

A STUDY OF THE LINES OF FLOW AND THE EQUIPOTENTIAL  
" LINES IN A PLATE CONDUCTOR

By

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## Introduction

In the field of science Conformal Mapping or Conformal Representation is a relatively new subject. However, since the beginning of the twentieth century the application of the mapping of conjugate or conformal functions to a wide variety of practical problems has been made. The chief application is to problems of flow--specifically, to the flow of gases, the flow of liquids, and the flow of electric currents. Since text books on Complex Variables and Conjugate Functions deal comprehensively with the theory of conformal mapping no effort is made herein to outline the fundamental principles of mapping.

This treatise will be confined to a discussion of the flow of electric currents. A thorough study of one particular type of problem, the flow of an electric current in a plate conductor around a circular, an elliptic, and a square aperture will be given with a mathematical analysis of this problem. In the problem considered, a linear source of current is assumed to be impressed at one end of the plate conductor and a linear sink is assumed to be at the other end. The current is induced uniformly from the source to the sink.

In the tables of calculations of this paper only the coordinates of points in the upper right quadrant are shown. In the graphical representations points in the other three quadrants were plotted from a knowledge of the symmetry of the problem and a study of the symmetry of the function which was used. The symbol "e", used in the tables, represents the base of the natural logarithm.

Reference to the Schwarz-Christoffel transformation will be made and an explanation of its failure in one type of problem will be cited.

So that some comparison might be made between the lines of flow of an electric current in a plate conductor and the lines in a magnetic field, a study of a magnetic field is made in Part V.

## Part I

## THE LINES OF FLOW AND EQUIPOTENTIAL LINES AROUND THE CIRCLE

$$x^2/1 + y^2/1 = 1$$

In the mapping of the function  $w = f(z) = f(x, jy)$  there must be a one-to-one correspondence; that is, for every value of  $z$  in the  $x, jy$  plane there is a corresponding value for  $w$  in the  $u, jv$  plane and there must be no points of discontinuity.

The continuous function  $z = w + k \sqrt{w^2 - 1}$  is a function which readily adapts itself to mapping from the  $w$  plane to the  $z$  plane.

If we consider the constant  $k = 1$  then  $z = w + \sqrt{w^2 - 1}$  where both  $z$  and  $w$  are complex. If  $w = u + jv$  then the function  $z$  becomes  $z = u + jv + \sqrt{(u^2 - v^2 - 1) + 2juv}$ . In computing  $z$  for each value of  $w$  there exists the necessity for the evaluation of the square root of a complex number by either one of two methods. Since the  $\log z = 1/2(\text{Log } t + j\theta_t)$ , this method of evaluation introduces unusually long computations in the solution of a single point only. Hence it is evident that the radical could be best calculated by the use of the form  $t e^{j\theta_t}$ . In subsequent calculations this is the method used. The resulting coordinates of each point then are expressed in both polar and rectangular form.

In the above case  $k$  is unity and computations are made for  $z$  corresponding to the straight lines  $w = jc$  parallel to the  $u$  axis, the lines of flow. Then the resulting lines of flow in the  $z$  plane will be determined for a conductor in which there exists a deleted circle

of unit radius. If the lines parallel to the  $jv$  axis,  $w = c$ , are mapped in the  $z$  plane, these lines will represent the equipotential lines. It should be observed that in the  $w$  plane the equipotential lines are perpendicular to the lines of flow and that in the transformation this equality of angles is preserved.

It should be observed further that  $R = m + jn = 2w$  gives a figure in the  $R$  plane similar to that in the  $w$  plane but greater in magnitude. This second translation is necessary for the establishment of the scale for the proper relationship between the two maps.

The lines of flow in a plate conductor become asymptotic to the lines of flow around a circular opening. The lines closest to the opening have the greatest amount of distortion and the lines farther from the opening flatten out. In a plate whose dimensions are large compared with the size of the opening the lines of flow become practically straight lines at a relatively short distance from the origin.

The necessary calculations for the mapping of the lines of flow and the equipotential lines in a plate conductor around a circle of unit radius are listed in Table 1. Figure 1 shows several of these lines plotted to scale. The  $R$  plane is superimposed on the  $z$  plane and the broken lines represent lines of flow and equipotential lines in the  $R$  plane.

Table 1

DATA FOR PLOTTING LINES OF FLOW AROUND THE CIRCLE

$$x^2/1 + y^2/1 = 1$$

$$z = w + \sqrt{w^2 - 1} = u + jv + \sqrt{(u^2 - v^2 - 1) + j2uv}$$

1. If  $v = 0$ ,  $z = u + j0 + \sqrt{(u^2 - 1) + j0}$

$$v = 0 \quad u = 0 \quad z = 0 + j1$$

$$v = 0 \quad u = 1 \quad z = 1.00 + j0$$

$$v = 0 \quad u = 2 \quad z = 3.73 + j0$$

$$v = 0 \quad u = 3 \quad z = 5.83 + j0$$

$$v = 0 \quad u = 4 \quad z = 7.87 + j0$$

$$v = 0 \quad u = 5 \quad z = 9.89 + j0$$

2. If  $v = \frac{1}{4}$ ,  $z = u + j\frac{1}{4} + \sqrt{(u^2 - 1.06) + j0.5u}$

$$v = \frac{1}{4} \quad u = 0 \quad z = 0 + j1.28$$

$$v = \frac{1}{4} \quad u = 1 \quad z = 1 + j\frac{1}{4} + 0.29 e^{j48^\circ 25'}$$

$$v = \frac{1}{4} \quad u = 2 \quad z = 2 + j\frac{1}{4} + 1.76 e^{j10^\circ 15'}$$

$$v = \frac{1}{4} \quad u = 3 \quad z = 3 + j\frac{1}{4} + 2.84 e^{j5^\circ 20'}$$

$$v = \frac{1}{4} \quad u = 4 \quad z = 4 + j\frac{1}{4} + 3.86 e^{j3^\circ 49'}$$

$$v = \frac{1}{4} \quad u = 5 \quad z = 5 + j\frac{1}{4} + 4.90 e^{j2^\circ 59'}$$

3. If  $v = \frac{1}{2}$ ,  $z = u + j\frac{1}{2} + \sqrt{(u^2 - 1.25) + ju}$

$$v = \frac{1}{2} \quad u = 0 \quad z = 0 + j1.61$$

$$v = \frac{1}{2} \quad u = 1 \quad z = 1 + j\frac{1}{2} + 1.01 e^{j52^\circ 01'}$$

$$v = \frac{1}{2} \quad u = 2 \quad z = 2 + j\frac{1}{2} + 2.04 e^{j18^\circ 01'}$$

$$v = \frac{1}{2} \quad u = 3 \quad z = 3 + j\frac{1}{2} + 2.88 e^{j10^\circ 35'}$$

$$v = \frac{1}{2} \quad u = 4 \quad z = 4 + j\frac{1}{2} + 3.90 e^{j7^\circ 35'}$$

$$v = \frac{1}{2} \quad u = 5 \quad z = 5 + j\frac{1}{2} + 4.92 e^{j5^\circ 57'}$$

4. If  $v = \frac{3}{4}$ ,  $z = u + j\frac{3}{4} + \sqrt{(u^2 - 1.56) + j1.5u}$

$v = \frac{3}{4}$     $u = 0$     $z = 0 + j1.99$

$v = \frac{3}{4}$     $u = 1$     $z = 1 + j\frac{3}{4} + 1.26 e^{j55^\circ 14'}$

