneric', 'Bill and Ted', and 'Batman and Robin'.

A further possible objection to the non-compositionality of names suggested above might be based on the following scenario. Here we have a man named John. There's John's son to the left. Next to him is John's son's son. In the crib upstairs John's son's son's son is sleeping. Etc. So, from an initial named person (or object) we're capable through a compositional procedure to procure names for an infinite number of other people (or objects). If the names of numbers work in a similar fashion based on an initial element (say, zero) and a generating function like " 's son" (the successor function in the case of numbers), then we

have all the composition we need: Why should '25' be composed out of '2' and '5'?

One way of responding to this type of objection is to again appeal to the uniformity constraint argued for in Chapter I. If we want '2' to make the same contribution for each of its appearances, then it should make the same contribution in '2' and '25.' If '2' is simply the name we apply the result of two applications of the successor function to zero, and '25' the name we apply to the result of twenty-five applications, the contribution of '2' in each isn't constant. This reply admittedly relies on my having established the desirability of the uniformity constraint. It would be better, though, if there was a reply that wasn't dependent on this earlier argumentation. I think there is such another response.

If number-words are composed as suggested by our objector as e.g. "the successor of the successor of zero" is 2, number-words turn out, then, to be definite descriptions and not names after all. This leaves this attempted account open to Paul Benacerraf's famous objection about the non-uniqueness of numbers (Benacerraf 1965): Which is *the* successor function that generates the numbers from zero? Should we repeatedly take, with Zermelo, the singleton of the previous number to generate the series of natural numbers \emptyset , $\{\emptyset\}$, $\{\{\emptyset\}\}$, etc. or is it taking the set of all predecessors that is our generator, as John von Neumann suggests, i.e., \emptyset , $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, etc.? Nothing in mathematical theory provides a means for deciding between the two choices, so if number-words compose in this fashion, their being names seems to be ruled out since there is no unique referent available and no way to establish one except by arbitrary fiat.

If number-words and numerals are taken to function semantically like names do, then, it looks like the compositionality we need in order to explain our comprehension of an infinite range of number-words is unavailable. Therefore, we'll have to move beyond this particular type of account.

4. A syncategorematic theory of number that takes number-words to disappear at the level of logical form—replaced by numerical quantifiers,²³ for example—is subject to many of the same objections raised in the previous section. Thus, the treatment here will be on the brief side. In the end, though, it will be clear that, given the compositionality requirement, the account will ultimately have to be abandoned as well.

The general proposal for how to offer a syncategorematic account of number-words begins with the following recursive definition of what Harold Hodes (1984) calls *numerical quantifiers*, written in the form $\exists_n x Xx$.

$$\exists_0 x Xx \Leftrightarrow \neg \exists x Xx$$

$$\exists_{n+1} x Xx \Leftrightarrow \exists x [Xx \wedge \neg \exists_n y (Xy \wedge x \neq y)].$$

Given this definition and the uniformity constraint, '2' reduces to $\exists_2 x Xx$ in statements involving it; '5', to $\exists_5 x Xx$; and so on.

Let's again ask about how this account will treat a syntactically complex

²³ (Hodes 1984) is a good example of this type of account. The numerical quantifiers to be employed in this type of account will be explained presently.

number-word like '25'—complex, to repeat, because we're trying to achieve compositionality. If '2' and '5' make exactly the contribution they would when appearing separately, we should expect that, instead of the hoped-for $\exists_{25x} Xx$, '25' reduces to $\exists_{2x} Xx \exists_{5x} Xx$ in statements involving it. Or, if, as in the last section, we take '25' to be shorthand for 'twenty-five', we'd get the also incorrect contribution of $\exists_{20x} Xx \exists_{5x} Xx$. The trouble, then, is exactly parallel.

Despite the similarity of the problems and objections between this syncategorematic account and the names treatment, however, there is at least one significant difference between the two. The previously considered objection that '25' ought not be understood as $\exists_{20x} Xx$ and $\exists_{5x} Xx$, but as $\exists_{20x} Xx$ -and- $\exists_{5x} Xx$, i.e., $\exists_{20x} Xx \wedge \exists_{5x} Xx$, is more effective in relation to the present theory. This last characterization comes very close to the desired contribution we're looking for from '25'. All that must be overlooked is the fact that some of the things satisfying the first existential claim can also satisfy the second, which is problematic because this allows the statement that, e.g., "Twenty-five people ate pizza," to be true even if only twenty people actually did. Although this is a real worry, the syncategorematic account nevertheless appears to be better off than the names treatment in this particular regard.

Despite this advantage, however, the account fails the compositionality requirement in another respect: There seems to be no way for it to explain how '4' in '400' can make the same contribution that it normally would. This is due to the fact that the numerical quantifier contributed by '4' in any statement can't itself

quantify over numerical quantifiers, as it would need to if it could make the required contribution to a statement like '400'. We need something like, $\exists_4 (\exists_{100}) X (\exists_{100})$, to get the necessary four $\exists_{100x} X$ s. This construction, however, is prohibited unless 'four' is allowed to somehow become of a higher type in certain locations, which would violate the earlier argued-for uniformity constraint.

So, although a syncategorematic account of number-words' contributions has a minor advantage over the names theory, the account will also have to be abandoned because of its failure to provide complete compositionality.

5. Thomas Hofweber (2005a) has offered a third, quite different alternative treatment of number-words that will now be considered. He suggests that philosophers of mathematics have too long overlooked Generalized Quantifier Theory (GQT), a common linguist's tool, and the handling of number-words it provides. If we take Hofweber's advice and look to GQT for our treatment of number-words, then we'll take them to be *determiners*.²⁴ I'll start by briefly reviewing Hofweber's proposal and then move on to an attempt to show that it ultimately will not pan out due to a failure of determiners to compose.

To get a better idea of the GQT claim that number-words are determiners, consider the fact that number-words like 'two' and 'four' very often play a role similar to quantifiers like 'all' and 'some'. Compare: "All dogs are rambunctious" and "Two dogs are rambunctious," or "Jupiter has some moons" and "Jupiter has

²⁴ A quick discussion of what determiners are according to GQT can be found in the following paragraph.

four moons." According to GQT, words like 'all' and 'some' are of semantic-type determiner: That is, they compose with predicates to form noun-phrases.²⁵ Common examples of determiners include 'every', 'all', 'some', 'the' and the like, and the following are some noun-phrases involving these determiners: 'every child', 'all students', 'the football team', etc. If number-words appear to behave just like these determiners, it's natural enough to suppose that they *just are* determiners. This is Hofweber's contention.

Keeping with our basic strategy, however, before doing anything else with this interesting suggestion we'll have to first ask whether Hofweber's number-determiners will allow a compositional semantics for number-words to be given. Consideration of several representative examples of determiners failing to compose below, however, will suggest that they will not.

As just mentioned, determiners in general appear to be ineligible for composition with other determiners. Consider the following nonsensical claims in which determiner composition is attempted.

- (3) *All some players at pizza.
- (4) *The every students were tired.
- (5) *Not all most men cried.

If 'four' is to make the same contribution to each of 'four books,' 'four hundred books,' and 'twenty-four books' and number-words are determiners, then

²⁵ Or equivalently, generalized quantifiers. Cf. (Barwise & Cooper 2002).

determiners will have to be able to compose with one another. But the above examples suggest that this just doesn't happen.²⁶

GQT doesn't overlook the fact that sentences like (6) - (8) below often occur, though, or the fact that the theory says that a determiner, like 'all', can't combine with another determiner, like 'eleven'. The theory therefore attempts to offer a way of avoiding the apparent determiner-determiner composition concerns. However, the way GQT theorists have chosen to solve this problem seems to go very much against the spirit of the theory.

