AN ARBITRARILY SHAPED SATELLITE

IN THE

RESTRICTED PROBLEM OF THREE BODIES

by

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LIST OF SYMBOLS

\( a_{ij} \)  
Elements of \([a]\)

\( A_3 \)  
Principal planar moment of inertia of the third body

\( B_3 \)  
Principal planar moment of inertia of the third body

\( c \)  
Cosine of the trigonometric argument following it

\( C_3 \)  
Polar moment of inertia of the third body

\( G \)  
Universal constant of gravitation

\( i \)  
Integer

\( Im \)  
Imaginary part

\( j \)  
Integer

\( k \)  
Integer

\( [l] \)  
Transformation matrix

\( L \)  
Lagrangian function

\( m \)  
Mass of a body

\( O \)  
Origin of a coordinate system

\( P \)  
Variable introduced into the analysis (page 13)

\( Q \)  
Variable introduced into the analysis (page 13)

\( Q_k \)  
Forcing term

\( r_{A3} \)  
Distance of center of mass of third body from body A

\( r_{B3} \)  
Distance of center of mass of third body from body B

\( R_3 \)  
Radial velocity fo the center of mass of third body

\( s \)  
Sine of the trigonometric argument following it

\( S_1 \)  
Size and shape factor of the third body

\( S_2 \)  
Shape factor of the third body
$S_3$  Constant introduced into the analysis (page 27)
t  Tangent of the trigonometric argument following it
t  Time
T  Kinetic energy of the system
$[T]$  Diagonal non-dimensionalizing matrix
$[TT]$  Diagonal non-dimensionalizing matrix
V  Potential energy of the system
$x_i$  Coordinate axis, also a state
X  Inertial coordinate axis
$X(x_i)$  Function of the states $x_i$
$y_i$  Coordinate axis, also perturbation in $x_i$
Y  Inertial coordinate axis
$z_i$  Coordinate axis
Z  Inertial coordinate axis
$\alpha, \beta$  Angles subtended by the center of mass of the third body with the line joining the two primaries
$\delta$  Small variation
$\theta$  Angular position of the center of mass of the third body
$\Theta$  Angular velocity of the center of mass of the third body
$\rho_3$  Radial position of the center of mass of the third body
$\rho_A$  Distance of body A from the barycenter of the system
$\rho_B$  Distance of body B from the barycenter of the system
$\phi$  Libration of the third body
$\phi_1$  Variable introduced into the analysis (page 13)
$\phi_2$  Variable introduced into the analysis (page 13)
$\epsilon, \delta, \phi$  Net pointing error
$\phi$  Libration velocity of the third body

$\psi$  Constant angular velocity of the earth moon system about its barycenter

$\lambda$  Eigenvalue of $[a]$

**SUBSCRIPTS**

<table>
<thead>
<tr>
<th>Subscript</th>
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<td>o</td>
<td>Initial condition</td>
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<td>Arc, e.g. $t^{-1}\theta = \arctan \theta$</td>
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1. **INTRODUCTION**

Just as Galileo's discovery of many "material centers of motion" was followed by the discovery of other "non-material center of motion", it is hoped that the placing of satellites around "material centers of motion" will be followed by their placement about other "non-material centers of motion". The closest such centers of motion are the five equilibrium points of the earth-moon system (Fig. 1). In particular, the placing of a station in orbit around a triangular libration point appears very attractive from numerous viewpoints.

If a point mass is placed in the close vicinity of the equilateral point $L_4$, its motion is similar to that of a double pendulum in that it is a superposition of two modes of motion- the long and short period modes. In general, the resulting orbit will describe Lassajau's figures; their shape will depend upon the initial conditions of the orbit. Initial conditions can be found such that only the short period mode can be excited. The resulting orbit will be elliptic and its minor axis will point towards the bary-center of the earth-moon system.

An interesting aspect of this libration point stems from the fact that the short period mode (28.62 days) has approximately the same period as that of the moon (27.32) days. Pederson (Ref. 1), with his third order theory showed that the period of any finite sized orbit (when compared to the period of an infinitesimally small orbit) becomes smaller for a short period mode while that of the long period
mode becomes bigger. Hence, the period of the short period mode approaches that of the moon and in fact, with a little control, could be made the same. This remarkable property of the earth moon system can be exploited to place the following stations in orbit about the libration point:

(i) Transcommunication Station: Such a station would make (-1) rotations per orbit. An axis would always point to the barycenter of the earth moon system (Fig. 2) and an antenna on it will be earth-pointing. This station could be used as a transcommunication station for interplanetary flights, especially for some communication eclipses.

(ii) Astronomical Station: Such a station would make (+2) rotations per orbit. A telescope on such a station will be "star-pointing" for an observer at the barycenter of the earth moon system (Fig. 3).

It appears that the study of libration dynamics of a station near the equilateral point is not only intriguing, but also justified.

DISCUSSION OF THE VARIOUS DISTURBING AND STABILIZING FORCES

A study of the stability and motion of earth satellites includes the effects of one or more of the following: Gravity gradient effects, aerodynamic effects, magnetic effects, electrostatic effects, electromagnetic interactions, effect of the solar radiation pressure, and the possible impact with meteorites. The nature and size of the satellite's orbit determines which of the above effects play a dominant role and hence, have to be taken into consideration.

However, for a station placed in orbit around an equilateral
point of the earth moon system, a complete study would include:
Gravity gradient effects of the earth and the moon, perturbations
due to the sun's gravitational field, effects of the solar radiation
pressure, and the possible impact with small objects.

The solar radiation pressure will cause disturbing torques on
the station. This problem could conceivably be circumvented by
making the external surface of the station spherical whereas the
internal mass distribution could be arranged so that the station has
the desired moment of inertias.

A study of the solar perturbation effect and the impact with
small objects is not covered in this work. Hence, the only effects
considered are the gravitational effects of the earth and the moon
on the motion of a non-symmetric station.

STATEMENT OF THE PROBLEM

The motion of an arbitrarily shaped body immersed in a gravi-
tational field of two bodies is studied. In particular, the system
is assumed to satisfy the conditions of a restricted three body
problem (These conditions are discussed in the next section).
Furthermore, the third body is in orbit about a stationary equilibrium
point in a system of two bodies (primaries).

Since the third body is of finite size, we must study the motion
of its center of mass and motion about its center of mass. To be more
elaborate, the nature and extent of coupling between the motion of
the center of mass and about the center of mass is investigated. In
other words, the magnitude of the gravity gradient torques which cause
the attitude motion is compared with the magnitude of the forces which cause the center of mass to describe its elliptic orbit.

Stability criteria for the third body are established. The pointing accuracy for such a body is studied.
2. HISTORICAL BACKGROUND

No thesis can be considered complete without a historical sketch and as such the following discussion ensues:

The many body problem was first precisely formulated by Newton. When the bodies involved are point masses, it can be stated as follows: Given at any instant of time, the position and velocities of three or more particles moving under their mutual gravitational forces, the masses also being known, calculate their positions and velocities at any other time. The simplest of the many body problem is the three body problem and this above has been a source of inspiration and frustration to many brilliant astronomers and mathematicians.

The only known closed form solutions of the three body problem are two particular solutions which were found by Lagrange in 1772 but were announced by him in his "Oeuvres" (Ref. 2) at a later date. In these solutions, the three bodies move such that the geometric form of the three body configuration does not change although the scale can change and the figure can rotate. The two cases found by Lagrange are: (i) the three bodies form the vertices of an equilateral triangle, (ii) the three bodies are collinear.

Three bodies (particles) of arbitrary mass could exist in such solutions if (i) the resultant force on each mass passed through the barycenter, (ii) the resultant force was directly proportional to the distance of each mass from the barycenter, (iii) the initial velocity
vectors were proportional in magnitude to the respective distances of the masses from the barycenter and made equal angles with the radius vectors to the masses from the barycenter. If there is no change in scale, the solutions are called stationary ones and the relative distances do not alter; the system also rotates in a plane about the barycenter with constant angular velocity. Given the positions of two bodies, the five equilibrium positions which can be taken up by the third body are called stationary, equilibrium, Laplacian, Eulerian, Lagrangian, libration points; or centers of libration. It is worth mentioning that although the two particular solutions are attributed to Lagrange, the first solution of a three body problem, i.e. that of three bodies which are moving on a fixed straight line and maintain a constant ratio of their distances was published by Euler in 1765 (Ref. 3). Euler's solution, though a particular solution, is really a special case of Lagrange's collinear solution.

Since no solutions, other than the two mentioned above were obtained, people turned their attention to a particular three body problem, now well known as the restricted three body problem. The birth date of the restricted problem is 1772 when Euler published his second lunar theory in his "Theoria Motum Lunae". The word "restricted" appears to have originated from Poincaré in his "Les Methodes Nouvelles de la Mecanique Celeste" when he calls the restricted problem "a particular case of the problem of three bodies" and later introduces the word "restreint".

Some of the important names associated with the restricted problem of three bodies are Lagrange (1772), Jacobi (1836), Hill (1878), Poincare (1899), Levi-Civita (1905), and Birkhoff (1915).
The restricted problem of three bodies can be defined as follows: Two bodies revolve around their center of mass in circular orbits under the influence of their mutual gravitational attraction and a third body (attracted by the previous two, but not influencing their motion) moves in the plane defined by the two revolving bodies. The restricted problem of three bodies is to describe the motion of this third body. The two revolving bodies are called the primaries.

The equilateral libration points, now commonly identified by \( L_4 \) and \( L_5 \), are of considerable interest as these points are stable if the ratio of the mass of the lighter primary to the sum of the masses of the primaries is less than 0.0385 and this is true for many situations occurring in the solar system. In particular, the value of this ratio in the earth moon case is approximately 0.01 and hence, motion about \( L_4 \) and \( L_5 \) is stable. Preferred references for the linearized treatment of motion around the triangular libration points are by Plummer (Ref. 4) and by Martin (Ref. 5). Nonlinear stability analyses of orbits of finite size around the triangular libration points have been performed by Deprit (Ref. 6) and by Pedersen (Ref. 1).

Unfortunately, literature on the libration dynamics of a station near the equilateral point is sparse and restrictive. Kane and Marsh (Ref. 7) studied the libration dynamics of a spinning symmetric satellite while Robinson (Ref. 8) investigated the libration period of a dumb-bell shaped satellite. This work presents the study of a communication station of arbitrary shape and internal mass distribution when placed in orbit about the equilateral point.
3. EQUATIONS OF MOTION

Two bodies A and B (typically the earth and the moon) are treated as point masses and revolve around their barycenter 0 with a constant angular velocity $\psi$ (Fig. 4). The distances $\rho_A$ and $\rho_B$ to the two bodies remain constant and are measured from the origin of an "inertial" co-ordinate system XYZ whose origin is at the barycenter and whose X and Y axes lie in the plane of motion of the two bodies.

