DESIGN CRITERIA AND EQUATIONS OF MOTION FOR THE DE-SPIN OF A VEHICLE BY THE RADIAL RELEASE OF WEIGHTS AND CABLES OF FINITE MASS

by

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Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute

in candidacy for the degree of

MASTER OF SCIENCE

in

Engineering Mechanics

March 1962

Blacksburg, Virginia

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IV. INTRODUCTION

In the area of space exploration fine pointing accuracy is required for the launch vehicle in order to inject a payload into either a successful orbit about the earth, a successful trajectory to reach celestial objects, or to maintain the stability of a vertical probe experiment to extreme altitudes. The use of spin stabilization in achieving this accuracy has proven very reliable for vehicles without final stage guidance systems. However, as the experiments become more complex, spin rates much lower than that needed for stabilization during the final boost stages and nonspinning platforms are required. One method available to achieve this reduction is by the use of retrojets acting tangentially on the circumference of payload. Another method is the use of weights on the end of flexible cables that are released from the weighted end and allowed to unwind by the action of centrifugal force until a predetermined position is reached. At this time the entire weight and cable system is jettisoned. This method is commonly called the "yo-yo" system and was first used by the Jet Propulsion Laboratory.

The accuracy in achieving a desired spin rate other than zero is dependent on the ability to predict the initial spin rate and the moment of inertia of the payload about the roll axis. However, the "yo-yo" system does not depend on pyrotechnics to the extent of retrojets and does not have the problem of providing the equal thrust required of retrojets. The time required to de-spin is of the order of tenths of seconds which would require large thrusts from retrojets for comparable performance. When zero spin rate is desired, greater simplicity is achieved since the yo-yo system is completely independent of initial spin rate.

The yo-yo system was studied in reference 1 for the case where the cable mass is considered negligible and is released from the payload tangentially or radially. In reference 2 the motion and expected forces are analyzed with a cable of finite mass that is included in the total mass by an approximation.

The purpose of this study is to find exact expressions for the length of cable of finite mass or the mass of attached weights required to reduce the initial spin rate to zero or by any fraction. The equations of motion are derived in order to examine the transient conditions so that an engineer may determine the time required to achieve the desired spin rate, inertial loads on the payload and support structure, and cable tensions that may be expected for a given physical configuration. The expression for a fractional reduction in spin rate is used in an analysis to determine the effect of errors in measuring the physical constants involved.

V. SYMBOL LIST

E	total energy of rotating system, ft-lb
F	cable tension, lb
H	angular momentum, ft-lb-sec
I	body roll moment of inertia, slug-ft^2
K	total mass of cables per unit length, $\sum_{i=1}^{n} K_i$, slugs/ft
l	distance along cable to differential element of cable mass, ft
L	length of flexible cable from the surface of the payload to the center of mass of the weights, ft
m_i	mass of the i^{th} cable, $K_i L$, slugs
M	total mass of weights, $\sum_{i=1}^{n} M_i$, slugs
n	number of weights and number of cables equal to or greater than two
R	radius of body from the center of rotation, ft
T	kinetic energy, ft-lb
V	linear velocity, ft/sec
X,Y	fixed coordinate system
x,y	rotating coordinate system in phase II
α	angle in phase II from x axis to extended cables
ψ	angle between the lines from the center of rotation to the point of cable contact on the surface and the original point of contact with the circumference of the i^{th} particle (see fig. 2)

ϕ angle between the lines from the center of rotation to the point of cable contact with the surface and a point fixed on the payload (see fig. 2)

θ angle between the X axis and a line from the center of rotation to a point fixed on the payload

λ ratio of final spin rate to initial spin rate

$\Delta\lambda$ error in residual spin rate, $\lambda_a - \lambda_d$

τ the total time required to de-spin

Subscripts

a actually achieved

c pertaining to cables

d desired

f final conditions

i i^{th} cable or weight

j j^{th} particle on i^{th} cable

M pertaining to attached masses

max maximum

o initial conditions

P pertaining to payload

VI. STATEMENT OF THE PROBLEM

The proper selection of a length of cable of known density for a given mass or a mass for a given cable length is required in order to obtain a fractional reduction of initial spin rate or to produce a nonspinning platform. This proper selection will transfer the required proportion of the initial angular momentum to the cable and weight of system at the instant of release, thereby providing the required reduction in spin rate of the payload.

The position for the release of the weights and cables is chosen at the instant they are colinear with a radius from the center of the payload which coincides with the center of rotation as seen in figure 1. This position is chosen to reduce the error in residual spin rate induced by a possible release of the cables at an angle other than $\alpha = 90^{\circ}$

This requires the analysis of the transient conditions in two phases. In the first phase the cables are unwinding from the circumference so that the cable is perpendicular to a radius from the center of the payload at the point of tangency. The difficulty in this phase is to correctly describe the rotational energy of the cables about their own centers of mass, since the centers of mass of the unwound cable section are constantly moving. Further complication comes from the fact the cable is not rotating about a fixed point on the cylinder as it unwinds. The second phase is handled in the same manner but is more straightforward due to the cable rotating about a fixed point on the rotating cylinder.

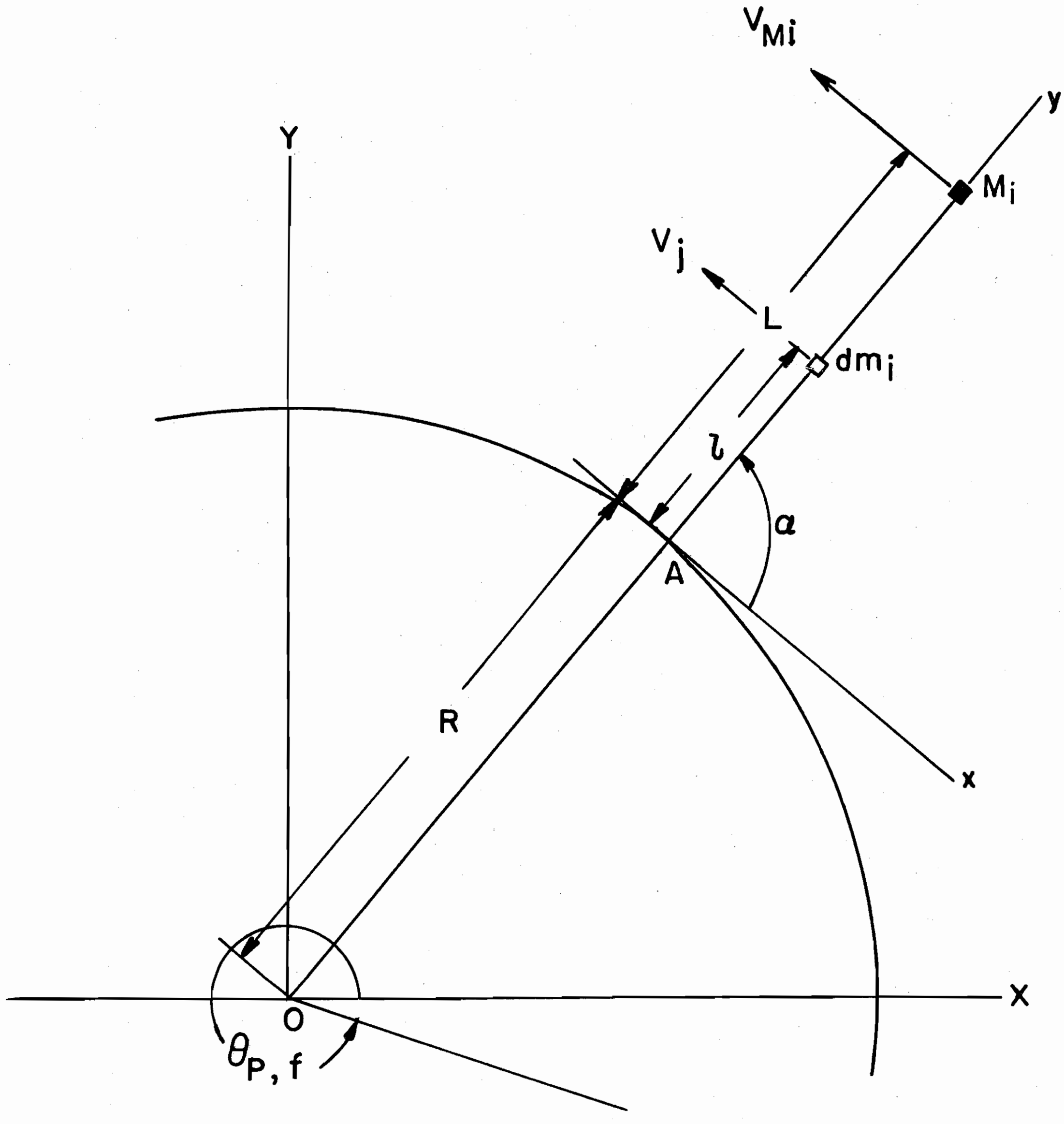


Figure 1.- Instant of release of weights and cables at $\alpha = 90^\circ$.

VII. ASSUMPTIONS

1. The system is conservative.
2. The effect of gravity is neglected.
3. The attached weights are considered point masses.
4. The cables are flexible, unstretchable, and of uniform mass per unit length.
5. The motion is two dimensional with no cross coupling.
6. The unwound portion of the cable is considered a straight line throughout both phases of the motion.

VIII. DERIVATION OF EQUATIONS TO DETERMINE LENGTH OF CABLE OR MASS REQUIRED TO OBTAIN ZERO RESIDUAL SPIN

In order to find the length of cable required to obtain zero residual spin in the payload, the initial and final angular momentum and kinetic energy are equated.

For the momentum balance

$$H_o = \sum_{i=1}^{n} H_{Mi,f} + \sum_{i=1}^{n} H_{Ci,f} \tag{1}$$

where $H_{Mi,f}$ is the final momentum of the i^{th} mass and $H_{Ci,f}$ is the final momentum of the i^{th} cable

$$H_o = \left(I + \sum_{i=1}^{n} M_i R^2 + \sum_{i=1}^{n} K_i L R^2\right)\dot{\theta}_{P,o} \tag{2}$$

$$\sum_{i=1}^{n} H_{Mi,f} = \sum_{i=1}^{n} M_i (L + R)^2 \dot{\theta}_{Mi,f} \tag{3}$$

When $\dot{\theta}_{P,f} = 0$,

$$\dot{\theta}_{Mi,f} = \frac{L\dot{\alpha}_{Mi,f}}{L + R} \tag{4}$$

To find the angular momentum of the cables, reference is made to figure 1. The angular momentum of the j^{th} particle on the i^{th} cable at the instant of release is

$$dH_{Ci,f} = (\ell + R)^2 \dot{\theta}_{Ci,f} dm_i$$

where $\ell + R$ is the distance of the j^{th} particle from the origin.

The rotation of the j^{th} particle about the origin at the final instant can be stated as

$$\dot{\theta}_{Cj,f} = \frac{l\dot{\alpha}_{Ci,f}}{l + R}$$

and the incremental mass as

$$dm_i = K_i\, dl$$

Therefore,

$$dH_{Ci,f} = K_i(l + R)^2 \frac{l\dot{\alpha}_{Ci,f}}{l + R}\, dl$$

$$H_{Ci,f} = K_i\dot{\alpha}_{Ci,f} \int_0^L (l + R)l\, dl$$

and

$$H_{C,f} = \sum_{i=1}^{n} H_{Ci,f} = \left[\sum_{i=1}^{n}\left(\frac{K_iL^3}{3} + \frac{K_iRL^2}{2}\right)\right]\dot{\alpha}_{Ci,f} \qquad (5)$$

Equation (5) gives the angular momentum of the cable about the origin which includes the angular momentum of the cable about its own center of mass.

With the assumption that the individual weights are of equal mass and that the individual cables are of equal length and mass per unit length,

$$\sum_{i=1}^{n} M_i = M$$

$$\sum_{i=1}^{n} K_i = K$$

With these relationships and by substituting equations (2) to (5) into equation (1), the following equation results:

$$(I + MR^2 + KLR^2)\dot{\theta}_{P,o} = \left[M(L + R)L + \frac{KL^3}{3} + \frac{KRL^2}{2}\right]\dot{\alpha}_f \tag{6}$$

where

$$\dot{\alpha}_f = \dot{\alpha}_{C,f} = \dot{\alpha}_{M,f}$$

The energy balance of the system is

$$T_o = \sum_{i=1}^{n} T_{Mi,f} + \sum_{i=1}^{n} T_{Ci,f} \tag{7}$$

where

$$T_o = \frac{1}{2}\left(I + \sum_{i=1}^{n} M_i R^2 + \sum_{i=1}^{n} K_i L R^2\right)\dot{\theta}_{P,o}{}^2 \tag{8}$$

and

$$T_{M,f} = \sum_{i=1}^{n} T_{Mi,f} = \frac{1}{2}\sum_{i=1}^{n} M_i(L + R)^2\dot{\theta}_{Mi,f}{}^2$$

or substituting the value previously obtained for $\dot{\theta}_{Mi,f}$ in equation (4) and summing, from $i = 1$ to n, leads to

$$T_{M,f} = \frac{ML^2\dot{\alpha}_f{}^2}{2} \tag{9}$$

To find the energy of the cables, refer again to figure 1. The incremental energy is

$$dT_{Ci,f} = \frac{1}{2}(l + R)^2\dot{\theta}_{Ci,f}{}^2 dm_i$$

Substituting previously determined values for these quantities

$$\sum_{i=1}^{n} dT_{Ci,f} = \frac{1}{2} \sum_{i=1}^{n} (l + R)^2 \frac{l^2 \dot{\alpha}_{Ci,f}^2 K_i}{(l + R)^2} dl$$

Integrating and summing the parts gives

$$T_{C,f} = \frac{KL^3}{6} \dot{\alpha}_f^2 \tag{10}$$

This represents the sum of the kinetic energies of the cables moving about the origin and the sum of the kinetic energies of the cables about their own center of mass. Therefore, substituting equations (8), (9), and (10) into equation (7) yields

$$\left(\frac{I + MR^2 + KLR^2}{2}\right)\dot{\theta}_{P,o}^2 = \left(\frac{ML^2}{2} + \frac{KL^3}{6}\right)\dot{\alpha}_f^2 \tag{11}$$

Solving equations (6) and (11) for $\frac{\dot{\theta}_{P,o}^2}{\dot{\alpha}_f^2}$ and equating the results yields

$$M + \frac{KL}{3} = \frac{\left[\frac{KL^2}{3} + \left(\frac{KR}{2} + M\right)L + MR\right]^2}{I + MR^2 + KLR^2}$$

or, in terms of L as a quartic equation,

$$\frac{K^2L^4}{9} + \left(\frac{K^2R}{3} + \frac{2}{3} KM\right)L^3 + \left(M^2 + \frac{2}{3} KMR - \frac{K^2R^2}{12}\right)L^2 + \left(2M^2R - \frac{KMR^2}{3} - \frac{IK}{3}\right)L - IM = 0 \tag{12}$$

Equation (12) is then solved for L. The solution for L will yield either one pair of complex roots (one negative real root and one positive real root), three negative roots and one positive root or two pairs of complex roots. From the physical situation the positive real root is the length required to achieve zero residual spin. An alternate method would be to choose a length of cable and solve equation (12) for the mass of the weights, since only M and M^2 terms are involved.