$v = \frac{3}{4}$     $u = 2$     $z = 2 + j\frac{3}{4} + 1.96 e^{j25^\circ 27'}$

$v = \frac{3}{4}$     $u = 3$     $z = 3 + j\frac{3}{4} + 2.94 e^{j15^\circ 35'}$

$v = \frac{3}{4}$     $u = 4$     $z = 4 + j\frac{3}{4} + 3.93 e^{j11^\circ 17'}$

$v = \frac{3}{4}$     $u = 5$     $z = 5 + j\frac{3}{4} + 4.96 e^{j8^\circ 52'}$

5. If  $v = 1$ ,  $z = u + j1 + \sqrt{(u^2 - 2) + j2u}$

$v = 1$     $u = 0$     $z = 0 + j2.41$

$v = 1$     $u = 1$     $z = 1 + j1 + 1.50 e^{j58^\circ 17'}$

$v = 1$     $u = 2$     $z = 2 + j1 + 2.11 e^{j31^\circ 43'}$

$v = 1$     $u = 3$     $z = 3 + j1 + 3.03 e^{j20^\circ 18'}$

$v = 1$     $u = 4$     $z = 4 + j1 + 4.01 e^{j14^\circ 53'}$

$v = 1$     $u = 5$     $z = 5 + j1 + 5.00 e^{j11^\circ 45'}$

6. If  $v = 2$ ,  $z = u + j2 + \sqrt{(u^2 - 5) + j4u}$

$v = 2$     $u = 0$     $z = 0 + j4.24$

$v = 2$     $u = 1$     $z = 1 + j2 + 2.38 e^{j67^\circ 30'}$

$v = 2$     $u = 2$     $z = 2 + j2 + 2.84 e^{j48^\circ 34'}$

$v = 2$     $u = 3$     $z = 3 + j2 + 3.55 e^{j35^\circ 47'}$

$v = 2$     $u = 4$     $z = 4 + j2 + 4.31 e^{j27^\circ 45'}$

$v = 2$     $u = 5$     $z = 5 + j2 + 5.31 e^{j22^\circ 30'}$

7. If  $v = 3$ ,  $z = u + j3 + \sqrt{(u^2 - 10) + j6u}$

$v = 3 \quad u = 0 \quad z = 0 + j6.16$

$v = 3 \quad u = 1 \quad z = 1 + j3 + 3.28 e^{j73^\circ 10'}$

$v = 3 \quad u = 2 \quad z = 2 + j3 + 3.66 e^{j58^\circ 17'}$

$v = 3 \quad u = 3 \quad z = 3 + j3 + 4.24 e^{j46^\circ 35'}$

$v = 3 \quad u = 4 \quad z = 4 + j3 + 4.97 e^{j37^\circ 59'}$

$v = 3 \quad u = 5 \quad z = 5 + j3 + 5.79 e^{j31^\circ 43'}$

8. If  $v = 4$ ,  $z = u + j4 + \sqrt{(u^2 - 17) + j8u}$

$v = 4 \quad u = 0 \quad z = 0 + j8.12$

$v = 4 \quad u = 1 \quad z = 1 + j4 + 4.23 e^{j76^\circ 43'}$

$v = 4 \quad u = 2 \quad z = 2 + j4 + 4.55 e^{j64^\circ 33'}$

$v = 4 \quad u = 3 \quad z = 3 + j4 + 5.03 e^{j54^\circ 13'}$

$v = 4 \quad u = 4 \quad z = 4 + j4 + 5.66 e^{j45^\circ 54'}$

$v = 4 \quad u = 5 \quad z = 5 + j4 + 6.39 e^{j39^\circ 21'}$

9. If  $v = 5$ ,  $z = u + j5 + \sqrt{(u^2 - 26) + j10u}$

$v = 5 \quad u = 0 \quad z = 0 + j10.09$

$v = 5 \quad u = 1 \quad z = 1 + j5 + 5.19 e^{j79^\circ 06'}$

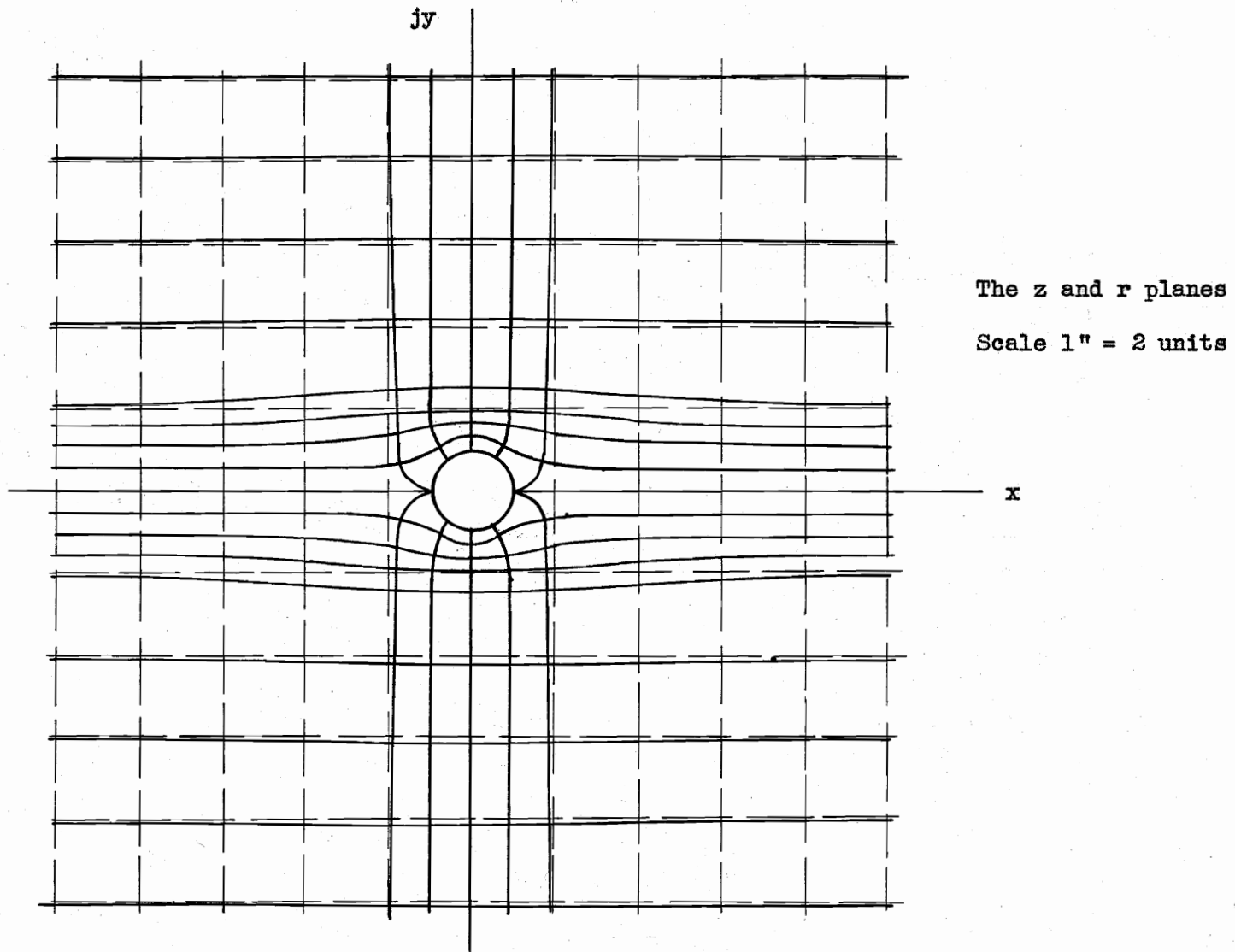
$v = 5 \quad u = 2 \quad z = 2 + j5 + 5.45 e^{j68^\circ 52'}$

