AN EVALUATION OF CLASSICAL AND Refined EQUIVALENT-SINGLE-LAYER LAMINATE THEORIES

by

Partha Bose

Thesis submitted to the Faculty of
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements
for the degree of

Master of Science
in
Engineering Mechanics

APPROVED:

J. N. Reddy
Dr. J. N. Reddy, Chairman

Dr. Robert A. Heller

Dr. Ronald D. Kriz

December 1995
Blacksburg, Virginia

Keywords: Laminated Plates, Finite Element
AN EVALUATION OF CLASSICAL AND REFINED EQUIVALENT-SINGLE-LAYER LAMINATE THEORIES
by
Partha Bose
Dr. J. N. Reddy, Chairman
Engineering Science and Mechanics Department

ABSTRACT

In this thesis, we study the static and free vibration response of symmetric and antisymmetric cross-ply laminated plates using different plate theories. Governing equations for two displacement-based third-order equivalent-single-layer theories have been developed. The first one is called the General Third-Order Theory (GTOT), and the second one is called the General Third-Order Theory of Reddy (GTTR). The displacement field of the second theory can be obtained from the first by imposing the condition of zero shear stresses at the bounding planes of the plate. The governing equations, analytical solutions, and finite element model of GTTR have been obtained in terms of tracers. Proceeding in this manner, the governing equations, analytical solutions, and finite element models of some lower-order plate theories fall out by just assigning appropriate values to the tracers (typically 1 or 0). While analytical and finite element solutions have been obtained for GTTR and its derivative cases, only finite element solutions have been obtained for GTOT. The analytical solutions are of two types. The Navier-type solution is for rectangular plates simply-supported on all four edges. In the Levy-type solution, two sides of the plate have to be simply-supported, while the remaining two sides can have any combination of free, clamped, or simply-supported boundary conditions. The results obtained from the different theories have been compared with exact solutions from existing literature. The response characteristics of the plates, like deflections, stresses, and frequencies, as well as the parameters affecting them have been studied. Some of the parameters investigated are span-to-thickness ratios, boundary conditions, loadings, and lamination schemes. The performance of the different theories in predicting plate responses have been evaluated.
Acknowledgements

At the onset, I wish to express my most sincere appreciation to Dr. J. N. Reddy, chairman of my thesis committee, for his guidance and patience throughout the preparation of this thesis.

I am also indebted to Dr. R. A. Heller and Dr. R. D. Kriz for agreeing to be on my committee.

My colleagues in the Engineering Science and Mechanics Department at Virginia Tech and in the Computational Mechanics group at Texas A&M University have been extremely cooperative and encouraging. I wish to thank all of them. Special thanks go to M. S. Ravisankar, without whose unselfish assistance this thesis would not have been complete.

Finally, I owe a deep sense of gratitude to my parents, who have always stood by me in whatever I have done, and who taught me to hope and work for a better tomorrow.
Contents

1 INTRODUCTION  ........................................ 1
   1.1 Motivation ......................................... 1
   1.2 Review of Literature  ................................ 2
       1.2.1 ESL Theories  .................................. 2
       1.2.2 Layerwise Theories  .............................. 7
       1.2.3 Continuum Based Theories  ...................... 9
   1.3 Objectives of the Present Study  ................... 9

2 GOVERNING EQUATIONS  ................................ 11
   2.1 Introduction ....................................... 11
   2.2 Lamina Constitutive Equations  ...................... 11
       2.2.1 Lamina Constitutive Equations for Plane Stress  15
   2.3 Kinematical Relations  ................................ 16
       2.3.1 Displacement Field for General Third-Order Theory  16
       2.3.2 Strain Field for General Third-Order Theory  ........ 17
       2.3.3 Displacement Field for General Third-Order Theory of Reddy  18
       2.3.4 Strain Field for General Third-Order Theory of Reddy  .... 20
   2.4 Laminate Constitutive Equations  .................... 20
   2.5 Equations of Motion  ................................ 22

3 ANALYTICAL SOLUTIONS  ................................ 28
   3.1 Introduction ....................................... 28
   3.2 Navier Solutions  .................................... 28
       3.2.1 Boundary Conditions  ............................. 30
       3.2.2 Assumed Displacements  .......................... 30
       3.2.3 Bending Analysis  ................................ 31
       3.2.4 Free Vibration Analysis  ......................... 32
3.3 Lévy Solutions ........................................ 32
   3.3.1 Assumed Displacements .......................... 32
   3.3.2 Solution Technique .............................. 33
   3.3.3 Closure ........................................ 39

4 FINITE ELEMENT SOLUTIONS 40
   4.1 Introduction ........................................ 40
   4.2 Weak Form .......................................... 40
   4.3 Interpolation Functions ............................ 43
   4.4 Finite Element Model ............................... 44

5 RESULTS AND DISCUSSION 47
   5.1 Introduction ........................................ 47
   5.2 Implementation ..................................... 47
      5.2.1 Analytical Solutions .......................... 47
      5.2.2 Finite Element Solutions ...................... 51
   5.3 Comparison with Exact Solutions .................. 55
   5.4 Static Analysis .................................... 68
   5.5 Free Vibration Analysis ............................ 107
   5.6 Conclusions ....................................... 115

A Navier Solution Coefficients 129
   A.1 Stiffness Matrix Coefficients ...................... 129
   A.2 Mass Matrix Coefficients ........................... 131

B Lévy Solution Coefficients 132
   B.1 Coefficients in Equation (3.14) .................... 132
   B.2 State Space Coefficients ........................... 135
      B.2.1 CLPT .......................................... 135
      B.2.2 FSCT .......................................... 136
      B.2.3 STTR .......................................... 138
      B.2.4 GTTR .......................................... 142

C Element Matrices 144
   C.1 Element Matrices for GTOT ......................... 144
      C.1.1 Element Stiffness Matrix ....................... 144
C.1.2 Element Mass Matrix ........................................ 147
C.1.3 Element Force Vector ........................................ 148
C.2 Element Matrices for GTTR .................................... 148
  C.2.1 Element Stiffness Matrix ................................. 148
  C.2.2 Element Mass Matrix ...................................... 150
  C.2.3 Element Force Vector .................................... 151
List of Figures

2.1 (a) Laminate Coordinates (b) Lamina Coordinates ................. 13

3.1 (a) Navier solution (b) Lévy solution .......................... 29

5.1 Boundary conditions for Lévy solution .......................... 50
5.2 Discretization for finite element solution ........................ 54
   (a) 8 × 8 mesh of 4-node elements in quarter-plate  
   (b) 16 × 8 mesh of 4-node elements in half-plate

5.3 Loading conditions (a) Uniform load (UL) (b) Line load (LL)  . 69

5.4 Nondimensional center deflection $\bar{w}$ vs span-to-thickness ratio $a/h$ for [0/90] square SSSS laminate using different theories .............. 82
5.5 Nondimensional center normal stress $\bar{\sigma}_z$ vs span-to-thickness ratio $a/h$ for [0/90] square SSSS laminate using different theories .............. 83
5.6 Nondimensional center deflection $\bar{w}$ vs span-to-thickness ratio $a/h$ for different layups of a square SSCC laminate using GTTR ........... 84
5.7 Nondimensional center normal stress $\bar{\sigma}_z$ vs span-to-thickness ratio $a/h$ for different layups of a square SSCC laminate using GTTR ........... 85
5.8 Nondimensional center deflection $\bar{w}$ vs aspect ratio $b/a$ for a [0/90/0] square SSCC laminate ($a/h = 5$) using different theories .......... 86
5.9 Nondimensional center normal stress $\bar{\sigma}_z$ vs aspect ratio $b/a$ for a [0/90/0] square SSCC laminate ($a/h = 5$) using different theories .......... 87
5.10 Nondimensional center deflection $\bar{w}$ vs span-to-thickness ratio $a/h$ for symmetric cross-ply square SSSS laminates with different number of layers $N$ using GTTR .............................................. 88
5.11 Nondimensional center normal stress $\bar{\sigma}_z$ vs span-to-thickness ratio $a/h$ for symmetric cross-ply square SSSS laminates with different number of layers $N$ using GTTR .............................................. 89
5.12 Nondimensional center deflection $\bar{w}$ vs span-to-thickness ratio $a/h$ for [0/90/0/90] square SSFF laminates with different modular ratios using GTTR ................................................................. 90
5.13 Nondimensional center normal stress $\bar{\sigma}_x$ vs span-to-thickness ratio $a/h$ for [0/90/0/90] square SSSS laminates with different modular ratios using GTTR ................................................................. 91
5.14 Nondimensional center deflection $\bar{w}$ vs span-to-thickness ratio $a/h$ for a [0/90] square laminate with different boundary conditions using GTTR 92
5.15 Nondimensional center normal stress $\bar{\sigma}_x$ vs span-to-thickness ratio $a/h$ for a [0/90] square laminate with different boundary conditions using GTTR ................................................................. 93
5.16 Nondimensional center deflection $\bar{w}$ vs modular ratio $E_1/E_2$ for antisymmetric cross-ply square SSSS laminates ($a/h = 5$) with different number of layers $N$ using GTTR ................................................................. 94
5.17 Nondimensional center normal stress $\bar{\sigma}_x$ vs modular ratio $E_1/E_2$ for antisymmetric cross-ply square SSSS laminates ($a/h = 5$) with different number of layers $N$ using GTTR ................................................................. 95
5.18 Variation of center normal stress $\sigma_x/g_0$ through the thickness of a [0/90] square SSCC laminate using different theories ................................................................. 96
5.19 Variation of center normal stress $\sigma_x/g_0$ through the thickness of a [0/90/0] square SSCC laminate using different theories ................................................................. 97
5.20 Variation of center normal stress $\sigma_x/g_0$ through the thickness of a [0/90/0] square laminate for different boundary conditions using GTTR ................................................................. 98
5.21 Variation of inplane shear stress $\tau_{xy}/g_0$ through the thickness of a [0/90] square SSCC laminate using different theories ................................................................. 99
5.22 Variation of inplane shear stress $\tau_{xy}/g_0$ through the thickness of a [0/90/0] square SSCC laminate using different theories ................................................................. 100
5.23 Variation of transverse shear stress $\tau_{xz}/g_0$ through the thickness of a [0/90] square SSSS laminate using different theories ................................................................. 101
5.24 Variation of transverse shear stress $\tau_{xz}/g_0$ through the thickness of a [0/90/0] square SSSS laminate using different theories ................................................................. 102
5.25 Variation of transverse shear stress $\tau_{xz}/g_0$ through the thickness of a [0/90/0] square laminate for different boundary conditions (SSFF,SSSF,SSCF) using GTTR ................................................................. 103

viii
5.26 Variation of transverse shear stress $\tau_{zz}/q_0$ through the thickness of a [0/90/0] square laminate for different boundary conditions (SSCS,SSCC,SSSS) using GTTR ................................................. 104
5.27 Variation of transverse normal stress $\sigma_z/q_0$ through the thickness of a [0/90/0] square SSSS laminate using different theories ................................ 105
5.28 Variation of transverse normal stress $\sigma_z/q_0$ through the thickness of a [0/90/0] square laminate for different boundary conditions using GTTR 106
5.29 Nondimensional fundamental frequency $\bar{\omega}$ vs span-to-thickness ratio $a/h$ for [0/90] square SSSS laminate using different theories ........ 109
5.30 Nondimensional fundamental frequency $\bar{\omega}$ vs span-to-thickness ratio $a/h$ for different layups of a square SSCC laminate using GTTR . . 110
5.31 Nondimensional fundamental frequency $\bar{\omega}$ vs aspect ratio $b/a$ for a [0/90/0] square SSSS laminate ($a/h = 5$) using different theories . . 111
5.32 Nondimensional fundamental frequency $\bar{\omega}$ vs span-to-thickness ratio $a/h$ for symmetric cross-ply square SSSS laminates with different number of layers $N$ using GTTR .................................................. 112
5.33 Nondimensional fundamental frequency $\bar{\omega}$ vs span-to-thickness ratio $a/h$ for [0/90/0/90] square SSSS laminates with different modular ratios using GTTR ............................................................. 113
5.34 Nondimensional fundamental frequency $\bar{\omega}$ vs modular ratio $E_1/E_2$ for antisymmetric cross-ply square SSSS laminates ($a/h = 5$) with different number of layers $N$ using GTTR ..................................................... 114
List of Tables

5.1 Material properties ................................................. 48
5.2 Boundary conditions for CLPT .................................... 51
5.3 Boundary conditions for FSDT ................................... 52
5.4 Boundary conditions for STTR ................................... 52
5.5 Boundary conditions for GTTR ................................... 53
5.6 Boundary conditions for GTOT ................................... 53
5.7 Transverse deflection \( (C_{11}w/hq_0) \) in an Orthotropic plate under uniform transverse load ................. 56
5.8 Normal stress \( (\sigma_x/q_0) \) in an orthotropic plate under uniform transverse load ........................................ 57
5.9 Normal stress \( (\sigma_y/q_0) \) in an orthotropic plate under uniform transverse load ........................................ 57
5.10 Shear stresses \( (\tau_{xz}/q_0) \) in an orthotropic plate under uniform transverse load ........................................ 58
5.11 Transverse deflection and stresses in a three-ply laminate under uniform transverse load \( (\beta = E_{x1}/E_{x2} = 1) \) ......................... 59
5.12 Transverse deflection and stresses in a three-ply laminate under uniform transverse load \( (\beta = E_{x1}/E_{x2} = 5) \) ......................... 60
5.13 Transverse deflection and stresses in a three-ply laminate under uniform transverse load \( (\beta = E_{x1}/E_{x2} = 10) \) ......................... 61
5.14 Transverse deflection and stresses in a three-ply laminate under uniform transverse load \( (\beta = E_{x1}/E_{x2} = 15) \) ......................... 62
5.15 Comparison of the lowest natural frequency of an orthotropic square plate: \( a/h = 10, \bar{\omega} = \omega h(\rho/C_{11})^{\frac{1}{2}} \) ......................... 63
5.16 Comparison of the second lowest natural frequency of an orthotropic square plate: \( a/h = 10, \bar{\omega} = \omega h(\rho/C_{11})^{\frac{3}{2}} \) ......................... 64
5.17 Comparison of the third lowest natural frequency of an orthotropic square plate: \(a/h = 10, \ \bar{\omega} = \omega h (\rho / C_{11})^{1/2} \) .................................................. 65

5.18 Effect of degree of orthotropy of individual layers on the fundamental frequency of simply-supported symmetric square laminates: \(a/h = 5, \ \bar{\omega} = 10 \times \omega (\rho h^2 / E_I)^{1/3} \) .................................................. 66

5.19 Effect of degree of orthotropy of individual layers on the fundamental frequency of simply-supported antisymmetric square laminates: \(a/h = 5, \ \bar{\omega} \approx 10 \times \omega (\rho h^2 / E_I)^{1/3} \) .................................................. 67

5.20 Center deflections \(\bar{w}\) of simply-supported [0/90] antisymmetric cross-ply square laminates under different load and boundary conditions: \(h/a = 0.15, \bar{w} = w \times 10^4 \) .................................................. 73

5.21 Center deflections \(\bar{w}\) of simply-supported [0/90] antisymmetric cross-ply square laminates under different load and boundary conditions: \(h/a = 0.2, \bar{w} = w \times 10^4 \) .................................................. 74

5.22 Center deflections \(\bar{w}\) of simply-supported [0/90] antisymmetric cross-ply square laminates under different load and boundary conditions: \(h/a = 0.3, \bar{w} = w \times 10^4 \) .................................................. 75

5.23 Center deflections \(\bar{w}\) of simply-supported [0/90/0] symmetric cross-ply square laminates under different load and boundary conditions: \(h/a = 0.15, \bar{w} = w \times 10^4 \) .................................................. 76

5.24 Center deflections \(\bar{w}\) of simply-supported [0/90/0] symmetric cross-ply square laminates under different load and boundary conditions: \(h/a = 0.2, \bar{w} = w \times 10^4 \) .................................................. 77

5.25 Center deflections \(\bar{w}\) of simply-supported [0/90/0] symmetric cross-ply square laminates under different load and boundary conditions: \(h/a = 0.3, \bar{w} = w \times 10^4 \) .................................................. 78

5.26 Center normal stresses \(\sigma_x\) for simply-supported [0/90/0] symmetric cross-ply square laminates under different load and boundary conditions: \(h/a = 0.15, \sigma_x = \sigma_x / 10 \) .................................................. 79

5.27 Center normal stresses \(\sigma_x\) for simply-supported [0/90/0] symmetric cross-ply square laminates under different load and boundary conditions: \(h/a = 0.2, \sigma_x = \sigma_x / 10 \) .................................................. 80
5.28 Center normal stresses $\sigma_z$ for simply-supported $[0/90/0]$ symmetric cross-ply square laminates under different load and boundary conditions: $h/a = 0.3, \sigma_z = \sigma_x/10$ ........................................ 81

5.29 Dimensionless fundamental frequencies of simply-supported antisymmetric cross-ply square laminates under different boundary conditions:

$\bar{\omega} = (\omega b^2/h)(\rho/E_2)^{1/3}$ .................................................. 108
Chapter 1

INTRODUCTION

1.1 Motivation

Composite materials have been used for a long time, although the particular nomenclature is of a very recent vintage. Israelites used straw to strengthen mud bricks. Ancient Egyptians rearranged wood to make plywood which had superior strength and resistance to thermal expansion [1]. The applications of composite materials have increased rapidly over the past two decades. They are now used in a variety of engineering structures, including automobiles, space and aircraft structures, underwater vehicles, electronic and medical devices, and sports equipment. The attractive features of most composite materials are their high strength-to-weight and stiffness-to-weight ratios. They also have good corrosion and wear resistance, excellent fatigue strength, high damping, and improved thermal stability over conventional materials like metals. More importantly, many of the properties of composite materials can be tailored to meet specific requirements.

Laminated composites are made up of layers of orthotropic materials that are bonded together. The principal material directions of each layer are oriented at different angles to the reference coordinates. As required in a design, by changing the material, orientation, or thickness, the designer can tailor the different properties of a laminate to suit a particular application [2]. The anisotropic behavior of composite laminates makes it a difficult material to analyse. Analysts of conventional materials, like metals, often find that they have to change their way of thinking while studying composite materials. The constitutive, failure, and damage models to be used are quite different. There are also phenomena which can be found only in composite
materials like delamination, fiber breakage and debonding, ply drop-off, matrix cracks, etc.

Most of the structural theories used till now to characterize the behavior of composite laminates fall into the category of “equivalent single layer (ESL) theories”. In these theories, the material properties of the constituent layers are “smeared” to form a hypothetical single layer whose properties are equivalent to the sum of its constituents. This category of theories have been found to be adequate in predicting global response characteristics of laminates, like maximum deflections, maximum stresses, fundamental frequencies, or critical buckling loads. The present study deals with a critical evaluation of various ESL theories in predicting global response characteristics of a composite laminate. In order to model local phenomena like delamination or matrix crack growth, some kind of layerwise theory or a continuum 3-D theory must be used.

1.2 Review of Literature

The subject of plate and shell theories continue to attract the attention of researchers judging by the number of publications in this area. Beams, plates, and shells being among the most common structural components in use today, their analysis is of considerable interest to engineers and scientists. The theories used for modeling laminated composite plates and shells fall into the following three categories [3].

1. Equivalent Single Layer (ESL) 2-D theories
2. Layerwise (LW) 2-D theories
3. Continuum based 3-D theories

Since we will be primarily concerned with the ESL theories, the literature related to that class of theories will be discussed in detail. The literature for layerwise and continuum based theories will also be discussed briefly.

1.2.1 ESL Theories

In the ESL theories, the displacements or stresses are expanded as a linear combination of the thickness coordinate and undetermined functions of position in the reference surface.

\[ \phi_i(x, y, z) = \sum_{j=0}^{N_i} \phi_i^j(x, y) z^j, \quad (i = 1, 2, 3) \]  (1.1)
where $N_i$ are the number of terms in the expansion. $\phi_i^j$ can be either displacements or stresses. This reduces the 3-D elasticity equations to 2-D equations in terms of thickness-averaged forces and moments.

In all the assumed displacement ESL theories, the displacements and their derivatives (and hence the strains) are continuous through the thickness of the plate. However, since the constitutive properties of each layer are different, the stresses are discontinuous at the layer interfaces. The principle of virtual displacements or the method of moments are used to derive the equations of motion.

**Classical Plate Theory**

The simplest ESL theory is the Classical Laminated Plate Theory (CLPT) which is based on the hypothesis of Kirchoff [4]. The hypothesis states that straight lines normal to the mid-plane of the plate before deformation remain straight and normal to the midsurface of the plate after deformation. They are also inextensible. This amounts to neglecting the transverse shear and transverse normal effects, i.e. the plate is assumed to be infinitely rigid in the transverse direction. The classical plate theory was originally developed for homogeneous isotropic plates and was later extended to laminated composite plates. The displacement field in CLPT is given by

\[
\begin{align*}
  u_1 &= u - z \frac{\partial w}{\partial x} \\
  u_2 &= v - z \frac{\partial w}{\partial y} \\
  u_3 &= w
\end{align*}
\]

where $u, v, w$ are the displacements of a point on the reference surface $(x, y, 0)$ of the plate.

The person who is credited with the first work on assumed displacement models is Basset [5]. Laminated plate theories based on CLPT are to be found in the works of Lekhnitskii [6], Stavsky [7], Reissner and Stavsky [8], Dong et al. [9], Bert and Mayberry [10], and Whitney and Leissa [11]. The transverse shear modulii of advanced composite materials are very low compared to the in-plane tensile modulus. Because of this, transverse shear deformation plays a very important role, especially in the case of thick plates. It has been seen that CLPT underpredicts deflections and overpredicts natural frequencies in the case of laminated plates.
First Order Shear Deformation Theory

The simplest theory to take into account transverse shear deformation was the First Order Shear Deformation Theory (FSDT). This theory accounts for linear variation of inplane displacements through the thickness. The transverse displacement is constant in the thickness direction. The displacement field is given by

\[
\begin{align*}
    u_1 &= u + z\phi_1 \\
    u_2 &= v + z\phi_2 \\
    u_3 &= w
\end{align*}
\]

(1.3)

where \(\phi_1\) and \(\phi_2\) are the rotations of a straight line in the \(xz\) and \(yz\) planes. In FSDT, the normals to the midplane remain straight, but not necessarily normal, to the deformed surface. Although the particular nomenclature, “first order shear deformation theory”, was introduced by Reddy [12], the original idea can be found in the works of Basset [5], Hildebrand et al. [13], Reissner [14], Hencky [15], and Mindlin [16]. As mentioned earlier, Basset [5] did the pioneering work in assumed displacement theories. The first papers on FSDT were from Reissner [14], Hencky [15], and Hildebrand et al. [13]. Mindlin [16] extended Hencky’s theory to include dynamic effects. Another early paper which included shear deformation effects in homogeneous plates was by Uflyand [17]. Attempts were first made to include shear deformation in non-homogeneous plates by Stavsky [18], Ambartsumyan [19], Yang et al. [20], Whitney [21], and Whitney and Pagano [22].

FSDT gives a state of constant shear strain through the thickness of the plate. However, according to 3-D elasticity theory, the shear strains vary quadratically. The so-called shear correction factors were introduced to compute equivalent shear forces. The value of the shear correction factors depend on the constituent ply properties, lamination scheme, geometry, and boundary conditions, among others [23, 24, 25].

FSDT has been referred to by different names in literature like the Hencky-Mindlin theory, the Reissner-Mindlin theory or sometimes, simply the Mindlin plate theory, although Mindlin was only responsible for extending Hencky’s theory to the dynamic case. There have been numerous papers which have used FSDT since the early 1970s, but it is not possible to mention all of them. Some papers dealing with finite element modeling of the theory will be mentioned later.
Higher Order Theories

The next step in the development of shear deformation plate theories were the higher order theories in which the displacements were expanded up to the quadratic or cubic powers of the thickness coordinate (see Nelson and Lorch [26], Sun and co-workers [27, 28, 29], Librescu [30], Jemeilita [31], Schmidt [32], Krishna Murty [33, 34, 35], Lo et al. [36, 37, 38], Levinson and co-workers [39, 40, 41], Murthy [42], Reddy [43, 44, 45], Bhimaraddi and Stevens [46], Di Sciuva [47], Stein [48], and Tessler [49, 50]). The third order theories of Reddy [43, 44, 45] will be examined in detail. The first one will be referred to as the Special Third Order Theory of Reddy (STTR). The displacement field in this theory is given by

\[
\begin{align*}
    u_1 &= u + z \left( \phi_1 - \frac{4}{3} \frac{z}{h} \right)^2 \left( \phi_1 + \frac{\partial w}{\partial x} \right) \\
    u_2 &= v + z \left( \phi_2 - \frac{4}{3} \frac{z}{h} \right)^2 \left( \phi_2 + \frac{\partial w}{\partial y} \right) \\
    u_3 &= w
\end{align*}
\]

(1.4)

This displacement field incorporates the condition that the transverse shear stresses vanish on the bounding surfaces of the plate. It is based on the displacement fields advanced earlier by Vlasov [51], Jemeilita [31], Schmidt [32], and Krishna Murty [33] for isotropic plates. After developing the theory based on the displacement field given in Eq. (1.4) in [52, 43], Reddy later extended it to include nonlinear effects [44, 45]. Higher order theories with similar displacement fields were advanced by Levinson [39], Murthy [42], and Bhimaraddi and Stevens [46]. Schmidt [32] also accounted for nonlinear strains. However, Schmidt and Levinson used the equilibrium equations of the first order theory in conjunction with the stresses evaluated from strains associated with the higher order theory. Murthy also used the same approach to obtain the equations of motion. Krishna Murty [35] suggested that the transverse deflection be split into two parts, one due to bending, and the other due to shear. A similar representation was given in the 1960s by Huffington [53] for beams. Of all the higher order theories, Reddy [43] was the first to obtain the equilibrium equations in a consistent manner using the principle of virtual displacements. This gives rise to self-adjoint equations, and the type and form of the boundary conditions fall out uniquely in the course of the derivation. Lewinski [54] investigated the mathematical aspects
of the theories, while Bielski and Telega [55] presented the existence of solutions to Reddy's nonlinear theory [45].

Lo et al. [36, 37] proposed a high order theory in which the inplane displacements are expanded up to the cubic power in $z$, and the transverse displacement is expanded up to the quadratic power in $z$. The displacement field is given by

$$
\begin{align*}
    u_1 &= u + z\phi_1 + z^2\psi_1 + z^3\theta_1 \\
    u_2 &= v + z\phi_2 + z^2\psi_2 + z^3\theta_2 \\
    u_3 &= w + z\psi_3 + z^2\theta_3
\end{align*}
$$

(1.5)

Finite element analysis of this theory will be carried out, since it can be viewed as the most general of all third order theories. Henceforth, it will be referred to as the General Third Order Theory (GTOT). Reddy [56] developed a modification of Lo's theory in which he accounted for the zero transverse shear stress conditions at the top and bottom of the plate. This theory will be referred to as the general third order theory of Reddy (GTTR). It also takes into account the stretching of the transverse normals. Analytical and finite element solutions of this theory (in a slightly modified form) will be carried out in the present study. GTTR contains as special cases STTR, FSDT, and CLPT. The displacement field for GTTR (in its unmodified form) is

$$
\begin{align*}
    u_1 &= u + z\phi_1 + z^2 \left( -\frac{1}{2}\frac{\partial \psi_3}{\partial x} \right) + z^3 \left[ -\frac{1}{3} \left\{ \frac{\partial \theta_3}{\partial x} + \frac{4}{h^2} \left( \phi_1 + \frac{\partial w}{\partial x} \right) \right\} \right] \\
    u_2 &= v + z\phi_2 + z^2 \left( -\frac{1}{2}\frac{\partial \psi_3}{\partial y} \right) + z^3 \left[ -\frac{1}{3} \left\{ \frac{\partial \theta_3}{\partial y} + \frac{4}{h^2} \left( \phi_2 + \frac{\partial w}{\partial y} \right) \right\} \right] \\
    u_3 &= w + z\psi_3 + z^2\theta_3
\end{align*}
$$

(1.6)

**Analytical Solutions and Finite Element Models**

Analytical solutions of the CLPT were developed by Whitney and Leissa [11], and of FSDT by Whitney [21, 57], Bert and Chen [58], Reddy [59], Reddy and Chao [60], Reddy, Khdeir, and Librescu [61], and Chaudhuri et al. [62, 63]. Analytical solutions of the higher order theories are due to Levinson and Cooke [46, 47], and the group consisting of Reddy and Librescu [59, 43, 64, 65, 66, 67, 68, 69, 70, 71, 72].

Numerous finite element models of the theories have been put forward. Of these, only a few representative contributions will be mentioned. For the first order theory,
they include the work of Hughes and co-workers [73, 74], Hinton and co-workers [75, 76] and Reddy and co-workers [77, 78, 79]. Hughes et al. [73] suggested reduced integration for shear terms to prevent “locking”. A C⁰ penalty finite element was developed by Reddy [77]. For the third order theories, Pan and Reddy [80], and Ren and Hinton [81] developed displacement finite element models of Reddy’s special third order theory [43], which we refer to as STTR. They used C¹ interpolation functions for the transverse displacement, and C⁰ interpolation functions for the other displacements. Averill and Reddy [82] analyzed plate finite elements based on STTR and FSDT for distortion sensitivity, accuracy, reliability and efficiency. Putcha and Reddy [83, 84] did a mixed finite element formulation of STTR. Doong [85] used the general third order theory (GTOT) of Lo et al. [36, 37] to study the stability and vibration of initially stressed thick plates. Other finite element models are due to Ren [86], Di Sciuva [87], Engblom and Ochoa [88], and Kant and co-workers [89, 90].

**Stress-Based and Mixed Theories**

In the stress-based theories, the stresses are expanded as a linear combination of z and unknown functions of the inplane coordinates. A stress-based formulation was first presented by Reissner [91, 14, 92]. He assumed linear variation of the inplane stresses through the thickness. The transverse stresses were obtained from the equations of equilibrium, which were in turn, derived using the principle of virtual forces. Other stress formulations are due to Panc [93, 94], and Kromm [95, 96]. Gol'denveizer [97, 98] generalized Reissner’s theory by replacing the linear distribution of stresses through the thickness by a distribution represented by an arbitrary function. Salerno and Goldberg [99], and Voyiadjis and Baluch [100, 101] also advanced theories which were modifications or extensions of Reissner’s theory to composite plates. Pryor et al. [102] conducted a finite element analysis of Reissner’s theory.

Although rarely used, a mixed formulation where both assumed displacement and assumed stress expansions are used was presented by Reissner [103, 104]. The governing equations are obtained using the mixed variational principle. Other works in this area are by Pagano [105] and Reissner’s later publications [106, 107].

**1.2.2 Layerwise Theories**

In the ESL theories, due to the single displacement expansions through the thickness of the laminate, the transverse strains are continuous through the thickness. When
the laminate is made of layers of dissimilar materials, the ESL theories do not give very good results. The same is true when it comes to accurately modeling local phenomena like delamination. In such cases, piecewise stress/displacement approximations in the thickness direction have been used in some theories. They are called Layer-Wise Theories, or are sometimes referred to in literature as Discrete Layer Theories [108]. Representative of this class of theories is the Generalized Layerwise Plate Theory (GLPT) of Reddy [109, 110, 111]. The displacement field is expanded as a linear combination of known layerwise continuous functions of the thickness coordinate $\phi^i_j$, and undetermined functions of position within each layer $U^i_j$.

$$u_i(x, y, z) = u^0_i(x, y) + \sum_{j=1}^{N_i} U^i_j(x, y)\phi^i_j(z) \quad (i = 1, 2, 3) \quad (1.7)$$

$N_i$ is the number of subdivisions (hypothetical layers) through the thickness. The approximation in Eq. (1.7) can be interpreted as the global semi-discrete finite element approximation of $u_i$s through the thickness. In that case, $\phi^i_j$ denote the global interpolation functions (they are piecewise continuous functions, defined only on two adjacent layers), and $U^i_j$ are the nodal values. On appropriate selection of the value of $N_i$, a lamina can have more than one element, or several laminae can be included in one element.

Quite a few other layer-wise theories have been proposed. Yu [112] and Durocher and Solecki [113] considered the case of a three-layer plate. Mau [114] used the first order theory with layerwise generalized displacements. Rehsfield and his colleagues [115, 116] advanced a layer-wise assumed stress theory, where the layer-wise stresses are expressed in terms of the stress and moment resultants using CLPT. Murakami [117], and Toledo and Murakami [118] based their work on Reissner’s mixed variational principle. Legendre polynomials were used for the stress and displacement expansions. Seide [119] also derived a theory for layer-wise linear displacements. Librescu [120, 121] presented a multilayer shell theory which included geometric nonlinearity. Pryor and Baker [122] presented a finite element model based on linear functions on each layer. Hinrichsen and Palazotto [123] used an idea similar to Reddy’s GLPT, but used cubic spline functions to account for the thickness variations. Other publications which fall into the category of discrete layer theories are due to Epstein and co-workers [124, 125] and Owen and Li [126, 127].
1.2.3 Continuum Based Theories

A 3-D elasticity solution for simply-supported homogeneous isotropic plates was presented by Vlasov [128]. This is perhaps the first work on continuum theories of plates. Later, with the development of composite materials, there was increased interest in the accurate prediction of the response characteristics of composite plates. Analytical solutions to the bending, free vibration, and stability of laminated plates were presented by Srinivas et al. [129, 130, 131, 132], Pagano [133, 134], Lee [135], Jones [136], Pagano and Hatfield [137], and Seide [138]. Lee and Reismann [139] studied the dynamic response of rectangular plates. Noor and Burton [140] presented an analytical solution for the stress and free vibration problems of rectangular multilayered anisotropic plates. Finite element three-dimensional models were developed by Lisitsyn and Krivenko [141], Putcha and Reddy [142], and and Liou and Sun [143]. The finite element modeling of 3-D theories is cumbersome, especially if one is dealing with transient or nonlinear problems. However, the analytical 3-D elasticity solutions are useful for the purpose of comparison with results obtained from other analyses performed using less rigorous theories.