For the case where the cable mass is considered negligible, the solution for L reduces to

$$L = R\left(\sqrt{\frac{I}{MR^2} + 1} - 1\right) \tag{13}$$

IX. DERIVATION OF EQUATIONS TO DETERMINE THE LENGTH OF CABLE OR MASS OF WEIGHTS REQUIRED TO OBTAIN ANY DESIRED RESIDUAL SPIN RATE

In order to find the length of cable or mass of the weight required to obtain a desired residual spin, the same procedure is adopted for the angular momentum and energy balance. In this case the initial angular velocity must be known to whatever end accuracy is desired.

The angular momentum may be written as

$$H_o = H_{M,f} + H_{C,f} + H_{P,f} \tag{14}$$

where

$$H_o = (I + MR^2 + KLR^2)\dot{\theta}_{P,o} \tag{15}$$

$$H_{P,f} = I\dot{\theta}_{P,f} \tag{16}$$

and

$$H_{M,f} = M(L + R)^2\dot{\theta}_{M,f}$$

and, since

$$\dot{\theta}_{M,f} = \frac{(L + R)\dot{\theta}_{P,f} + L\dot{\alpha}_f}{L + R} \tag{17}$$

the equation for $H_{M,f}$ is

$$H_{M,f} = M(L + R)^2\dot{\theta}_{P,f} + ML(L + R)\dot{\alpha}_f \tag{18}$$

By referring to figure 1, the incremental momentum of the cable can be seen to be

$$dH_{C,f} = K(l + R)^2 \frac{(l + R)\dot{\theta}_{P,f} + l\dot{\alpha}_f}{l + R} dl$$

Therefore,

$$H_{C,f} = \int_0^L \left(K\dot{\theta}_{P,f}l^2 + 2KR\dot{\theta}_{P,f}l + KR^2\dot{\theta}_{P,f} + K\dot{\alpha}_f l^2 + KR\dot{\alpha}_f l\right)dl$$

$$= \left(\frac{KL^3}{3} + KRL^2 + KR^2L\right)\dot{\theta}_{P,f} + \left(\frac{KL^3}{3} + \frac{KRL^2}{2}\right)\dot{\alpha}_f \tag{19}$$

Substituting equations (15), (16), (18), and (19) into equation (14) yields

$$(I + MR^2 + KLR^2)\dot{\theta}_{P,o} = \left[I + MR^2 + \frac{KL^3}{3} + (M + KR)L^2 + (2MR + KR^2)L\right]\dot{\theta}_{P,f}$$

$$+ \left[\frac{KL^3}{3} + \left(M + \frac{KR}{2}\right)L^2 + MRL\right]\dot{\alpha}_f \tag{20}$$

For the energy

$$T_o = T_{M,f} + T_{C,f} + T_{P,f} \tag{21}$$

where

$$T_o = \left(\frac{I + MR^2 + KLR^2}{2}\right)\dot{\theta}_{P,o}^{\ 2} \tag{22}$$

$$T_{P,f} = \frac{I\dot{\theta}_{P,f}^{\ 2}}{2} \tag{23}$$

$$T_{M,f} = \frac{M\left[(L + R)\dot{\theta}_{P,f} + L\dot{\alpha}_f\right]^2}{2} \tag{24}$$

The energy of an increment of the cable is

$$dT_{C,f} = \frac{K}{2}\left[(l + R)\dot{\theta}_{P,f} + l\dot{\alpha}_f\right]^2 dl$$

Therefore,

$$T_{C,f} = \frac{K}{2}\int_0^L \left[(R^2 + l^2 + 2RL)\dot{\theta}_{P,f}^2 + (2l^2 + 2RL)\dot{\theta}_{P,f}\dot{\alpha}_f + l^2\dot{\alpha}_f^2\right]dl$$

$$= \frac{1}{2}\left[\left(KR^2L + \frac{KL^3}{3} + KRL^2\right)\dot{\theta}_{P,f}^2 + \left(\frac{2KL^3}{3} + KRL^2\right)\dot{\theta}_{P,f}\dot{\alpha}_f + \frac{KL^3}{3}\dot{\alpha}_f^2\right] \quad (25)$$

Substituting equations (22) to (25) into equation (21) yields

$$(I + MR^2 + KLR^2)\dot{\theta}_{P,o}^2 = \left[MR^2 + I + (2MR + KR^2)L + (KR + M)L^2 + \frac{KL^3}{3}\right]\dot{\theta}_{P,f}$$

$$+ \left[2MR + (2M + KR)L + \frac{2KL^2}{3}\right]L\dot{\theta}_{P,f}\dot{\alpha}_f + \left(M + \frac{KL}{3}\right)L^2\dot{\alpha}_f^2$$

(26)

Equations (20) and (26) are then solved simultaneously for L or M for any desired ratio of λ, where $\lambda = \frac{\dot{\theta}_{P,f}}{\dot{\theta}_{P,o}}$. In order to simplify this solution, numerical coefficients should be computed to solve these equations.

For the cases considered here, the mass of the weights were chosen and then the energy and momentum equations were solved simultaneously for the length of cable by the Newton-Raphson iterative method on the IBM 7090.

Solving equations (20) and (26) simultaneously for λ, gives

$$\lambda = \frac{A}{B}\left(1 - \sqrt{\frac{C^2(B - A)}{A(BD - C^2)}}\right) \quad (27)$$

where

$$A = I + MR^2 + KLR^2$$

$$B = I + MR^2 + L^2\left(M + \frac{KL}{3}\right) + 2RL\left(M + \frac{KL}{2}\right) + KR^2L$$

$$C = L^2\left(M + \frac{KL}{3}\right) + RL\left(M + \frac{KL}{2}\right)$$

$$D = L^2\left(M + \frac{KL}{3}\right)$$

Equation (27) may be used to solve for λ if the length of cable and mass of weights are known or assumed. This expression may also be solved to obtain the error introduced in λ for a given error in the moment of inertia and length of cable.

X. DERIVATION OF THE EQUATIONS OF TRANSIENT MOTION

Lagrange's mechanics are used to obtain the equations of motion for the transient conditions. The motion is considered in two separate phases. In the first phase the cables are completely unwound from the payload until they are extended perpendicular to a radius through the center of rotation of the payload. The second phase starts at the end of phase I with the cables and weights revolving about the point of attachment to the payload. Phase II continues until the cables and weights are colinear with a radius from the center of the payload at which time they are considered released. The equations for the transient motion in both phase I and phase II were programed for and solved on the IBM 7090 electronic data processing system.

Phase I

During phase I, the X- and Y-axes are fixed in space with their origin coinciding with the center of rotation of the payload. As shown in figure 2, line OB rotates with the payload with an angular velocity of $\dot{\theta}$, and $\dot{\phi}$ is the rate at which the point of tangency (point A) of the cable moves from the original position of the weight at point B. The total energy consists of the rotation of the payload about the origin, and the motion of the weights and cables about the origin:

$$E = \Sigma T = T_P + T_M + T_C \tag{28}$$

where

$$T_P = \left[\frac{I + (KL - KR\phi)R^2}{2}\right]\dot{\theta}^2 \tag{29}$$

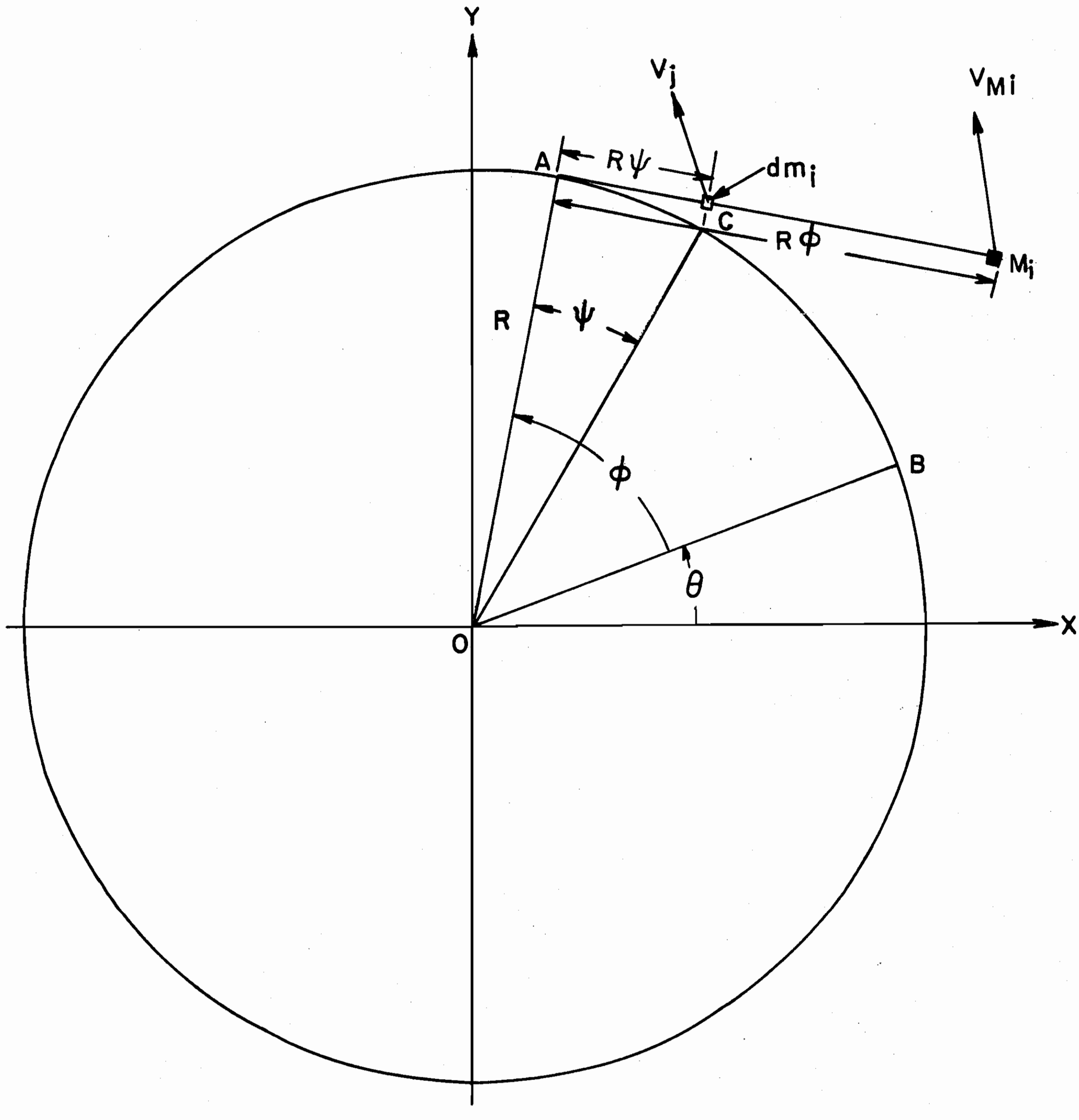


Figure 2. - Motion during phase I.

$$T_M = \sum_{i=1}^{n} \frac{M_i}{2}\left(\dot{X}_{Mi}^2 + \dot{Y}_{Mi}^2\right) \tag{30}$$

In order to produce a symmetric problem where the center of mass of the system is located at the origin of the fixed coordinate system, the total number of weights (and hence, the total number of cables used) is equal to or greater than 2. Therefore,

$$X_{Mi} = R \cos(\theta + \phi) + R\phi \sin(\theta + \phi)$$

$$Y_{Mi} = R \sin(\theta + \phi) - R\phi \cos(\theta + \phi)$$

which yields

$$\dot{X}_{Mi}^2 + \dot{Y}_{Mi}^2 = R^2\phi^2(\dot{\theta} + \dot{\phi})^2 + R^2\dot{\theta}^2 = V_{Mi}^2 \tag{31}$$

Substituting equation (31) into equation (30) yields

$$T_M = \sum_{i=1}^{n} \frac{M_i}{2}\left[R^2\phi^2(\dot{\theta} + \dot{\phi})^2 + R^2\dot{\theta}^2\right] \tag{32}$$

To find the energy of the cables, refer to figure 2; thus

$$dT_{Ci} = \frac{\dot{X}_j^2 + \dot{Y}_j^2}{2}\, dm_i \tag{33}$$

The variable ψ is now introduced which is the angle measured from line OA to line OC. (See fig. 2.) Point C is the point of original contact of the j^{th} particle on the circumference of the payload. The mass of the j^{th} particle on the i^{th} cable is then

$$dm_i = K_i R\, d\psi \tag{34}$$

where $R\,d\psi$ is the incremental length of cable that has unwound. The location of the j^{th} particle on the i^{th} cable relative to the fixed coordinate system is

$$X_j = R\psi \sin(\theta + \phi) + R \cos(\theta + \phi)$$

$$Y_j = -R\psi \cos(\theta + \phi) + R \sin(\theta + \phi)$$

The velocity is then

$$\dot{X}_j = R\psi(\dot{\theta} + \dot{\phi})\cos(\theta + \phi) + R\dot{\psi} \sin(\theta + \phi) - R(\dot{\theta} + \dot{\phi})\sin(\theta + \phi)$$

$$\dot{Y}_j = R\psi(\dot{\theta} + \dot{\phi})\sin(\theta + \phi) - R\dot{\psi} \cos(\theta + \phi) + R(\dot{\theta} + \dot{\phi})\cos(\theta + \phi)$$