To resolve the concerns about combining determiners in statements like the following,

- (6) All eleven players ate pizza.
- (7) The two students were tired.
- (8) Not all six men cried.

GQT takes 'the two' in 'the two students' to be its own distinct determiner. That is, it's not treated as being composed of the determiners 'the' and 'two'. However, again, this goes against the very idea that has made GQT so unique and successful. The theory is thought to be such a great innovation and advance

²⁶ Cf. (Ionin & Matushansky *MSa*, *MSb*). According to the technical construction rules of GQT, determiners don't compose with one another. For an expression of GQT, e.g., $\langle A, B \rangle$, A is the type of the expression's arguments and B is the value it returns. *e* means type entity and *t* means type truth-value. So, as an example, a predicate has type $\langle e, t \rangle$: It accepts entities as arguments and returns truth-value T or F depending on whether or not the given *e* is in the extension of the predicate in question. A determiner has type $\langle \langle e, t \rangle, \langle e, \langle e, t \rangle \rangle$. So, given a predicate argument, a noun-phrase of type $\langle e, \langle e, t \rangle \rangle$ is returned, which can't take a determiner as an argument. Neither can the determiner compose with another determiner straightaway since it only accepts predicate argument. Therefore, we can see that determiners won't compose with other determiners.

beyond the first-order treatment of noun-phrases like 'two men' and 'all men' because it provides a way of determining the meaning of complex noun-phrases based solely on the contribution of their simpler parts.²⁷ So, to already have 'the' be a part of the theory that makes a specific contribution to noun-phrases in which it appears, and for this to be true of 'two' also, but to then form a new determiner for 'the two' would appear to be a step in the wrong direction. I'd suggest, then, making the change from the determiner treatment of number-words to the modifier account I'll eventually present below and that this change would allow GQT to better comply with its own apparent goals.

There are other (non-composition related) concerns about a determiner treatment of number-words that may be worth briefly considering. As mentioned in Chapter I, number-words often occur in both attributive positions and as apparently singular-terms, as in (1_{att}) and (1_{sing}) respectively.

(1_{att}) Jupiter has four moons

(1_{sing}) The number of Jupiter's moons is four.

However, if a word like 'two' is a determiner wherever it occurs, then its occurrence in a numerical statement like (1_{sing}), results in a split noun-phrase, which is for the most part disallowed in languages such as English.²⁸

²⁷ Cf. (Hofweber 2005a: 186-187)

²⁸ Split NPs are apparently more commonly encountered in languages like German, however. Cf. (De Kuthy 2001) (Direct translation of the German followed by its grammatical English counterpart):

"Books written has she yet none." ⇒ "She hasn't written any books yet."

"The children have all an ice cream got." ⇒ "All the children got an ice cream."

Even in the few cases when the relocation of a *non*-numerical determiner is allowable in English, the form of the resulting shifted sentence doesn't seem to share any similarities with a supposedly number-determiner-involving sentence. Consider the following few examples.²⁹

- (9) *a.* I ate two bagels. / Two is the number of bagels I ate.
b. I ate both bagels. / *Both is how many bagels I ate.
c. / The bagels have both been eaten by me.
- (10) *a.* Two students play basketball / The number of students playing is two.
b. All students play basketball / *The quantity of students playing is all.
c. / The students have all played basketball.
- (11) *a.* Two people are funny. / Two is the number of funny people.
b. Some people are funny./ *Some is the amount of funny people.
c. / ?

So, besides the main concern about a determiner treatment of number-words failing to allow for a compositional semantics, there seem to be other considerations that should also make us question the proposal as well.

6. Byeong-uk Yi (1999 & 2002) is the most prominent defender of the claim that number-words should be considered as predicates.³⁰ The failure of this treatment of number-words to meet the compositionality requirement will quickly become evident, however. Therefore, once again, the account will have to be abandoned.

Yi's account suggests that whenever an ascription of number is made, the

²⁹ These examples draw on some provided in (Kuhn 1999).

³⁰ See also (Partee 2002).

property *being n thing(s)* is attributed to some plurality. To see why numerical properties can't provide for compositionality, we can consider any complex cardinal-involving property; say, *being 25 years old*, for example. The distinctions considered in previous sections between whether the numerical properties compose as 20 and 5 or 2 and 5 are irrelevant here. If each of the composing numerals represents a numerical property, there can *never* be any bearer of the composed property: Nothing is ever both 20 and 5 of the same thing, nor could anything be 2 and 5 of the same thing. Therefore, there is no way for Yi's numerical properties to make their usual contributions to the semantics of a complex cardinal in an attempt to provide a compositional semantics for the numbers.³¹ Thus, the account fails our basic requirement.

7. Given what's been said thus far, number-words' being of the semantic-type modifier is almost the only remaining option. The argument is therefore basically a "What else?"-argument, which, though not ideal, should nevertheless not be shied away from because of its not being apodictic.

By now, the general idea of compositionality and its importance should be clear, so all that should be required is to show that modifiers do compose for this type to be preferred in our treatment of number-words in natural language. Fortunately, examples of non-numerical modifiers combining with one another

³¹ If we also want to examine the GQT treatment of predicates as in the case of Hofweber's determiners, we can see that according to the theory's rules predicates can't provide for a compositional semantics of our number-words either. As mentioned in note 26, predicates have type $\langle e, t \rangle$, so they can only accept type e , or entity, as an argument; not another predicate. Thus, composition fails.

are plentiful, and can even become quite complex. Consider the following lunch special, for instance:

(12) Garlicky Pork Sausage Stuffed Crisp Fried Maryland Soft Shell Crab³²

Furthermore, not only are adjectival modifiers commonly composed with one another, but adverbial modifiers can do the same.³³ E.g.,

(13) Jump very quickly with caution today.

So, modifiers seem to be just what we've been looking for as far as being eligible for composition is concerned. If numbers are modifiers, then they'll compose in the required manner. So, as an example, given some, say, cards, we can form the modified predicate *being 52 cards* in the following manner: *being 52 cards* reduces to *being 5 10s and 2 1s cards*.

I've thus far confined the GQT account of the (failed) composition of other semantic-types to footnotes, but perhaps it will be useful to see how the modifier account is supposed to technically achieve its compositionality.³⁴

³² From a posting on the "Language Log" 27 May 2006. This was apparently a menu item at a restaurant somewhere near Boston.

³³ This will actually end up being an unfortunate fact because it will force a decision between adjectival and adverbial modifiers; this choice will be made in the following chapter.

³⁴ See (Ionin & Matushansky *MSa*, *MSb*) for full details.

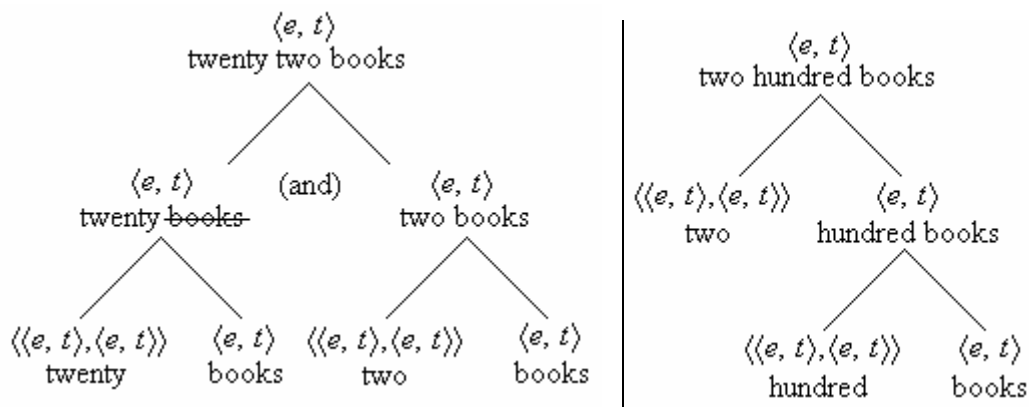


Figure 1. Two types of numerical composition with modifiers.