1.3 Objectives of the Present Study

In the present study, we shall evaluate the performance of some displacement based ESL theories in predicting deflections, stresses, and natural frequencies in laminated composite plates. The theories used are the commonly used Classical Laminated Plate Theory (CLPT), First Order Shear Deformation Theory (FSDT), the Special Third Order Theory of Reddy (STTR), the General Third Order Theory of Reddy based on [56, 144] which includes transverse normal stresses (GTTR), and the general third order theory proposed by Lo et al. (GTOT). Actually GTTR can be obtained from GTOT by imposing the condition that the transverse shear stresses become zero at the top and bottom of the plate. In this work, all commonly used assumed displacement, single layer theories are unified as special cases of the general third order theory of Reddy by the introduction of so-called “tracers”. By assigning different values to these tracers (0 or 1), one can obtain the displacement field for either GTTR, STTR, FSDT, or CLPT. While this kind of unified kinematical relations have been presented before [56, 144], what is novel about this study is that the equations of motion, analytical solutions, and finite element models have been derived for the
comprehensive unified theory, and the other theories simply fall out from this by proper assignment of tracers.

The mathematical formulation of the theories are carried out in Chapter 2. The constitutive equations and kinematical relations are presented and the dynamic equations of equilibrium are derived in a variationally consistent manner. Chapter 3 gives the analytical solutions to the comprehensive theory and its special cases. The analytical solutions include the Navier solutions in which all four sides of a rectangular plate are simply supported and the Levy solutions in which two sides of the plate are simply supported and the remaining two sides can have any arbitrary combination of free, clamped, and simply supported boundary conditions. In Chapter 4, the finite element models of the unified theory and of GTOT are constructed. The analytical and numerical results are presented in Chapter 5. The different parameters investigated are the orientation of the layers, span-to-thickness ratio, boundary conditions, loading, and degree of anisotropy. The results obtained from the ESL theories are also compared with the exact solutions from continuum 3-D theories.
Chapter 2

GOVERNING EQUATIONS

2.1 Introduction

In this chapter, the kinematical relations, constitutive equations and the governing equations of motion are presented. As stated in the Chapter 1, an effort is made to come up with a comprehensive theory, which encompasses all the lower order theories. The lower order theories can be obtained from the main theory by changing the values of some quantities, called tracers. All the commonly used plate theories, as well as the refined theories presented in this chapter, fall into the category of the so-called “Equivalent single-layer theories”. In an Equivalent single-layer (ESL) theory, the heterogeneous laminated composite plate is treated as a statically equivalent, single, homogeneous layer. The 3-D problem is thus reduced to a 2-D one.

2.2 Lamina Constitutive Equations

The linear constitutive equations for the \( k \)-th orthotropic lamina are given by

\[
\begin{bmatrix}
\sigma'_{1} \\
\sigma'_{2} \\
\sigma'_{3} \\
\sigma'_{4} \\
\sigma'_{5} \\
\sigma'_{6}
\end{bmatrix} =
\begin{bmatrix}
C'_{11} & C'_{12} & C'_{13} & 0 & 0 & 0 \\
C'_{12} & C'_{22} & C'_{23} & 0 & 0 & 0 \\
C'_{13} & C'_{23} & C'_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C'_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C'_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C'_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon'_{1} \\
\epsilon'_{2} \\
\epsilon'_{3} \\
\epsilon'_{4} \\
\epsilon'_{5} \\
\epsilon'_{6}
\end{bmatrix}
\]

(2.1)

In the above equation, the coordinate system is the principal material coordinate system \((x'_1, x'_2, x'_3) = (x', y', z')\). However, in general, the coordinate system \((x_1, x_2, x_3) = (x, y, z)\) of a problem will not coincide with the material coordinate system. Hence
it is necessary to determine the elastic properties with respect to the \((x, y, z)\) coordinate system when the elastic stiffnesses are known relative to the \((x', y', z')\) coordinate system. The above equation can be written in the form

\[
\{\sigma'\} = [C']\{\epsilon'\} \quad (2.2)
\]

The stresses are transformed from the primed coordinate system to the unprimed coordinate system as

\[
\begin{bmatrix}
\sigma'_1 \\
\sigma'_2 \\
\sigma'_3 \\
\sigma'_4 \\
\sigma'_5 \\
\sigma'_6
\end{bmatrix} =
\begin{bmatrix}
c^2 & s^2 & 0 & 0 & 0 & 2cs \\
s^2 & c^2 & 0 & 0 & 0 & -2cs \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & c & -s & 0 \\
0 & 0 & 0 & s & c & 0 \\
-cs & cs & 0 & 0 & 0 & c^2 - s^2
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix} \quad (2.3)
\]

or

\[
\{\sigma'\} = [T_\sigma]\{\sigma\} \quad (2.4)
\]

where \(c = \cos \theta\) and \(s = \sin \theta\). \(\theta\) is the angle between the primed \((x', y')\) and unprimed \((x, y)\) coordinate systems (see Figure 2.1).

Similarly, the strains are transformed from the primed coordinate system to the unprimed coordinate system as

\[
\begin{bmatrix}
\epsilon'_1 \\
\epsilon'_2 \\
\epsilon'_3 \\
\epsilon'_4 \\
\epsilon'_5 \\
\epsilon'_6
\end{bmatrix} =
\begin{bmatrix}
c^2 & s^2 & 0 & 0 & 0 & cs \\
s^2 & c^2 & 0 & 0 & 0 & -cs \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & c & -s & 0 \\
0 & 0 & 0 & s & c & 0 \\
-2cs & 2cs & 0 & 0 & 0 & c^2 - s^2
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 \\
\epsilon_4 \\
\epsilon_5 \\
\epsilon_6
\end{bmatrix} \quad (2.5)
\]

or

\[
\{\epsilon'\} = [T_\epsilon]\{\epsilon\} \quad (2.6)
\]

Putting Eq. (2.4) and Eq. (2.6) into Eq. (2.2), we get

\[
[T_\sigma]\{\sigma\} = [C'][T_\epsilon]\{\epsilon\} \quad (2.7)
\]

Pre-multiplying Eq. (2.7) by \(T_\sigma^{-1}\), we obtain
Figure 2.1: (a) Laminate Coordinates (b) Lamina Coordinates
\[ \{\sigma\} = [T_{\sigma}^{-1}] [C'] [T_\epsilon] \{\epsilon\} \quad (2.8) \]

Hence

\[ [C] = [T_{\sigma}^{-1}] [C'] [T_\epsilon] \quad (2.9) \]

Written in explicit form, we get the components \( C_{ij} \) of matrix \([C]\) as

\[
\begin{align*}
C_{11} &= C'_{11}c^4 + 2c^2s^2(C'_{12} + 2C'_{66}) + C'_{22}s^4 \\
C_{12} &= c^2s^2(C'_{11} + C'_{22} - 4C'_{66}) + C'_{12}(c^2 + s^4) \\
C_{13} &= C'_{13}c^2 + C'_{23}s^2 \\
C_{16} &= cs \left[ C'_{11}c^2 - C'_{22}s^2 - (C'_{12} + 2C'_{66})(c^2 - s^2) \right] \\
C_{22} &= C'_{11}s^4 + 2c^2s^2(C'_{12} + 2C'_{66}) + C'_{22}c^4 \\
C_{23} &= C'_{13}s^2 + C'_{23}c^2 \\
C_{26} &= cs \left[ C'_{11}s^2 - C'_{22}c^2 + (C'_{12} + 2C'_{66})(c^2 - s^2) \right] \\
C_{33} &= C'_{33} \\
C_{36} &= cs(C'_{13} - C'_{23}) \\
C_{44} &= C'_{44}c^2 + C'_{55}s^2 \\
C_{45} &= cs(C'_{44} - C'_{55}) \\
C_{55} &= C'_{44}s^2 + C'_{55}c^2 \\
C_{66} &= c^2s^2(C'_{11} + C'_{22} - 2C'_{12}) + C'_{66}(c^2 - s^2)^2 \quad (2.10)
\]

In the laminate coordinates then, the stress-strain relations can be written in matrix form as

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\
C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\
C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\
0 & 0 & 0 & C_{44} & C_{45} & 0 \\
0 & 0 & 0 & C_{45} & C_{55} & 0 \\
C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 \\
\epsilon_4 \\
\epsilon_5 \\
\epsilon_6
\end{bmatrix} \quad (2.11)
\]
2.2.1 Lamina Constitutive Equations for Plane Stress

A plane stress state is one in which the out-of-plane stresses are neglected. For a lamina in the \((x, y)\) plane, this is defined by setting

\[
\sigma_z = 0 \quad \tau_{yz} = 0 \quad \tau_{xz} = 0
\]  

(2.12)

For a lamina of constant thickness and made of an orthotropic material, the plane stress constitutive equations in the principal material coordinates are

\[
\begin{bmatrix}
\sigma'_1 \\
\sigma'_2 \\
\sigma'_6
\end{bmatrix} =
\begin{bmatrix}
Q'_{11} & Q'_{12} & 0 \\
Q'_{12} & Q'_{22} & 0 \\
0 & 0 & Q'_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon'_1 \\
\epsilon'_2 \\
\epsilon'_6
\end{bmatrix}
\]  

(2.13)

and

\[
\begin{bmatrix}
\sigma'_4 \\
\sigma'_5
\end{bmatrix} =
\begin{bmatrix}
Q'_{44} & 0 \\
0 & Q'_{55}
\end{bmatrix}
\begin{bmatrix}
\epsilon'_4 \\
\epsilon'_5
\end{bmatrix}
\]  

(2.14)

where \(Q'_{ij}\) are the plane stress reduced stiffnesses and are given by

\[
\begin{align*}
Q'_{11} & = \frac{E_1}{1 - \nu_{12}\nu_{21}} \\
Q'_{12} & = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} \\
Q'_{22} & = \frac{E_2}{1 - \nu_{12}\nu_{21}} \\
Q'_{44} & = G_{23} \\
Q'_{55} & = G_{13} \\
Q'_{66} & = G_{12}
\end{align*}
\]  

(2.15)

By following a method similar to the 3-D stress state, we can get the stress-strain relations in the laminate coordinates as

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_6
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{12} & Q_{22} & Q_{26} \\
Q_{16} & Q_{26} & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_6
\end{bmatrix}
\]  

(2.16)

and

\[
\begin{bmatrix}
\sigma_4 \\
\sigma_5
\end{bmatrix} =
\begin{bmatrix}
Q_{44} & Q_{45} \\
Q_{45} & Q_{55}
\end{bmatrix}
\begin{bmatrix}
\epsilon_4 \\
\epsilon_5
\end{bmatrix}
\]  

(2.17)

where \(Q_{ij}\) are the transformed reduced stiffnesses. The relations between the \(Q_{ij}\)'s and the \(Q'_{ij}\)'s are exactly the same as the relations between their counterpart \(C_{ij}\)'s and \(C'_{ij}\)'s in the 3-D stress state (Eq. 2.10).
2.3 Kinematical Relations

2.3.1 Displacement Field for General Third-Order Theory

In a single-layer theory, the displacements (or stresses) are expanded as a linear combination of the thickness coordinate \( z \) and a set of undetermined functions, which depend upon the coordinates \((x, y)\) of the reference surface.

\[
\phi_i(x, y, z, t) = \sum_{j=1}^{N} \phi_i^j(x, y, t)z^j \quad (i = 1, 2, 3)
\]  
(2.18)

We consider a general third-order laminated plate theory in which the inplane displacements \((u_1, u_2)\) are expanded up to the cubic term in the thickness term \( z \), and the transverse displacement \( u_3 \) is expanded up to the quadratic term in \( z \). This is done to take into account the parabolic variation of the transverse shear stresses through the thickness of the plate.

\[
\begin{align*}
    u_1 &= u + z\phi_1 + z^2\psi_1 + z^3\theta_1 \\
    u_2 &= v + z\phi_2 + z^2\psi_2 + z^3\theta_2 \\
    u_3 &= w + z\psi_3 + z^2\theta_3
\end{align*}
\]  
(2.19)

where \( u, v, w, \phi_1, \phi_2, \psi_1, \psi_2, \psi_3, \theta_1, \theta_2, \theta_3 \) are unknown functions of the planar coordinates \((x, y)\) and time \( t \). Here, \((u, v, w)\) denote the displacements of a point on the mid-surface of the plate \((z = 0)\). \( \phi_x \) and \( \phi_y \) are the rotations of the transverse normal in the \( xz \) and \( yz \) planes. The term \( \psi_3 \) can be interpreted as the stretching of the transverse normal. For the remaining higher-order terms, a physical interpretation is not immediately apparent, but can be taken to be higher-order rotations or extensions. Lo et al. did an analytical study of the behavior of elastic plates based on the general third-order theory given above.

A number of special cases can be derived from the above theory:

- If we assume that the transverse normals are inextensible \((\psi_3 = 0, \theta_3 = 0)\), the number of unknowns reduce to nine. In this case, the plane stress assumption may be used.

- A second-order theory in which \( \theta_1 \) and \( \theta_2 \) are set to zero. In this case, there are nine unknowns. An extension of this is when, additionally, \( \theta_3 \) is set to zero.
If we incorporate the transverse normal inextensibility condition, then $\psi_3 = 0$, and the number of unknowns are seven. For this case also, the plane stress reduced stiffnesses can be used.

- A theory in which the inplane displacements $(u_1, u_2)$ are expanded up to the linear power of $z$ and the transverse displacement $(u_3)$ is expanded up to the quadratic power of $z$.

- A third-order theory in which the second-order terms are eliminated by satisfying the stress-free boundary conditions on $\sigma_{xz}$ and $\sigma_{yx}$ on the surfaces of the plate and assuming inextensibility of transverse normals [43, 44]. The number of unknown functions are eight.

- A class of theories in which the transverse shear stresses $(\sigma_{yz}, \sigma_{zx})$ are made to vanish on the top and bottom of the plate $(z = \pm \frac{t}{2})$, but inextensibility of transverse normals is not assumed. This class of theories will be discussed in a subsequent section.

### 2.3.2 Strain Field for General Third-Order Theory

The linear strain field for the general third order theory (Eq. 2.19) is obtained from the strain-displacement relations as

\[
\begin{align*}
\epsilon_1 &= \frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} + z \frac{\partial \phi_1}{\partial x} + z^2 \frac{\partial \psi_1}{\partial x} + z^3 \frac{\partial \theta_1}{\partial x} \\
\epsilon_2 &= \frac{\partial u_2}{\partial x} = \frac{\partial v}{\partial y} + \frac{\partial \phi_2}{\partial y} + z^2 \frac{\partial \psi_2}{\partial y} + z^3 \frac{\partial \theta_2}{\partial y} \\
\epsilon_3 &= \frac{\partial u_3}{\partial z} = \psi_3 + 2z \theta_3 \\
\epsilon_4 &= \frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial \phi_3} = \left( \phi_2 + \frac{\partial w}{\partial y} \right) + z \left( 2\psi_2 + \frac{\partial \psi_3}{\partial y} \right) + z^2 \left( 3\theta_2 + \frac{\partial \theta_3}{\partial y} \right) \\
\epsilon_5 &= \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \left( \phi_1 + \frac{\partial w}{\partial x} \right) + z \left( 2\psi_1 + \frac{\partial \psi_3}{\partial x} \right) + z^2 \left( 3\theta_1 + \frac{\partial \theta_3}{\partial x} \right) \\
\epsilon_6 &= \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} = \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + z \left( \frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \right) + z^2 \left( \frac{\partial \psi_1}{\partial y} + \frac{\partial \psi_2}{\partial x} \right) \\
&\quad + z^3 \left( \frac{\partial \theta_1}{\partial y} + \frac{\partial \theta_2}{\partial x} \right) \quad (2.20)
\end{align*}
\]
The strains can be expressed in a compact form as

$$\begin{align*}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6 
\end{bmatrix} &=
\begin{bmatrix}
\varepsilon_1^{(0)} \\
\varepsilon_2^{(0)} \\
\varepsilon_3^{(0)} \\
\varepsilon_4^{(0)} \\
\varepsilon_5^{(0)} \\
\varepsilon_6^{(0)} 
\end{bmatrix} + z \begin{bmatrix}
\varepsilon_1^{(1)} \\
\varepsilon_2^{(1)} \\
\varepsilon_3^{(1)} \\
\varepsilon_4^{(1)} \\
\varepsilon_5^{(1)} \\
\varepsilon_6^{(1)} 
\end{bmatrix} + z^2 \begin{bmatrix}
\varepsilon_1^{(2)} \\
\varepsilon_2^{(2)} \\
\varepsilon_3^{(2)} \\
\varepsilon_4^{(2)} \\
\varepsilon_5^{(2)} \\
\varepsilon_6^{(2)} 
\end{bmatrix} + z^3 \begin{bmatrix}
\varepsilon_1^{(3)} \\
\varepsilon_2^{(3)} \\
\varepsilon_3^{(3)} \\
\varepsilon_4^{(3)} \\
\varepsilon_5^{(3)} \\
\varepsilon_6^{(3)} 
\end{bmatrix}
\end{align*}
$$

(2.21)

where

$$\begin{align*}
\varepsilon_1^{(0)} &= \frac{\partial u}{\partial x}, & \varepsilon_1^{(1)} &= \frac{\partial \phi_1}{\partial x}, & \varepsilon_1^{(2)} &= \frac{\partial \psi_1}{\partial x}, & \varepsilon_1^{(3)} &= \frac{\partial \theta_1}{\partial x}, \\
\varepsilon_2^{(0)} &= \frac{\partial v}{\partial y}, & \varepsilon_2^{(1)} &= \frac{\partial \phi_2}{\partial y}, & \varepsilon_2^{(2)} &= \frac{\partial \psi_2}{\partial y}, & \varepsilon_2^{(3)} &= \frac{\partial \theta_2}{\partial y}, \\
\varepsilon_3^{(0)} &= \psi_3, & \varepsilon_3^{(1)} &= 2\theta_3, & \varepsilon_3^{(2)} &= 0, & \varepsilon_3^{(3)} &= 0, \\
\varepsilon_4^{(0)} &= \phi_2 + \frac{\partial w}{\partial y}, & \varepsilon_4^{(1)} &= 2\psi_2 + \frac{\partial \psi_3}{\partial y}, & \varepsilon_4^{(2)} &= 3\theta_2 + \frac{\partial \theta_3}{\partial y}, & \varepsilon_4^{(3)} &= 0, \\
\varepsilon_5^{(0)} &= \phi_1 + \frac{\partial w}{\partial x}, & \varepsilon_5^{(1)} &= 2\psi_1 + \frac{\partial \psi_3}{\partial x}, & \varepsilon_5^{(2)} &= 3\theta_1 + \frac{\partial \theta_3}{\partial x}, & \varepsilon_5^{(3)} &= 0, \\
\varepsilon_6^{(0)} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, & \varepsilon_6^{(1)} &= \frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x}, & \varepsilon_6^{(2)} &= \frac{\partial \psi_1}{\partial y} + \frac{\partial \psi_2}{\partial x}, & \varepsilon_6^{(3)} &= \frac{\partial \theta_1}{\partial y} + \frac{\partial \theta_2}{\partial x}
\end{align*}
$$

(2.22)

### 2.3.3 Displacement Field for General Third-Order Theory of Reddy

The displacement field for the general third order theory of Reddy (GTTR) will be derived in this section. In a particular lamina, the transverse shear stresses are given by (see Reddy [56, 144]).

$$\begin{align*}
\begin{bmatrix}
\sigma_4 \\
\sigma_5 
\end{bmatrix} &=
\begin{bmatrix}
C_{44} & C_{45} \\
C_{45} & C_{55}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_4 \\
\varepsilon_5 
\end{bmatrix}
\end{align*}
$$

(2.23)

If the transverse shear stresses $\sigma_4$ and $\sigma_5$ are to vanish at the bounding planes of the plate (at $z = \pm \frac{h}{2}$), the transverse shear strains $\varepsilon_4$ and $\varepsilon_5$ should also vanish there. That is

$$\varepsilon_4 \left( x, y, \pm \frac{h}{2} \right) = \varepsilon_5 \left( x, y, \pm \frac{h}{2} \right) = 0$$

(2.24)
Consider the displacement field for the general third order theory given in Eq. (2.19). Substituting Eq. (2.24) in that equation leads to the conditions

\[
\psi_1 = -\frac{1}{2} \frac{\partial \psi_3}{\partial x}, \quad \theta_1 = -\frac{1}{3} \left[ \frac{\partial \theta_3}{\partial x} + \frac{4}{h^2} \left( \phi_1 + \frac{\partial w}{\partial x} \right) \right]
\]

\[
\psi_2 = -\frac{1}{2} \frac{\partial \psi_3}{\partial y}, \quad \theta_2 = -\frac{1}{3} \left[ \frac{\partial \theta_3}{\partial y} + \frac{4}{h^2} \left( \phi_2 + \frac{\partial w}{\partial y} \right) \right]
\]

Putting the above conditions in Eq. (2.19) leads to the following displacement field

\[
u_1 = u + z\phi_1 + z^2 \left( -\frac{1}{2} \frac{\partial \psi_3}{\partial x} \right) + z^3 \left[ -\frac{1}{3} \left( \frac{\partial \theta_3}{\partial x} + \frac{4}{h^2} \left( \phi_1 + \frac{\partial w}{\partial x} \right) \right) \right]
\]

\[
u_2 = v + z\phi_2 + z^2 \left( -\frac{1}{2} \frac{\partial \psi_3}{\partial y} \right) + z^3 \left[ -\frac{1}{3} \left( \frac{\partial \theta_3}{\partial y} + \frac{4}{h^2} \left( \phi_2 + \frac{\partial w}{\partial y} \right) \right) \right]
\]

\[
u_3 = w + z\psi_3 + z^2 \theta_3
\]

(2.26)

In order to simplify analysis, the following substitution of variables is made

\[
\zeta = \theta_3 + \frac{4}{h^2} w
\]

(2.27)

The introduction of \( \zeta \) prevents the occurrence of some redundant boundary conditions in the equations of motion. It also simplifies the finite element modeling using this theory. The number of variables requiring \( C^1 \)-continuity is reduced from three to two.

Also, at this point, we introduce some symbols called tracers into the displacement field of Eq. (2.26). As these tracers assume different values, the displacement field reduces to different special cases.

\[
u_1 = u + z \left[ 1 - \frac{4}{3} \left( \frac{z}{h} \right)^2 \right] \phi_1 + \lambda z^2 \left( -\frac{1}{2} \frac{\partial \psi_3}{\partial x} \right) + \gamma z^3 \left( -\frac{1}{3} \frac{\partial \zeta}{\partial x} \right)
\]

\[
u_2 = v + z \left[ 1 - \frac{4}{3} \left( \frac{z}{h} \right)^2 \right] \phi_2 + \lambda z^2 \left( -\frac{1}{2} \frac{\partial \psi_3}{\partial y} \right) + \gamma z^3 \left( -\frac{1}{3} \frac{\partial \zeta}{\partial y} \right)
\]

\[
u_3 = w + z\psi_3 + z^2 \left( \zeta - c_1 w \right)
\]

(2.28)

where \( c_1 = 4/h^2 \).

The different theories that can be obtained by assigning different values to the tracers are

- \( \chi = \lambda = \gamma = 0, \quad \phi_1 = -\partial w/\partial x, \quad \phi_2 = -\partial w/\partial y \): Classical Plate Theory (CLPT)
• \( \chi = \lambda = \gamma = 0 \): First Order Shear Deformation Theory (FSDT)

• \( \chi = 1, \quad \lambda = 0, \quad \gamma = 1, \quad \zeta = c_1 w \): Special Third-Order Theory of Reddy (STTR)

• \( \chi = \lambda = \gamma = 1 \): General Third-Order Theory of Reddy which has provision for transverse normal stresses in its kinematics (GTTR)

### 2.3.4 Strain Field for General Third-Order Theory of Reddy

The strain field associated with the displacement field in Eq. (2.28) is of the same form as in Eq. (2.21) with the different strain components as

\[
\begin{align*}
\varepsilon_1^{(0)} &= \frac{\partial u}{\partial x}, & \varepsilon_1^{(1)} &= \frac{\partial \phi_1}{\partial x}, & \varepsilon_1^{(2)} &= -\frac{\lambda}{2} \frac{\partial^2 \psi_3}{\partial x^2}, & \varepsilon_1^{(3)} &= -\frac{1}{3} \left( \chi c_1 \frac{\partial \phi_1}{\partial x} + \gamma \frac{\partial^2 \zeta}{\partial x^2} \right) \\
\varepsilon_2^{(0)} &= \frac{\partial v}{\partial y}, & \varepsilon_2^{(1)} &= \frac{\partial \phi_2}{\partial y}, & \varepsilon_2^{(2)} &= -\frac{\lambda}{2} \frac{\partial^2 \psi_3}{\partial y^2}, & \varepsilon_2^{(3)} &= -\frac{1}{3} \left( \chi c_1 \frac{\partial \phi_2}{\partial y} + \gamma \frac{\partial^2 \zeta}{\partial y^2} \right) \\
\varepsilon_3^{(0)} &= \chi \psi_3, & \varepsilon_3^{(1)} &= 2\gamma \left( \zeta - c_1 w \right), & \varepsilon_3^{(2)} &= 0, & \varepsilon_3^{(3)} &= 0 \\
\varepsilon_4^{(0)} &= \phi_2 + \frac{\partial w}{\partial y}, & \varepsilon_4^{(1)} &= 0, & \varepsilon_4^{(2)} &= -c_1 \left( \chi \phi_2 + \gamma \frac{\partial w}{\partial y} \right), & \varepsilon_4^{(3)} &= 0 \\
\varepsilon_5^{(0)} &= \phi_1 + \frac{\partial w}{\partial x}, & \varepsilon_5^{(1)} &= 0, & \varepsilon_5^{(2)} &= -c_1 \left( \chi \phi_1 + \gamma \frac{\partial w}{\partial x} \right), & \varepsilon_5^{(3)} &= 0 \\
\varepsilon_6^{(0)} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, & \varepsilon_6^{(1)} &= \frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x}, & \varepsilon_6^{(2)} &= -\lambda \frac{\partial^2 \psi_3}{\partial x \partial y}, & \varepsilon_6^{(3)} &= -\frac{1}{3} \left[ \chi c_1 \left( \frac{\partial \phi_1}{\partial y} + \frac{\partial \phi_2}{\partial x} \right) + 2\gamma \frac{\partial^2 \zeta}{\partial x \partial y} \right]
\end{align*}
\]

### 2.4 Laminate Constitutive Equations

First we define the overall laminate stiffnesses \( A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, G_{ij} \) and \( H_{ij} \).

\[
(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, G_{ij}, H_{ij}) = \int_{-h/2}^{h/2} C_{ij}^{(k)} \left( 1, z, z^2, z^3, z^4, z^5, z^6 \right) dz \quad (i, j = 1, 2, 3, 4, 5, 6)
\]

If we write the \( A_{ij}, B_{ij}, \) etc., in terms of the ply stiffnesses \( C_{ij}^{(k)} \) and the ply coordinates \( z_k \) and \( z_{k+1} \), we have
\[(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, G_{ij}, H_{ij}) = \frac{1}{n} \sum_{k=1}^{N} C_{ij}^{(k)} (z_k^n - z_k^m) \quad (n = 1, 2, 3, 4, 5, 6, 7) \quad (2.31)\]

For the plane stress case, we substitute the \(C_{ij}^{(k)}\)'s in the above equations with the plane stress reduced stiffnesses \(Q_{ij}^{(k)}\). The stress resultants \(N_i, M_i, P_i, S_i\) are defined as
\[
(N_i, M_i, P_i, S_i) = \int_{-h/2}^{h/2} \sigma_i(1, z, z^2, z^3) dz = \sum_{k=1}^{N} \int_{z_k}^{z_k+1} \sigma_i^{(k)}(1, z, z^2, z^3) dz \quad (2.32)\]

If we use Eq. (2.21)
\[
\{\epsilon\} = \{\epsilon^{(0)}\} + z\{\epsilon^{(1)}\} + z^2\{\epsilon^{(2)}\} + z^3\{\epsilon^{(3)}\} \quad (2.33)\]

and the constitutive relations (2.16) and (2.17), the stress resultants (2.32) can be written in compact form as
\[
\begin{bmatrix}
\{N\} \\
\{M\} \\
\{P\} \\
\{S\}
\end{bmatrix} =
\begin{bmatrix}
\end{bmatrix}
\begin{bmatrix}
\{\epsilon^{(0)}\} \\
\{\epsilon^{(1)}\} \\
\{\epsilon^{(2)}\} \\
\{\epsilon^{(3)}\}
\end{bmatrix} \quad (2.34)\]

For cross-ply laminates, the plate stiffnesses with subscripts (16), (26), (36), and (45) are zero. Also, for symmetric laminates we have, \(B_{ij} = E_{ij} = G_{ij} = 0\), and for antisymmetric cross-ply laminates, \(B_{12} = B_{66} = E_{12} = E_{66} = G_{12} = G_{66} = 0\).

We introduce the plate inertias \(I_i\) at this point.

\[
(I_1, I_2, I_3, I_4, I_5, I_6, I_7) = \int_{-h/2}^{h/2} \rho^{(k)}(1, z, z^2, z^3, z^4, z^5, z^6) dz
\]

\[
= \sum_{k=1}^{N} \int_{z_k}^{z_k+1} \rho^{(k)}(1, z, z^2, z^3, z^4, z^5, z^6) dz \quad (2.35)\]

If the material for all the layers are identical, that is if the density \(\rho^{(k)}\) is the same for all \(k\), then we have

\[
I_2 = I_6 = 0 \\
I_1 = \rho h, \quad I_3 = \frac{1}{12} \rho h^3, \quad I_5 = \frac{1}{80} \rho h^5, \quad I_7 = \frac{1}{448} \rho h^7 \quad (2.36)\]
2.5 Equations of Motion

The Hamilton's principle for an elastic body is

$$\int_{t_1}^{t_2} (\delta U + \delta V - \delta K) \, dt = 0 \tag{2.37}$$

where $\delta U$ is the virtual strain energy, $\delta V$ is the virtual work done by external forces, and $\delta K$ is the virtual kinetic energy. $\delta U$, $\delta V$, and $\delta K$ are given by

$$\delta U = \int_\Omega \int_{-h/2}^{h/2} (\sigma_i \delta \epsilon_i) \, dz \, dx \, dy$$

$$= \int_\Omega \left( N_i \delta \epsilon_i^{(0)} + M_i \delta \epsilon_i^{(1)} + P_i \delta \epsilon_i^{(2)} + S_i \delta \epsilon_i^{(3)} \right) \, dz \, dx \, dy \tag{2.38}$$

$$\delta V = - \int_\Omega \left[ q(x,y) \delta u_i \right] \, dz \, dx \, dy \tag{2.39}$$

$$\delta K = \int_\Omega \int_{-h/2}^{h/2} \rho \left( \ddot{u}_j \delta \dot{u}_j \right) \, dz \, dx \, dy \tag{2.40}$$

For $i = 1, 2, \ldots, 6$, $j = 1, 2, 3$.

For the general third-order theory, we put Eq. (2.19) and Eq. (2.22) for kinematic relations into the expressions for $\delta U$, $\delta V$, and $\delta K$. Using all of them in the Hamilton's principle, integrating the resulting equation by parts and collecting the relevant components, we get the following equations of motion of the general third-order theory (GTOT),

$$\delta u : \quad \frac{\partial N_1}{\partial x} + \frac{\partial N_6}{\partial y} = I_1 \dddot{u} + I_2 \dddot{\phi}_1 + I_3 \dddot{\psi}_1 + I_4 \dddot{\theta}_1$$

$$\delta v : \quad \frac{\partial N_6}{\partial x} + \frac{\partial N_2}{\partial y} = I_1 \dddot{v} + I_2 \dddot{\phi}_2 + I_3 \dddot{\psi}_2 + I_4 \dddot{\theta}_2$$

$$\delta w : \quad \frac{\partial N_2}{\partial x} + \frac{\partial N_4}{\partial y} + q = I_1 \dddot{w} + I_2 \dddot{\psi}_3 + I_3 \dddot{\theta}_3$$

$$\delta \phi_1 : \quad \frac{\partial M_1}{\partial x} + \frac{\partial M_6}{\partial y} = N_3 = I_2 \dddot{\phi}_1 + I_3 \dddot{\psi}_1 + I_4 \dddot{\psi}_1 + I_5 \dddot{\theta}_1$$

$$\delta \phi_2 : \quad \frac{\partial M_6}{\partial x} + \frac{\partial M_2}{\partial y} = N_4 = I_2 \dddot{\phi}_2 + I_3 \dddot{\psi}_2 + I_4 \dddot{\psi}_2 + I_5 \dddot{\theta}_2$$

$$\delta \psi_1 : \quad \frac{\partial P_1}{\partial x} + \frac{\partial P_6}{\partial y} - 2M_5 = I_3 \dddot{\psi}_1 + I_4 \dddot{\theta}_1 + I_5 \dddot{\psi}_1 + I_6 \dddot{\theta}_1$$

$$\delta \psi_2 : \quad \frac{\partial P_6}{\partial x} + \frac{\partial P_2}{\partial y} - 2M_4 = I_3 \dddot{\psi}_2 + I_4 \dddot{\theta}_2 + I_5 \dddot{\psi}_2 + I_6 \dddot{\theta}_2$$

22
\[ \delta \psi_3 : \quad \frac{\partial M_5}{\partial x} + \frac{\partial M_4}{\partial y} - N_3 + q_1 = I_2 \ddot{w} + I_3 \ddot{\psi}_3 + I_4 \ddot{\theta}_3 \]
\[ \delta \theta_1 : \quad \frac{\partial S_1}{\partial x} + \frac{\partial S_6}{\partial y} - 3 P_5 = I_4 \ddot{u} + I_5 \ddot{\phi}_1 + I_6 \ddot{\psi}_1 + I_7 \ddot{\theta}_1 \]
\[ \delta \theta_2 : \quad \frac{\partial S_6}{\partial x} + \frac{\partial S_2}{\partial y} - 3 P_4 = I_4 \ddot{\psi}_2 + I_5 \ddot{\phi}_2 + I_6 \ddot{\psi}_2 + I_7 \ddot{\theta}_2 \]
\[ \delta \theta_3 : \quad \frac{\partial P_3}{\partial x} + \frac{\partial P_1}{\partial y} - 2 M_3 + q_2 = I_3 \ddot{w} + I_4 \ddot{\psi}_3 + I_5 \ddot{\theta}_3 \]

(2.41)

where \( q \) is the transverse load on the top surface and \( q_1 = qh/2, \quad q_2 = qh^2/4 \).