However, ψ must increase at the same rate as ϕ; therefore, $\dot{\psi} = \dot{\phi}$ and

$$\dot{X}_j^2 + \dot{Y}_j^2 = R^2\psi^2(\dot{\theta} + \dot{\phi})^2 + R^2\dot{\theta}^2 = V_j^2 \tag{35}$$

Substituting equations (34) and (35) into equation (33) gives

$$dT_{Ci} = \frac{K_i R}{2}\left[R^2\psi^2(\dot{\theta} + \dot{\phi})^2 + R^2\dot{\theta}^2\right]d\psi \tag{36}$$

Therefore, the total kinetic energy of the cables is

$$T_C = \sum_{i=1}^{n} \int_0^{\phi} \frac{K_i R^3}{2}\left[(\dot{\theta} + \dot{\phi})^2\psi^2 + \dot{\theta}^2\right]d\psi$$

Performing the required integration and letting $\sum_{i=1}^{n} K_i = K$ yields

$$T_C = \frac{KR^3\phi}{2}\left[\frac{\phi^2(\dot{\theta} + \dot{\phi})^2}{3} + \dot{\theta}^2\right] \tag{37}$$

Substituting equations (29), (32), and (37) into equation (28) and letting $\sum_{i=1}^{n} M_i = M$

$$T = \left[\frac{I + (KL - KR\phi)R^2}{2}\right]\dot{\theta}^2 + \frac{M}{2}\left[R^2\phi^2(\dot{\theta} + \dot{\phi})^2 + R^2\dot{\theta}^2\right]$$

$$+ \frac{KR^3\phi}{2}\left[\frac{\phi^2(\dot{\theta} + \dot{\phi})^2}{3} + \dot{\theta}^2\right] \qquad (38)$$

Now Lagrange's equations are used to determine the equations of motion for the system; since the potential energy is zero, and the system is conservative, Lagrange's equation for θ reduces to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = 0 \qquad (39)$$

Performing the required operations on equation (38) as indicated by equation (39) yields

$$\left[\frac{I}{R^2} + KL + \phi^2\left(M + \frac{KR\phi}{3}\right) + M\right]\ddot{\theta} + \phi^2\left[M + \frac{KR\phi}{3}\right]\ddot{\phi} + \phi\dot{\phi}(\dot{\theta} + \dot{\phi})(2M + KR\phi) = 0 \qquad (40)$$

Lagrange's equation for ϕ reduces to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = 0$$

Thus equation (38) yields

$$\left[\ddot{\theta} + \ddot{\phi}\right]\left[\phi\left(M + \frac{KR\phi}{3}\right)\right] + \left(M + \frac{KR\phi}{2}\right)(\dot{\phi}^2 - \dot{\theta}^2) = 0 \qquad (41)$$

Equations (40) and (41) are solved simultaneously to obtain the transient conditions in phase I.

Phase II

In phase II the cables have completely unwound with $R\phi = L =$ Constant. As shown by figure 3, the cables and weights now rotate about the fixed point A on the cylinder. Point A is chosen as the origin of the x, y coordinate system which is rotating about the origin (point O) of the X, Y coordinate system which is fixed. The weights and cables rotate about point A until they are colinear with a radius from the center of the payload. At this moment the cables and weights are considered released. The total energy for this phase is

$$E = \Sigma T = T_P + T_M + T_C \tag{42}$$

where

$$T_P = \frac{I\dot{\theta}^2}{2} \tag{43}$$

$$T_M = \sum_{i=1}^{n} \frac{M_i}{2}(\dot{X}_{Mi}^2 + \dot{Y}_{Mi}^2) \tag{44}$$

Also,

$$X_{Mi} = L\sin(\theta + \phi + \alpha) + R\cos(\theta + \phi)$$
$$Y_{Mi} = -L\cos(\theta + \phi + \alpha) + R\sin(\theta + \phi)$$

which yields

$$\dot{X}_{Mi}^2 + \dot{Y}_{Mi}^2 = L^2(\dot{\theta} + \dot{\alpha})^2 + R^2\dot{\theta}^2 + 2RL\dot{\theta}(\dot{\theta} + \dot{\alpha})\sin\alpha \tag{45}$$

Substituting equation (45) into equation (44)

$$T_M = \sum_{i=1}^{n} \frac{M_i}{2}\left[L^2(\dot{\theta} + \dot{\alpha})^2 + R^2\dot{\theta}^2 + 2RL\dot{\theta}(\dot{\theta} + \dot{\alpha})\sin\alpha\right] \tag{46}$$

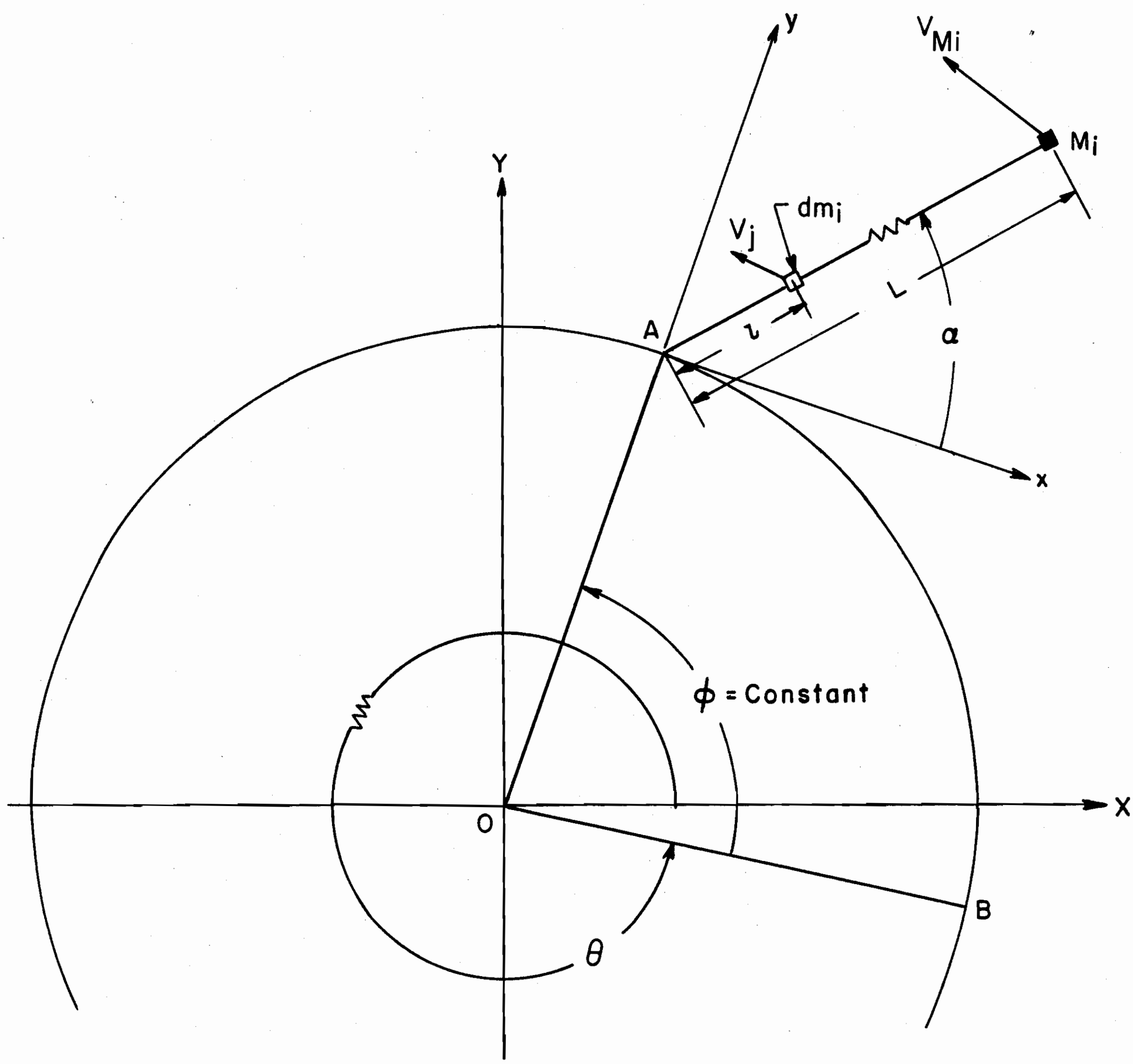


Figure 3.- Motion during phase II.

To find the energy of the cables, refer to figure 3. The location of the j^{th} particle on the i^{th} cable is

$$X_j = R\cos(\theta + \phi) + l\sin(\theta + \phi + \alpha)$$

$$Y_j = R\sin(\theta + \phi) - l\cos(\theta + \phi + \alpha)$$

Thus,

$$\dot{X}_j^2 + \dot{Y}_j^2 = R^2\dot{\theta}^2 + l^2(\dot{\theta} + \dot{\alpha})^2 + 2Rl\dot{\theta}(\dot{\theta} + \dot{\alpha})\sin\alpha$$

The mass of the j^{th} particle on the i^{th} cable

$$dm_i = K_i\, dl$$

Then, the energy of the j^{th} particle is

$$dT_{Ci} = \frac{K_i}{2}(\dot{X}_j^2 + \dot{Y}_j^2)dl$$

Therefore, the total kinetic energy of the cables is

$$T_C = \int_0^L \sum_{i=1}^{n} \frac{K_i}{2}\left[R^2\dot{\theta}^2 + l^2(\dot{\theta} + \dot{\alpha})^2 + 2RL\dot{\theta}(\dot{\theta} + \dot{\alpha})\sin\alpha\right]dl$$

$$= \sum_{i=1}^{n} \frac{K_i}{2}\left[LR^2\dot{\theta}^2 + \frac{L^3(\dot{\theta} + \dot{\alpha})^2}{3} + RL^2\dot{\theta}(\dot{\theta} + \dot{\alpha})\sin\alpha\right] \qquad (47)$$

Again, let

$$\sum_{i=1}^{n} M_i = M$$

$$\sum_{i=1}^{n} K_i = K$$

Substituting equations (43), (46), and (47) into equation (42) yields

$$T = \frac{I\dot{\theta}^2}{2} + \frac{M}{2}\left[L^2(\dot{\theta} + \dot{\alpha})^2 + R^2\dot{\theta}^2 + 2RL\dot{\theta}(\dot{\theta} + \dot{\alpha})\sin\alpha\right]$$

$$+ \frac{KL}{2}\left[R^2\dot{\theta}^2 + \frac{L^2(\dot{\theta} + \dot{\alpha})^2}{3} + RL\dot{\theta}(\dot{\theta} + \dot{\alpha})\sin\alpha\right] \tag{48}$$

Equation (48) is then used in Lagrange's equations to obtain the equations of motion. Lagrange's equation for θ reduces to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = 0 \tag{49}$$

Performing the required operations on equation (48) as indicated by equation (49) yields

$$\left[I + R^2(M + KL) + L^2\left(M + \frac{KL}{3}\right) + 2\left(M + \frac{KL}{2}\right)RL\sin\alpha\right]\ddot{\theta}$$

$$+ \left[L^2\left(M + \frac{KL}{3}\right) + \left(M + \frac{KL}{2}\right)RL\sin\alpha\right]\ddot{\alpha} + RL\dot{\alpha}\cos\alpha\left[\left(M + \frac{KL}{2}\right)(2\dot{\theta} + \dot{\alpha})\right] = 0 \tag{50}$$

In like manner, Lagrange's equation for α reduces to

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\alpha}}\right) - \frac{\partial T}{\partial \alpha} = 0 \tag{51}$$

and equation (48) becomes

$$\left[L^2\left(m + \frac{KL}{3}\right) + \left(M + \frac{KL}{2}\right)RL\sin\alpha\right]\ddot{\theta} + \left[L^2\left(m + \frac{KL}{3}\right)\right]\ddot{\alpha} - RL\dot{\theta}^2\left(m + \frac{KL}{2}\right)\cos\alpha = 0$$

Equations (50) and (52) are then solved simultaneously to obtain the transient conditions in phase II.

XI. DISCUSSION OF RESULTS

The length of cable required for any spin-reduction ratio is shown to be independent of the initial spin rate. However, the accuracy of the residual spin is no better than the prediction of the initial spin after a value of λ has been selected that is not equal to zero. If the desired residual spin is zero, then the length is completely independent of the initial spin. The length varies with the parameter $\frac{I}{MR^2}$ for a constant value of KL/M and λ. The effect on λ due to an increase in the ratio of cable weight to attached weight is significant. An example of this significance is seen in figure 4. This plot shows the variation of cable length with KL/M to achieve various values of λ for a payload configuration of $\frac{I}{MR^2} = 200$. If a value of $\lambda = 0.10$ is desired, a cable length of approximately 11.85 feet is needed if the cable weight is considered negligible and R is assumed to be 1 foot. However, if the cable weight is one-half the weight of the attached masses and the same cable length is used, then a value of λ of approximately 0.019 is achieved as seen from the plot in figure 4 of λ against KL/M.

