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Particle Shape Corrections to Twersky's Formalism for Multiple Scattering in a Random Particulate Medium

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(ABSTRACT)

In the past forty years much work has been done in the area of multiple scattering effects on the propagation of electromagnetic waves in a random particulate medium. This work is important to wave propagation in the atmosphere, the planetary sphere and the ocean. Current research is aimed at high frequencies (gigahertz to terahertz). At these frequencies, multiple scattering effects become very important since the wavelength reduces to the size of a particle.

The purpose of this thesis is to augment the Twersky theory of multiple scattering in a random particulate medium. Most applications of Twersky's work use a far-field approximation and a point-particle assumption. At high frequencies, particle sizes may be large relative to a wavelength; therefore, the point-particle assumption is inaccurate.

Under a low-density approximation, this thesis introduces a scattering

operator, which defines closed equations for the fields due to multiple scattering. The low-density approximation holds for many media (e.g. clouds and rain). The scattering operator may be solved for various particle shapes, eliminating the need for the point-particle assumption.

Acknowledgements

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1. INTRODUCTION

Optical waves in the atmosphere, acoustical waves in the ocean, and radio waves in the ionosphere are examples of waves that are scattered by the medium in which they propagate. The scattering results in fluctuations of the amplitude, phase, angle-of-arrival and the polarization of the wave. These fluctuations can provide information about the properties of the medium. The fluctuations can also degrade propagating signals. This has led to numerous applications in physics and engineering and to extensive research into scattering in random media over the past forty years. For instance, in radar and remote sensing systems, researchers want to detect the characteristics of the medium, as in weather prediction and probing of the earth, ocean or other planets. In communications, they wish to build systems which overcome the noise, fading and other problems caused by scattering.

In a random or stochastic medium, the properties of the medium vary irregularly with either space and/or time. In the atmosphere, we can assume that the conductivity is zero and the magnetic permeability is constant; therefore, only random fluctuations of the dielectric permittivity (refractive index) affect the passage of electromagnetic waves [1]. Since the velocity of the wave is much larger than the velocity of the particles, time variations of the permittivity can be ignored [2]; therefore we only need to examine spatial fluctuations of the dielectric permittivity.

The relative dielectric permittivity has a mean value $\bar{\epsilon}$ (assumed stationary over the propagation path) and a fluctuating part $\bar{\epsilon}\ \delta\epsilon_r(\vec{r})$.

$$\epsilon(\vec{r}) = \bar{\epsilon}\left[1 + \delta\epsilon_r(\vec{r})\right] \tag{1.1}$$

The governing scalar wave equation for a medium with a fluctuating permittivity is given by

$$(\ \nabla^2 + k^2\ )\ \Psi(\ \vec{r}) = -k^2\ [\epsilon_r(\vec{r}) - 1]\ \Psi(\vec{r}) \tag{1.2}$$

where k is the effective propagation constant given by

$$k = \bar{\epsilon}\ k_0,$$

and

$$\epsilon_r(\vec{r})\ = 1 + \frac{\delta\epsilon_r(\vec{r})}{\bar{\epsilon}}$$

The derivation of Equation (1.2) is given in Appendix A.

There are two forms of scattering, and they are classified by the size of the scattering mechanism relative to a wavelength. The scattering mechanism may be due to a change in the refractive index or to particles which have different properties than the background medium. In a "turbulent," "continuous random," or "weakly random" medium, the wavelength is much smaller than a typical scale size of the scattering mechanism (e.g. the distance over which the refractive index varies by an appreciable percentage). For example, temperature fluctuations in the atmosphere cause smooth random changes of the dielectric permittivity, and this causes the twinkling of stars (scintillation). In a "turbid," "discrete random," or "random particulate" medium, the scatterers are about the same size or smaller than a wavelength, or they have sharp discontinuities in refractive index. For example, aerosols or water droplets cause fading when waves propagate through clouds or rain.

When a wave encounters a particle or eddy, it undergoes reflection and diffraction. A wave may interact with only one particle or it may have multiple interactions with many particles. Multiple scattering effects become important over large propagation distances relative to particle size or in a dense medium.

This thesis examines multiple scattering of electromagnetic waves in a random particulate medium. The principal formulations upon which it draws are Twersky's multiple-scattered amplitude method [3, 13, 14], Waterman's T-matrix formalism [4], and Frisch's diagram method [5]. In particular, this thesis relates to the Twersky formalism.

Many applications of Twersky's formalism use a far-field representation and an approximation that the scatterers are point-particles. However, current research is aimed at high frequencies (gigahertz to terahertz), and for these frequencies, particles are the same size or larger than a wavelength, which makes the point-particle assumption inaccurate. Using a low-density approximation, this thesis derives a scattering operator $\bar{s}_\alpha$, which defines closed equations for the fields due to multiple scattering. It may be solved for various particle shapes. Therefore, the scattering operator provides corrections to the Twersky far-field approximation based on the shape of the particles. The low-density approximation is applicable for many media such as clouds or normal rainfall [6, 7, 8].

This thesis contains five chapters. Chapter 2 reviews previous relevant work in the area of multiple scattering in a random medium. In particular, the chapter summarizes the Twersky formalism and gives a brief introduction to stochastic equations and Twersky's formulas in a random medium.

Chapter 3 contains the derivation of the scattering operator. The derivation starts with the wave equation in a random medium. Conversion of the equation to Dirac operator notation [9] allows a transition operator t_α to be defined more easily. The application of a low-density approximation to the transition operator produces the scattering operator $\bar{s}_\alpha$. This operator can be solved in principle from a closed equation which describes the multiple scattering effects on the wave. This new formulation under a point-particle assumption is equivalent to Twersky's equations in the far-field.

Chapter 4 calculates the scattering operator for spherical, oblate and prolate spheroidal particles. The corrections to the far-field approximation are only negligible when the particle size is much smaller than a wavelength. Chapter 5 summarizes this thesis and suggests future work.

2. LITERATURE SURVEY

In the late nineteenth century, Lord Rayleigh first looked at the phenomenon of particle scattering in his theory of the color of sky [10]. His work involved only single scattering from and no interactions between particles. Since then numerous authors have examined the subject of scattering from particles. There are many good review articles such as by Barabanenkov, et. al. [11], Frisch [5], and Ishimaru [12].

Foldy is the first author to introduce a full formalism for the multiple scattering of a random distribution of particles [13]. He applies configurational averaging to a medium of uncorrelated, isotropic, point scatterers. Through a heuristic approximation, he obtains equations for the first two statistical moments. Lax expands Foldy's work to include anisotropic scattering and pairwise correlation between particles [14]. Lax introduces the "quasicrystalline approximation" to obtain a finite set of equations [15]. Twersky further extends this work by deriving a set of expanded equations that describe the physical picture of the various processes of multiple scattering [16, 17, 3, 18]. Twersky later extends his work to include pair correlation between particles [19-22]. Brown further examines the range of validity of the Twersky formalism in his articles [23, 24].

Many other authors have worked on multiple scattering effects. Keller approaches the scattering problem in a slightly different way by using a perturbation scheme or smoothing method [2]. Frisch uses the Feynman diagram method to define equations for the coherent field and the correlation function, which are equivalent to the Dyson and Bethe-Salpeter equations [5]. Waterman and Truell introduce a scattering operator, 'T' to describe the scattering characteristics of a particle [4]. Varadan, et. al. [25, 26, 27] utilize the T-matrix formalism to find the response from various shapes and configurations of paricles. Tsang and Kong examine the coherent potential in quantum mechanical scattering, which is analogous to electromagnetic scattering [28]. Tsang, et. al. [29] and Ding and Tsang [30] utilize the quasicrytalline approximation to find mean-field equations for various configurations.

The majority of the work described so far has resulted in solutions for scalar fields. Tsolakis, et. al. derive equations for the first two statistical moments for a vector wave including pair-correlation between particles [31], and Lang, et. al. found a general vector solution for the mean-fields in a layer of arbitrarily shaped particles [32].

This thesis examines the original Twersky formalism with no correlation between particles. Section 2.1 gives a brief summary of the Twersky formalism, and Section 2.2 describes these equations in a random medium by applying stochastic theory.

2.1 Twersky Formalism[3]

Figure 2.1.1 shows the notation of this thesis. The observation point is located at $\vec{r}$ and the center of particle α is at $\vec{r}_\alpha$. The symbol v_α represents the volume of particle α. The arguments of the operators $T_\alpha(\vec{r}_1, \vec{r}_2)$ and $t_\alpha(\vec{r}_1, \vec{r}_2)$ refer to the transition of the wave from location $\vec{r}_1$ to $\vec{r}_2$.

We consider a medium made up of N scatterers located at $\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N$. The symbol $\Psi_0(\vec{r})$ represents the incident wave at $\vec{r}$ if there were no particles present. Therefore the cumulative field at $\vec{r}$ becomes the sum of the directly incident wave and all the scattered waves.

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + \sum_{\alpha=1}^{N} \mathcal{T}_\alpha(\vec{r}, \vec{r}_\alpha) \tag{2.1.1}$$

Here $\mathcal{T}_\alpha$ stands for the scattered wave at $\vec{r}$ due to scatterer α where α is located at $\vec{r}_\alpha$. The scattered wave can be written symbolically as

$$\mathcal{T}_\alpha = T_\alpha(\vec{r}, \vec{r}_\alpha)\, \Psi_\alpha \tag{2.1.2}$$

where the symbolic operator T_α is operating on Ψ_α, the effective field felt by particle α at location $\vec{r}_\alpha$.

The effective field at α is also made up of incident and scattered components:

$$\Psi_\alpha = \Psi_0(\vec{r}_\alpha) + \sum_{\beta \neq \alpha} T_\beta(\vec{r}_\alpha, \vec{r}_\beta)\, \Psi_\beta. \tag{2.1.3}$$

This sets up a recursive relationship. Iteratively substituting Equation (2.1.3) into Equation (2.1.1) produces the following equation.

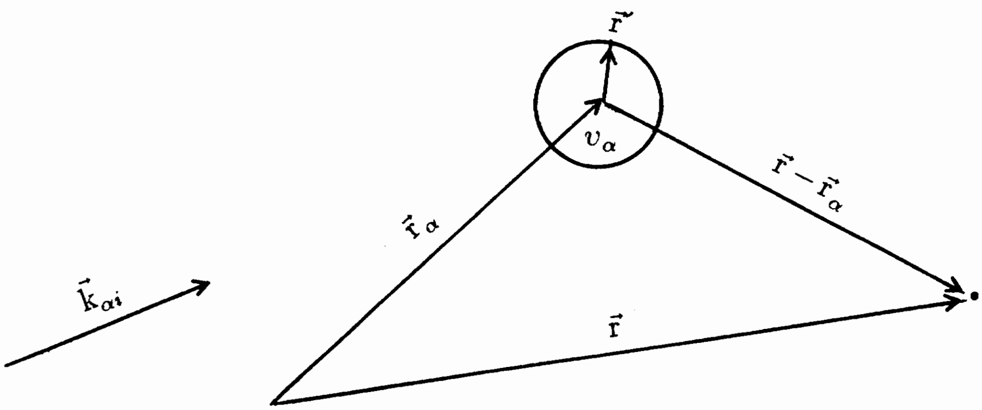


Figure 2.1.1. Geometry of the problem.