The primary variables (i.e. generalized displacements) and secondary variables (i.e. generalized forces) of the theory are

**primary variables:** \( u_n, u_s, w, \phi_n, \phi_s, \psi_n, \psi_s, \psi_z, \theta_n, \theta_s, \theta_z \)

**secondary variables:** \( N_n, N_{ns}, N_z, M_n, M_{ns}, P_n, P_{ns}, M_s, S_n, S_{ns}, P_z \)

(2.42)

Proceeding in a similar manner, we use Eq. (2.28) and Eq. (2.29) in Hamilton’s principle to get the equations of motion for the general third-order theory of Reddy (GTTR).
\[ \frac{\lambda}{2} \ddot{I}_4 \left( \frac{\partial \ddot{\phi}_1}{\partial x} + \frac{\partial \ddot{\phi}_2}{\partial y} \right) + \lambda \ddot{I}_2 \ddot{w} + \lambda^2 I_3 \dddot{\psi}_3 - \frac{\lambda^2}{4} I_5 \left( \frac{\partial^2 \ddot{\psi}_3}{\partial x^2} + \frac{\partial^2 \ddot{\psi}_3}{\partial y^2} \right) + \lambda \gamma I_4 \dddot{\zeta} \]

\[ \frac{\lambda \gamma}{6} I_6 \left( \frac{\partial^2 \ddot{\zeta}}{\partial x^2} + \frac{\partial^2 \ddot{\zeta}}{\partial y^2} \right) \]

\[ \delta \zeta = \frac{\gamma}{3} \left[ \left( \frac{\partial^2 S_1}{\partial x^2} + 2 \frac{\partial^2 S_2}{\partial x \partial y} + \frac{\partial^2 S_2}{\partial y^2} \right) - 6 M_3 \right] + q_2 = \frac{\gamma}{3} I_4 \left( \frac{\partial \ddot{u}}{\partial x} + \frac{\partial \ddot{v}}{\partial y} \right) + \frac{\gamma}{3} I_5 \left( \frac{\partial \ddot{\phi}_1}{\partial x} + \frac{\partial \ddot{\phi}_2}{\partial y} \right) + \gamma \ddot{I}_3 \dddot{w} + \lambda \gamma I_4 \dddot{\psi}_3 - \frac{\lambda \gamma}{6} I_6 \left( \frac{\partial^2 \ddot{\psi}_3}{\partial x^2} + \frac{\partial^2 \ddot{\psi}_3}{\partial y^2} \right) + \gamma^2 I_5 \dddot{\zeta} - \frac{\gamma^2}{9} I_7 \left( \frac{\partial^2 \ddot{\zeta}}{\partial x^2} + \frac{\partial^2 \ddot{\zeta}}{\partial y^2} \right) \quad (2.43) \]

where

\[ \ddot{I}_1 = I_1 - 2 \gamma c_1 I_3 + \gamma^2 c_2 I_5 \]
\[ \ddot{I}_2 = I_2 - \chi c_2 I_4 \]
\[ \ddot{I}_3 = I_3 - \gamma c_1 I_4 \]
\[ \ddot{I}_4 = I_4 - \gamma c_1 I_5 \]
\[ \ddot{I}_5 = I_5 - \chi c_2 I_7 \]
\[ c_1 = \frac{4}{h^2} \]
\[ c_2 = \frac{4}{3h^2} \]
\[ d_1 = c_1^2 \]
\[ d_2 = c_2^2 \]

(2.44)

The primary variables and secondary variables of the theory are

primary variables: \( u, u_n, w, \phi_n, \phi_3, \psi_3, \frac{\partial \psi_3}{\partial n}, \zeta, \frac{\partial \zeta}{\partial n} \)

secondary variables: \( N_n, N_n, N_2, M_n, M_n, Q_z, P_n, R_z, S_n \)

(2.45)

Note that the special third-order theory of Reddy (STTR) is a special case of GTR (set \( \psi_3 = 0 \), and \( \zeta = c_1 w \)). Since we shall find out analytical solutions for the general third order theory of Reddy and its particular derivative cases, we write the equilibrium equations of this theory (Eq. 2.43) in terms of the displacements. We shall consider the case of symmetric and anti-symmetric cross ply plates in which all the laminae are of the same material.
\[\delta u:\]

\[
A_{11} \frac{\partial^2 u}{\partial x^2} + A_{12} \frac{\partial^2 v}{\partial x \partial y} + A_{13} \lambda \frac{\partial \psi_3}{\partial x} + B_{11} \frac{\partial^2 \phi_1}{\partial x^2} + 2B_{13} \gamma \left( \frac{\partial \zeta}{\partial x} - c_1 \frac{\partial w}{\partial x} \right) - \frac{1}{2} \lambda D_{11} \frac{\partial^3 \psi_3}{\partial x^3} - \frac{1}{2} \lambda D_{12} \frac{\partial^3 \psi_3}{\partial x \partial y^2} - \frac{1}{3} E_{11} \left( \chi c_1 \frac{\partial^2 \phi_1}{\partial x^2} + \gamma \frac{\partial^3 \zeta}{\partial x^3} \right) + A_{66} \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} \right) - \lambda D_{66} \frac{\partial^3 \psi_3}{\partial x \partial y^2} = I_1 \ddot{u} - \frac{1}{2} \lambda I_3 \frac{\partial \psi_3}{\partial x} \quad (2.46)
\]

\[\delta v:\]

\[
A_{66} \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} \right) - \lambda D_{66} \frac{\partial^3 \psi_3}{\partial x \partial y^2} + A_{12} \frac{\partial^2 u}{\partial x \partial y} + A_{22} \frac{\partial^2 v}{\partial y^2} + \lambda A_{23} \frac{\partial \psi_3}{\partial y} + B_{22} \frac{\partial^2 \phi_2}{\partial y^2} + 2B_{23} \gamma \left( \frac{\partial \zeta}{\partial y} - c_1 \frac{\partial w}{\partial y} \right) \frac{1}{2} \lambda D_{12} \frac{\partial^3 \psi_3}{\partial x^2 \partial y} - \frac{1}{2} \lambda D_{22} \frac{\partial^3 \psi_3}{\partial y^3} - \frac{1}{3} E_{22} \left( \chi c_1 \frac{\partial^2 \phi_2}{\partial y^2} + \gamma \frac{\partial^3 \zeta}{\partial y^3} \right) = I_1 \ddot{v} - \frac{1}{2} \lambda I_3 \frac{\partial \psi_3}{\partial y} \quad (2.47)
\]

\[\delta w:\]

\[
A_{55} \left( \frac{\partial \phi_1}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) + A_{44} \left( \frac{\partial \phi_2}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) - c_1 D_{55} \left( \chi \frac{\partial \phi_1}{\partial x} + \gamma \frac{\partial^2 w}{\partial x^2} \right) - c_1 D_{44} \left( \chi \frac{\partial \phi_2}{\partial y} + \gamma \frac{\partial^2 w}{\partial y^2} \right) - \gamma c_1 D_{55} \left( \frac{\partial \phi_1}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) - \gamma c_1 D_{44} \left( \frac{\partial \phi_2}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) + \gamma c_2 F_{55} \left( \chi \frac{\partial \phi_1}{\partial x} + \gamma \frac{\partial^2 w}{\partial x^2} \right) + \gamma c_2 F_{44} \left( \chi \frac{\partial \phi_2}{\partial y} + \gamma \frac{\partial^2 w}{\partial y^2} \right) + 2 \gamma c_1 \left[ B_{13} \frac{\partial u}{\partial x} \right] + B_{23} \frac{\partial v}{\partial y} + \lambda B_{33} \psi_3 + D_{13} \frac{\partial \phi_1}{\partial x} + D_{23} \frac{\partial \phi_2}{\partial y} + 2 \gamma D_{33} (\zeta - c_1 w) - \frac{1}{2} \lambda E_{13} \frac{\partial^2 \psi_3}{\partial x^2} - \frac{1}{2} \lambda E_{23} \frac{\partial^2 \psi_3}{\partial y^2} - \frac{1}{3} F_{13} \left( \chi c_1 \frac{\partial \phi_1}{\partial x} + \gamma \frac{\partial^2 \zeta}{\partial x^2} \right) - \frac{1}{3} F_{23} \left( \chi c_1 \frac{\partial \phi_2}{\partial x} + \gamma \frac{\partial^2 \zeta}{\partial y^2} \right) \right] + q = I_1 \ddot{w} + \gamma I_3 \dot{\zeta} \quad (2.48)
\]
\[ \delta \phi_1 : \]
\[
\frac{1}{2} \lambda^2 E_{11} \frac{\partial^3 \psi_3}{\partial x^3} - \frac{1}{3} F_{11} \left( \chi c_1^2 \frac{\partial^2 \phi_1}{\partial x^2} + \gamma \frac{\partial^2 \zeta}{\partial x^2} \right) - \frac{1}{3} F_{12} \left( \chi c_1^2 \frac{\partial^2 \phi_2}{\partial x^2} + \gamma \frac{\partial^2 \phi_2}{\partial x^2} \right) - \frac{2}{3} \gamma F_{66} \frac{\partial^3 \zeta}{\partial x \partial y^2}
\]
\[
+ D_{66} \left( \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial x^2} \right) - \chi d_1 F_{66} \left( \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial x^2} \right) - \frac{2}{3} \gamma F_{66} \frac{\partial^3 \zeta}{\partial x \partial y^2}
\]
\[
- \chi d_1 \left( \frac{E_{11} \frac{\partial^2 u}{\partial x^2} + \lambda E_{13} \frac{\partial^2 \phi_1}{\partial x^2} + F_{11} \frac{\partial^2 \phi_1}{\partial x^2} + F_{12} \frac{\partial^2 \phi_2}{\partial x^2} + 2 \gamma F_{13} \left( \frac{\partial \zeta}{\partial x} - c_1 \frac{\partial w}{\partial x} \right) \right)
\]
\[
- \frac{1}{2} \lambda G_{11} \frac{\partial^3 \psi_3}{\partial y^3} - \frac{1}{3} H_{11} \left( \chi c_1^2 \frac{\partial^2 \phi_1}{\partial x^2} + \gamma \frac{\partial^2 \zeta}{\partial x^2} \right) - \frac{1}{3} H_{12} \left( \chi c_1^2 \frac{\partial^2 \phi_2}{\partial x^2} + \gamma \frac{\partial^2 \phi_2}{\partial x^2} \right) - \frac{2}{3} \gamma H_{66} \frac{\partial^3 \zeta}{\partial x \partial y^2}
\]
\[
- c_1 \frac{\partial^2 u}{\partial x^2} + \lambda c_1 \frac{\partial^2 \phi_1}{\partial x^2} + F_{11} \frac{\partial^2 \phi_1}{\partial x^2} + F_{12} \frac{\partial^2 \phi_2}{\partial x^2} + 2 \gamma F_{13} \left( \frac{\partial \zeta}{\partial x} - c_1 \frac{\partial w}{\partial x} \right) \right)
\]
\[
= \tilde{I}_3 \phi_1 - \frac{1}{3} \gamma I_5 \frac{\partial \zeta}{\partial x} \quad (2.49)
\]

\[ \delta \phi_2 : \]
\[
+ D_{66} \left( \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial x^2} \right) - \chi d_1 F_{66} \left( \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial x^2} \right) - \frac{2}{3} \gamma F_{66} \frac{\partial^3 \zeta}{\partial x^2 \partial y^2}
\]
\[
B_{22} \frac{\partial^2 v}{\partial y^2} + \lambda D_{23} \frac{\partial \psi_3}{\partial y} + D_{12} \frac{\partial^2 \phi_1}{\partial x \partial y} + D_{22} \frac{\partial^2 \phi_2}{\partial x \partial y} + 2 \gamma D_{23} \left( \frac{\partial \zeta}{\partial y} - c_1 \frac{\partial w}{\partial y} \right)
\]
\[
- \frac{1}{2} \lambda E_{22} \frac{\partial^3 \psi_3}{\partial y^3} - \frac{1}{3} F_{22} \left( \chi c_1^2 \frac{\partial^2 \phi_1}{\partial x \partial y} + \gamma \frac{\partial^2 \zeta}{\partial x \partial y} \right) - \frac{1}{3} F_{23} \left( \chi c_1^2 \frac{\partial^2 \phi_2}{\partial x \partial y} + \gamma \frac{\partial^2 \phi_2}{\partial x \partial y} \right) - \frac{2}{3} \gamma H_{66} \frac{\partial^3 \zeta}{\partial x \partial y^2}
\]
\[
- \chi d_1 \left[ F_{66} \left( \frac{\partial^2 \phi_1}{\partial x \partial y} + \frac{\partial^2 \phi_2}{\partial x \partial y} \right) - \chi d_1 H_{66} \left( \frac{\partial^2 \phi_1}{\partial x \partial y} + \frac{\partial^2 \phi_2}{\partial x \partial y} \right) - \frac{2}{3} \gamma H_{66} \frac{\partial^3 \zeta}{\partial x \partial y^2} \right]
\]
\[
E_{22} \frac{\partial^2 v}{\partial y^2} + \lambda E_{23} \frac{\partial \psi_3}{\partial y} + F_{12} \frac{\partial^2 \phi_1}{\partial x \partial y} + F_{22} \frac{\partial^2 \phi_2}{\partial x \partial y} + 2 \gamma F_{23} \left( \frac{\partial \zeta}{\partial y} - c_1 \frac{\partial w}{\partial y} \right)
\]
\[
- \frac{1}{2} \lambda G_{22} \frac{\partial^3 \psi_3}{\partial y^3} - \frac{1}{3} H_{22} \left( \chi c_1^2 \frac{\partial^2 \phi_1}{\partial x \partial y} + \gamma \frac{\partial^2 \zeta}{\partial x \partial y} \right) - \frac{1}{3} H_{23} \left( \chi c_1^2 \frac{\partial^2 \phi_2}{\partial x \partial y} + \gamma \frac{\partial^2 \phi_2}{\partial x \partial y} \right) \right)
\]
\[
- A_{44} \left( \phi_2 + \frac{\partial w}{\partial y} \right) + c_1 D_{44} \left( \phi_2 + \gamma \frac{\partial w}{\partial y} \right) + \chi c_1 D_{44} \left( \phi_2 + \frac{\partial w}{\partial y} \right)
\]
\[
= \tilde{I}_3 \phi_2 - \frac{1}{3} \gamma I_5 \frac{\partial \zeta}{\partial y} \quad (2.50)
\]
\[ \delta \psi_3 : \]
\[ \frac{1}{2} \lambda \left[ D_{11} \frac{\partial^3 u}{\partial x^3} + D_{12} \frac{\partial^3 v}{\partial x^2 \partial y} + \lambda D_{13} \frac{\partial^2 \psi_3}{\partial x^3} + E_{11} \frac{\partial^3 \phi_1}{\partial x^3} + 2 \gamma E_{13} \left( \frac{\partial^2 \zeta}{\partial x^2} - c_1 \frac{\partial^2 w}{\partial x^2} \right) \right] \\
- \frac{1}{2} \lambda F_{11} \frac{\partial^4 \psi_3}{\partial x^4} - \frac{1}{2} \lambda F_{12} \frac{\partial^4 \psi_3}{\partial x^2 \partial y^2} - \frac{1}{3} G_{11} \left( \chi c_1 \frac{\partial^3 \phi_1}{\partial x^3} + \gamma \frac{\partial^4 \zeta}{\partial x^4} \right) \\
+ 2 D_{66} \frac{\partial^3 u}{\partial x \partial y^2} + D_{66} \frac{\partial^3 v}{\partial x^2 \partial y} + \frac{1}{2} \lambda F_{12} \frac{\partial^4 \psi_3}{\partial x^2 \partial y^2} = \frac{1}{2} \lambda F_{12} \frac{\partial^4 \psi_3}{\partial y^4} - \frac{1}{2} \lambda F_{22} \frac{\partial^4 \psi_3}{\partial y^4} \\
+ \frac{1}{3} G_{22} \left( \chi c_1 \frac{\partial^3 \phi_2}{\partial y^3} + \gamma \frac{\partial^4 \zeta}{\partial y^4} \right) - \lambda \left[ A_{13} \frac{\partial \phi_1}{\partial x} + A_{23} \frac{\partial \phi_2}{\partial y} + \lambda A_{33} \psi_3 + B_{13} \frac{\partial \phi_1}{\partial x} \\
+ B_{23} \frac{\partial \phi_2}{\partial y} + 2 \gamma B_{33} (\zeta - c_1 w) - \frac{1}{2} \lambda D_{13} \frac{\partial^2 \psi_3}{\partial x^2} - \frac{1}{2} \lambda D_{23} \frac{\partial^2 \psi_3}{\partial y^2} \\
- \frac{1}{3} \left( c_1 \frac{\partial \psi_3}{\partial x} + \gamma \frac{\partial^2 \zeta}{\partial y^2} \right) - \frac{1}{3} E_{23} \left( \chi c_1 \frac{\partial \phi_2}{\partial x} + \gamma \frac{\partial^2 \zeta}{\partial y^2} \right) \right] = \frac{1}{2} \lambda I_3 \left( \frac{\partial^2 \psi_3}{\partial x^2} + \frac{\partial^2 \psi_3}{\partial y^2} \right) \] (2.51)

\[ \delta \zeta : \]
\[ \frac{1}{3} \gamma \left[ E_{11} \frac{\partial^3 u}{\partial x^3} + \lambda E_{13} \frac{\partial^2 \psi_3}{\partial x^3} + F_{11} \frac{\partial^3 \phi_1}{\partial x^3} + F_{12} \frac{\partial^3 \phi_2}{\partial x^2 \partial y} + 2 \gamma F_{13} \left( \frac{\partial^2 \zeta}{\partial x^2} - c_1 \frac{\partial^2 w}{\partial x^2} \right) \right] \\
- \frac{1}{2} \lambda G_{11} \frac{\partial^4 \psi_3}{\partial x^4} - \frac{1}{3} H_{11} \left( \chi c_1 \frac{\partial^3 \phi_1}{\partial x^3} + \gamma \frac{\partial^4 \zeta}{\partial x^4} \right) - \frac{1}{3} H_{12} \left( \chi c_1 \frac{\partial^3 \phi_2}{\partial x^2 \partial y} + \gamma \frac{\partial^4 \zeta}{\partial x^2 \partial y^2} \right) \\
+ 2 F_{66} \left( \frac{\partial^3 \phi_1}{\partial x \partial y^2} + \frac{\partial^3 \phi_2}{\partial x^2 \partial y} \right) - 2 \chi d_1 H_{60} \left( \frac{\partial^3 \phi_1}{\partial x \partial y^2} + \frac{\partial^3 \phi_2}{\partial x^2 \partial y} - \frac{4}{3} \gamma H_{66} \frac{\partial^4 \zeta}{\partial y^4} \right) \\
E_{23} \frac{\partial^2 v}{\partial y^3} + \lambda E_{23} \frac{\partial^2 \psi_3}{\partial y^3} + F_{23} \frac{\partial^3 \phi_1}{\partial x \partial y^2} + F_{22} \frac{\partial^3 \phi_2}{\partial y^3} + 2 \gamma F_{23} \left( \frac{\partial^2 \zeta}{\partial y^2} - c_1 \frac{\partial^2 w}{\partial y^2} \right) \\
- \frac{1}{2} \lambda G_{22} \frac{\partial^4 \psi_3}{\partial y^4} - \frac{1}{3} H_{12} \left( \chi c_1 \frac{\partial^3 \phi_1}{\partial x \partial y^2} + \gamma \frac{\partial^4 \zeta}{\partial x \partial y^2} \right) - \frac{1}{3} H_{22} \left( \chi c_1 \frac{\partial^3 \phi_2}{\partial y^3} + \gamma \frac{\partial^4 \zeta}{\partial y^3} \right) \right] \\
- 2 \gamma \left[ B_{13} \frac{\partial \phi_1}{\partial x} + B_{23} \frac{\partial \phi_2}{\partial y} + \lambda B_{33} \psi_3 + D_{13} \frac{\partial \phi_1}{\partial x} + D_{23} \frac{\partial \phi_2}{\partial y} + 2 \gamma D_{33} (\zeta - c_1 w) \right] \\
- \frac{1}{2} \lambda E_{13} \frac{\partial^2 \psi_3}{\partial x^2} - \frac{1}{2} \lambda E_{23} \frac{\partial^2 \psi_3}{\partial y^2} - \frac{1}{3} F_{13} \left( \chi c_1 \frac{\partial \phi_1}{\partial x} + \gamma \frac{\partial^2 \zeta}{\partial x^2} \right) - \frac{1}{3} F_{23} \left( \chi c_1 \frac{\partial \phi_2}{\partial y} + \gamma \frac{\partial^2 \zeta}{\partial y^2} \right) \\
+ \gamma \frac{\partial^2 \zeta}{\partial y^2} \right] = \frac{1}{3} \gamma I_5 \left( \frac{\partial \phi_1}{\partial x} + \frac{\partial \phi_2}{\partial y} \right) + \gamma^2 I_5 \zeta - \frac{1}{9} \gamma^2 I_7 \left( \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right) \] (2.52)
Chapter 3

ANALYTICAL SOLUTIONS

3.1 Introduction

In this chapter, we find the exact solutions to the set of partial differential equations of motion enumerated in the previous chapter. Since it is almost impossible to find the exact analytical solutions for any arbitrary set of geometries, lamination schemes and boundary conditions, we restrict ourselves to specific cases of the above parameters. For problems to be solved by the Navier method, we take a rectangular plate simply-supported on all four sides. For problems to be solved by the Lévy method, two edges of the rectangular plate are assumed to be on simple supports, while the remaining two edges can have any combination of free, clamped, or simply-supported boundary conditions (see Figure 3.1). A sufficiently wide variety of problems are considered, and their results can be used to validate other more general methods to solve similar problems, like the finite element method.

An exact solution to a problem satisfies the governing equations of the problem at every point of the domain, as well as satisfying the boundary conditions. Both the Navier and Lévy methods assume solutions in the form of an infinite series expansion. In both cases, however, the series can be truncated after a few terms to get a sufficiently accurate solution.

3.2 Navier Solutions

We consider the cases of symmetric and anti-symmetric cross-ply rectangular plates which are simply-supported on all sides. The dependent unknowns \( u, v, w, \phi_1, \phi_2, \psi_3, \zeta \) and the load term \( q \) (in the case of bending analysis) are expanded in a double
Figure 3.1: (a) Xavier solution (b) Lévy solution
trigonometric series in terms of unknown coefficients. The trigonometric functions of the series may be either sine or cosine functions depending upon the particular boundary conditions of the problem. The series expansions of the generalized displacements are then substituted into the governing equations of motion. This results in a set of simultaneous algebraic equations in the undetermined coefficients, which can be easily solved in the case of bending analysis. For free vibration analysis, the substitution leads to a generalized eigenvalue problem, which can be solved for the eigenvalues and eigenvectors.

### 3.2.1 Boundary Conditions

The Navier solutions are developed for the following set of simply-supported boundary conditions (for a rectangular plate of dimension $a$ by $b$).

\begin{align*}
u(x, 0) &= v(x, b) = 0, \quad N_2(x, 0) = N_2(x, b) = 0, \\
v(0, y) &= v(a, y) = 0, \quad N_1(0, y) = N_1(a, y) = 0, \\
w(x, 0) &= w(x, b) = w(0, y) = w(a, y) = 0, \\
\phi_1(x, 0) &= \phi_1(x, b) = 0, \quad M_2(x, 0) = M_2(x, b) = 0, \\
\phi_2(0, y) &= \phi_2(a, y) = 0, \quad M_1(0, y) = M_1(a, y) = 0, \\
\psi_3(x, 0) &= \psi_3(x, b) = \psi_3(0, y) = \psi_3(a, y) = 0, \\
\zeta(x, 0) &= \zeta(x, b) = \zeta(0, y) = \zeta(a, y) = 0.
\end{align*}

(3.1)

### 3.2.2 Assumed Displacements

The following displacement field satisfies the above boundary conditions.

\begin{align*}
u(x, y) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} \cos \alpha x \sin \beta y \\
v(x, y) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn} \sin \alpha x \cos \beta y \\
w(x, y) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin \alpha x \sin \beta y \\
\phi_1(x, y) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} X_{mn} \cos \alpha x \sin \beta y
\end{align*}
\[
\phi_2(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn} \sin \alpha x \cos \beta y \\
\psi_2(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Z^{(1)}_{mn} \sin \alpha x \sin \beta y \\
\zeta(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Z^{(2)}_{mn} \sin \alpha x \sin \beta y
\] (3.2)

where \( \alpha = m\pi/a \) and \( \beta = n\pi/b \).

### 3.2.3 Bending Analysis

The distributed transverse load \( q \) is expanded in a double trigonometric series.

\[
q(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Q_{mn} \sin \alpha x \sin \beta y
\] (3.3)

\( Q_{mn} \) can be evaluated for different types of loading conditions. For example

\[
Q_{mn} = \begin{cases} 
(16g_0)/(mn\pi^2) & \text{for uniformly distributed load (UL)} \\
(4P/ab)\sin(m\pi/2)\sin(n\pi/2) & \text{for point load at the center (PL)} 
\end{cases}
\] (3.4)

where \( m = 1, 3, 5, \ldots \) and \( n = 1, 3, 5, \ldots \).

Substituting Eq. (3.2) and Eq. (3.4) into the static equilibrium equations of Chapter 2, for each \( (m, n) \), we get a set of equations of the form

\[
[S_{ij} \{\Delta_i\}] = \{F_i\}
\] (3.5)

where

\[
\Delta_i = \begin{bmatrix} U_{mn} & V_{mn} & W_{mn} & X_{mn} & Y_{na} & Z^{(1)}_{mn} & Z^{(2)}_{mn} \end{bmatrix}^T
\] (3.6)

\[
F_i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & Q^{(1)}_{mn} & Q^{(2)}_{mn} \end{bmatrix}^T
\] (3.7)

\[
Q^{(1)}_{mn} = Q_{mn}(h/2) \\
Q^{(2)}_{mn} = Q_{mn}(h^2/4)
\] (3.8)

and the coefficients \( S_{ij} = S_{ji} \) are given in Appendix A.1.

Equation (3.5) is solved for the undetermined coefficients \((U_{mn}, V_{mn}, \ldots \text{etc.})\) for each \( (m, n) \). The actual displacements \((u, v, \ldots \text{etc.})\) are then obtained from Eq. (3.2). Note that the equations have been developed for the special comprehensive third order theory (GTTR). All the other theories (CLPT, FSDT, STTR) can be obtained from GTTR by assigning different values to the tracers.
3.2.4 Free Vibration Analysis

For free vibration analysis, we assume that the displacements are of the form

\[
\begin{pmatrix}
  u(x, y, t) \\
v(x, y, t) \\
w(x, y, t) \\
\phi_1(x, y, t) \\
\phi_2(x, y, t) \\
\psi_3(x, y, t) \\
\zeta(x, y, t)
\end{pmatrix} = e^{-i\omega t} \begin{pmatrix}
  u(x, y) \\
v(x, y) \\
w(x, y) \\
\phi_1(x, y) \\
\phi_2(x, y) \\
\psi_3(x, y) \\
\zeta(x, y)
\end{pmatrix} (3.9)
\]

where \( \omega \) is the frequency of natural vibration.

On substituting the above relations in the dynamic equilibrium equations of Chapter 2, we get

\[
([S_{ij}] - \omega^2_{mn} [M_{ij}]) \{\Delta_j\} = \{0\} (3.10)
\]

The coefficients \( M_{ij} = M_{ji} \) are given in Appendix A.2. The above equations can be solved for eigenvalues and eigenvectors for each \((m, n)\).

3.3 Lévy Solutions

In this case, we consider a rectangular plate which is simply-supported at \( y = 0 \) and \( y = b \). The other two sides \( (x = -a/2 \text{ and } x = a/2) \) can have a combination of free, clamped, and simply supported boundary conditions. The generalized displacements are expressed as a product of undetermined functions of \( x \) and known trigonometric functions of \( y \). The trigonometric functions (sine or cosine) are chosen such that the simply supported boundary conditions at \( y = 0 \) and \( y = b \) are identically satisfied.

3.3.1 Assumed Displacements

\[
v(x, y) = \sum_{m=1}^{\infty} U_m(x) \sin \beta y
\]
\[
v(x, y) = \sum_{n=1}^{\infty} V_n(x) \cos \beta y
\]
\[
w(x, y) = \sum_{m=1}^{\infty} W_m(x) \sin \beta y
\]
\[ \phi_1(x, y) = \sum_{m=1}^{\infty} X_m(x) \sin \beta y \]
\[ \phi_2(x, y) = \sum_{m=1}^{\infty} Y_m(x) \cos \beta y \]
\[ \psi_3(x, y) = \sum_{m=1}^{\infty} Z^{(1)}_m(x) \sin \beta y \]
\[ \zeta(x, y) = \sum_{m=1}^{\infty} Z^{(2)}_m(x) \sin \beta y \]

(3.11)

where \( \beta = m \pi / b \).

### 3.3.2 Solution Technique

Unlike the Navier method, for the Lévy method, the solution technique is similar for static and free vibration problems. The procedure will be described in the subsequent paragraphs. First, the distributed transverse load \( q \) is expanded in a manner similar to the displacements.

\[ q(x, y) = \sum_{m=1}^{\infty} Q_m(x) \sin \beta y \]

(3.12)

As before, \( Q_m \) can be evaluated for different types of loading conditions.

\[ Q_m = \begin{cases} 
(4q_0) / (m \pi) & \text{for uniformly distributed load (UL)} \\
(2P/b) \sin(m \pi / 2) & \text{for line load along the centerline } y = b/2 \text{ (LL)} 
\end{cases} \]

(3.13)

where \( m = 1, 3, 5, \ldots \).

Second, the time-dependent displacements are expressed in exactly the same manner as in the Navier method (see Eq. (3.9)). Of course, the generalized displacements \( u, v, \ldots \) etc. are defined differently in this case. Substitution of this displacement field into the dynamic equilibrium equations gives rise to a system of ordinary differential equations in \( x \) of the form

\[ e_1 U_m + e_2 U_m'' + e_3 V_m'' + e_4 W_m'' + e_5 X_m'' + e_6 Z^{(1)}_m'' + e_7 Z^{(1)}_m'' + e_8 Z^{(2)}_m'' + e_9 Z^{(2)}_m'' = 0 \]
\[ e_{10} U_m' + e_{11} V_m' + e_{12} V_m'' + e_{13} W_m' + e_{14} Y_m' + e_{15} Z^{(1)}_m' + e_{16} Z^{(1)}_m' + e_{17} Z^{(2)}_m' = 0 \]
\[ e_{18} U_m' + e_{19} V_m' + e_{20} W_m' + e_{21} W_m'' + e_{22} X_m' + e_{23} Y_m' + e_{24} Z_m(1) + e_{25} Z_m(1)' = 0 \]
\[
\begin{align*}
e_{26} + Z_m(2) + e_{27}Z_m(2)'' &= 0 \\
e_{28}U_m'' + e_{29}W_m' + e_{30}X_m + e_{31}X_m'' + e_{32}X_m' + e_{33}Z_m'(1)' + e_{34}Z_m'(1)'' + e_{35}Z_m'(2)' + e_{36}Z_m'(2)''' &= 0 \\
e_{37}V_m + e_{38}W_m + e_{39}X_m' + e_{40}Y_m + e_{41}Y_m'' + e_{42}Z_m'(1) + e_{43}Z_m'(2) + e_{44}Z_m'(2)'' &= 0 \\
e_{45}U_m' + e_{46}U_m'' + e_{47}V_m + e_{48}V_m'' + e_{49}W_m + e_{50}W_m'' + e_{51}X_m' + e_{52}X_m'' + e_{53}Y_m + e_{54}Z_m'(1) + e_{55}Z_m'(1)'' + e_{56}Z_m'(1)''' + e_{57}Z_m'(2) + e_{58}Z_m'(2)'' + e_{59}Z_m'(2)''' + Q_m &= 0 \\
e_{60}U_m' + e_{61}U_m'' + e_{62}V_m + e_{63}V_m'' + e_{64}W_m + e_{65}W_m'' + e_{66}X_m' + e_{67}X_m'' + e_{68}Y_m + e_{69}Z_m'(1) + e_{70}Z_m'(1)'' + e_{71}Z_m'(1)''' + e_{72}Z_m'(2) + e_{73}Z_m'(2)'' + e_{74}Z_m'(2)''' + Q_m &= 0 \\
\end{align*}
\] 

(3.14)

where

\[
Q_m^{(1)} = Q_m(h/2) \quad Q_m^{(2)} = Q_m(h^2/4)
\] 

(3.15)

and the coefficients \( e_i \) are given in Appendix B.1. The primes over the variables indicate differentiation with respect to \( x \).

The next step is to modify Eq. (3.14) to make it suitable for the state-space approach. We try to bring the highest powers of each variable to the left hand side and express it in terms of its lower powers as well as the lower powers of the other variables. This will convert the set of equations to a tractable form which can be solved by the state-space procedure. We introduce a new set of variables.
\[
\begin{pmatrix}
\Theta_1 \\
\Theta_2 \\
\Theta_3 \\
\Theta_4 \\
\Theta_5 \\
\Theta_6 \\
\Theta_7 \\
\Theta_8 \\
\Theta_9 \\
\Theta_{10} \\
\Theta_{11} \\
\Theta_{12} \\
\Theta_{13} \\
\Theta_{14} \\
\Theta_{15} \\
\Theta_{16} \\
\Theta_{17} \\
\Theta_{18}
\end{pmatrix} =
\begin{pmatrix}
U_m \\
U'_m \\
V_m \\
V'_m \\
W_m \\
W'_m \\
X_m \\
X'_m \\
Y_m \\
Y'_m \\
Z^{(1)}_m \\
Z^{(1)'}_m \\
Z^{(1)''}_m \\
Z^{(2)}_m \\
Z^{(2)'}_m \\
Z^{(2)''}_m
\end{pmatrix}
\] (3.16)

Equation 3.14 is written in the form as described in the previous paragraph.

\[
\{\Theta'\} = [A] \{\Theta\} + \{\Gamma\}
\] (3.17)

\{\Gamma\} is the load vector after manipulation. Obviously, for a free vibration problem, \{\Gamma\} = 0. The general structure of the matrix \([A]\) and the vector \{\Gamma\} are given in Appendix B.2.4. Their components are not written out explicitly since they are very long. However, the actual effort involved in manipulating Eq. (3.14) to bring it to the form of Eq. (3.17) is not too much. Note that the equations above have been derived for the most general case of the general third order theory (GTTR). The other theories, viz. CLPT, FSDT, and STTR, can be derived from GTTR quite easily. Also, the definition of \(\Theta\)’s change with the different theories as do the form of \([A]\) and \{\Gamma\}. \{\Theta\} is given below for the different cases and \([A]\) and \{\Gamma\} are given explicitly for CLPT, FSDT, and STTR in Appendix B.2.