Investigation of the error in residual spin due to errors in the physical constants of moment of inertia about roll axis and cable length shows the largest error is from the measured inertia. This error due to the expected error from measuring the inertia is greater than that for the length. The error in residual spin (within ±5 percent for all the cases considered) is shown plotted against the error in inertia in figure 5. The positive values of $\Delta I/I$ correspond to inertias that were measured lower than the actual inertia and the negative values of $\Delta I/I$ correspond

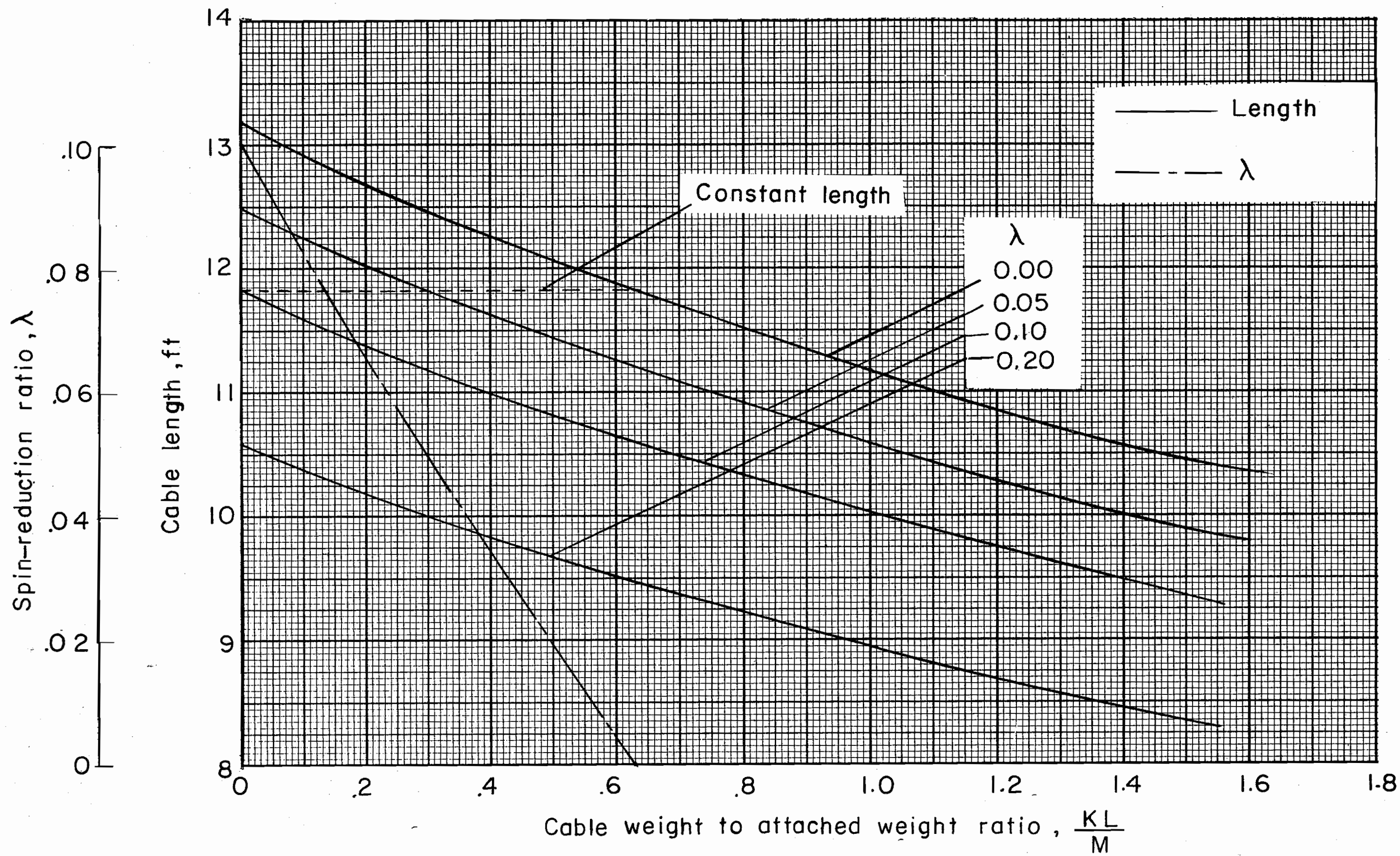


Figure 4.- Cable lengths and spin-reduction ratios against the cable weight to attached weight ratios for $\frac{I}{MR^2} = 200$.

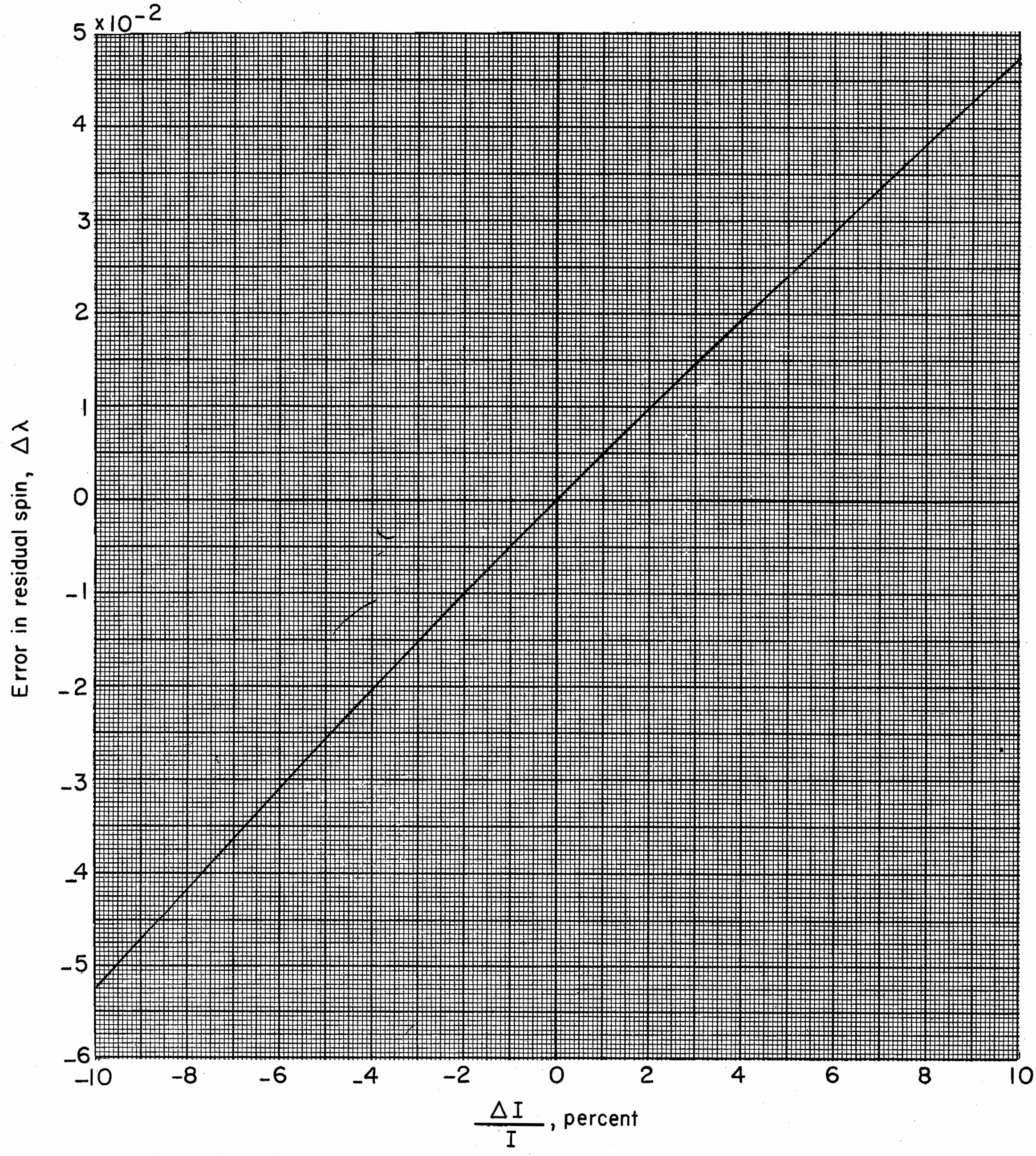


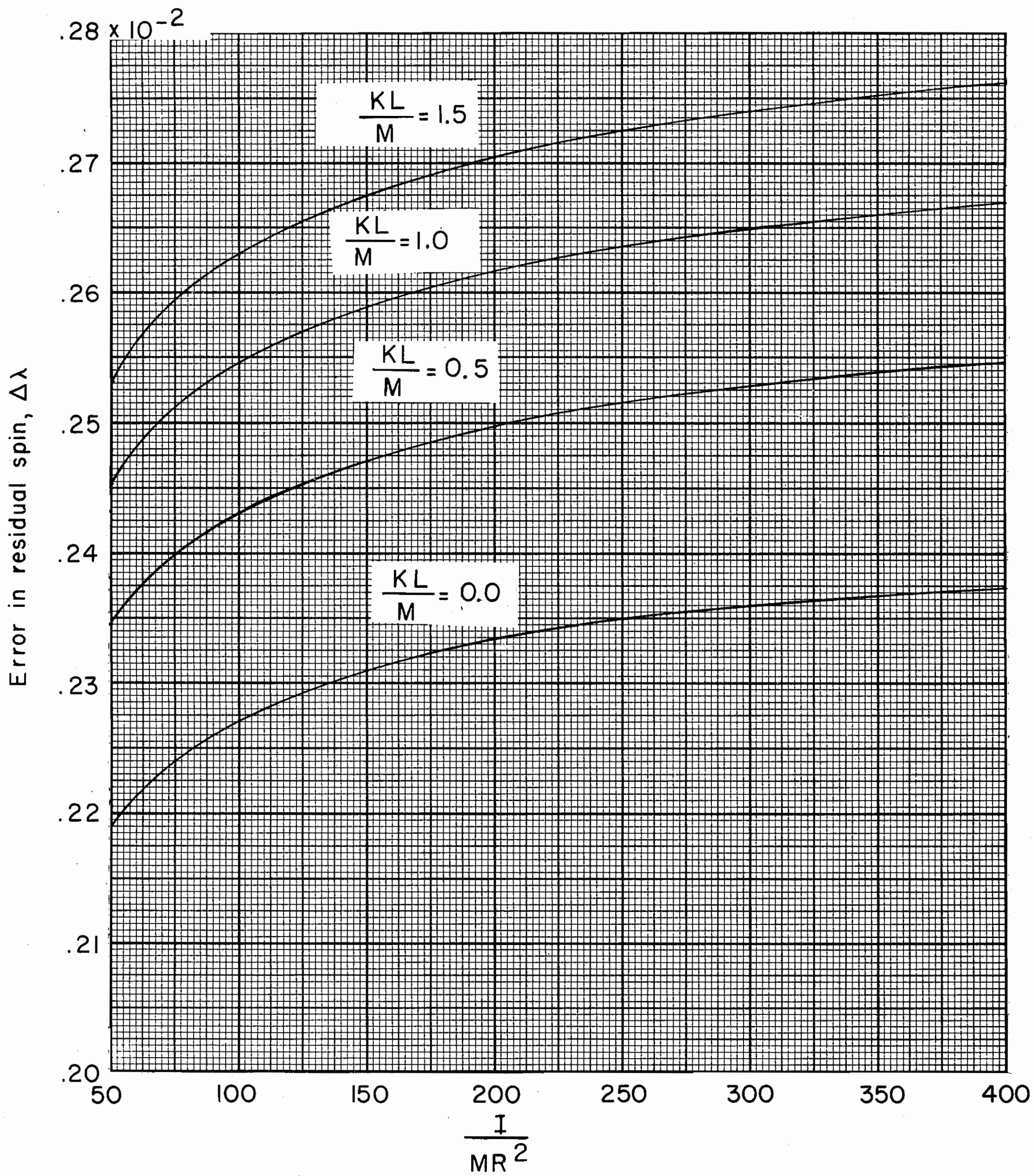
Figure 5.- Error in residual spin plotted against error in moment of inertia.

to inertias that were measured higher than the actual inertia. When the inertia is measured too high, the corresponding cable length is too long for a given case. If the desired λ is at or near zero, this error can produce a residual spin opposite in sense to the initial spin. The slope of the curve in figure 5 is approximately 1:200 so that an error of 5 percent in inertia produces an error of $\Delta\lambda \approx 2.5 \times 10^{-2}$, where $\Delta\lambda = \lambda_a - \lambda_d$.

Figure 6 shows the error in residual spin plotted against I/MR^2 for an error of ± 0.25 percent in length of cable and for various ratios of cable weight to attached weight (KL/M) and spin-reduction ratios (λ). The error in residual spin is shown to decrease as λ is increased.

