AN ARBITRARILY SHAPED SATELLITE

IN THE

RESTRICTED PROBLEM OF THREE BODIES

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Thesis submitted to the Graduate Faculty of the

Virginia Polytechnic Institute and State University

in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in

Aerospace Engineering

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July 1974

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ACKNOWLEDGMENTS

The author would like to take this opportunity to express his deep gratitude to Dr. F. H. Lutze Jr. for his advice during the course of this work. He thanks Dr. E. M. Cliff and Dr. R. P. McNitt for acting on his advisory committee.

Fully aware of their constant sacrifices and encouragement, the author thanks his father and mother who have been a continuing source of inspiration and moral support.

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LIST OF SYMBOLS

a_{ij}	Elements of [a]
A_3	Principal planar moment of inertia of the third body
B_3	Principal planar moment of inertia of the third body
c	Cosine of the trignometric argument following it
C_3	Polar moment of inertia of the third body
G	Universal constant of gravitation
i	Integer
Im	Imaginary part
j	Integer
k	Integer
[1]	Transformation matrix
L	Lagrangian function
m	Mass of a body
O	Origin of a coordinate system
P	Variable introduced into the analysis (page 13)
Q	Variable introduced into the analysis (page 13)
Q_k	Forcing term
r_{A3}	Distance of center of mass of third body from body A
r_{B3}	Distance of center of mass of third body from body B
R_3	Radial velocity fo the center of mass of third body
s	Sine of the trignometric argument following it
S_1	Size and shape factor of the third body
S_2	Shape factor of the third body

S_3	Constant introduced into the analysis (page 27)
t	Tangent of the trignometric argument following it
t	Time
T	Kinetic energy of the system
[T]	Diagonal non-dimensionalizing matrix
[TT]	Diagonal non-dimensionalizing matrix
V	Potential energy of the system
x_i	Coordinate axis, also a state
X	Inertial coordinate axis
$X(x_i)$	Function of the states x_i
y_i	Coordinate axis, also perturbation in x_i
Y	Inertial coordinate axis
z_i	Coordinate axis
Z	Inertial coordinate axis
α, β	Angles subtended by the center of mass of the third body with the line joining the two primaries
δ	Small variation
θ	Angular position of the center of mass of the third body
Θ	Angular velocity of the center of mass of the third body
ρ_3	Radial position of the center of mass of the third body
ρ_A	Distance of body A from the barycenter of the system
ρ_B	Distance of body B from the barycenter of the system
ϕ	Libration of the third body
ϕ_1	Variable introduced into the analysis (page 13)
ϕ_2	Variable introduced into the analysis (page 13)
ϵ $\delta\phi$	Net pointing error

$\dot{\Phi}$	Libration velocity of the third body
ψ	Constant angular velocity of the earth moon system about its barycenter
λ	Eigenvalue of [a]

SUBSCRIPTS

A	Body A
B	Body B
D	A dumb-bell shaped station
ℓ	Long period mode
no	Non-equilibrium position
o	Equilateral triangle position
oo	Equilibrium position
s	Short period mode
3	Third body

SUBERSCRIPTS

o	Initial condition
-1	Arc, e. g. $t^{-1}\theta$ = arctan θ

1. INTRODUCTION

Just as Galileo's discovery of many "material centers of motion" was followed by the discovery of other "non-material center of motion", it is hoped that the placing of satellites around "material centers of motion" will be followed by their placement about other "non-material centers of motion". The closest such centers of motion are the five equilibrim points of the earth-moon system (Fig. 1). In particular, the placing of a station in orbit around a triangular libration point appears very attractive from numerous viewpoints.

If a point mass is placed in the close vicinity of the equilateral point L_4, its motion is similar to that of a double pendulum in that it is a superposition of two modes of motion- the long and short period modes. In general, the resulting orbit will describe Lassajau's figures; their shape will depend upon the initial conditions of the orbit. Initial conditions can be found such that only the short period mode can be excited. The resulting orbit will be elliptic and its minor axis will point towards the bary-center of the earth-moon system.