CLPT:

\[
\begin{align*}
\Theta_1 &= U_m, & \Theta_2 &= U'_m, & \Theta_3 &= V_m, & \Theta_4 &= V'_m, \\
\Theta_5 &= W_m, & \Theta_6 &= W'_m, & \Theta_7 &= W''_m, & \Theta_8 &= W'''_m
\end{align*}
\] (3.18)
FSDT:
\[
\begin{align*}
\Theta_1 &= U_m, \quad \Theta_2 = U'_m, \quad \Theta_3 = V_m, \quad \Theta_4 = V'_m, \quad \Theta_5 = W_m, \\
\Theta_6 &= W'_m, \quad \Theta_7 = X_m, \quad \Theta_8 = X'_m, \quad \Theta_9 = Y_m, \quad \Theta_{10} = Y'_m
\end{align*}
\] (3.19)

STTR:
\[
\begin{align*}
\Theta_1 &= U_m, \quad \Theta_2 = U'_m, \quad \Theta_3 = V_m, \quad \Theta_4 = V'_m, \quad \Theta_5 = W_m, \quad \Theta_6 = W'_m, \\
\Theta_7 &= W''_m, \quad \Theta_8 = W'''_m, \quad \Theta_9 = X_m, \quad \Theta_{10} = X'_m, \quad \Theta_{11} = Y_m, \quad \Theta_{12} = Y'_m
\end{align*}
\] (3.20)

State Space Solution and Imposition of Boundary Conditions

The solution to the state space Eq. (3.17) is given by [145, 146]
\[
\{\Theta(x)\} = \left[e^{Ax}\right] \{K\} + \left[e^{Ax}\right] \int \left[e^{-A\xi}\right] \{\Gamma\} d\xi
\] (3.21)

where \(\{K\}\) is a constant vector which is to be determined from the boundary conditions of the problem and \(e^{Ax}\) is given by
\[
\left[e^{Ax}\right] = [T] \begin{bmatrix} e^{\lambda_1 x} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 x} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n x} \end{bmatrix} [T]^{-1}
\] (3.22)

\(\lambda_i\) denotes the distinct eigenvalues of \([A]\), and \([T]\) is the matrix of distinct eigenvectors of \([A]\). The value of \(n\) is 8 for CLPT, 10 for FSDT, 12 for STTR, and 18 for GTTR. The second term on the right hand side of Eq. (3.21) is present only for bending problems.

Equation (3.21) is then substituted into the boundary conditions associated with the two edges \(x = \pm a/2\). In the case of bending analysis, this results in a system of non-homogeneous equations which can be solved for \(\{K\}\).
\[
[P] \{K\} + \{R\} = \{0\}
\] (3.23)

The \(K_i\)s can then be put back into Eq. (3.21) to get \(U_m, V_m, W_m, \ldots\) etc.

In the case of free vibration analysis, we get a system of homogeneous equations (\(\{R\}\) is not present). For a non-trivial solution to the homogeneous equations, the determinant of \([P]\) should be zero.
\[
|P| = 0
\] (3.24)
Equation (3.24) gives the natural frequencies for each value of $m$.

**Boundary conditions**

The boundary conditions for simply supported (S), clamped (C), and free (F) at the edges $x = \pm a/2$ for the different theories are described below:

**CLPT**

\[
\begin{align*}
S & : \quad v = w = N_1 = M_1 = 0 \\
C & : \quad u = v = w = \frac{\partial w}{\partial x} = 0 \\
F & : \quad N_1 = N_6 = M_1 = \left(\frac{\partial M_1}{\partial x} + 2\frac{\partial M_6}{\partial y}\right) = 0
\end{align*}
\]

(3.25)

**FSDT**

\[
\begin{align*}
S & : \quad v = w = \phi_2 = N_1 = M_1 = 0 \\
C & : \quad u = v = w = \phi_1 = \phi_2 = 0 \\
F & : \quad N_1 = N_6 = N_5 = M_1 = M_6 = 0
\end{align*}
\]

(3.26)

**STTR**

\[
\begin{align*}
S & : \quad v = w = \phi_2 = N_1 = M_1 = S_1 = 0 \\
C & : \quad u = v = w = \frac{\partial w}{\partial x} = \phi_1 = \phi_2 = 0 \\
F & : \quad N_1 = N_6 = \left[(N_5 - c_1 P_5) + d_1 \left(\frac{\partial S_1}{\partial x} + \frac{\partial S_6}{\partial y}\right)\right] = M_1 = S_1 = (M_6 - d_1 S_6) = 0
\end{align*}
\]

(3.27)

**GTTR**

\[
\begin{align*}
S & : \quad v = w = \phi_2 = \psi_3 = \zeta = N_1 = M_1 = P_1 = S_1 = 0 \\
C & : \quad u = v = w = \phi_1 = \phi_2 = \psi_3 = \frac{\partial \psi_3}{\partial x} = \zeta = \frac{\partial \zeta}{\partial x} = 0 \\
F1 & : \quad N_1 = N_6 = (N_5 - c_1 P_5) = M_1 = S_1 = (M_6 - d_1 S_6) = P_1 = \left(\frac{\partial P_1}{\partial x} + \frac{\partial P_6}{\partial y}\right)
\end{align*}
\]
\[
= \left( \frac{\partial S_1}{\partial x} + \frac{\partial S_6}{\partial y} \right) = 0
\]  

(3.28)

Computational Aspects

Some computational problems encountered while implementing the Lévy method and their suggested solutions are discussed here.

There are a multitude of available methods for calculating the matrix exponential \([e^{Ax}]\). The method which immediately attracts attention is one in which it is evaluated in the form of an infinite series [147].

\[
[e^{Ax}] = \begin{bmatrix} 1 \\ A \\ \frac{x^2}{2!} [A]^2 \\ \frac{x^3}{3!} [A]^3 \\ \cdots \end{bmatrix}
\]  

(3.29)

Although this procedure is easy to program and does not involve complex numbers, the series will diverge if even one of the eigenvalues of \([A]\) fall on the right half of the complex \(z\)-plane.

The closed form expression for the matrix exponential follows from the Cayley-Hamilton Theorem [146]. Due to the sparse nature of the \([A]\) matrix, the matrix \([P]\) in Eq. (3.23) obtained after imposing the boundary conditions is often ill-conditioned and computer overflow or underflow is a common occurrence. This problem is overcome in the following manner [148]. Equation (3.21) is rewritten in the form

\[
\{\Theta(x)\} = [T] \begin{bmatrix} e^{\lambda_1 x} & \cdots & 0 \\ 0 & \ddots & e^{\lambda_n x} \end{bmatrix} [T]^{-1} \{K\} + [e^{Ax}] \int [e^{-A\xi}] \{\Gamma\} d\xi
\]

or

\[
\{\Theta(x)\} = [T] \begin{bmatrix} e^{\lambda_1 x} & \cdots & 0 \\ 0 & \ddots & e^{\lambda_n x} \end{bmatrix} \{\tilde{K}\} + [e^{Ax}] \int [e^{-A\xi}] \{\Gamma\} d\xi
\]  

(3.30)

where \(\{\tilde{K}\} = [T]^{-1} \{K\}\). In effect, we will be solving for \(\{\tilde{K}\}\) instead of \(\{K\}\). After imposing the boundary conditions, we arrive at an equation similar to Eq. (3.23).

\[
[Q] \{\tilde{K}\} + \{R\} = \{0\}
\]  

(3.31)

But, in this case, the matrix \([Q]\) is not ill-conditioned and can be easily handled. In the bending case, after solving for \(\{\tilde{K}\}\), \(\{K\}\) can be obtained by

\[
\{K\} = [T] \{\tilde{K}\}
\]  

(3.32)

38
For free vibration problems, \(|P| = 0\) (Eq. (3.24)) implies
\[
| [Q] [T]^{-1} | = 0
\]
or
\[
| Q | / | T | = 0
\]  \hspace{1cm} (3.33)

However, it should be kept in mind that while, in the previous case, \([P]\) and \([K]\) were both real matrices, after the transformation, \([Q]\) and \([\tilde{K}]\) will, in general, both be complex.

A few more mathematical "tricks" were employed to facilitate computation. From simple integral calculus it is known that
\[
\int_0^T e^{a(T-t)} dt = \int_0^T e^{at} dt
\]
Extending the above result to the case where \(a\) is a matrix, Eq. (3.21) can be simplified to
\[
\{ \Theta(x) \} = [e^{Ax}] \{ K \} + \int [e^{A\xi}] \{ \Gamma \} d\xi
\]  \hspace{1cm} (3.34)

In another scenario, difficulties might arise during the computation of the eigenvalues and eigenvectors of \([A]\) since the diagonal elements of \([A]\) are always all zero. This can be circumvented by adding a constant non-zero number to the diagonal elements of \([A]\). The eigenvalues of \([A]\) can be obtained by subtracting the same number from the eigenvalues of the new matrix. The eigenvectors are the same in both cases.

### 3.3.3 Closure

In closing, we make a few remarks about the Lévy method. Although the Lévy method is more general than the Navier method in its range of applications, it has two limitations.

1. Two opposite edges of the plate have to be simply supported. In the present study, the simple supports are assumed to be at the edges \(y = 0\) and \(y = b\).

2. The shape of the loading function should be the same for all sections parallel to the other two edges. In our case, the loading function should have the same shape parallel to the \(y\)-axis.

But, the convergence in the Lévy method is much faster compared to the Navier solutions. In almost all the cases, very accurate results are obtained in the Lévy method by considering only the first few terms.
Chapter 4

FINITE ELEMENT SOLUTIONS

4.1 Introduction

Though the analytical solutions presented in the previous chapter are useful for the purpose of comparison, their scope is limited to particular geometries, loads, and boundary conditions. For a more general treatment of complex problems, one has to turn to an approximate numerical method. Although its development has been fairly recent, the finite element method has quickly established itself as the premier numerical method for a wide class of problems from solid and fluid mechanics to heat transfer and electromagnetics. In this chapter, we develop the finite element models of the different laminated plate theories.

4.2 Weak Form

The first step is the development of the weak form of the equations of motion. First we consider the equilibrium equations of the general third order theory (Eq. 2.41). For each of the equations, we move all the terms to the right hand side, leaving only zero on the left hand side. In the next step, we multiply the entire equation by a weight function \( \varphi \), and integrate it over the domain \( \Omega^e \) of a typical element. For the \( i \)-th equation, we have

\[
0 = \int_{\Omega^e} \varphi_i \left[ \text{RHS of modified equation (i)} \right] dx dy
\]  

(4.1)

The above statement is called the \textit{weighted-integral} or \textit{weighted-residual} statement equivalent to the original equation of motion [149]. The displacements \( \Delta_i \) in the equation are approximated by different order polynomials. Equation (4.1) implies
that the error in the differential equations (Eq. 2.41) due to the approximation in the solution is zero in the weighted-integral sense.

Next, Eq. (4.1) is integrated by parts. This transfers the derivative from the displacements to the weight functions, and results in weaker continuity requirements on the \( \Delta \)'s. The weighted-integral statement so obtained is called the weak form [149]. The weak form over the domain of a typical element for the general third order theory is given by

\[
0 = \int_{\Omega_e} \left[ \varphi_{1,x} N_1 + \varphi_{1,y} N_0 + \varphi_1 \left\{ I_1 \ddot{u} + I_2 \ddot{\phi}_1 + I_3 \ddot{v}_1 + I_4 \ddot{\theta}_1 \right\} \right] dxdy - \oint_{\Gamma_e} \varphi_1 N_n ds
\]

\[
0 = \int_{\Omega_e} \left[ \varphi_{2,x} N_6 + \varphi_{2,y} N_2 + \varphi_2 \left\{ I_1 \ddot{v} + I_2 \ddot{\phi}_2 + I_3 \ddot{v}_2 + I_4 \ddot{\theta}_2 \right\} \right] dxdy - \oint_{\Gamma_e} \varphi_2 N_n ds
\]

\[
0 = \int_{\Omega_e} \left[ \varphi_{3,x} N_5 + \varphi_{3,y} N_4 + \varphi_3 \left\{ -q + I_1 \ddot{w} + I_2 \ddot{\psi}_3 + I_3 \ddot{\theta}_3 \right\} \right] dxdy - \oint_{\Gamma_e} \varphi_3 N_n ds
\]

\[
0 = \int_{\Omega_e} \left[ \varphi_{4,x} M_1 + \varphi_{4,y} M_6 + \varphi_4 \left\{ N_5 + I_2 \ddot{u} + I_3 \ddot{\phi}_1 + I_4 \ddot{v}_1 + I_5 \ddot{\theta}_1 \right\} \right] dxdy - \oint_{\Gamma_e} \varphi_4 M_n ds
\]

\[
0 = \int_{\Omega_e} \left[ \varphi_{5,x} M_6 + \varphi_{5,y} M_2 + \varphi_5 \left\{ N_4 + I_2 \ddot{v} + I_3 \ddot{\phi}_2 + I_4 \ddot{v}_2 + I_5 \ddot{\theta}_2 \right\} \right] dxdy - \oint_{\Gamma_e} \varphi_5 M_n ds
\]

\[
0 = \int_{\Omega_e} \left[ \varphi_{6,x} P_1 + \varphi_{6,y} P_6 + \varphi_6 \left\{ 2M_5 + I_3 \ddot{u} + I_4 \ddot{\phi}_1 + I_5 \ddot{v}_1 + I_6 \ddot{\theta}_1 \right\} \right] dxdy - \oint_{\Gamma_e} \varphi_6 P_n ds
\]

\[
0 = \int_{\Omega_e} \left[ \varphi_{7,x} P_6 + \varphi_{7,y} P_2 + \varphi_7 \left\{ 2M_4 + I_3 \ddot{v} + I_4 \ddot{\phi}_2 + I_5 \ddot{v}_2 + I_6 \ddot{\theta}_2 \right\} \right] dxdy - \oint_{\Gamma_e} \varphi_7 P_n ds
\]

\[
0 = \int_{\Omega_e} \left[ \varphi_{8,x} M_5 + \varphi_{8,y} M_4 + \varphi_8 \left\{ -q_1 + I_2 \ddot{w} + I_3 \ddot{v}_3 + I_4 \ddot{\theta}_3 \right\} \right] dxdy - \oint_{\Gamma_e} \varphi_8 M_n ds
\]

\[
0 = \int_{\Omega_e} \left[ \varphi_{9,x} S_1 + \varphi_{9,y} S_6 + \varphi_9 \left\{ 3P_5 + I_4 \ddot{u} + I_5 \ddot{\phi}_1 + I_6 \ddot{v}_1 + I_7 \ddot{\theta}_1 \right\} \right] dxdy - \oint_{\Gamma_e} \varphi_9 S_n ds
\]

\[
0 = \int_{\Omega_e} \left[ \varphi_{10,x} S_6 + \varphi_{10,y} S_2 + \varphi_{10} \left\{ 3P_4 + I_4 \ddot{v} + I_5 \ddot{\phi}_2 + I_6 \ddot{v}_2 + I_7 \ddot{\theta}_2 \right\} \right] dxdy - \oint_{\Gamma_e} \varphi_{10} S_n ds
\]

\[
0 = \int_{\Omega_e} \left[ \varphi_{11,x} P_5 + \varphi_{11,y} P_4 + \varphi_{11} \left\{ -q_2 + I_3 \ddot{w} + I_4 \ddot{\psi}_3 + I_5 \ddot{\theta}_3 \right\} \right] dxdy - \oint_{\Gamma_e} \varphi_{11} P_n ds
\]

\[
(4.2)
\]
where $\Gamma_i$ is the boundary of the typical element, $(\cdot)_x$ denotes differentiation with respect to $x$, and $(\cdot)_y$ denotes differentiation with respect to $y$.

The weak form for the general third order theory of Reddy (GTTR) can be constructed in a similar manner from Eq. (2.43). In this case, since two of the seven equations are of the second order, integration by parts is performed twice on those two equations.

\[
0 = \int_{\Omega^r} \left[ \varphi_{1,x} N_1 + \varphi_{1,y} N_6 + \varphi_1 \left\{ I_1 \ddot{u} + I_2 \ddot{\phi}_1 - \frac{1}{2} \lambda I_3 \ddot{\psi}_{3,x} - \frac{1}{3} \gamma I_4 \ddot{\zeta}_{x} \right\} \right] dxdy \\
- \oint_{\Gamma^r} \varphi_1 N_x ds \\
0 = \int_{\Omega^r} \left[ \varphi_{2,x} N_6 + \varphi_{2,y} N_2 + \varphi_2 \left\{ I_1 \ddot{u} + I_2 \ddot{\phi}_2 - \frac{1}{2} \lambda I_3 \ddot{\psi}_{3,y} - \frac{1}{3} \gamma I_4 \ddot{\zeta}_{y} \right\} \right] dxdy \\
- \oint_{\Gamma^r} \varphi_2 N_y ds \\
0 = \int_{\Omega^r} \left[ (\varphi_{3,x} N_5 + \varphi_{3,y} N_4) - \gamma c_1 (\varphi_{3,x} P_5 + \varphi_{3,y} P_4) - \varphi_1 (2 \gamma c_1 M_3 + q) \\
+ \varphi_3 \left\{ \ddot{I}_1 \ddot{u} + \lambda \ddot{I}_2 \ddot{\psi}_3 + \gamma \ddot{I}_3 \ddot{\zeta} \right\} \right] dxdy \\
- \oint_{\Gamma^r} \varphi_3 N_z ds \\
0 = \int_{\Omega^r} \left[ (\varphi_{4,x} M_1 + \varphi_{4,y} M_6) - \chi d_1 (\varphi_{4,x} S_1 + \varphi_{4,y} S_6) + \varphi_4 (N_5 - \chi c_1 P_5) \\
+ \varphi_4 \left\{ I_2 \ddot{u} + I_3 \ddot{\phi}_1 - \frac{1}{2} \lambda I_4 \ddot{\psi}_{3,x} - \frac{1}{3} \gamma I_5 \ddot{\zeta}_{x} \right\} \right] dxdy \\
- \oint_{\Gamma^r} \varphi_4 M_{z} ds \\
0 = \int_{\Omega^r} \left[ (\varphi_{5,x} M_6 + \varphi_{5,y} M_2) - \chi d_1 (\varphi_{5,x} S_6 + \varphi_{5,y} S_2) + \varphi_5 (N_4 - \chi c_1 P_4) \\
+ \varphi_5 \left\{ I_2 \ddot{u} + I_3 \ddot{\phi}_2 - \frac{1}{2} \lambda I_4 \ddot{\psi}_{3,y} - \frac{1}{3} \gamma I_5 \ddot{\zeta}_{y} \right\} \right] dxdy \\
- \oint_{\Gamma^r} \varphi_5 M_{z} ds \\
0 = \frac{1}{2} \lambda \int_{\Omega^r} \left[ - (\varphi_{6,xx} P_1 + 2 \varphi_{6,xy} P_6 + \varphi_{6,yy} P_2) + \varphi_6 \left\{ 2 N_3 + I_3 (\ddot{u}_x + \ddot{v}_y) \\
+ \ddot{I}_4 \left( \ddot{\phi}_{1,x} + \ddot{\phi}_{2,y} \right) + 2 \ddot{I}_2 \ddot{v} + 2 \lambda I_3 \ddot{\psi}_3 - \frac{1}{2} \lambda M_5 \left( \ddot{\psi}_{3,xx} + \ddot{\psi}_{3,yy} \right) + 2 \gamma I_4 \ddot{\zeta} \\
- \frac{1}{3} \gamma I_6 \left( \ddot{\zeta}_{xx} + \ddot{\zeta}_{yy} \right) \right\} \right] dxdy \\
+ \frac{1}{2} \lambda \oint_{\Gamma^r} \left( - \varphi_{6,xx} P_n - \varphi_{6,yy} P_{ns} + \varphi_6 Q_x \right) ds \\
0 = \frac{1}{3} \gamma \int_{\Omega^r} \left[ - (\varphi_{7,xx} S_1 + 2 \varphi_{7,xy} S_6 + \varphi_{7,yy} S_2) + \varphi_7 \left\{ 6 M_3 + I_4 (\ddot{u}_x + \ddot{v}_y) \\
+ \ddot{I}_5 \left( \ddot{\phi}_{1,x} + \ddot{\phi}_{2,y} \right) + 3 \ddot{I}_3 \ddot{w} + 3 \gamma I_4 \ddot{\psi}_3 - \frac{1}{2} \lambda M_6 \left( \ddot{\psi}_{3,xx} + \ddot{\psi}_{3,yy} \right) + 3 \gamma I_5 \ddot{\zeta} \\
- \frac{1}{3} \gamma I_7 \left( \ddot{\zeta}_{xx} + \ddot{\zeta}_{yy} \right) \right\} \right] dxdy \\
+ \frac{1}{3} \gamma \oint_{\Gamma^r} \left( - \varphi_{7,xx} S_n - \varphi_{7,yy} S_{ns} + \varphi_7 R_x \right) ds \\
(4.3)
Here, \((xx), (yy),\) and \((xy)\) denote respectively double derivative with respect to \(x\), double derivative with respect to \(y\), and the cross derivative.

Now that we have the weak form for both the theories, it is easy to identify the primary and secondary variables from the boundary terms. The sixth and seventh equations of Eq. (4.3) give rise to three boundary terms each due to the fact that they were integrated by parts twice. The essential and natural boundary conditions of the theories are

**General Third Order Theory (GTOT)**

\[
\begin{align*}
\text{essential} &: \quad \text{specify} \quad u_n, u_s, w, \phi_n, \phi_s, \psi_n, \psi_s, \psi_z, \theta_n, \theta_s, \theta_z \\
\text{natural} &: \quad \text{specify} \quad N_n, N_{ns}, N_z, M_n, M_{ns}, P_n, P_{ns}, M_z, S_n, S_{ns}, P_z
\end{align*}
\]  

(4.4)

**General Third Order Theory of Reddy (GTTR)**

\[
\begin{align*}
\text{essential} &: \quad \text{specify} \quad u_n, u_s, w, \phi_n, \phi_s, \psi_3, \frac{\partial \psi_3}{\partial n}, \frac{\partial \psi_3}{\partial s}, \frac{\partial \xi}{\partial n}, \frac{\partial \zeta}{\partial s} \\
\text{natural} &: \quad \text{specify} \quad N_n, N_{ns}, N_z, M_n, M_{ns}, Q_z, P_n, P_{ns}, R_z, S_n, S_{ns}
\end{align*}
\]  

(4.5)

### 4.3 Interpolation Functions

An examination of the weak forms of the two theories show that for the general third order theory, continuity of only the primary variables are required across the inter-element boundaries. Hence it is enough for an element to be \(C^0\) continuous. For the special third order theory, the first derivatives of some primary variables also need to be continuous across the elements. Hence the nodal degrees of freedom should include these first derivatives, i.e. \(C^1\) continuity of the element is required. This is true for three of the special cases of the special third order theory, GTTR, STTR, and CLPT. For FSDT, however, \(C^0\) continuity is enough.

The nodal degrees of freedom used for the different theories are given below.

**CLPT**: \(u, v, w, \partial w/\partial x, \partial w/\partial y, \partial^2 w/\partial x \partial y\)
\[
\text{FSDT: } u, v, w, \phi_1, \phi_2 \\
\text{STTR: } u, v, w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x \partial y}, \phi_1, \phi_2 \\
\text{GTTR: } u, v, w, \phi_1, \phi_2, \psi_3, \frac{\partial \psi_3}{\partial x}, \frac{\partial \psi_3}{\partial y}, \frac{\partial^2 \psi_3}{\partial x \partial y}, \\
\zeta, \frac{\partial \zeta}{\partial x}, \frac{\partial \zeta}{\partial y}, \frac{\partial^2 \zeta}{\partial x \partial y} \\
\text{GTOT: } u, v, w, \phi_1, \phi_2, \psi_1, \psi_2, \psi_3, \theta_1, \theta_2, \theta_3 \tag{4.6}
\]

All the generalized displacements in GTOT and \(u, v, \phi_1, \phi_2\) in the other theories (also \(w\) in GTTR) can be interpolated by the Lagrange or serendipity family of interpolation functions. That is

\[
\Delta(x, y, t) = \sum_{j=1}^{n} \Delta_j(t) \psi_j(x, y) \tag{4.7}
\]

where \(\Delta\) can be any of the above-mentioned displacements, \(\Delta_j\) are the nodal values of \(\Delta\), and \(\psi_j\) are the interpolation functions. The value of \(n\) can be four for linear Lagrange element, eight for eight-node serendipity element, or nine for quadratic Lagrange element.

The variables \(w\) (in the case of CLPT and STTR) and \(\psi_3\) and \(\zeta\) in the case of GTTR are interpolated by Hermite cubic interpolation functions. The cross derivative term is also taken as a nodal degree of freedom. Such an element is called a conforming element [149]. If \(\hat{\Delta}\) represents any of the displacements mentioned above then

\[
\hat{\Delta}(x, y, t) = \sum_{j=1}^{m} \hat{\Delta}_j(t) \hat{\psi}_j(x, y) \tag{4.8}
\]

\(\hat{\Delta}_j\) are the nodal values of \(\hat{\Delta}\) and \(\hat{\psi}_j\) are the Hermite cubic interpolation functions. For a four-node element, there will be four degrees of freedom at each node thus making the value of \(m\) to be 16.

### 4.4 Finite Element Model

For the finite element model, the weak forms are first written out explicitly in terms of the generalized displacements. Then we substitute the proper interpolation functions for the generalized displacements (as discussed in the previous section) in the weak form. The weight functions \(\varphi_i\) for each of the equations are substituted by the corresponding interpolation functions (\(\psi_i\) or \(\hat{\psi}_i\)). This leads to a set of algebraic equations for each element, the \(i\)-th equation of which is given by
where \( \alpha = 1, 2, \ldots, p \). The value of \( p \) is 11 for the general third order theory, and 7 for the special third order theory. It represents the number of primary variables in the problem. The element stiffness matrix \([K]\) and element mass matrix \([M]\) each have \( p \times p \) submatrices. The size of a submatrix \([K^{\alpha \beta}]\) or \([M^{\alpha \beta}]\) is \( n(\alpha) \times n(\beta) \). \( n(\alpha) = n(\beta) = 4 \) or 8 or 9, depending on whether the corresponding primary variable is interpolated by linear Lagrange functions, serendipity functions or quadratic Lagrange functions. \( n(\alpha) = n(\beta) = 16 \), if the corresponding primary variable is interpolated by Hermite cubic interpolation functions. The element force vector \( \{f\} \) and the vector of secondary variables \( \{Q\} \) have \( p \) subvectors, one corresponding to each primary variable. The size of each subvector is \( n(\alpha) \times 1 \).

In matrix notation, the system of finite element equations for each element can be written as

\[
[K^e] \{\Delta_e\} + [M^e] \{\Delta^e\} = \{f^e\} + \{Q^e\}
\]

(4.10)

For the general third order theory, if we write Eq. (4.10) in terms of the component submatrices, we will get

\[
\begin{bmatrix}
[K^{11}] & \cdots & \cdots & [K^{1,11}] \\
\vdots & \ddots & \vdots & \vdots \\
[K^{11,1}] & \cdots & \cdots & [K^{11,11}]
\end{bmatrix}
\begin{bmatrix}
\{u\} \\
\{\theta_3\}
\end{bmatrix}
+ 
\begin{bmatrix}
[M^{11}] & \cdots & \cdots & [M^{1,11}] \\
\vdots & \ddots & \vdots & \vdots \\
[M^{11,1}] & \cdots & \cdots & [M^{11,11}]
\end{bmatrix}
\begin{bmatrix}
\{\ddot{u}\} \\
\{\ddot{\theta}_3\}
\end{bmatrix}
= 
\begin{bmatrix}
\{f^1\} \\
\vdots \\
\{f^{11}\}
\end{bmatrix}
+ 
\begin{bmatrix}
\{Q^1\} \\
\vdots \\
\{Q^{11}\}
\end{bmatrix}
\]

(4.11)

The element stiffness and mass matrices in Eq. (4.11) are of order \( 11n \times 11n \), where \( n \) is the number of nodes per element. The elements of \([K^{\alpha \beta}]\), \([M^{\alpha \beta}]\), and \( \{f^\alpha\} \) are given explicitly in Appendix C.1.

The element stiffness matrix for a four-node element using the special third order theory can be written as
\[ [K] = \begin{bmatrix}
(K^{11})_{(4 \times 4)} & (K^{12})_{(4 \times 4)} & (K^{13})_{(4 \times 4)} & (K^{14})_{(4 \times 4)} & (K^{15})_{(4 \times 4)} & (K^{16})_{(4 \times 16)} & (K^{17})_{(4 \times 16)} \\
(K^{21})_{(4 \times 4)} & (K^{22})_{(4 \times 4)} & (K^{23})_{(4 \times 4)} & (K^{24})_{(4 \times 4)} & (K^{25})_{(4 \times 4)} & (K^{26})_{(4 \times 4)} & (K^{27})_{(4 \times 4)} \\
(K^{31})_{(4 \times 4)} & (K^{32})_{(4 \times 4)} & (K^{33})_{(4 \times 4)} & (K^{34})_{(4 \times 4)} & (K^{35})_{(4 \times 4)} & (K^{36})_{(4 \times 4)} & (K^{37})_{(4 \times 4)} \\
(K^{41})_{(4 \times 1)} & (K^{42})_{(4 \times 1)} & (K^{43})_{(4 \times 1)} & (K^{44})_{(4 \times 1)} & (K^{45})_{(4 \times 1)} & (K^{46})_{(4 \times 1)} & (K^{47})_{(4 \times 1)} \\
(K^{51})_{(16 \times 16)} & (K^{52})_{(16 \times 16)} & (K^{53})_{(16 \times 16)} & (K^{54})_{(16 \times 16)} & (K^{55})_{(16 \times 16)} & (K^{56})_{(16 \times 16)} & (K^{57})_{(16 \times 16)} \\
(K^{61})_{(16 \times 16)} & (K^{62})_{(16 \times 16)} & (K^{63})_{(16 \times 16)} & (K^{64})_{(16 \times 16)} & (K^{65})_{(16 \times 16)} & (K^{66})_{(16 \times 16)} & (K^{67})_{(16 \times 16)} \\
(K^{71})_{(16 \times 16)} & (K^{72})_{(16 \times 16)} & (K^{73})_{(16 \times 16)} & (K^{74})_{(16 \times 16)} & (K^{75})_{(16 \times 16)} & (K^{76})_{(16 \times 16)} & (K^{77})_{(16 \times 16)} 
\end{bmatrix} \]

The size of each submatrix is shown over that particular submatrix. The element mass matrix \([M^e]\) has the same structure, but it is not symmetric. The elements of each submatrix are stated in explicit form in Appendix C.2. Note that the stiffness matrices for CLPT, FSDT, and STTR can be derived from those of GTTR by using tracers.
Chapter 5

RESULTS AND DISCUSSION

5.1 Introduction

Three computer programs were written to solve a number of bending and free vibration problems in laminated plates. The first program was for finding the solution by Navier method, the second for finding the solution by Lévy method, and the third was a finite element program which can solve a general laminated composite plate problem for arbitrary loadings, geometries and boundary conditions.

Three different materials were used for the numerical examples. Their properties are listed in Table 5.1. The first material is Aragonite. Exact solutions for a number of cases were given for this material by Srinivas et al. [129, 130, 131, 132]. Results obtained from the different theories have been compared with these exact solutions.

The second material is graphite/epoxy. The majority of numerical results given in this chapter are for this material. Noor [140] gave exact solutions to free vibration problems for a high modulus composite, which we shall call Material 3. We compare results of free vibration from the different theories with the exact solutions given by Noor.

5.2 Implementation

5.2.1 Analytical Solutions

The theoretical basis for Navier and Lévy solutions have been explained in Chapter 3. In both these methods, the solutions are approximated in the form of an infinite series which is truncated after a few terms. For the Navier method, $m = n = 25$ gives sufficiently accurate results. In the Lévy method, the convergence is extremely fast
<table>
<thead>
<tr>
<th>Table 5.1: Material properties</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Aragonite</strong></td>
</tr>
<tr>
<td>$C_{11}$</td>
</tr>
<tr>
<td>$C_{12}$</td>
</tr>
<tr>
<td>$C_{13}$</td>
</tr>
<tr>
<td>$C_{22}$</td>
</tr>
<tr>
<td>$C_{23}$</td>
</tr>
<tr>
<td>$C_{33}$</td>
</tr>
<tr>
<td>$C_{44}$</td>
</tr>
<tr>
<td>$C_{55}$</td>
</tr>
<tr>
<td>$C_{66}$</td>
</tr>
<tr>
<td><strong>Graphite-epoxy</strong></td>
</tr>
<tr>
<td>$E_1$</td>
</tr>
<tr>
<td>$E_2$</td>
</tr>
<tr>
<td>$E_3$</td>
</tr>
<tr>
<td>$G_{23}$</td>
</tr>
<tr>
<td>$G_{13}$</td>
</tr>
<tr>
<td>$G_{12}$</td>
</tr>
<tr>
<td>$\nu_{23}$</td>
</tr>
<tr>
<td>$\nu_{13}$</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
</tr>
<tr>
<td><strong>Material 3</strong></td>
</tr>
<tr>
<td>$E_1$</td>
</tr>
<tr>
<td>$E_2$</td>
</tr>
<tr>
<td>$E_3$</td>
</tr>
<tr>
<td>$G_{23}$</td>
</tr>
<tr>
<td>$G_{13}$</td>
</tr>
<tr>
<td>$G_{12}$</td>
</tr>
<tr>
<td>$\nu_{23}$</td>
</tr>
<tr>
<td>$\nu_{13}$</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
</tr>
</tbody>
</table>
as mentioned in the concluding part of Chapter 3. Taking only the first four terms gives very good results in all the cases.