Modifiers have GQT type $\langle \langle e, t \rangle, \langle e, t \rangle \rangle$, so, given a predicate (or verb) to compose with, a number modifier will return a predicate (or verb) that is again eligible to compose with another number modifier and compositionality is thus achieved.

8. Since modifiers can either be adjectival, as in "a blue sweater," or adverbial, as in "ran quickly," we'll eventually need to make a choice about which is more appropriate for the semantic-type of number-words. Consideration of a number of paradigm modifier-involving statements, though, doesn't seem to settle the question in itself.

- (14) *a.* I wore a blue shirt. / Blue is the color of the shirt I wore.
b. I wore two shirts. / Two is the number of the shirts I wore.

- (15) *a.* I went to the store speedily. / Speedily is how I went to the store.
b. I went to the store two times. / Two times is how often I went to the store.
- (16) *a.* A stone statue is preferable. / Stone is the preferable type of statue.
b. Two statues are preferable. / Two is the number of statues I prefer.

In the statements in (14) - (16), an adjectival treatment may appear *slightly* more natural, but adverbs seem to do just about as well.

If we could have our choice between the two, with respect to the overall goal of finding a non-objectual theory of number, an adverbial treatment might be preferable. That is, an adjectival modifier like 'blue', for example, appears to be more plausibly treatable as some kind of object—say, the set of all blue things—than does an adverb like quickly (the set of all quickly things?).³⁵ So, an adverbial account would have at least this *prima facie* advantage. Yet, numbers simply don't seem to behave like adverbs at all. It appears to be much more likely that, in a claim like "Jupiter has four moons," we're ascribing some property to Jupiter; and this is how most philosophers have tried to understand these claims. There are, then, considerations pulling in both directions here. The settling of this matter is therefore best postponed until the next chapter.

³⁵ This is the basic idea behind the supposed ontological parsimony of "adverbial theories of X" in general. The adverbial theory of perception of (Chisholm 1957), for example, attempts to get rid of troublesome objectual sense-data by appealing to things like being appeared to redly and such.

III. AN ADVERBIAL THEORY OF NUMBER ASCRIPTIONS.

If we are asked why we count, we are tempted to say that we count in order to find out the number of things.

—Wittgenstein, *Lectures on the Foundations of Mathematics*.

1. In the last chapter, I suggested that the semantics of number-words is (and must be for creatures with finite intellects) compositional and that this compositionality can't be explained if number-words are treated as names for abstract objects, syncategoremata, determiners, or predicates. This left only the semantic-type modifier as a plausible candidate for a number-word's role in natural language. Also in the last chapter, I began to outline a further choice that had to be made about a number-word's semantic-type before we could go on to investigate the content of empirical ascriptions of number: If number-words are modifiers, are they adjectival or adverbial modifiers? The present chapter will argue that, despite initial appearances to the contrary, number-words are *adverbial* modifiers.

The argument, in outline, will run as follows. If number-words are adjectival modifiers, then there must be bearers of the numerically modified properties that they variously form. But, as a matter of fact, there don't seem to be any such eligible candidates amongst everyday objects. In the *Grundlagen*, Frege sought to solve this apparent problem by proposing that not objects, but *concepts*, were the bearers of numerical properties. His selected bearers, *viz.* concepts, come close to doing the required work, but ironically his own "relativity of number

argument" ultimately shows concepts to fall short of achieving their task. At this point, then, the possibility of an adjectival modifier account of number appears slim. By noting the role of counting procedures in resolving the trouble with Frege's initial proposal, an alternative account of the content of empirical number ascriptions emerges, a natural consequence of which is that number-words should be treated as adverbial modifiers. Thus, it will be concluded that number-words are in fact adverbial modifiers.

2. To repeat: We've already decided that number-words are going to have to be treated as modifiers, but we're not yet settled on whether these modifiers are adjectival or adverbial. At first glance, observing statements like (1) and (2), adjectival modifiers appear to be the more natural choice.

(1) I ate onion bagels. / I ate two bagels.

(2) I went quickly. / I went twice.

In general, when making an ascription of number, it seems as if it's the noun-phrase, not the verb, that's modified, e.g. in (1) 'bagels' becomes 'two bagels'. However, a closer look at the supposition that number-words function adjectivally raises some serious concerns. Consider the following inferences:

(3) I ate onion bagels \therefore I ate something onion.

(4) I own blue shirts \therefore I own something blue.

Each of these inferences seems to be a paragon of right reasoning. Yet, when the

switch from statements involving common adjectival modifiers to statements involving numerical modifiers is made, things quickly go awry. Consider:

(5) I ate two bagels \therefore I ate something two.

(6) I own six shirts \therefore I own something six.³⁶

These problem inferences involving adjectival number modifiers can be seen as a special case of the first big worry for an adjectival modifier treatment of number, what I'll call *Russell's Dilemma*: No matter what we take as the subject of an empirical ascription of number, there doesn't seem to be anything that can be the bearer of a numerical property such as *being two philosophers*—the subject is either *one* thing or *there is no single subject* that could bear the property.³⁷ The inferences in (5) and (6) above seem to fail because there *can't be* anything that is *two*. Further examination of this dilemma and the problems it raises for adjectival modifiers will be the subject of the next section.

3. Russell struggled with the dilemma introduced in the last section primarily in his early work, *The Principles of Mathematics*. He found that with regards to claims like (7) below there didn't seem to be any possible logical subject.³⁸

(7) Jones and Brown are two philosophers.

³⁶ (Hugly and Sayward 2006: 219)

³⁷ (Laycock 2006: 76)

³⁸ See (Russell 1903: Chapter VI, esp. § 70-71)

That is, there is *prima facie* no eligible bearer of the property *being two philosophers*. Henry Laycock, from whom the phrase 'Russell's Dilemma' is derived, offers a clear exposition of the worry.

If we suppose that [(7)] has a single compound subject, then, trivially, that subject has to be one, not two. If however we suppose that [(7)] does not have a single compound subject—that all there is is merely the individual Brown on the one hand, and the individual Jones on the other—then once again we have nothing of which 'are two' is predicable (2006: 76).

This dilemma led Russell to say some quite bizarre things, and the strange positions that philosophers have been inclined to adopt in order to deal with the dilemma don't peak with Russell, as a review of some representative solutions in the following paragraphs will reveal.

It's important to note here that whether or not *we* agree that Russell's Dilemma is a real problem, a solution is required for anyone who takes number-words to be adjectival modifiers. If 'two' modifies the predicate '*___ are books*' into the predicate '*___ are two books*', there must be something that can be *two books*. Considering several attempts to resolve this trouble and their failures will set the stage for the adverbial account that I'll offer near the close of the chapter.

First, let's look at Russell's own response to the dilemma. He begins by distinguishing between a 'class as many' and a 'class as one'.

In a class as many, the component terms, though they have some kind of unity, have less than is required for a whole. They have, in fact, just so much unity as is required to make them many, and not enough to prevent them from remaining many (Russell 1903: §70).

So, the class as many has enough unity to allow it to bear numerical properties,

but not enough to make it one and thus not the bearer of any numerical property greater than one. In other words, Jones and Brown can be treated as having some kind of unity in order for them to be capable of instantiating properties, but they're not unified enough to count as a single entity Jones-and-Brown. While Russell is clearly trying hard to handle the worry here, his reply appears to border on incoherence (and at the very least looks desperately *ad hoc*): What could be meant by a unity that is not quite a unity? And how could something that is not quite a something be the bearer of *any* properties?³⁹

Another solution to the dilemma has been suggested by Peter Simons, who takes note of the trouble and offers his theory of *manifolds* as a way of avoiding it. For Simons, number is a property of a manifold, and he gives the following characterization of what manifolds are.