$v = 5 \quad u = 3 \quad z = 3 + j5 + 5.87 e^{j59^\circ 46'}$

$v = 5 \quad u = 4 \quad z = 4 + j5 + 6.42 e^{j51^\circ 01'}$

$v = 5 \quad u = 5 \quad z = 5 + j5 + 7.07 e^{j45^\circ 34'}$

It should be observed that to locate various points on any one equipotential line the corresponding values for  $z$  in each of the nine sections listed above must be used.



Horizontal Lines: Lines of Flow  
 Vertical Lines: Equipotential Lines

Fig. 1



## Part II

## THE LINES OF FLOW AND EQUIPOTENTIAL LINES AROUND THE ELLIPSE

$$x^2/1 + y^2/4 = 1$$

In working with the function  $z = w + k\sqrt{w^2 - 1}$  it is seen that the only change resulting from a change in the constant  $k$  is a decrease or increase, that is a dilation or contraction, in the vertical component, the  $iy$  component of a number, depending upon whether the constant is less than unity or greater than unity. In the specific example shown below the value of  $k$  was chosen as two and the resulting lines of flow were those for a plate conductor in which there exists a deleted oblate ellipse of the form equated above.

The data from which these lines of flow and the resulting equipotential lines are found in Table 2 and the lines, plotted to scale are shown in Figure 2. The  $R$  plane,  $R = m + jn = 3w$ , in this case is again superimposed on the  $z$  plane.

It should be observed that the only difference in this case and the one in Part I is that the lines of flow around this particular elliptic aperture are "pushed" farther away from the origin vertically, that they flatten out and become almost straight lines as the distance from the  $x$  axis is increased.

Table 2

DATA FOR PLOTTING LINES OF FLOW AROUND THE ELLIPSE

$$x^2/1 + y^2/4 = 1$$

$$z = w + 2\sqrt{w^2 - 1} = u + jv + 2\sqrt{(u^2 - v^2 - 1) + j2uv}$$

1. If  $v = 0$ ,  $z = u + j0 + 2\sqrt{(u^2 - 1) + j0}$

$$v = 0 \quad u = 0 \quad z = 0.00 + j2$$

$$v = 0 \quad u = 1 \quad z = 1.00 + j0$$

$$v = 0 \quad u = 2 \quad z = 5.46 + j0$$

$$v = 0 \quad u = 3 \quad z = 8.66 + j0$$

$$v = 0 \quad u = 4 \quad z = 11.75 + j0$$

$$v = 0 \quad u = 5 \quad z = 14.79 + j0$$

2. If  $v = \frac{1}{4}$ ,  $z = u + j\frac{1}{4} + 2\sqrt{(u^2 - 1.06) + j0.5u}$

$$v = \frac{1}{4} \quad u = 0 \quad z = 0 + j2.31$$

$$v = \frac{1}{4} \quad u = 1 \quad z = 1 + j\frac{1}{4} + 0.58 e^{j48^\circ 25'}$$

$$v = \frac{1}{4} \quad u = 2 \quad z = 2 + j\frac{1}{4} + 3.52 e^{j10^\circ 15'}$$

$$v = \frac{1}{4} \quad u = 3 \quad z = 3 + j\frac{1}{4} + 5.68 e^{j5^\circ 20'}$$

$$v = \frac{1}{4} \quad u = 4 \quad z = 4 + j\frac{1}{4} + 7.72 e^{j3^\circ 49'}$$

$$v = \frac{1}{4} \quad u = 5 \quad z = 5 + j\frac{1}{4} + 9.80 e^{j2^\circ 59'}$$

3. If  $v = \frac{1}{2}$ ,  $z = u + j\frac{1}{2} + 2\sqrt{(u^2 - 1.25) + ju}$

$$v = \frac{1}{2} \quad u = 0 \quad z = 0 + j2.72$$

$$v = \frac{1}{2} \quad u = 1 \quad z = 1 + j\frac{1}{2} + 2.02 e^{j52^\circ 01'}$$

$$v = \frac{1}{2} \quad u = 2 \quad z = 2 + j\frac{1}{2} + 4.08 e^{j18^\circ 01'}$$

$$v = \frac{1}{2} \quad u = 3 \quad z = 3 + j\frac{1}{2} + 5.76 e^{j10^\circ 35'}$$

$$v = \frac{1}{2} \quad u = 4 \quad z = 4 + j\frac{1}{2} + 7.80 e^{j7^\circ 35'}$$

$$v = \frac{1}{2} \quad u = 5 \quad z = 5 + j\frac{1}{2} + 9.84 e^{j5^\circ 57'}$$

4. If  $v = \frac{3}{4}$ ,  $z = u + j\frac{3}{4} + 2\sqrt{(u^2 - 1.56) + j1.5u}$

$v = \frac{3}{4}$     $u = 0$     $z = 0 + j3.25$

$v = \frac{3}{4}$     $u = 1$     $z = 1 + j\frac{3}{4} + 2.52 e^{j55^\circ} 14'$

$v = \frac{3}{4}$     $u = 2$     $z = 2 + j\frac{3}{4} + 3.92 e^{j25^\circ} 27'$

$v = \frac{3}{4}$     $u = 3$     $z = 3 + j\frac{3}{4} + 5.88 e^{j15^\circ} 35'$

$v = \frac{3}{4}$     $u = 4$     $z = 4 + j\frac{3}{4} + 7.86 e^{j11^\circ} 17'$