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + \sum_{\alpha=1}^{N} T_\alpha(\vec{r},\, \vec{r}_\alpha)\, \Psi_0(\vec{r}_\alpha)$$

$$+ \sum_{\alpha=1}^{N} \sum_{\beta=1,\beta\neq\alpha}^{N} T_\alpha(\vec{r},\, \vec{r}_\alpha)\, T_\beta(\vec{r}_\alpha,\, \vec{r}_\beta)\, \Psi_0(\vec{r}_\beta)$$

$$+ \sum_{\alpha=1}^{N} \sum_{\beta=1,\beta\neq\alpha}^{N} \sum_{\gamma=1,\gamma\neq\beta}^{N} T_\alpha(\vec{r},\, \vec{r}_\alpha)\, T_\beta(\vec{r}_\alpha,\, \vec{r}_\beta)\, T_\gamma(\vec{r}_\beta,\, \vec{r}_\gamma)\, \Psi_0(\vec{r}_\gamma)$$

$$+ \ldots \tag{2.1.4}$$

The first summation represents all single scatterings (Figure 2.1.2a). The second summation term represents all double scatterings (Figure 2.1.2b).

The third summation may be divided into two parts.

$$\sum_{\alpha=1}^{N} \sum_{\beta=1,\beta\neq\alpha}^{N} \sum_{\gamma=1,\gamma\neq\beta}^{N} T_\alpha(\vec{r},\, \vec{r}_\alpha)\, T_\beta(\vec{r}_\alpha,\, \vec{r}_\beta)\, T_\gamma(\vec{r}_\beta,\, \vec{r}_\gamma)\, \Psi_0(\vec{r}_\gamma) =$$

$$+ \sum_{\alpha=1}^{N} \sum_{\beta=1,\beta\neq\alpha}^{N} \sum_{\gamma=1,\gamma\neq\beta,\gamma\neq\alpha}^{N} T_\alpha(\vec{r},\, \vec{r}_\alpha)\, T_\beta(\vec{r}_\alpha,\, \vec{r}_\beta)\, T_\gamma(\vec{r}_\beta,\, \vec{r}_\gamma)\, \Psi_0(\vec{r}_\gamma)$$

$$+ \sum_{\alpha=1}^{N} \sum_{\beta=1,\beta\neq\alpha}^{N} T_\alpha(\vec{r},\, \vec{r}_\alpha)\, T_\beta(\vec{r}_\alpha,\, \vec{r}_\beta)\, T_\alpha(\vec{r}_\beta,\, \vec{r}_\alpha)\, \Psi_0(\vec{r}_\alpha) \tag{2.1.5}$$

The first term in Equation (2.1.5) involves those waves that go through a particle only once, (Figure 2.1.2c). The second term involves those waves scattered by the same particle more than one time, (Figure 2.1.2d). Twersky retains only the scattering chains that involve all different particles; this is known as the Twersky assumption. Frisch notes that the back-and-forth scattering ignored by Twersky is completely incoherent for low-density media, such as rain [5].

These fields and operators are very difficult to calculate; therefore Twersky examines them in the far-field, where $|\vec{r} - \vec{r}_\alpha|$ approaches infinity. The fields and operators become

$$\Psi_\alpha \simeq e^{i\vec{k}_{\alpha i}\cdot(\vec{r}-\vec{r}_\alpha)} \tag{2.1.6}$$

$$T_\alpha \simeq f(\vec{k}_{\alpha f},\, \vec{k}_{\alpha i})\, \frac{e^{ik\,|\,\vec{r}-\vec{r}_\alpha\,|}}{|\,\vec{r}-\vec{r}_\alpha\,|} \tag{2.1.7}$$

where the k-vector for the incident wave is $\vec{k}_{\alpha i}$, and for the wave propagating from particle α to the observation point $\vec{r}$ is

$$\vec{k}_{\alpha f} = k\,\frac{\vec{r}-\vec{r}_\alpha}{|\,\vec{r}-\vec{r}_\alpha\,|}$$

and $f(\vec{k}_{\alpha f},\, \vec{k}_{\alpha i})$ is the "scattering amplitude". This approximation assumes a distant observation point and that the particles are far enough apart that they have no effect on each other (uncorrelated). Twersky further assumes that the distances between particles and observation point are large enough that the particles appear as points relative to each other and to the observer (in the far-field). Most applications of Twersky's equations use the far-field or point-particle approximation.

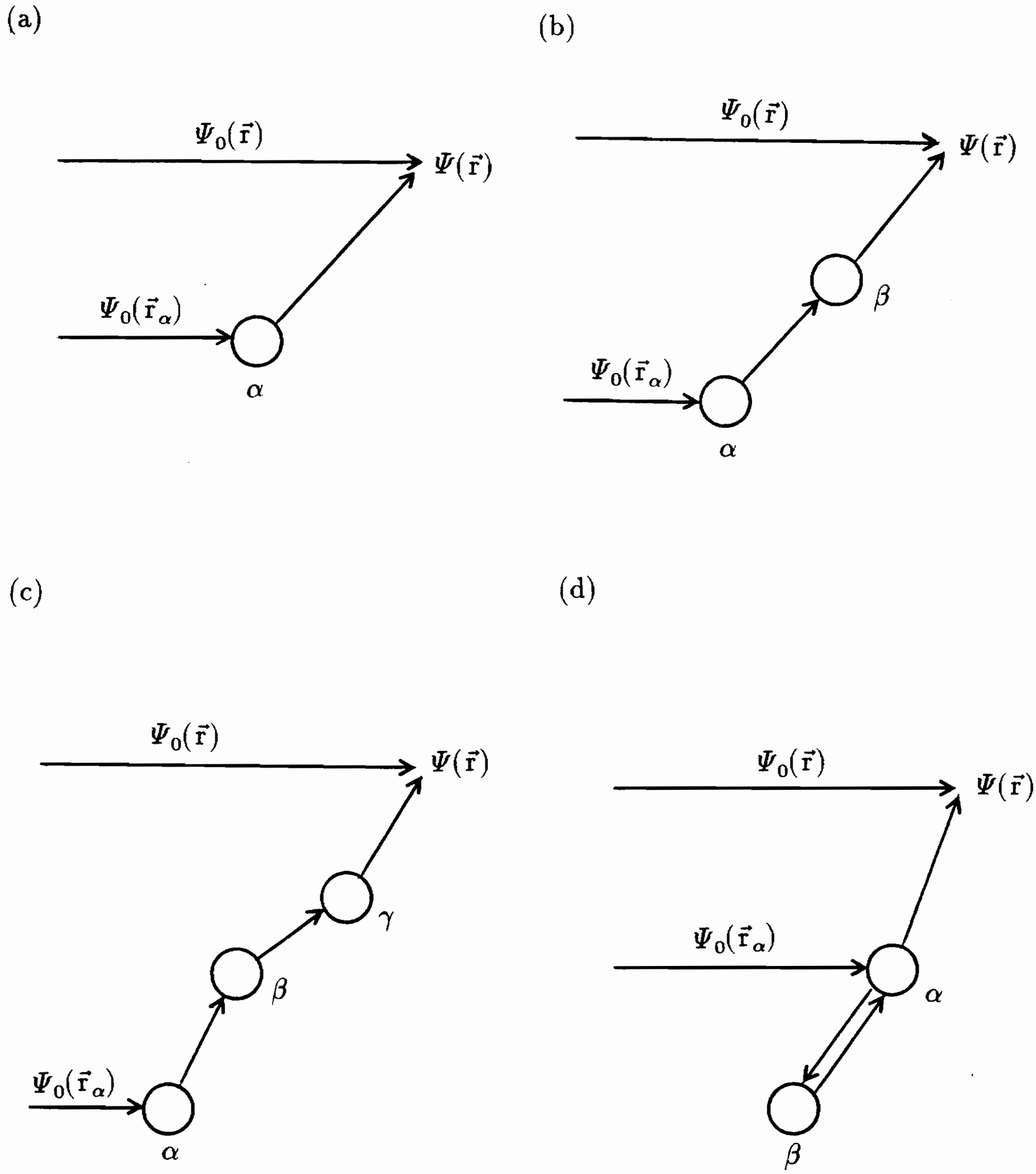


Figure 2.1.1. (a) Single scattering. (b) Double scattering. (c) Triple scattering through different particles. (d) Triple scattering when the propagation path goes through the same particle more than once [12].

2.2 Random Media[3, 12]

This section presents a basic introduction to stochastic equations and their application to Twersky's formalism.

Almost all situations in science and engineering may be viewed as having a stochastic component since everything is subject to random variations, such as noise or thermal fluctuations. If $f(\underline{1}, \underline{2}, \ldots, \underline{\alpha} \ldots, \underline{N})$ is a random function, then an ensemble of random functions can be specified by an appropriate probability distribution function $W(\underline{1}, \underline{2}, \ldots, \underline{\alpha} \ldots, \underline{N})$, where $\underline{\alpha}$ stands for all properties of scatterer "α" such as location, shape, orientation or dielectric constant. The ensemble average of f is given by

$$< f > = \int \int \cdots \int f \, W(\underline{1}, \underline{2}, \ldots, \underline{\alpha} \ldots, \underline{N}) \, \mathrm{d}\underline{1} \, \mathrm{d}\underline{2} \cdots \mathrm{d}\underline{\alpha} \cdots \mathrm{d}\underline{N} \qquad (2.2.1)$$

For a particulate medium, if the density is low and the particles small relative to their separation, we can assume the location and characteristics of each scatterer are independent of other particles, and the probability distribution function may be written in the following way:

$$W(\underline{1}, \underline{2}, \ldots, \underline{\alpha} \ldots, \underline{N}) = w(\underline{1}) \, w(\underline{2}) \cdots w(\underline{\alpha}) \cdots w(\underline{N}) \qquad (2.2.2)$$

where $w(\underline{\alpha})$ is the one-particle distribution function. Twersky divides $\mathrm{d}\underline{\alpha}$ into two parts:

$$\mathrm{d}\underline{\alpha} = \mathrm{d}^3\mathrm{r}_\alpha \, \mathrm{d}\xi_\alpha \qquad (2.2.3)$$

where $\mathrm{d}^3\mathrm{r}_\alpha$ represents the volume integral and $\mathrm{d}\xi_\alpha$ designates all other particle characteristics. Then $w(\underline{\alpha})$ can be written:

$$w(\underline{\alpha}) \equiv w(\vec{\mathrm{r}}_\alpha, \xi_\alpha). \qquad (2.2.4)$$

By integrating with respect to ξ_α, the ensemble average becomes

$$<f> = \int\int\cdots\int [f]_\xi \, w(\vec{r}_1)\, w(\vec{r}_2)\cdots w(\vec{r}_\alpha)\cdots w(\vec{r}_N)\, d^3r_1\cdots d^3r_N \quad (2.2.5)$$

where $[f]_\xi$ represents the average f with respect to the average characteristics ξ.

Twersky uses the "particle" or "number" density ρ in his equations. Since $w(\vec{r}_\alpha)\, d^3r_\alpha$ is the probability of finding scatterer α within an incremental volume $d^3\vec{r}_\alpha$ and $\rho(\vec{r}_\alpha)\, d^3r_\alpha$ is the number of scatterers in $d^3\vec{r}_\alpha$, the particle density is related to the the probability density by

$$w(\vec{r}_\alpha)\, d^3r_\alpha = \frac{\rho(\vec{r}_\alpha)\, d^3r_\alpha}{N} \quad (2.2.6)$$

where N is the total number of scatterers in V. Therefore, the ensemble average of f in terms of the number density is

$$<f> = \int\int\cdots\int [f]_\xi \, \frac{\rho(\vec{r}_1)\, \rho(\vec{r}_2)\cdots \rho(\vec{r}_\alpha)\cdots \rho(\vec{r}_N)}{N^N}\, d^3r_1\cdots d^3r_N. \quad (2.2.7)$$

By definition,

$$\int w(\vec{r}_\alpha)\, d^3r_\alpha = \int \frac{\rho(\vec{r}_\alpha)\, d^3r_\alpha}{N} = 1 \quad (2.2.8)$$

If we assume that $[f]_\xi$ depends only on the location of the scatterer, then

$$< f(\vec{r}_\alpha) > = \int [f(\vec{r}_\alpha)]_\xi \, \frac{\rho(\vec{r}_1)}{N}\, d^3r_\alpha. \quad (2.2.9)$$

By applying these stochastic formulas, Twersky found equations for the coherent field and the correlation function in a random particulate medium [12]. The Foldy-Twersky integral equation for the coherent equation in the compact form is

$$< \Psi(\vec{r}) > \; = \Psi_0(\vec{r}) + \int d^3r' \; T_\alpha(\vec{r}, \vec{r}') < \Psi_\alpha > \rho(\vec{r}'), \qquad (2.2.10)$$

and the Twersky integral equation for the correlation function is

$$< \Psi(\vec{r}_1) \, \Psi^*(\vec{r}_2) > \; = \; < \Psi(\vec{r}_1) > \; < \Psi^*(\vec{r}_2) >$$
$$+ \int d^3r' \; T_\alpha(\vec{r}_1, \vec{r}') \; T_\alpha(\vec{r}_2, \vec{r}') \; < |\Psi_\alpha|^2 > \; \rho(\vec{r}'). \qquad (2.2.11)$$

These equations are equivalent to the first-order smoothing approximation to the Dyson equation and to the Bethe-Salpeter equation found by Frisch.