I take...manifolds to stand to plural terms as individuals stand to singular...We might say that 'manifold' is the plural of 'individual' (1982a: 165-166).

There is no difference between the manifold, and the several individuals, despite the fact that we can talk about *a* manifold, and indeed can count manifolds to some extent as though they were individuals (1982a: 166).

Here, again, the way out of the dilemma appears to land us in more obscurity. Analogies and vague gesturing at the idea of a manifold, are not enough to provide the means to clear up the difficult question of what it is that instantiates numerical properties. Furthermore, just like Russell's solution to the dilemma,

³⁹ It even becomes difficult to state objections to the position: We end up saying things like, This *something* is not a thing. Maybe we just need to occasionally invoke Wittgenstein here: "It is not a *something*, but not a *nothing* either!" (1953: §304).

Simons' manifolds are *many*, not one, and so are just as unable to exemplify any proposed property—the numerical property of an ascription of number in particular—as were Russell's unities that weren't quite unities.

An inability to speak literally about the subject of the bearers of numerical properties seems to be a common factor of the proposed solutions to Russell's Dilemma canvassed so far. We've seen Russell suggest that the class as many has "some kind of unity," but exactly what kind this is we cannot say. Next, Simons claimed that we can "count manifolds to some extent as though they were individuals," while nonetheless emphasizing that they are not *really* individuals. A final example of figurative speech in this arena appears in Byeong-uk Yi's treatment of the bearers of numerical properties. He claims that,

[A] property is *plurally instantiated* by some things (e.g., John and Carol), if it is instantiated by them, so to speak, taken together (2002: 70).

Yet, Yi emphasizes that "taken together" *cannot be taken literally* nearly every time the phrase is mentioned.⁴⁰

The general worry with these responses is becoming clear and has been forcefully put by Uwe Meixner: Each author's own characterization of the bearers of numerical properties makes it apparent that there simply are *no such things* as the bearers of these properties.⁴¹ Russell's 'classes as many' don't have enough unity to count as an entity; Simons's manifolds are nothing but the individual

⁴⁰ See (Yi 2002: 70 and 83) for example.

⁴¹ (Meixner 1997: 213)

constituents of the manifold; and Yi's talk of "taking together" is nothing but a manner of speaking. I take it that the failure of these very bright authors to do anything more than resort to a *façon de parler* when trying to deal with our dilemma suggests that the wrong approach is somehow being taken.

Something like this dissatisfaction with previous writers' comments on the bearers of numerical properties motivated Frege to propose an alternative to the idea that number is a property of external things. His own account takes number not to be a property of things, but of concepts. In the next section, as a transition to the account I'll offer in §5, I'll quickly review Frege's solution to the worry and explain why an objection raised to it by Palle Yourgrau (2002) ultimately does it in. The lessons learned from attempting to correct Frege's account, while maintaining something of its spirit, will lead into the proposal of our new alternative.

4. Although not strictly concerned with solving Russell's Dilemma, Frege did strongly oppose treating number as a property of external objects. Instead of not being able to find any bearer of numerical properties, he found too many: Any material object could seemingly be ascribed a great many numerical properties. This is basically Frege's *Relativity of Number* argument against numbers being ascribed to external objects.

Frege asks us to consider an everyday object like a deck of cards. Depending on how we think about it (as *one* deck or as *fifty-two* cards) it will be

ascribed a different number. The first reason to doubt that number is a property of external objects like this one that this fact suggests is that if number is really a property of objects, then we're able to change an object's properties merely by an act of thought. However, the properties of material objects don't seem to be changeable just by being thought of under different concepts.⁴² We can't change the shape of a box nor its color merely by thinking about it differently for example. Number, then appears to be quite different in this sense from other real properties, and Frege concludes from this that number is therefore *not* a property of material objects.⁴³

The deeper concern Frege has with external objects as bearers of numerical properties is that the question, "How many?" is indeterminate for any external object, and it's this fact that is responsible for our ability to change the number ascribed to an object by thinking of it under different conceptions. Since "How many?"-questions are incomplete we can substitute "How many (cards)?" or "How many (decks)?" to get answers of fifty-two and one respectively with respect to a single object. For Frege, though, we must be able to ask *unambiguously*, "How many?" of whatever ultimately is ascribed numerical properties.

⁴² (Frege 1884*b*: §22)

⁴³ Although Frege is usually given credit as the originator of this argument it seems to have been known at least to Berkeley as well:

That number is entirely the creature of the mind, even though other qualities be allowed to exist without, will be evident to whoever considers, that the same thing bears a different denomination of number, as the mind views it in different respects (1710: §12).

Berkeley draws very different conclusions from this argument, however. He goes on to suggest that this relativity shows that number is subjective, something that Frege would vehemently deny.

Instead of being a property of objects, Frege takes number to be a property of concepts, e.g. *being a deck of cards* or *being a card in the deck*.⁴⁴ Given the concept *card in the deck*, Frege thinks that we now *can* simply ask "How many?" and ask a determinate question whose answer is 52. Each of the concepts mentioned has an objective number that is (without going into the details of the account) completely determined by its particular extension.⁴⁵ Thus, there is just one thing falling under the concept *being a deck of cards* (on, say, the table), and there are fifty-two things falling under the concept *being a card in the deck*, and this is a property of each concept independently of any of our perspectives. Each concept is therefore ascribed this respective number, and concepts are the objective bearers of these numerical properties.

This proposal of Frege's seems to solve both Russell's Dilemma and the Relativity of Number worry in one fell swoop by abandoning the notion of numerical properties being ascribed to objects in favor of a concept-ascription model. It does this while also allowing us to maintain the initial appeal of an adjectival type of account. Further, Frege felt that his theory had the additional benefit of explaining how number ascriptions can be objective (since a concept's number is strictly determined by its extension). And this objectivity could make his proposed theory of number, therefore, more "suitable for science," as he had hoped it would be.⁴⁶

⁴⁴ (Frege 1884b: §46)

⁴⁵ *Ibid*, §72.

⁴⁶ *Ibid*, §57.

Frege undeniably made many ingenious suggestions about number like this one in the *Grundlagen*. However, as Palle Yourgrau has noted, he may have failed to appreciate the full applicability of one of these suggestions, i.e., his relativity of number argument. Yourgrau's main objection asks, Why should we think that number is a property of a concept any more than an object?

Concepts seemed to Frege better candidates for the bearers of numerical properties because they have the extensions they do essentially.⁴⁷ When we considered the deck of cards mentioned above as an external object, it wasn't clear whether we should count it as having just one part (the deck itself) or as having fifty-two parts (the various cards). This question doesn't arise with the concept *being a card in the deck*, however. Therefore, the "How many?"-question that Frege noticed was indeterminate with respect to material objects appears to become determinate when the question is asked about a concept whose very identity is tied-up with its extension. The concept, *being a card in the deck*, has an extension of fifty-two members independent of any human choice.

As it turns out, however, on Frege's theory the property ascribed to the concept *being a card in the deck* is really something like, *being 52-membered*. And then, as Yourgrau puts it, "If we are allowed in the case of sets to construe the number question as 'really': How many (elements)?, then we could just as well construe Frege's famous question about the deck of cards as: How many (cards)?"

⁴⁷ Frege believed that concepts with seemingly variable extensions really had their extensions invariably because they were all implicitly given a timestamp. E.g. The concept *Citizens of Germany* would appear to have a constantly changing extension, but Frege suggested that the concept as presented is incomplete. Its full formulation must be something like *Citizens of Germany on 27 November 1910*. Cf. (Frege 1884b: §46) and (Frege 1897: 236)

(2002: 358).