An interesting aspect of this libration point stems from the fact that the short period mode (28.62 days) has approximately the same period as that of the moon (27.32) days. Pederson (Ref. 1), with his third order theory showed that the period of any finite sized orbit (when compared to the period of an infinitesimally small orbit) becomes smaller for a short period mode while that of the long period mode becomes bigger. Hence, the period of the short period mode approaches that of the moon and in fact, with a little control, could be made the same. This remarkable property of the earth moon system can be exploited to place the following stations in orbit about the libration point:

(i) Transcommunication Station: Such a station would make (-1) rotations per orbit. An axis would always point to the barycenter of the earth moon system (Fig. 2) and an antenna on it will be earth-pointing. This station could be used as a transcommunication station for interplanetary flights, especially for some communication eclipses.

(ii) Astronomical Station: Such a station would make (+2) rotations per orbit. A telescope on such a station will be "star-pointing" for an observer at the barycenter of the earth moon system (Fig. 3).

It appears that the study of libration dynamics of a station near the equilateral point is not only intriguing, but also justified.

DISCUSSION OF THE VARIOUS DISTURBING AND STABILIZING FORCES

A study of the stability and motion of earth satellites includes the effects of one or more of the following: Gravity gradient effects, aerodynamic effects, magnetic effects, electrostatic effects, electromagnetic interactions, effect of the solar radiation pressure, and the possible impact with meteorites. The nature and size of the satellite's orbit determines which of the above effects play a dominant role and hence, have to be taken into consideration.

However, for a station placed in orbit around an equilateral point of the earth moon system, a complete study would include: Gravity gradient effects of the earth and the moon, perturbations due to the sun's gravitational field, effects of the solar radiation pressure, and the possible impact with small objects.

The solar radiation pressure will cause disturbing torques on the station. This problem could conceivably be circumvented by making the external surface of the station spherical whereas the internal mass distribution could be arranged so that the station has the desired moment of inertias.

A study of the solar perturbation effect and the impact with small objects is not covered in this work. Hence, the only effects considered are the gravitational effects of the earth and the moon on the motion of a non-symmetric station.

STATEMENT OF THE PROBLEM

The motion of an arbitrarily shaped body immersed in a gravitational field of two bodies is studied. In particular, the system is assumed to satisfy the conditions of a restricted three body problem (These conditions are discussed in the next section). Furthermore, the third body is in orbit about a stationary equilibrium point in a system of two bodies (primaries).

Since the third body is of finite size, we must study the motion of its center of mass and motion about its center of mass. To be more elaborate, the nature and extent of coupling between the motion of the center of mass and about the center of mass is investigated. In other words, the magnitude of the gravity gradient torques which cause the attitude motion is compared with the magnitude of the forces which cause the center of mass to describe its elliptic orbit.

Stability criteria for the third body are established. The pointing accuracy for such a body is studied.

2. HISTORICAL BACKGROUND

No thesis can be considered complete without a historical sketch and as such the following discussion ensues:

The many body problem was first precisely formulated by Newton. When the bodies involved are point masses, it can be stated as follows: Given at any instant of time, the position and velocities of three or more particles moving under their mutual gravitational forces, the masses also being known, calculate their positions and velocities at any other time. The simplest of the many body problem is the three body problem and this above has been a source of inspiration and frustration to many brilliant astronomers and mathematicians.

The only known closed form solutions of the three body problem are two particular solutions which were found by Lagrange in 1772 but were announced by him in his "OEuvres" (Ref. 2) at a later date. In these solutions, the three bodies move such that the geometric form of the three body configuration does not change although the scale can change and the figure can rotate. The two cases found by Lagrange are: (i) the three bodies form the verticies of an equilateral triangle, (ii) the three bodies are collinear.

Three bodies (particles) of arbitrary mass could exist in such solutions if (i) the resultant force on each mass passed through the barycenter, (ii) the resultant force was directly proportional to the distance of each mass from the barycenter, (iii) the initial velocity vectors were proportional in magnitude to the respective distances of the masses from the barycenter and made equal angles with the radius vectors to the masses from the barycenter. If there is no change in scale, the solutions are called stationary ones and the relative distances do not alter; the system also rotates in a plane about the barycenter with constant angular velocity. Given the positions of two bodies, the five equilibrium positions which can be taken up by the third body are called stationary, equilibrium, Laplacian, Eulerian, Lagrangian, libration points; or centers of libration. It is worth mentioning that although the two particular solutions are attributed to Lagrange, the first solution of a three body problem, i.e. that of three bodies which are moving on a fixed straight line and maintain a constant ratio of their distances was published by Euler in 1765 (Ref. 3). Euler's solution, though a particular solution, is really a special case of Lagrange's collinear solution.