$v = \frac{3}{4}$     $u = 5$     $z = 5 + j\frac{3}{4} + 9.92 e^{j8^\circ} 52'$

5. If  $v = 1$ ,  $z = u + j1 + 2\sqrt{(u^2 - 2) + j2u}$

$v = 1$     $u = 0$     $z = 0 + j3.83$

$v = 1$     $u = 1$     $z = 1 + j1 + 3.00 e^{j58^\circ} 17'$

$v = 1$     $u = 2$     $z = 2 + j1 + 4.22 e^{j31^\circ} 43'$

$v = 1$     $u = 3$     $z = 3 + j1 + 6.06 e^{j29^\circ} 18'$

$v = 1$     $u = 4$     $z = 4 + j1 + 8.02 e^{j14^\circ} 53'$

$v = 1$     $u = 5$     $z = 5 + j1 + 10.01 e^{j11^\circ} 45'$

6. If  $v = 2$ ,  $z = u + j2 + 2\sqrt{(u^2 - 5) + j4u}$

$v = 2$     $u = 0$     $z = 0 + j6.48$

$v = 2$     $u = 1$     $z = 1 + j2 + 4.78 e^{j67^\circ} 30'$

$v = 2$     $u = 2$     $z = 2 + j2 + 5.68 e^{j48^\circ} 34'$

$v = 2$     $u = 3$     $z = 3 + j2 + 7.10 e^{j35^\circ} 47'$

$v = 2$     $u = 4$     $z = 4 + j2 + 8.62 e^{j27^\circ} 45'$

$v = 2$     $u = 5$     $z = 5 + j2 + 10.64 e^{j22^\circ} 30'$

7. If  $v = 3$ ,  $z = u + j3 + 2 \sqrt{(u^2 - 10) + j6u}$

$v = 3 \quad u = 0 \quad z = 0 + j9.16$

$v = 3 \quad u = 1 \quad z = 1 + j3 + 6.56 e^{j73^\circ} 10'$

$v = 3 \quad u = 2 \quad z = 2 + j3 + 7.32 e^{j58^\circ} 17'$

$v = 3 \quad u = 3 \quad z = 3 + j3 + 8.48 e^{j46^\circ} 35'$

$v = 3 \quad u = 4 \quad z = 4 + j3 + 9.94 e^{j37^\circ} 59'$

$v = 3 \quad u = 5 \quad z = 5 + j3 + 11.58 e^{j31^\circ} 43'$

8. If  $v = 4$ ,  $z = u + j4 + 2 \sqrt{(u^2 - 17) + j8u}$

$v = 4 \quad u = 0 \quad z = 0 + j12.24$

$v = 4 \quad u = 1 \quad z = 1 + j4 + 8.46 e^{j76^\circ} 43'$

$v = 4 \quad u = 2 \quad z = 2 + j4 + 9.10 e^{j64^\circ} 33'$

$v = 4 \quad u = 3 \quad z = 3 + j4 + 10.06 e^{j54^\circ} 13'$

$v = 4 \quad u = 4 \quad z = 4 + j4 + 11.32 e^{j45^\circ} 54'$

$v = 4 \quad u = 5 \quad z = 5 + j4 + 12.78 e^{j39^\circ} 21'$

9. If  $v = 5$ ,  $z = u + j5 + 2 \sqrt{(u^2 - 26) + j10u}$

$v = 5 \quad u = 0 \quad z = 0 + j15.18$

$v = 5 \quad u = 1 \quad z = 1 + j5 + 10.38 e^{j79^\circ} 06'$

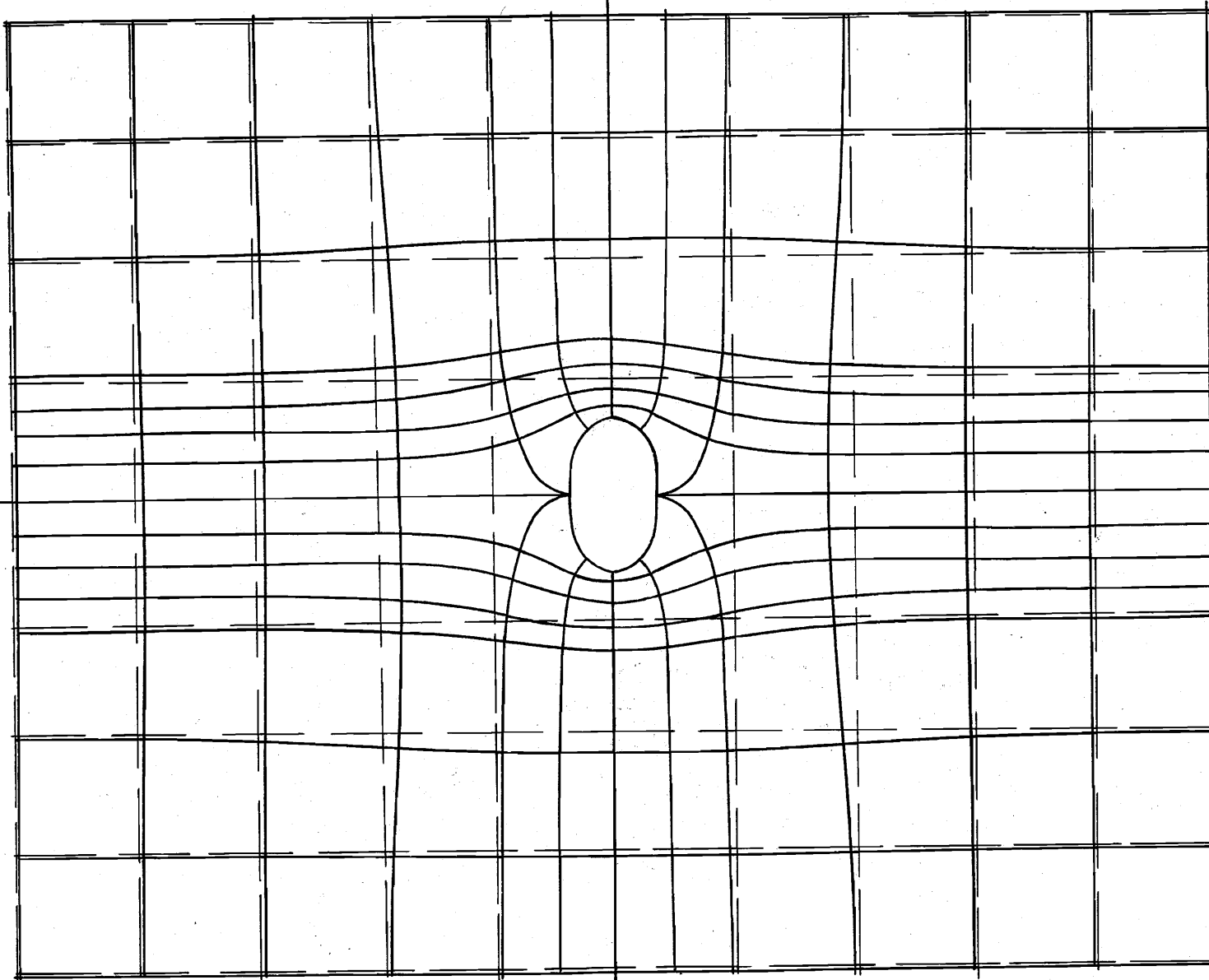
$v = 5 \quad u = 2 \quad z = 2 + j5 + 10.90 e^{j68^\circ} 52'$

$v = 5 \quad u = 3 \quad z = 3 + j5 + 11.74 e^{j59^\circ} 46'$

$v = 5 \quad u = 4 \quad z = 4 + j5 + 12.84 e^{j52^\circ} 01'$

$v = 5 \quad u = 5 \quad z = 5 + j5 + 14.14 e^{j45^\circ} 34'$

It should be observed that to locate various points on any one equipotential line the corresponding values for  $z$  in each of the nine sections listed above must be used.



The z and r planes  
Scale 1" = 2 units

Horizontal Lines: Lines of Flow

Vertical Lines: Equipotential Lines

Fig. 2

## Part III

## THE LINES OF FLOW AND EQUIPOTENTIAL LINES AROUND THE ELLIPSE

$$x^2/1 + y^2/\frac{1}{4} = 1$$

It was stated in Part II that a change in the constant  $k$  changed the  $iy$  component only in the function  $z = w + k \sqrt{w^2 - 1}$  and no contraction or expansion was affected along the  $x$  axis. If the constant is arbitrarily chosen to be one-half, the function  $z = w + 0.5 \sqrt{w^2 - 1}$  will give the lines of flow and the equipotential lines around the prolate ellipse of the form shown above. The calculations were carried out in the same manner as that outlined in Part I. The effect of the use of the constant  $k$  less than unity is to draw the lines of flow in closer to the origin vertically. The  $R$  plane,  $R = m + jn = 1.5w$ , in this case is again superimposed on the  $z$  plane so that the amount of distortion due to the elliptic aperture may be clearly seen.