For problems which have been solved by the Navier method, all four sides of the plate are simply-supported. For analytical solutions by the Lévy method, two parallel edges have to be simply-supported, and the remaining two edges can have any combination of free, clamped, or simply-supported boundary conditions. The different sets of boundary conditions for which results have been obtained are shown in Figure 5.1. For example, the acronym SSCF refers to a plate which is simply-supported at \( y = 0 \) and \( y = b \), clamped at \( x = -a/2 \), and free at \( x = +a/2 \). A total of six such boundary conditions have been investigated.
Figure 5.1: Boundary conditions for Lévy solution
5.2.2 Finite Element Solutions

The different boundary conditions used (including the symmetry boundary conditions) are shown in tabular form for the different theories in Table 5.2 - 5.6. Symm.X refers to symmetry boundary conditions along lines parallel to the x-axis. Simple.Y refers to simply-supported boundary conditions along edges parallel to the y-axis. Symm.Y, Simple.X, Clamped.Y, and Free.Y have similar connotations. For bending analysis of SSSS, SSCC, SSFF plates, quarter-plate models have been used. For bending analysis of SSCS, SCCF, and SSSF plates, half-plate models have been used. For free vibration analysis, half-plate models have been used for all the six sets of boundary conditions. The discretization used for a quarter-plate model is a $8 \times 8$ mesh of 4-node elements for CLPT, STTR, and GTTR. For FSDT and GTOT, two meshes have been used for the quarter-plate model; one a $8 \times 8$ mesh of linear Lagrange elements, and the other a $4 \times 4$ mesh of quadratic Lagrange elements (referred to in the tables as FEM.L and FEM.Q respectively). For half-plate models, the discretizations used are a $16 \times 8$ mesh of 4-node elements for all the theories. Additionally, a $8 \times 4$ mesh of 9-node quadratic Lagrange elements were used for FSDT and GTOT (see Figure 5.2).

<table>
<thead>
<tr>
<th>Boundary Condition</th>
<th>Specified primary variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symm.X</td>
<td>$v = \partial w/\partial y = \partial^2 w/\partial x \partial y = 0$</td>
</tr>
<tr>
<td>Symm.Y</td>
<td>$u = \partial w/\partial x = \partial^2 w/\partial x \partial y = 0$</td>
</tr>
<tr>
<td>Simple.X</td>
<td>$u = w = \partial w/\partial x = 0$</td>
</tr>
<tr>
<td>Simple.Y</td>
<td>$v = w = \partial w/\partial y = 0$</td>
</tr>
<tr>
<td>Clamped.X</td>
<td>All PVs specified zero</td>
</tr>
<tr>
<td>Free.X</td>
<td>No PVs specified</td>
</tr>
</tbody>
</table>

Table 5.2: Boundary conditions for CLPT
Table 5.3: Boundary conditions for FSDT

<table>
<thead>
<tr>
<th>Specified primary variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symm_X</td>
</tr>
<tr>
<td>Symm_Y</td>
</tr>
<tr>
<td>Simple_X</td>
</tr>
<tr>
<td>Simple_Y</td>
</tr>
<tr>
<td>Clamped_X</td>
</tr>
<tr>
<td>Free_X</td>
</tr>
</tbody>
</table>

Table 5.4: Boundary conditions for STTR

<table>
<thead>
<tr>
<th>Specified primary variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symm_X</td>
</tr>
<tr>
<td>Symm_Y</td>
</tr>
<tr>
<td>Simple_X</td>
</tr>
<tr>
<td>Simple_Y</td>
</tr>
<tr>
<td>Clamped_X</td>
</tr>
<tr>
<td>Free_X</td>
</tr>
</tbody>
</table>
Table 5.5: Boundary conditions for GTTR

<table>
<thead>
<tr>
<th>Specified primary variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symm.X</td>
</tr>
<tr>
<td>$v = \phi_2 = \frac{\partial \psi_3}{\partial y} = \frac{\partial^2 \psi_3}{\partial x \partial y} = \frac{\partial \zeta}{\partial y} = \frac{\partial^2 \zeta}{\partial x \partial y} = 0$</td>
</tr>
<tr>
<td>Symm.Y</td>
</tr>
<tr>
<td>$u = \phi_1 = \frac{\partial \psi_3}{\partial x} = \frac{\partial^2 \psi_3}{\partial x \partial y} = \frac{\partial \zeta}{\partial x} = \frac{\partial^2 \zeta}{\partial x \partial y} = 0$</td>
</tr>
<tr>
<td>Simple.X</td>
</tr>
<tr>
<td>$u = w = \psi_1 = \psi_2 = \frac{\partial \psi_3}{\partial x} = \zeta = \frac{\partial \zeta}{\partial x} = 0$</td>
</tr>
<tr>
<td>Simple.Y</td>
</tr>
<tr>
<td>$v = w = \phi_2 = \psi_3 = \frac{\partial \psi_3}{\partial y} = \zeta = \frac{\partial \zeta}{\partial y} = 0$</td>
</tr>
<tr>
<td>Clamped.X</td>
</tr>
<tr>
<td>All PVs specified zero</td>
</tr>
<tr>
<td>Free.X</td>
</tr>
<tr>
<td>No PVs specified</td>
</tr>
</tbody>
</table>

Table 5.6: Boundary conditions for GTOT

<table>
<thead>
<tr>
<th>Specified primary variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>Symm.X</td>
</tr>
<tr>
<td>$v = \phi_2 = \psi_2 = \theta_2 = 0$</td>
</tr>
<tr>
<td>Symm.Y</td>
</tr>
<tr>
<td>$u = \phi_1 = \psi_1 = \theta_1 = 0$</td>
</tr>
<tr>
<td>Simple.X</td>
</tr>
<tr>
<td>$u = w = \phi_1 = \psi_1 = \psi_3 = \theta_1 = \theta_3 = 0$</td>
</tr>
<tr>
<td>Simple.Y</td>
</tr>
<tr>
<td>$v = w = \phi_2 = \psi_2 = \psi_3 = \theta_2 = \theta_3 = 0$</td>
</tr>
<tr>
<td>Clamped.X</td>
</tr>
<tr>
<td>All PVs specified zero</td>
</tr>
<tr>
<td>Free.X</td>
</tr>
<tr>
<td>No PVs specified</td>
</tr>
</tbody>
</table>

53
Figure 5.2: Discretization for finite element solution  
(a) $8 \times 8$ mesh of 4-node elements in quarter-plate  
(b) $16 \times 8$ mesh of 4-node elements in half-plate
5.3 Comparison with Exact Solutions

The results obtained from the different theories were compared with the exact solutions given by Srinivas and Rao [130] (Tables 5.7-5.17), and Noor [140] (Table 5.18 and 5.19). In all the cases, the plate is assumed to be simply-supported on all four sides. The dimension of the plate is $a \times b$.

Table 5.7 gives the transverse deflections $w$ at the mid-point of the plate ($x/a = 0.5, y/b = 0.5, z/h = 0$). Table 5.8 and 5.9 give the normal stresses $\sigma_x$ and $\sigma_y$ respectively at the center of the top surface of the plate ($x/a = 0.5, y/b = 0.5, z/h = -0.5$). Table 5.10 gives the shear stresses $\tau_{xz}$ at the center of an edge ($x/a = 0, y/b = 0.5, z/h = 0$). For deflections, GTTR gives the best results. STTR and FSDT values are almost the same, while CLPT values are least accurate. The error in the values predicted by CLPT also increases as the plate thickness increases. For the normal stresses $\sigma_x$ and $\sigma_y$, the values given by GTTR and STTR are almost the same, both being very close to the exact value. FSDT values are slightly worse, but still quite good. For shear stress $\tau_{xz}$, GTTR gives the best results, followed by STTR. FSDT values are not good, while CLPT values are uniformly zero since it does not take into account effects of shear deformation.

In Tables 5.11-5.14, the transverse deflection $w$, and stresses $\sigma_x$, $\sigma_y$, and $\tau_{xz}$ are given for a square three-ply laminate with the top and bottom plies being of equal thickness and made up of identical material ($h/a = 0.1, h_1/h = 0.1, h_2/h = 0.8, h_3/h \approx 0.1$). The modular ratio between the middle ply and the outer plies ($\beta = E_{x1}/E_{x2}$) is varied from 1 to 15. When $\beta = 1$, the laminate essentially consists of a single layer. The deflections and stresses are calculated at the same points as in Tables 5.7-5.10. Additionally, the variation of the stresses through the thickness of the plate is shown. For deflection, GTTR and STTR give the most accurate results, with the error increasing with increasing value of $\beta$. For normal stresses, the values predicted by GTTR and STTR are quite close to the exact values. With increasing $\beta$, the errors do not increase as much as in the case of deflections. For shear stress $\tau_{xz}$, while the GTTR and STTR values are good at the mid-surface, they do not compare very well with the exact solutions at the interfaces. This is because the stress values have been evaluated from the constitutive equations, and not from the equilibrium equations. FSDT, of course, predicts a constant shear stress state through the thickness of the plate.
Tables 5.15, 5.16, and 5.17 give the natural frequencies of the first three antisymmetric modes of free vibration of a homogeneous plate. GTTR gives the best results, followed closely by STTR and FSDT. Table 5.18 and 5.19 give the fundamental frequencies of free vibration for symmetric and antisymmetric cross-ply plates. The plates have different number of layers, and varying degrees of orthotropy of individual layers, i.e., different values of $E_l/E_t$. Here, $E_l$ refers to the modulus in the longitudinal (fiber) direction, and $E_t$ refers to the modulus in the transverse direction. In most cases, GTTR gives the most accurate results. However for $N = 3$ in the case of symmetric plates, and $N = 2$ in the case of antisymmetric plates, FSDT gives the best results. CLPT constantly overpredicts the frequency values.

Table 5.7: Transverse deflection ($C_{11}w/hq_0$) in an Orthotropic plate under uniform transverse load

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>$h/a$</th>
<th>$c_{11}w/hq_0$</th>
<th>Exact</th>
<th>GTTR</th>
<th>STTR</th>
<th>FSDT</th>
<th>CLPT</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.05</td>
<td>21542.0</td>
<td>21543.0</td>
<td>21544.2</td>
<td>21544.3</td>
<td>21210.6</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>1408.5</td>
<td>1408.5</td>
<td>1409.0</td>
<td>1409.0</td>
<td>1325.7</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.14</td>
<td>387.23</td>
<td>387.25</td>
<td>387.55</td>
<td>387.60</td>
<td>345.08</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.05</td>
<td>10443.0</td>
<td>10443.6</td>
<td>10446.8</td>
<td>10446.8</td>
<td>10250.7</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>688.57</td>
<td>688.60</td>
<td>689.54</td>
<td>689.57</td>
<td>640.67</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.14</td>
<td>191.07</td>
<td>191.09</td>
<td>191.61</td>
<td>191.64</td>
<td>166.77</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.05</td>
<td>2048.7</td>
<td>2048.8</td>
<td>2051.5</td>
<td>2051.5</td>
<td>1989.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>139.08</td>
<td>139.09</td>
<td>139.83</td>
<td>139.85</td>
<td>124.31</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.14</td>
<td>39.790</td>
<td>39.801</td>
<td>40.215</td>
<td>40.231</td>
<td>32.359</td>
<td></td>
</tr>
</tbody>
</table>
Table 5.8: Normal stress ($\sigma_x/q_0$) in an orthotropic plate under uniform transverse load

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>$h/a$</th>
<th>Exact</th>
<th>GTTR</th>
<th>STTR</th>
<th>FSDT</th>
<th>CLPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.05</td>
<td>262.67</td>
<td>262.67</td>
<td>262.67</td>
<td>262.08</td>
<td>262.26</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>65.975</td>
<td>65.977</td>
<td>65.978</td>
<td>65.392</td>
<td>65.564</td>
</tr>
<tr>
<td></td>
<td>0.14</td>
<td>33.862</td>
<td>33.864</td>
<td>33.865</td>
<td>33.279</td>
<td>33.451</td>
</tr>
<tr>
<td>1</td>
<td>0.05</td>
<td>144.31</td>
<td>144.31</td>
<td>144.32</td>
<td>143.91</td>
<td>144.30</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>36.021</td>
<td>36.024</td>
<td>36.034</td>
<td>35.623</td>
<td>35.098</td>
</tr>
<tr>
<td></td>
<td>0.14</td>
<td>18.346</td>
<td>18.348</td>
<td>18.358</td>
<td>17.948</td>
<td>18.417</td>
</tr>
<tr>
<td>0.5</td>
<td>0.05</td>
<td>40.657</td>
<td>40.658</td>
<td>40.708</td>
<td>40.525</td>
<td>40.860</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>10.925</td>
<td>10.026</td>
<td>10.074</td>
<td>9.893</td>
<td>10.215</td>
</tr>
<tr>
<td></td>
<td>0.14</td>
<td>5.0364</td>
<td>5.0389</td>
<td>5.0842</td>
<td>4.9052</td>
<td>5.2118</td>
</tr>
</tbody>
</table>

Table 5.9: Normal stress ($\sigma_y/q_0$) in an orthotropic plate under uniform transverse Load

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>$h/a$</th>
<th>Exact</th>
<th>GTTR</th>
<th>STTR</th>
<th>FSDT</th>
<th>CLPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.05</td>
<td>79.545</td>
<td>79.558</td>
<td>79.337</td>
<td>79.230</td>
<td>79.119</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>20.204</td>
<td>20.221</td>
<td>20.038</td>
<td>19.891</td>
<td>19.780</td>
</tr>
<tr>
<td></td>
<td>0.14</td>
<td>10.515</td>
<td>10.334</td>
<td>10.350</td>
<td>10.203</td>
<td>10.092</td>
</tr>
<tr>
<td>1</td>
<td>0.05</td>
<td>87.080</td>
<td>87.100</td>
<td>86.990</td>
<td>86.826</td>
<td>86.486</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>22.210</td>
<td>22.232</td>
<td>22.123</td>
<td>21.959</td>
<td>21.622</td>
</tr>
<tr>
<td></td>
<td>0.14</td>
<td>11.615</td>
<td>11.637</td>
<td>11.529</td>
<td>11.365</td>
<td>11.031</td>
</tr>
<tr>
<td>0.5</td>
<td>0.05</td>
<td>54.279</td>
<td>54.303</td>
<td>54.284</td>
<td>54.097</td>
<td>53.838</td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>13.888</td>
<td>13.912</td>
<td>13.895</td>
<td>13.708</td>
<td>13.460</td>
</tr>
<tr>
<td></td>
<td>0.14</td>
<td>7.2794</td>
<td>7.3055</td>
<td>7.2909</td>
<td>7.1038</td>
<td>6.8671</td>
</tr>
</tbody>
</table>
Table 5.10: Shear stresses ($\tau_{xz}/q_0$) in an orthotropic plate under uniform transverse load

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>$h/a$</th>
<th>(-\tau_{xz}/q_0)</th>
<th>Exact</th>
<th>GTTR</th>
<th>STTR</th>
<th>FSDT</th>
<th>CLPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.05</td>
<td>14.048</td>
<td>14.149</td>
<td>14.187</td>
<td>11.411</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>6.9266</td>
<td>6.9852</td>
<td>7.0290</td>
<td>5.7032</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.14</td>
<td>4.8782</td>
<td>4.9368</td>
<td>4.9816</td>
<td>4.0716</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>0.05</td>
<td>10.873</td>
<td>11.069</td>
<td>11.048</td>
<td>8.8999</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>5.3411</td>
<td>5.3987</td>
<td>5.4462</td>
<td>4.4367</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.14</td>
<td>3.7313</td>
<td>3.7882</td>
<td>3.8383</td>
<td>3.1566</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>0.5</td>
<td>0.05</td>
<td>6.2434</td>
<td>6.3098</td>
<td>6.3524</td>
<td>5.1430</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.10</td>
<td>2.9573</td>
<td>3.0125</td>
<td>3.0676</td>
<td>2.3332</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.14</td>
<td>1.9987</td>
<td>2.0522</td>
<td>2.1128</td>
<td>1.7749</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
Table 5.11: Transverse deflection and stresses in a three-ply laminate under uniform transverse load ($\beta = E_{x1}/E_{x2} = 1$)

<table>
<thead>
<tr>
<th>Source</th>
<th>Exact</th>
<th>GTTR</th>
<th>STTR</th>
<th>FSDT</th>
<th>CLPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-wE_{x2}/hq_0$</td>
<td>688.58</td>
<td>688.60</td>
<td>689.54</td>
<td>689.57</td>
<td>640.67</td>
</tr>
<tr>
<td>$\sigma_z/q_0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Top ply at top surface</td>
<td>36.021</td>
<td>36.023</td>
<td>36.034</td>
<td>35.623</td>
<td>36.098</td>
</tr>
<tr>
<td>Bottom ply at bottom surface</td>
<td>-35.937</td>
<td>-35.940</td>
<td>-36.034</td>
<td>-35.623</td>
<td>-36.098</td>
</tr>
<tr>
<td>$\sigma_y/q_0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Top ply at interface</td>
<td>17.669</td>
<td>17.671</td>
<td>17.580</td>
<td>17.567</td>
<td>17.297</td>
</tr>
<tr>
<td>Mid ply at upper interface</td>
<td>17.669</td>
<td>17.671</td>
<td>17.580</td>
<td>17.567</td>
<td>17.297</td>
</tr>
<tr>
<td>Mid ply at lower interface</td>
<td>-17.631</td>
<td>-17.633</td>
<td>-17.580</td>
<td>-17.567</td>
<td>-17.297</td>
</tr>
<tr>
<td>Bottom ply at interface</td>
<td>-17.631</td>
<td>-17.633</td>
<td>-17.580</td>
<td>-17.567</td>
<td>-17.297</td>
</tr>
<tr>
<td>$-\tau_{xz}/q_0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mid ply at upper interface</td>
<td>2.4029</td>
<td>1.9435</td>
<td>1.9606</td>
<td>4.4367</td>
<td>0</td>
</tr>
<tr>
<td>Mid ply at midsurface</td>
<td>5.3411</td>
<td>5.3987</td>
<td>5.4462</td>
<td>4.4367</td>
<td>0</td>
</tr>
<tr>
<td>Mid ply at lower interface</td>
<td>1.9826</td>
<td>1.9435</td>
<td>1.9606</td>
<td>4.4367</td>
<td>0</td>
</tr>
</tbody>
</table>
Table 5.12: Transverse deflection and stresses in a three-ply laminate under uniform transverse load \((\beta = \frac{E_{x1}}{E_{x2}} = 5)\)

<table>
<thead>
<tr>
<th>Source</th>
<th>(\sigma_x / q_0)</th>
<th>(\sigma_y / q_0)</th>
<th>(-\tau_{xy} / q_0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-wE_{x2}/hq_0)</td>
<td>Exact</td>
<td>GTTR</td>
<td>STTR</td>
</tr>
<tr>
<td></td>
<td>258.97</td>
<td>258.75</td>
<td>257.03</td>
</tr>
<tr>
<td>(\sigma_x / q_0)</td>
<td>60.353</td>
<td>60.389</td>
<td>60.348</td>
</tr>
<tr>
<td>Top ply at top surface</td>
<td>46.623</td>
<td>47.057</td>
<td>46.989</td>
</tr>
<tr>
<td>Mid ply at lower interface</td>
<td>-46.426</td>
<td>-46.824</td>
<td>-46.989</td>
</tr>
<tr>
<td>Bottom ply at interface</td>
<td>-60.155</td>
<td>-60.155</td>
<td>-60.348</td>
</tr>
<tr>
<td>(\sigma_y / q_0)</td>
<td>38.491</td>
<td>38.634</td>
<td>38.435</td>
</tr>
<tr>
<td>Top ply at top surface</td>
<td>30.097</td>
<td>30.389</td>
<td>30.226</td>
</tr>
<tr>
<td>Top ply at interface</td>
<td>6.1607</td>
<td>6.0779</td>
<td>6.0452</td>
</tr>
<tr>
<td>Mid ply at upper interface</td>
<td>-6.6574</td>
<td>-6.0570</td>
<td>-6.0452</td>
</tr>
<tr>
<td>Bottom ply at interface</td>
<td>-35.715</td>
<td>-38.530</td>
<td>-38.435</td>
</tr>
<tr>
<td>(-\tau_{xy} / q_0)</td>
<td>3.7194</td>
<td>1.6895</td>
<td>1.6981</td>
</tr>
<tr>
<td>Mid ply at upper interface</td>
<td>4.3641</td>
<td>4.6929</td>
<td>4.7169</td>
</tr>
<tr>
<td>Mid ply at midsurface</td>
<td>3.2675</td>
<td>1.6895</td>
<td>1.6981</td>
</tr>
<tr>
<td>Mid ply at lower interface</td>
<td>3.2675</td>
<td>1.6895</td>
<td>1.6981</td>
</tr>
</tbody>
</table>
Table 5.13: Transverse deflection and stresses in a three-ply laminate under uniform transverse load. ($\beta = E_{z1}/E_{z2} = 10$)

<table>
<thead>
<tr>
<th>$\beta = E_{z1}/E_{z2}$</th>
<th>Source</th>
<th>Exact</th>
<th>GTTR</th>
<th>STTR</th>
<th>FSDT</th>
<th>CLPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-wE_{z2}/hq_0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Top ply at top surface</td>
<td>65.332</td>
<td>65.407</td>
<td>65.335</td>
<td>65.272</td>
<td>66.947</td>
<td></td>
</tr>
<tr>
<td>Top ply at interface</td>
<td>48.857</td>
<td>50.939</td>
<td>49.937</td>
<td>52.217</td>
<td>53.557</td>
<td></td>
</tr>
<tr>
<td>Mid ply at upper interface</td>
<td>4.9030</td>
<td>5.0039</td>
<td>4.9937</td>
<td>5.2217</td>
<td>5.3557</td>
<td></td>
</tr>
<tr>
<td>Mid ply at lower interface</td>
<td>-4.8600</td>
<td>-4.9740</td>
<td>-4.9937</td>
<td>-5.2217</td>
<td>-5.3557</td>
<td></td>
</tr>
<tr>
<td>Bottom ply at bottom surface</td>
<td>-65.083</td>
<td>-65.197</td>
<td>-65.335</td>
<td>-65.272</td>
<td>-66.947</td>
<td></td>
</tr>
<tr>
<td>$\sigma_y/q_0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Top ply at top surface</td>
<td>43.566</td>
<td>43.427</td>
<td>43.200</td>
<td>41.290</td>
<td>40.099</td>
<td></td>
</tr>
<tr>
<td>Top ply at interface</td>
<td>33.413</td>
<td>33.787</td>
<td>33.606</td>
<td>33.032</td>
<td>32.079</td>
<td></td>
</tr>
<tr>
<td>Mid ply at upper interface</td>
<td>3.4995</td>
<td>3.3787</td>
<td>3.3606</td>
<td>3.3032</td>
<td>3.2079</td>
<td></td>
</tr>
<tr>
<td>Bottom ply at interface</td>
<td>-33.756</td>
<td>-33.653</td>
<td>-33.606</td>
<td>-33.032</td>
<td>-32.079</td>
<td></td>
</tr>
<tr>
<td>Bottom ply at bottom surface</td>
<td>-43.098</td>
<td>-43.294</td>
<td>-43.200</td>
<td>-41.290</td>
<td>-40.099</td>
<td></td>
</tr>
<tr>
<td>$-\tau_{xz}/q_0$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Mid ply at upper interface</td>
<td>3.9285</td>
<td>1.5274</td>
<td>1.5332</td>
<td>1.5785</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Mid ply at midsurface</td>
<td>4.0959</td>
<td>4.2427</td>
<td>4.2589</td>
<td>1.5785</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>Mid ply at lower interface</td>
<td>3.5154</td>
<td>1.5274</td>
<td>1.5332</td>
<td>1.5785</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>
Table 5.14: Transverse deflection and stresses in a three-ply laminate under uniform transverse load ($\beta = E_{x1}/E_{x2} = 15$)

<table>
<thead>
<tr>
<th>Source</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-wE_{x2}/hq_0$</td>
<td></td>
</tr>
<tr>
<td>121.72</td>
<td></td>
</tr>
<tr>
<td>114.41</td>
<td></td>
</tr>
<tr>
<td>114.52</td>
<td></td>
</tr>
<tr>
<td>94.626</td>
<td></td>
</tr>
<tr>
<td>81.802</td>
<td></td>
</tr>
<tr>
<td>$\sigma_x/q_0$</td>
<td></td>
</tr>
<tr>
<td>66.787</td>
<td></td>
</tr>
<tr>
<td>66.945</td>
<td></td>
</tr>
<tr>
<td>66.857</td>
<td></td>
</tr>
<tr>
<td>67.287</td>
<td></td>
</tr>
<tr>
<td>69.135</td>
<td></td>
</tr>
<tr>
<td>48.299</td>
<td></td>
</tr>
<tr>
<td>50.394</td>
<td></td>
</tr>
<tr>
<td>50.274</td>
<td></td>
</tr>
<tr>
<td>53.830</td>
<td></td>
</tr>
<tr>
<td>55.308</td>
<td></td>
</tr>
<tr>
<td>3.2379</td>
<td></td>
</tr>
<tr>
<td>3.3596</td>
<td></td>
</tr>
<tr>
<td>3.3516</td>
<td></td>
</tr>
<tr>
<td>3.5886</td>
<td></td>
</tr>
<tr>
<td>3.6872</td>
<td></td>
</tr>
<tr>
<td>-3.2009</td>
<td></td>
</tr>
<tr>
<td>-3.3374</td>
<td></td>
</tr>
<tr>
<td>-3.3516</td>
<td></td>
</tr>
<tr>
<td>-3.5886</td>
<td></td>
</tr>
<tr>
<td>-3.6872</td>
<td></td>
</tr>
<tr>
<td>-48.028</td>
<td></td>
</tr>
<tr>
<td>-50.062</td>
<td></td>
</tr>
<tr>
<td>-50.274</td>
<td></td>
</tr>
<tr>
<td>-53.830</td>
<td></td>
</tr>
<tr>
<td>-55.308</td>
<td></td>
</tr>
<tr>
<td>-66.513</td>
<td></td>
</tr>
<tr>
<td>-66.615</td>
<td></td>
</tr>
<tr>
<td>-66.857</td>
<td></td>
</tr>
<tr>
<td>-67.287</td>
<td></td>
</tr>
<tr>
<td>-69.135</td>
<td></td>
</tr>
<tr>
<td>$\sigma_y/q_0$</td>
<td></td>
</tr>
<tr>
<td>46.424</td>
<td></td>
</tr>
<tr>
<td>45.810</td>
<td></td>
</tr>
<tr>
<td>45.569</td>
<td></td>
</tr>
<tr>
<td>42.723</td>
<td></td>
</tr>
<tr>
<td>41.410</td>
<td></td>
</tr>
<tr>
<td>34.955</td>
<td></td>
</tr>
<tr>
<td>35.315</td>
<td></td>
</tr>
<tr>
<td>35.129</td>
<td></td>
</tr>
<tr>
<td>34.179</td>
<td></td>
</tr>
<tr>
<td>33.123</td>
<td></td>
</tr>
<tr>
<td>2.4941</td>
<td></td>
</tr>
<tr>
<td>2.3543</td>
<td></td>
</tr>
<tr>
<td>2.3419</td>
<td></td>
</tr>
<tr>
<td>2.2786</td>
<td></td>
</tr>
<tr>
<td>2.2085</td>
<td></td>
</tr>
<tr>
<td>-2.3476</td>
<td></td>
</tr>
<tr>
<td>-2.3444</td>
<td></td>
</tr>
<tr>
<td>-2.3419</td>
<td></td>
</tr>
<tr>
<td>-2.2786</td>
<td></td>
</tr>
<tr>
<td>-2.2085</td>
<td></td>
</tr>
<tr>
<td>-35.353</td>
<td></td>
</tr>
<tr>
<td>-35.166</td>
<td></td>
</tr>
<tr>
<td>-35.129</td>
<td></td>
</tr>
<tr>
<td>-34.179</td>
<td></td>
</tr>
<tr>
<td>-33.123</td>
<td></td>
</tr>
<tr>
<td>-46.821</td>
<td></td>
</tr>
<tr>
<td>-45.663</td>
<td></td>
</tr>
<tr>
<td>-45.569</td>
<td></td>
</tr>
<tr>
<td>-42.723</td>
<td></td>
</tr>
<tr>
<td>-41.410</td>
<td></td>
</tr>
<tr>
<td>$-t_{x2}/q_0$</td>
<td></td>
</tr>
<tr>
<td>3.9559</td>
<td></td>
</tr>
<tr>
<td>1.4035</td>
<td></td>
</tr>
<tr>
<td>1.4080</td>
<td></td>
</tr>
<tr>
<td>1.1025</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3.9638</td>
<td></td>
</tr>
<tr>
<td>3.8985</td>
<td></td>
</tr>
<tr>
<td>3.9111</td>
<td></td>
</tr>
<tr>
<td>1.1625</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
<tr>
<td>3.5768</td>
<td></td>
</tr>
<tr>
<td>1.4035</td>
<td></td>
</tr>
<tr>
<td>1.4080</td>
<td></td>
</tr>
<tr>
<td>1.1625</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

62
Table 5.15: Comparison of the lowest natural frequency of an orthotropic square plate: \( a/h = 10, \ \bar{\omega} = \omega h (\rho/C_{11})^{\frac{1}{4}} \)

<table>
<thead>
<tr>
<th>( m )</th>
<th>( n )</th>
<th>Exact</th>
<th>GTTR</th>
<th>STTR</th>
<th>FSDT</th>
<th>CLPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.0474</td>
<td>0.0474</td>
<td>0.0474</td>
<td>0.0474</td>
<td>0.0493</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>0.1033</td>
<td>0.1033</td>
<td>0.1032</td>
<td>0.1032</td>
<td>0.1098</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0.1188</td>
<td>0.1188</td>
<td>0.1188</td>
<td>0.1187</td>
<td>0.1327</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.1694</td>
<td>0.1694</td>
<td>0.1693</td>
<td>0.1692</td>
<td>0.1924</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>0.1888</td>
<td>0.1888</td>
<td>0.1884</td>
<td>0.1884</td>
<td>0.2070</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0.2150</td>
<td>0.2181</td>
<td>0.2180</td>
<td>0.2178</td>
<td>0.2671</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>0.2475</td>
<td>0.2476</td>
<td>0.2471</td>
<td>0.2469</td>
<td>0.2879</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0.2624</td>
<td>0.2625</td>
<td>0.2623</td>
<td>0.2619</td>
<td>0.3248</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>0.2969</td>
<td>0.2969</td>
<td>0.2960</td>
<td>0.2959</td>
<td>0.3371</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.3319</td>
<td>0.3320</td>
<td>0.3320</td>
<td>0.3311</td>
<td>0.4471</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.3320</td>
<td>0.3321</td>
<td>0.3315</td>
<td>0.3310</td>
<td>0.4172</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0.3476</td>
<td>0.3476</td>
<td>0.3466</td>
<td>0.3463</td>
<td>0.4152</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0.3070</td>
<td>0.3708</td>
<td>0.3706</td>
<td>0.3696</td>
<td>0.5018</td>
</tr>
</tbody>
</table>
Table 5.16: Comparison of the second lowest natural frequency of an orthotropic square plate: $a/h = 10, \quad \tilde{\omega} = \omega h (\rho / C_{11})^{\frac{1}{2}}$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>Exact</th>
<th>GTTR</th>
<th>STTR</th>
<th>FSDT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1.3077</td>
<td>1.3055</td>
<td>1.3086</td>
<td>1.3159</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1.3331</td>
<td>1.3339</td>
<td>1.3339</td>
<td>1.3410</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.4205</td>
<td>1.4213</td>
<td>1.4215</td>
<td>1.4283</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1.4316</td>
<td>1.4324</td>
<td>1.4323</td>
<td>1.4393</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>1.3765</td>
<td>1.3773</td>
<td>1.3772</td>
<td>1.3841</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1.5777</td>
<td>1.5786</td>
<td>1.5788</td>
<td>1.5857</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1.4596</td>
<td>1.4604</td>
<td>1.4603</td>
<td>1.4671</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1.5651</td>
<td>1.5659</td>
<td>1.5657</td>
<td>1.5727</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>1.4372</td>
<td>1.4379</td>
<td>1.4379</td>
<td>1.4445</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1.7179</td>
<td>1.7187</td>
<td>1.7186</td>
<td>1.7265</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1.5737</td>
<td>1.5745</td>
<td>1.5744</td>
<td>1.5812</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1.5068</td>
<td>1.5076</td>
<td>1.5076</td>
<td>1.5142</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1.6940</td>
<td>1.6948</td>
<td>1.6947</td>
<td>1.7022</td>
</tr>
</tbody>
</table>
Table 5.17: Comparison of the third lowest natural frequency of an orthotropic square plate: $a/h = 10$, $\tilde{\omega} = \omega h (\rho/C_{11})^{\frac{1}{2}}$

<table>
<thead>
<tr>
<th>$m$</th>
<th>$n$</th>
<th>Exact</th>
<th>GTTR.</th>
<th>STTR</th>
<th>FSDT</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1.6530</td>
<td>1.6542</td>
<td>1.6550</td>
<td>1.6646</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>1.7160</td>
<td>1.7178</td>
<td>1.7209</td>
<td>1.7305</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1.6805</td>
<td>1.6818</td>
<td>1.6827</td>
<td>1.6921</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1.7509</td>
<td>1.7528</td>
<td>1.7561</td>
<td>1.7653</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>1.8115</td>
<td>1.8143</td>
<td>1.8208</td>
<td>1.8306</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>1.7394</td>
<td>1.7347</td>
<td>1.7361</td>
<td>1.7450</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1.8523</td>
<td>1.8552</td>
<td>1.8620</td>
<td>1.8715</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>1.8195</td>
<td>1.8215</td>
<td>1.8253</td>
<td>1.8341</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>1.9306</td>
<td>1.9349</td>
<td>1.9461</td>
<td>1.9560</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>1.8548</td>
<td>1.8564</td>
<td>1.8586</td>
<td>1.8657</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1.9289</td>
<td>1.9320</td>
<td>1.9391</td>
<td>1.9480</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1.9749</td>
<td>1.9793</td>
<td>1.9906</td>
<td>2.0002</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>1.9447</td>
<td>1.9469</td>
<td>1.9511</td>
<td>1.9587</td>
</tr>
</tbody>
</table>
Table 5.18: Effect of degree of orthotropy of individual layers on the fundamental frequency of simply-supported symmetric square laminates: \(a/h = 5\), \(\bar{\omega} = 10 \times \omega(\rho h^2/E_T)^{\frac{1}{3}}\)