Since no solutions, other than the two mentioned above were obtained, people turned their attention to a particular three body problem, now well known as the restricted three body problem. The birth date of the restricted problem is 1772 when Euler published his second lunar theory in his "Theoria Motunm Lunae". The word "restricted" appears to have originated from Poincaré in his "Les Methodes Nouvelles de la Mecanique Celeste" when he calls the restricted problem "a particular case of the problem of three bodies" and later introduces the word "restreint".

Some of the important names associated with the restricted problem of three bodies are Lagrange (1772), Jacobi (1836), Hill (1878), Poincare (1899), Levi-Civita (1905), and Birkhoff (1915).

The restricted problem of three bodies can be defined as follows: Two bodies revolve around their center of mass in circular orbits under the influence of their mutual gravitational attraction and a third body (attracted by the previous two, but not influencing their motion) moves in the plane defined by the two revolving bodies. The restricted problem of three bodies is to describe the motion of this third body. The two revolving bodies are called the primaries.

The equilateral libration points, now commonly identified by L_4 and L_5, are of considerable interest as these points are stable if the ratio of the mass of the lighter primary to the sum of the masses of the primaries is less than 0.0385 and this is true for many situations occuring in the solar system. In particular, the value of this ratio in the earth moon case is approximately 0.01 and hence, motion about L_4 and L_5 is stable. Preferred references for the linearized treatment of motion around the triangular libration points are by Plummer (Ref. 4) and by Martin (Ref. 5). Nonlinear stability analyses of orbits of finite size around the triangular libration points have been performed by Deprit (Ref. 6) and by Pedersen (Ref. 1).

Unfortunately, literature on the libration dynamics of a station near the equilateral point is sparse and restrictive. Kane and Marsh (Ref. 7) studied the libration dynamics of a spinning symmetric satellite while Robinson (Ref. 8) investigated the libration period of a dumb-bell shaped satellite. This work presents the study of a communication station of arbitrary shape and internal mass distribution when placed in orbit about the equilateral point.

3. EQUATIONS OF MOTION

Two bodies A and B (typically the earth and the moon) are treated as point masses and revolve around their barycenter 0 with a constant angular velocity $\dot{\psi}$(Fig. 4). The distances ρ_A and ρ_B to the two bodies remain constant and are measured from the origin of an "inertial" co-ordinate system XYZ whose origin is at the barycenter and whose X and Y axes lie in the plane of motion of the two bodies.

The position of the center of mass of the third body (typically a space station) is defined by its distance ρ_3 from the point o and by the angle that the radius vector $\bar{\rho}_3$ makes with the line joining the point masses A and B. The vectors $\bar{r}_{A3}$ and $\bar{r}_{B3}$ which locate the center of mass of the third body from the bodies A and B respectively, make angles α and β with the line joining the bodies A and B. The two bodies A and B and the third body form a restricted three body problem.

The center of mass of the third body is the origin o (Fig. 3) of three systems of axes. The axes x_o, y_o, z_o form the principal axes of the non-spherical body. A_3, B_3, and C_3 are the moments of inertia of the third body about the principal axes x_o, y_o, z_o. The libration of this body in the xy plane is defined by the angle which the x_o axis makes with line passing through its center of mass and parallel to the line joining A and B. The other two axes systems are given by x_1, y_1, z_1 which is a system of axes with origin at o such that the direction of the axis x_1 is taken along the line joining A and o and by x_2, y_2, z_2 which is a system of axes with origin at o such that the direction of the x_2 axis is taken along the line joining B and o.

It should be noted that the co-ordinate system used is a polar co-ordinate system. This is in contrast to the conventionally used cartesian system. The polar system used is later found advantageous in analyzing the pointing errors. As a historical note it may be added that a cartesian system with the origin at the barycenter, and with the heavier primary placed on the left side of the origin, belongs to Moulton's school of thought.

The equations of motion are derived using a Lagrangian formulation. The potential energy terms in the Lagrangian L use a spherical model for the primaries. The potential energy for the third (non-symmetric) body is obtained by using MacCullagh's formula (Ref. 9) which can be obtained by expanding the potential function in a Legendre polynomial series and retaining the first three terms.