The objection to concepts as the bearers of numerical properties can perhaps be clarified by a brief example. For Frege, the referent of a concept, like *is a member of the Ramones*, is essentially the set of all the Ramones,⁴⁸ i.e., {Joey, Johnny, Marky, Dee Dee}. And this treatment of concepts in terms of their extensions is not peculiar to Frege but is fairly commonly accepted method, though often slightly modified by taking the concept's extension to range across all possible worlds (*cf.* e.g. Lewis 1999: 9) to account for (arguably) accidentally coextensive concepts such as, *has a heart* and *has a liver*. But, if a concept-phrase like *is a member of the Ramones* or *is a moon of Jupiter* has as its referent some particular set, then we seem to land right back in the grips of the relativity of number argument. Being given {Joey, Johnny, Marky, Dee Dee} and being asked "How many?" is no better than being given a deck of cards and being asked "How many?" In both cases, one answer might be more forthcoming than another (i.e., four and fifty-two respectively), but in neither case is the "How many?" question yet determinate. Why couldn't the set of the Ramones be considered in regards to the question of "How many (guitar players)?" This number is just as legitimately ascribed to the concept as it would be in the case of simply considering the band itself and asking, "How many?"⁴⁹ Frege seems to privilege "How many (members)?" or "How many (elements)?" as what the number question is *really*

⁴⁸ *Cf.* (Frege 1891) for example.

⁴⁹ *Cf.* (Yourgrau 2002: 358).

about, but this leaves him no principled reason (to repeat) to rule out "How many (cards)?" as the *real* question a person asking about the number of a certain deck of cards is after.

Since we, with Frege, ruled out external objects as the bearers of numerical properties because it seemed to be up to us to determine the number they bore in a way that a material object's real properties weren't similarly determinable, we must also rule out concepts as the bearers of these properties if they are shown to be variable in this way as well. This is just what Yourgrau's objection shows, though, so we'll have to do without the help of concepts as playing the role of *bearers* of numerical properties.

At this point, then, we appear to be running short of options for what can bear the necessary properties if number-words are to be treated as adjectival modifiers. I think we can best move forward from here by abandoning the adjectival approach and, instead, pursuing the suggestion that number-words are adverbial modifiers. The proposal will assume that concepts are still important in ascriptions of number, since they supply the unit by which we count, but that they are not themselves the bearers of numerical properties.

5. If what's been said so far is correct, then number-words are modifiers but not adjectival modifiers. They are, therefore, adverbial modifiers. As such, we now need to determine what kind of verb(s) they modify. At first glance, though, there don't appear to be any likely candidates.

Consider the diverse use of number-words in the statements below.

- (8) Jupiter has four moons.
- (9) Those are six beers.
- (10) Two men walked into the bar.

None of the verbs in these sentences seem to be modified by any of the respective number-words. In fact, the very notion of a number-word modifying *any* verb sounds confused; e.g., To the question "How did she run?" Could we possibly respond, "Two"? The aim of the present section is to show that there is a common verb that can be taken to be modified by a number-word in each of the above claims, and that there is nothing confused about the idea.

Despite the fact that there is no explicit verb shared by statements (8) - (10), there does seem to be an implicit common verb form: *to count*. Given a non-mass-term subject or predicate, since mass-terms like 'water' or 'gold' aren't numerable, each of the above claims can be reformulated as making the following type of assertion:

- (#) *Atomized predicate is/are to be (completely) counted to N.*

The "atomized predicate" here determines which things are to be counted as steps in the counting procedure, and the phrase "to *N*" is supposed to specify how the counting is to be done. I'll suggest that this understanding of empirical claims of number best comports with all the findings that have come before and is therefore

the one that should be accepted.

On this account, then, (8) - (10) are most perspicuously stated as,

(8') Jupiter's moons are counted to four.

(9') Those beers are counted to six.

(10') The men that walked into the bar are counted to two.

Given this understanding of empirical number claims, the problem of the bearers of numerical properties is avoided because the numerical property is a characteristic of the counting *act*, not any "class as many," or "manifold," or other such strange entity. Furthermore, Frege's important insight that the concept an object is considered under is an essential ingredient for answering any "How many?"-question is respected by allowing a concept to specify the counting unit. Finally, each of the number-words functions as an adverbial modifier and can therefore allow for a compositional semantics of complex cardinals.⁵⁰

Turning to statements involving complex number-words like the following,

(11) I have twenty five books.

(12) Three hundred tyrants ruled.

these statements will be construed compositionally by this account as,

(11') My books are counted to twenty and five.

⁵⁰ "to four" may appear to be prepositional, not adverbial; this complaint will be addressed shortly.

(12') The ruling tyrants are counted to three hundreds.

Thus, we've found an account of number ascriptions that meets all our previously determined desired characteristics.

The most serious obstacle for this suggested treatment of empirical number ascriptions to overcome is accounting for the truth-value of uncounted or uncountable groups of objects. Consider for example,

(13) There are 83 people walking across the Drill Field right now.

(14) There are 500 billion stars.

Unless we give up the Law of Excluded Middle for numerical claims like these (and that would certainly be quite a drastic move), (13) and (14) are either true or false right now. But by hypothesis, the people walking across the Drill Field at the moment have not been counted by anyone, and the counting of all the stars in the sky is what Bertrand Russell has called "medically" impossible—that is, it couldn't be carried out in the course of any one person's life. The problem, then, is to find some way to ground either the truth or falsity of these claims.

An account of number ascription like Frege's is able to avoid this kind of worry because he supposed that any concept had a numerical property independent of whether or not people *even existed*. There is just something objective about the world that determines the property that any given concept bears. Since an ascription of number on the account I'm presenting here depends

on the act of counting, though, this exact move is not available. However, a similar type of response seems to be possible. There is no numerical property that any group of countable objects has independent of anyone's counting on this account because I've argued that numbers aren't ascribed to groups of objects at all. There is though, I think, an objective fact about the world, that a being with our type of mind either would or would not count the stars in the sky through 500 billion, as well as whether or not a person with our type of mind either would or would not count the people walking across the Drill Field to 83. And on this fact, the truth or falsity of claims like (13) and (14) must depend. Admittedly, this may feel like sweeping the problem under the rug of counterfactuals, but I presently see no better way for dealing with the concern. If what has been said up to now is on the right track, then something like a counterfactual claim about would-be counters does look like the only available option for accounting for the truth or falsity of (13), (14), and the like though.

Finally, beside any of the theoretical advantages that I'm suggesting this adverbial account has, there is some empirical evidence that "How many?"-questions are most basically answered with an implicit use of a counting verb. Heike Wiese, for example, reports the following finding:

[W]hen you ask a three-year-old how many toys there are in her room, she might just say 'one, two, three, four, five', counting the toys without using the last counting word as an answer. If you ask again, chances are that she will...repeat the counting procedure...Karen Fuson reports that in one study some children recounted sets of blocks as often as seven times (2003: 169).

This suggests that among the modes of thinking basic to the human mind, there is

a capacity for the activity of counting. That is, the mind has an idea of what it's like to consider something as one and then another and then another, etc. When ascribing numerical properties, then, it's not the things themselves that bear the property of number, but the way I carry out the act of counting them that does. Therefore, there is a sense in which Berkeley was right to say that number is a "creature of the mind":⁵¹ Number is a creature of the mind because it's minds' acts of counting that are described by number-modifiers; however, number is not a creature of the mind in the (pejorative) relativistic sense that Berkeley thought he showed them to be.

Before moving on, let me address the concern that in "counted to four", for example, the verb is not modified at all but is complemented by the prepositional phrase, "to four". This objection would claim that counting to four is analogous to, for example, running to work.