<table>
<thead>
<tr>
<th>Number of layers</th>
<th>Source</th>
<th>(E_t/E_T)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>Exact</td>
<td>2.6474</td>
</tr>
<tr>
<td></td>
<td>GTTR</td>
<td>2.6286</td>
</tr>
<tr>
<td></td>
<td>STTR</td>
<td>2.6211</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>2.6258</td>
</tr>
<tr>
<td></td>
<td>CLPT</td>
<td>2.9198</td>
</tr>
<tr>
<td>5</td>
<td>Exact</td>
<td>2.6587</td>
</tr>
<tr>
<td></td>
<td>GTTR</td>
<td>2.6416</td>
</tr>
<tr>
<td></td>
<td>STTR</td>
<td>2.6340</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>2.6337</td>
</tr>
<tr>
<td></td>
<td>CLPT</td>
<td>2.9198</td>
</tr>
<tr>
<td>7</td>
<td>Exact</td>
<td>2.6640</td>
</tr>
<tr>
<td></td>
<td>GTTR</td>
<td>2.6460</td>
</tr>
<tr>
<td></td>
<td>STTR</td>
<td>2.6384</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>2.6376</td>
</tr>
<tr>
<td></td>
<td>CLPT</td>
<td>2.9198</td>
</tr>
</tbody>
</table>
Table 5.19: Effect of degree of orthotropy of individual layers on the fundamental frequency of simply-supported antisymmetric square laminates: $a/h = 5$, $\tilde{\omega} = 10 \times \omega(h^2/E_T)^{\frac{1}{2}}$

<table>
<thead>
<tr>
<th>Number of layers</th>
<th>Source</th>
<th>3</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Exact</td>
<td>2.5031</td>
<td>2.7938</td>
<td>3.0698</td>
<td>3.2705</td>
<td>3.4250</td>
</tr>
<tr>
<td></td>
<td>GTTR</td>
<td>2.4936</td>
<td>2.8011</td>
<td>3.1331</td>
<td>3.4060</td>
<td>3.6384</td>
</tr>
<tr>
<td></td>
<td>STTR</td>
<td>2.4868</td>
<td>2.7955</td>
<td>3.1284</td>
<td>3.4020</td>
<td>3.6348</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>2.4834</td>
<td>2.7757</td>
<td>3.0824</td>
<td>3.3285</td>
<td>3.5333</td>
</tr>
<tr>
<td></td>
<td>CLPT</td>
<td>2.7082</td>
<td>3.0968</td>
<td>3.5422</td>
<td>3.9335</td>
<td>4.2884</td>
</tr>
<tr>
<td>4</td>
<td>Exact</td>
<td>2.6182</td>
<td>3.2578</td>
<td>3.7622</td>
<td>4.0660</td>
<td>4.2719</td>
</tr>
<tr>
<td></td>
<td>GTTR</td>
<td>2.6080</td>
<td>3.2863</td>
<td>3.8583</td>
<td>4.2208</td>
<td>4.4747</td>
</tr>
<tr>
<td></td>
<td>STTR</td>
<td>2.6003</td>
<td>3.2782</td>
<td>3.8506</td>
<td>4.2139</td>
<td>4.4686</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>2.6017</td>
<td>3.2898</td>
<td>3.8754</td>
<td>4.2479</td>
<td>4.5083</td>
</tr>
<tr>
<td></td>
<td>CLPT</td>
<td>2.8676</td>
<td>3.8877</td>
<td>4.9907</td>
<td>5.8900</td>
<td>6.6690</td>
</tr>
<tr>
<td>6</td>
<td>Exact</td>
<td>2.6440</td>
<td>3.3657</td>
<td>3.9359</td>
<td>4.2783</td>
<td>4.5091</td>
</tr>
<tr>
<td></td>
<td>GTTR</td>
<td>2.6299</td>
<td>3.3700</td>
<td>3.9745</td>
<td>4.3483</td>
<td>4.6060</td>
</tr>
<tr>
<td></td>
<td>STTR</td>
<td>2.6223</td>
<td>3.3621</td>
<td>3.9672</td>
<td>4.3419</td>
<td>4.6005</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>2.6228</td>
<td>3.3673</td>
<td>3.9771</td>
<td>4.3531</td>
<td>4.6106</td>
</tr>
<tr>
<td></td>
<td>CLPT</td>
<td>2.8966</td>
<td>4.0215</td>
<td>5.2234</td>
<td>6.1963</td>
<td>7.0359</td>
</tr>
<tr>
<td>10</td>
<td>Exact</td>
<td>2.6583</td>
<td>3.4250</td>
<td>4.0337</td>
<td>4.4011</td>
<td>4.6498</td>
</tr>
<tr>
<td></td>
<td>GTTR</td>
<td>2.6413</td>
<td>3.4128</td>
<td>4.0339</td>
<td>4.4140</td>
<td>4.6745</td>
</tr>
<tr>
<td></td>
<td>STTR</td>
<td>2.6337</td>
<td>3.4651</td>
<td>4.0270</td>
<td>4.4079</td>
<td>4.6692</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>2.6335</td>
<td>3.4053</td>
<td>4.0255</td>
<td>4.4023</td>
<td>4.6577</td>
</tr>
<tr>
<td></td>
<td>CLPT</td>
<td>2.9115</td>
<td>4.0888</td>
<td>5.3397</td>
<td>6.3489</td>
<td>7.2184</td>
</tr>
</tbody>
</table>
5.4 Static Analysis

In the first part of this section, the results are presented in the form of tables (Tables 5.20-5.28). In this part, the analytical and finite element solutions for center deflections \( w \) in a [0/90] square laminate, and for center deflections and transverse normal stresses \( \sigma_x \) in a [0/90] and [0/90/0] square laminate are given. The solutions are evaluated using the different theories, under different boundary and loading conditions. The dimensions of the laminate are taken to be 10 inch \( \times \) 10 inch, and results are presented for three different values of \( h/a \), 0.15, 0.2, and 0.3. The material is graphite-epoxy (see Table 5.1 for properties). The two different loading conditions are shown in Figure 5.3. UL is a uniformly distributed load over the entire plate, while LL is a line load along the centerline of the plate parallel to the \( x \)-axis. The magnitude of the loads are 1000 lb/sq.in. for UL and 1000 lb/in. for LL. The elements and discretizations used have been explained in Section 2.2.

For both deflections and normal stresses, GTTR and GTOT give the best results followed closely by STTR. FSDT gives good results for transverse deflections even in the case of thick plates, but its values for normal stresses in thick plates are not very good. As is well documented, CLPT does not give good results in the case of thick plates for either \( w \) or \( \sigma_x \). Due to the fine discretization employed, the finite element solutions are very close to the analytical solutions for each theory.

In the second part of this section, the results are presented graphically. Nondimensional center deflections, \( \bar{w} \), \( (\bar{w} = 100wh^3E_2/a^4q_0) \), and nondimensional center normal stresses, \( \bar{\sigma}_x \), \( (\bar{\sigma}_x = 10\sigma_xh^2/a^2q_0) \) are plotted against different parameters like span-to-thickness ratio \( (a/h) \), aspect ratio \( (b/a) \), and degree of orthotropy \( (E_1/E_2) \), for different stacking sequences, number of layers, boundary conditions, and using the different theories. The \( \sigma_x \) stress has been taken to be at the center of the topmost layer of the laminate. Towards the end of this section, we present some graphs which show the variation of \( \sigma_x \), \( \tau_{xy} \), \( \tau_{xz} \), and \( \sigma_z \) through the thickness of the plate. In all the above examples, the load is an uniformly distributed transverse load over the entire plate (UL). Unless mentioned otherwise, the material is graphite-epoxy.

In Figure 5.4 and 5.5, \( \bar{w} \) and \( \bar{\sigma}_x \) have been plotted against the span-to-thickness ratio, \( a/h \), for a [0/90] square laminate with SSSS boundary condition. For CLPT, \( \bar{w} \) remains constant with \( a/h \). All other theories give very close results for \( \bar{w} \). In the case of \( \bar{\sigma}_x \), the CLPT and FSDT values remain almost constant with \( a/h \). For \( a/h \)
Figure 5.3: Loading conditions  (a) Uniform load (UL)  (b) Line load (LL)
greater than 7.5, all theories give very similar results. For low \( a/h \), GTTR and STTR curves exhibit a high negative gradient. The GTOT curve is much flatter.

In Figure 5.6 and 5.7, \( \bar{w} \) and \( \bar{\sigma}_x \) have been plotted against \( a/h \) for a square SSCC laminate using GTTR for three different layups, [0], [0/90], and [0/90/0]. For very thick plates (\( a/h < 5 \)), the deflections are almost the same for all three. For moderately thick and thin plates, the deflections for [0] and [0/90/0] laminates are very close to each other, while it is higher for the [0/90] laminate. The center normal stress is highest for [0/90] antisymmetric laminate, while it is almost the same for [0] and [0/90/0] laminates.

In Figure 5.8 and 5.9, \( \bar{w} \) and \( \bar{\sigma}_x \) have been plotted against the aspect ratio, \( b/a \), for a square [0/90/0] SSCC laminate with \( a/h = 5 \). It is seen that both \( \bar{w} \) and \( \bar{\sigma}_x \) remain almost constant with increasing aspect ratio. For \( \bar{w} \), CLPT gives very low values, while FSDT gives slightly higher values. STTR and GTTR values are very close to each other. For \( \bar{\sigma}_x \), CLPT and FSDT values are close and they are lower than STTR and GTTR values, which are almost the same. The GTOT values have not been plotted in both the cases because they are almost exactly the same as the GTTR values.

In Figure 5.10 and 5.11, \( \bar{w} \) and \( \bar{\sigma}_x \) have been plotted against the span-to-thickness ratio, \( a/h \), for symmetric cross-ply square SSSS laminates with 3, 7, and 11 layers. The deflections are highest for \( N = 3 \), and becomes almost constant with an increase in the number of layers. The \( \bar{\sigma}_x \) values are very close for all three.

In Figure 5.12, \( \bar{w} \) has been plotted against \( a/h \) for a square SSFF [0/90/0/90] laminate with different modular ratios. In Figure 5.13, \( \bar{\sigma}_x \) has been plotted against \( a/h \) for the same laminate but with SSSS boundary condition. The material properties are those of Material 3 in Table 5.1. With increase in modular ratio, \( E_1/E_2 \), the deflections decrease and the normal stresses increase. With increasing modular ratio, \( \bar{w} \) seems to reach a limiting lower value, while \( \bar{\sigma}_x \) seems to reach a limiting upper value.

In Figure 5.14 and 5.15, \( \bar{w} \) and \( \bar{\sigma}_x \) have been plotted against \( a/h \) for a square [0/90] laminate using GTTR for different boundary conditions. The deflections are lowest for SSCC laminates, and highest for SSFF laminates. The normal stresses are highest for SSSS laminates, and lowest for SSFF laminates.

In Figure 5.16 and 5.17, \( \bar{w} \) and \( \bar{\sigma}_x \) have been plotted against the modular ratio, \( E_1/E_2 \), for a square SSSS antisymmetric cross-ply laminate with \( a/h = 5 \). The number of layers considered are \( N = 2, 6, \) and 10. The material considered is Material 3 in
Table 5.1. \( \bar{w} \) decreases with increasing modular ratio. This was also observed in Figure 5.12. \( \bar{w} \) also decreases with increasing number of layers. This trend, again, was observed for symmetric cross-ply laminates in Figure 5.10. However, in this case, for an antisymmetric cross-ply laminate, the difference in deflections between \( N = 2 \) and \( N = 6 \) is quite substantial. The values for \( N = 6 \) and \( N = 10 \) are close to each other. The normal stresses increase with increase in modular ratio (also observed in Figure 5.13). The stress values are highest for \( N = 2 \), and significantly lower for \( N = 6 \) and \( N = 10 \). \( \sigma_z \), like \( \bar{w} \), seems to reach a limiting value with increase in the number of layers.

In the remaining figures of this section, the span-to-thickness ratio, \( a/h \), is taken as 5. Figure 5.18 and 5.19 give the variation of \( \sigma_z \) stress through the thickness of a [0/90] SSCC and a [0/90/0] SSCC laminate respectively. The stresses have been evaluated at the center of the plate \( (x/a = 0.5, y/b = 0.5) \). It is seen that the variation in stress is much more pronounced in the 0 degree ply compared to the 90 degree ply. Also, in Figure 5.19, while CLPT, FSDT, and STTR give zero stresses at the mid-surface, GTTR and GTOT do not (although the stress values are very close to zero). Figure 5.20 gives the variation of \( \sigma_z \) stress through the thickness of a [0/90/0] laminate using GTTR for different boundary conditions. The values are highest in magnitude for SSSS and lowest for SSFF, with the other boundary conditions falling in between.

Figure 5.21 and 5.22 give the variation of \( \tau_{xy} \) stress through the thickness of a [0/90] SSCC and a [0/90/0] SSCC laminate respectively. The stresses have been evaluated at \( (x/a = 0.75, y/b = 0.75) \). Both the graphs show an almost linear variation of the inplane shear stress, changing in sign as it crosses the mid-surface.

Figure 5.23 and 5.24 give the variation of \( \tau_{xz} \) stress through the thickness of a [0/90] SSSS and a [0/90/0] SSSS laminate respectively. The \( \tau_{xz} \) stresses have been evaluated at the same location as the \( \tau_{xy} \) stresses, at \( (x/a = 0.75, y/b = 0.75) \). The variation in magnitude is less in the 90 degree ply compared to the 0 degree ply. STTR, GTTR, and GTOT correctly predict the quadratic variation within each layer, whereas FSDT gives constant values in each layer. Figure 5.25 and 5.26 gives the variation of \( \tau_{zz} \) stress through the thickness of a [0/90/0] laminate using GTTR for different boundary conditions. Note that the scales used for \( \tau_{xz} \) are different for the two graphs. The laminates involving free boundary conditions have much lower stress values than the rest.
Figure 5.27 gives the variation of $\sigma_z$ stress through the thickness of a [0/90/0] SSSS laminate. The location is the center of the plate. The variation is almost linear through the thickness, with a change in sign in between. Figure 5.28 gives the variation of $\sigma_z$ stress through the thickness of a [0/90/0] laminate using GTTR for different boundary conditions. The $\sigma_z$ values are all very close to each other in the 0 degree plies. In the 90 degree plies, the boundary conditions involving free edges exhibit a steep gradient in their stress values. The SSSS, SSCC, and SCS values are essentially the same. Also, the values at the mid-surface are almost the same for all six sets of boundary conditions.
Table 5.20: Center deflections $\bar{w}$ of simply-supported [0/90] antisymmetric cross-ply square laminates under different load and boundary conditions: $h/a = 0.15$, $\bar{w} = w \times 10^3$

<table>
<thead>
<tr>
<th>Load</th>
<th>Source</th>
<th>Type of solution</th>
<th>Boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>SSSS</td>
</tr>
<tr>
<td>GTOT</td>
<td>FEM.Q</td>
<td>517.996</td>
<td>276.916</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>509.308</td>
<td>271.808</td>
</tr>
<tr>
<td></td>
<td>Exact</td>
<td>515.727</td>
<td>267.373</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>512.575</td>
<td>261.749</td>
</tr>
<tr>
<td>UL</td>
<td>STTR</td>
<td>523.121</td>
<td>273.781</td>
</tr>
<tr>
<td></td>
<td>Exact</td>
<td>521.607</td>
<td>271.040</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>528.116</td>
<td>290.271</td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>528.648</td>
<td>290.551</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>528.191</td>
<td>299.063</td>
</tr>
<tr>
<td></td>
<td>Exact</td>
<td>441.034</td>
<td>174.391</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>440.439</td>
<td>173.168</td>
</tr>
<tr>
<td></td>
<td>CLPT</td>
<td>89.0559</td>
<td>51.0527</td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>87.8542</td>
<td>50.3176</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>87.4431</td>
<td>48.5869</td>
</tr>
<tr>
<td></td>
<td>Exact</td>
<td>87.7009</td>
<td>48.3439</td>
</tr>
<tr>
<td>LL</td>
<td>STTR</td>
<td>88.6627</td>
<td>49.7565</td>
</tr>
<tr>
<td></td>
<td>Exact</td>
<td>89.2436</td>
<td>49.9226</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>89.1016</td>
<td>53.3286</td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>92.2062</td>
<td>54.6579</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>92.2414</td>
<td>54.5059</td>
</tr>
<tr>
<td></td>
<td>Exact</td>
<td>72.1285</td>
<td>30.1901</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>71.8663</td>
<td>29.9775</td>
</tr>
</tbody>
</table>
Table 5.21: Center deflections $\bar{w}$ of simply-supported [0/90] antisymmetric cross-ply square laminates under different load and boundary conditions: $h/a = 0.2, \bar{w} = w \times 10^4$

<table>
<thead>
<tr>
<th>Load</th>
<th>Source</th>
<th>Type of solution</th>
<th>Boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>SSSS</td>
</tr>
<tr>
<td></td>
<td>FEM_L</td>
<td>241.441</td>
<td>145.831</td>
</tr>
<tr>
<td>GTTR</td>
<td>Exact</td>
<td>242.360</td>
<td>143.948</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>241.683</td>
<td>138.282</td>
</tr>
<tr>
<td>UL</td>
<td>Exact</td>
<td>244.985</td>
<td>143.898</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>246.618</td>
<td>142.555</td>
</tr>
<tr>
<td>STTR</td>
<td>Exact</td>
<td>251.648</td>
<td>157.041</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>251.786</td>
<td>157.191</td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>251.774</td>
<td>157.213</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>251.618</td>
<td>157.555</td>
</tr>
<tr>
<td>FSDT</td>
<td>Exact</td>
<td>186.061</td>
<td>73.5713</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>185.810</td>
<td>73.0552</td>
</tr>
<tr>
<td>CLPT</td>
<td>Exact</td>
<td>42.8409</td>
<td>27.6154</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>42.6595</td>
<td>27.5168</td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>41.7723</td>
<td>26.2295</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>42.3626</td>
<td>26.1251</td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>42.7467</td>
<td>26.7977</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>43.1342</td>
<td>26.7958</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>44.4738</td>
<td>29.5071</td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>45.4636</td>
<td>30.5325</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>45.5128</td>
<td>30.5473</td>
</tr>
<tr>
<td></td>
<td>Exact</td>
<td>30.4292</td>
<td>12.7364</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>30.3186</td>
<td>12.6468</td>
</tr>
</tbody>
</table>
Table 5.22: Center deflections $\ddot{w}$ of simply-supported $[0/90]$ antisymmetric cross-ply square laminates under different load and boundary conditions: $h/a = 0.3, \ddot{w} = w \times 10^6$

<table>
<thead>
<tr>
<th>Load</th>
<th>Source</th>
<th>Type of solution</th>
<th>Boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>SSFF</td>
</tr>
<tr>
<td>UL</td>
<td>GTOT</td>
<td>FEM.Q</td>
<td>93.1362</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FEM.L</td>
<td>92.8403</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Exact</td>
<td>92.0886</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FEM</td>
<td>92.1474</td>
</tr>
<tr>
<td></td>
<td>STTR</td>
<td>Exact</td>
<td>95.5751</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FEM</td>
<td>95.4117</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>Exact</td>
<td>98.8961</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FEM.Q</td>
<td>98.9687</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FEM.L</td>
<td>99.1177</td>
</tr>
<tr>
<td></td>
<td>CLPT</td>
<td>Exact</td>
<td>55.1293</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FEM</td>
<td>55.0549</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Exact</td>
<td>16.0020</td>
</tr>
<tr>
<td></td>
<td>STTR</td>
<td>Exact</td>
<td>17.0053</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FEM.Q</td>
<td>19.0244</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FEM.L</td>
<td>19.0682</td>
</tr>
</tbody>
</table>
Table 5.23: Center deflections $\hat{w}$ of simply-supported $[0/90/0]$ symmetric cross-ply square laminates under different load and boundary conditions: $h/a = 0.15, \hat{w} = w \times 10^4$

<table>
<thead>
<tr>
<th>Load</th>
<th>Source</th>
<th>Type of solution</th>
<th>Boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>SSSS</td>
</tr>
<tr>
<td>GTOT</td>
<td>FEM.Q</td>
<td>352.245</td>
<td>201.402</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>349.185</td>
<td>200.117</td>
</tr>
<tr>
<td>GTTR</td>
<td>Exact</td>
<td>351.617</td>
<td>194.590</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>351.002</td>
<td>192.302</td>
</tr>
<tr>
<td>STTR</td>
<td>Exact</td>
<td>355.161</td>
<td>195.070</td>
</tr>
<tr>
<td>UL</td>
<td>FEM</td>
<td>355.244</td>
<td>192.829</td>
</tr>
<tr>
<td>FSDT</td>
<td>Exact</td>
<td>345.609</td>
<td>197.990</td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>345.813</td>
<td>198.179</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>346.152</td>
<td>198.663</td>
</tr>
<tr>
<td>CLPT</td>
<td>Exact</td>
<td>229.028</td>
<td>53.1624</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>229.055</td>
<td>53.1885</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Load</th>
<th>Source</th>
<th>Type of solution</th>
<th>Boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>SSSS</td>
</tr>
<tr>
<td>GTOT</td>
<td>FEM.Q</td>
<td>66.0826</td>
<td>42.0033</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>65.4267</td>
<td>41.5902</td>
</tr>
<tr>
<td>GTTR</td>
<td>Exact</td>
<td>65.2329</td>
<td>40.1324</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>65.4576</td>
<td>40.2432</td>
</tr>
<tr>
<td>STTR</td>
<td>Exact</td>
<td>66.5903</td>
<td>40.9559</td>
</tr>
<tr>
<td>LL</td>
<td>FEM</td>
<td>67.3967</td>
<td>41.4982</td>
</tr>
<tr>
<td>FSDT</td>
<td>Exact</td>
<td>65.1123</td>
<td>41.4776</td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>66.7091</td>
<td>43.0740</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>66.6992</td>
<td>43.0491</td>
</tr>
<tr>
<td>CLPT</td>
<td>Exact</td>
<td>41.6534</td>
<td>12.7945</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>41.6711</td>
<td>12.8977</td>
</tr>
</tbody>
</table>
Table 5.24: Center deflections $\bar{w}$ of simply-supported [0/90/0] symmetric cross-ply square laminates under different load and boundary conditions: $h/a = 0.2, \bar{w} = w \times 10^4$

<table>
<thead>
<tr>
<th>Load</th>
<th>Source</th>
<th>Type of solution</th>
<th>Boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>SSSS</td>
</tr>
<tr>
<td>GTOT</td>
<td>FEM.Q</td>
<td>185.576</td>
<td>126.286</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>184.718</td>
<td>125.832</td>
</tr>
<tr>
<td>GTTR</td>
<td>Exact</td>
<td>185.282</td>
<td>120.408</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>185.396</td>
<td>119.594</td>
</tr>
<tr>
<td>STTR</td>
<td>Exact</td>
<td>188.344</td>
<td>120.849</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>188.414</td>
<td>119.588</td>
</tr>
<tr>
<td>UL</td>
<td>FSDT</td>
<td>Exact</td>
<td>182.011</td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>182.148</td>
<td>126.638</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>182.500</td>
<td>127.093</td>
</tr>
<tr>
<td>CLPT</td>
<td>Exact</td>
<td>96.6214</td>
<td>22.4279</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>96.6326</td>
<td>22.4389</td>
</tr>
<tr>
<td>LL</td>
<td>GTOT</td>
<td>FEM.Q</td>
<td>34.9732</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>34.8992</td>
<td>25.5056</td>
</tr>
<tr>
<td></td>
<td>Exact</td>
<td>34.6047</td>
<td>24.2636</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>34.8516</td>
<td>24.4559</td>
</tr>
<tr>
<td>STTR</td>
<td>Exact</td>
<td>35.7848</td>
<td>25.0037</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>36.2897</td>
<td>25.3514</td>
</tr>
<tr>
<td>FSDT</td>
<td>Exact</td>
<td>34.8615</td>
<td>26.0017</td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>36.0250</td>
<td>27.1659</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>36.6008</td>
<td>27.2026</td>
</tr>
<tr>
<td>CLPT</td>
<td>Exact</td>
<td>17.5725</td>
<td>5.39766</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>17.5800</td>
<td>5.44122</td>
</tr>
</tbody>
</table>
Table 5.25: Center deflections $\bar{w}$ of simply-supported $[0/90/0]$ symmetric cross-ply square laminates under different load and boundary conditions: $h/a = 0.3, \bar{w} = w \times 10^4$

<table>
<thead>
<tr>
<th>Load</th>
<th>Source</th>
<th>Type of solution</th>
<th>Boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>SSSS</td>
</tr>
<tr>
<td>GTOT</td>
<td>FEM.Q</td>
<td>83.1366</td>
<td>66.8417</td>
</tr>
<tr>
<td>FEM.L</td>
<td>83.0203</td>
<td>66.7606</td>
<td>325.109</td>
</tr>
<tr>
<td>GTTR</td>
<td>Exact</td>
<td>83.1862</td>
<td>62.0734</td>
</tr>
<tr>
<td>FEM</td>
<td>83.3670</td>
<td>61.9013</td>
<td>327.624</td>
</tr>
<tr>
<td>STTR</td>
<td>Exact</td>
<td>85.8345</td>
<td>62.7392</td>
</tr>
<tr>
<td>FEM</td>
<td>85.8300</td>
<td>62.2308</td>
<td>340.465</td>
</tr>
<tr>
<td>UL</td>
<td>FSDT</td>
<td>Exact</td>
<td>82.7761</td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>82.8604</td>
<td>70.0597</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>83.1223</td>
<td>70.3613</td>
</tr>
<tr>
<td>CLPT</td>
<td>Exact</td>
<td>28.6286</td>
<td>6.64539</td>
</tr>
<tr>
<td>FEM</td>
<td>28.6319</td>
<td>6.64856</td>
<td>259.258</td>
</tr>
<tr>
<td>LL</td>
<td>GTOT</td>
<td>FEM.Q</td>
<td>15.5263</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>15.6377</td>
<td>13.0444</td>
</tr>
<tr>
<td>GTTR</td>
<td>Exact</td>
<td>15.5921</td>
<td>12.2305</td>
</tr>
<tr>
<td>FEM</td>
<td>15.7051</td>
<td>12.3187</td>
<td>54.4442</td>
</tr>
<tr>
<td>FSDT</td>
<td>Exact</td>
<td>16.3060</td>
<td>14.2661</td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>17.0604</td>
<td>15.0265</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>17.1087</td>
<td>15.0712</td>
</tr>
<tr>
<td>CLPT</td>
<td>Exact</td>
<td>5.26667</td>
<td>1.59931</td>
</tr>
<tr>
<td>FEM</td>
<td>5.20888</td>
<td>1.61221</td>
<td>41.4735</td>
</tr>
</tbody>
</table>
Table 5.26: Center normal stresses $\bar{\sigma}_x$ for simply-supported [0/90/0] symmetric cross-ply square laminates under different load and boundary conditions: $h/a = 0.15, \bar{\sigma}_x = \sigma_x/10$

<table>
<thead>
<tr>
<th>Load</th>
<th>Source</th>
<th>Type of solution</th>
<th>Boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>SSSS</td>
<td>SCC</td>
</tr>
<tr>
<td>GTOT</td>
<td>FEM.Q</td>
<td>1889.11</td>
<td>709.589</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>1857.04</td>
<td>696.314</td>
</tr>
<tr>
<td>GTTR</td>
<td>Exact</td>
<td>1913.36</td>
<td>723.159</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>1889.14</td>
<td>709.748</td>
</tr>
<tr>
<td>STTR</td>
<td>Exact</td>
<td>1918.64</td>
<td>708.770</td>
</tr>
<tr>
<td>UL</td>
<td>FEM</td>
<td>1918.45</td>
<td>707.512</td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>2013.60</td>
<td>802.016</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>2003.40</td>
<td>794.235</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>1991.37</td>
<td>789.210</td>
</tr>
<tr>
<td>CLPT</td>
<td>Exact</td>
<td>2148.26</td>
<td>825.448</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>2149.88</td>
<td>827.754</td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>350.959</td>
<td>159.038</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>341.651</td>
<td>153.068</td>
</tr>
<tr>
<td>GTTR</td>
<td>Exact</td>
<td>350.319</td>
<td>157.402</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>362.941</td>
<td>172.904</td>
</tr>
<tr>
<td>STTR</td>
<td>Exact</td>
<td>345.302</td>
<td>140.885</td>
</tr>
<tr>
<td>LL</td>
<td>FEM</td>
<td>348.863</td>
<td>145.198</td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>363.311</td>
<td>156.861</td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>360.019</td>
<td>154.626</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>355.762</td>
<td>151.944</td>
</tr>
<tr>
<td>CLPT</td>
<td>Exact</td>
<td>386.013</td>
<td>101.235</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>388.239</td>
<td>194.000</td>
</tr>
</tbody>
</table>
Table 5.27: Center normal stresses $\bar{\sigma}_x$ for simply-supported [0/90/0] symmetric cross-ply square laminates under different load and boundary conditions: $h/a = 0.2, \bar{\sigma}_x = \sigma_x/10$

<table>
<thead>
<tr>
<th>Load</th>
<th>Source</th>
<th>Type of solution</th>
<th>SSSS</th>
<th>SCCC</th>
<th>SSFF</th>
<th>SSCS</th>
<th>SSCF</th>
<th>SSSF</th>
</tr>
</thead>
<tbody>
<tr>
<td>GTOT</td>
<td>FEM.Q</td>
<td>973.641</td>
<td>320.041</td>
<td>30.2642</td>
<td>653.548</td>
<td>256.610</td>
<td>459.510</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>960.459</td>
<td>314.217</td>
<td>31.6600</td>
<td>647.906</td>
<td>241.614</td>
<td>450.267</td>
<td></td>
</tr>
<tr>
<td>GTTR</td>
<td>Exact</td>
<td>992.114</td>
<td>334.057</td>
<td>30.8988</td>
<td>630.800</td>
<td>308.397</td>
<td>478.964</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>982.709</td>
<td>327.325</td>
<td>19.4407</td>
<td>644.871</td>
<td>274.358</td>
<td>462.094</td>
<td></td>
</tr>
<tr>
<td>STTR</td>
<td>Exact</td>
<td>995.369</td>
<td>344.509</td>
<td>17.6550</td>
<td>635.385</td>
<td>327.724</td>
<td>473.740</td>
<td></td>
</tr>
<tr>
<td>UL</td>
<td>FEM</td>
<td>995.066</td>
<td>343.802</td>
<td>17.9600</td>
<td>647.690</td>
<td>291.056</td>
<td>469.563</td>
<td></td>
</tr>
<tr>
<td>FSDT</td>
<td>Exact</td>
<td>1085.14</td>
<td>431.937</td>
<td>20.2023</td>
<td>733.613</td>
<td>262.910</td>
<td>516.305</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>1079.65</td>
<td>427.733</td>
<td>20.2709</td>
<td>751.250</td>
<td>209.910</td>
<td>499.454</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>1072.81</td>
<td>424.712</td>
<td>20.6861</td>
<td>750.503</td>
<td>200.040</td>
<td>493.030</td>
<td></td>
</tr>
<tr>
<td>CLPT</td>
<td>Exact</td>
<td>1208.39</td>
<td>464.315</td>
<td>32.3017</td>
<td>680.166</td>
<td>280.892</td>
<td>629.531</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>1209.31</td>
<td>465.612</td>
<td>32.7585</td>
<td>690.654</td>
<td>259.913</td>
<td>627.118</td>
<td></td>
</tr>
<tr>
<td>GTOT</td>
<td>FEM.Q</td>
<td>172.668</td>
<td>58.9037</td>
<td>9.7681</td>
<td>117.795</td>
<td>33.9851</td>
<td>83.3899</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>170.031</td>
<td>57.6618</td>
<td>9.07930</td>
<td>116.507</td>
<td>32.0467</td>
<td>81.5028</td>
<td></td>
</tr>
<tr>
<td>GTTR</td>
<td>Exact</td>
<td>176.810</td>
<td>62.2996</td>
<td>6.07885</td>
<td>116.326</td>
<td>43.2240</td>
<td>86.1107</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>174.305</td>
<td>61.3136</td>
<td>0.07714</td>
<td>117.201</td>
<td>36.8403</td>
<td>83.7726</td>
<td></td>
</tr>
<tr>
<td>STTR</td>
<td>Exact</td>
<td>188.970</td>
<td>67.9291</td>
<td>6.66829</td>
<td>118.186</td>
<td>43.1874</td>
<td>87.4451</td>
<td></td>
</tr>
<tr>
<td>LL</td>
<td>FEM</td>
<td>181.575</td>
<td>70.8969</td>
<td>9.9552</td>
<td>123.380</td>
<td>34.8713</td>
<td>88.8165</td>
<td></td>
</tr>
<tr>
<td>FSDT</td>
<td>Exact</td>
<td>195.984</td>
<td>82.6924</td>
<td>7.51739</td>
<td>135.614</td>
<td>32.2051</td>
<td>95.7561</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FEM.Q</td>
<td>194.187</td>
<td>81.5892</td>
<td>7.61716</td>
<td>138.217</td>
<td>24.3347</td>
<td>91.8081</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FEM.L</td>
<td>191.814</td>
<td>80.2266</td>
<td>7.53731</td>
<td>137.054</td>
<td>23.4689</td>
<td>90.0119</td>
<td></td>
</tr>
<tr>
<td>CLPT</td>
<td>Exact</td>
<td>217.132</td>
<td>107.570</td>
<td>11.6526</td>
<td>137.876</td>
<td>23.9371</td>
<td>115.618</td>
<td></td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>218.384</td>
<td>109.158</td>
<td>12.8193</td>
<td>140.920</td>
<td>19.7495</td>
<td>115.979</td>
<td></td>
</tr>
</tbody>
</table>

80
Table 5.28: Center normal stresses $\sigma_x$ for simply-supported $[0/90/0]$ symmetric cross-ply square laminates under different load and boundary conditions: $h/a = 0.3, \sigma_x = \sigma_x/10$