ρ_3, θ, and ϕ are the three generalized co-ordinates used. ρ_3 and θ describe the position of the center of mass of the station whereas ϕ describes its attitude.

The Lagrangian is given by

$$L = T - V$$

$$\begin{aligned} L = {} & \tfrac{1}{2} m_A \rho_A^2 \dot{\psi}^2 + \tfrac{1}{2} m_B \rho_B^2 \dot{\psi}^2 + \tfrac{1}{2} m_3 \rho_3^2 (\dot{\psi} + \dot{\theta})^2 + \tfrac{1}{2} m_3 \dot{\rho}_3^2 \\ & + \tfrac{1}{2} C_3 (\dot{\psi} + \dot{\phi})^2 + \frac{G m_A m_B}{\rho_A + \rho_B} + \frac{G m_A m_3}{r_{A3}} + \frac{G m_B m_3}{r_{B3}} \\ & + \frac{G m_A}{2 r_{A3}^3} (A_3 + B_3 + C_3 - 3 I_{x_1 x_1}) \\ & + \frac{G m_B}{2 r_{B3}^3} (A_3 + B_3 + C_3 - 3 I_{x_2 x_2}) \end{aligned} \tag{3-1}$$

where $I_{x_1x_1}$ and $I_{x_2x_2}$ are the moment of inertias of the third body about the axes x_1 and x_2 respectively.

Using the relations

$$r_{A3}^2 = \rho_A^2 + \rho_3^2 + 2\rho_A\rho_3 \quad c\theta \tag{3.2}$$

$$r_{B3}^2 = \rho_B^2 + \rho_3^2 - 2\rho_B\rho_3 \quad c\theta \tag{3.3}$$

equation 3-1 can be written in the form

$$\begin{aligned} L = {} & \tfrac{1}{2} m_A\rho_A^2\dot{\psi}^2 + \tfrac{1}{2} m_B\rho_B^2\dot{\psi}^2 + \tfrac{1}{2} m_3\rho_3^2 \ (\dot{\psi} + \dot{\theta})^2 + \tfrac{1}{2} m_3\dot{\rho}_3^2 \\ & + \tfrac{1}{2} C_3(\dot{\psi} + \dot{\phi})^2 + \frac{Gm_Am_B}{\rho_A+\rho_B} + \frac{Gm_Am_3}{r_{A3}} + \frac{Gm_Bm_3}{r_{B3}} \\ & + \frac{Gm_A}{2(\rho_A^2+\rho_3^2+2\rho_A\rho_3 c\,\theta)^{3/2}} (A_3 + B_3 + C_3 - 3I_{x_1x_1}) \\ & + \frac{Gm_B}{2(\rho_B^2+\rho_3^2-2\rho_B\rho_3 c\,\theta)^{3/2}} (A_3 + B_3 + C_3 - 3I_{x_2x_2}) \end{aligned} \tag{3.4}$$

The terms $I_{x_1x_1}$ and $I_{x_2x_2}$ in the Lagrangian are functions of ρ_3, θ, ϕ and hence should be written explicitly in terms of these quantities. To this end, the inertia matrices in the x_1, y_1, z_1 and x_2, y_2, z_2 system of axes are written in terms of the principal axes system.

If $[I]$ is the principal inertia matrix, $[I_i]$ the inertia matrix in a system which has been given a planar rotation from the principal axis system can be obtained from the relation

$$[I_i] = [\ell_i][I][\ell_i]^T \qquad i = 1, 2 \tag{3.5}$$

where the transformation matrices $[\ell_i]$ are given by

$$[\ell_1] = \begin{bmatrix} c(\alpha-\phi) & s(\alpha-\phi) & 0 \\ -s(\alpha-\phi) & c(\alpha-\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.6}$$

$$[\ell_2] = \begin{bmatrix} c(\pi-\beta-\phi) & s(\pi-\beta-\phi) & 0 \\ -s(\pi-\beta-\phi) & c(\pi-\beta-\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.7}$$