The position of the center of mass of the third body (typically a space station) is defined by its distance $\rho_3$ from the point 0 and by the angle that the radius vector $\mathbf{r}_3$ makes with the line joining the point masses A and B. The vectors $\mathbf{r}_A$ and $\mathbf{r}_B$ which locate the center of mass of the third body from the bodies A and B respectively, make angles $\alpha$ and $\beta$ with the line joining the bodies A and B. The two bodies A and B and the third body form a restricted three body problem.

The center of mass of the third body is the origin 0 (Fig. 3) of three systems of axes. The axes $x_0, y_0, z_0$ form the principal axes of the non-spherical body. $A_3, B_3,$ and $C_3$ are the moments of inertia of the third body about the principal axes $x_0, y_0, z_0$. The libration of this body in the xy plane is defined by the angle which the $x_0$ axis makes with a line passing through its center of mass and parallel to the line joining A and B. The other two axes systems are given by $x_1, y_1, z_1$ which is a system of axes with origin at 0 such that the direction of the axis $x_1$ is taken along the line joining A and 0 and by $x_2, y_2, z_2$ which is
a system of axes with origin at \( o \) such that the direction of the \( x \) axis is taken along the line joining \( B \) and \( o \).

It should be noted that the co-ordinate system used is a polar co-ordinate system. This is in contrast to the conventionally used cartesian system. The polar system used is later found advantageous in analyzing the pointing errors. As a historical note it may be added that a cartesian system with the origin at the barycenter, and with the heavier primary placed on the left side of the origin, belongs to Moulton's school of thought.

The equations of motion are derived using a Lagrangian formulation. The potential energy terms in the Lagrangian \( L \) use a spherical model for the primaries. The potential energy for the third (non-symmetric) body is obtained by using MacCullagh's formula (Ref. 9) which can be obtained by expanding the potential function in a Legendre polynomial series and retaining the first three terms.

\( \rho_3, \, \theta, \, \text{and} \, \phi \) are the three generalized co-ordinates used. \( \rho_3 \) and \( \theta \) describe the position of the center of mass of the station whereas \( \phi \) describes its attitude.

The Lagrangian is given by

\[
L = T - V
\]

\[
L = \frac{1}{2} m_A \rho_A^2 \psi^2 + \frac{1}{2} m_B \rho_B^2 \psi^2 + \frac{1}{2} m_3 \rho_3^2 (\dot{\psi}^2 + \dot{\theta}^2) + \frac{1}{2} m_3 \dot{\rho}_3^2 \\
+ \frac{1}{2} C_3 (\dot{\psi} + \dot{\phi})^2 + \frac{\text{Gm}_A m_B}{\rho_A + \rho_B} + \frac{\text{Gm}_A m_A}{r_{A3}} + \frac{\text{Gm}_B m_3}{r_{B3}} \\
+ \frac{\text{Gm}_A}{2r_{A3}^3} (A_3 + B_3 + C_3 - 3I_{11}x_{11}) \\
+ \frac{\text{Gm}_B}{2r_{B3}^3} (A_3 + B_3 + C_3 - 3I_{22}x_{22})
\]

\[
(3-1)
\]
where $I_{x_1x_1}$ and $I_{x_2x_2}$ are the moment of inertias of the third body about the axes $x_1$ and $x_2$ respectively.

Using the relations

$$r_{A3}^2 = \rho_A^2 + \dot{\rho}_3^2 + 2\rho_A \dot{\rho}_3 \ c \theta$$

(3.2)

$$r_{B3}^2 = \rho_B^2 + \dot{\rho}_3^2 - 2\rho_B \dot{\rho}_3 \ c \theta$$

(3.3)

equation 3.1 can be written in the form

$$L = \frac{1}{2} m_A \rho_A^2 \dot{\psi}^2 + \frac{1}{2} m_B \rho_B^2 \dot{\phi}^2 + \frac{1}{2} m_3 \rho_3^2 (\dot{\psi} + \dot{\phi})^2 + \frac{1}{2} m_3 \dot{\theta}^2$$

$$+ \frac{1}{6} C_3 (\dot{\psi} + \dot{\phi})^2 + \frac{Gm_A m_B}{\rho_A + \rho_B} + \frac{Gm_A m_3}{r_{A3}} + \frac{Gm_B m_3}{r_{B3}}$$

$$+ \frac{Gm_A}{2(\rho_A^2 + \rho_3^2 + 2\rho_A \rho_3 \ c \theta)^{3/2}} (A_3 + B_3 + C_3 - 3I_{x_1x_1})$$

$$+ \frac{Gm_B}{2(\rho_B^2 + \rho_3^2 + 2\rho_B \rho_3 \ c \theta)^{3/2}} (A_3 + B_3 + C_3 - 3I_{x_2x_2})$$

(3.4)

The terms $I_{x_1x_1}$ and $I_{x_2x_2}$ in the Lagrangian are functions of $\rho_3$, $\theta$, $\phi$ and hence should be written explicitly in terms of these quantities.

To this end, the inertia matrices in the $x_1$, $y_1$, $z_1$ and $x_2$, $y_2$, $z_2$ system of axes are written in terms of the principal axes system.

If $[I]$ is the principal inertia matrix, $[I_i]$ the inertia matrix in a system which has been given a planar rotation from the principal axis system can be obtained from the relation

$$[I_i] = [t_i] [I] [t_i]^T$$

(3.5)

where the transformation matrices $[t_i]$ are given by
\[
\begin{bmatrix}
c(\alpha-\phi) & s(\alpha-\phi) & 0 \\
-s(\alpha-\phi) & c(\alpha-\phi) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  
(3.6)

\[
\begin{bmatrix}
c(\pi-\beta-\phi) & s(\pi-\beta-\phi) & 0 \\
-s(\pi-\beta-\phi) & c(\pi-\beta-\phi) & 0 \\
0 & 0 & 1
\end{bmatrix}
\]  
(3.7)

Also, from the geometry of the problem

\[ t \alpha = \frac{\rho_3 s \theta}{\rho_A + \rho_3 c \theta} \]  
(3.8)

\[ t \beta = \frac{\rho_3 s \theta}{\rho_B - \rho_3 c \theta} \]  
(3.9)

Substituting equations 3.8 and 3.9 in equations 3.6 and 3.7 respectively and the result in 3.5 yields equations 3.10 and 3.11

\[
[I_1] = \begin{bmatrix}
A_3 c^2 [t^{-1} \left( \frac{\rho_3 s \theta}{\rho_A + \rho_3 c \theta} - \phi \right)] & (B_3 - A_3) s [t^{-1} \left( \frac{\rho_3 s \theta}{\rho_A + \rho_3 c \theta} - \phi \right)] & 0 \\
B_3 s^2 [t^{-1} \left( \frac{\rho_3 s \theta}{\rho_A + \rho_3 c \theta} - \phi \right)] & x c [t^{-1} \left( \frac{\rho_3 s \theta}{\rho_A + \rho_3 c \theta} - \phi \right)] & 0 \\
(B_3 - A_3) s [t^{-1} \left( \frac{\rho_3 s \theta}{\rho_A + \rho_3 c \theta} - \phi \right)] & A_3 s^2 [t^{-1} \left( \frac{\rho_3 s \theta}{\rho_A + \rho_3 c \theta} - \phi \right)] & 0 \\
x c [t^{-1} \left( \frac{\rho_3 s \theta}{\rho_A + \rho_3 c \theta} - \phi \right)] + B_3 c^2 [t^{-1} \left( \frac{\rho_3 s \theta}{\rho_A + \rho_3 c \theta} - \phi \right)] & 0 & c_3
\end{bmatrix}
\]  
(3.10)
\[
[I_2] = \begin{bmatrix}
A_3 c^2 \left[ \pi t^{-1} \left( \frac{\rho_3 s \Theta}{\rho_B - \rho_3 c \Theta} \right) - \phi \right] & (B_3 - A_3) c \left[ \pi t^{-1} \left( \frac{\rho_3 s \Theta}{\rho_B - \rho_3 c \Theta} \right) - \phi \right] \\
+ B_3 c^2 \left[ \pi t^{-1} \left( \frac{\rho_3 s \Theta}{\rho_B - \rho_3 c \Theta} \right) - \phi \right] & A_3 c^2 \left[ \pi t^{-1} \left( \frac{\rho_3 s \Theta}{\rho_B - \rho_3 c \Theta} \right) - \phi \right] \\
x c \left[ \pi t^{-1} \left( \frac{\rho_3 s \Theta}{\rho_B - \rho_3 c \Theta} \right) - \phi \right] & + B_3 c^2 \left[ \pi t^{-1} \left( \frac{\rho_3 s \Theta}{\rho_B - \rho_3 c \Theta} \right) - \phi \right]
\end{bmatrix}
\]

(3.11)

Using equations 3.4, 3.10, and 3.11, the Lagrangian can now be written as

\[
L = \frac{1}{2} m_A \dot{\rho}_A \dot{\psi} + \frac{1}{2} m_B \dot{\rho}_B \dot{\psi} + \frac{1}{2} m_3 \dot{\rho}_3^2 (\dot{\psi} + \dot{\Theta})^2 + \frac{1}{2} m_3 \dot{\psi}^2 \\
+ \frac{1}{2} C_3 (\dot{\psi} + \dot{\Theta})^2 + \frac{G m_B}{\rho_A + \rho_B} + \frac{G m_3}{\rho_A + \rho_3} \\
+ \frac{G m_B m_3}{(\rho_A^2 + \rho_B^2 + \rho_3^2 c \Theta) \sqrt{2}} \begin{bmatrix}
A_3 + B_3 + C_3 - B_3 - A_3 \right) c^2 \left[ \pi t^{-1} \left( \frac{\rho_3 s \Theta}{\rho_A + \rho_3 c \Theta} \right) - \phi \right] \\
A_3 + B_3 + C_3 - B_3 - A_3 \right) c^2 \left[ \pi t^{-1} \left( \frac{\rho_3 s \Theta}{\rho_B - \rho_3 c \Theta} \right) - \phi \right]
\end{bmatrix}
\]

(3.12)