Table 3 lists the data from which the lines of flow and the equipotential lines were plotted to scale and Figure 3 gives a graphical representation of these lines. In a plate conductor with a deleted prolate ellipse the lines of flow of an electric current are not distorted to the same degree as in the case of the circle and these lines of flow flatten out and become almost straight lines at an even shorter distance from the origin.

Table 3

DATA FOR PLOTTING LINES OF FLOW AROUND THE ELLIPSE

$$x^2/1 + y^2/4 = 1$$

$$z = w + 0.5\sqrt{w^2 - 1} = u + jv + 0.5\sqrt{(u^2 - v^2 - 1) + j2uv}$$

1. If  $v = 0$ ,  $z = u + j0 + 0.5\sqrt{(u^2 - 1) + j0}$

$$v = 0 \quad u = 0 \quad z = 0 + j0.5$$

$$v = 0 \quad u = 1 \quad z = 1.00 + j0$$

$$v = 0 \quad u = 2 \quad z = 2.87 + j0$$

$$v = 0 \quad u = 3 \quad z = 4.41 + j0$$

$$v = 0 \quad u = 4 \quad z = 5.94 + j0$$

$$v = 0 \quad u = 5 \quad z = 7.45 + j0$$

2. If  $v = \frac{1}{4}$ ,  $z = u + j\frac{1}{4} + 0.5\sqrt{(u^2 - 1.06) + j0.5u}$

$$v = \frac{1}{4} \quad u = 0 \quad z = 0 + j0.97$$

$$v = \frac{1}{4} \quad u = 1 \quad z = 1 + j\frac{1}{4} + 0.15 e^{j48^\circ 25'}$$

$$v = \frac{1}{4} \quad u = 2 \quad z = 2 + j\frac{1}{4} + 0.88 e^{j10^\circ 15'}$$

$$v = \frac{1}{4} \quad u = 3 \quad z = 3 + j\frac{1}{4} + 1.42 e^{j5^\circ 20'}$$

$$v = \frac{1}{4} \quad u = 4 \quad z = 4 + j\frac{1}{4} + 1.93 e^{j3^\circ 49'}$$

$$v = \frac{1}{4} \quad u = 5 \quad z = 5 + j\frac{1}{4} + 2.45 e^{j2^\circ 59'}$$

3. If  $v = \frac{1}{2}$ ,  $z = u + j\frac{1}{2} + 0.5\sqrt{(u^2 - 1.25) + ju}$

$$v = \frac{1}{2} \quad u = 0 \quad z = 0 + j1.05$$

$$v = \frac{1}{2} \quad u = 1 \quad z = 1 + j\frac{1}{2} + 0.51 e^{j52^\circ 01'}$$

$$v = \frac{1}{2} \quad u = 2 \quad z = 2 + j\frac{1}{2} + 1.02 e^{j18^\circ 01'}$$

$$v = \frac{1}{2} \quad u = 3 \quad z = 3 + j\frac{1}{2} + 1.44 e^{j10^\circ 35'}$$

$$v = \frac{1}{2} \quad u = 4 \quad z = 4 + j\frac{1}{2} + 1.95 e^{j7^\circ 35'}$$

$$v = \frac{1}{2} \quad u = 5 \quad z = 5 + j\frac{1}{2} + 2.46 e^{j5^\circ 57'}$$

$$4. \text{ If } v = \frac{3}{4}, z = u + j\frac{3}{4} + 0.5 \sqrt{(u^2 - 1.56) + j1.5u}$$

$$v = \frac{3}{4} \quad u = 0 \quad z = 0 + j1.38$$

$$v = \frac{3}{4} \quad u = 1 \quad z = 1 + j\frac{3}{4} + 0.63 e^{j55^\circ} 14'$$

$$v = \frac{3}{4} \quad u = 2 \quad z = 2 + j\frac{3}{4} + 0.98 e^{j25^\circ} 27'$$

$$v = \frac{3}{4} \quad u = 3 \quad z = 3 + j\frac{3}{4} + 1.47 e^{j15^\circ} 35'$$

$$v = \frac{3}{4} \quad u = 4 \quad z = 4 + j\frac{3}{4} + 1.97 e^{j11^\circ} 17'$$

$$v = \frac{3}{4} \quad u = 5 \quad z = 5 + j\frac{3}{4} + 2.48 e^{j8^\circ} 52'$$