3. MULTIPLE SCATTERING

In this chapter, a scattering operator will be derived to describe the multiple scattering effects on the field. This operator results from a low-density approximation applied to the exact multiple scattering formulation, which is derived in Section 3.1. In Section 3.3, the new formulation under a point-particle assumption is shown to be equivalent to the Twersky equations in the far-field.

3.1 Transition Operator Derivation

This section defines a transition operator t_α which describes repeated scatterings due to one particle. A compact, exact set of equations results from writing the field equations in terms of this transition operator.

We consider a medium made up of N scatterers located at $\vec{r}_1, \vec{r}_2, \ldots \vec{r}_N$.

The wave equation inside the particles is

$$(\nabla^2 + k^2)\ \Psi(\vec{r}) = -k^2[\epsilon_r(\vec{r}) - 1]\ \Psi(\vec{r}). \tag{3.1.1}$$

The scalar integral equation for the field in a particulate medium can be written via Green's functions as

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + k^2 \int d^3r'\ g(\vec{r},\ \vec{r}')\ [\epsilon_r(\vec{r}') - 1]\ \Psi(\vec{r}') \tag{3.1.2}$$

where the free space Green's function is

$$g(\vec{r},\ \vec{r}') = \frac{e^{ik\,|\,\vec{r} - \vec{r}'\,|}}{4\pi\,|\,\vec{r} - \vec{r}'\,|}$$

and the relative permittivity is given by

$$\epsilon_r(\vec{r}') = \begin{cases} \epsilon_\alpha & \text{when } r' \in v_\alpha \ \ \text{(inside particle } \alpha\text{)} \\ 1 & \text{otherwise} \ \ \text{(outside the particle)} \end{cases}.$$

The scalar integral equation for the field, Equation (3.1.2), is made up of a free-space incident wave Ψ_0 and a scattered wave. It can be rewritten in terms of a new variable $\zeta_\alpha(\vec{r}')$; specifically,

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + \sum_\alpha \int d^3r'\ g(\vec{r},\ \vec{r}')\ \zeta_\alpha(\vec{r}')\ \Psi(\vec{r}') \tag{3.1.3}$$

where

$$\zeta_\alpha(\vec{r}') \equiv k^2\ v_\alpha\ (\epsilon_\alpha - 1)\ \theta_\alpha(\vec{r}' - \vec{r}_\alpha) \tag{3.1.4}$$

and

$$\theta_\alpha(\vec{r}' - \vec{r}_\alpha) \equiv \begin{cases} \frac{1}{v_\alpha} & \text{for } \vec{r}' \in v_\alpha \\ 0 & \text{otherwise} \end{cases} \tag{3.1.5}$$

The function θ_α represents a volume step function that is zero outside of particle α and equal to $(v_\alpha)^{-1}$ inside the particle.

To simplify the derivation of the transition operator, we convert the equations to the Dirac operator notation. By making the following transformations into Dirac notation,

$$\Psi(\vec{r}) \rightarrow |\Psi\rangle$$

$$\int d^3\vec{r}' g(\vec{r}, \vec{r}') \rightarrow g$$

$$\zeta_\alpha(\vec{r}) \rightarrow \zeta_\alpha$$

the integral equation, Equation (3.1.3), becomes

$$|\Psi\rangle = |\Psi_0\rangle + \sum_\alpha g\,\zeta_\alpha\,|\Psi\rangle. \tag{3.1.6}$$

To find the transition operator, we examine scattering due to a single particle α.

$$|\Psi\rangle = |\Psi_0\rangle + g\,\zeta_\alpha\,|\Psi\rangle \tag{3.1.7}$$

Equation (3.1.7) describes a recursive relationship. The transition operator t_α may be defined by iterating Equation (3.1.7).

$$|\Psi\rangle = |\Psi_0\rangle + g\,\zeta_\alpha\,|\Psi_0\rangle + (g\,\zeta_\alpha)^2\,|\Psi_0\rangle + (g\,\zeta_\alpha)^3\,|\Psi_0\rangle \dots$$

$$= |\Psi_0\rangle + \sum_{m=1}^{\infty} (g\,\zeta_\alpha)^m\,|\Psi_0\rangle$$

$$= |\Psi_0\rangle + t_\alpha\,|\Psi_0\rangle \tag{3.1.8}$$

Therefore in Dirac notation, the transition operator t_α is

$$t_\alpha \equiv \sum_{m=1}^{\infty} (g\,\zeta_\alpha)^m = g\,\zeta_\alpha\,(1 + t_\alpha). \tag{3.1.9}$$

Equation (3.1.9) shows that physically, the transition operator is an infinite sum of all orders $m = (1, 2, \dots \infty)$ of scatterings.

Now we will write the equation for the field as a function of the transition operator by finding a relationship for $(g\,\zeta_\alpha)$ in terms of t_α. By rearranging Equation (3.1.9), we obtain the relationship:

$$g\ \zeta_\alpha = t_\alpha\ (1 + t_\alpha)^{-1}. \tag{3.1.10}$$

Substitution of Equation (3.1.10) into the field equation for many particles, Equation (3.1.6), results in a new field equation that is a function of the transition operator:

$$|\Psi\rangle = |\Psi_0\rangle + \sum_\alpha t_\alpha\ (1 + t_\alpha)^{-1}\ |\Psi\rangle. \tag{3.1.11}$$

To manipulate Equation (3.1.11) into the Twersky form, we need to define a new field $|\Psi_\alpha\rangle$:

$$|\Psi_\alpha\rangle \equiv (1 + t_\alpha)^{-1}\ |\Psi\rangle. \tag{3.1.12}$$

In terms of $|\Psi_\alpha\rangle$, Equation (3.1.11) becomes

$$|\Psi\rangle = |\Psi_0\rangle + \sum_\alpha t_\alpha\ |\Psi_\alpha\rangle. \tag{3.1.13}$$

To put $|\Psi_\alpha\rangle$ in a form that gives it physical meaning, we substitute Equation (3.1.13) into Equation (3.1.12):

$$|\Psi_\alpha\rangle = (1 + t_\alpha)^{-1}\,|\Psi_0\rangle + (1 + t_\alpha)^{-1}\ t_\alpha\,|\Psi_\alpha\rangle + (1 + t_\alpha)^{-1} \sum_{\beta \neq \alpha} t_\beta\,|\Psi_\beta\rangle. \tag{3.1.14}$$

If we multiply Equation 3.1.14 by $(1 + t_\alpha)$, i.e.,

$$(1 + t_\alpha)\,|\Psi_\alpha\rangle = |\Psi_0\rangle + t_\alpha\,|\Psi_\alpha\rangle + \sum_{\beta \neq \alpha} t_\beta\,|\Psi_\beta\rangle, \tag{3.1.15}$$

and rearrange Equation (3.1.15), then the final equation for $|\Psi_\alpha\rangle$ is

$$|\Psi_\alpha\rangle = |\Psi_0\rangle + \sum_{\beta \neq \alpha} t_\beta\,|\Psi_\beta\rangle. \tag{3.1.16}$$

Equation (3.1.16) shows that physically $|\Psi_\alpha\rangle$ represents the cumulative field felt by particle α. This cumulative field has a free space component, $|\Psi_0\rangle$, and a scattered component due to scatterings from all particles except itself.

A summary of the field equations and the transition operator in their final form is given below.

$$|\Psi\rangle = |\Psi_0\rangle + \sum_{\alpha} t_\alpha \; |\Psi_\alpha\rangle \tag{3.1.13}$$

$$|\Psi_\alpha\rangle = |\Psi_0\rangle + \sum_{\beta \neq \alpha} t_\beta \; |\Psi_\beta\rangle \tag{3.1.16}$$

where

$$t_\alpha \equiv g \; \zeta_\alpha \; (1 + t_\alpha) \tag{3.1.9}$$

In coordinate language, these equations are equivalent to the following:

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + \sum_{\alpha} \int d^3r' \; t_\alpha(\vec{r}, \vec{r}') \; \Psi_\alpha(\vec{r}') \tag{3.1.17}$$

$$\Psi_\alpha(\vec{r}') = \Psi_0(\vec{r}') + \sum_{\beta \neq \alpha} \int d^3r'' \; t_\beta(\vec{r}', \vec{r}'') \; \Psi_\beta(\vec{r}'') \tag{3.1.18}$$

where

$$t_\alpha(\vec{r}, \vec{r}') = g(\vec{r}, \vec{r}') \; \zeta_\alpha(\vec{r}') + \int d^3r'' \; g(\vec{r}, \vec{r}'') \; \zeta_\alpha(\vec{r}'') \; t_\alpha(\vec{r}'', \vec{r}'). \tag{3.1.19}$$

Equations (3.1.17) - (3.1.19) for the fields and transition operator are exact since no approximations have been made.

We may simplify the equations of the fields and transition operator by noting that by definition ζ_α is a function of the volume step function θ_α:

$$\zeta_\alpha(\vec{r}') \equiv k^2 \; v_\alpha \; (\epsilon_\alpha - 1) \; \theta_\alpha(\vec{r}' - \vec{r}_\alpha) \tag{3.1.4}$$

where

$$\theta_\alpha(\vec{r}' - \vec{r}_\alpha) \equiv \begin{cases} \frac{1}{v_\alpha} & \text{for } \vec{r}' \in v_\alpha \\ 0 & \text{otherwise} \end{cases} . \tag{3.1.5}$$

By looking at Equations (3.1.17) - (3.1.19), we observe that the transition operator and therefore the fields are functions of ζ_α and consequently θ_α. Since the volume step function is zero except inside the particle, the limits of integration in the field and transition equations reduce to the volume of the particles instead of all space.