Also, from the geometry of the problem

$$t\,\alpha = \frac{\rho_3 s\theta}{\rho_A+\rho_3 c\theta} \tag{3.8}$$

$$t\,\beta = \frac{\rho_3 s\theta}{\rho_B-\rho_3 c\theta} \tag{3.9}$$

Substituting equations 3.8 and 3.9 in equations 3.6 and 3.7 respectively and the result in 3.5 yields equations 3.10 and 3.11

$$[I_1] = \left[\begin{array}{c:c:c} \begin{array}{l} A_3 c^2\{t^{-1}(\frac{\rho_3 s\theta}{\rho_A+\rho_3 c\theta}) - \phi\} \\ + B_3 s^2\{t^{-1}(\frac{\rho_3 s\theta}{\rho_A+\rho_3 c\theta}) - \phi\} \end{array} & \begin{array}{l} (B_3-A_3) s\{t^{-1}(\frac{\rho_3 s\theta}{\rho_A+\rho_3 c\theta}) - \phi\} \\ x\ c\{t^{-1}(\frac{\rho_3 s\theta}{\rho_A+\rho_3 c\theta}) - \phi\} \end{array} & 0 \\ \hdashline \begin{array}{l} (B_3-A_3) s\ \{t^{-1}(\frac{\rho_3 s\theta}{\rho_A+\rho_3 c\theta}) - \phi\} \\ x\ c\{t^{-1}(\frac{\rho_3 s\theta}{\rho_A+\rho_3 c\theta}) - \phi\} \end{array} & \begin{array}{l} A_3 s^2\{t^{-1}(\frac{\rho_3 s\theta}{\rho_A+\rho_3 c\theta}) - \phi\} \\ + B_3 c^2\{t^{-1}(\frac{\rho_3 s\theta}{\rho_A+\rho_3 c\theta}) - \phi\} \end{array} & 0 \\ \hdashline 0 & 0 & C_3 \end{array}\right]$$

(3.10)

$$[I_2] = \left[\begin{array}{c|c|c} A_3c^2\{\pi - t^{-1}(\frac{\rho_3 s\theta}{\rho_B - \rho_3 c\theta}) - \phi\} & (B_3-A_3)s\{\pi - t^{-1}(\frac{\rho_3 s\theta}{\rho_B - \rho_3 c\theta}) - \phi\} & \\ + B_3 s^2\{\pi - t^{-1}(\frac{\rho_3 s\theta}{\rho_B - \rho_3 c\theta}) - \phi\} & \times c\{\pi - t^{-1}(\frac{\rho_3 s\theta}{\rho_B - \rho_3 c\theta}) - \phi\} & 0 \\ \hline (B_3-A_3)s\{\pi - t^{-1}(\frac{\rho_3 s\theta}{\rho_B - \rho_3 c\theta}) - \phi\} & A_3 s^2\{\pi - t^{-1}(\frac{\rho_3 s\theta}{\rho_B - \rho_3 c\theta}) - \phi\} & \\ \times c\{\pi - t^{-1}(\frac{\rho_3 s\theta}{\rho_B - \rho_3 c\theta}) - \phi\} & + B_3 c^2\{\pi - t^{-1}(\frac{\rho_3 s\theta}{\rho_B - \rho_3 c\theta}) - \phi\} & 0 \\ \hline 0 & 0 & C_3 \end{array}\right] \tag{3.11}$$

Using equations 3.4, 3.10, and 3.11, the Lagrangian can now be **written** as

$$\begin{aligned} L = {} & \tfrac{1}{2} m_A \rho_A^2 \dot{\psi}^2 + \tfrac{1}{2} m_B \rho_B^2 \dot{\psi}^2 + \tfrac{1}{2} m_3 \rho_3^2 (\dot{\psi} + \dot{\theta})^2 + \tfrac{1}{2} m_3 \dot{\rho}_3^2 \\ & + \tfrac{1}{2} C_3 (\dot{\psi} + \dot{\phi})^2 + \frac{Gm_A m_B}{\rho_A + \rho_B} + \frac{Gm_A m_3}{(\rho_A^2 + \rho_3^2 + 2\rho_B\rho_3 c\theta)^{\frac{1}{2}}} \\ & + \frac{Gm_B m_3}{(\rho_B^2 + \rho_3^2 - 2\rho_B\rho_3 c\theta)^{\frac{1}{2}}} \\ & + \frac{Gm_A}{2(\rho_A^2 + \rho_3^2 + 2\rho_A\rho_3 c\theta)^{3/2}} \left\{ A_3 + B_3 + C_3 - 3B_3 + 3(B_3 - A_3)c^2\{t^{-1}(\frac{\rho_3 s\theta}{\rho_A + \rho_3 c\theta}) - \phi\} \right\} \\ & + \frac{Gm_B}{2(\rho_B^2 + \rho_3^2 - 2\rho_B\rho_3 c\theta)^{3/2}} \left\{ A_3 + B_3 + C_3 - 3B_3 + 3(B_3 - A_3)c^2\{\pi - t^{-1}(\frac{\rho_3 s\theta}{\rho_B - \rho_3 c\theta}) - \phi\} \right\} \end{aligned} \tag{3.12}$$