Knowing the Lagrangian, the equations of motion can now be deduced and can be shown to be given by the relations
\[ \theta + \frac{2 \rho_3}{\rho_3} (\dot{\psi} + \dot{\phi}) - \frac{G_{m A}}{2 \rho_3^{2/3} r_{A3}} \left\{ \rho_A \rho_3 s \theta (2 + \frac{3p}{m_3 r_{A3}}) - 3s_1 \frac{\rho_A \rho_3 c \theta + \rho_3^2}{r_{A3}^{2/3}} s2 \phi_1 \right\} \]

\[ - \frac{G_{m B}}{2 \rho_3^{2/3} r_{B3}} \left\{ - \rho_B \rho_3 s \theta (2 + \frac{3q}{m_3 r_{B3}}) + 3s_1 \frac{\rho_B \rho_3 c \theta - \rho_3^2}{r_{B3}^{2/3}} s2 \phi_2 \right\} = 0 \]

(3.13)

\[ \rho_3 (\dot{\psi} + \dot{\phi})^2 + \frac{G_{m A}}{r_{A3}^{3/2}} \left\{ (\rho_3 + \rho_A c \theta) (2 + \frac{3p}{m_3 r_{A3}^{2/3}}) + 3s_1 \frac{\rho_A s \theta}{r_{A3}^{2/3}} s2 \phi_1 \right\} \]

\[ + \frac{G_{m B}}{2 r_{B3}^{3/2}} \left\{ (\rho_3 - \rho_B c \theta) (2 + \frac{3q}{m_3 r_{B3}^{2/3}}) - 3s_1 \frac{\rho_B s \theta}{r_{B3}^{2/3}} s2 \phi_2 \right\} = 0 \]

(3.14)

\[ \phi - 3G s_2 \left\{ \frac{m_A}{2 r_{A3}^{3/2}} s2 \phi_1 + \frac{m_B}{2 r_{B3}^{3/2}} s2 \phi_2 \right\} = 0 \]

(3.15)

where \( r_{A3} \) and \( r_{B3} \) are defined by equations (3.2) and (3.3) and

\[ \phi_1 = t^{-1} \left( \frac{\rho_3 s \theta}{\rho_A + \rho_3 c \theta} \right) - \phi \]

(3.16)

\[ \phi_2 = \pi - t^{-1} \left( \frac{\rho_3 s \theta}{\rho_B - \rho_3 c \theta} \right) - \phi \]

(3.17)

\[ p = A_3 + B_3 + C_3 - 3B_3^2 + 3(B_3 - A_3)c^2 \phi_1 \]

(3.18)

\[ q = A_3 + B_3 + C_3 - 3B_3^2 + 3(B_3 - A_3)c^2 \phi_2 \]

(3.19)

\[ s_1 = \frac{(B_3 - A_3)/m_3}{C_3} \]

(3.20)

\[ s_2 = \frac{(B_3 - A_3)/C_3}{C_3} \]

(3.21)

From equations 3.13 through 3.15, 3.20 and 3.21, it will be observed
that the motion depends upon two parameters $S_1$ and $S_2$, both of which are independent of the mass of the third body. $S_1$ and $S_2$ may respectively be called size and shape parameters. $S_1$ depends upon the dimensions and geometric configuration of the third body whereas $S_2$ depends solely on its geometric configuration. It should also be noted that when $S_1 = S_2 = 0$; i.e. for a symmetric body, equations 3.13 and 3.14 reduce to the polar equations of motion for the classical restricted three body problem.

A solution to the highly non-linear equations of motion (equations 3.13 through 3.15), though illuminating, does not seem plausible. Hence the equations of motion must first be linearized about an equilibrium position before any meaningful results can be obtained.
4. LINEARIZED EQUATIONS OF MOTION

Before proceeding into the linearized analysis, it should be noted that when a non-symmetric body of a finite size (finite values of principal moment of inertias), but of negligible mass, is placed in a system of two bodies rotating about each other, the equilibrium positions will not be exactly at the equilibrium points of a corresponding symmetric body. These differences for the equilateral point \( L_4 \) are now calculated. The theory developed for this investigation is valid for all equilibrium points, and can be outlined as follows:

Let a set of \( 2N \) first order differential equations describe the motion of an \( N \)-degree of freedom dynamical system.

\[
\dot{x}_i = X_i(x_1, x_2, \ldots, x_m) \quad i = 1, 2, \ldots, m \tag{4.1}
\]

where \( m = 2N \) and \( x_i \) are the generalized coordinates (which for one case include actual coordinates and velocities). At an equilibrium point, the generalized velocities should be zero.

\[
\left. \dot{x}_i \right|_{x_i = x_{i,oo}} = X_i(x_1, x_2, \ldots, x_m) \left|_{x_i = x_{i,oo}} \right. = 0 \quad i = 1, 2, \ldots, m \tag{4.2}
\]

where the subscript 'oo' refers to an equilibrium point. The left hand side of the second half of equation (4.2) can be replaced by a Taylor Series expansion about a point close to the equilibrium point and denoted by the subscript 'no'. Retaining first order terms in the Taylor Series expansion
\[ X_i(x_1, x_2, \ldots, x_m) + \sum_{j=1}^{m} a_{ij} y_j = 0 \quad i = 1, 2, \ldots, m \quad (4.3) \]

where \( a_{ij} = \left. \frac{\partial X(x_1, x_2, \ldots, x_m)}{\partial x_j} \right|_{x_i = x_{i0}} \quad i, j = 1, 2, \ldots, m \quad (4.4) \)

and where the \( y_j \) are the perturbations in \( x_j \).

Applying equations (4.3) and (4.4) to this problem in which the non-equilibrium point (which is close to the equilibrium point) is the equilateral triangle point \( L_4 \) denoted by the subscript 'o'.

Equation (4.3) can now be written in the form

\[ \sum_{j=1}^{m} a_{ij} y_j = -X_i(x_1, x_2, \ldots, x_m) \quad i = 1, 2, \ldots, m \quad (4.5) \]

where the \( a_{ij} \) are given by equation (4.4). The values of \( y_j \) which are a linear approximation to the difference between the equilibrium positions of a non-symmetric and symmetric body can now be obtained from the matrix equation

\[ \{y\} = -[a]^{-1} \{x\} \quad \{x\} \rightarrow \{x_{i0}\} \quad (4.6) \]

The problem now is to calculate the coefficients \( a_{ij} \). However, before doing so, it should be noted that the theory developed is for a set of first order differential equations of motion while the derived equations of motion (equations (3.13), (3.14), and (3.15)) are a set of second order differential equations. By a process of ablimination, these three equations are now replaced by a set of six first order differential equations which are given by
\[ \begin{align*}
\dot{\phi} &= -\frac{2R_3}{\rho_3} (\dot{\psi} + \Theta) + \frac{Gm_A}{2^\frac{2}{3} \rho_3^2 r_{A3}} \left\{ \rho_A \rho_3 \Theta (2 + \frac{3p}{m_3 r_{A3}^2}) - 3s_1 \frac{\rho_A \rho_3 c\Theta}{r_{A3}^2} \right\} \\
&\quad + \frac{Gm_B}{2^\frac{2}{3} \rho_3 r_{B3}} \left\{ -\rho_B \rho_3 \Theta (2 + \frac{3Q}{m_3 r_{B3}^2}) + 3 \frac{\rho_A \rho_3 c\Theta - c_3}{r_{B3}^2} \right\} \\
\dot{R}_3 &= \rho_3 (\dot{\psi} + \Theta)^2 - \frac{Gm_A}{2r_{A3}} \left\{ (\rho_3 + \rho_A c\Theta) (2 + \frac{3p}{m_3 r_{A3}^2}) + 3s_1 \frac{\rho_A s\Theta}{r_{A3}^2} \right\} \\
&\quad - \frac{Gm_B}{2r_{B3}} \left\{ (\rho_3 - \rho_B c\Theta) (2 + \frac{3Q}{m_3 r_{B3}^2}) - 3s_1 \frac{\rho_B s\Theta}{r_{B3}^2} \right\} \\
\dot{\Theta} &= 3Gs_2 \frac{m_A}{2r_{A3}^3} s2\phi_1 + \frac{m_B}{2r_{B3}^3} s2\phi_2 \\
\dot{\rho}_3 &= R_3 \\
\dot{\phi} &= \dot{\psi}
\end{align*} \]

From equations (4.7) through (4.12), the thirty six elements of \([a]\) can be found and after due simplification can be shown to be given by

\[ a_{\Theta\Theta} = -\frac{2R_3}{\rho_3} \quad (4.13) \]

\[ a_{\Theta R_3} = -2(\dot{\psi} + \Theta)/\rho_3 \quad (4.14) \]

\[ a_{\Theta\phi} = 0 \quad (4.15) \]

\[ a_{\Theta\Theta} = \frac{Gm_A}{r_{A3}^2 \rho_3} \left\{ \frac{2}{2^\frac{2}{5} \rho_3^2 r_{A3}} \left( (\rho_3 + \rho_A c\Theta)^2 \right) \right\} \\
&\quad -3 \frac{\rho_A s\Theta}{2r_{A3}^2} \left\{ 3s_1 (\rho_3 + \rho_A c\Theta)s2\phi_1 - \frac{2p}{m_3} \rho_A s\Theta \right\} \]
\begin{align*}
\frac{2}{r_{A3}^2} & \left[ \frac{2}{r_{A3}^2} - \frac{2}{\rho_3 r_{A3}} \left( \rho_3 + \rho_A c^\theta \right) \right] \\
- \frac{2}{r_{A3}} & \left[ \frac{2}{r_{A3}^2} - \frac{2}{\rho_3 r_{A3}} \left( \rho_3 + \rho_A c^\theta \right) \right] \\
- \frac{3G_m}{r_{B3}^2} & \left[ \frac{r_{B3}^2 \rho_3 c^\theta - 3 (\rho_B c^\theta)^2}{2 \rho_3 r_{B3}^5} \right] \left( 2 + \frac{3Q}{m_3 r_{B3}^2} \right) \\
+ \frac{3G_m}{2r_{B3}^7} & \left[ \frac{3s_1 (\rho_3 - \rho_B c^\theta) s \delta_2 - 2Q}{m_3 \rho_B s^\delta} \right] \\
- \frac{3G_m}{2r_{B3}^7} & \left[ \frac{2}{r_{B3}^2} + 2 (\rho_3 - \rho_B c^\theta) \right] \\
+ \frac{3G_m}{2r_{B3}^7} & \left[ \frac{2}{r_{B3}^2} - 3 (\rho_B c^\theta)^2 \right] \\
- \frac{2R_3}{\rho_3^2} (\gamma + \Theta) & - \frac{2}{\rho_3^2} \left[ \frac{r_{A3}^2 \rho_3 s^\theta + 3 \rho_A \rho_3 s^\theta (\rho_3 + \rho_A c^\theta)}{2 \rho_3^2 r_{A3}^5} \right] \left( 2 + \frac{3P}{m_3 r_{A3}^2} \right) \\
- \frac{3G_m}{2r_{A3}^7} & \left[ \frac{3s_1 \rho_3 s \delta_1 + \frac{2P}{m_3} (\rho_3 + \rho_A c^\theta)}{r_{A3}^2} \right] \\
+ \frac{3G_m}{2r_{A3}^7} & \left[ \frac{r_{A3}^2 \rho_A c^\theta + 2 \rho_3 (\rho_3 + \rho_A c^\theta) (\rho_A c^\theta + \rho_3)}{2 \rho_3^2 r_{A3}^5} \right] \\
- \frac{3G_m}{2r_{A3}^7} & \left[ \frac{2 \rho_A s \delta_1 c^\theta - 3 (\rho_3 + \rho_A c^\theta) s \delta_1}{r_{A3}^2} \right] \\
+ \frac{3G_m}{2r_{B3}^5} & \left[ \frac{r_{B3}^2 \rho_B s^\theta + 3 \rho_B \rho_3 s^\theta (\rho_3 - \rho_B c^\theta)}{2 \rho_3^2 r_{B3}^5} \right] \left( 2 + \frac{3Q}{m_3 r_{B3}^2} \right)
\end{align*}
\[ -3 G_m \frac{\rho_B^2 \rho_3 \rho_{3 \theta}}{2 \rho_3^5 r_{B3}^7} \left\{ 3 S_1 \frac{\rho_B^2 \rho_3 \rho_{3 \theta} \rho_{3 \phi}^2}{2 \rho_3^5 r_{B3}^7} - \frac{2 \rho_B}{m_3^2} (\rho_3 - \rho_B \rho_{3 \theta}) \right\} \]