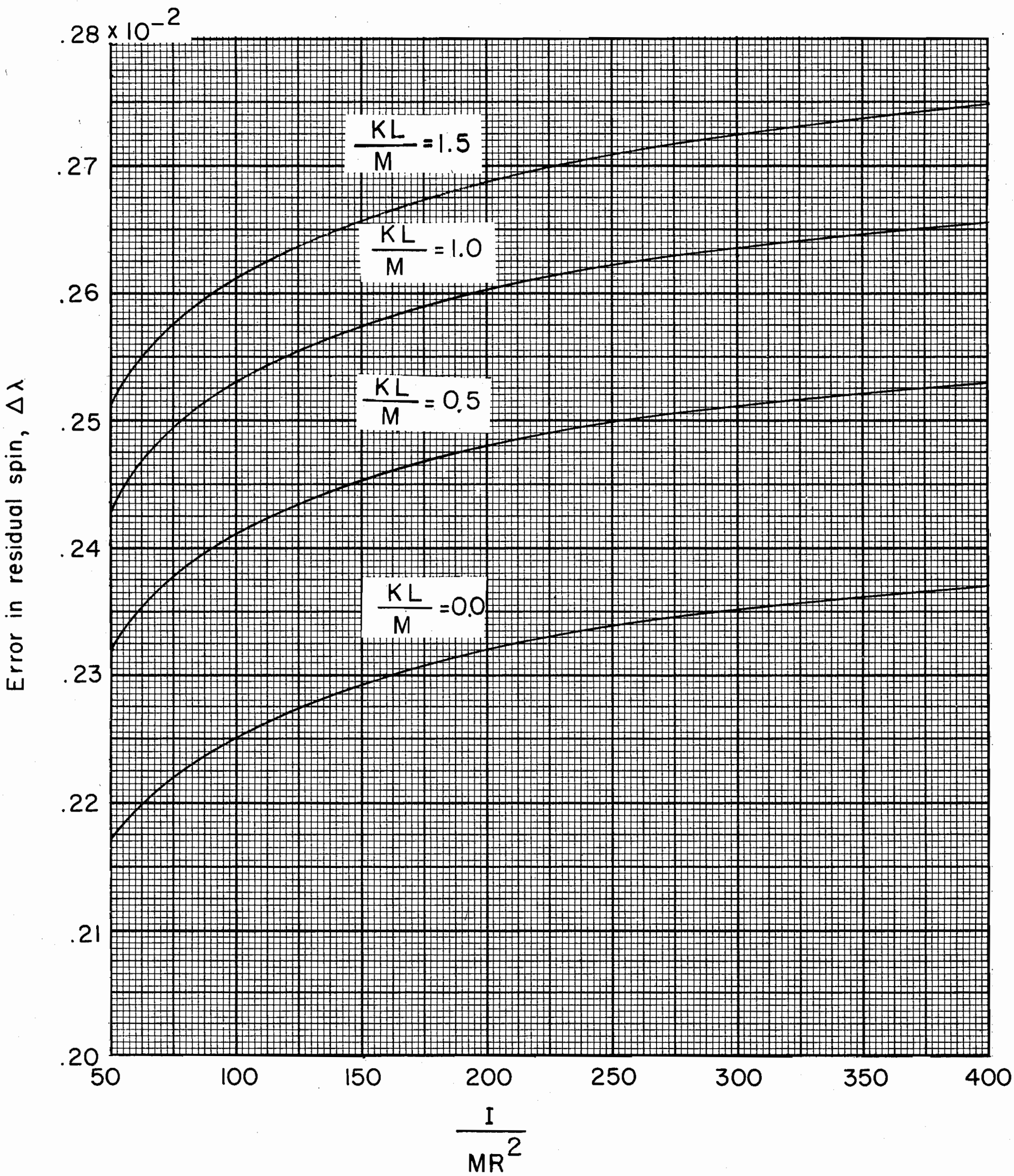
Examination of the transient conditions reveals that an increase in the parameter $\frac{I}{MR^2}$ results in a decrease in the error in the residual spin due to a release of the cables and weights at an angle $\alpha \neq 90^{\circ}$. Figure 7 shows the error in the residual spin due to a release of the weights and cables at an error in release of $\Delta\alpha = \pm 20^{\circ}$ from $\alpha = 90^{\circ}$. This error is shown plotted against $\frac{I}{MR^2}$ for various values of KL/M and is less than 0.01 for most cases considered. The decrease in error with an increase in $\frac{I}{MR^2}$ is due to a reduced slope of δ against α as α approaches 90°. This decrease in error is also achieved with lighter cables.

The effects of all the previous errors discussed can be minimized by proper staging. An example of this staging would be to reduce the large initial spin rate to an intermediate fraction of the desired reduction value. The second stage of de-spin would then have the final spin rate of the first stage as its initial spin rate. Therefore, for



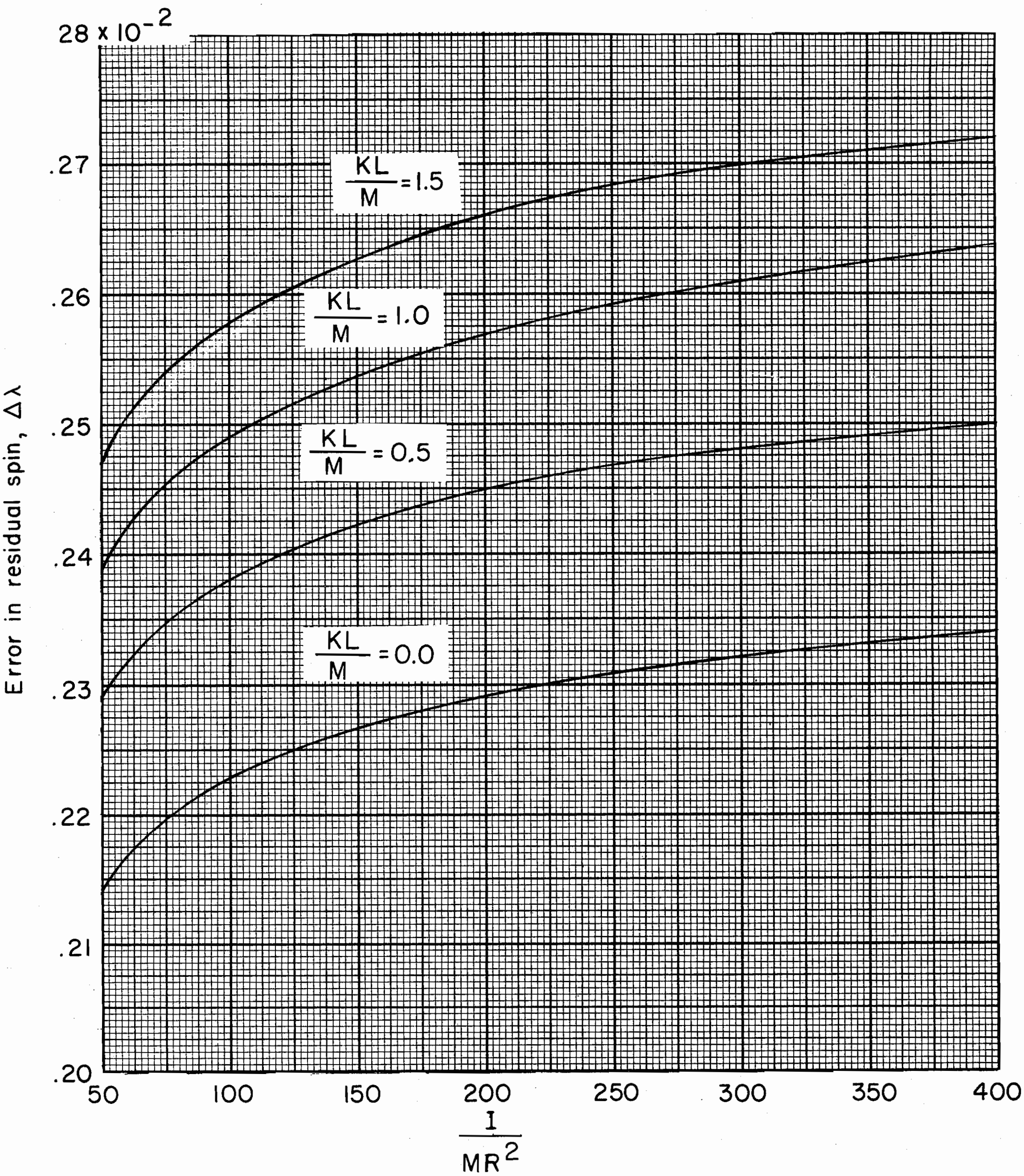
(a) $\lambda = 0.00$.

Figure 6.- Error in residual spin due to $\frac{\Delta L}{L} = \pm 0.25$ percent plotted against $\frac{I}{MR^2}$ for various values of $\frac{KL}{M}$ and λ.



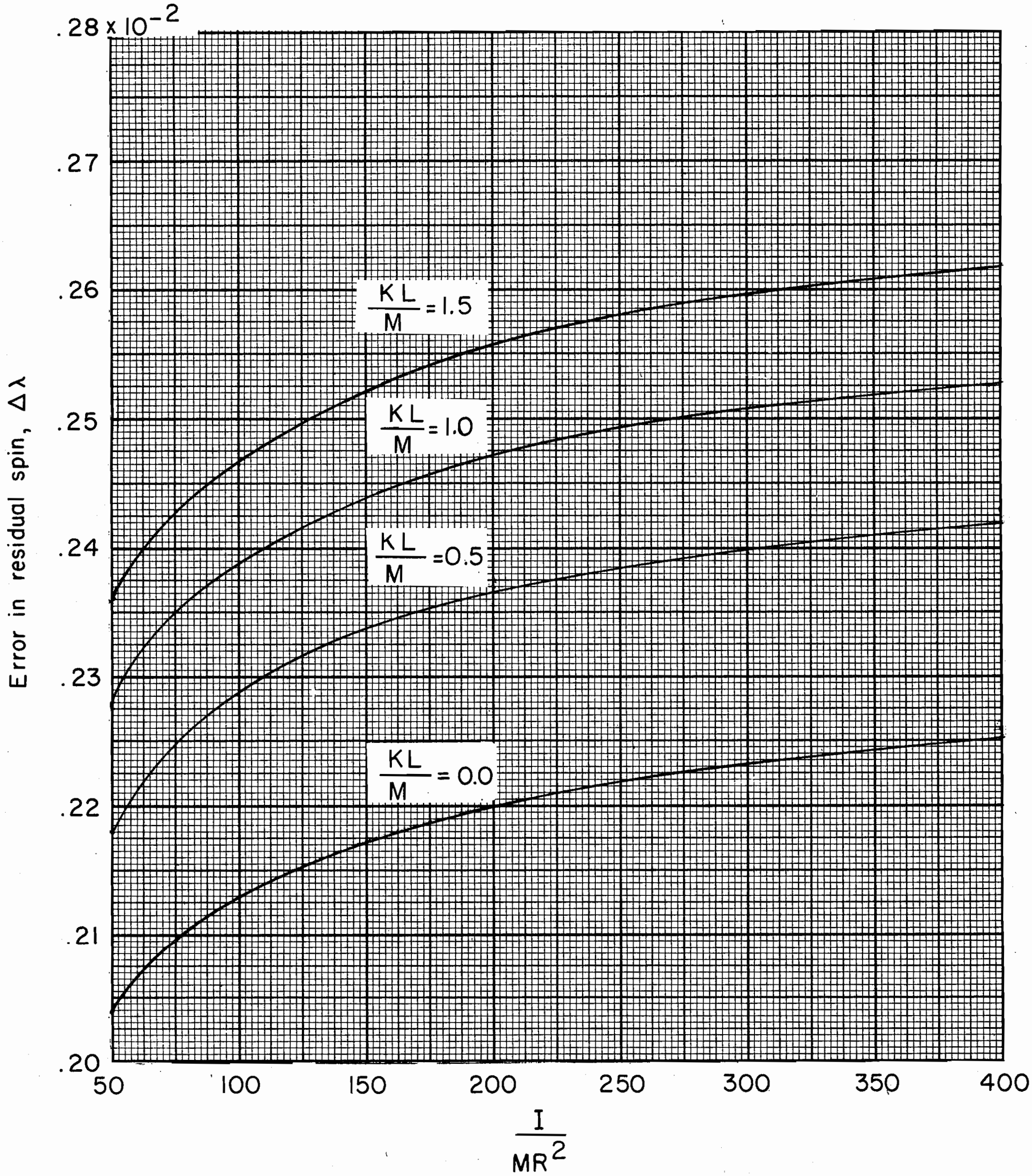
(b) $\lambda = 0.05$.

Figure 6.- Continued.



(c) $\lambda = 0.10$.

Figure 6.- Continued.



(d) $\lambda = 0.20$.

Figure 6.- Concluded.