Knowing the Lagrangian, the equations of motion can now be deduced and can be shown to be given by the relations

$$\ddot{\theta} + \frac{2\dot{\rho}_3}{\rho_3}(\dot{\psi}+\dot{\theta}) - \frac{Gm_A}{2\rho_3^2 r_{A3}^3}\left\{\rho_A\rho_3 s\theta(2 + \frac{3P}{m_3 r_{A3}^2}) - 3S_1 \frac{\rho_A\rho_3 c\theta + \rho_3^2}{r_{A3}^2} s2\phi_1\right\}$$

$$- \frac{Gm_B}{2\rho_3^2 r_{B3}^3}\left\{-\rho_B\rho_3 s\theta(2 + \frac{3Q}{m_3 r_{B3}^2}) + 3S_1 \frac{\rho_B\rho_3 c\theta - \rho_3^2}{r_{B3}^2} s2\phi_2\right\} = 0$$

(3.13)

$$\ddot{\rho}_3 - \rho_3(\dot{\psi} + \dot{\theta})^2 + \frac{Gm_A}{2r_{A3}^3}\left\{(\rho_3+\rho_A c\theta)(2 + \frac{3P}{m_3 r_{A3}^2}) + 3S_1 \frac{\rho_A s\theta}{r_{A3}^2} s2\phi_1\right\}$$

$$+ \frac{Gm_B}{2r_{B3}^3}\left\{(\rho_3 - \rho_B c\theta)(2 + \frac{3Q}{m_3 r_{B3}^2}) - 3S_1 \frac{\rho_B s\theta}{r_{B3}^2} s2\phi_2\right\} = 0$$

(3.14)

$$\ddot{\phi} - 3GS_2\left\{\frac{m_A}{2r_{A3}^3} s2\phi_1 + \frac{m_B}{2r_{B3}^3} s2\phi_2\right\} = 0$$

(3.15)

where r_{A3} and r_{B3} are defined by equations (3.2) and (3.3) and

$$\phi_1 = t^{-1}\left(\frac{\rho_3 s\theta}{\rho_A+\rho_3 c\theta}\right) - \phi \qquad (3.16)$$

$$\phi_2 = \pi - t^{-1}\left(\frac{\rho_3 s\theta}{\rho_B - \rho_3 c\theta}\right) - \phi \qquad (3.17)$$

$$P = A_3+B_3+C_3-3B_3+3(B_3-A_3)c^2\ \phi_1 \qquad (3.18)$$

$$Q = A_3+B_3+C_3-3B_3+3(B_3-A_3)c^2\ \phi_2 \qquad (3.19)$$

$$S_1 = (B_3-A_3)/m_3 \qquad (3.20)$$

$$S_2 = (B_3-A_3)/C_3 \qquad (3.21)$$

From equations 3.13 through 3.15, 3.20 and 3.21, it will be observed that the motion depends upon two parameters S_1 and S_2, both of which are independant of the mass of the third body. S_1 and S_2 may respectively be called size and shape parameters. S_1 depends upon the dimensions and geometric configuration of the third body whereas S_2 depends solely on its geometric configuration. It should also be noted that when $S_1=S_2=0$; i.e. for a symmetric body, equations 3.13 and 3.14 reduce to the polar equations of motion for the classical restricted three body problem.

A solution to the highly non-linear equations of motion (equations 3.13 through 3.15), though illuminating, does not seem plausible. Hence the equations of motion must first be linearized about an equilibrium position before any meaningful results can be obtained.