\[ -3 G_m \frac{\rho_3 \rho_{3 \theta} \rho_{3 \phi}^2}{2 \rho_3^5 r_{B3}^7} \left\{ \frac{2 \rho_B^2 \rho_{3 \theta} \rho_{3 \phi}^2}{2 \rho_3^5 r_{B3}^7} + 2 \rho_3 (\rho_3 - \rho_B \rho_{3 \theta}) (\rho_B \rho_{3 \theta} - \rho_3) \right\} \]

\[ -3 G_m \frac{\rho_3 (\rho_B \rho_{3 \theta}^2 - \rho_3)}{2 \rho_3^5 r_{B3}^7} \left\{ 2 \rho_B \rho_{3 \theta} \rho_{3 \phi}^2 + 3 (\rho_3 - \rho_B \rho_{3 \theta}) \rho_{3 \phi}^2 \right\} \quad \text{(4.17)} \]

\[ a_{\rho \phi} = \frac{3 G_m S_1}{2 \rho_3^5 r_{A3}^7} \left\{ \frac{3 \rho_B^2 \rho_3 \rho_{3 \phi}^2}{2 \rho_3^5 r_{A3}^7} + 2 \rho_3 (\rho_B \rho_{3 \phi}^2 + \rho_3) \rho_{3 \phi} \right\} \]

\[ - \frac{3 G_m S_1}{2 \rho_3^5 r_{B3}^7} \left\{ 3 \rho_B^2 \rho_3 \rho_{3 \phi}^2 + 2 \rho_3 (\rho_B \rho_{3 \phi}^2 - \rho_3) \rho_{3 \phi} \right\} \quad \text{(4.18)} \]

\[ a_{\rho \theta} = 2 \rho_3 (\dot{\psi} + \Theta) \quad \text{(4.19)} \]

\[ a_{R_3 \theta} = a_{R_3 \phi} = 0 \quad \text{(4.20)} \]

\[ a_{R_3 \rho} = \frac{G_m \rho_B^2}{A_3^3} \left\{ \frac{\rho_3 + \rho_A \rho_{3 \theta}}{r_{A3}^5} \left\{ \frac{2 \rho_3}{r_{A3}^5} - \frac{3 \rho_3 (\rho_3 + \rho_A \rho_{3 \theta})}{2 \rho_3^5 r_{A3}^7} \right\} \right\} \left\{ 2 + \frac{3 \rho}{m_3^2} \rho_A \rho_{3 \theta} \right\} \]

\[ + \frac{3}{2} G_m \rho_B^2 \rho_{3 \phi} \left\{ \frac{1}{r_{A3}^7} \left\{ \frac{2 \rho_3}{r_{A3}^5} - \frac{3 \rho_3 (\rho_3 + \rho_A \rho_{3 \theta})}{2 \rho_3^5 r_{A3}^7} \right\} \right\} \left\{ \frac{2 \rho_3}{r_{A3}^7} - \frac{2 \rho}{m_3^2} \rho_A \rho_{3 \theta} \right\} \]

\[ - \frac{3}{2} G_m \rho_B^2 \rho_{3 \phi} S_1 \left\{ \frac{1}{r_{A3}^7} \left\{ \frac{2 \rho_3}{r_{A3}^5} - \frac{3 \rho_3 (\rho_3 + \rho_A \rho_{3 \theta})}{2 \rho_3^5 r_{A3}^7} \right\} \right\} s^2 \phi_1 \]

\[ - \frac{3}{2} G_m \rho_B^2 \rho_{3 \phi} S_1 s^2 \theta \left\{ \frac{2 (\rho_3 + \rho_A \rho_{3 \theta})}{r_{A3}^7} - \frac{3 \rho_3 \rho_{3 \theta} s^2 \phi_1}{r_{A3}^7} \right\} \]
\[-Gm_B\rho_B \delta^\theta \left\{ \frac{r_{B3}^2 - 3\rho_3 (\rho_3 - \rho_B c\theta)}{2r_{B3}^5} \right\} \left(2 + \frac{3Q}{m_3 r_{B3}}\right) \]

\[+ \frac{3}{2} Gm_B \rho_B^3 \left\{ \frac{\rho_3 - \rho_B c\theta}{r_{B3}^2} \right\} \left\{ 3S_1 (\rho_3 - \rho_B c\theta) s2\phi_2 + \frac{2Q}{m_3} \rho_B s\theta \right\} \]

\[+ \frac{3}{2} Gm_B \rho_B S_1 \left\{ \frac{r_{B3}^2 c\theta - 2\rho_B \rho_3 s^2 \theta}{r_{B3}^7} \right\} s2\phi_2 \]

\[+ \frac{3}{2} Gm_B \rho_B^3 S_1 \left\{ \frac{2(\rho_3 - \rho_B c\theta)c2\phi_2 - 3\rho_B s\theta s2\phi_2}{r_{B3}^2} \right\} \]

\[= (\psi + \Theta)^2 - Gm_A \left\{ \frac{r_{A3}^2 - 3(\rho_3 + \rho_A c\theta)^2}{2r_{A3}^5} \right\} \left(2 + \frac{3P}{m_3 r_{A3}}\right) \]

\[+ \frac{3}{2} Gm_A \left\{ \frac{\rho_3 + \rho_A c\theta}{r_{A3}^7} \right\} \left\{ 3S_1 \rho_A s\theta s2\phi_1 + \frac{2P}{m_3} (\rho_3 + \rho_A c\theta) \right\} \]

\[+ 3Gm_A S_1 \left\{ \frac{\rho_A s\theta (\rho_3 + \rho_A c\theta) s2\phi_1}{r_{A3}^2} \right\} \]

\[+ \frac{3}{2} Gm_A S_1 \left\{ \frac{\rho_A s\theta}{r_{A3}^7} \right\} \left\{ 3(\rho_3 + \rho_A c\theta)s2\phi_1 - 2\rho_A s\theta c2\phi_1 \right\} \]

\[-Gm_B \left\{ \frac{r_{B3}^2 - 3(\rho_3 - \rho_B c\theta)^2}{2r_{B3}^5} \right\} \left(2 + \frac{3Q}{m_3 r_{B3}}\right) \]

\[+ \frac{3}{2} Gm_B \left\{ \frac{\rho_3 - \rho_B c\theta}{r_{B3}^7} \right\} \left\{ -3S_1 \rho_B s\theta s2\phi_2 + \frac{2Q}{m_3} (\rho_3 - \rho_B c\theta) \right\} \]

\[-3Gm_B S_1 \left\{ \frac{\rho_B s\theta (\rho_3 - \rho_B c\theta) s2\phi_2}{r_{B3}^7} \right\} \]
\[-\frac{3}{2} \, \text{Gm}_B \, S \, \frac{\rho_B \, s \theta}{r_{B3}} \left\{ 3(\rho_B - \rho_B \, c \theta) \, s^{2} \phi_2 + 2\rho_B \, s \theta \, c^{2} \phi_2 \right\} \]

\[a_{R_3} \phi = -\frac{3 \text{Gm}_B \, S \, \frac{1}{2} \, r_{A3}}{\rho_B \, s \theta} \left\{ 3(\rho_B + \rho_B \, c \theta) \, s^{2} \phi_1 - 2\rho_B \, s \theta \, c^{2} \phi_1 \right\} \]

\[-\frac{3 \text{Gm}_B \, S \, \frac{1}{2} \, r_{B3}}{2} \, 3(\rho_B - \rho_B \, c \theta) \, s^{2} \phi_2 + 2\rho_B \, s \theta \, c^{2} \phi_2 \]

\[a_{\phi \phi} = 0 \]

\[a_{R_3} \phi = 0 \]

\[a_{\phi \phi} = 0 \]

\[a_{\phi \theta} = -3 \text{Gm}_2 \, \frac{m_A \, \rho_3}{r_{A3}} \left\{ \frac{3}{2} \, \rho_A \, s \theta \, c^{2} \phi_1 - (\rho_3 + \rho_A \, c \theta) \, c^{2} \phi_1 \right\} \]

\[-3 \text{Gm}_2 \, \frac{m_B \, \rho_3}{r_{B3}} \left\{ \frac{3}{2} \, \rho_B \, s \theta \, c^{2} \phi_2 - (\rho_3 - \rho_B \, c \theta) \, c^{2} \phi_2 \right\} \]

\[a_{R_3} \rho = -3 \text{Gm}_2 \, \frac{m_A}{r_{A3}} \left\{ \frac{3}{2} \, (\rho_3 + \rho_A \, c \theta) \, s^{2} \phi_1 - \rho_A \, s \theta \, c^{2} \phi_1 \right\} \]

\[-3 \text{Gm}_2 \, \frac{m_B}{r_{B3}} \left\{ \frac{3}{2} \, (\rho_3 - \rho_B \, c \theta) \, s^{2} \phi_2 + \rho_B \, s \theta \, c^{2} \phi_2 \right\} \]

\[a_{\phi \phi} = -3 \text{Gm}_2 \, \left\{ \frac{m_A}{r_{A3}} \, c^{2} \phi_1 + \frac{m_B}{r_{B3}} \, c^{2} \phi_2 \right\} \]

\[a_{\theta \theta} = a_{R_3} \rho = a_{\phi \phi} = 1 \]
\[ a \theta R_3 = a \theta \dot{\phi} = a \dot{\theta} = a \rho_3 = a \phi = 0 \quad (4.31) \]

\[ a \rho_3 = a \rho_3 = a \rho_3 \dot{\phi} = a \rho_3 \rho_3 = a \phi = 0 \quad (4.32) \]

\[ a \dot{\phi} = a \phi R_3 = a \phi \dot{\theta} = a \phi \rho_3 = a \phi = 0 \quad (4.33) \]

where \( r_{A3} \) and \( r_{B3} \) are defined by equations (3.2) and (3.3) and \( \phi_1, \phi_2, p, \)

\( Q, S_1, S_2 \) are defined by equations (3.10) through (3.21).