To simplify the integral in the transition operator, we factor $\zeta_\alpha(\vec{r}')$ so it is the product of the volume step function θ_α and a new constant $\bar{\zeta}_\alpha$;

$$\zeta_\alpha(\vec{r}') \equiv \bar{\zeta}_\alpha \theta_\alpha(\vec{r}' - \vec{r}_\alpha) \tag{3.1.20}$$

$$\bar{\zeta}_\alpha \equiv k^2 \, v_\alpha \, (\epsilon_\alpha - 1). \tag{3.1.21}$$

By using Equation (3.120) for $\zeta_\alpha(\vec{r}')$, the expression for the transition operator, Equation (3.1.19), becomes

$$t_\alpha(\vec{r}, \vec{r}') = \bar{\zeta}_\alpha \, g(\vec{r}, \vec{r}') \, \theta_\alpha(\vec{r}' - \vec{r}_\alpha) + \bar{\zeta}_\alpha \int d^3r'' \, g(\vec{r}, \vec{r}'') \, \theta_\alpha(\vec{r}'' - \vec{r}_\alpha) \, t_\alpha(\vec{r}'', \vec{r}'). \tag{3.1.22}$$

By integrating Equation (3.1.22), the transition operator becomes

$$t_\alpha(\vec{r}, \vec{r}') = \bar{\zeta}_\alpha \, g(\vec{r}, \vec{r}') \, \theta_\alpha(\vec{r}' - \vec{r}_\alpha) + \frac{\bar{\zeta}_\alpha}{v_\alpha} \int_{v_\alpha} d^3r'' \, g(\vec{r}, \vec{r}'') \, t_\alpha(\vec{r}'', \vec{r}'). \tag{3.1.23}$$

To simplify the integral in the field equations, we also factor the transition operator into a product of the volume step function θ_α and a new transition operator $\bar{t}_\alpha$.

$$t_\alpha(\vec{r}, \vec{r}') \equiv \bar{\zeta}_\alpha \, \bar{t}_\alpha(\vec{r}, \vec{r}') \, \theta_\alpha(\vec{r}' - \vec{r}_\alpha). \tag{3.1.24}$$

The combination of Definition (3.1.24) and the expression for the transition operator, Equation (3.1.23), results in the following formula for the new transition operator:

$$\bar{t}_\alpha(\vec{r}, \vec{r}') = g(\vec{r}, \vec{r}') + \bar{\zeta}_\alpha \int d^3r'' \, g(\vec{r}, \vec{r}'') \, \theta_\alpha(\vec{r}'' - \vec{r}_\alpha) \, \bar{t}_\alpha(\vec{r}'', \vec{r}') \qquad \text{for } \vec{r}' \in v_\alpha \tag{3.1.25}$$

Integration of Equation (3.1.25) gives the simplified new transition operator equation

$$\bar{t}_\alpha(\vec{r}, \vec{r}') = g(\vec{r}, \vec{r}') + \frac{\bar{\zeta}_\alpha}{v_\alpha} \int_{v_\alpha} d^3r'' \, g(\vec{r}, \vec{r}'') \, \bar{t}_\alpha(\vec{r}'', \vec{r}') \qquad \text{for } \vec{r}' \in v_\alpha. \tag{3.1.26}$$

Next we substitute the new transition operator, Equation (3.1.26), into the exact form of the fields, Equations (3.1.17) and (3.1.18):

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + \sum_{\alpha} \overline{\zeta}_\alpha \int d^3r' \, \overline{t}_\alpha(\vec{r}, \vec{r}') \, \theta_\alpha(\vec{r}' - \vec{r}_\alpha) \, \Psi_\alpha(\vec{r}') \tag{3.1.27}$$

$$\Psi_\alpha(\vec{r}') = \Psi_0(\vec{r}') + \sum_{\beta \neq \alpha} \overline{\zeta}_\beta \int d^3r'' \, \overline{t}_\beta(\vec{r}', \vec{r}'') \, \theta_\beta(\vec{r}'' - \vec{r}_\beta) \, \Psi_\beta(\vec{r}''). \tag{3.1.28}$$

Integration of Equations (3.1.27) and (3.1.28) gives the following set of simplified equations.

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + \sum_{\alpha} \frac{\overline{\zeta}_\alpha}{v_\alpha} \int_{v_\alpha} d^3r' \, \overline{t}_\alpha(\vec{r}, \vec{r}') \, \Psi_\alpha(\vec{r}') \tag{3.1.29}$$

$$\Psi_\alpha(\vec{r}') = \Psi_0(\vec{r}') + \sum_{\beta \neq \alpha} \frac{\overline{\zeta}_\beta}{v_\beta} \int_{v_\beta} d^3r'' \, \overline{t}_\beta(\vec{r}', \vec{r}'') \, \Psi_\beta(\vec{r}'') \tag{3.1.30}$$

where

$$\overline{t}_\alpha(\vec{r}_1, \vec{r}_2) = g(\vec{r}_1, \vec{r}_2) + \frac{\overline{\zeta}_\alpha}{v_\alpha} \int_{v_\alpha} d^3r'' \, g(\vec{r}_1, \vec{r}'') \, \overline{t}_\alpha(\vec{r}'', \vec{r}_2) \qquad \text{for } \vec{r}' \in v_\alpha \tag{3.1.26}$$

and

$$\overline{\zeta}_\alpha \equiv k^2 \, v_\alpha \, (\epsilon_\alpha - 1) \tag{3.1.20}$$

Equations (3.1.29), (3.1.30) and (3.1.26) are the general equations for the fields and the transition operator in a random particulate medium with multiple scattering.

3.2 Low-density Approximation and the Scattering Operator Derivation

In fog, clouds, normal rainfall and many other media, the concentration of water droplets is sufficiently small for a low-density approximation. If the density is low enough then the correlation between particles should have little effect and can be ignored. The low-density approximation allows us to derive a scattering operator, which defines a closed equation for the scattering effects.

The low-density approximation requires that the dimension of a particle be much less than the length of the propagation path to another particle:

$$l_\alpha \ll \mathrm{R}_\alpha$$

where

$$l_\alpha \sim (v_\alpha)^{1/3}$$

$$\mathrm{R}_\alpha \sim |\vec{\mathrm{r}} - \vec{\mathrm{r}}_\alpha|$$

The symbol l_α approximately represents the diameter of particle α and R_α represents the length of the propagation path. Also we assume that the density is low enough that the observation point $\vec{\mathrm{r}}$ is almost always outside of the particles. Notation in this section assumes that all primed variables are inside a particle.

For the case where

$$|\vec{\mathrm{r}} - \vec{\mathrm{r}}_\alpha| \gg |\vec{\mathrm{r}}' - \vec{\mathrm{r}}_\alpha|$$

the following Green's function approximation applies.

$$\mathrm{g}(\vec{\mathrm{r}}, \vec{\mathrm{r}}') \simeq \mathrm{g}(\vec{\mathrm{r}}, \vec{\mathrm{r}}_\alpha)\, \mathrm{e}^{-\mathrm{i}\vec{\mathrm{k}}_{\alpha f} \cdot (\vec{\mathrm{r}}' - \vec{\mathrm{r}}_\alpha)} \tag{3.2.1}$$

Here, the k-vector for the wave propagating from particle α to the observation point is given by

$$\vec{k}_{\alpha f} = k \frac{\vec{r} - \vec{r}_\alpha}{|\vec{r} - \vec{r}_\alpha|} \tag{3.2.2}$$

We will use this Green's function in the transition operator equation in order to define the scattering operator.

We begin by iterating the equation for the transition operator, Equation (3.1.26) derived in Section 3.1.

$$\bar{t}_\alpha(\vec{r}, \vec{r}') = g(\vec{r}, \vec{r}') + \frac{\bar{\zeta}_\alpha}{v_\alpha}\int_{v_\alpha} d^3r''\, g(\vec{r}, \vec{r}'')\, \bar{t}_\alpha(\vec{r}'', \vec{r}')$$

$$= g(\vec{r}, \vec{r}') + \frac{\bar{\zeta}_\alpha}{v_\alpha}\int_{v_\alpha} d^3r''\, g(\vec{r}, \vec{r}'')\left[g(\vec{r}'', \vec{r}') + \frac{\bar{\zeta}_\alpha}{v_\alpha}\int_{v_\alpha} d^3r'''\, g(\vec{r}'', \vec{r}''')\, \bar{t}_\alpha(\vec{r}''', \vec{r}')\right]$$

$$\bar{t}_\alpha(\vec{r}, \vec{r}') = g(\vec{r}, \vec{r}') + \frac{\bar{\zeta}_\alpha}{v_\alpha}\int_{v_\alpha} d^3r''\, g(\vec{r}, \vec{r}'')\, g(\vec{r}'', \vec{r}')$$

$$+\left(\frac{\bar{\zeta}_\alpha}{v_\alpha}\right)^2\int_{v_\alpha} d^3r'' \int_{v_\alpha} d^3r'''\, g(\vec{r}, \vec{r}'')\;\; g(\vec{r}'', \vec{r}''')\, g(\vec{r}''', \vec{r}') + \ldots \tag{3.2.3}$$

Substitution of the Green's function, Equation (3.2.1), into the equation for the transition operator, Equation (3.2.3), yields the following relationship:

$$\bar{t}_\alpha(\vec{r}, \vec{r}') = g(\vec{r}, \vec{r}_\alpha)\, e^{-i\vec{k}_{\alpha f}\cdot(\vec{r}' - \vec{r}_\alpha)}$$

$$+ \frac{\bar{\zeta}_\alpha}{v_\alpha}\int_{v_\alpha} d^3r''\, g(\vec{r}, \vec{r}_\alpha)\, e^{-i\vec{k}_{\alpha f}\cdot(\vec{r}'' - \vec{r}_\alpha)}\, g(\vec{r}'', \vec{r}_\alpha)\, e^{-i\vec{k}_{\alpha f}\cdot(\vec{r}' - \vec{r}_\alpha)}$$

$$+\left(\frac{\bar{\zeta}_\alpha}{v_\alpha}\right)^2\int_{v_\alpha} d^3r'' \int_{v_\alpha} d^3r'''\, g(\vec{r}, \vec{r}_\alpha)\, e^{-i\vec{k}_{\alpha f}\cdot(\vec{r}'' - \vec{r}_\alpha)} g(\vec{r}'', \vec{r}_\alpha)\, e^{-i\vec{k}_{\alpha f}\cdot(\vec{r}''' - \vec{r}_\alpha)}$$

$$\cdot g(\vec{r}''', \vec{r}_\alpha)\, e^{-i\vec{k}_{\alpha f}\cdot(\vec{r}' - \vec{r}_\alpha)} + \ldots \tag{3.2.4}$$

We can factor the Green's function $g(\vec{r}, \vec{r}')$ out of Equation (3.2.4):

$$\bar{t}_\alpha(\vec{r}, \vec{r}') = g(\vec{r}, \vec{r}_\alpha)\, e^{-i\vec{k}_{\alpha f}\cdot(\vec{r}' - \vec{r}_\alpha)} \{1 + \frac{\bar{\zeta}_\alpha}{v_\alpha}\int_{v_\alpha} d^3r''\, g(\vec{r}'', \vec{r}_\alpha)\, e^{-i\vec{k}_{\alpha f}\cdot(\vec{r}'' - \vec{r}_\alpha)}$$

$$+\left(\frac{\bar{\zeta}_\alpha}{v_\alpha}\right)^2 \int_{v_\alpha} d^3r'' \int_{v_\alpha} d^3r'''\, g(\vec{r}'', \vec{r}_\alpha)\, e^{-i\vec{k}_{\alpha f}\cdot(\vec{r}'' - \vec{r}_\alpha)} g(\vec{r}''', \vec{r}_\alpha)\, e^{-i\vec{k}_{\alpha f}\cdot(\vec{r}''' - \vec{r}_\alpha)}$$

$$+ \ldots \} \tag{3.2.5}$$

where

$$g(\vec{r}, \vec{r}') \simeq g(\vec{r}, \vec{r}_\alpha)\, e^{-i\vec{k}_{\alpha f}\cdot(\vec{r}' - \vec{r}_\alpha)}. \tag{3.2.1}$$

We will define the scattering operator $\bar{s}_\alpha$ as the part that remains after factoring out the Green's funtion:

$$\bar{t}_\alpha(\vec{r}, \vec{r}') \equiv g(\vec{r}, \vec{r}_\alpha)\, e^{-i\vec{k}_{\alpha f}\cdot(\vec{r}' - \vec{r}_\alpha)}\, \bar{s}_\alpha$$

$$= g(\vec{r}, \vec{r}')\, \bar{s}_\alpha \tag{3.2.6}$$

Hence the scattering operator in Equation (3.2.6) is

$$\bar{s}_\alpha = 1 + \frac{\bar{\zeta}_\alpha}{v_\alpha}\int_{v_\alpha} d^3r''\, g(\vec{r}'', \vec{r}_\alpha)\, e^{-i\vec{k}_{\alpha f}\cdot(\vec{r}'' - \vec{r}_\alpha)}\, \bar{s}_\alpha \tag{3.2.7}$$

The rearrangement of Equation (3.2.7) results in the closed form of the scattering operator.

$$\bar{s}_\alpha = \left[1 - \frac{\bar{\zeta}_\alpha}{v_\alpha}\int_{v_\alpha} d^3r''\, g(\vec{r}'', \vec{r}_\alpha)\, e^{-i\vec{k}_{\alpha f}\cdot(\vec{r}'' - \vec{r}_\alpha)}\right]^{-1} \tag{3.2.8}$$

The scattering operator $\bar{s}_\alpha$ given by Equation (3.2.8) describes the multiple scattering effects and can be solved for various particle shapes.