4. LINEARIZED EQUATIONS OF MOTION

Before proceeding into the linearized analysis, it should be noted that when a non-symmetric body of a finite size (finite values of principal moment of inertias), but of negligible mass, is placed in a system of two bodies rotating about each other, the equilibrium positions will not be exactly at the equilibrium points of a corresponding symmetric body. These differences for the equilateral point L_4 are now calculated. The theory developed for this investigation is valid for all equilibrium points, and can be outlined as follows:

Let a set of 2N first order differential equations describe the motion of an N-degree of freedom dynamical system.

$$\dot{x}_i = X_i(x_1, x_2, \ldots, x_m) \qquad i = 1,2,\ldots,m \tag{4.1}$$

where m = 2N and x_i are the generalized coordinates (which for one case include actual coordinates and velocities). At an equilibrium point, the generalized velocities should be zero.

$$\dot{x}_i \big|_{x_i = x_{i_{oo}}} = X_i(x_1, x_2, \ldots, x_m)\big|_{x_i = x_{i_{oo}}} = 0 \quad i = 1,2,\ldots,m \tag{4.2}$$

where the subscript 'oo' refers to an equilibrium point. The left hand side of the second half of equation (4.2) can be replaced by a Taylor Series expansion about a point close to the equilibrium point and denoted by the subscript 'no'. Retaining first order terms in the Taylor Series expansion

$$X_i(x_1,x_2,\ldots,x_m)\Big|_{x_i=x_{i_{no}}} + \sum_{j=1}^{m} a_{ij} \Big|_{x_i=x_{i_{no}}} y_j = 0 \quad i=1,2,\ldots,m \tag{4.3}$$

$$\text{where } a_{ij} = \frac{\partial X(x_1,x_2,\ldots,x_m)}{\partial x_j}\Bigg|_{x_i=x_{i_{no}}} \quad i,j = 1,2,\ldots,m \tag{4.4}$$

and where the y_j are the perturbations in x_j.

Applying equations (4.3) and (4.4) to this problem in which the non-equilibrium point (which is close to the equilibrium point) is the equilateral triangle point L_4 denoted by the subscript 'o'.

Equation (4.3) can now be written in the form

$$\sum_{j=1}^{m} a_{ij} \Big|_{x_i=x_{i_o}} y_j = -X_i(x_1,x_2,\ldots x_m)\Big|_{x_i=x_{i_o}} \quad i = 1,2,\ldots,m \tag{4.5}$$

where the a_{ij} are given by equation (4.4). The values of y_j which are a linear approximation to the difference between the equilibrium positions of a non-symmetric and symmetric body can now be obtained from the matrix equation

$$\{y\} = -[a]^{-1}_{x=x_o} \{X\}_{x=x_o} \tag{4.6}$$

The problem now is to calculate the coefficients a_{ij}. However, before doing so, it should be noted that the theory developed is for a set of first order differential equations of motion while the derived equations of motion (equations (3.13), (3.14), and (3.15)) are a set of second order differential equations. By a process of ablimination, these three equations are now replaced by a set of six first order differential equations which are given by

$$\dot{\Theta} = -\frac{2R_3}{\rho_3}(\dot{\psi}+\Theta) + \frac{Gm_A}{2\rho_3^2 r_{A3}^3}\left\{\rho_A\rho_3 s\theta\left(2+\frac{3P}{m_3 r_{A3}^2}\right) - 3S_1\frac{\rho_A\rho_3 c\theta + \rho_3^2}{r_{A3}^2}\, s2\phi_1\right\}$$

$$+\frac{Gm_B}{2\rho_3^2 r_{B3}^3}\left\{-\rho_B\rho_3 s\theta\left(2+\frac{3Q}{m_3 r_{B3}^2}\right) + 3\;{}_1\frac{\rho_A\rho_3 c\theta - \rho_3^2}{r_{B_3}^2}\right\} \tag{4.7}$$

$$\dot{R}_3 = \rho_3(\dot{\psi}+\Theta)^2 - \frac{Gm_A}{2r_{A3}^3}\left\{(\rho_3+\rho_A c\theta)\left(2+\frac{3P}{m_3 r_{A3}^2}\right) + 3S_1\frac{\rho_A s\theta}{r_{A3}^2}\, s2\phi_1\right\}$$

$$-\frac{Gm_B}{2r_{B3}^3}\left\{(\rho_3-\rho_B c\theta)\left(2+\frac{3Q}{m_3 r_{B3}^2}\right) - 3S_1\frac{\rho_B s\theta}{r_{B3}^2}\, s2\phi_2\right\} \tag{4.8}$$