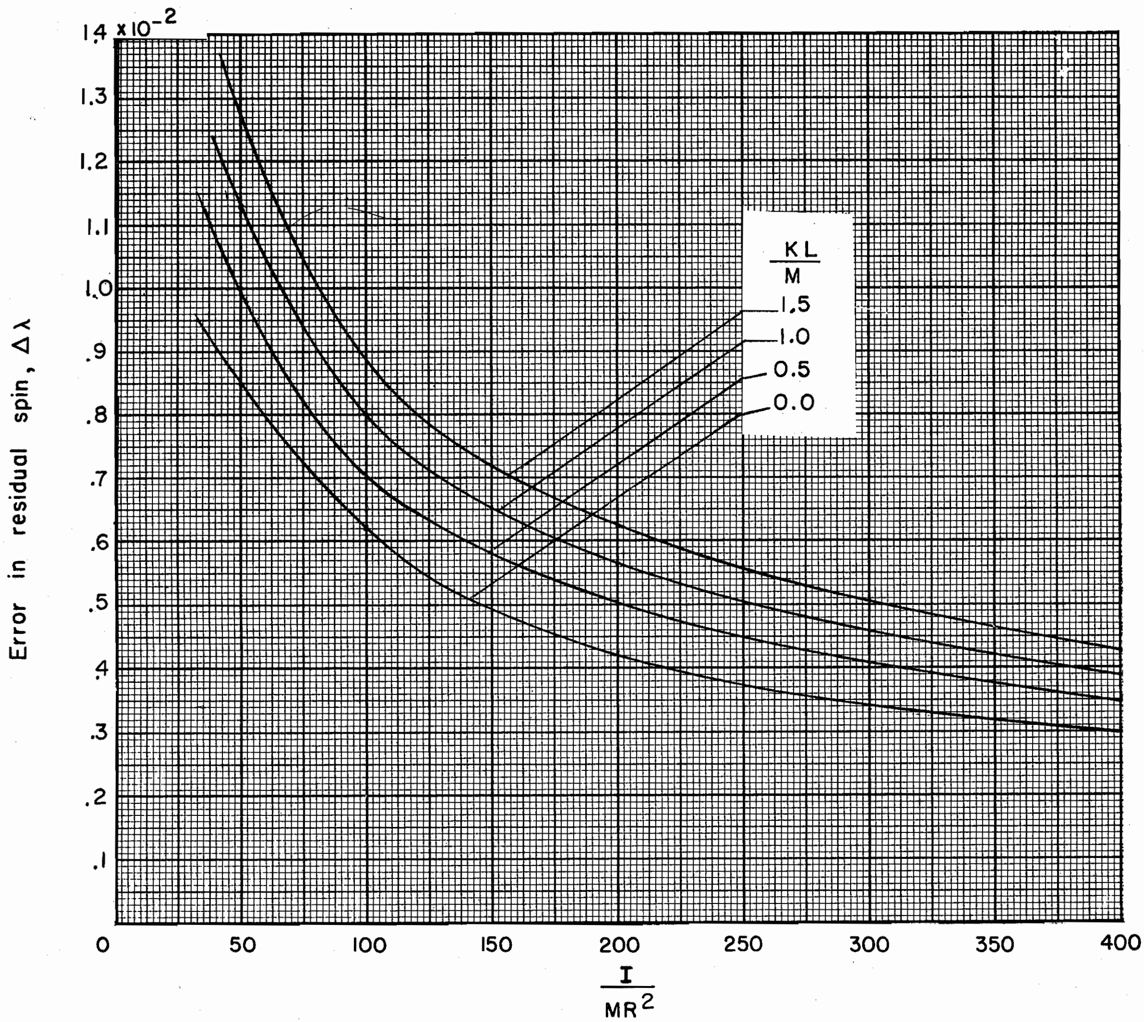
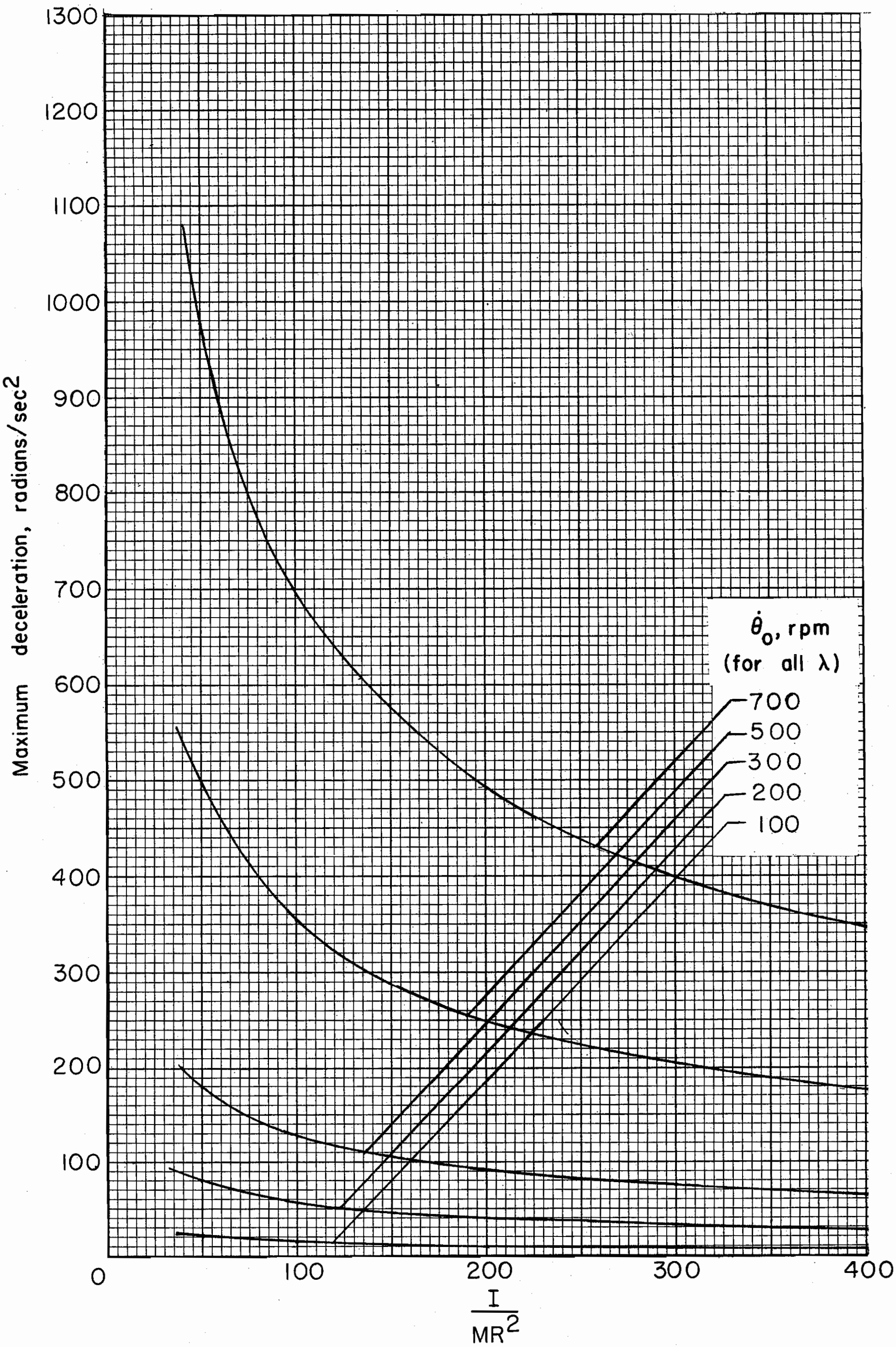


Figure 7.- Error in residual spin rate against $\frac{I}{MR^2}$ for an error in cable release of $\Delta\alpha = \pm 20^{\circ}$ from $\alpha = 90^{\circ}$.

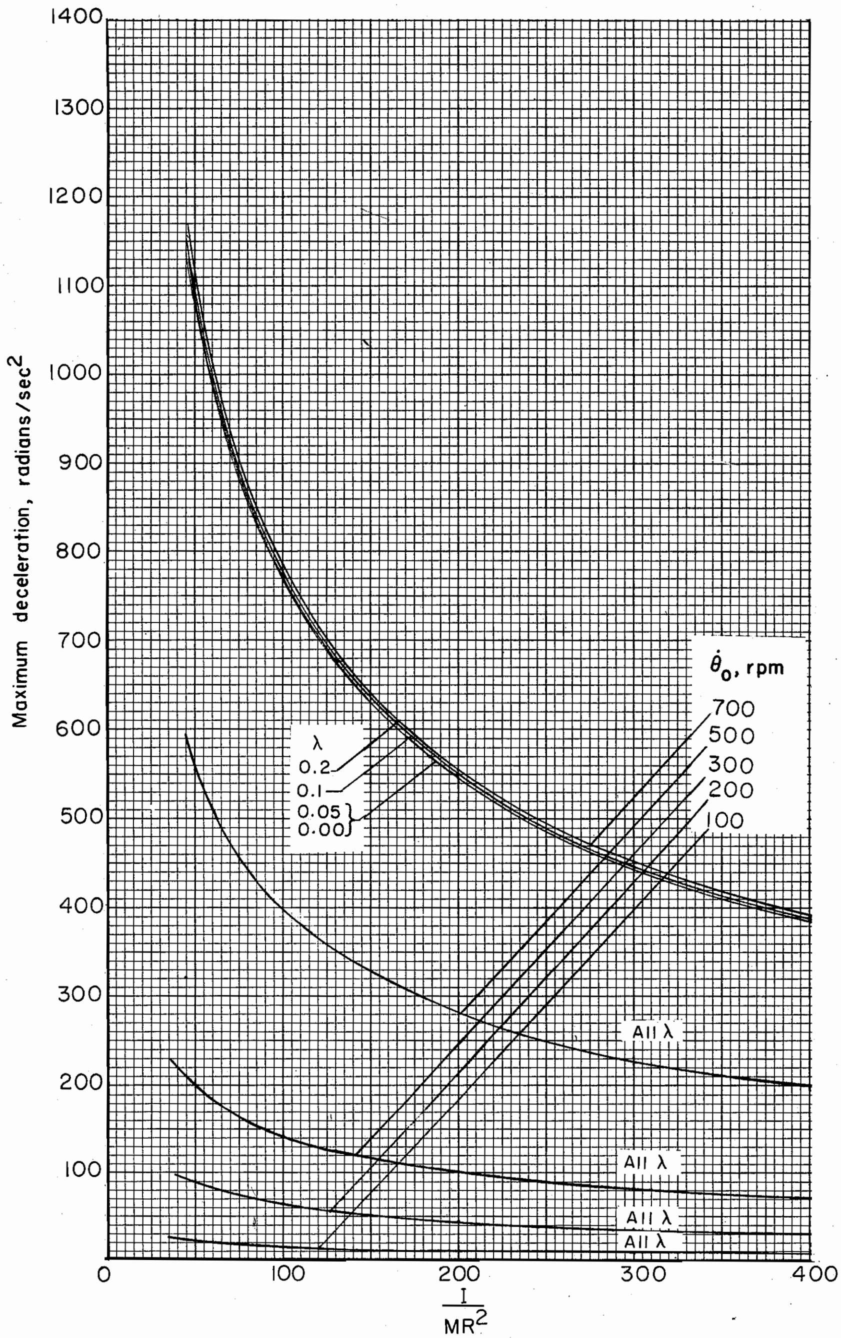
any error in the physical constants or in the release angle, the total error produced on the residual spin is substantially reduced. Care must be exercised when the first-stage spin reduction is close to zero. The tolerance on this reduction can produce a spin in the opposite sense for the second stage. This type of spin would result in an undesirable perturbation.

The maximum deceleration experienced by a payload is seen in figure 8 for various values of $\dot{\theta}_o$ and KL/M. It is seen that the use of lighter attached masses and lower initial spin rates reduces the maximum deceleration experienced by the payload. Where noticeable, the effect of λ on the maximum deceleration is indicated in the figure. The variation of maximum deceleration with λ is explained by the fact that when the length of cable is shortened to achieve a λ for constant initial conditions, the mass of the cable is effectively increased in order to keep KL/M constant. This increase in cable mass increases the total effective mass acting to decelerate the payload; thus, there is a larger resulting deceleration. For the case of $\frac{KL}{M} = 0$, this effect is not seen. The effect of maintaining the density of the cable constant while shortening the cable length in order to achieve $\lambda \neq 0$ is seen in figure 9. This figure shows the time history of the deceleration through phases I and II on a payload with $\frac{I}{MR^2} = 100$ and $\dot{\theta}_o = 700$ rpm. For different values of λ, the payload experiences the same peak deceleration until the time of transition from phase I to phase II occurs near the time of maximum deceleration. At this time, it is possible for the maximum deceleration to occur during phase II. An extreme case of $\lambda = 0.675$ is



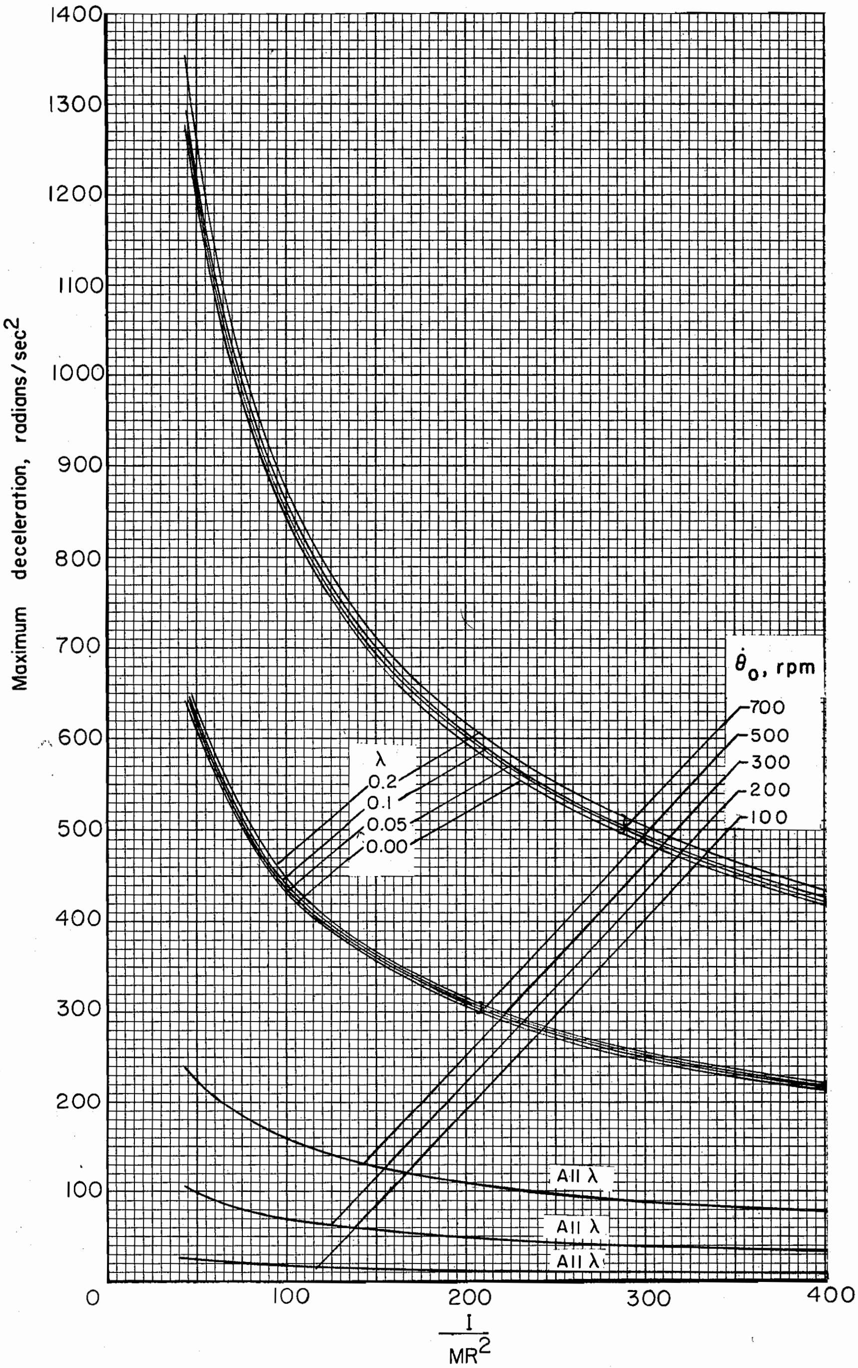
(a) $\frac{KL}{M} = 0.0$.

Figure 8.- Maximum payload deceleration plotted against $\frac{I}{MR^2}$ for various values of $\frac{KL}{M}$ and $\dot{\theta}_o$.