The matrix \([a]\) in equation (4.6) has to be evaluated at the equilateral triangular point and as such the elements \( a_{ij} \) are now evaluated at the equilateral triangle point.

\[ a \dot{\theta} \bigg|_0 = 0 \quad (4.34) \]

\[ a \dot{R}_3 \bigg|_0 = -2 \frac{\dot{\psi}}{\rho_3} \quad (4.35) \]

\[ a \dot{\phi} \bigg|_0 = 0 \quad (4.36) \]

\[ a \theta \bigg|_0 = \frac{\left( \frac{(\rho_A + \rho_B)^2}{2 \rho_3 \rho_3 (\rho_A + \rho_B)} \right)^2}{2 \rho_3 \rho_3 (\rho_A + \rho_B)^5} \left( \frac{[S_{3}^{3} + 3(B_{3} - A_{3})\pi^2 (\pi/3 - \phi_{o})]}{m^3 (\rho_A + \rho_B)^2} \right) \]

\[ - 3Gm_A \left( \frac{\rho_A s^2}{2 (\rho_A + \rho_B)} \right) \left( \frac{3S_1 (\rho_3 + \rho_A s^2 (\pi/3 - \phi_{o})) - 2 \rho_A s^2}{m^3} \right) \left( \frac{3S_3 + 3(B_3 - A_3) \pi^2 (\pi/3 - \phi_{o})}{m^3} \right) \]

\[ + 3Gm_A \left( \frac{\rho_A s^2 (\pi/3 - \phi_{o})}{2 \rho_3 (\rho_A + \rho_B)^2} \right) \left( \frac{(\rho_A + \rho_B)^7}{2 (\rho_A + \rho_B)} \right) \]
\[-3 G m_A \left\{ \frac{\rho_{3o} + \rho_A c th_o}{2 (\rho_A + \rho_B)^7} \right\} \left\{ 2 (\rho_{3o} + \rho_A c th_o) c^2 (\pi/3 - \phi_o) + 3 \rho_A s th_o s2 (\pi/3 - \phi_o) \right\} \]
\[-G m_B \left\{ \frac{2 (\rho_A + \rho_B)^2 \rho_B \rho_{3o} c th_o - 3 (\rho_B \rho_{3o} s th_o)^2}{2 \rho_{3o} (\rho_A + \rho_B)^5} \right\} \left\{ 3 \left[ \frac{S_3 + 3 (B_3 - A_3)}{m_3 (\rho_A + \rho_B)^2} \right] c^2 (2\pi/3 - \phi_o) \right\} \]
\[+ 3 G m_B \left\{ \frac{\rho_B s th_o}{2 (\rho_A + \rho_B)^7} \right\} \left\{ 3 s_1 (\rho_{3o} - \rho_B c th_o) s2 (2\pi/3 - \phi_o) - \frac{2 \rho_B s th_o}{m_3} \left[ \frac{S_3 + 3 (B_3 - A_3)}{m_3 (\rho_A + \rho_B)^2} \right] c^2 (2\pi/3 - \phi_o) \right\} \]
\[-3 G m_B S_1 \left\{ \frac{\rho_B \rho_{3o} s th_o}{2 (\rho_A + \rho_B)^7} \right\} \left\{ (\rho_A + \rho_B)^2 + 2 (\rho_{3o} - \rho_B c th_o)^2 \right\} \]
\[+ 3 G m_B S_1 \left\{ \frac{\rho_{3o} - \rho_B c th_o}{2 (\rho_A + \rho_B)^7} \right\} \left\{ 2 (\rho_{3o} - \rho_B c th_o) c^2 (2\pi/3 - \phi_o) + 3 \rho_B s th_o s2 (2\pi/3 - \phi_o) \right\} \]

\[\alpha o = -G m_A \rho_A \left\{ \frac{(\rho_A + \rho_B)^2}{2 \rho_{3o} (\rho_A + \rho_B)^5} \right\} \left\{ 3 s_1 \rho_A s th_o s2 (\pi/3 - \phi_o) + \frac{2 (\rho_{3o} + \rho_A c th_o)}{m_3} \left[ \frac{S_3 + 3 (B_3 - A_3)}{m_3 (\rho_A + \rho_B)^2} \right] c^2 (\pi/3 - \phi_o) \right\} \]
\[-3 G m_A \left\{ \frac{\rho_A \rho_{3o} s th_o}{2 \rho_{3o} (\rho_A + \rho_B)^7} \right\} \left\{ (\rho_A + \rho_B)^2 + 2 (\rho_{3o} + \rho_A c th_o)^2 \right\} \]
\[+3 G m_A S_1 \left\{ \frac{s2 (\pi/3 - \phi_o)}{2 \rho_{3o} (\rho_A + \rho_B)^7} \right\} \left\{ (\rho_A + \rho_B)^2 + 2 (\rho_{3o} + \rho_A c th_o)^2 \right\} \]
\[-3 G m_A S_1 \left\{ \frac{\rho_{3o} (\rho_{3o} + \rho_A c th_o)}{2 \rho_{3o} (\rho_A + \rho_B)^7} \right\} \left\{ 2 \rho_A s th_o c2 (\pi/3 - \phi_o) - 3 (\rho_{3o} + \rho_A c th_o) s2 (\pi/3 - \phi_o) \right\} \]
\[+ G m_B \rho_B \left\{ \frac{(\rho_A + \rho_B)^2 s th_o}{2 \rho_{3o} (\rho_A + \rho_B)^5} \right\} \left\{ 3 \left[ \frac{S_3 + 3 (B_3 - A_3)}{m_3 (\rho_A + \rho_B)^2} \right] c^2 (2\pi/3 - \phi_o) \right\} \]
\[
-3Gm_B \left\{ \frac{\rho_B \rho_{3o} s\theta_o}{2 \rho^2 (\rho_A + \rho_B)} \right\} \left\{ 3 S_1 \rho_B s\theta_o s_2 (2\pi/3 - \phi_o) - \frac{2 (\rho_{3o} - \rho_B c\theta_o)}{m^3} \{S_3 + 3(B_3 - A_3)c^2 (2\pi/3 - \phi_o) \} \right\} \\
-3Gm_B S_1 \left\{ \frac{s_2 (2\pi/3 - \phi_o)}{2 \rho^2 (\rho_A + \rho_B)} \right\} \left\{ (\rho_A + \rho_B)^2 \rho_B c\theta_o - 2 \rho_{3o} (\rho_{3o} - \rho_B c\theta_o)^2 \right\} \\
+ 3Gm_B S_1 \left\{ \frac{\rho_{3o} (\rho_{3o} - \rho_B c\theta_o)}{2 \rho^2 (\rho_A + \rho_B)} \right\} \left\{ 2 \rho_B s\theta_o c_2 (2\pi/3 - \phi_o) + 3 (\rho_{3o} - \rho_B c\theta_o)^2 s_2 (2\pi/3 - \phi_o) \right\} \\
+ \frac{3Gm S_1}{2 \rho^5 (\rho_A + \rho_B)} \left\{ 3 \rho_B s\theta_o s_2 (2\pi/3 - \phi_o) + 2 (\rho_{3o} - \rho_B c\theta_o)^2 c_2 (2\pi/3 - \phi_o) \right\} \\
- \frac{3Gm S_1}{2 \rho^5 (\rho_A + \rho_B)} \left\{ 3 \rho_B s\theta_o s_2 (2\pi/3 - \phi_o) - 2 (\rho_{3o} - \rho_B c\theta_o)^2 c_2 (2\pi/3 - \phi_o) \right\} \\
\]

\[
a_{\theta|_o} = \frac{3Gm S_1}{2 \rho^5 (\rho_A + \rho_B)} \left\{ 3 \rho_B s\theta_o s_2 (2\pi/3 - \phi_o) + 2 (\rho_{3o} - \rho_B c\theta_o)^2 c_2 (2\pi/3 - \phi_o) \right\} \\
\]

\[
a_{R_3 \theta|_o} = 2 \rho_{3o} \psi \\
\]

\[
a_{R_3 R_3|_o} = 0 \\
\]

\[
a_{R_3 \phi|_o} = 0 \\
\]

\[
a_{R_3 \theta|_o} = Gm_A \rho_A s\theta_o \left\{ \frac{(\rho_A + \rho_B)^2 - 3 \rho_{3o} (\rho_{3o} - \rho_A c\theta_o)}{2(\rho_A + \rho_B)^5} \right\} \left\{ 2 + \frac{3S_3 + 3(B_3 - A_3)c^2 (\pi/3 - \phi_o)}{m^3 (\rho_A + \rho_B)^2} \right\} \\
+ \frac{3}{2} Gm_A \rho_{3o} \left\{ \frac{\rho_{3o} + \rho_A c\theta_o}{(\rho_A + \rho_B)} \right\} \left\{ 3S_1 (\rho_A + \rho_A c\theta_o) s_2 (\pi/3 - \phi_o) - \frac{2 \rho_A s\theta_o}{m_3} \{S_3 + 3(B_3 - A_3)c^2 (\pi/3 - \phi_o) \} \right\}
\]
\[-\frac{3}{2} \text{Gm}_A \rho_A S_1 \left\{ \frac{(\rho_A + \rho_B)^2 c_o + 2 \rho_A \rho_3 o s^2 \theta_o}{(\rho_A + \rho_B)^7} \right\} \sin(\pi/3 - \phi_o) \]

\[-\frac{3}{2} \text{Gm}_A \rho_A \rho_3 o S_1 \sin\theta_o \left\{ \frac{2(\rho_3 o + \rho_A c\theta_o) c^2 (\pi/3 - \phi_o) - 3 \rho_A s\theta_o \sin(\pi/3 - \phi_o)}{(\rho_A + \rho_B)^7} \right\} \]