We can obtain closed formulas for the fields by rewriting the field equations given in Section 3.1 in terms of the scattering operator $\bar{s}_\alpha$. To obtain field equations that are functions of the scattering operator, we substitute the definition of the scattering operator, Equation (3.2.6), into the field equations, Equations (3.1.29) and (3.1.30); specifically,

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + \sum_\alpha \frac{\bar{\zeta}_\alpha}{v_\alpha} \bar{s}_\alpha \int_{v_\alpha} d^3r' \, g(\vec{r}, \vec{r}') \, \Psi_\alpha(\vec{r}') \tag{3.2.9}$$

$$\Psi_\alpha(\vec{r}') = \Psi_0(\vec{r}') + \sum_{\beta \neq \alpha} \frac{\bar{\zeta}_\beta}{v_\beta} \bar{s}_\beta \int_{v_\beta} d^3r'' \, g(\vec{r}', \vec{r}'') \, \Psi_\beta(\vec{r}'') \tag{3.2.10}$$

where

$$\bar{s}_\alpha = \left[1 - \frac{\bar{\zeta}_\alpha}{v_\alpha} \int_{v_\alpha} d^3r'' \, g(\vec{r}''', \vec{r}_\alpha) \, e^{-i\vec{k}_{\alpha f} \cdot (\vec{r}''' - \vec{r}_\alpha)} \right]^{-1} \tag{3.2.8}$$

In order to find the scattering amplitude f_α, we need to approximate the form of Ψ_α. We will assume that near particle α, Ψ_α will have the following form:

$$\Psi_\alpha(\vec{r}') \simeq \Psi_\alpha(\vec{r}_\alpha) \, e^{i(\vec{k}_{\alpha i})_{\text{eff}} \cdot (\vec{r}' - \vec{r}_\alpha)} \tag{3.2.11}$$

where $(\vec{k}_{\alpha i})_{\text{eff}}$ is the effective k-vector for the incident wave. By substituting the approximation for Ψ_α, Equation (3.2.11), and the Green's function, Equation (3.2.1), into the equation for the field, Equation (3.2.9), the field equation becomes

$$\Psi(\vec{r}) \simeq \Psi_0(\vec{r}) + \sum_\alpha \frac{\bar{\zeta}_\alpha}{v_\alpha} \bar{s}_\alpha \int_{v_\alpha} d^3r' \, g(\vec{r}, \vec{r}_\alpha) \, e^{-i\vec{k}_{\alpha f} \cdot (\vec{r}' - \vec{r}_\alpha)} \, \Psi_\alpha(\vec{r}_\alpha) \, e^{i(\vec{k}_{\alpha i})_{\text{eff}} \cdot (\vec{r}' - \vec{r}_\alpha)}$$

$$= \Psi_0(\vec{r}) + \sum_\alpha \frac{\bar{\zeta}_\alpha}{v_\alpha} \bar{s}_\alpha \, g(\vec{r}, \vec{r}_\alpha) \, \Psi_\alpha(\vec{r}_\alpha) \int_{v_\alpha} d^3r' \, e^{i((\vec{k}_{\alpha i})_{\text{eff}} - \vec{k}_{\alpha f}) \cdot (\vec{r}' - \vec{r}_\alpha)} \tag{3.2.12}$$

Therefore, the scattering amplitude is

$$f_\alpha(\vec{K}_\alpha) = \frac{1}{4\pi} \frac{\bar{\zeta}_\alpha}{v_\alpha} \bar{s}_\alpha \int_{v_\alpha} d^3r' \, e^{i\vec{K}_\alpha \cdot (\vec{r}' - \vec{r}_\alpha)} \tag{3.2.13}$$

where

$$\vec{K}_\alpha = (\vec{k}_{\alpha i})_{\text{eff}} - \vec{k}_{\alpha f}, \tag{3.2.14}$$

and the field equation in terms of the scattering amplitude is

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + \sum_\alpha 4\pi\ g(\vec{r},\ \vec{r}_\alpha)\ f_\alpha(\vec{K}_\alpha)\ \Psi_\alpha(\vec{r}_\alpha) \tag{3.2.15}$$

or

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + \sum_\alpha\ f_\alpha(\vec{K}_\alpha)\ \frac{e^{ik\,|\vec{r} - \vec{r}_\alpha|}}{|\vec{r} - \vec{r}_\alpha|}\ \Psi_\alpha(\vec{r}_\alpha). \tag{3.2.15b}$$

3.3 Point-particle approximation

By applying a point-particle approximation to the equations derived in Section 3.2, we can obtain the Twersky equations, given in Section 2.1, for the far-field case, where $|\vec{r} - \vec{r}_\alpha|$ approaches infinity.

We begin with the point-particle approximation by letting the volume of the particles approach zero. In the limit as the volume goes to zero, the volume step function θ_α becomes the Dirac delta function δ:

$$\lim_{v_\alpha \to 0} \theta_\alpha(\vec{r}',\ \vec{r}_\alpha) = \delta(\vec{r}' - \vec{r}_\alpha). \tag{3.1.1.1}$$

We substitute the delta function for the step function θ_α in the field equations derived in Section 3.1, Equations (3.1.27) and (3.1.28).

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + \sum_\alpha \bar{\zeta}_\alpha \int d^3r'\ \bar{t}_\alpha(\vec{r},\ \vec{r}')\ \delta(\vec{r}' - \vec{r}_\alpha)\ \Psi_\alpha(\vec{r}') \tag{3.3.2}$$

$$\Psi_\alpha(\vec{r}') = \Psi_0(\vec{r}') + \sum_{\beta \neq \alpha} \bar{\zeta}_\beta \int d^3r''\ \bar{t}_\beta(\vec{r}',\ \vec{r}'')\ \delta(\vec{r}'' - \vec{r}_\beta)\ \Psi_\beta(\vec{r}'') \tag{3.3.3}$$

Integration of Equations (3.3.2) and (3.3.3) yields the following simple equations.

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + \sum_{\alpha} \bar{\zeta}_\alpha \, \bar{t}_\alpha(\vec{r}, \vec{r}_\alpha) \, \Psi_\alpha(\vec{r}_\alpha) \tag{3.3.4a}$$

$$\Psi_\alpha(\vec{r}_\alpha) = \Psi_0(\vec{r}_\alpha) + \sum_{\beta \neq \alpha} \bar{\zeta}_\beta \, \bar{t}_\beta(\vec{r}_\alpha, \vec{r}_\beta) \, \Psi_\beta(\vec{r}_\beta) \tag{3.3.5a}$$

Since the particles are points, Equations (3.3.4a) and (3.3.5b) may be simplified by dropping the arguments for Ψ_α and Ψ_β:

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + \sum_{\alpha} \bar{\zeta}_\alpha \, \bar{t}_\alpha(\vec{r}, \vec{r}_\alpha) \, \Psi_\alpha \tag{3.3.4b}$$

$$\Psi_\alpha = \Psi_0(\vec{r}_\alpha) + \sum_{\beta \neq \alpha} \bar{\zeta}_\beta \, \bar{t}_\beta(\vec{r}_\alpha, \vec{r}_\beta) \, \Psi_\beta. \tag{3.3.5b}$$

Under the point-particle assumption, the transition operator definition, Equation (3.1.23) from Section 3.1, becomes

$$t_\alpha(\vec{r}, \vec{r}') = \bar{\zeta}_\alpha \, \bar{t}_\alpha(\vec{r}, \vec{r}') \, \delta(\vec{r}' - \vec{r}_\alpha)$$

$$= \bar{\zeta}_\alpha \, \bar{t}_\alpha(\vec{r}, \vec{r}_\alpha). \tag{3.3.6a}$$

Therefore,

$$t_\alpha(\vec{r}, \vec{r}') = t_\alpha(\vec{r}, \vec{r}_\alpha). \tag{3.3.6b}$$

Substitution of Equation (3.3.6b) into the field equations, Equations (3.3.4b) and (3.3.5b), yields the following equations for the fields:

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + \sum_{\alpha} t_\alpha(\vec{r}, \vec{r}_\alpha) \, \Psi_\alpha \tag{3.3.7}$$

$$\Psi_\alpha = \Psi_0(\vec{r}_\alpha) + \sum_{\beta \neq \alpha} t_\beta(\vec{r}_\alpha, \vec{r}_\beta) \, \Psi_\beta. \tag{3.3.8}$$

Next we apply the point-particle approximation to the equation for the transition operator in the low-density case, Equation (3.2.6).

$$\bar{t}_\alpha(\vec{r}, \vec{r}') = g(\vec{r}, \vec{r}_\alpha)\, e^{-i\vec{k}_{\alpha f}\cdot(\vec{r}' - \vec{r}_\alpha)}\, \bar{s}_\alpha \tag{3.2.6}$$

Therefore, from Equation (3.3.6), the transition operator is

$$t_\alpha(\vec{r}, \vec{r}_\alpha) = \bar{\zeta}_\alpha\, g(\vec{r}, \vec{r}_\alpha)\, \bar{s}_\alpha \tag{3.3.9}$$

where

$$\bar{s}_\alpha = \left[1 - \frac{\bar{\zeta}_\alpha}{v_\alpha}\int_{v_\alpha} d^3r''\, g(\vec{r}'', \vec{r}_\alpha)\, e^{-i\vec{k}_{\alpha f}\cdot(\vec{r}'' - \vec{r}_\alpha)}\right]^{-1} \tag{3.2.8}$$

Finally, the transition operator equation may be written in terms of the scattering amplitude f_α:

$$t_\alpha(\vec{r}, \vec{r}_\alpha) = f(\vec{k}_{\alpha f}, \vec{k}_{\alpha i})\, \frac{e^{ik\,|\vec{r} - \vec{r}_\alpha|}}{|\vec{r} - \vec{r}_\alpha|} \tag{3.3.10}$$

where $\vec{k}_{\alpha f}$ is the k-vector for the wave propagating from particle α to the observation point $\vec{r}$, and $\vec{k}_{\alpha i}$ is the k-vector for the incident wave.

The following is a summary of the fields and transition operator for the point-particle approximation.

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + \sum_\alpha t_\alpha(\vec{r}, \vec{r}_\alpha)\, \Psi_\alpha \tag{3.3.7}$$

$$\Psi_\alpha = \Psi_0(\vec{r}_\alpha) + \sum_{\beta \neq \alpha} t_\beta(\vec{r}_\alpha, \vec{r}_\beta)\, \Psi_\beta \tag{3.3.8}$$

$$t_\alpha(\vec{r}, \vec{r}_\alpha) = f(\vec{k}_{\alpha f}, \vec{k}_{\alpha i})\, \frac{e^{ik\,|\vec{r} - \vec{r}_\alpha|}}{|\vec{r} - \vec{r}_\alpha|} \tag{3.3.10}$$

These equations are identical to the ones given in section 2.1 for the Twersky formalism in the far-field.