This is simply a mistaken comparison. The complaint presupposes that 'four' refers to an *object*, just as 'work' refers to some place of business that I'm trying to get to in time. However, what we've found in Chapter II at least, is that number-words do not function as names, and in fact that they can't function as names if we're to be able to explain our grasp of an infinite variety of them. If 'four' doesn't name the abstract object four, then the analogy between counting to four and running to work is inappropriate.

A better analogy might be between "running a short distance" and

⁵¹ (Berkeley 1710: §12)

"counting to four". There are certain ways of running: quickly, in large leaps, and for a brief span. And these ways apparently correspond to different ways of counting: quickly, by tens, and to four, for example. So, if we keep in mind that an examination of the functioning of number-words has (at least initially) shown them not to be names for abstract objects, the claim that "counting to four" is an adverb-modified version of 'to count' is not objectionable in the present sense.

6. Up to this point, we've been primarily concerned with statements of number in what I've called in Chapter I their *attributive* sense. This form of a numerical statement is contrasted with those in which number-words function as apparently singular-terms. The examples previously given to demonstrate the difference between the two forms were,

(1_{att}) Jupiter has four moons.

(1_{sing}) The number of Jupiter's moons is four.

Since we have come to the conclusion that number-words are modifiers in Chapter II and we've also decided that a *uniform* account of number is to be preferred,⁵² a treatment of statements like (1_{sing}) in which number-words function as modifiers will also have to be given. As a brief preview, the basic idea of the treatment of singular-term number-words to be given in the next chapter will run as follows.

⁵² Reminder: A uniform account will have a number-word like 'four' making the same contribution to statements of number in either of these forms.

Statements like (1_{sing}) appear to be more closely aligned with the statements of arithmetic than do those similar to (1_{att}) : In ' $2 + 2 = 4$ ' for example, each of the terms involved is singular and the '=' seems to be essentially the "'is' of identity" that is in use in (1_{sing}) . This *prima facie* greater likeness between (1_{sing}) and arithmetical statements is somewhat curious, given that (1_{sing}) and (1_{att}) evidently are truth-functionally equivalent: Why shouldn't they be equally similar? The relation that I'll suggest exists between (1_{att}) , (1_{sing}) , and the claims of arithmetic is that statements like (1_{sing}) are, in effect, "bridge laws" that take empirical ascriptions of number into the "language-game" of arithmetic. Elements of Wittgenstein's philosophy of mathematics, along with some aspects of mathematical fictionalism, will play a role in making the relationship between these three types of statement clearer.

For a perhaps helpful pictorial representation of the general idea to be pursued, see below.

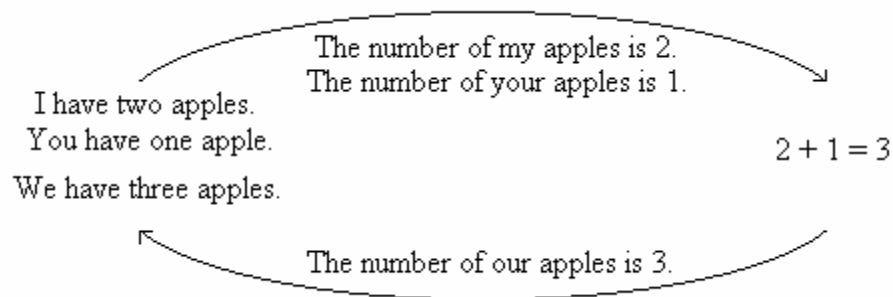


Figure 2. The proposed relationship between attributive, singular-term, and symbolic statements.

IV. ADVERBIAL NUMBERS AND ARITHMETIC.

"Can you do addition?" the White Queen asked. "What's one and one and one and one and one and one and one and one and one?" "I don't know," said Alice. "I lost count."

—Lewis Carroll, *Through the Looking Glass*.

1. As noted at the end of last chapter, it is a curious fact that number-words as they occur in mathematical statements appear to be more like those that show up in singular-term involving statements than those in attributive statements of number. The exemplars of each type of claim throughout have been,

(1_{att}) Jupiter has four moons.

(1_{sing}) The number of Jupiter's moons is four.

This chapter will attempt to answer several important questions regarding this perhaps surprising fact: First, (i) Why should singular-terms be (or appear to be) more appropriate for use in the statements of arithmetic? (ii) Why does language provide these apparently different logical forms for any statement of number? and finally, (iii) How can an account of the claims of arithmetic be given that comports well with the findings of the previous chapters?

Some insight into question (i) will be gained by examining a problem common to theories of arithmetic that don't make use of singular-term number-words. The consideration of (ii) will explain the existence of these distinct forms

of numerical statements by expanding on the "bridge law" theory suggested at the close of Chapter III. And the chapter will close by answering (iii) with an account of the statements of arithmetic that draws on aspects of Wittgenstein's remarks on the foundations of mathematics.

2. To see why the singular-term usage of number-words has seemed more fundamental to the claims of arithmetic, we'll consider an account of these claims that doesn't make use of them.

Harold Hodes (1984) offers the following theory about the relation between (1_{att}) , (1_{sing}) , and the statements of arithmetic. He suggests that a number-word ' n ' always contributes the *numerical quantifier*, $\exists_n x Xx$, to the logical form of numerical statements involving it.⁵³ If we want to say *either* (1_{att}) or (1_{sing}) , we are essentially stating that $\exists_4 x Jx$; that there are seven days in a week (or that the number of days in the week is seven) $\exists_7 x DIWx$; and so on. For Hodes, when number-words like 'four' appear to be singular-terms in statements like (1_{sing}) or " $2 + 2 = 4$ ", they're only "coding" these second-order numerical quantifiers, which are "notationally messy and logically complex," in easier to handle objectual terms. According to this account, then, the statements of arithmetic are, at bottom, disguised statements of second-order logic,⁵⁴ and " $2 + 2 = 4$ ", for example, has the logical form,

⁵³ Numerical quantifiers are given the recursive definition stated in Chapter II.

⁵⁴ Or, third-order predications of second-order quantifiers: " 4 is square" \Rightarrow (Square Ω)[$\exists_4 x Xx$].

$$\mathbf{H1} \quad \exists X \exists Y [(\exists_2 x Xx \wedge \exists_2 x Yx \wedge \neg \exists x (Xx \wedge Yx)) \rightarrow \exists_4 x (Xx \vee Yx)].$$

We're now in position to appreciate one reason why it's better for our theory of number to respect the *prima facie* singular-term uses of number-words in these arithmetic statements: Hodes's account, and those similar to it in not making use of real singular-term number-words, are subject to what we can call the *finite domain objection*. Suppose there's only one object: Then on Hodes's account "2 + 2 = 5" is true. The antecedent of H1 will be false in a one-object world, ensuring the truth of the whole.⁵⁵

The problem here is that "2 + 2 = 4" is on its face an identity statement, but an account like Hodes's can't capture this logical form. If numbers are syncategorematic and really disappear at the level of logical form in singular-term involving statements, they can't be identical to anything there. A Hodes-style account is therefore forced to take the logical form of the statements of arithmetic to be conditional, and it's this move that ultimately ushers in the finite domain concerns. A singular-term number-word account appears to offer relief from this problem because it has number-words purporting to refer to objects and allows for

⁵⁵ Hodes recognizes this problem and tries to solve it by suggesting that every world accesses another possible world containing more objects than it itself has. If that's the case, even if our world only has three objects, it accesses another with four objects, and that world in turn accesses one with five, which gives us access to a five-object world as well, and so on (assuming a modal logic as strong as S4). Thus, Hodes suggests reformulating "2 + 2 = 4" as,

$$\exists X \exists Y \square [((\exists_2 x Xx \wedge \exists_2 x Yx \wedge \neg \exists x (Xx \wedge Yx)) \rightarrow \exists_4 x (Xx \vee Yx)].$$

Whether it's our world or not, on this understanding of the claim, some world or another will falsify "2 + 2 = 5."—The suggestion seems to me problematic, especially if we're interested in pursuing a nominalistically acceptable theory (*pace* Lewis?), but here is not the place to address these worries.