<table>
<thead>
<tr>
<th>Load</th>
<th>Source</th>
<th>Type of solution</th>
<th>Boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>SSSS</td>
</tr>
<tr>
<td>GTOT</td>
<td>FEM,Q</td>
<td>348.581</td>
<td>87.9982</td>
</tr>
<tr>
<td></td>
<td>FEM,L</td>
<td>344.681</td>
<td>86.2224</td>
</tr>
<tr>
<td>STTR</td>
<td>Exact</td>
<td>360.856</td>
<td>91.3736</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>358.320</td>
<td>89.5469</td>
</tr>
<tr>
<td>STTR</td>
<td>Exact</td>
<td>359.121</td>
<td>96.2008</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>358.772</td>
<td>95.7166</td>
</tr>
<tr>
<td>FSDT</td>
<td>Exact</td>
<td>438.498</td>
<td>173.347</td>
</tr>
<tr>
<td></td>
<td>FEM,Q</td>
<td>436.311</td>
<td>171.697</td>
</tr>
<tr>
<td></td>
<td>FEM,L</td>
<td>433.361</td>
<td>170.369</td>
</tr>
<tr>
<td>CLPT</td>
<td>Exact</td>
<td>537.064</td>
<td>206.362</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>537.471</td>
<td>206.939</td>
</tr>
<tr>
<td>GTOT</td>
<td>FEM,Q</td>
<td>61.9769</td>
<td>16.1048</td>
</tr>
<tr>
<td></td>
<td>FEM,L</td>
<td>61.0269</td>
<td>15.5878</td>
</tr>
<tr>
<td>STTR</td>
<td>Exact</td>
<td>64.2224</td>
<td>16.9090</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>64.1179</td>
<td>17.3335</td>
</tr>
<tr>
<td>FSDT</td>
<td>Exact</td>
<td>64.6119</td>
<td>19.6445</td>
</tr>
<tr>
<td></td>
<td>FEM,Q</td>
<td>66.1283</td>
<td>21.2389</td>
</tr>
<tr>
<td></td>
<td>FEM,L</td>
<td>77.8648</td>
<td>31.8005</td>
</tr>
<tr>
<td>CLPT</td>
<td>Exact</td>
<td>96.5031</td>
<td>47.8089</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>97.0598</td>
<td>48.5149</td>
</tr>
</tbody>
</table>
Figure 5.4: Nondimensional center deflection $\bar{w}$ vs span-to-thickness ratio $a/h$ for [0/90] square SSSS laminate using different theories
Figure 5.5: Nondimensional center normal stress $\sigma_x$ vs span-to-thickness ratio $a/h$ for $[0/90]$ square SSSS laminate using different theories
Figure 5.6: Nondimensional center deflection $\bar{w}$ vs span-to-thickness ratio $a/h$ for different layups of a square SSCC laminate using GTTR.
Figure 5.7: Nondimensional center normal stress $\sigma_x$ vs span-to-thickness ratio $a/h$ for different layups of a square SSCC laminate using GTTR
Figure 5.8: Nondimensional center deflection $\tilde{w}$ vs aspect ratio $b/a$ for a $[0/90/0]$ square SSCC laminate ($a/h = 5$) using different theories
Figure 5.9: Nondimensional center normal stress $\sigma_x$ vs aspect ratio $b/a$ for a [0/90/0] square SSCC laminate ($a/h = 5$) using different theories.
Figure 5.10: Nondimensional center deflection $\bar{w}$ vs span-to-thickness ratio $a/h$ for symmetric cross-ply square SSSS laminates with different number of layers $N$ using GTTR.
Figure 5.11: Nondimensional center normal stress $\sigma_x$ vs span-to-thickness ratio $a/h$ for symmetric cross-ply square SSSS laminates with different number of layers $N$ using GTTR
Figure 5.12: Nondimensional center deflection $\bar{w}$ vs span-to-thickness ratio $a/h$ for [0/90/0/90] square SSFF laminates with different modular ratios using GTTR
Figure 5.13: Nondimensional center normal stress $\sigma_x$ vs span-to-thickness ratio $a/h$ for [0/90/0/90] square SSSS laminates with different modular ratios using GTTR
Figure 5.14: Nondimensional center deflection $\bar{w}$ vs span-to-thickness ratio $a/h$ for a [0/90] square laminate with different boundary conditions using GTTR.
Figure 5.15: Nondimensional center normal stress $\sigma_z$ vs span-to-thickness ratio $a/h$ for a $[0/90]$ square laminate with different boundary conditions using GTTR.
Figure 5.16: Nondimensional center deflection $\bar{w}$ vs modular ratio $E_1/E_2$ for antisymmetric cross-ply square SSSS laminates ($a/h = 5$) with different number of layers $N$ using GTTR.
Figure 5.17: Nondimensional center normal stress $\bar{w}$ vs modular ratio $E_1/E_2$ for antisymmetric cross-ply square SSSS laminates $(a/h = 5)$ with different number of layers $N$ using GTTR.
Figure 5.18: Variation of center normal stress $\sigma_z/q_0$ through the thickness of a $[0/90]$ square SSCC laminate using different theories.
Figure 5.19: Variation of center normal stress $\sigma_x/q_0$ through the thickness of a [0/90/0] square SSCC laminate using different theories
Figure 5.20: Variation of center normal stress $\sigma_z/q_0$ through the thickness of a [0/90/0] square laminate for different boundary conditions using GTTR
Figure 5.21: Variation of inplane shear stress $\tau_{xz}/q_0$ through the thickness of a [0/90] square SSCC laminate using different theories.
Figure 5.22: Variation of inplane shear stress $\tau_{xy}/q_0$ through the thickness of a [0/90/0] square SSCC laminate using different theories
Figure 5.23: Variation of transverse shear stress $\tau_{xz}/q_0$ through the thickness of a [0/90] square SSSS laminate using different theories.
Figure 5.24: Variation of transverse shear stress $\tau_{xz}/q_0$ through the thickness of a [0/90/0] square SSSS laminate using different theories
Figure 5.25: Variation of transverse shear stress $\tau_{xz}/g_0$ through the thickness of a $[0/90/0]$ square laminate for different boundary conditions (SSFF, SSSF, SSCF) using GTTR.
Figure 5.26: Variation of transverse shear stress $\tau_{xz}/q_0$ through the thickness of a [0/90/0] square laminate for different boundary conditions (SSCS,SSCC,SSSS) using GTTR.
Figure 5.27: Variation of transverse normal stress $\sigma_z/q_0$ through the thickness of a [0/90/0] square SSSS laminate using different theories.
Figure 3.28: Variation of transverse normal stress $\sigma_z/q_0$ through the thickness of a [0/90/0] square laminate for different boundary conditions using GTTR
5.5 Free Vibration Analysis

Results of free vibration analysis are presented in this section. The dimension of the plate is the same as in static analysis, and the material properties are those of graphite-epoxy unless otherwise mentioned. Table 5.29 gives the lowest natural frequency of vibration of antisymmetric square plates under different boundary conditions.

In Figure 5.29, the nondimensional fundamental frequency, $\bar{\omega}$, ($\bar{\omega} = \omega(pa^4/E_2h^2)^{\frac{1}{2}}$) is plotted against $a/h$ for square SSSS [0/90] laminates. Apart from CLPT, which gives consistently higher values, all the other theories give quite close results.

In Figure 5.30, $\bar{\omega}$ is plotted against $a/h$ for three different layups [0], [0/90], and [0/90/0], of square SSCC laminates. [0/90] has the lowest frequency among the three. The values for [0] and [0/90/0] laminates are very close.

In Figure 5.31, $\bar{\omega}$ is plotted against the aspect ratio, $b/a$ of a square SSSS [0/90/0] laminate with $a/h = 5$. GTTR, STTR, and FSDT values are close to each other. $\bar{\omega}$ is higher for low values of $b/a$, but flattens out and remains almost constant for $b/a > 2$.

In Figure 5.32, the nondimensional fundamental frequency is plotted against $a/h$ for different number of layers $N$ of a square symmetric cross-ply SSSS laminate. $\bar{\omega}$ is the lowest for $N = 3$, and reaches a limiting upper value as the number of layers increase.

For Figure 5.33 and Figure 5.34, the material properties are those of Material 3 in Table 5.1. In Figure 5.33, $\bar{\omega}$ is plotted against $a/h$ for a square [0/96/0/90] SSSS laminate for different modular ratios $E_1/E_2$. The frequencies are lowest for $E_1/E_2 = 5$, and increases with increasing modular ratio.

In Figure 5.34, $\bar{\omega}$ is plotted against $E_1/E_2$ for square antisymmetric cross-ply SSSS laminates with $a/h = 5$ using GTTR. $\bar{\omega}$ reaches an upper limit with increase in the number of layers $N$, and also increases with increase in modular ratio, trends which were also evident in the last two plots.
Table 5.29: Dimensionless fundamental frequencies of simply-supported antisymmetric cross-ply square laminates under different boundary conditions: $\tilde{\omega} = (\omega b^2 / h)(\rho / E_2)^{1/2}$

<table>
<thead>
<tr>
<th>No of layers</th>
<th>$h/a$</th>
<th>Source</th>
<th>Boundary condition</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>SSSS</td>
<td>SCC</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>GTTR</td>
<td>9.0959</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CLPT</td>
<td>10.7207</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>GTTR</td>
<td>6.5792</td>
</tr>
<tr>
<td></td>
<td></td>
<td>STTR</td>
<td>6.5729</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CLPT</td>
<td>9.3590</td>
</tr>
<tr>
<td>10</td>
<td>0.2</td>
<td>GTTR</td>
<td>11.6861</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
<td>GTTR</td>
<td>7.1302</td>
</tr>
<tr>
<td></td>
<td></td>
<td>STTR</td>
<td>7.1237</td>
</tr>
<tr>
<td></td>
<td></td>
<td>FSDT</td>
<td>6.9550</td>
</tr>
<tr>
<td></td>
<td></td>
<td>CLPT</td>
<td>16.5366</td>
</tr>
</tbody>
</table>
Figure 5.29: Nondimensional fundamental frequency $\bar{\omega}$ vs span-to-thickness ratio $a/h$ for $[0/90]$ square SSSS laminate using different theories
Figure 5.30: Nondimensional fundamental frequency $\tilde{\omega}$ vs span-to-thickness ratio $a/h$ for different layups of a square SSCC laminate using GTTR.
Figure 5.31: Nondimensional fundamental frequency $\bar{\omega}$ vs aspect ratio $b/a$ for a [0/90/0] square SSSS laminate ($a/h = 5$) using different theories
Figure 5.32: Nondimensional fundamental frequency $\tilde{\omega}$ vs spar-to-thickness ratio $a/h$ for symmetric cross-ply square SSSS laminates with different number of layers $N$ using GTTR
Figure 5.33: Nondimensional fundamental frequency $\bar{\omega}$ vs span-to-thickness ratio $a/h$ for [0/90/0/90] square SSSS laminates with different modular ratios using GTTR.
Figure 5.34: Nondimensional fundamental frequency $\bar{\omega}$ vs modular ratio $E_1/E_2$ for antisymmetric cross-ply square SSSS laminates ($a/h = 5$) with different number of layers $N$ using GTTR.
5.6 Conclusions

The inability of CLPT to model thick plates is well known. Since thick plates have been considered in most of the numerical examples, that shortcoming of CLPT is well evident. It gives lower values for deflections, and higher values for frequencies. It gives higher values for some stresses, and lower values for some others. FSDT gives reasonably good results for deflections and frequencies, even for thick plates. But the error in the inplane stress values predicted by FSDT increases as the plate gets thicker. The transverse shear stress values given by FSDT are quite inaccurate for all thicknesses. GTTR and GTOT give the most accurate results in almost every case. GTTR has the added advantage that it takes into account the zero transverse shear stress conditions at the top and bottom of the plate. Overall, the STTR values are always very close to those predicted by GTTR and GTOT. Considering the fact that getting solutions using STTR requires less computational effort than doing the same using GTTR and GTOT, STTR seems to possess the optimal performance among the ESL theories considered in this study.
Bibliography


121


Appendix A

Navier Solution Coefficients

A.1 Stiffness Matrix Coefficients

\[ S_{11} = -\left(\alpha^2 A_{11} + \beta^2 A_{66}\right) \]
\[ S_{12} = -\alpha \beta (A_{12} + A_{66}) \]
\[ S_{13} = -2\gamma \alpha c_1 B_{13} \]
\[ S_{14} = -\alpha^2 (B_{11} - \chi d_1 E_{11}) \]
\[ S_{15} = 0 \]
\[ S_{16} = \lambda \alpha \left(A_{13} + \frac{1}{2} \alpha^2 D_{11} + \frac{1}{2} \beta^2 D_{12} + \beta^2 D_{66}\right) \]
\[ S_{17} = \gamma \alpha \left(2B_{13} + \frac{1}{3} \alpha^2 E_{11}\right) \]
\[ S_{22} = -\left(\alpha^2 A_{66} + \beta^2 A_{22}\right) \]
\[ S_{23} = -2\gamma \beta c_1 B_{23} \]
\[ S_{24} = 0 \]
\[ S_{25} = -\beta^2 (B_{22} - \chi d_1 E_{22}) \]
\[ S_{26} = \lambda \beta \left(A_{23} + \frac{1}{2} \alpha^2 D_{12} + \frac{1}{2} \beta^2 D_{22} + \alpha^2 D_{66}\right) \]
\[ S_{27} = \gamma \beta \left(2B_{23} + \frac{1}{3} \beta^2 E_{22}\right) \]
\[ S_{33} = -\alpha^2 A_{55} - \beta^2 A_{44} + 2\gamma \alpha^2 c_1 D_{55} + 2\gamma \beta^2 c_1 D_{44} - \gamma^2 \alpha^2 c_2 F_{55} \]
\[ -\gamma^2 \beta^2 c_2 F_{44} - 4\gamma^2 c_2 D_{33} \]
\[ S_{34} = \alpha \left( -A_{55} + \chi c_1 D_{55} + \gamma c_1 D_{55} - \chi \gamma c_2 F_{55} - 2 \gamma c_1 D_{13} + \frac{2}{3} \chi \gamma c_2 F_{13} \right) \]
\[ S_{35} = \beta \left( -A_{44} + \chi c_1 D_{44} + \gamma c_1 D_{44} - \chi \gamma c_2 F_{44} - 2 \gamma c_1 D_{23} + \frac{2}{3} \chi \gamma c_2 F_{23} \right) \]
\[ S_{36} = \lambda \gamma c_1 \left( 2B_{33} + \alpha^2 E_{13} + \beta^2 E_{23} \right) \]
\[ S_{37} = \gamma^2 c_1 \left( 4D_{33} + \frac{2}{3} \alpha^2 F_{13} + \frac{2}{3} \beta^2 F_{23} \right) \]

\[ S_{44} = -A_{55} + 2\chi c_1 D_{55} - \chi^2 c_2 F_{55} - \alpha^2 D_{11} - \beta^2 D_{66} + 2\chi \alpha^2 d_1 F_{11} + 2\chi \beta^2 d_1 F_{66} - \chi^2 \alpha^2 d_2 H_{11} - \chi^2 \beta^2 d_2 H_{66} \]
\[ S_{45} = \alpha \beta \left( -D_{12} - D_{66} + 2\chi d_1 F_{12} + 2\chi d_1 F_{66} - \chi^2 d_2 H_{12} - \chi^2 d_2 H_{66} \right) \]
\[ S_{46} = \lambda \alpha \left( B_{13} + \frac{1}{2} \alpha^2 E_{13} - \chi d_1 E_{13} - \frac{1}{2} \chi \alpha^2 d_1 G_{11} \right) \]
\[ S_{47} = \gamma \alpha \left( 2D_{13} + \frac{1}{3} \alpha^2 F_{13} + \frac{1}{3} \beta^2 F_{12} + \frac{2}{3} \beta^2 F_{66} - 2\chi d_1 F_{13} - \frac{1}{3} \chi \alpha^2 d_1 H_{11} - \frac{1}{3} \chi \beta^2 d_1 H_{12} - \frac{2}{3} \chi \beta^2 d_1 H_{66} \right) \]

\[ S_{55} = -A_{44} + 2\chi c_1 D_{44} - \chi^2 c_2 F_{44} - \alpha^2 D_{66} - \beta^2 D_{22} + 2\chi \alpha^2 d_1 F_{66} + 2\chi \beta^2 d_1 F_{22} - \chi^2 \alpha^2 d_2 H_{66} - \chi^2 \beta^2 d_2 H_{22} \]
\[ S_{56} = \lambda \beta \left( B_{23} + \frac{1}{2} \beta^2 E_{23} - \chi d_1 E_{23} - \frac{1}{2} \chi \beta^2 d_1 G_{22} \right) \]
\[ S_{57} = \gamma \beta \left( 2D_{23} + \frac{1}{3} \alpha^2 F_{23} + \frac{1}{3} \beta^2 F_{22} + \frac{2}{3} \alpha^2 F_{66} - 2\chi d_1 F_{23} - \frac{1}{3} \chi \alpha^2 d_1 H_{12} - \frac{1}{3} \chi \beta^2 d_1 H_{22} - \frac{2}{3} \chi \alpha^2 d_1 H_{66} \right) \]

\[ S_{66} = \frac{1}{2} \lambda^2 \left( -2A_{33} - 2\alpha^2 D_{13} - 2\beta^2 D_{23} - \frac{1}{2} \alpha^4 F_{11} - \alpha^2 \beta^2 F_{12} - \frac{1}{2} \beta^4 F_{22} - 2\alpha^2 \beta^2 F_{66} \right) \]
\[ S_{67} = \frac{1}{6} \lambda \gamma \left( -12B_{33} - 8\alpha^2 E_{13} - 8\beta^2 E_{23} - \alpha^4 G_{11} - \beta^4 G_{22} \right) \]
\[ S_{77} = \frac{1}{3} \gamma^2 \left( -12D_{33} - 4\alpha^2 F_{13} - 4\beta^2 F_{23} - \frac{1}{3} \alpha^4 H_{11} - \frac{2}{3} \alpha^2 \beta^2 H_{12} - \frac{1}{3} \beta^4 H_{22} - \frac{4}{3} \alpha^2 \beta^2 H_{66} \right) \]

(A.1)
A.2 Mass Matrix Coefficients

\begin{align*}
M_{11} &= I_1 \\ M_{16} &= -\frac{1}{2} \lambda \alpha I_3 \\
M_{22} &= I_1 \\ M_{26} &= -\frac{1}{2} \lambda \beta I_3 \\
M_{33} &= I_I \\ M_{37} &= \gamma I_3 \\
M_{44} &= I_3 \\ M_{47} &= -\frac{1}{3} \gamma \alpha I_5 \\
M_{55} &= I_3 \\ M_{57} &= -\frac{1}{3} \gamma \beta I_5 \\
M_{66} &= \lambda^2 \left(I_3 + \frac{1}{4} \alpha^2 I_5 + \frac{1}{4} \beta^2 I_5 \right) \\
M_{77} &= \gamma^2 \left(I_5 + \frac{1}{9} \alpha^2 I_7 + \frac{1}{9} \beta^2 I_7 \right) \quad (A.2)
\end{align*}
Appendix B

Levy Solution Coefficients

B.1 Coefficients in Equation (3.14)

\[ e_1 = -\beta^2A_{66} + \omega_m^2I_1 \]
\[ e_2 = A_{11} \]
\[ e_3 = -\beta(A_{12} + A_{66}) \]
\[ e_4 = -2\gamma c_1B_{13} \]
\[ e_5 = B_{11} - \chi d_1E_{11} \]
\[ e_6 = \lambda \left( A_{13} + \frac{1}{2}\beta^2D_{12} + \beta^2D_{66} - \frac{1}{2}\omega_m^2I_3 \right) \]
\[ e_7 = -\frac{1}{2}\lambda D_{11} \]
\[ e_8 = 2\gamma B_{13} \]
\[ e_9 = -\frac{1}{3}\gamma E_{11} \]
\[ e_{10} = -e_3 \]
\[ e_{11} = -\beta^2A_{22} + \omega_m^2I_1 \]
\[ e_{12} = A_{66} \]
\[ e_{13} = -2\gamma c_1B_{23} \]
\[ e_{14} = -\beta^2(B_{22} - \chi d_1E_{22}) \]
\[ e_{15} = \lambda\beta \left( A_{23} + \frac{1}{2}\beta^2D_{22} - \frac{1}{2}\omega_m^2I_3 \right) \]
\[ e_{16} = -\lambda\beta \left( \frac{1}{2}D_{12} + D_{66} \right) \]
\[ e_{17} = \gamma\beta \left( 2B_{23} + \frac{1}{3}\beta^2E_{22} \right) \]
\[ e_{18} = -e_4 \]
\[ e_{19} = -2\gamma c_1 \beta B_{23} \]
\[ e_{20} = \beta^2 \left( -A_{44} + 2\gamma c_1 D_{44} - \gamma^2 c_2 F_{44} \right) - 4\gamma^2 c_2 D_{33} + \omega_m^2 \bar{I}_1 \]
\[ e_{21} = A_{55} - 2\gamma c_1 D_{55} + \gamma^2 c_2 F_{55} \]
\[ e_{22} = A_{55} - (\chi + \gamma) c_1 D_{55} + \chi^2 c_2 F_{55} + 2\gamma c_1 (D_{13} - \chi d_1 F_{13}) \]
\[ e_{23} = \beta \left[ -A_{44} + (\chi + \gamma) c_1 D_{44} - \chi^2 c_2 F_{44} - 2\gamma c_1 (D_{23} - \chi d_1 F_{23}) \right] \]
\[ e_{24} = \lambda \gamma c_1 \left( 2B_{33} + \beta^2 E_{23} \right) \]
\[ e_{25} = -\lambda \gamma c_1 E_{13} \]
\[ e_{26} = \gamma^2 c_1 \left( 4D_{33} + \frac{2}{3} \beta^2 F_{23} \right) + \gamma \omega_m^2 \bar{I}_3 \]
\[ e_{27} = -2\gamma^2 d_1 F_{13} \]
\[ e_{28} = e_5 \]
\[ e_{29} = -e_{22} \]
\[ e_{30} = -A_{55} + 2\chi c_1 D_{55} - \chi^2 c_2 F_{55} + \beta^2 \left( -D_{66} + 2\chi d_1 F_{66} - \chi^2 d_2 H_{66} \right) + \omega_m^2 \bar{I}_3 \]
\[ e_{31} = D_{11} - 2\chi d_1 F_{11} + \chi^2 d_2 H_{11} \]
\[ e_{32} = \beta \left[ -(D_{12} + D_{66}) + 2\chi d_1 (F_{12} + F_{66}) - \chi^2 d_2 (H_{12} + H_{66}) \right] \]
\[ e_{33} = \lambda (B_{13} - \chi d_1 E_{13}) \]
\[ e_{34} = -\frac{1}{2} \lambda (E_{11} - \chi d_1 G_{11}) \]
\[ e_{35} = \gamma \left[ 2(D_{13} - \chi d_1 F_{13}) + \frac{1}{3} \beta^2 (F_{12} + 2F_{66}) - \frac{1}{3} \chi \beta^2 d_1 (H_{12} + 2H_{66}) - \frac{1}{3} \omega_m^2 \bar{I}_5 \right] \]
\[ e_{36} = -\frac{1}{3} \gamma (F_{11} - \chi d_1 H_{11}) \]
\[ e_{37} = -\beta^2 (B_{22} - \chi d_1 E_{22}) \]
\[ e_{38} = e_{23} \]
\[ e_{39} = -e_{32} \]
\[ e_{40} = -A_{44} + 2\chi c_1 D_{44} - \chi^2 c_2 F_{44} + \beta^2 \left( -D_{22} + 2\chi d_1 F_{22} - \chi^2 d_2 H_{22} \right) + \omega_m^2 \bar{I}_3 \]
\[ e_{41} = D_{66} - 2\chi d_1 F_{66} + \chi^2 d_2 H_{66} \]
\[ e_{42} = \lambda \beta \left[ (B_{23} - \chi d_1 E_{23}) + \frac{1}{2} \beta^2 (E_{22} - \chi d_1 G_{22}) \right] \]
\[ e_{43} = \gamma \beta \left[ 2(D_{23} - \chi d_1 F_{23}) + \frac{1}{3} \beta^2 (F_{22} - \chi d_1 H_{22}) - \frac{1}{3} \omega_m^2 \bar{I}_5 \right] \]
\[ e_{44} = \frac{1}{3} \gamma \beta \left[ -(F_{12} + 2F_{66}) + \chi d_1 (H_{12} + 2H_{66}) \right] \]
\[ e_{45} = -\lambda \left[ A_{13} + \frac{1}{2} \beta^2 (D_{12} + 2D_{66}) + \frac{1}{2} \omega_m^2 I_3 \right] \]
\[ e_{46} = D_{11} \]
\[ e_{47} = \lambda \beta \left( A_{23} + \frac{1}{2} \beta^2 D_{22} + \frac{1}{2} \omega_m^2 I_3 \right) \]
\[ e_{48} = e_{16} \]
\[ e_{49} = e_{24} \]
\[ e_{50} = e_{25} \]
\[ e_{51} = -e_{33} \]
\[ e_{52} = -e_{34} \]
\[ e_{53} = e_{42} \]
\[ e_{54} = -\lambda^2 \left( A_{33} + \beta^2 D_{23} + \frac{1}{4} \beta^4 F_{22} - \omega_m^2 I_3 - \frac{1}{4} \omega_m^2 \lambda^2 I_5 \right) \]
\[ e_{55} = \lambda^2 \left[ D_{13} + \frac{1}{2} \beta^2 (F_{12} + 2F_{66}) - \frac{1}{4} \omega_m^2 I_5 \right] \]
\[ e_{56} = -\frac{1}{4} \lambda^2 F_{11} \]
\[ e_{57} = -\lambda \gamma \left( 2B_{33} + \frac{4}{3} \beta^2 E_{23} + \frac{1}{6} \beta^4 G_{22} \right) \]
\[ e_{58} = \frac{4}{3} \lambda \gamma E_{13} \]
\[ e_{59} = -\frac{1}{6} \lambda \gamma G_{11} \]
\[ e_{60} = -e_8 \]
\[ e_{61} = -e_9 \]
\[ e_{62} = e_{17} \]
\[ e_{63} = e_{26} \]
\[ e_{64} = e_{27} \]
\[ e_{65} = -e_{35} \]
\[ e_{66} = -e_{36} \]
\[ e_{67} = e_{43} \]
\[ e_{68} = e_{44} \]
\[ e_{69} = e_{57} \]
\[ e_{70} = e_{58} \]
\[ e_{71} = e_{59} \]

134
\[ e_{72} = -\gamma^2 \left( 4D_{33} + \frac{4}{3} \beta^2 F_{23} + \frac{1}{9} \beta H_{22} - \omega_m^2 I_5 - \frac{1}{9} \omega_m^2 \beta^2 I_7 \right) \]
\[ e_{73} = \frac{1}{9} \gamma^2 \left[ 12F_{13} + 2\beta^2 (H_{12} + 2H_{66}) - \omega_m^2 I_7 \right] \]
\[ e_{74} = -\frac{1}{9} \gamma^2 H_{11} \]  \hspace{1cm} (B.1)

**B.2 State Space Coefficients**

**B.2.1 CLPT**

\[
[A] = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_1 & 0 & 0 & C_2 & 0 & 0 & C_3 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & C_4 & C_5 & 0 & C_6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & C_7 & C_8 & 0 & C_9 & 0 & C_{10} & 0 \\
\end{bmatrix} \quad (B.2)
\]

\[
\{ \Gamma \} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & C_0 Q_n \end{bmatrix}^T \quad (B.3)
\]

\[
C_1 = -e_1/e_2 \\
C_2 = -e_3/e_2 \\
C_3 = -e_4/e_2 \\
C_4 = -e_5/e_7 \\
C_5 = -e_6/e_7 \\
C_6 = -e_8/e_7 \\
C_7 = e_{15}/e_{14} \\
C_8 = e_{16}/e_{14} \\
C_9 = e_{17}/e_{14} \\
C_{10} = e_{18}/e_{14} \\
C_0 = -1/e_{14} \\
e_1 = -\beta^2 A_{66} + \omega_m^2 I_1 \\
e_2 = A_{11} \\
e_3 = -\beta (A_{12} + A_{66})
\]

135
\begin{align*}
e_4 &= -B_{11} \\
e_5 &= -e_3 \\
e_6 &= -\beta^2 A_{22} + \omega_m^2 I_1 \\
e_7 &= A_{66} \\
e_8 &= \beta^3 B_{22} \\
e_9 &= e_4 \\
e_{10} &= -e_8 \\
e_{11} &= \beta^4 D_{22} + \omega_m^2 \left( I_1 - \beta^2 I_3 \right) \\
e_{12} &= -2\beta^2 \left( D_{12} + 2D_{66} \right) + \omega_m^2 I_3 \\
e_{13} &= D_{11} \\
e_{14} &= (e_2 e_{13} - e_4 e_9) / e_2 \\
e_{15} &= (e_1 e_7 e_9 - e_3 e_5 e_9) / e_2 e_7 \\
e_{16} &= -(e_2 e_7 e_{10} + e_3 e_6 e_9) / e_2 e_7 \\
e_{17} &= -(e_2 e_7 e_{11} + e_3 e_8 e_9) / e_2 e_7 \\
e_{18} &= -e_{12}
\end{align*}

(B.4)

### B.2.2 FSDT

\[
[A] = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_1 & 0 & 0 & C_2 & 0 & C_3 & C_4 & 0 & 0 & C_5 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & C_6 & C_7 & 0 & 0 & 0 & 0 & C_8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & C_9 & 0 & 0 & C_{10} & C_{11} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{12} & 0 & 0 & C_{13} & 0 & C_{14} & C_{15} & 0 & 0 & C_{16} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & C_{17} & 0 & C_{18} & 0 & 0 & C_{19} & C_{20} & 0 \end{bmatrix}
\]  

(B.5)

\[
\{\Gamma\} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & C_m Q_m & 0 & 0 & 0 & 0
\end{bmatrix}^T
\]  

(B.6)

\begin{align*}
C_1 &= (-e_1 e_{27} + e_4 e_{23}) / e_2 e_{27} \\
C_2 &= (-e_3 e_{27} + e_4 e_{24}) / e_2 e_{27}
\end{align*}
\[ C_3 = (e_4e_{25})/e_2e_{27} \]
\[ C_4 = (e_4e_{26})/e_2e_{27} \]
\[ C_5 = (e_4e_{28})/e_2e_{27} \]
\[ C_6 = -e_5/e_7 \]
\[ C_7 = -e_6/e_7 \]
\[ C_8 = -e_8/e_7 \]
\[ C_9 = -e_9/e_{10} \]
\[ C_{10} = -e_{11}/e_{10} \]
\[ C_{11} = -e_{12}/e_{10} \]
\[ C_{12} = -e_{23}/e_{27} \]
\[ C_{13} = -e_{24}/e_{27} \]
\[ C_{14} = -e_{25}/e_{27} \]
\[ C_{15} = -e_{26}/e_{27} \]
\[ C_{16} = -e_{28}/e_{27} \]
\[ C_{17} = -e_{18}/e_{22} \]
\[ C_{18} = -e_{19}/e_{22} \]
\[ C_{19} = -e_{20}/e_{22} \]
\[ C_{20} = -e_{21}/e_{22} \]
\[ C_0 = -1/e_{10} \]
\[ e_1 = -\beta^2 A_{66} + \omega_m^2 I_1 \]
\[ e_2 = A_{11} \]
\[ e_3 = -\beta (A_{12} + A_{66}) \]
\[ e_4 = B_{11} \]
\[ e_5 = -e_3 \]
\[ e_6 = -\beta^2 A_{22} + \omega_m^2 I_1 \]
\[ e_7 = A_{66} \]
\[ e_8 = -\beta^2 B_{22} \]
\[ e_9 = -\beta^2 A_{44} + I_1 \omega_m^2 \]
\[ e_{10} = A_{55} \]
\[ e_{11} = A_{55} \]
\[ e_{12} = -\beta A_{44} \]
\[ e_{13} = e_4 \]
\[ e_{14} = -e_{11} \]
\[ e_{15} = -\beta^2 D_{66} - A_{55} + \omega_m^2 I_3 \]
\[ e_{16} = D_{11} \]
\[ e_{17} = -\beta (D_{12} + D_{96}) \]
\[ e_{18} = e_8 \]
\[ e_{19} = e_{12} \]
\[ e_{20} = -e_{17} \]
\[ e_{21} = -\beta^2 D_{22} - A_{44} + \omega_m^2 I_3 \]
\[ e_{22} = D_{66} \]  \hspace{1cm} \text{(B.7)}