\[-\text{Gm}_B \rho_B s\theta_o \left\{ \frac{(\rho_A + \rho_B)^2 - 3 \rho_3 o (\rho_3 o - \rho_B c\theta_o)}{2(\rho_A + \rho_B)^5} \right\} \left\{ \frac{3(S_3 + 3(B_3 - A_3) c^2 (\pi/3 - \phi_o))}{m_3(\rho_A + \rho_B)^2} \right\} \]

\[+ \frac{3}{2} \text{Gm}_B \rho_3 o \left\{ \frac{\rho_3 o - \rho_B c\theta_o}{(\rho_A + \rho_B)^7} \right\} \left\{ \frac{3S_1 (\rho_3 o - \rho_B c\theta_o) \sin(2\pi/3 - \phi_o) + \frac{2 \rho_B s\theta_o}{m_3} \{S_3 + 3(B_3 - A_3) c^2 (\pi/3 - \phi_o)\}}{(\rho_A + \rho_B)^7} \right\} \]

\[+ \frac{3}{2} \text{Gm}_B \rho_3 o S_1 \sin\theta_o \left\{ \frac{2(\rho_3 o - \rho_B c\theta_o) c^2 (\pi/3 - \phi_o) - 3 \rho_B s\theta_o \sin(2\pi/3 - \phi_o)}{(\rho_A + \rho_B)^7} \right\} \]

\[+ \frac{3}{2} \text{Gm}_B \rho_3 o S_1 \sin\theta_o \left\{ \frac{2(\rho_3 o - \rho_B c\theta_o) c^2 (\pi/3 - \phi_o) - 3 \rho_B s\theta_o \sin(2\pi/3 - \phi_o)}{(\rho_A + \rho_B)^7} \right\} \]

\[\alpha_{R3 o} = \psi^2 - \text{Gm}_A \left\{ \frac{(\rho_A + \rho_B)^2 - 3 (\rho_3 o + \rho_A c\theta_o)^2}{2(\rho_A + \rho_B)^5} \right\} \left\{ \frac{3(S_3 + 3(B_3 - A_3) c^2 (\pi/3 - \phi_o))}{m_3(\rho_A + \rho_B)^2} \right\} \]

\[+ \frac{3}{2} \text{Gm}_A \left\{ \frac{\rho_3 o + \rho_A c\theta_o}{(\rho_A + \rho_B)^7} \right\} \left\{ \frac{3S_1 A_3 \sin(\pi/3 - \phi_o) + \frac{2(\rho_3 o + \rho_A c\theta_o)}{m_3} \{S_3 + 3(B_3 - A_3) c^2 (\pi/3 - \phi_o)\}}{(\rho_A + \rho_B)^7} \right\} \]

\[+ 3 \text{Gm}_A \sin\theta_o \left\{ \frac{2(\rho_3 o + \rho_A c\theta_o) \sin(\pi/3 - \phi_o)}{(\rho_A + \rho_B)^7} \right\} \]

\[+ \frac{3}{2} \text{Gm}_A \sin\theta_o \left\{ \frac{2(\rho_3 o + \rho_A c\theta_o) \sin(\pi/3 - \phi_o) - 2 \rho_A s\theta_o \sin(\pi/3 - \phi_o)}{(\rho_A + \rho_B)^7} \right\} \]
\[
\begin{align*}
\text{Results:} \\
\frac{3Gm}{2}(\frac{\rho_A + \rho_B}{2})^5 \left\{ \frac{1}{2} \left[ \frac{S_3 + 3(B_3 - A_3)c^2(2\pi/3 - \phi_o)}{m_3(\rho_A + \rho_B)^2} \right] \\
+ \frac{3}{2} \frac{\rho_3 - \rho_B c^\Theta_o}{(\rho_A + \rho_B)^7} \left[ -3S_1 s^theta o s^2(2\pi/3 - \phi_o) + (\frac{2\rho_3 - \rho_B c^\Theta_o}{m_3})[S_3 + 3(B_3 - A_3)c^2(2\pi - \phi_o)] \right] \\
- 3Gm \times \left\{ \frac{\rho_B s^\Theta o (\rho_3 o - \rho_B c^\Theta o)s^2(2\pi/3 - \phi_o)}{(\rho_A + \rho_B)^7} \right\} \\
- \frac{3}{2} \frac{\rho_B s^\Theta o}{(\rho_A + \rho_B)^7} \left[ (\rho_3 o - \rho_B c^\Theta o)s^2(2\pi/3 - \phi_o) + 2\rho_B s^\Theta o c^2(2\pi/3 - \phi_o) \right] \right\} \\
(4.44)
\end{align*}
\]

\[
\begin{align*}
\alpha_{R_3} &= -\frac{3Gm \times 1}{2(\rho_A + \rho_B)^5} \left\{ 3(\rho_3 o + \rho_A c^\Theta o)s^2(2\pi/3 - \phi_o) - 2\rho_B s^\Theta o c^2(2\pi/3 - \phi_o) \right\} \\
- \frac{3Gm \times 1}{2(\rho_A + \rho_B)^5} \left\{ 3(\rho_3 o - \rho_B c^\Theta o)s^2(2\pi/3 - \phi_o) + 2\rho_B s^\Theta o c^2(2\pi/3 - \phi_o) \right\} \\
(4.45)
\end{align*}
\]

\[
\begin{align*}
a_{\phi o} &= 0 \\
(4.46) \\
\alpha_{\phi R_3} &= 0 \\
(4.47) \\
a_{\phi o} &= 0 \\
(4.48)
\end{align*}
\]

\[
\begin{align*}
a_{\phi o} &= -3Gm (\frac{\rho_3 o + \rho_A c^\Theta o}{(\rho_A + \rho_B)^5}) \left\{ \frac{3}{2} \rho_B s^\Theta o s^2(\pi/3 - \phi_o) - (\rho_3 o + \rho_A c^\Theta o)c^2(\pi/3 - \phi_o) \right\} \\
- 3Gm (\frac{\rho_3 o - \rho_B c^\Theta o}{(\rho_A + \rho_B)^5}) \left\{ \frac{3}{2} \rho_B s^\Theta o s^2(2\pi/3 - \phi_o) - (\rho_3 o - \rho_B c^\Theta o)c^2(2\pi/3 - \phi_o) \right\} \\
(4.50)
\end{align*}
\]
\[
a_{\theta \phi} \big|_o = -\frac{3G S_2}{(\rho_A + \rho_B)^3} \left\{ m_A c^2 (\pi / 3 - \phi_0) + m_B c^2 (2\pi / 3 - \phi_0) \right\}
\]

(4.51)

\[
a_{\theta \theta} \big|_o = a_{\phi \phi} \big|_o = 1
\]

(4.52)

\[
a_{\theta R_3} \big|_o = a_{\phi R_3} \big|_o = a_{\theta \theta} \big|_o = a_{\theta R_3} \big|_o = a_{\phi \phi} \big|_o = 1
\]

(4.53)

\[
a_{\rho_3 \theta} \big|_o = a_{\rho_3 \phi} \big|_o = a_{\rho_3 \theta} \big|_o = a_{\rho_3 \rho_3} \big|_o = a_{\rho_3 \phi} \big|_o = 0
\]

(4.54)

\[
a_{\phi \theta} \big|_o = a_{\phi R_3} \big|_o = a_{\phi \theta} \big|_o = a_{\phi \rho_3} \big|_o = a_{\phi \phi} \big|_o = 0
\]

(4.55)

where \( S_1 \) and \( S_2 \) are given by (3.20) and (3.21) and where

\[
S_3 = (A_3 + B_3 + C_3) - 3B_3
\]

(4.56)

\( \rho_{30} \) and \( \theta_0 \) are the values of \( \rho_3 \) and \( \theta \) at the equilateral triangle position and \( \phi_0 \) measures the libration of the third body when placed at the equilateral triangle point.

\( \rho_{30} \) and \( \theta_0 \) can be calculated from the geometry of the figure and are given by

\[
\rho_{30} = \frac{\{(\rho_B - \rho_A)^2 + 3(\rho_B + \rho_A)^2\}^{1/3}}{2}
\]

(4.57)

\[
\theta_0 = \sqrt{-1} \frac{\rho_B - \rho_A}{\{(\rho_B - \rho_A)^2 + 3(\rho_B + \rho_A)^2\}^{1/3}}
\]

(4.58)

\( \phi_0 \) can be obtained by substituting (4.57) and (4.58) into (3.15) and setting \( \phi \) equal to zero, and is given by
\[
\phi_o = \frac{1}{2} t^{-\frac{1}{2}} \left( \frac{m_B s \left( \frac{\pi}{3} \right)}{m_A - m_B c \left( \frac{\pi}{3} \right)} \right) + \left( \frac{\pi}{3} \right) \tag{4.59}
\]

or

\[
\phi_o = \frac{1}{2} t^{-\frac{1}{2}} \left( \frac{m_B s \left( \frac{\pi}{3} \right)}{m_A - m_B c \left( \frac{\pi}{3} \right)} \right) - \left( \frac{\pi}{6} \right) \tag{4.60}
\]

Knowing the earth moon distance, and the mass of the earth and the moon, the values of \(\theta_o\), \(\rho_{30}\), and \(\phi_o\) can be calculated. These are given by

\[
\theta_o = 1.0578 \text{ radians} \tag{4.61}
\]

\[
\rho_{30} = 3.329 \times 10^5 \text{ km.} \tag{4.62}
\]

\[
\phi_o = 1.0578 \text{ and } (1.0578 - \frac{\pi}{2}) \text{ radians} \tag{4.63}
\]

It may be noted that the value of \(\phi_o\) obtained (Fig. 5) agrees exactly with the value \((60^036')\) given by Robinson (Ref. 8) for a dumb-bell shaped station.

The position \(\phi_o = (1.0578 - \frac{\pi}{2})\) will later be shown to be unstable and hence for the immediate analysis we use the stable position

\[
\phi_o = 1.0578 \text{ radians} \tag{4.64}
\]

At this stage, the differences between the true equilibrium position and the equilateral position can be found by using equation 4.6 after 4.61, 4.62, and 4.64 have been substituted in equations 4.34 through 4.55. These differences were calculated for a dumb-bell shaped station for which the differences can be expected to be a maximum (being zero for a symmetric station). These were found to be negligibly small and hence we can take the equilibrium position to be the equilateral point.