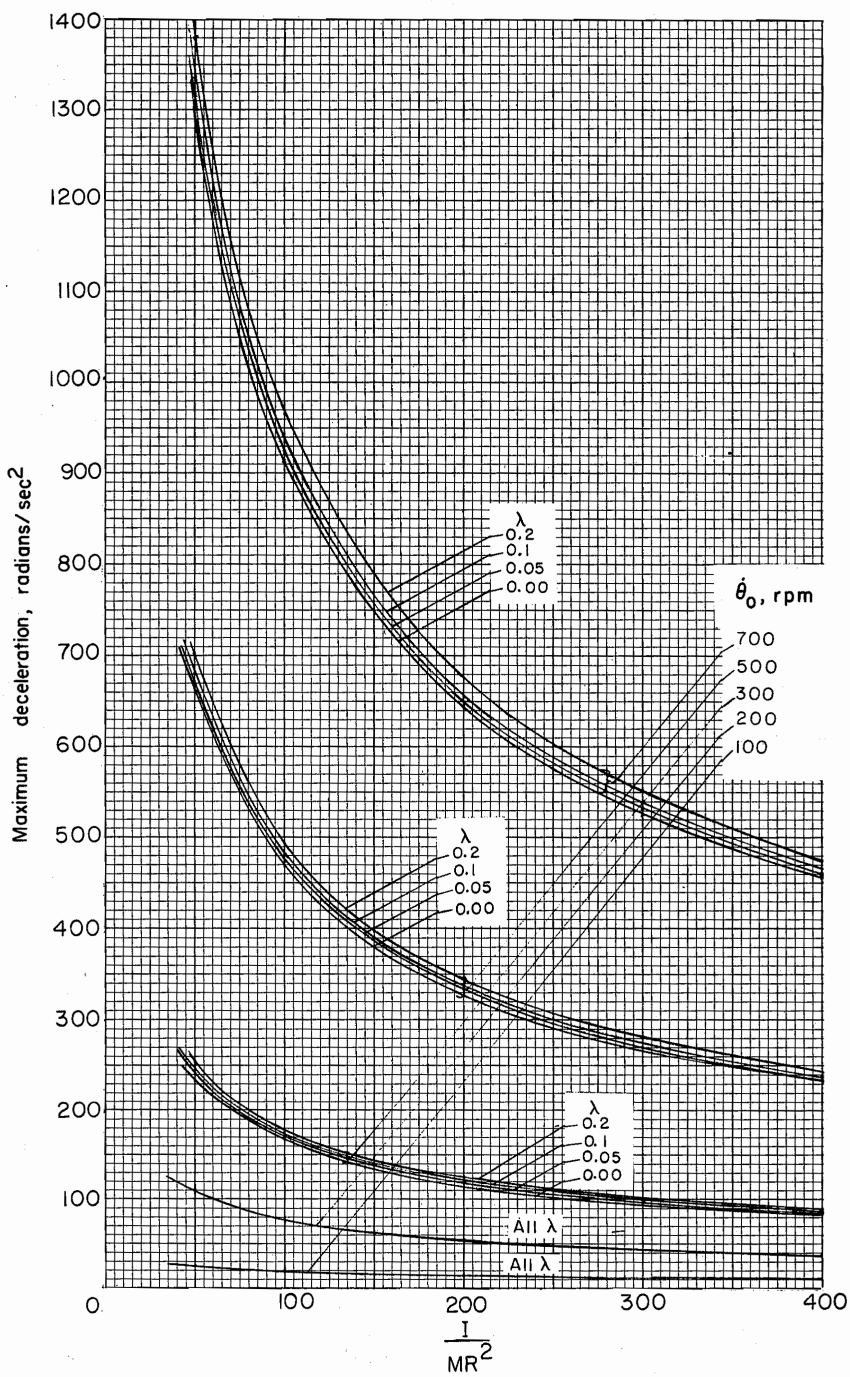
(b) $\frac{KL}{M} = 0.5$.

Figure 8.- Continued.



(c) $\frac{KL}{M} = 1.0.$

Figure 8.- Continued.



(d) $\frac{KL}{M} = 1.5.$

Figure 8.- Concluded.

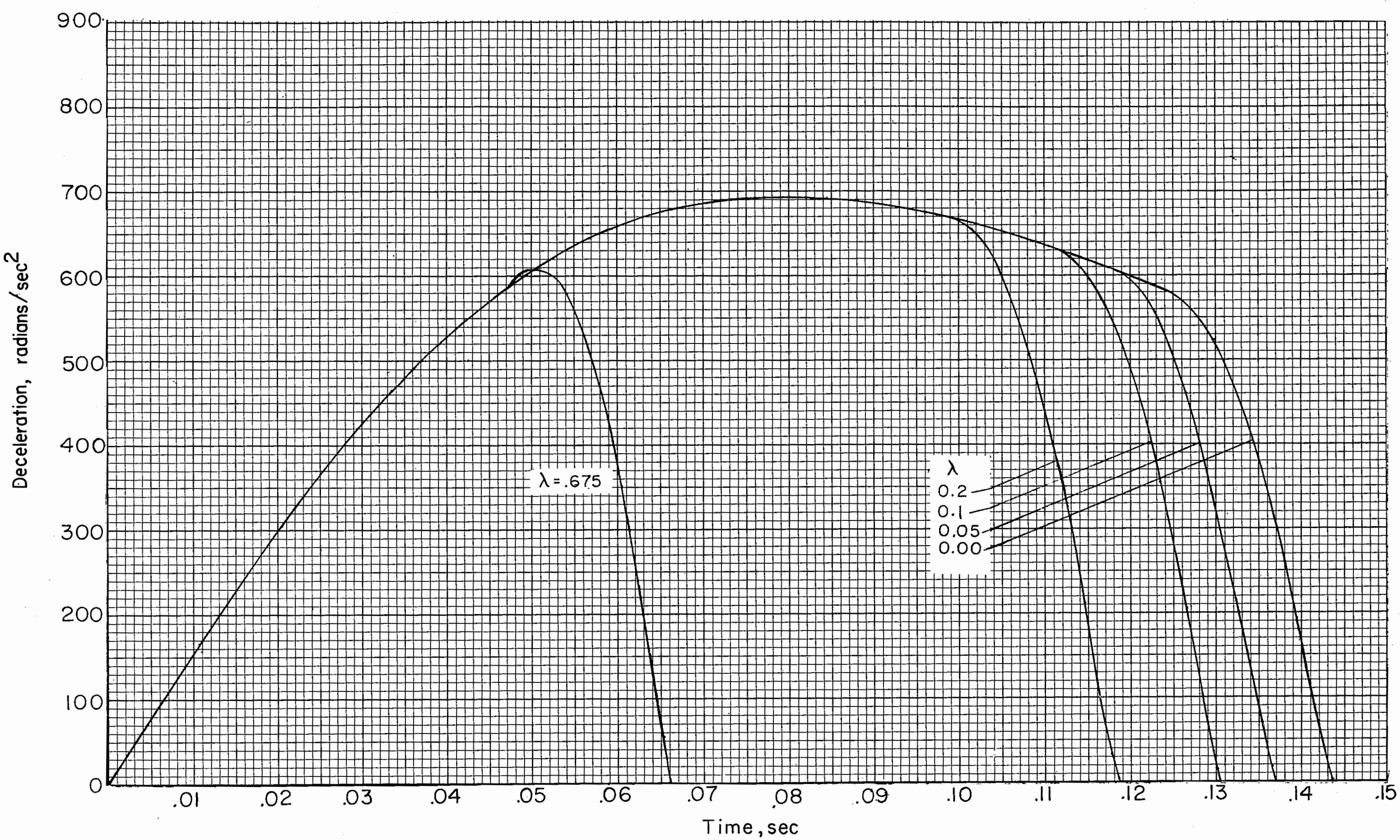


Figure 9.- Payload deceleration against time at various spin reduction ratios for $\frac{I}{MR^2} = 100$, $\frac{KL}{M} = 0$, and $\dot{\theta}_o = 700$ rpm.

shown to have a lower maximum deceleration and to occur before the maximum deceleration of the other values of λ. This fact can be utilized to lower the value of maximum deceleration experienced by a payload for a given set of initial conditions by de-spinning in two stages. The first stage would de-spin to a value of λ, with a transition from phase I to phase II occurring before the maximum deceleration. The residual spin from this stage is then the initial spin for the second stage.

The time required to de-spin for any value of λ and given initial conditions is inversely proportional to the initial spin; therefore,

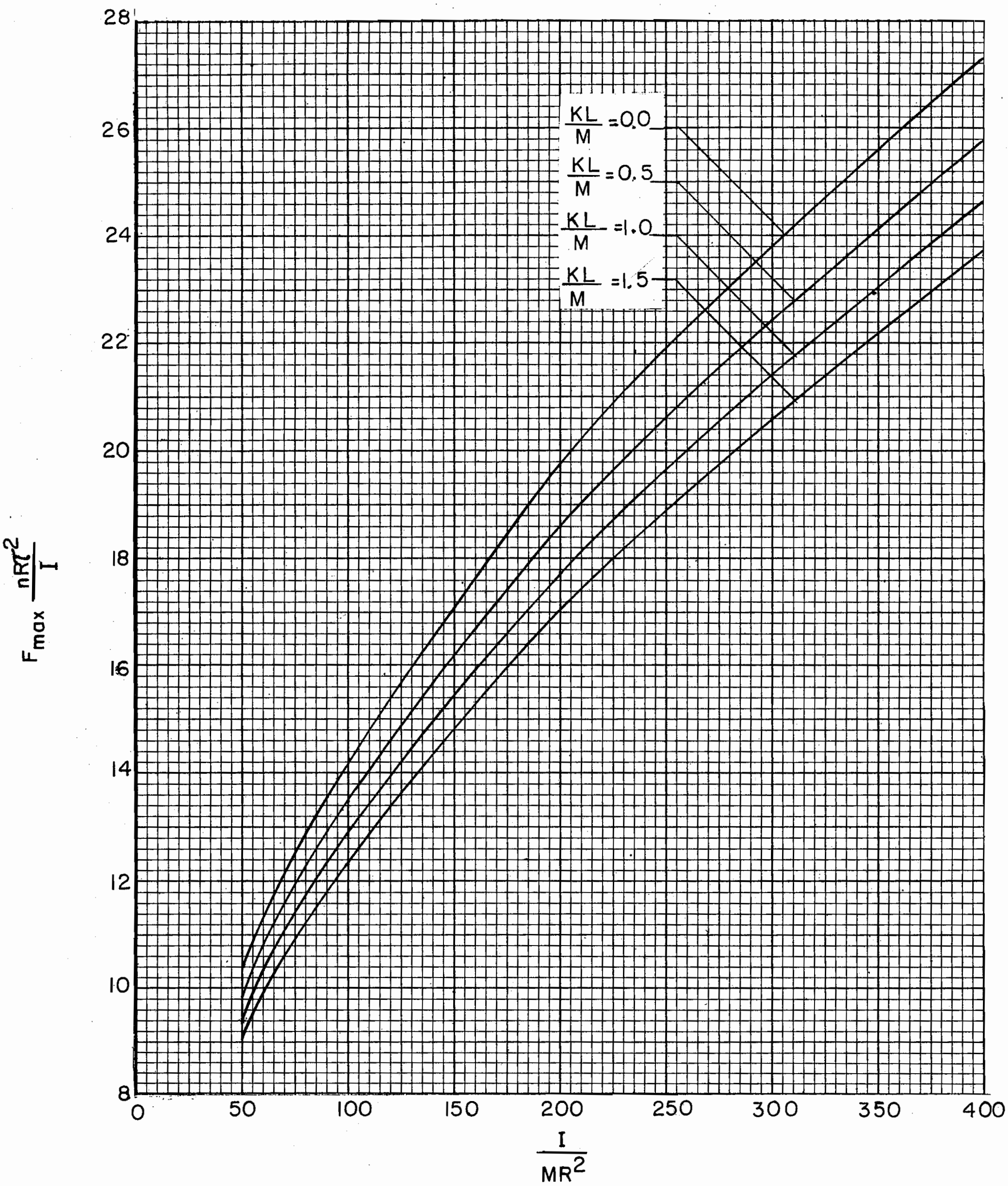
$$\tau\dot{\theta}_o = \text{Constant}$$

As the length of cable is increased, the time required to de-spin to the desired λ is also increased. This time is approximated in the following manner. The results of investigating the transient conditions indicate that, in phase I, the rate $\dot{\phi}$ of the cable unwinding is almost a constant value equal to the initial spin rate. The total rate of rotation relative to the fixed coordinate system is considered constant and approximately equal to the initial spin rate. Therefore, the total number of radians traveled relative to the fixed coordinate system for a given configuration is considered equal to the total angle in phase I plus the angle traveled in phase II; that is,

$$\tau\dot{\theta}_o = \frac{L}{R} + \arctan\frac{L}{R}$$

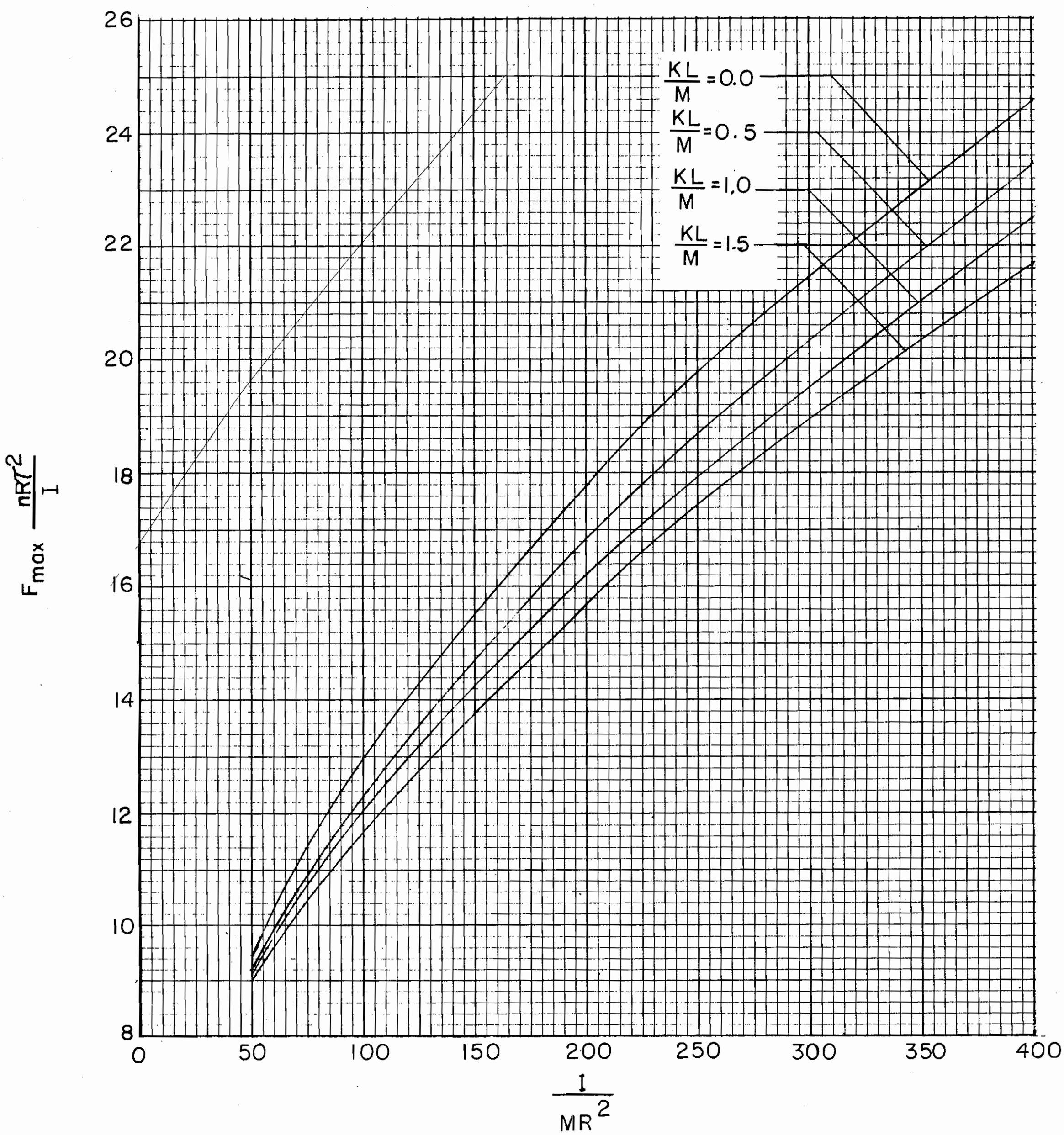
This expression yields the total time required to de-spin to within 0.5 percent of the actual values with L either chosen or computed for a given configuration.