" $2 + 2 = 4$ " to be a legitimate identity claim. If we can provide an account of number able to preserve the logical form of an identity statement for the claims of arithmetic (whether or not number-words ultimately turn out to refer to objects), then it must be preferred over one that takes these claims to be conditionals because of this serious problem.⁵⁶ We are, therefore, going to want an account of number in which statements that appear to be identity claims can be legitimately taken as such.

It's also worth noting here how unnatural singular-termed statements like (1_{sing}), "The number of Jupiter's moons is four", sound as they stand, and that statements of this type are more frequently encountered in a slightly different form. We do say things like, e.g.,

- (2) The number of Jupiter's moons is the number of cardinal directions.
- (3) The number of planets is greater than the number of Stooges.

Here again we interestingly see the apparently singular-term number-words being used in statements closely resembling familiar mathematical claims: Comparatives like (2) and (3), seem to be based on the truth of the facts that $4 = 4$ and $9 > 3$ respectively. Attempting to put (2) and (3) into an attributive form results in statements of questionable grammaticalness.

- (2') ?Jupiter has the number of cardinal directions moons.

⁵⁶ Providing a better account of the conditional involved in H1 is also an option, but this (certainly necessary) project can't be undertaken simply to make better sense of an account of number.

(3') ?The planets are more than the Stooges.

Once again, then, it appears to be the case that singular-term number-words are somehow more fundamental to mathematical statements.

To begin answering why this might be the case, we'll need to first consider some of the important differences between empirical ascriptions of number, on the one hand, and the statements of arithmetic on the other.

3. The distinction between empirical ascriptions of number and the claims of arithmetic is treated in great detail in numerous of Wittgenstein's writings about the foundations of mathematics.⁵⁷ As Crispin Wright points out,

No question receives more attention in *RFM* [*Remarks on the Foundations of Mathematics*] than that of the nature of the distinction between calculation, or proof in general, and experiment; to no question does Wittgenstein's thought revert more often (1980: 318).

Some of Wittgenstein's observations on this subject closely relate to the account I'll present in §4, so it will be helpful to consider a few of them presently.

In §11 of *RFM*, the following interesting question is posed.

|||||
|||||

Is this pattern a proof of $27 + 16 = 43$, because one reaches '27' if one counts the strokes on the left-hand side, '16' on the right-hand side, and '43' when one counts the whole row?

⁵⁷ This marks another big departure from Fregean ideas in Wittgenstein's later philosophy. Frege seemed to think that the difference between statements like (1_{att}) and (1_{sing}) was insignificant:

Since what concerns us here is to define a concept of number that is useful for science, we should not be put off by the attributive form in which number also appears in our everyday use of language. This can always be avoided. For example, the proposition "Jupiter has four moons" can be converted into "The number of Jupiter's moons is four" (Frege 1884*b*: §57).

Pursuing the answer to this question sheds some light on the distinction between empirical and mathematical statements currently being considered.

What Wittgenstein is asking us to pay attention to with the above question is the crucial role that he believes is played by the proposition, "I must have miscalculated." In fact, Wittgenstein claims that this proposition is "really the key to an understanding of the 'foundations' of mathematics" (1976: 221).

According to Wittgenstein,⁵⁸ presented with the above pattern of strokes, anyone can simply count each grouping as well as the combination of the two and come up with the following empirical claims: (i) There are twenty-seven strokes on the left-hand side; (ii) There are sixteen strokes on the right-hand side; and (iii) There are forty-three strokes altogether. And, as a matter of fact, when people engage in this kind of counting activity their results "practically always agree."⁵⁹ However, he notes that the supposed mathematical fact that ' $27 + 16 = 43$ ' isn't based on this empirical agreement. That is, the relation between this general agreement and ' $27 + 16 = 43$ ' can't be that of an empirical generalization. The critical proposition that "I must have miscalculated" helps in explaining why.

If a given counter knows that there are 27 strokes on the left-hand side of the above diagram and 16 on the right-hand side, then, unless she counts 43 altogether, she'll come to the conclusion, "I must have miscalculated," not that

⁵⁸ As always, care must be taken not to portray Wittgenstein as "advancing theses." In fact, he starts out his lectures on the foundations of mathematics with the explicit warning that any new interpretation that he may seem to offer is merely to be treated as "gas to expel old gas" (1976: 14). Nevertheless, he does make a number of very interesting suggestions repeatedly throughout his mathematical works, and even if *he* doesn't want them to be taken too seriously, I think there are some benefits to be gained in doing so.

⁵⁹ (Wittgenstein 1976: 154)

"Maybe $27 + 16 \neq 43$." But, if $27 + 16 = 43$ is somehow based in experience or agreement, it can't play this normative role of making people sure that they've miscounted if their results don't comply with it. Instead, a "miscounting" would simply be one piece of evidence against the oft-confirmed generalization that $27 + 16 = 43$, enough of which could conceivably make us decide that $27 + 16$ is not 43. This giving up of ' $27 + 16 = 43$ ', however, is not really an option,⁶⁰ and this shows that empirical evidence doesn't bear on the claims of arithmetic as it clearly must on empirical ascriptions of number.

Wittgenstein's discussion of these issues is relevant to the present theory because I've taken empirical ascriptions of number to be based on modifications of the act of counting. However, since I'm in agreement with the conclusion that, because of these claims' normative force, the statements of arithmetic can't also be based simply on an empirical counting procedure, I need some sort of story similar to Wittgenstein's that can account for the difference between these types of statement. To this, I'll now turn.

4. As has been mentioned, the most natural way of understanding statements like (1_{sing}), 'The number of Jupiter's moons is four,' and ' $2 + 2 = 4$ ' takes 'four' to name the abstract object four. This, first of all, is due to the fact that it looks like the 'is' in (1_{sing}) can only be the "is of identity." Secondly, I argued in §2 of this chapter that unless the statements of arithmetic are taken to be identity statements, they

⁶⁰ This would take some argument, of course, as Quine has shown.

will be subject to the finite domain objection. However, based on Chapter II and the further uniformity constraint on our theory, numbers must be taken to be modifiers in all contexts, including (1_{sing}) and the statements of arithmetic. But, since modifiers have a semantic-value but lack a referent (because they're incomplete or "unsaturated" to use Frege's term), these claims are *not* claims about the identity of any objects. Thus, the fact that modifiers do not refer to objects is problematic if we're looking for an account of arithmetic claims that are identity statements. This is because, on a modifier theory, we have no well-defined notion of what it means to say that something *is* (identical to) (the modifier) four.