The equations of motion (4.1) can now be expanded (linearized) in a Taylor's series about an equilibrium point denoted by the subscript 'oo'
which, in general, is different from the equilateral point denoted by the subscript 'o'. The linearized equations of motion are given by

$$\{\dot{y}\} = \{a\}_{oo} \{y\}$$  \hspace{1cm} (4.65)

where

$$\{y\} = [\delta \Theta \delta R_3 \delta \phi \delta \Theta \delta R_3 \delta \phi]^T$$

and

$$[a] = \begin{bmatrix}
[\alpha_{\Theta} & \cdots & \cdots & \alpha_{\phi} \\
\cdots & \cdots & \cdots & \cdots \\
\alpha_{\phi} & \cdots & \cdots & \alpha_{\phi}
\end{bmatrix}$$  \hspace{1cm} (4.66)

The elements of $[a]$ are given by equations 4.13 through 4.33. Since we have shown that the equilibrium position is very close to the equilateral point, we can replace the 'oo' in 4.65 by 'o'. Hence the linearized equations are given by

$$\{\dot{y}\} = [a]_o \{y\}$$  \hspace{1cm} (4.67)

where the elements of $[a]_o$ are given by equations 4.36 through 4.55.

It is worth noting that the effects of the non-symmetric nature of the third body are given by those terms which contain the size and shape factors $S_2$ and $S_3$. If these are set equal to zero, we would be left with the equations of motion for a spherical body in a restricted three body problem.
5. NON-DIMENSIONALISATION OF THE EQUATIONS OF MOTION

The linearized equations of motion as given by equation 4.67 are in a dimensional form. In order to make the results more tractable, it will be advantageous to non-dimensionalise the equations of motion. The non-dimensionalising scheme is based on a "Newtonian formulation of Kepler’s third law”.

\[
\dot{\psi}^2 = \frac{G(m_A + m_B)}{(\rho_A + \rho_B)^3} \tag{5.1}
\]

The non-dimensional equations of motion are given by

\[
\{\ddot{y}\} = [TT]^{-1}[T]^{-1}[a][T]^{-1}\{y\} \tag{5.2}
\]

It should be noted that the dot above the \(y\) in equation 5.2 and in 5.5 (which follows) denotes the differentiation of \(y\) with respect to non-dimensional time.

\([TT]\) is a 2nx2n diagonal matrix given by

\[
[TT] = \begin{bmatrix}
\dot{\psi} & 0 \\
0 & \dot{\psi}
\end{bmatrix} \tag{5.3}
\]

and \([T]\) is another diagonal matrix given by

\[
[T] = \begin{bmatrix}
1/\dot{\psi} & 0 & 1/\dot{\psi} & 0 \\
0 & 1/(\rho_A + \rho_B)\dot{\psi} & 1 & 0 \\
1/\dot{\psi} & 0 & 1/(\rho_A + \rho_B) & 1
\end{bmatrix} \tag{5.4}
\]

Equation 5.2 can now be written as

\[
\{\ddot{y}\} = [a]_o\{y\} \tag{5.5}
\]

where the states \(\{y\}\) and consequently \([a]_o\) are non-dimensional.
6. STABILITY, RESONANCE, AND THE SOLUTION TO THE
EQUATIONS OF MOTION

In accordance with classical results which can be found in a good
textbook on space mechanics (example, Ref. 10), we can expect \( \rho_3 \) and \( \theta \)
to vary such that the center of mass describes an elliptic orbit about
the equilateral point. The semiminor axis of the resulting orbit will
point towards the barycenter of the system (Fig. 1). These results are
valid for a point mass. In this problem, we are including the possible
effect of the libration motion (of a non-symmetric station) on the motion
of the center of mass. Though this effect is very small, it is conceivable
that if the libration motion is periodic (which depends on the sign of the
shape factor \( S_2 \)), and furthermore has the same period as that of the
center of mass, resonance will occur.

Before investigating the possibility of the resonant condition, it
would be fruitful to find out the condition under which the libration
motion is periodic and hence not unstable. In general, \(-1 \leq S_2 \leq 1\) (6.1)
The libration motion can be examined for periodic motion by solving the
linearized libration equation. It will be observed that the case
\(-1 \leq S_2 < 0\) is equivalent to that of a spring mass system with the spring
having a negative stiffness. If \(0 < S_2 \leq 1\), the solution is periodic
and will be shown to be given by equation 6.5. Hence we can state the
first stability rule

\[ B_3 > A_3 \text{ for stability} \]

(6.2)
Now in order to look for resonant behavior, the solution to the
non-dimensional and linearized equations have to be found. The eigen-
values $[\alpha]$ are found (for any $(m_B/(m_A+m_B))<0.0385$) to be three complex
conjugate pairs with no real parts $(\text{Im } \lambda_1 = -\text{Im } \lambda_2 > 0, \text{Im } \lambda_3 = -\text{Im } \lambda_4 > \text{Im } \lambda_5 = -\text{Im } \lambda_6 > 0)$. However, from previous knowledge, we know that
for no libration motion (for a spherical station) the position of the
center of mass (i.e., $\delta \theta$ and $\delta \rho_3$) varies harmonically. Also if the station
remains at $L_4$, its libration motion (i.e., $\delta \phi$) is harmonic for $B_3 \neq A_3$.
However for our case, coupling between the motions exist due to the
$\alpha_{\theta}$, $\alpha_{\rho_3}$, $\alpha_{\theta \rho_3}$, and $\alpha_{\rho_3 \phi}$ terms. Hence each of the solutions (for $\delta \theta$
$\delta \rho_3$, and $\delta \phi$) can be written out in terms of a harmonic solution (with
two constants of integration) for the homogeneous part and a term (say
$Q_k$, $k = 1, 2, 3$) due to the coupling or forcing term. Since these coupling
terms are themselves harmonic, all the $Q_k$ will contain a $\left(\frac{\lambda_i^2 - \lambda_j^2}{\lambda_i - \lambda_j}\right)$ term
in the denominator where $i \neq j; i, j = 1, 3, 5$. If $\lambda_i = \lambda_j$ a resonant
condition occurs. Based on the above arguments the solutions can be
shown to be given by

$$
\delta \theta = \delta \theta^0 \cos \lambda_s t + \frac{\delta \dot{\theta}^0}{\lambda_s} \sin \lambda_s t + \frac{a_{\rho_3}}{\lambda_s - \lambda_\phi} \left(\delta \phi^0 \cos \lambda_\phi t + \frac{\delta \dot{\phi}^0}{\lambda_\phi} \sin \lambda_\phi t\right) \quad (6.3)
$$

$$
\delta \rho_3 = \delta \rho_3^0 \cos \lambda_s t + \frac{\delta \dot{\rho}_3^0}{\lambda_s} \sin \lambda_s t + \frac{a_{\theta \rho_3}}{\lambda_s - \lambda_\phi} \left(\delta \phi^0 \cos \lambda_\phi t + \frac{\delta \dot{\phi}^0}{\lambda_\phi} \sin \lambda_\phi t\right) \quad (6.4)
$$

$$
\delta \phi = \delta \phi^0 \cos \lambda_\phi t + \frac{\delta \dot{\phi}^0}{\lambda_\phi} \sin \lambda_\phi t + \frac{a_{\theta \phi}}{\lambda_\phi - \lambda_s} \left(\delta \theta^0 \cos \lambda_s t + \frac{\delta \dot{\theta}^0}{\lambda_s} \sin \lambda_s t\right) + \frac{a_{\rho_3 \phi}}{\lambda_\phi - \lambda_s} \left(\delta \rho_3^0 \cos \lambda_s t + \frac{\delta \dot{\rho}_3^0}{\lambda_s} \sin \lambda_s t\right) \quad (6.5)
$$
In the above expressions, the superscript 'o' stands for initial conditions which have been chosen so that the long period mode is eliminated. \( \lambda_s \) and \( \lambda_\phi \) are two of the eigenvalues of \([a]\). \( \lambda_s \) is the frequency of the short period mode and \( \lambda_\phi \) is the libration frequency which depends on the shape of the station and hence on the shape factor \( S_2 \).

For a particular value of \( S_2 \), the libration frequency can be equal to the frequency of the short period mode. Resonance occurs and the amplitudes of motion may become large. This resonant condition can be obtained by equating the frequency of the short period mode to the libration frequency.

\[
\lambda_s^2 = \lambda_\phi^2 = S_2 \lambda_D^2
\]  

(6.6)

\( \lambda_D \) is the libration frequency of a dumb-bell shaped station which is one of the eigenvalues of \([a]\) when \( A_3 \) is equal to zero.

Substituting the value of \( S_2 \) from equation 3.21 into 6.6, we obtain

\[
\frac{B_3}{A_3} = \frac{\lambda_\phi^2 + \lambda_s^2}{\lambda_\phi^2 - \lambda_s^2}
\]  

(6.7)

Taking the value of \( \mu \) for the earth moon system to be 0.01235, we obtain

\[
\frac{B_3}{A_3} = 1.9201 \text{ for resonance}
\]  

(6.8)

The stability results obtained can be conveniently plotted in an inertia domain by drawing stability boundaries (Fig. 6). The motion is stable in the space where \( B_3 > A_3 \) except along the resonance line. In
practice, any points close to the line will correspond to station inertias which will cause large amplitudes of motion. Hence the station inertias should be chosen such that they are away (to any desired degree) from the line, but in the stability region defined by equation 6.2.

It can be shown that for any value of \( S_2 \), \( a_{\theta\theta} \) and \( a_{\psi\delta} \) are many orders of magnitude smaller than \( a_{\varphi\varphi} \) and \( a_{\psi\rho_3} \). In fact \( a_{\theta\theta} \) are much smaller than any of the other terms in equations 3.3 through 3.5. As such, the coupling terms in equations 3.3 and 3.4 can be neglected as long as the station inertias are away from the resonant condition. This shall be done in the sequel. Hence we can conclude that if the libration motion is kept finite, the motion of the center of mass will also be finite. In other words, stability criteria are more restrictive for libration motion than for motion of the center of mass.
7. INITIAL CONDITIONS AND POINTING ERRORS

The selection of initial conditions can hardly be overemphasized, for their appropriate selection is crucial for the ensuing motion. The four constants of motion in equations 6.3 and 6.4 are not arbitrary. They have been chosen to completely eliminate the long period mode, and the relationship between them can be obtained by using equations 6.3 and 6.4 with equation 4.67. If the initial position for the center of mass of the station is chosen, the velocities that must be imparted to it can be shown to be given by the following relations

\[
\delta \dot{\theta}^0 = \frac{a_R \delta \theta^0 + (a_R \rho_3 + \lambda_s^2) \delta \rho_3^0}{a_R \rho_3} \quad (7.1)
\]

\[
\delta \dot{\rho}_3^0 = \frac{(a_R \rho_3 + \lambda_s^2) \delta \theta^0 + a_R \delta \rho_3 \delta \rho_3^0}{a_R \rho_3} \quad (7.2)
\]

The libration motion (given by equation 6.5) is a superposition of two periodic motions one having the period of the short period made and one with the period of libration. The initial libration position and velocity are selected to be zero, i.e. \( \delta \theta^0 = \delta \theta^o = 0 \). These correspond to an initial configuration such that the center of mass is on a semiminor axis (Fig 1) and one initial orientation is such that the station points towards the barycenter of the earth-moon system.