$$5. \text{ If } v = 1, z = u + j1 + 0.5 \sqrt{(u^2 - 2) + j2u}$$

$$v = 1 \quad u = 0 \quad z = 0 + j1.71$$

$$v = 1 \quad u = 1 \quad z = 1 + j1 + 0.75 e^{j58^\circ} 17'$$

$$v = 1 \quad u = 2 \quad z = 2 + j1 + 1.06 e^{j31^\circ} 43'$$

$$v = 1 \quad u = 3 \quad z = 3 + j1 + 1.52 e^{j20^\circ} 18'$$

$$v = 1 \quad u = 4 \quad z = 4 + j1 + 2.01 e^{j14^\circ} 53'$$

$$v = 1 \quad u = 5 \quad z = 5 + j1 + 2.50 e^{j11^\circ} 45'$$

$$6. \text{ If } v = 2, z = u + j2 + 0.5 \sqrt{(u^2 - 5) + j4u}$$

$$v = 2 \quad u = 0 \quad z = 0 + j3.12$$

$$v = 2 \quad u = 1 \quad z = 1 + j2 + 1.19 e^{j67^\circ} 30'$$

$$v = 2 \quad u = 2 \quad z = 2 + j2 + 1.42 e^{j48^\circ} 34'$$

$$v = 2 \quad u = 3 \quad z = 3 + j2 + 1.78 e^{j35^\circ} 47'$$

$$v = 2 \quad u = 4 \quad z = 4 + j2 + 2.16 e^{j27^\circ} 45'$$

$$v = 2 \quad u = 5 \quad z = 5 + j2 + 2.66 e^{j22^\circ} 30'$$



7. If  $v = 3$ ,  $z = u + j3 + 0.5 \sqrt{(u^2 - 10) + j6u}$

$$v = 3 \quad u = 0 \quad z = 0 + j4.58$$

$$v = 3 \quad u = 1 \quad z = 1 + j3 + 1.64 e^{j73^\circ} 10'$$

$$v = 3 \quad u = 2 \quad z = 2 + j3 + 1.83 e^{j58^\circ} 17'$$

$$v = 3 \quad u = 3 \quad z = 3 + j3 + 2.12 e^{j46^\circ} 35'$$

$$v = 3 \quad u = 4 \quad z = 4 + j3 + 2.49 e^{j37^\circ} 59'$$

$$v = 3 \quad u = 5 \quad z = 5 + j3 + 2.90 e^{j31^\circ} 43'$$

8. If  $v = 4$ ,  $z = u + j4 + 0.5 \sqrt{(u^2 - 17) + j8u}$

$$v = 4 \quad u = 0 \quad z = 0 + j6.06$$

$$v = 4 \quad u = 1 \quad z = 1 + j4 + 2.12 e^{j76^\circ} 43'$$

$$v = 4 \quad u = 2 \quad z = 2 + j4 + 2.28 e^{j64^\circ} 33'$$

$$v = 4 \quad u = 3 \quad z = 3 + j4 + 2.52 e^{j54^\circ} 13'$$

$$v = 4 \quad u = 4 \quad z = 4 + j4 + 2.83 e^{j45^\circ} 54'$$

$$v = 4 \quad u = 5 \quad z = 5 + j4 + 3.19 e^{j39^\circ} 21'$$

9. If  $v = 5$ ,  $z = u + j5 + 0.5 \sqrt{(u^2 - 26) + j10u}$

$$v = 5 \quad u = 0 \quad z = 0 + j7.55$$

$$v = 5 \quad u = 1 \quad z = 1 + j5 + 2.60 e^{j79^\circ} 06'$$

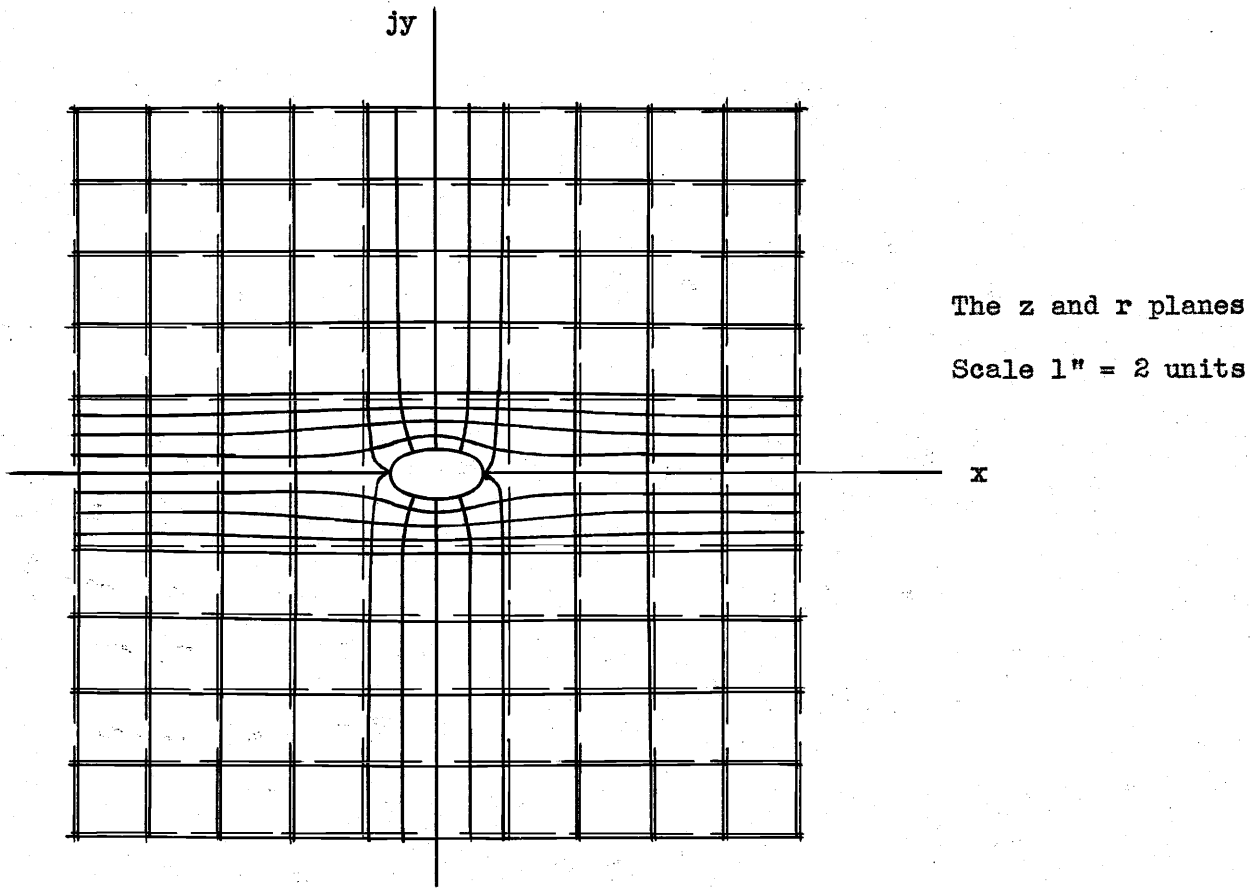
$$v = 5 \quad u = 2 \quad z = 2 + j5 + 2.73 e^{j68^\circ} 52'$$

$$v = 5 \quad u = 3 \quad z = 3 + j5 + 2.94 e^{j59^\circ} 46'$$

$$v = 5 \quad u = 4 \quad z = 4 + j5 + 3.21 e^{j52^\circ} 01'$$

$$v = 5 \quad u = 5 \quad z = 5 + j5 + 3.54 e^{j45^\circ} 34'$$

It should be observed that to locate various points on any one equipotential line the corresponding values for  $z$  in each of the nine sections listed above must be used.



Horizontal Lines: Lines of Flow  
Vertical Lines: Equipotential Lines

Fig. 3

## Part IV

## THE LINES OF FLOW AROUND THE SQUARE

The Schwarz-Christoffel transformation of the form

$$\frac{dw}{dz} = K(z - x_1)^{a_1/\pi - 1} (z - x_2)^{a_2/\pi - 1} (z - x_3)^{a_3/\pi - 1} \dots (z - x_n)^{a_n/\pi - 1} \quad (A)$$

where  $w = u + jv$  and  $z = x + jy$  as derived in most standard text books on Conjugate Functions gives the transformation of any polygon of  $n$  sides in the  $z$  plane to a straight line in the  $w$  plane. The interior angles through which the sides of the polygon must be rotated to produce a straight line in the  $w$  plane are denoted by  $a_1, a_2, \dots, a_n$  and the distances of the several vertices of the polygon from the  $jy$  axis in the  $z$  plane are denoted by  $x_1, x_2, \dots, x_n$ .

An attempt to find mathematically the lines of flow in a plate conductor with a deleted unit square in the center of the conductor (see Fig. 4) was made by the application of the Schwarz-Christoffel transformation.