Identity is supposed to be a relation that holds only between a *thing* and itself or characterized by Leibniz's Law: "If x is y , then x and y share all the same properties" with x and y ranging over first-order *objects*. Therefore, our customary ways of handling identity claims fail to address such claims occurring between numerical modifiers. This, of course, doesn't mean that these claims are nonsensical or without purpose, however. There is a clear sense in which we could say that '2 + 2' and '4' are the same modifier; they both make the same contribution to any statement involving them, even if one is often a bit more awkward. The same goes for 'the number of Jupiter's moons' and 'four'. Whenever one can accurately be said to modify a predicate so can the other, although elegance is again sacrificed. Even statement that will hurt any logician's eyes like " $\forall x = \neg \exists x \neg$ " seems to be in some sense true, despite there being no criteria for

applying the identity sign between these non-well-formed expressions. We simply see these (perhaps) philosophically bizarre identity statements throughout everyday life: e.g., all soccer fans know that '2 yellows = 1 red.' What we find, then, is that the instances in which we want to apply the concept of identity outrun the instances in which we have clear rules established for doing so. When this happens, *we're forced to improvise*. What I suggest is that we follow the Wittgensteinian idea⁶¹ that this improvising is accomplished by the *inventing* of rules necessary to allow for our desired number-talk: These rules take the form of the statements of arithmetic.⁶²

If this is the case, then, e.g. 'four' and '4' do maintain their modifier semantic-type wherever they appear; however, they'll play very different roles in claims like (1_{att}) and, say, '2 + 2 = 4'. 'Four' in "Jupiter has four moons" asserts something about how the moons of Jupiter have previously been counted. As 'four' occurs in the claims of arithmetic, on the other hand, it is simply part of a rule that asserts something about how it is to be *used*. Therefore, '2 + 2 = 4', on this account, should be understood as basically an imperative that has as its explicit content something like, "'2 + 2' is to be the same modifier as '4'" or "Let 2 + 2 be 4."⁶³ We could call the 'is' involved in such claims the "'is' of is-to-be." In this manner, then, the normativity of arithmetic claims noted by Wittgenstein with

⁶¹ See (Wright 1980: 153-166) for further details of his take on Wittgenstein.

⁶² If '2 + 2 = 4' is going to be treated as a rule, then it's technically not a statement, I suppose. It will be easier to talk about this material without continually keeping this distinction in view, but I think it is worth noting.

⁶³ Cf. (Wittgenstein 1976: 276).

his "I must have miscalculated"-discussion is built directly into the subject-matter's formulation.

This treatment of the occurrences of number-words in statements like (1_{sing}) and the claims of arithmetic is also able to make sense out of the apparent singular-term uses there. Instead of 'four' referring to four and allowing for the identity-claim status of these statements, 'four' is about the modifier itself and how it is to be used. This allows these identity-claims to avoid the Hodes-style conditional treatment that led to the finite domain objection. In (1_{att}) and the like, on the other hand, the focus of the statement is on the counting of the items picked out by the atomizing predicate involved, and not on the modifier itself.

Now, since we're accustomed to thinking about identity in terms of the identity of *objects*, it's natural to suppose that we're able to gain a working understanding of the way these rules for the identity of non-objects (such as modifiers) should be formed by treating them *as if* they were objects. Thus, the application of this new "'is' of is-to-be" to number-word involving statements may be facilitated by treating number modifiers (as far as we can) as if they were objects. The pretense allows for the importation of much of the conventional knowledge about identity to otherwise uncertain applications. Besides helping in this rule-forming process, this object-pretense seems to be one likely source of the idea that numbers are some kind object.⁶⁴

⁶⁴ The account being presented here shares many similarities with Stephen Yablo's brand of fictionalism, *figuralism*. See (Yablo 2000, 2001, 2002, 2005). One significant respect in which the accounts are similar is that the claims of arithmetic are not taken as being *true*; for Yablo because they are merely a figurative way of speaking, for the Wittgenstein inspired account here because

If we were very different kinds of creatures, ones that never accidentally miscounted, or that had a much greater cognitive capacity, or that never got distracted in the midst of a long counting job, perhaps we could get by with counting alone—without the need to formulate any of the simplifying modifier-rules of arithmetic. We're not such animals, however, and we do seem to need to rely on helpful representations. If we know that there is an effective method to get from the modifier-rules we use in arithmetic to facts about the counting of actual objects, we can perform mechanical operations on the symbols without consideration of what they stand for. And this allows us to avoid the question, "What is a modifier in the rule-providing statements of arithmetic?" It's simply a bad question on this account. Therefore, the addition of modifier-rules that enable us to expand the concept of identity to encompass these new semantic-types gives us a useful new "way of speaking" about the identity of non-objects that makes representing the world more efficient. This extension of the sense of the 'is' of identity may seem like a mere confusion which ought to be correctable, but the un-(explicitly)-noticed advantages that this expanded language provides once it's in use helps explain why it continues to stick around. We don't want language to go on holiday, but maybe we can let it take a business-trip.

they are essentially commands. Although I'm making some use of pretense in my account, as Yablo does (more prominently) in his, the different role this pretense plays in each theory is an important difference between the two. For Yablo, the claims of arithmetic resemble figurative language so much, that these claims simply are figurative; no one thinks numbers really exist, they just provide a way of speaking. The role for pretense in the present theory is only to provide a means to aid in formulating the rules of modifier identity claims, which in turn provide the way of speaking with numbers that makes representation of complex facts simpler.

The account provided thus far about empirical ascriptions of number such as, (1_{att}) "Jupiter has four moons," and the claims of arithmetic like "2 + 2 = 4" ends up suggesting a natural reason for why language provides us with the (1_{sing}) in addition to (1_{att}).⁶⁵ Statements like (1_{sing}) function essentially as bridge laws that allow for the application of arithmetic to the world around us. The same 'is' that was found to be at play in the claims of arithmetic will now enforce rules such as "Let the number of Jupiter's moons be four,"⁶⁶ which inform us which of the rules of arithmetic are to be employed when considering the moons of Jupiter. We might say about these rules that they connect our other rules with life.⁶⁷ Again though, despite its seemingly different usage, 'four' functions only as a modifier here, so the account conveniently remains uniform throughout.

5. I'd like to close with a quick summary of the full theory of number outlined in this work:

— The account of number presented here took as its starting point the need for

⁶⁵ And I think this is for more interesting reasons than *focus-shifting*: cf. (Hofweber 2005b). For Hofweber, we have both (1_{att}) and (1_{sing}) as part of our language so we can answer certain questions with better emphases.

—Which planet has four moons? Answer: (1_{att})
 —How many moons does Jupiter have? Answer: (1_{sing})

⁶⁶ Again, comparison with Yablo's figuralism is interesting here. Yablo suggests that arithmetic is derivable from the following rule:

If $*\exists_n x Fx*$ then $*\text{there is a thing } n = \text{the number of } F_S*$.

Where $*S*$, means "It is to be supposed that S. See (Yablo 2002). The comparing and contrasting of Wittgenstein's mathematical rules suggestion and Yablo's figuralism would, I think, be a very interesting project, but it is beyond the scope of the present work.

⁶⁷ Cf. (Wittgenstein 1974: 293).

- semantic compositionality in accounting for our understanding of an infinite amount of number-words. This compositionality was found to only be accomplishable when number-words were taken to be modifiers.
- After deciding on a modifier account of number, a further decision between adjectival and adverbial modifiers had to be made. Because of the difficulty in determining any possible bearers of numerically modified properties, adverbial modifiers were settled on over an adjectival treatment.
 - This forced the reconsideration of statements like "Jupiter has four moons" to discover what verb might occur in each of them that was plausibly eligible for modification.
 - I then suggested that every empirical ascription of number could be reformulated into a statement containing the verb "to count", which is therefore the subject of the numerical modification. After these empirical aspects of the theory were dealt with, I turned to the consideration of the statements of arithmetic.
 - I first argued that all the statements of arithmetic appear to, and ought to be taken to, involve the singular-term use of number-words like 'four' rather than their attributive uses. However, since number-words had been determined to stand for modifiers, and modifiers being non-referential, these apparent singular-terms couldn't function like names. Instead, number-words in these

mathematical statements seemed to be singular because they were free from the verb they usually modify. In the statements of arithmetic, which set out the rules for employing these modifiers, the modifiers appear alone.

— ' $2 + 2 = 4$ ' on our account, then, says basically "Let $2 + 2$ be 4" and it can be legitimately said that the claims of arithmetic are not *about* numbers, but are instead about *how to use numerical modifiers*.

— Finally, the alternative form of common empirical number ascriptions like (1_{att}), that is, (1_{sing}), was taken to function as a bridge law that allowed for the application of arithmetic rules, which only mention modifiers, to the objects encountered in everyday life, thus securing the applicability of mathematics.

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