4. RESULTS

In this chapter, the scattering operator $\bar{s}_\alpha$ derived in Chapter 3, will be solved for spherical and spheroidal particles. The results show that these corrections to Twersky's far-field approximation are not negligible unless the particle size is much smaller than a wavelength.

From Chapter 3, the formula for the scattering operator $\bar{s}_\alpha$ is

$$\bar{s}_\alpha = \left[1 - \frac{\bar{\zeta}_\alpha}{v_\alpha} \int_{v_\alpha} d^3r' \, g(\vec{r}', \vec{r}_\alpha) \, e^{-i\vec{k}_{\alpha f} \cdot (\vec{r}' - \vec{r}_\alpha)} \right]^{-1} . \tag{3.2.8}$$

$\bar{I}$ will designate the integral part of Equation (3.2.8):

$$\bar{I} = \frac{\bar{\zeta}_\alpha}{v_\alpha} \int_{v_\alpha} d^3r' \, g(\vec{r}', \vec{r}_\alpha) \, e^{-i\vec{k}_{\alpha f} \cdot (\vec{r}' - \vec{r}_\alpha)} \tag{4.1}$$

where

$$\bar{s}_\alpha = [1 - \bar{I}]^{-1} . \tag{4.2}$$

To use the far-field or point-particle assumption, the scattering operator $\bar{s}_\alpha$ must be one; therefore, $\bar{I}$ must must be negligible .

The integral $\bar{\mathrm{I}}$ will be solved for spherical particles in Section 4.1, and for oblate and prolate spheroidal particles in Section 4.2. These particle shapes are reasonable approximations for the types of particles found in the atmosphere [6].

4.1 Spherical Particles

We let the radius of the spherical particle α be a. To simplify the calculations, we assume that the particle is located at the origin:

$$\vec{r}_\alpha = 0.$$

Therefore the integral $\bar{\mathrm{I}}$, Equation (4.1), in spherical coordinates becomes

$$\bar{\mathrm{I}} = \frac{\bar{\zeta}_\alpha}{v_\alpha} \iiint_{v_\alpha} d\phi \; d\theta \; dr'' r''^2 \sin\theta \; g(\vec{r}'') \; e^{-i\vec{k}_{\alpha f} \cdot \vec{r}''}$$

$$= \frac{\bar{\zeta}_\alpha}{v_\alpha} \iiint_{v_\alpha} d\phi \; d\theta \; dr'' r''^2 \sin\theta \; \frac{e^{ikr''}}{4\pi \, r''} \, e^{-i\vec{k}_{\alpha f} \cdot \vec{r}''}. \tag{4.1.1}$$

Simplification of the exponential in Equation (4.1.1) produces the following relation.

$$-i\vec{k}_{\alpha f} \cdot \vec{r}'' = -ik \frac{\vec{r}}{r} \cdot \vec{r}'' = -ik \; r'' \cos\theta. \tag{4.1.2}$$

Next we substitute Equation (4.1.2) for the exponential in $\bar{\mathrm{I}}$ and integrate $\bar{\mathrm{I}}$, Equation (4.1.1), over the volume of the sphere.

$$\bar{\mathrm{I}} = \frac{1}{4\pi} \frac{\bar{\zeta}_\alpha}{v_\alpha} \int_0^{2\pi} d\phi \; \int_0^{a} dr'' \int_0^{\pi} d\theta \; \sin\theta \; r'' e^{ikr''} \; e^{-ikr''\cos\theta} \tag{4.1.3a}$$

We make the following substitution, in order to have all varibles and constants dimensionless:

$$r'' = k\ t,$$

$$dr'' = k\ dt;$$

therefore,

$$\bar{I} = \frac{1}{4\pi}\frac{\bar{\zeta}_\alpha}{v_\alpha}\int_0^{2\pi} d\phi \int_0^{ka} dt \int_0^{\pi} d\theta\ \sin\theta\ \frac{t}{k^2}\ e^{it}\ e^{-it\ \cos\theta}. \qquad (4.1.3b)$$

Integration over ϕ yields the following equation.

$$\bar{I} = \frac{\bar{\zeta}_\alpha}{v_\alpha}\frac{1}{2}\frac{1}{k^2}\int_0^{ka} dt \int_0^{\pi} d\theta\ t\ e^{it}\ \sin\theta\ e^{-ikt\ \cos\theta} \qquad (4.1.3c)$$

By letting

$$u = \cos\theta,$$

$$du\ = -\sin\theta\ d\theta,$$

Equation (4.1.3c) becomes

$$\bar{I} = \frac{\bar{\zeta}_\alpha}{v_\alpha}\frac{1}{2}\frac{1}{k^2}\int_0^{ka} dt \int_{-1}^{1} du\ t\ e^{it}\ e^{-it\ u}. \qquad (4.1.3d)$$

Next we integrate over u.

$$\bar{I} = \frac{\bar{\zeta}_\alpha}{v_\alpha}\frac{1}{2}\frac{1}{k^2}\int_0^{ka} dt\ -i\left(1 - e^{i2t}\right) \qquad (4.1.3.e)$$

Integration over t produces the final result:

$$\bar{I} = \frac{\bar{\zeta}_\alpha}{v_\alpha}\frac{1}{2}\frac{1}{k^2}\ i\left[ka - \frac{1}{2i}\left(e^{i2ka} - 1\right)\right]$$

$$= \frac{\bar{\zeta}_\alpha}{v_\alpha}\frac{1}{4}\frac{1}{k^2}\left[i2ka\left(1 - \frac{\sin\ (2ka)}{2ka}\right) + (1 - \cos(2ka))\right]. \qquad (4.1.3f)$$

Since

$$\overline{\zeta}_\alpha \equiv k^2\, v_\alpha\, (\epsilon_\alpha - 1), \tag{3.1.21}$$

Equation (4.1.3f) for $\overline{\mathrm{I}}$ becomes

$$\overline{\mathrm{I}} = \frac{(\epsilon_\alpha - 1)}{4}\left[\mathrm{i}2\mathrm{ka}\left(1 - \frac{\sin\,(2\mathrm{ka})}{2\mathrm{ka}}\right) + (1 - \cos(2\mathrm{ka}))\right]. \tag{4.1.4}$$

We can obtain the scattering operator $\overline{\mathrm{s}}_\alpha$ for a spherical particle by substituting $\overline{\mathrm{I}}$, Equation (4.1.4), into Equation (4.2).

$$\overline{\mathrm{s}}_\alpha = \left\{1 - \frac{(\epsilon_\alpha - 1)}{4}\left[\mathrm{i}2\mathrm{ka}\left(1 - \frac{\sin\,(2\mathrm{ka})}{2\mathrm{ka}}\right) + (1 - \cos(2\mathrm{ka}))\right]\right\}^{-1} \tag{4.1.5}$$

We may neglect the scattering operator correction when $\overline{\mathrm{s}}_\alpha$ approaches one; therefore, $\overline{\mathrm{I}}$ must approach zero. From Equation (4.1.4), $\overline{\mathrm{I}}$ becomes zero when

$$\frac{\sin\,(2\mathrm{ka})}{2\mathrm{ka}} \rightarrow 1$$

and

$$\cos\,(2\mathrm{ka}) \rightarrow 1.$$

For these two conditions to be met, it is necessary that

$$2\mathrm{ka} \ll 1,$$

or

$$4\pi \mathrm{a} \ll \lambda.$$

Therefore, for the scattering operator corrections to be negligible for spherical particles, the particle size must be much smaller than a wavelength (e.g., point-particles).

4.2 Spheroidal Particles

The equation for an ellipsoid in Cartesian coordinates is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \tag{4.2.1}$$

where x is the major axis, and y is the minor axis (a>b>c). In spherical coordinates, the equation for the ellipsoid becomes

$$\frac{r^2 \sin^2\theta \cos^2\phi}{a^2}+\frac{r^2 \sin^2\theta \sin^2\phi}{b^2}+\frac{r^2 \cos^2\theta}{c^2}=1. \tag{4.2.2}$$

In this section, $\bar{\mathrm{I}}$ will be solved for oblate and prolate spheroids. A spheroid is an ellipsoid that has two axes of the same length. An oblate spheroid has a=b and is an ellipse that has been rotated about its minor axis, and a prolate spheroid has b=c and is an ellipse that has been rotated about its major axis.

4.2.1 Oblate spheroids

The equation for an oblate spheroid (a=b) in spherical coordinates, from Equation (4.2.2), is

$$r^2\left(\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{c^2}\right)=1 \tag{4.2.1.1}$$

The integral equation $\bar{\mathrm{I}}$ for the prolate spheroid is

$$\bar{\mathrm{I}}=\frac{\bar{\zeta}_\alpha}{v_\alpha}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{R} dr''\, r''^2 \sin\theta\, g(\vec{r}'')\, e^{-i\vec{k}_{\alpha f}\cdot\vec{r}''} \tag{4.2.1.2}$$

where

$$kR=\left[\frac{\sin^2\theta}{(ka)^2}+\frac{\cos^2\theta}{(kc)^2}\right]^{-\frac{1}{2}}. \tag{4.2.1.3}$$

Again for simplicity we assume that the particle is located at the origin:

$$\vec{r}_\alpha = 0.$$

Therefore as in Section 4.1, Equation (4.1.2), the exponential in the integrand of $\bar{I}$ becomes

$$\vec{k}_{\alpha f} \cdot \vec{r}'' = k\ r''\cos\theta. \tag{4.1.2}$$

Now we integrate $\bar{I}$, Equation (4.2.1.2). To have the variables and constants dimensionless, we make the following substitution:

$$r'' = k\ t,$$

$$dr'' = k\ dt;$$

therefore,

$$\bar{I} = \frac{\bar{\zeta}_\alpha}{v_\alpha} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{kR} dt\ \frac{t}{k^2} \sin\theta\ e^{it(1-\cos\theta)}. \tag{4.2.1.4a}$$

Integration over ϕ yields the expression

$$\bar{I} = \frac{\bar{\zeta}_\alpha}{v_\alpha} \frac{2\pi}{k^2} \int_0^{\pi} d\theta\ \sin\theta \int_0^{kR} dt\ t\ e^{it(1-\cos\theta)}. \tag{4.2.1.4b}$$

We refer to integral tables [33] to solve the integral over t:

$$\int dx\ x\ e^{ax} = \frac{e^{ax}}{a^2}\ (ax - 1). \tag{4.2.1.5}$$

Therefore, integration of Equation (4.2.1.4b) over t and substitution of Equation (4.2.1.3) for kR produces the integral

$$\bar{I} = \frac{\bar{\zeta}_\alpha}{v_\alpha} \frac{2\pi}{k^2} \int_0^{\pi} d\theta\ \frac{-\sin\theta}{(1-\cos\theta)^2} \left\{1 + e^{ikR(1-\cos\theta)} \left[ikR(1-\cos\theta) - 1\right]\right\}$$

$$= \frac{\overline{\zeta}_\alpha}{v_\alpha}\frac{2\pi}{k^2}\int_0^\pi d\theta \, \frac{-\sin\theta}{(1-\cos\theta)^2}\left\{\exp\left[\frac{i(1-\cos\theta)}{\left(\frac{\sin^2\theta}{(ka)^2}+\frac{\cos^2\theta}{(kc)^2}\right)^{\frac{1}{2}}}\right]\left[\frac{i(1-\cos\theta)}{\left(\frac{\sin^2\theta}{(ka)^2}+\frac{\cos^2\theta}{(kc)^2}\right)^{\frac{1}{2}}}-1\right]+1\right\}. \tag{4.2.1.4c}$$