**B.2.3 STTR**

\[ [A] = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_1 & 0 & 0 & C_2 & 0 & C_3 & 0 & C_4 & C_5 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_7 & C_8 & 0 & C_9 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & C_{11} & C_{12} & 0 & C_{13} & 0 & C_{14} & 0 & 0 & C_{15} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
C_{17} & 0 & 0 & C_{18} & 0 & C_{19} & 0 & C_{20} & C_{21} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & C_{23} & 0 & C_{24} & 0 & C_{25} & 0 & 0 & C_{26} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 
\end{bmatrix} \]  \hspace{1cm} \text{(B.8)}

\[ \{\Gamma\} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & C_0 Q_m & 0 & 0 & 0 
\end{bmatrix}^T \]  \hspace{1cm} \text{(B.9)}

\[ C_1 = e_{33} \]
\[ C_2 = e_{34} \]
\[ C_3 = e_{35} \]
\[ C_4 = e_{36} \]
\begin{align*}
C_5 &= e_{37} \\
C_6 &= e_{38} \\
C_7 &= e_{45} \\
C_8 &= e_{46} \\
C_9 &= e_{47} \\
C_{10} &= e_{48} \\
C_{11} &= -e_{54}/e_{58} \\
C_{12} &= -e_{55}/e_{58} \\
C_{13} &= -e_{56}/e_{58} \\
C_{14} &= -e_{57}/e_{58} \\
C_{15} &= -e_{59}/e_{58} \\
C_{16} &= -e_{60}/e_{58} \\
C_{17} &= e_{39} \\
C_{18} &= e_{40} \\
C_{19} &= e_{41} \\
C_{20} &= e_{42} \\
C_{21} &= e_{43} \\
C_{22} &= e_{44} \\
C_{23} &= e_{49} \\
C_{24} &= e_{50} \\
C_{25} &= e_{51} \\
C_{26} &= e_{52} \\
C_{27} &= e_{53} \\
C_0 &= -1/e_{58} \\
e_1 &= -\beta^2 A_{66} + \omega_m^2 I_1 \\
e_2 &= A_{11} \\
e_3 &= -\beta (A_{12} + A_{66}) \\
e_4 &= -d_1 E_{11} \\
e_5 &= B_{11} - d_1 E_{11}
\end{align*}
\[ e_6 = -e_3 \]
\[ e_7 = -\beta^2 A_{22} + \omega_m^2 I_1 \]
\[ e_8 = A_{66} \]
\[ e_9 = d_1 \beta^3 E_{22} \]
\[ e_{10} = -\beta^2 (B_{22} - d_1 E_{22}) \]
\[ e_{11} = -e_4 \]
\[ e_{12} = e_9 \]
\[ e_{13} = \beta^2 (-A_{44} + 2c_1 D_{44} - c_2 F_{44}) - \beta^4 d_2 H_{22} + \omega_m^2 \left( I_1 + 2c_1 I_3 + c_2 I_5 + \beta^2 d_2 I_7 \right) \]
\[ e_{14} = (A_{55} - 2c_1 D_{55} + c_2 F_{55}) + 2\beta^2 d_2 (H_{12} + 2H_{66}) - \omega_m^2 d_2 I_7 \]
\[ e_{15} = -d_2 H_{11} \]
\[ e_{16} = (A_{55} - 2c_1 D_{55} + c_2 F_{55}) - \beta^2 d_1 (F_{12} + 2F_{66}) + \beta^2 d_2 (H_{12} + 2H_{66}) + \omega_m^2 d_1 I_5 \]
\[ e_{17} = d_1 F_{11} - d_2 H_{11} \]
\[ e_{18} = \beta (-A_{44} + 2c_1 D_{44} - c_2 F_{44}) + d_1 \beta^3 (F_{22} - d_1 H_{22}) - \omega_m^2 \beta d_1 I_5 \]
\[ e_{19} = -\beta d_1 (F_{12} + 2F_{66}) + \beta d_2 (H_{12} + 2H_{66}) \]
\[ e_{20} = e_5 \]
\[ e_{21} = -e_{16} \]
\[ e_{22} = -e_{17} \]
\[ e_{23} = (-A_{55} + 2c_1 D_{55} - c_2 F_{55}) + \beta^2 (-D_{66} + 2d_1 F_{66} - d_2 H_{66}) + \omega_m^2 I_3 \]
\[ e_{24} = D_{11} - 2d_1 F_{11} + d_2 H_{11} \]
\[ e_{25} = \beta [- (D_{12} + D_{66}) + 2d_1 (F_{12} + F_{66}) - d_2 (H_{12} + H_{66})] \]
\[ e_{26} = e_{10} \]
\[ e_{27} = e_{18} \]
\[ e_{28} = e_{19} \]
\[ e_{29} = -e_{25} \]
\[ e_{30} = (-A_{44} + 2c_1 D_{44} - c_2 F_{44}) + \beta^2 (-D_{22} + 2d_1 F_{22} - d_2 H_{22}) + \omega_m^2 I_3 \]
\[ e_{31} = D_{66} - 2d_1 F_{66} + d_2 H_{66} \]
\[ e_{32} = e_2 e_{24} - e_5 e_{20} \]
\[ e_{33} = (-e_1 e_{32} - e_1 e_5 e_{20}) / e_2 e_{32} \]
\[ e_{34} = (-e_3e_{32} - e_3e_5e_{20})/e_2e_{32} \]
\[ e_{35} = e_5e_{21}/e_{32} \]
\[ e_{36} = (e_2e_5e_{22} - e_4e_5e_{20} - e_4e_{32})/e_2e_{32} \]
\[ e_{37} = e_5e_{23}/e_{32} \]
\[ e_{38} = e_5e_{25}/e_{32} \]
\[ e_{39} = e_1e_{20}/e_{32} \]
\[ e_{40} = e_3e_{20}/e_{32} \]
\[ e_{41} = -e_2e_{21}/e_{32} \]
\[ e_{42} = (e_4e_{20} - e_2e_{22})/e_{32} \]
\[ e_{43} = -e_2e_{23}/e_{32} \]
\[ e_{44} = -e_2e_{25}/e_{32} \]
\[ e_{45} = -e_6/e_8 \]
\[ e_{46} = -e_7/e_8 \]
\[ e_{47} = -e_9/e_8 \]
\[ e_{48} = -e_{10}/e_8 \]
\[ e_{49} = -e_{26}/e_{31} \]
\[ e_{50} = -e_{27}/e_{31} \]
\[ e_{51} = -e_{28}/e_{31} \]
\[ e_{52} = -e_{29}/e_{31} \]
\[ e_{53} = -e_{30}/e_{31} \]
\[ e_{54} = e_{11}e_{33} + e_{17}e_{39} + e_{11}e_{34}e_{46} + e_{17}e_{40}e_{46} + e_{19}e_{40} + e_{11}e_{38}e_{49} + e_{17}e_{44}e_{49} \]
\[ e_{55} = e_{12} + e_{11}e_{34}e_{46} + e_{17}e_{40}e_{46} + e_{19}e_{40} + e_{11}e_{38}e_{49} + e_{17}e_{44}e_{49} \]
\[ e_{56} = e_{13} + e_{11}e_{34}e_{47} + e_{17}e_{40}e_{47} + e_{19}e_{50} + e_{11}e_{38}e_{50} + e_{17}e_{44}e_{50} \]
\[ e_{57} = e_{14} + e_{11}e_{35} + e_{17}e_{41} + e_{19}e_{51} + e_{11}e_{38}e_{51} + e_{17}e_{44}e_{51} \]
\[ e_{58} = e_{15} + e_{11}e_{36} + e_{17}e_{42} \]
\[ e_{59} = e_{16} + e_{11}e_{37} + e_{17}e_{43} + e_{19}e_{52} + e_{11}e_{38}e_{52} + e_{17}e_{44}e_{52} \]
\[ e_{60} = e_{18} + e_{11}e_{34}e_{48} + e_{17}e_{40}e_{48} + e_{19}e_{53} + e_{11}e_{38}e_{53} + e_{17}e_{44}e_{53} \]  
(B.10)
B.2.4 GTTR

\[
[A] = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_1 & 0 & 0 & C_2 & 0 & C_3 & C_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{10} & C_{11} & 0 & C_{12} & 0 & 0 & 0 & C_{13} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & C_{17} & C_{18} & 0 & C_{19} & 0 & 0 & C_{20} & C_{21} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
C_{26} & 0 & 0 & C_{27} & 0 & C_{28} & C_{29} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & C_{35} & 0 & C_{36} & 0 & 0 & C_{37} & C_{38} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{42} & C_{43} & 0 & C_{44} & 0 & 0 & C_{45} & C_{46} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{51} & C_{52} & 0 & C_{53} & 0 & 0 & C_{54} & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_5 & 0 & C_6 & 0 & C_7 & 0 & C_8 & 0 & C_9 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{14} & 0 & C_{15} & 0 & C_{16} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{22} & 0 & C_{22} & 0 & C_{24} & 0 & C_{25} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{30} & 0 & C_{31} & 0 & C_{32} & 0 & C_{33} & 0 & C_{34} & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{39} & 0 & 0 & 0 & C_{40} & 0 & C_{41} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & C_{47} & 0 & C_{48} & 0 & C_{49} & 0 & C_{50} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & C_{56} & 0 & C_{57} & 0 & C_{58} & 0 & C_{59} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\Rightarrow
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_5 & 0 & C_6 & 0 & C_7 & 0 & C_8 & 0 & C_9 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{14} & 0 & C_{15} & 0 & C_{16} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{22} & 0 & C_{22} & 0 & C_{24} & 0 & C_{25} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
C_{30} & 0 & C_{31} & 0 & C_{32} & 0 & C_{33} & 0 & C_{34} & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & C_{39} & 0 & 0 & 0 & C_{40} & 0 & C_{41} & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & C_{47} & 0 & C_{48} & 0 & C_{49} & 0 & C_{50} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & C_{56} & 0 & C_{57} & 0 & C_{58} & 0 & C_{59} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\Rightarrow
(B.11)
\[ \{ \Gamma \} = \begin{bmatrix} 0 & 0 & 0 & 0 & C_{01} Q_m & 0 & 0 & 0 & 0 & 0 & C_{02} Q_m & 0 & 0 & 0 & C_{03} Q_m \end{bmatrix}^T \]

(B.12)
Appendix C

Element Matrices

C.1 Element Matrices for GTOT

C.1.1 Element Stiffness Matrix

\[ K_{ij}^{11} = \int_{r'} \left( A_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + A_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy \]

\[ K_{ij}^{12} = \int_{r'} \left( A_{12} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + A_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} \right) dxdy \]

\[ K_{ij}^{13} = 0 \]

\[ K_{ij}^{14} = \int_{r'} \left( B_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) dxdy \]

\[ K_{ij}^{15} = 0 \]

\[ K_{ij}^{16} = \int_{r'} \left( D_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + D_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy \]

\[ K_{ij}^{17} = \int_{r'} \left( D_{12} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + D_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} \right) dxdy \]

\[ K_{ij}^{18} = \int_{r'} \left( A_{13} \frac{\partial \psi_i}{\partial x} \psi_j \right) dxdy \]

\[ K_{ij}^{19} = \int_{r'} \left( E_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) dxdy \]

\[ K_{ij}^{1,10} = 0 \]

\[ K_{ij}^{1,11} = \int_{r'} \left( 2B_{13} \frac{\partial \psi_i}{\partial x} \psi_j \right) dxdy \]

\[ K_{ij}^{22} = \int_{r'} \left( A_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + A_{66} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) dxdy \]
\[ K_{ij}^{23} = 0 \]
\[ K_{ij}^{24} = 0 \]
\[ K_{ij}^{25} = \int_{\Omega'} \left( B_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) dx \]
\[ K_{ij}^{26} = \int_{\Omega'} \left( D_{12} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + D_{66} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) dx \]
\[ K_{ij}^{27} = \int_{\Omega'} \left( D_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + D_{66} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) dx \]
\[ K_{ij}^{28} = \int_{\Omega'} \left( A_{23} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx \]
\[ K_{ij}^{29} = 0 \]
\[ K_{ij}^{2,10} = \int_{\Omega'} \left( E_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx \]
\[ K_{ij}^{2,11} = \int_{\Omega'} \left( 2B_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx \]
\[ K_{ij}^{33} = \int_{\Omega'} \left( A_{55} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + A_{44} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx \]
\[ K_{ij}^{34} = \int_{\Omega'} \left( A_{55} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) dx \]
\[ K_{ij}^{35} = \int_{\Omega'} \left( A_{44} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx \]
\[ K_{ij}^{36} = \int_{\Omega'} \left( 2B_{55} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) dx \]
\[ K_{ij}^{37} = \int_{\Omega'} \left( 2B_{44} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) dx \]
\[ K_{ij}^{38} = \int_{\Omega'} \left( B_{55} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + B_{44} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx \]
\[ K_{ij}^{39} = \int_{\Omega'} \left( 3D_{55} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) dx \]
\[ K_{ij}^{3,10} = \int_{\Omega'} \left( 3D_{44} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx \]
\[ K_{ij}^{3,11} = \int_{\Omega'} \left( D_{55} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + D_{44} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx \]
\[ K_{ij}^{44} = \int_{\Omega'} \left( A_{55} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + D_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + D_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx \]
\[ K_{ij}^{45} = K_{ij}^{17} \]
\[ K_{ij}^{46} = \int_{\Omega'} \left( E_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + 2B_{55} \psi_i \psi_j \right) dx \, dy \]
\[ K_{ij}^{47} = 0 \]
\[ K_{ij}^{48} = \int_{\Omega'} \left( B_{13} \frac{\partial \psi_i}{\partial x} \psi_j + B_{55} \psi_i \frac{\partial \psi_j}{\partial x} \right) dx \, dy \]
\[ K_{ij}^{49} = \int_{\Omega'} \left( 3D_{55} \psi_i \psi_j + F_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + F_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx \, dy \]
\[ K_{ij}^{4,16} = \int_{\Omega'} \left( F_{12} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} + F_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} \right) dx \, dy \]
\[ K_{ij}^{4,11} = \int_{\Omega'} \left( 2D_{13} \frac{\partial \psi_i}{\partial x} \psi_j + D_{55} \psi_i \frac{\partial \psi_j}{\partial x} \right) dx \, dy \]
\[ K_{ij}^{55} = \int_{\Omega'} \left( A_{44} \psi_i \psi_j + D_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + D_{66} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) dx \, dy \]
\[ K_{ij}^{56} = 0 \]
\[ K_{ij}^{57} = \int_{\Omega'} \left( E_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + 2B_{44} \psi_i \psi_j \right) dx \, dy \]
\[ K_{ij}^{58} = \int_{\Omega'} \left( B_{23} \frac{\partial \psi_i}{\partial y} \psi_j + B_{44} \psi_i \frac{\partial \psi_j}{\partial y} \right) dx \, dy \]
\[ K_{ij}^{59} = \int_{\Omega'} \left( F_{12} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} + F_{66} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} \right) dx \, dy \]
\[ K_{ij}^{5,10} = \int_{\Omega'} \left( 3D_{44} \psi_i \psi_j + F_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + F_{66} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) dx \, dy \]
\[ K_{ij}^{5,11} = \int_{\Omega'} \left( 2D_{23} \frac{\partial \psi_i}{\partial y} \psi_j + D_{44} \psi_i \frac{\partial \psi_j}{\partial y} \right) dx \, dy \]
\[ K_{ij}^{66} = \int_{\Omega'} \left( 4D_{55} \psi_i \psi_j + F_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + F_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dx \, dy \]
\[ K_{ij}^{67} = K_{ij}^{4,10} \]
\[ K_{ij}^{68} = \int_{\Omega'} \left( D_{13} \frac{\partial \psi_i}{\partial x} \psi_j + 2D_{55} \psi_i \frac{\partial \psi_j}{\partial x} \right) dx \, dy \]
\[ K_{ij}^{69} = \int_{\Omega'} \left( G_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + 6E_{55} \psi_i \psi_j \right) dx \, dy \]
\[ K_{ij}^{6,10} = 0 \]
\[ K_{ij}^{6,11} = \int_{\Omega'} \left( 2E_{13} \frac{\partial \psi_i}{\partial x} \psi_j + 2E_{55} \psi_i \frac{\partial \psi_j}{\partial x} \right) dx \, dy \]
\[ K_{ij}^{77} = \int_{\Omega'} \left( 4A_{44} \psi_i \psi_j + F_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + F_{66} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) dx \, dy \]
\[
\begin{align*}
K_{ij}^{78} &= \int_{\Omega_e} \left( D_{23} \frac{\partial \psi_i}{\partial y} \psi_j + 2D_{44} \psi_i \frac{\partial \psi_j}{\partial y} \right) dxdy \\
K_{ij}^{79} &= 0 \\
K_{ij}^{7,10} &= \int_{\Omega_e} \left( G_{22} \frac{\partial \psi_i}{\partial y} \psi_j + 6E_{44} \psi_i \psi_j \right) dxdy \\
K_{ij}^{7,11} &= \int_{\Omega_e} \left( 2E_{23} \frac{\partial \psi_i}{\partial y} \psi_j + 2E_{44} \psi_i \frac{\partial \psi_j}{\partial y} \right) dxdy \\
K_{ij}^{88} &= \int_{\Omega_e} \left( A_{33} \psi_i \psi_j + D_{55} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + D_{44} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy \\
K_{ij}^{89} &= \int_{\Omega_e} \left( E_{13} \psi_i \frac{\partial \psi_j}{\partial x} + 3E_{55} \frac{\partial \psi_i}{\partial x} \psi_j \right) dxdy \\
K_{ij}^{8,10} &= \int_{\Omega_e} \left( E_{23} \psi_i \frac{\partial \psi_j}{\partial y} + 3E_{44} \frac{\partial \psi_i}{\partial y} \psi_j \right) dxdy \\
K_{ij}^{8,11} &= \int_{\Omega_e} \left( 2B_{33} \psi_i \psi_j + E_{56} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + E_{44} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy \\
K_{ij}^{99} &= \int_{\Omega_e} \left( 9F_{55} \psi_i \psi_j + H_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + H_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy \\
K_{ij}^{9,10} &= \int_{\Omega_e} \left( H_{12} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + H_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} \right) dxdy \\
K_{ij}^{9,11} &= \int_{\Omega_e} \left( 2F_{13} \frac{\partial \psi_i}{\partial x} \psi_j + 3F_{55} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) dxdy \\
K_{ij}^{10,10} &= \int_{\Omega_e} \left( 9F_{44} \psi_i \psi_j + H_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + H_{66} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) dxdy \\
K_{ij}^{10,11} &= \int_{\Omega_e} \left( 2F_{23} \frac{\partial \psi_i}{\partial y} \psi_j + 3F_{44} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy \\
K_{ij}^{11,11} &= \int_{\Omega_e} \left( 4D_{33} \psi_i \psi_j + F_{55} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + F_{44} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) dxdy \quad \text{(C.1)}
\end{align*}
\]

C.1.2 Element Mass Matrix

Only the non-zero submatrices are enumerated.

\[
\begin{align*}
M_{ij}^{11} &= M_{ij}^{22} = M_{ij}^{33} = \int_{\Omega_e} (I_1 \psi_i \psi_j) dxdy \\
M_{ij}^{16} &= M_{ij}^{27} = M_{ij}^{3,11} = M_{ij}^{44} = M_{ij}^{55} = M_{ij}^{88} = \int_{\Omega_e} (I_3 \psi_i \psi_j) dxdy \\
M_{ij}^{49} &= M_{ij}^{5,10} = M_{ij}^{66} = M_{ij}^{77} = M_{ij}^{11,11} = \int_{\Omega_e} (I_5 \psi_i \psi_j) dxdy \\
M_{ij}^{99} &= M_{ij}^{10,10} = \int_{\Omega_e} (I_7 \psi_i \psi_j) dxdy \quad \text{(C.2)}
\end{align*}
\]
C.1.3 Element Force Vector

Only the non-zero subvectors are enumerated.

\[ f_i^1 = \int_{\Omega^e} q_1 \psi_i \, dx \, dy, \quad f_i^5 = \int_{\Omega^e} q_1 \psi_i \, dx \, dy, \quad f_i^{11} = \int_{\Omega^e} q_2 \psi_i \, dx \, dy \]  
\[(C.3)\]

where \( q_1 = q h / 2 \), and \( q_2 = q h^2 / 4 \).

C.2 Element Matrices for GTTR

C.2.1 Element Stiffness Matrix

\[ K_{ij}^{11} = \int_{\Omega^e} \left( A_{12} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + A_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) \, dx \, dy \]
\[ K_{ij}^{12} = \int_{\Omega^e} \left( A_{12} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + A_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} \right) \, dx \, dy \]
\[ K_{ij}^{13} = \int_{\Omega^e} \left( \left( B_{11} - \chi d_{11} E_{11} \right) \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) \, dx \, dy \]
\[ K_{ij}^{14} = \int_{\Omega^e} \left( \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} \right) \, dx \, dy \]
\[ K_{ij}^{15} = 0 \]
\[ K_{ij}^{16} = \lambda \int_{\Omega^e} \left( A_{12} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} - \frac{1}{2} D_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial^2 \psi_j}{\partial x^2} - \frac{1}{2} D_{12} \frac{\partial \psi_i}{\partial y} \frac{\partial^2 \psi_j}{\partial y^2} - D_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial^2 \psi_j}{\partial x \partial y} \right) \, dx \, dy \]
\[ K_{ij}^{17} = \gamma \int_{\Omega^e} \left( 2 B_{13} \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} - \frac{1}{3} E_{11} \frac{\partial \psi_i}{\partial x} \frac{\partial^2 \psi_j}{\partial x^2} \right) \, dx \, dy \]
\[ K_{ij}^{22} = \int_{\Omega^e} \left( A_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + A_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial x} \right) \, dx \, dy \]
\[ K_{ij}^{23} = \int_{\Omega^e} \left( -2 \gamma c_{1} B_{23} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) \, dx \, dy \]
\[ K_{ij}^{24} = 0 \]
\[ K_{ij}^{25} = \int_{\Omega^e} \left( (B_{22} - \chi d_{11} E_{22}) \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right) \, dx \, dy \]
\[ K_{ij}^{26} = \lambda \int_{\Omega^e} \left( A_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} - \frac{1}{2} D_{12} \frac{\partial \psi_i}{\partial y} \frac{\partial^2 \psi_j}{\partial y^2} - \frac{1}{2} D_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial^2 \psi_j}{\partial y^2} - D_{66} \frac{\partial \psi_i}{\partial y} \frac{\partial^2 \psi_j}{\partial x \partial y} \right) \, dx \, dy \]
\[ K_{ij}^{27} = \gamma \int_{\Omega^e} \left( 2 B_{23} \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} - \frac{1}{3} E_{22} \frac{\partial \psi_i}{\partial y} \frac{\partial^2 \psi_j}{\partial y^2} \right) \, dx \, dy \]
\[ K_{ij}^{33} = \int_{\Omega^e} \left[ (A_{55} - 2\gamma c_1 D_{55} + \gamma^2 c_2 F_{55}) \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + \left( A_{44} - 2\gamma c_1 D_{44} + \gamma^2 c_2 F_{44} \right) \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + 4\gamma^2 c_2 D_{33} \psi_i \psi_j \right] dxdy \]

\[ K_{ij}^{34} = \int_{\Omega^e} \left[ \{A_{55} - (\chi + \gamma) c_1 D_{55} + \chi \gamma c_2 F_{55}\} \frac{\partial \psi_i}{\partial x} \psi_j - 2\gamma c_1 (D_{13} - \chi d_1 F_{13}) \psi_i \frac{\partial \psi_j}{\partial x} \right] dxdy \]

\[ K_{ij}^{35} = \int_{\Omega^e} \left[ \{A_{44} - (\chi + \gamma) c_1 D_{44} + \chi \gamma c_2 F_{44}\} \frac{\partial \psi_i}{\partial y} \psi_j - 2\gamma c_1 (D_{23} - \chi d_1 F_{23}) \psi_i \frac{\partial \psi_j}{\partial y} \right] dxdy \]

\[ K_{ij}^{36} = \lambda \gamma c_1 \int_{\Omega^e} \left[ -2D_{33} \psi_i \hat{\phi}_j + E_{13} \psi_i \frac{\partial^2 \hat{\phi}_j}{\partial x^2} + E_{23} \psi_i \frac{\partial^2 \hat{\phi}_j}{\partial y^2} \right] dxdy \]

\[ K_{ij}^{37} = 2\gamma^2 c_1 \int_{\Omega^e} \left[ -2D_{33} \psi_i \hat{\phi}_j + \frac{1}{3} F_{13} \psi_i \frac{\partial^2 \hat{\phi}_j}{\partial x^2} + \frac{1}{3} F_{23} \psi_i \frac{\partial^2 \hat{\phi}_j}{\partial y^2} \right] dxdy \]

\[ K_{ij}^{34} = \int_{\Omega^e} \left[ (D_{11} - 2\chi d_1 F_{11} + \chi^2 d_2 H_{11}) \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + \left( D_{66} - 2\chi d_1 F_{66} + \chi^2 d_2 H_{66} \right) \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + \left( A_{55} - 2\gamma c_1 D_{55} + \chi^2 c_2 F_{55} \right) \psi_i \psi_j \right] dxdy \]

\[ K_{ij}^{35} = \int_{\Omega^e} \left[ (D_{12} - 2\chi d_1 F_{12} + \chi^2 d_2 H_{12}) \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial y} + \left( D_{66} - 2\chi d_1 F_{66} + \chi^2 d_2 H_{66} \right) \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} \right] dxdy \]

\[ K_{ij}^{36} = \lambda \int_{\Omega^e} \left[ (B_{13} - \chi d_1 E_{13}) \frac{\partial \psi_i}{\partial x} \hat{\phi}_j - \frac{1}{2} \left( E_{11} - \chi d_1 G_{11} \right) \frac{\partial \psi_i}{\partial x} \frac{\partial^2 \hat{\phi}_j}{\partial x^2} \right] dxdy \]

\[ K_{ij}^{37} = \gamma \int_{\Omega^e} \left[ 2(D_{13} - \chi d_1 F_{13}) \frac{\partial \psi_i}{\partial x} \hat{\phi}_j - \frac{1}{3} (F_{11} - \chi d_1 H_{11}) \frac{\partial \psi_i}{\partial x} \frac{\partial^2 \hat{\phi}_j}{\partial x^2} - \frac{1}{3} (F_{12} - \chi d_1 H_{12}) \frac{\partial \psi_i}{\partial x} \frac{\partial^2 \hat{\phi}_j}{\partial y^2} - \frac{2}{3} (F_{66} - \chi d_1 H_{66}) \frac{\partial \psi_i}{\partial y} \frac{\partial^2 \hat{\phi}_j}{\partial x \partial y} \right] dxdy \]

\[ K_{ij}^{35} = \int_{\Omega^e} \left[ (D_{66} - 2\chi d_1 F_{66} + \chi^2 d_2 H_{66}) \frac{\partial \psi_i}{\partial x} \frac{\partial \psi_j}{\partial x} + \left( D_{22} - 2\chi d_1 F_{22} + \chi^2 d_2 H_{22} \right) \frac{\partial \psi_i}{\partial y} \frac{\partial \psi_j}{\partial y} + \left( A_{44} - 2\gamma c_1 D_{44} + \chi^2 c_2 F_{44} \right) \psi_i \psi_j \right] dxdy \]

\[ K_{ij}^{36} = \lambda \int_{\Omega^e} \left[ (B_{23} - \chi d_1 E_{23}) \frac{\partial \psi_i}{\partial y} \hat{\phi}_j - \frac{1}{2} \left( E_{22} - \chi d_1 G_{22} \right) \frac{\partial \psi_i}{\partial y} \frac{\partial^2 \hat{\phi}_j}{\partial y^2} \right] dxdy \]
\[ K_{ij}^{37} = \gamma \int_{\Omega^e} \left[ 2(D_{23} - \chi d_1 F_{23}) \frac{\partial \ddot{\phi}_i}{\partial y} \ddot{\phi}_j - \frac{1}{3} (F_{12} - \chi d_1 H_{12}) \frac{\partial \ddot{\phi}_i}{\partial y} \frac{\partial^2 \ddot{\phi}_j}{\partial x^2} \right. \\
\left. - \left( \frac{1}{3} (F_{22} - \chi d_1 H_{22}) \frac{\partial \ddot{\phi}_i}{\partial y} \frac{\partial^2 \ddot{\phi}_j}{\partial y^2} - \frac{2}{3} (F_{66} - \chi d_1 H_{66}) \frac{\partial \ddot{\phi}_i}{\partial x} \frac{\partial^2 \ddot{\phi}_j}{\partial x \partial y} \right) \right] dxdy \]

\[ K_{ij}^{66} = \frac{\lambda^2}{2} \int_{\Omega^e} \left[ 2A_{33} \ddot{\phi}_i \ddot{\phi}_j - D_{13} \left( \ddot{\phi}_i \frac{\partial^2 \ddot{\phi}_j}{\partial x^2} + \frac{\partial^2 \ddot{\phi}_i}{\partial x^2} \ddot{\phi}_j \right) - D_{23} \left( \ddot{\phi}_i \frac{\partial^2 \ddot{\phi}_j}{\partial y^2} + \frac{\partial^2 \ddot{\phi}_i}{\partial y^2} \ddot{\phi}_j \right) \right. \\
\left. + \frac{1}{2} \left( F_{11} \frac{\partial^2 \ddot{\phi}_i}{\partial x^2} \frac{\partial^2 \ddot{\phi}_j}{\partial x^2} + F_{12} \frac{\partial^2 \ddot{\phi}_i}{\partial y^2} \frac{\partial^2 \ddot{\phi}_j}{\partial y^2} + F_{13} \frac{\partial^2 \ddot{\phi}_i}{\partial x^2} \frac{\partial^2 \ddot{\phi}_j}{\partial y^2} + F_{22} \frac{\partial^2 \ddot{\phi}_i}{\partial y^2} \frac{\partial^2 \ddot{\phi}_j}{\partial y^2} \right. \\
\left. + 2F_{66} \frac{\partial^2 \ddot{\phi}_i}{\partial x \partial y} \frac{\partial^2 \ddot{\phi}_j}{\partial x \partial y} \right) \right] dxdy \]

\[ K_{ij}^{37} = \lambda \gamma \int_{\Omega^e} \left[ 2B_{33} \ddot{\phi}_i \ddot{\phi}_j - E_{13} \left( \frac{\partial^2 \ddot{\phi}_i}{\partial x^2} \ddot{\phi}_j + \frac{1}{3} \ddot{\phi}_i \frac{\partial^2 \ddot{\phi}_j}{\partial x^2} \right) - E_{23} \left( \frac{\partial^2 \ddot{\phi}_i}{\partial y^2} \ddot{\phi}_j + \frac{1}{3} \ddot{\phi}_i \frac{\partial^2 \ddot{\phi}_j}{\partial y^2} \right) \right. \\
\left. + \frac{1}{6} \left( G_{11} \frac{\partial^2 \ddot{\phi}_i}{\partial x^2} \frac{\partial^2 \ddot{\phi}_j}{\partial x^2} + G_{22} \frac{\partial^2 \ddot{\phi}_i}{\partial y^2} \frac{\partial^2 \ddot{\phi}_j}{\partial y^2} \right) \right] dxdy \]

\[ K_{ij}^{37} = \frac{\gamma^2}{3} \int_{\Omega^e} \left[ 12D_{33} \ddot{\phi}_i \ddot{\phi}_j - 2D_{13} \left( \ddot{\phi}_i \frac{\partial^2 \ddot{\phi}_j}{\partial x^2} + \frac{\partial^2 \ddot{\phi}_i}{\partial x^2} \ddot{\phi}_j \right) - 2D_{23} \left( \ddot{\phi}_i \frac{\partial^2 \ddot{\phi}_j}{\partial y^2} + \frac{\partial^2 \ddot{\phi}_i}{\partial y^2} \ddot{\phi}_j \right) \right. \\
\left. + \frac{1}{3} \left( H_{11} \frac{\partial^2 \ddot{\phi}_i}{\partial x^2} \frac{\partial^2 \ddot{\phi}_j}{\partial x^2} + H_{12} \frac{\partial^2 \ddot{\phi}_i}{\partial x^2} \frac{\partial^2 \ddot{\phi}_j}{\partial y^2} + H_{13} \frac{\partial^2 \ddot{\phi}_i}{\partial x^2} \frac{\partial^2 \ddot{\phi}_j}{\partial y^2} + H_{22} \frac{\partial^2 \ddot{\phi}_i}{\partial y^2} \frac{\partial^2 \ddot{\phi}_j}{\partial y^2} \right) \right. \\
\left. + \frac{4}{3} H_{66} \frac{\partial^2 \ddot{\phi}_i}{\partial x \partial y} \frac{\partial^2 \ddot{\phi}_j}{\partial x \partial y} \right] dxdy \]

(C.4)

### C.2.2 Element Mass Matrix

Only the non-zero submatrices are enumerated.

\[ M_{ij}^{11} = \int_{\Omega^e} (I_1 \psi_i \psi_j) \ dxdy \]

\[ M_{ij}^{16} = \int_{\Omega^e} \left( -\frac{1}{2} \lambda I_3 \psi_i \frac{\partial \ddot{\phi}_j}{\partial x} \right) \ dxdy \]

\[ M_{ij}^{22} = M_{ij}^{11} \]

\[ M_{ij}^{26} = \int_{\Omega^e} \left( -\frac{1}{2} \lambda I_3 \psi_i \frac{\partial \ddot{\phi}_j}{\partial y} \right) \ dxdy \]

\[ M_{ij}^{33} = \int_{\Omega^e} (I_1 \psi_i \psi_j) \ dxdy \]

\[ M_{ij}^{37} = \int_{\Omega^e} (\gamma I_3 \psi_i \psi_j) \ dxdy \]

150
\[ M_{ij}^{44} = \int_{\Omega} \left( \hat{I}_3 \hat{\psi}_i \hat{\psi}_j \right) dxdy \]
\[ M_{ij}^{47} = \int_{\Omega} \left( -\frac{1}{3} \gamma \hat{I}_5 \hat{\psi}_i \frac{\partial \hat{\phi}_j}{\partial x} \right) dxdy \]
\[ M_{ij}^{55} = M_{ij}^{44} \]
\[ M_{ij}^{57} = \int_{\Omega} \left( -\frac{1}{3} \gamma \hat{I}_5 \hat{\psi}_i \frac{\partial \hat{\phi}_j}{\partial y} \right) dxdy \]
\[ M_{ij}^{61} = \int_{\Omega} \left( \frac{1}{2} \lambda \hat{I}_3 \hat{\phi}_i \frac{\partial \hat{\psi}_j}{\partial y} \right) dxdy \]
\[ M_{ij}^{62} = \int_{\Omega} \left( \frac{1}{2} \lambda \hat{I}_3 \hat{\phi}_i \frac{\partial \hat{\psi}_j}{\partial x} \right) dxdy \]
\[ M_{ij}^{66} = \lambda^2 \int_{\Omega} \left[ I_5 \hat{\phi}_i \hat{\phi}_j - \frac{1}{4} I_5 \left( \hat{\phi}_i \frac{\partial^2 \hat{\phi}_j}{\partial x^2} + \hat{\phi}_i \frac{\partial^2 \hat{\phi}_j}{\partial y^2} \right) \right] dxdy \]
\[ M_{ij}^{73} = M_{ji}^{37} \]
\[ M_{ij}^{71} = \int_{\Omega} \left( \frac{1}{3} \gamma \hat{I}_5 \hat{\phi}_i \frac{\partial \hat{\psi}_j}{\partial x} \right) dxdy \]
\[ M_{ij}^{75} = \int_{\Omega} \left( \frac{1}{3} \gamma \hat{I}_5 \hat{\phi}_i \frac{\partial \hat{\psi}_j}{\partial y} \right) dxdy \]
\[ M_{ij}^{77} = \gamma^2 \int_{\Omega} \left[ I_5 \hat{\phi}_i \hat{\phi}_j - \frac{1}{9} I_7 \left( \hat{\phi}_i \frac{\partial^2 \hat{\phi}_j}{\partial x^2} + \hat{\phi}_i \frac{\partial^2 \hat{\phi}_j}{\partial y^2} \right) \right] dxdy \] (C.5)

### C.2.3 Element Force Vector

Only the non-zero subvectors are enumerated.

\[ f_i^6 = \int_{\Omega} q_1 \psi_i dxdy, \quad f_i^7 = \int_{\Omega} q_2 \psi_i dxdy \] (C.6)

where \( q_1 = qh/2 \), and \( q_2 = qh^2/4 \).
Vita

The author was born in Calcutta, India. He completed his school education in the same city. In 1986, he joined Indian Institute of Technology (IIT), Bombay. After graduating from IIT in 1990 with a Bachelor's degree in Mechanical Engineering, he worked for one year with Shaw Wallace and Company, India, as a management trainee. In Fall 1991, he joined the M.S. program in Engineering Mechanics at Virginia Polytechnic Institute and State University. From the Summer of 1994 onward, he has been enrolled as a Ph.D. student in Business Administration at the University of Rochester.

Partha Bose