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We are now in a position to analyze the pointing errors of a
transcommunication station.

As soon as the station moves away from its initial orbit position,
pointing errors of two kinds will occur. One of them is due to the
effect of gravity gradients which will cause the station to rotate
about its center of mass. This error is given by the value of $\delta \phi(t)$.
Even if this error was not present, another kind of pointing error
would occur due to the finite size of the orbit. This error is given
by $[-\delta \theta(t)]$. Hence, at any position in the orbit, the total pointing
error is given by

$$\xi \delta \phi(t) = \delta \phi(t) - \delta \theta(t)$$

(7.3)

where $\xi$ stands for an error in the variable which follows it. From
equations 6.4 and 6.5 (remembering the remarks made at the end of
last chapter and that $\delta \phi^o = \delta \theta^o = 0$), the total pointing error
can be shown to be given by

$$\xi \delta \phi(t) = \frac{a \delta \theta}{\lambda^2 - \lambda^2_s} \left[ \delta \theta^o \cos \lambda_s t + \frac{\delta \theta^o}{\lambda_s} \sin \lambda_s t \right]$$

$$+ \frac{a \delta \rho_3}{\lambda^2 - \lambda^2_s} \left[ \delta \rho_3^o \cos \lambda_s t + \frac{\delta \rho_3^o}{\lambda_s} \sin \lambda_s t \right]$$

$$- (\delta \theta^o \cos \lambda_s t + \frac{\delta \theta^o}{\lambda_s} \sin \lambda_s t)$$

(7.4)

Knowing the nature and magnitude of the pointing error, a suitable
control system can be designed to minimize the error.
8. DUMB-BELL SHAPEED STATION

The analytical results obtained are now checked for their validity by studying the dynamics of a dumb-bell shaped station. This is done because some of the results obtained for a dumb-bell shaped station have been obtained by Robinson (Ref. 8). Furthermore, the gravity gradient effect is a maximum for a dumb-bell shaped station.

The following station parameters were used for the study.

\[ m_3 = 1.0 \times 10^3 \text{ Kg} \]  \hspace{1cm} (8.1)

\[ A_3 = 0 \]  \hspace{1cm} (8.2)

\[ B_3 = 625 \times 10^3 \text{ Kg. m}^2 \]  \hspace{1cm} (8.3)

\[ C_3 = 625 \times 10^3 \text{ Kg. m}^2 \]  \hspace{1cm} (8.4)

The following non-dimensional results are obtained

\[
[a] = \begin{bmatrix}
0 & -2.012 & 0 & 0.2744 \times 10^{-1} & -0.1555 \times 10^{-1} & 0.3274 \times 10^{-19} \\
1.988 & 0 & 0 & -0.1536 \times 10^{-1} & 2.973 & 0.9926 \times 10^{-16} \\
0 & 0 & 0 & 2.923 & 0.2347 \times 10^{-1} & -2.999 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 
\end{bmatrix}
\]  \hspace{1cm} (8.5)

It should be noted that since \( a_{\theta \phi} \) and \( a_{R3 \phi} \) are much smaller than
\( a_{\varpi A}, a_{\Theta P_3}, a_{R_3 \Theta}, a_{R_3 \rho_3} \), the libration motion does not affect the motion of the center of mass. Since \( a_{\varpi A} \) and \( a_{\Theta P_3} \) are of comparable magnitude to \( a_{\Phi \rho_3} \), the motion of the center of mass does effect the libration motion. Furthermore, since \( a_{\varpi A} \) and \( a_{R_3 \Theta} \) are much smaller than \( a_{\varpi A} \) and \( a_{\Phi \rho_3} \) (all of which are the coupling terms which can cause resonant behavior), stability criteria are more restrictive for libration motion than for motion of the center of mass.

The eigenvalues of \([a]\) are

\[
\lambda_{1,2} = \pm 1.7319i \quad \text{where } i = \sqrt{-1} \quad (8.6)
\]
\[
\lambda_{3,4} = \pm 0.95429i \quad (8.9)
\]
\[
\lambda_{5,6} = \pm 0.29877i \quad (8.10)
\]

Classical results (Ref. 10) show that \( \lambda_3 \) and \( \lambda_5 \) correspond to the frequency of short and long period modes. Hence \( \lambda_1 \) must be the frequency of libration of a dumb-bell shaped station. Hence we can write

\[
\lambda_{\Phi D} = \lambda_1 = 1.7319i \quad (8.11)
\]
\[
\lambda_5 = \lambda_3 = 0.95429i \quad (8.12)
\]
\[
\lambda_{\perp} = \lambda_5 = 0.29877i \quad (8.13)
\]

where \( \lambda_{\perp} \) is the frequency of the long period mode which is eliminated by appropriately selecting the initial conditions. The value of \( \lambda_{\Phi D} \) corresponds to a libration period of 15.8 days which is of the "order of 16 days" which was given by Robinson (Ref. 8). It should be noted that the libration frequency of any other station can be obtained from
the relation

\[ \lambda_0 = 1.7319 \sqrt{S} \]  

(8.14)

The initial conditions chosen are given

\[ \delta\theta^o = 0 \]  

(8.15)

\[ \delta p_3^o = 0.001 \]  

(8.16)

\[ \delta R^o = 0 \]  

(8.17)

\[ \delta R^o = 0 \]  

(8.18)

The calculated initial conditions (velocities that should be imparted to the center of mass) are given by

\[ \delta\dot{\theta}^o = 1.9535 \times 10^{-3} \]  

(8.19)

\[ \delta\dot{p}_3^o = -0.7727 \times 10^{-5} \]  

(8.20)

The pointing errors for various orbit positions are tabulated in Table I.
9. CONCLUSIONS

Based on this work, the following conclusions can be drawn:

EQUILIBRIUM POSITION
Equilibrium position is negligibly affected by the non-symmetric shape.
Stations of all shapes point to the barycenter of the system.

EXTENT OF COUPLING BETWEEN THE EQUATIONS OF MOTION
Motion of the center of mass is negligibly affected by libration motion
provided the station inertias are significantly different away from the
resonant condition. The libration motion is affected by center of mass
motion.

INITIAL CONDITIONS
Proper selection of initial conditions enables:

Center of Mass to move in a short period mode.
Libration motion of any station to have the same period as that of the
short period mode.

STABILITY
For stability, $B_3 > A_3$.
Stability criteria are more restrictive for the libration motion than
for the motion of the center of mass.

POINTING ERRORS
Pointing errors due to the center of mass not being at the equilateral
point act in opposition (and hence reduce) the pointing errors due to
the finite size of the orbit.

The initial conditions were chosen so that the above two pointing errors are of the same period - that of the short period mode.
TABLE I

POINTING ERRORS

<table>
<thead>
<tr>
<th>Location in orbit</th>
<th>Due to gravity gradient effects $= \delta \phi$ (Radians)</th>
<th>Due to finite size of orbit $= -\delta \phi$ (Radians)</th>
<th>Total $\delta \phi$ (Radians)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{4}$</td>
<td>$-2.8645 \times 10^{-3}$</td>
<td>$2.0471 \times 10^{-3}$</td>
<td>$-0.8174 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\frac{1}{2}$</td>
<td>$-1.1236 \times 10^{-5}$</td>
<td>$0.0$</td>
<td>$-1.1236 \times 10^{-5}$</td>
</tr>
<tr>
<td>$\frac{3}{4}$</td>
<td>$2.8645 \times 10^{-3}$</td>
<td>$-2.0471 \times 10^{-3}$</td>
<td>$0.8174 \times 10^{-3}$</td>
</tr>
<tr>
<td>$1$</td>
<td>$1.1236 \times 10^{-5}$</td>
<td>$0.0$</td>
<td>$1.1236 \times 10^{-5}$</td>
</tr>
</tbody>
</table>
Fig. 1. Equilibrium Points of the Earth-Moon System.
FIG. 2. TRANSCOMMUNICATION STATION
FIG. 3. ASTRONOMICAL STATION

INITIAL POSITION AND ORIENTATION

POSITION AND ORIENTATION AFTER QUARTER ORBIT

(a)

(b)

- EARTH
- MOON
- BARYCENTER OF SYSTEM
Fig. 5  EQUILIBRIUM ATTITUDE FOR AN ARBITRARILY SHAPED STATION
$I = \tan^{-1} \left( \frac{\phi_D + \phi_S + 2}{(1 + \phi_D - \lambda_S)} \right)$

$\theta = 62.9^\circ$

Fig. 6: Stability Boundaries

Internal Resonance Line

Large Amplitudes of Motion

Stable

Unstable
REFERENCES


VITA

Ashok Kumar Saxena was born on October 10, 1951 in Ajmere, India. Having won an "All India Merit Scholarship" to study in a boarding school, he attended The Lawrence School, Sanawar from 1958 to 1965. He graduated by obtaining the second position in the school.

In 1966 he joined Jadavpur University, Calcutta to study towards a Bachelor's degree in Mechanical Engineering. During his undergraduate work, he received tuition waivers from time to time. At the Intermediate Engineering Examination he stood first in all the departments of Engineering and was consequently awarded a Silver Medal for his performance. He is a recipient of a "J. N. Tata Memorial Scholarship for Higher Studies abroad".

After leaving Jadavpur University in 1971, he joined Virginia Polytechnical Institute and State University where he has been doing graduate work in the department of Aerospace Engineering. During this period he has been a Teaching Assistant, Research Assistant, and as an Instructor has taught a course on Astromechanics. He is a member of the honor society of Phi Kappa Phi.

Ashok Kumar Saxena
AN ARBITRARILY SHAPED SATELLITE IN THE
RESTRICTED PROBLEM OF THREE BODIES

Ashok Kumar Saxena

Abstract

It is proposed to place a space station in a short period orbit about the L₄ triangular point in the earth moon system for the purpose of communication and possibly astronomical observations. Within limits of a linear analysis, an exhaustive and general approach to the dynamical problem is presented. The equations of motion for a finite sized body of arbitrary shape and internal mass distribution are derived. The phenomenon of resonance and stability are discussed. Stability boundaries for such a space station are established. Initial conditions and pointing errors are analyzed.