The tensile stress experienced by the cable during the de-spin is a design consideration that must be accounted for in choosing a cable which will determine a value of $\frac{KL}{M}$. The maximum force experienced by the cable can occur in either phase I or II and is shown as a function of $\frac{I}{MR^2}$ in figure 10. The maximum force is plotted in the dimensionless form $F_{max} \frac{nR\tau^2}{I}$ where τ is the total elapsed time from initial release to jettison of cables and weights. This force is for one of n cables of equal density symmetrically spaced around the circumference.



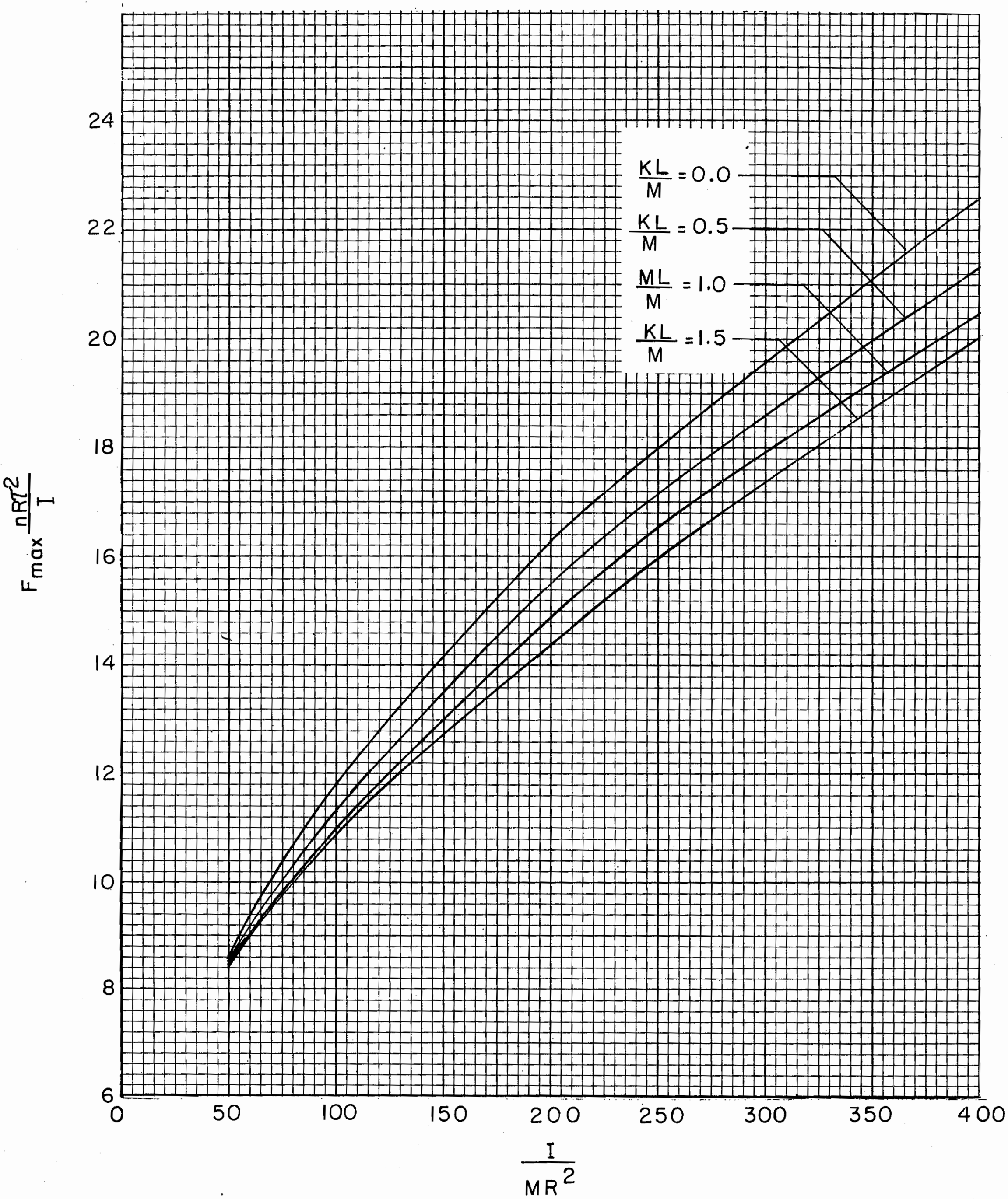
(a) $\lambda = 0.00$.

Figure 10.- Variation of $F_{max} \frac{nR\tau^2}{I}$ with $\frac{I}{MR^2}$ for various values of $\frac{KL}{M}$ and λ.



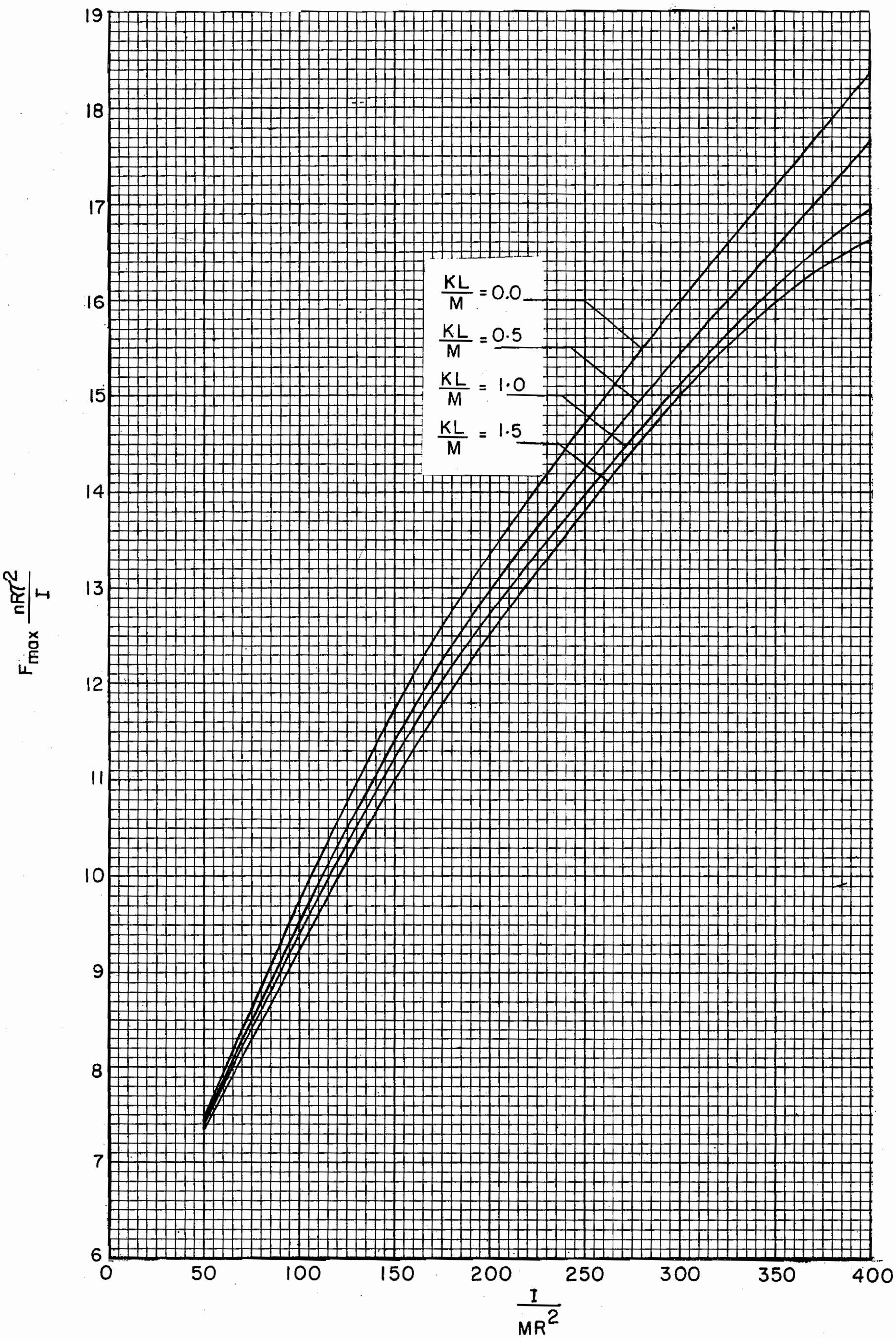
(b) $\lambda = 0.05$.

Figure 10.- Continued.



(c) $\lambda = 0.10$.

Figure 10.- Continued.



(d) $\lambda = 0.20$.

Figure 10.- Concluded.

XII. CONCLUSIONS

The equations of motion derived for the de-spinning of a rigid body by utilizing weights on the ends of unwinding flexible cables show the following:

1. The length of cable of given density or mass of attached weights may be found from the initial and final angular momentum and energy equations to produce any desired spin-reduction ratio for a spinning body of known inertia and no acting friction.

2. The length of cable or mass used to de-spin a given body is a function of the spin-reduction ratio and is independent of initial spin rate. The accuracy of desired residual spin is directly dependent on the initial spin.

3. The error in the spin-reduction ratio due to an error in length of cable or measured inertia may be predicted, with the largest error produced by the large expected error in inertia.

4. In order to obtain accurate results, the weight of the cable must be considered.

5. The error in the residual spin due to a release of the cables and weights at an error in release angle of $\pm 20^\circ$ from the desired angle of 90° is less than 0.01 for most cases considered. This error can be made negligible by a properly designed release mechanism. This error is also reduced by using lighter cables and a higher ratio of payload inertia to initial inertia $\frac{I}{MR^2}$ which corresponds to lighter attached weights.

6. Greater accuracy in residual spin can be obtained by proper staging of the total de-spin reduction.

7. The maximum payload deceleration and the maximum cable tension can occur in either first or second phase. The maximum deceleration and tension can be decreased by:

(a) Lower initial spin rate

(b) Higher value of $\frac{I}{MR^2}$

(c) Staging the de-spin operation so that the maximum deceleration for the first stage occurs before the maximum deceleration for the configuration.

8. An expression for the total time required to de-spin to any residual spin rate is presented as a function of the cable length. This time is within 0.5 percent of the actual time required.

XIII. ACKNOWLEDGMENT

Acknowledgment is made to the National Aeronautics and Space Administration for allowing the author to use a portion of a published report as a subject for this master's thesis.

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XV. VITA

The author, Donald Gordon Eide, was born May 26, 1928 in New York City. He graduated from Peter Stuyvesant High School in New York City in January 1945. He was drafted into the Army in October 1950 and was discharged from active duty as a 2d Lt. in January 1954. He received the degree of Bachelor of Science in Aeronautical Engineering from the University of Oklahoma in June 1957. After a short stay with the Martin Company of Baltimore, Maryland he resigned to accept a position with the National Advisory Committee for Aeronautics and has remained with them through their transition to the National Aeronautics and Space Administration at Langley Air Force Base, Hampton, Va.

DESIGN CRITERIA AND EQUATIONS OF MOTION FOR THE DE-SPIN
OF A VEHICLE BY THE RADIAL RELEASE OF WEIGHTS AND
CABLES OF FINITE MASS

by

Donald G. Eide

ABSTRACT

The equations of motion are derived for the de-spinning of a rigid-body payload by the use of weights attached to the ends of unwinding cables of finite mass that are released when colinear with a radius of the payload. Expressions for the length of cable of a given mass per unit length or mass of attached weights required to de-spin a body to zero rotational speed are derived. The energy and momentum balance equations of initial and final conditions are presented which must be solved simultaneously for either the length of cable or mass required to de-spin a payload to a spin rate other than zero. The effect of cable mass is significant on the desired spin-reduction ratio. The effects on the residual spin due to an error in body moment of inertia, cable length, and release of cables and weights at an angle other than the desired 90° are shown. The expected errors in inertia produce the largest errors in residual spin.

The maximum payload deceleration and maximum cable tension are presented for various nondimensional parameters. The effect of the spin-reduction ratio on the peak deceleration is discussed. Reduction in the maximum payload deceleration and cable tension may be obtained by proper staging of the de-spin, lowering the initial spin rate, and using lighter

weights. An expression for the total time required to de-spin is presented and found to be within 0.5 percent of the actual time required.