By letting

$$u = \cos\,\theta,$$

$$du = -\sin\theta\; d\theta,$$

Equation (4.2.1.4c) becomes

$$\overline{I} = \frac{\overline{\zeta}_\alpha}{v_\alpha}\frac{2\pi}{k^2}\int_{-1}^{1} du \, \frac{-1}{(1-u)^2}\left\{\exp\left[\frac{i(1-u)}{\left(\frac{(1-u^2)}{(ka)^2}+\frac{u^2}{(kc)^2}\right)^{\frac{1}{2}}}\right]\left[\frac{i(1-u)}{\left(\frac{(1-u^2)}{(ka)^2}+\frac{u^2}{(kc)^2}\right)^{\frac{1}{2}}}-1\right]+1\right\}$$

$$= \frac{\overline{\zeta}_\alpha}{v_\alpha}\frac{2\pi}{k^2}\int_{-1}^{1} du \, \frac{1}{(1-u)^2}\cdot\left\{\exp\left[\frac{ika(1-u)}{\sqrt{1+u^2\left(\left(\frac{a}{c}\right)^2-1\right)}}\right]\left[1-\frac{ika(1-u)}{\sqrt{1+u^2\left(\left(\frac{a}{c}\right)^2-1\right)}}\right]-1\right\}. \tag{4.2.1.4d}$$

Since

$$\overline{\zeta}_\alpha \equiv k^2\, v_\alpha\,(\epsilon_\alpha - 1), \tag{3.1.21}$$

Equation (4.2.1.4d) for $\overline{I}$ becomes

$$\overline{I} = 2\pi\,(\epsilon_\alpha - 1)\int_{-1}^{1} du \, \frac{1}{(1-u)^2}$$

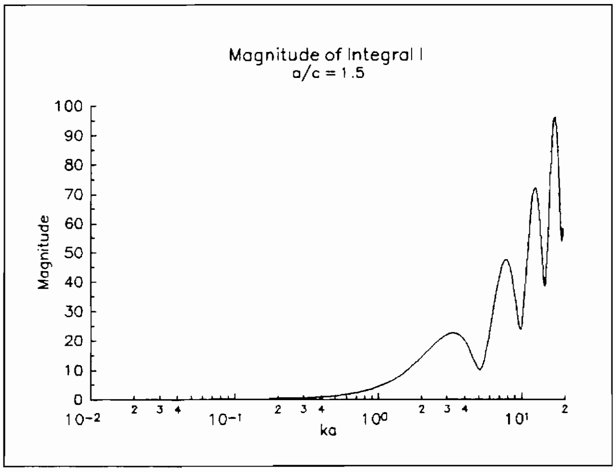
$$\cdot\left\{\exp\left[\frac{ika(1-u)}{\sqrt{1+u^2\left(\left(\frac{a}{c}\right)^2-1\right)}}\right]\left[1-\frac{ika(1-u)}{\sqrt{1+u^2\left(\left(\frac{a}{c}\right)^2-1\right)}}\right]-1\right\}. \tag{4.2.1.6}$$

From Equation (4.2.1.6), we observe that as (ka) approaches zero as in the point-particle case, $\bar{I}$ approaches zero. Therefore, for oblate spheroidal particles, the scattering operator correction becomes negligible as the particle size becomes much smaller than a wavelength. To solve Equation (4.2.1.6) for the case where (ka) is not small, we use numerical integration.

A plot of $\bar{I}$ versus (ka) is given in Figures 4.2.1.1 - 4.2.1.3. We calculated $\bar{I}$ for axis ratios of

$$\frac{a}{c} = 1.5,\ 2,\ 3.$$

The graphs show that the magnitude of the integral $\bar{I}$ increases with increasing (ka), and for (ka) less than one, the integral $\bar{I}$ is very small. It also shows that as (ka) increases, $\bar{I}$ begins to oscillate and approaches a purely imaginary number. In physical terms, as the particle increases in size, it becomes a plane boundary, and therefore, we obtain a phase change at this boundary. Therefore for oblate spheroids, the scattering operator is only negligible when the particles are small relative to a wavelength.



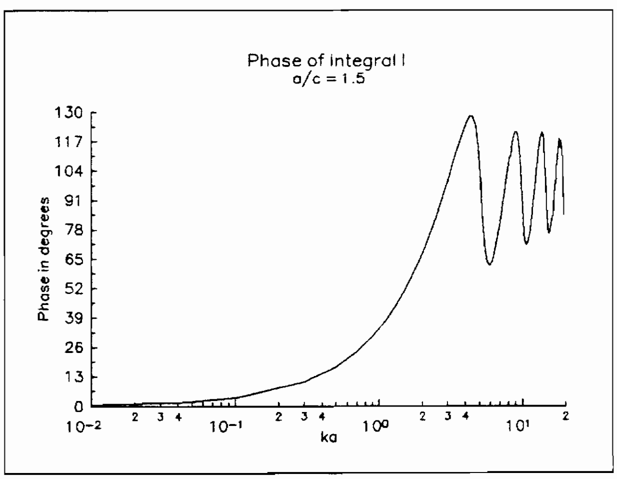
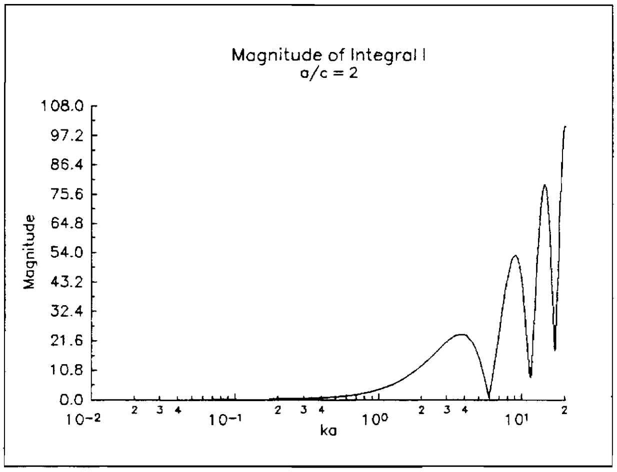


Figure 4.2.1.1 Graph of the integral $\bar{I}$ for oblate spheroids (a/c = 1.5)



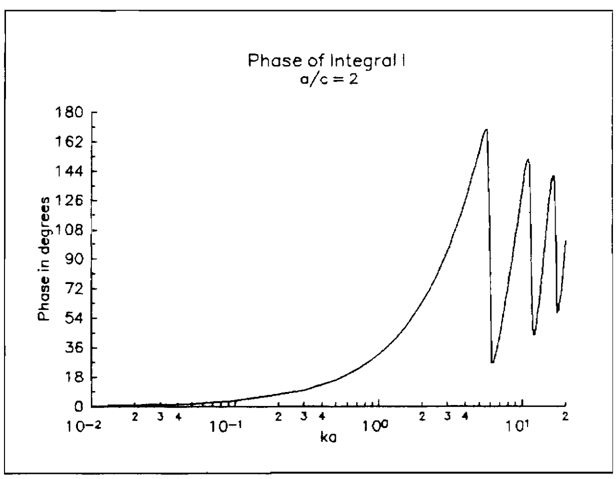
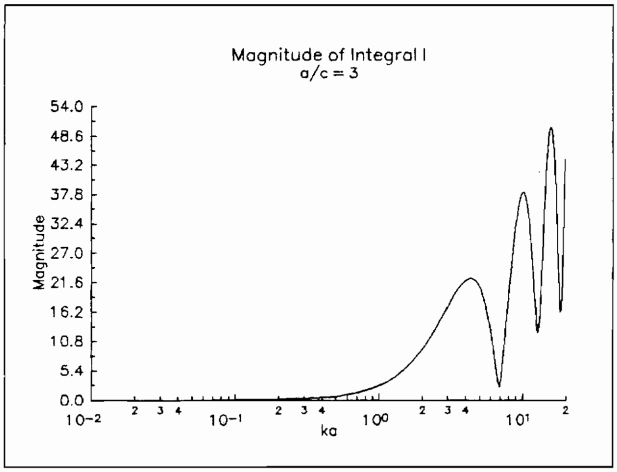


Figure 4.2.1.2 Graph of the integral $\bar{I}$ for oblate spheroids (a/c = 2)



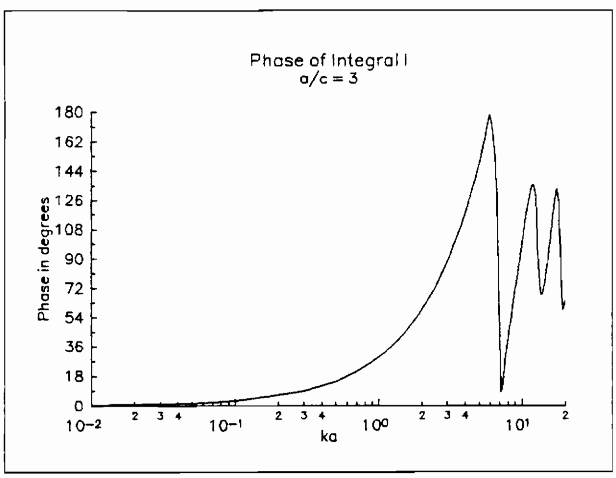


Figure 4.2.1.3 Graph of the integral $\bar{\mathrm{I}}$ for oblate spheroids (a/c = 3)

4.2.2 Prolate spheroids

The equation for a prolate spheroid (b=c) in spherical coordinates, from Equation (4.2.2), is

$$r^2 \sin^2\theta \left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2} \right) + \frac{r^2 \cos^2\theta}{b^2} = 1. \tag{4.2.2.1}$$

Therefore the integral $\bar{I}$ for the prolate spheroid is

$$\bar{I} = \frac{\bar{\zeta}_\alpha}{v_\alpha} \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \int_0^{R} dr'' r''^2 \sin\theta \, g(\vec{r}''') \, e^{-i\vec{k}_{\alpha f} \cdot \vec{r}'''} \tag{4.2.2.2}$$

where

$$kR = \left[\frac{\kappa}{k^2} \sin^2\theta + \frac{\cos^2\theta}{(kb)^2} \right]^{-\frac{1}{2}} \tag{4.2.2.3}$$

and

$$\frac{\kappa}{k^2} = \left[\frac{\cos^2\phi}{(ka)^2} + \frac{\sin^2\phi}{(kb)^2} \right]. \tag{4.2.2.4}$$

Again to simplify the calculations, we will assume that the particle is located at the origin:

$$\vec{r}_\alpha = 0.$$

Therefore, as in Section 4.1, Equation (4.1.2), the exponential in the integrand of $\bar{I}$ becomes

$$\vec{k}_{\alpha f} \cdot \vec{r}''' = k \, r'' \cos\theta. \tag{4.1.2}$$

Now we integrate Equation (4.2.2.2). First we perform the following substitution for r'' to make the variables and constants dimensionless.

$$r'' = k \, t,$$

$$dr'' = k \, dt,$$

We have, then,

$$\bar{\mathrm{I}} = \frac{\bar{\zeta}_\alpha}{v_\alpha}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \int_0^{\mathrm{R}} \mathrm{dr''r''e^{ikr''}sin\theta\ e^{-ikr''cos\theta}}$$

$$= \frac{\bar{\zeta}_\alpha}{v_\alpha}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta\ \sin\theta \int_0^{\mathrm{kR}} \mathrm{dt}\ \frac{\mathrm{t}}{\mathrm{k}^2}\ \mathrm{e}^{\mathrm{it}(1-\cos\theta)}. \tag{4.2.2.5a}$$

To integrate over t, we use Equation (4.2.26) from integral tables [33].

$$\int \mathrm{dx\ x\ e^{ax}} = \frac{\mathrm{e^{ax}}}{\mathrm{a}^2}\,(\mathrm{ax}-1) \tag{4.2.2.6}$$