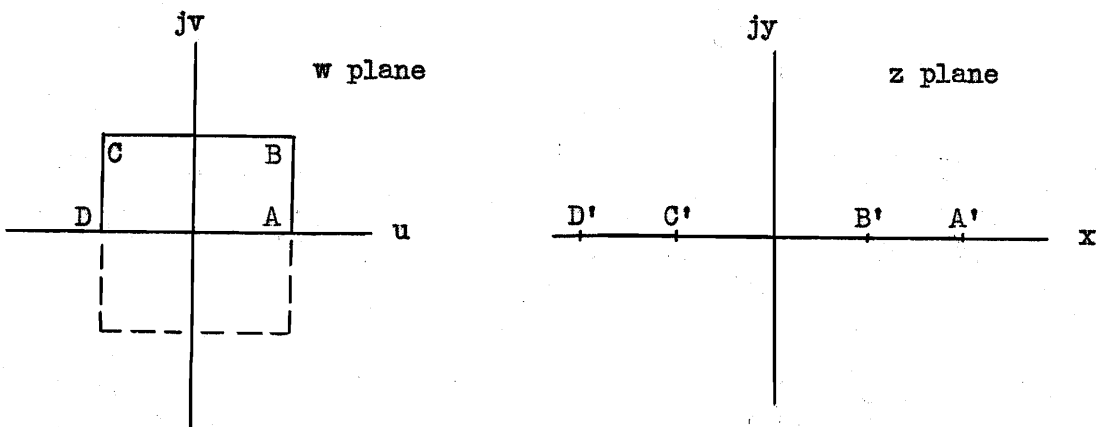


Fig. 4

Equation (A) becomes  $w = K (z - 2)^{-\frac{1}{2}}(z - 1)^{\frac{1}{2}}(z + 1)^{\frac{1}{2}}(z + 2)^{-\frac{1}{2}}dz$

or,  $w = K \left( \sqrt{\frac{z^2 - 1}{z^2 - 4}} dz \right) \text{----(B)}$ . A simple trigonometric substitution shows equation (B) to be an elliptic integral the solution to which admits of a discontinuity when  $z = jy$ , that is whenever  $x = 0$  and  $y = c$ .

A square of two units, rotated through forty five degrees, was studied(see Fig. 5). Equation (A) became  $w = K \left( \sqrt{\frac{z^2}{z^2 - 1}} dz \right)$ . The solution of this equation likewise runs into an elliptic integral which is also discontinuous at the origin.

Combinations of several known transformations were attempted, particularly those for flow around the right angle, but none of these could be made to satisfy the initial prerequisite conditions which necessitate that A go into A', B into B', and C into C'. Experimentally we have found out that the lines of flow do form a smooth, symmetrical curve but a pure mathematical derivation of its equation by the Schwarz-Christoffel transformation is not admitted.

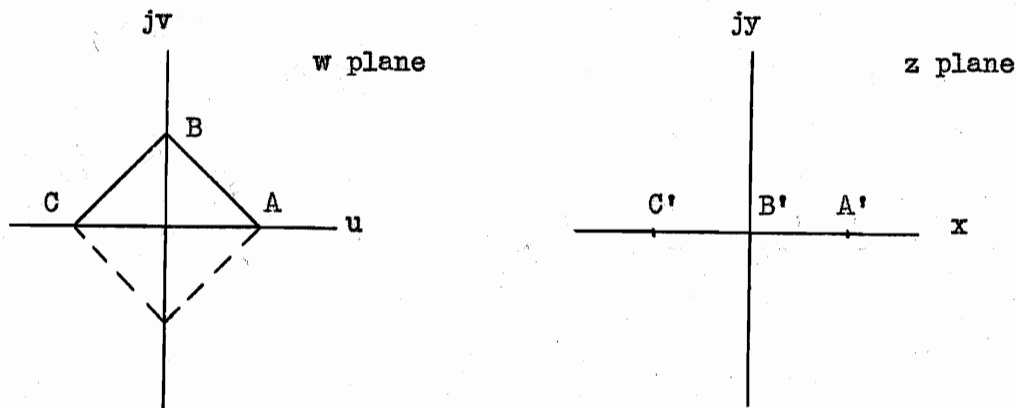


Fig. 5

## Part V

## THE MAGNETIC FIELD

So that there might be some definite basis of comparison between the lines of flow of an electric current in a plate conductor and the lines of a magnetic field two wrought iron plates 4" x 5" x  $\frac{1}{4}$ " were wound with 350 turns of No. 20 wire and magnetized by the current from five Edison cells connected in series. The field between the two electromagnets was sprinkled with iron filings and a portion of the field was blocked out first by a right circular cylinder made from a nonmagnetic material and the field was photographed.

The field was next disturbed by the insertion of an oblate and then a prolate elliptic cylinder and then by a parallelepiped with a square cross section. Figures 6a, 6b, 6c, and 6d show the fields in the order named. It is evident from the photographs that the lines were not disturbed by the insertion of a nonmagnetic obstacle in the field. Since a magnetic field cannot be insulated, the magnetic lines do not conform to the theory for the lines of flow of an electric current in a plate conductor in which certain portions have been deleted or insulated.

Four plate conductors, 3" x 3", one with a deleted circle, one with a deleted oblate ellipse, one with a deleted prolate ellipse, and one with a deleted square were inserted between the magnets. Iron filings were sprinkled on a zelonite square placed over each conductor in the order named. When the plates became magnetized the filings assumed the positions shown in Figures 7a, 7b, 7c, and 7d respectively. These lines of flow compare favorably with those lines calculated

mathematically. The plate conductors were imperfect in consistency and perfect contact with the magnets could not be accomplished. In addition to the mechanical difficulties which arose, the wrought iron plate conductors could not be strongly magnetized due to the limitation in the strength of the magnets and the distance between the poles. The lines of flow shown in Figures 7a, 7b, 7c, and 7d, do, however, show a strong tendency to conform to the theoretical lines discussed in the preceding sections.

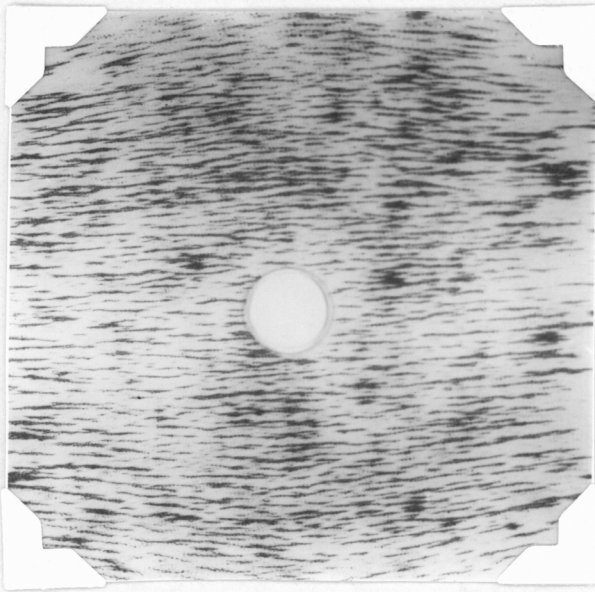


Fig. 6a

Magnetic Field Interrupted  
by a Circular Cylinder

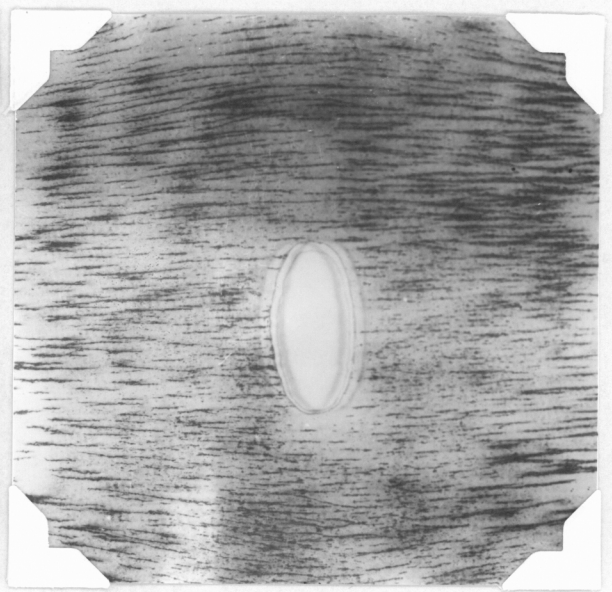


Fig. 6b

Magnetic Field Interrupted by  
an Oblate Cylinder (Elliptic)