Therefore, Equation (4.2.2.5a) after integrating over t and substituting in for kR, Equation (4.2.2.3) becomes

$$\bar{\mathrm{I}} = \frac{\bar{\zeta}_\alpha}{v_\alpha}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta\ \frac{-\sin\theta}{\mathrm{k}^2(1-\cos\theta)^2}\left\{\mathrm{e}^{\mathrm{i}(1-\cos\theta)\mathrm{kR}}\left[\mathrm{i}(1-\cos\theta)\mathrm{kR}-1\right]+1\right\}$$

$$= \frac{\bar{\zeta}_\alpha}{v_\alpha}\frac{1}{\mathrm{k}^2}\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta\ \frac{-\sin\theta}{(1-\cos\theta)^2}$$

$$\cdot\left\{\exp\left(\frac{\mathrm{i}(1-\cos\theta)}{\left[\frac{\kappa}{\mathrm{k}^2}\sin^2\theta+\frac{\cos^2\theta}{(\mathrm{kb})^2}\right]^{\frac{1}{2}}}\right)\left[\frac{\mathrm{i}(1-\cos\theta)}{\left[\frac{\kappa}{\mathrm{k}^2}\sin^2\theta+\frac{\cos^2\theta}{(\mathrm{kb})^2}\right]^{\frac{1}{2}}}-1\right]+1\right\}. \tag{4.2.2.5b}$$

By letting

$$\mathrm{u} = \cos\theta,$$

$$\mathrm{du} = -\sin\theta\ \mathrm{d}\theta,$$

Equation (4.2.2.5b) becomes

$$\bar{\mathrm{I}} = \frac{\bar{\zeta}_\alpha}{v_\alpha}\frac{1}{\mathrm{k}^2}\int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{du}\, \frac{-1}{(1-\mathrm{u})^2}$$

$$\cdot\left\{\exp\!\left(\frac{\mathrm{i}(1-\mathrm{u})}{\left[\frac{\kappa}{\mathrm{k}^2}\,(1-\mathrm{u}^2)+\frac{\mathrm{u}^2}{(\mathrm{kb})^2}\right]^{\frac{1}{2}}}\right)\left[\frac{\mathrm{i}(1-\mathrm{u})}{\left[\frac{\kappa}{\mathrm{k}^2}\,(1-\mathrm{u}^2)+\frac{\mathrm{u}^2}{(\mathrm{kb})^2}\right]^{\frac{1}{2}}}-1\right]+1\right\}$$

$$= \frac{\bar{\zeta}_\alpha}{v_\alpha}\frac{1}{\mathrm{k}^2}\int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{du}\, \frac{1}{(1-\mathrm{u})^2}$$

$$\cdot\left\{\exp\!\left(\frac{\mathrm{i}(1-\mathrm{u})}{\sqrt{\frac{\kappa}{\mathrm{k}^2}+\mathrm{u}^2\!\left(\left(\frac{\mathrm{a}}{\mathrm{b}}\right)^2-\frac{\kappa}{\mathrm{k}^2}\right)}}\right)\left[1-\frac{\mathrm{i}(1-\mathrm{u})}{\sqrt{\frac{\kappa}{\mathrm{k}^2}+\mathrm{u}^2\!\left(\left(\frac{\mathrm{a}}{\mathrm{b}}\right)^2-\frac{\kappa}{\mathrm{k}^2}\right)}}\right]-1\right\}. \qquad (4.2.2.5c)$$

We substitute Equation (4.2.2.4) for κ:

$$\bar{\mathrm{I}} = \frac{\bar{\zeta}_\alpha}{v_\alpha}\frac{1}{\mathrm{k}^2}\int_0^{2\pi} \mathrm{d}\phi \int_{-1}^{1} \mathrm{du}\, \frac{1}{(1-\mathrm{u})^2}\left\{\exp\!\left(\frac{\mathrm{ika}(1-\mathrm{u})}{\sqrt{1+\left(\left(\frac{\mathrm{a}}{\mathrm{b}}\right)^2-1\right)(\sin^2\phi+\mathrm{u}^2\cos^2\phi)}}\right)\right.$$

$$\left.\cdot\left[1-\frac{\mathrm{ika}(1-\mathrm{u})}{\sqrt{1+\left(\left(\frac{\mathrm{a}}{\mathrm{b}}\right)^2-1\right)(\sin^2\phi+\mathrm{u}^2\cos^2\phi)}}\right]-1\right\}. \qquad (4.2.2.5d)$$

Since

$$\bar{\zeta}_\alpha \equiv \mathrm{k}^2\, v_\alpha\,(\epsilon_\alpha - 1), \qquad (3.1.21)$$

Equation (4.2.1.5d) for $\bar{\mathrm{I}}$ becomes

$$\bar{I} = (\epsilon_\alpha - 1)\int_0^{2\pi} d\phi \int_{-1}^{1} du \, \frac{-1}{(1-u)^2}\left\{\exp\left(\frac{ika(1-u)}{\sqrt{1+\left(\left(\frac{a}{b}\right)^2 - 1\right)(\sin^2\phi + u^2\cos^2\phi)}}\right)\right.$$

$$\left.\cdot\left[1 - \frac{ika(1-u)}{\sqrt{1+\left(\left(\frac{a}{b}\right)^2 - 1\right)(\sin^2\phi + u^2\cos^2\phi)}}\right] - 1\right\} \tag{4.2.2.7}$$

From Equation (4.2.2.7), we observe that as (ka) approaches zero as in the point-particle case, $\bar{I}$ approaches zero, and the scattering operator $\bar{s}_\alpha$, Equation (4.2) approaches one. Therefore, for prolate spheroidal particles, the scattering operator correction becomes negligible as the particle size becomes much smaller than a wavelength. To solve Equation (4.2.1.7) for the case where (ka) is not small, numerical integration can be used. The function oscillates very rapidly as (ka) becomes larger; therefore, the integration should be performed very carefully.

5. CONCLUSIONS

Most applications of Twersky's formalism for multiple scattering in a random particulate medium assume a point-particle approximation. At high frequencies, the point-particle approximation becomes inaccurate since particles appear large relative to a wavelength. This thesis augments the Twersky formalism by making corrections for particle shape. This is done by defining a scattering operator $\bar{s}_\alpha$, which describes the multiple scattering effects. The scattering operator can be solved for various particle shapes and can then be used to define closed equations for the scattered fields.

Chapter 3 derives the scattering operator under a low-density approximation. In Chapter 4, we perform simple calculations of the scattering operator for spheroidal particles. The scattering operator may be solved for any particle shape by integrating over the particle volume; therefore, for future work, further calculations can be made for different particle shapes, for instance ellipsoids.

In this work, the stochastic forms of the field equations have not been derived for the scattering operator formulation. The scattering operator is already in the correct integral form for the mean-field equations. Therefore, the scattering operator formulation is very applicable to future work involving mean-field equations.

APPENDIX A. The wave equation derivation in random media

The derivation begins with the time-harmonic Maxwell's equations in a source-free region [34].

$$\nabla \times \vec{E}(\vec{r}) = -i\omega\mu_0\vec{H}(\vec{r}) \tag{A.1}$$

$$\nabla \times \vec{H}(\vec{r}) = i\omega\epsilon(\vec{r})\epsilon_0\vec{E}(\vec{r}) \tag{A.2}$$

$$\nabla \cdot \vec{B}(\vec{r}) = 0 \tag{A.3}$$

$$\nabla \cdot \vec{D}(\vec{r}) = 0 \tag{A.4}$$

Here, μ_0 is the free space magnetic permeability, ϵ_0 is the free-space dielectric permittivity and $\epsilon(\vec{r})$ is the relative permittivity of the medium. The electric field vector $\vec{E}(\vec{r}, t)$ and the magnetic field vector $\vec{H}(\vec{r}, t)$ are sinusoidal with respect to time:

$$\vec{E}(\vec{r}, t) = \mathrm{Re}[\vec{E}(\vec{r})\ e^{-i\omega t}] \tag{A.5}$$

$$\vec{H}(\vec{r}, t) = \mathrm{Re}[\vec{H}(\vec{r})\ e^{-i\omega t}]. \tag{A.6}$$

The following equations define the magnetic flux density $\vec{B}(\vec{r})$ and the electric flux density $\vec{D}(\vec{r})$ for an isotropic medium.

$$\vec{D}(\vec{r}) = \epsilon(\vec{r})\epsilon_0\ \vec{E}(\vec{r}) \tag{A.7}$$

$$\vec{B}(\vec{r}) = \mu_0\ \vec{H}(\vec{r}) \tag{A.8}$$

To begin the derivation, we take the curl of (A.1) to obtain

$$\nabla \times \nabla \times \vec{E} = \nabla \times (\text{-}i\omega\mu_0\vec{H}) = \text{-}i\omega\mu_0 \nabla \times \vec{H}. \tag{A.9}$$

Substituting Equation (A.2) into Equation (A.9) yields the equation,

$$\nabla \times \nabla \times \vec{E} = \text{-}i\omega\mu_0(i\omega\epsilon(\vec{r})\epsilon_0\vec{E}) = k_0^2\ \epsilon(\vec{r})\vec{E}, \tag{A.10}$$

where $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ is the free-space wave number. By using the vector identity

$$\nabla \times \nabla \times \vec{F} = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}, \tag{A.11}$$

Equation (A.10) becomes

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2\vec{E} = k_0^2\ \epsilon(\vec{r})\vec{E}. \tag{A.12}$$

By using Equation (A.4) in (A.12), we get the wave equation

$$\nabla^2\vec{E} = \text{-}k_0^2\ \epsilon(\vec{r})\vec{E}. \tag{A.13}$$

The relative dielectric permittivity can be represented by its mean value of $\bar{\epsilon}$ and a fluctuating part, $\bar{\epsilon}\delta\epsilon_r$, about this mean:

$$\epsilon(\vec{r}) = \bar{\epsilon}\left[1 + \delta\epsilon_r(\vec{r})\right]. \tag{A.14}$$

Substituting Equation (A.14) for the permittivity in the wave equation, Formula (A.13), results in the relationship

$$\nabla^2\vec{E} - k^2\ \vec{E} = k^2\ [\epsilon_r(\vec{r}) - 1]\ \vec{E} \tag{A.15}$$

where the effective propagation constant of the medium is

$$k^2 = \bar{\epsilon}\ k_0^2$$

and

$$\epsilon_r(\vec{r}) = 1 + \frac{\delta\epsilon_r(\vec{r})}{\bar{\epsilon}}. \tag{A.16}$$

The left hand side is the wave equation in free-space, and the right hand side represents a pseudo-source term. A solution of this equation via Green's functions is

$$\vec{E}(\vec{r}) = \vec{E}_0(\vec{r}) + k^2 \int d^3r'\ \bar{\bar{G}}\,(\vec{r},\ \vec{r}')\ [\epsilon_r(\vec{r}') - 1]\ \vec{E}(\vec{r}') \tag{A.16}$$

where the dyadic Green's function is

$$\overline{\overline{G}}(\vec{r}, \vec{r}') = (\overline{\overline{I}} + \frac{\nabla\nabla}{k^2}) \frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|} \tag{A.17}$$

and $\vec{E}_0(\vec{r})$, the solution to the homogeneous equation, is the field received at $\vec{r}$ without any effect by the medium. The second part of Equation (A.16) represents the scattered field.

This thesis examines scalar fields. Therefore Equations (A.15), (A.16) and (A.17) become

$$\nabla^2\Psi(\vec{r}) - k^2\,\Psi(\vec{r}) = k^2\,[\epsilon_r(\vec{r}) - 1]\,\Psi(\vec{r}) \tag{A.18}$$

$$\Psi(\vec{r}) = \Psi_0(\vec{r}) + k^2 \int d^3r'\, g(\vec{r}, \vec{r}')\,[\epsilon_r(\vec{r}') - 1]\,\Psi(\vec{r}') \tag{A.19}$$

$$g(\vec{r}, \vec{r}') = \frac{e^{ik|\vec{r}-\vec{r}'|}}{4\pi|\vec{r}-\vec{r}'|}. \tag{A.20}$$

where $\vec{r}$ defines an orthogonal coordinate system.

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