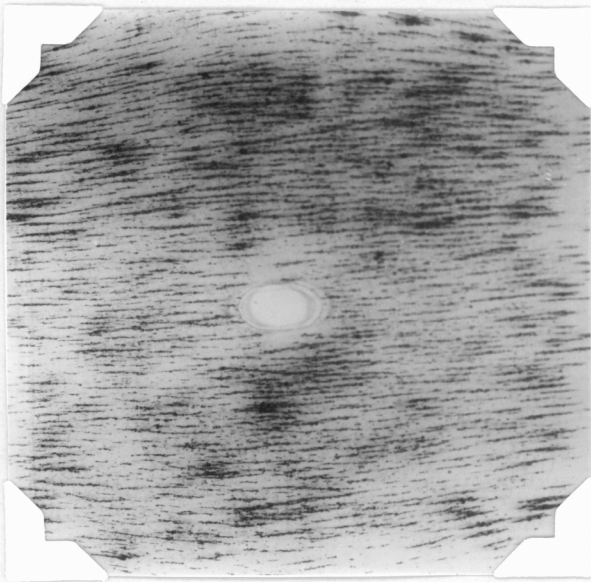


Fig. 6c

Magnetic Field Interrupted by  
a Prolate Elliptic Cylinder

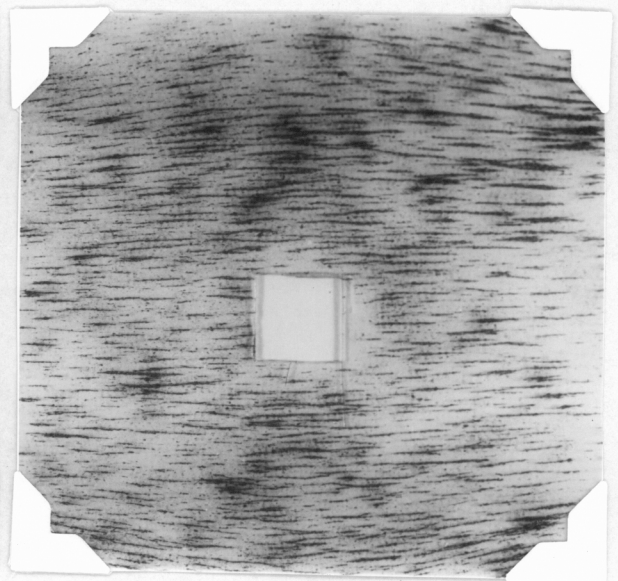


Fig. 6d

Magnetic Field Interrupted by  
a Square Parallelepiped

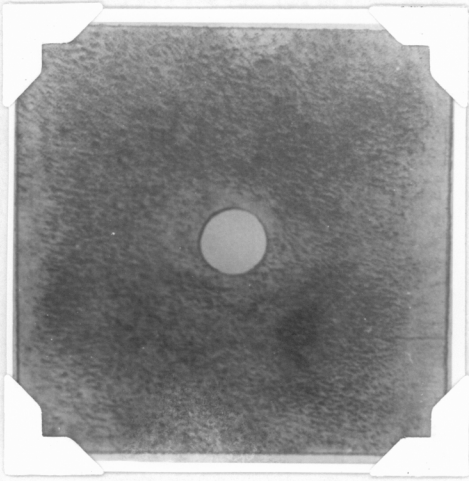


Fig. 7a

Lines of Flow Around the  
Circle

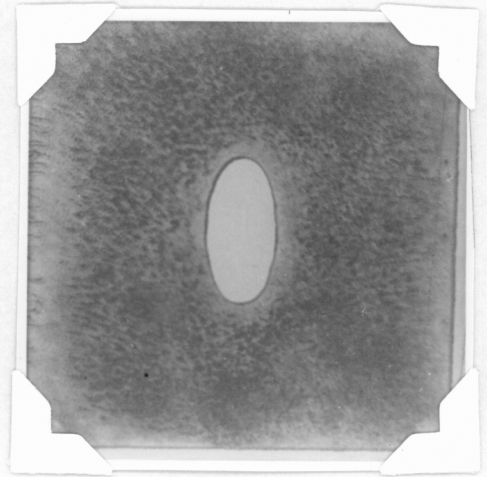


Fig. 7b

Lines of Flow Around the  
Oblate Ellipse

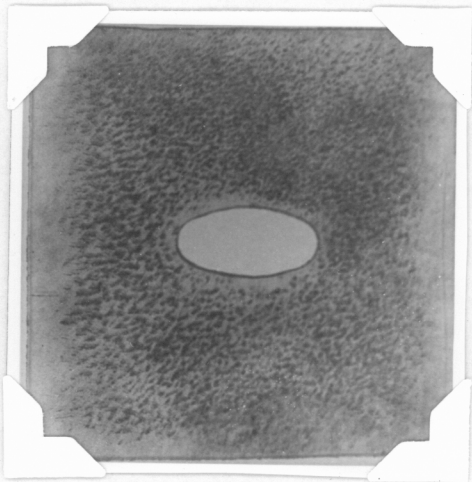


Fig. 7c

Lines of Flow Around the  
Prolate Ellipse

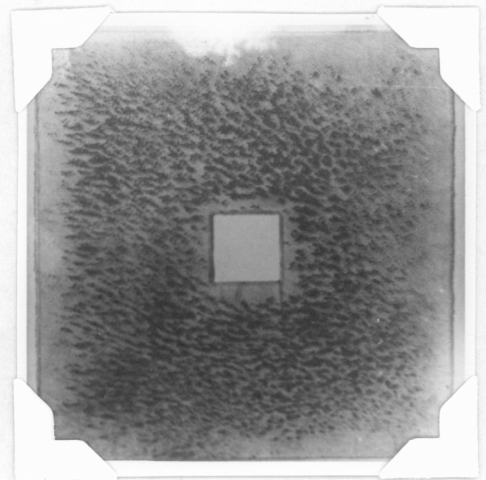


Fig. 7d

Lines of Flow Around the  
Square



## Conclusions

1. The function  $z = w + k\sqrt{w^2 - 1}$  will represent in the  $z$  plane the lines of flow of an electric current in a plate conductor around a circle of unit radius if the constant  $k$  is unity and around an ellipse of the form  $x^2/1 + y^2/k^2 = 1$  which is oblate if the constant  $k$  is greater than unity and prolate if the constant  $k$  is less than unity.

2. The lines of flow of an electric current approach straight lines in the  $z$  plane as  $y$  increases and the smaller the vertical axis of the deleted section the smaller will be the distortion of the lines of flow.

3. The finding of the lines of flow of an electric current in a plate conductor around a square section does not admit of a mathematical solution.

4. The magnetic lines in a field which has been partially blocked out do not conform to "bending" as do the lines of flow of an electric current in a plate conductor in which sections of certain families of geometric figures have been deleted.

5. Lines of flow in a plate conductor with certain deleted geometric sections do conform with the theoretical lines mathematically calculated and plotted when the conductor is placed in the magnetic field.

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