$$\dot{\Phi} = 3GS_2\left(\frac{m_A}{2r_{A3}^3}\, s2\phi_1 + \frac{m_B}{2r_{B3}^3}\, s2\phi_2\right) \tag{4.9}$$

$$\dot{\theta} = \Theta \tag{4.10}$$

$$\dot{\rho}_3 = R_3 \tag{4.11}$$

$$\dot{\phi} = \Phi \tag{4.12}$$

From equations (4.7) through (4.12), the thirty six elements of [a] can be found and after due simplification can be shown to be given by

$$a_{\Theta\Theta} = -\frac{2R_3}{\rho_3} \tag{4.13}$$

$$a_{\Theta R_3} = -2(\dot{\psi}+\Theta)/\rho_3 \tag{4.14}$$

$$a_{\Theta\Phi} = 0 \tag{4.15}$$

$$a_{\Theta\theta} = Gm_A\left\{\frac{r_{A3}^2\rho_A\rho_3 c\theta + 3(\rho_A\rho_3 s\theta)^2}{2\rho_3^2 r_{A3}^5}\right\}\left(2+\frac{3P}{m_3 r_{A3}^2}\right)$$

$$-3Gm_A\frac{\rho_A s\theta}{2r_{A3}^7}\left\{3S_1(\rho_3+\rho_A c\theta)s2\phi_1 - \frac{2P}{m_3}\rho_A s\theta\right\}$$

$$+ \; 3Gm_A S_1 \frac{\rho_A s\theta \, s2\phi_1}{2\rho_3 r_{A3}^7} \left\{ r_{A3}^2 - 2(\rho_3 + \rho_A c\theta) \right\}$$

$$- \; 3Gm_A S_1 \frac{\rho_3 + \rho_A c\theta}{2 r_{A3}^7} \left\{ 2(\rho_3 + \rho_A c\theta) c2\phi_1 + 3\rho_A s\theta \, s2\phi_1 \right\}$$

$$- \; Gm_B \left\{ \frac{r_{B3}^2 \rho_B \rho_3 c\theta - 3(\rho_B \rho_3 s\theta)^2}{2\rho_3^2 \; r_{B3}^5} \right\} (2 + \frac{3Q}{m_3 r_{B3}^2})$$

$$+ \; 3Gm_B \frac{\rho_B s\theta}{2 r_{B3}^7} \left\{ 3S_1 (\rho_3 - \rho_B c\theta) s2\phi_2 - \frac{2Q}{m_3} \rho_B s\theta \right\}$$

$$- \; 3Gm_B S_1 \frac{\rho_B s\theta \, s2\phi_2}{2\rho_3 r_{B3}^7} \left\{ r_{B3}^2 + 2(\rho_3 - \rho_B c\theta) \right\}$$

$$+ \; 3Gm_B S_1 \frac{\rho_3 - \rho_B c\theta}{2 r_{B3}^7} \left\{ 2(\rho_3 - \rho_B c\theta) c2\phi_2 + 3\rho_B s\theta \, s2\phi_2 \right\} \qquad (4.16)$$

$$a_{\Theta\rho_3} = \frac{2R_3}{\rho_3^2} (\dot{\psi} + \dot{\Theta}) - Gm_A \left\{ \frac{r_{A3}^2 \rho_A s\theta + 3\rho_A \rho_3 s\theta (\rho_3 + \rho_A c\theta)}{2\rho_3^2 \; r_{A3}^5} \right\} (2 + \frac{3P}{m_3 r_{A3}^2})$$

$$- \; 3Gm_A \frac{\rho_A \rho_3 s\theta}{2\rho_3^2 \; r_{A3}^7} \left\{ 3S_1 \rho_A s\theta \, s2\phi_1 + \frac{2P}{m_3} (\rho_3 + \rho_A c\theta) \right\}$$

$$+ \; 3Gm_A S_1 \frac{s2\phi_1}{2\rho_3^2 r_{A3}^7} \left\{ r_{A3}^2 \rho_A c\theta + 2\rho_3 (\rho_3 + \rho_A c\theta)(\rho_A c\theta + \rho_3) \right\}$$

$$- \; 3Gm_A S_1 \frac{\rho_3 (\rho_A c\theta + \rho_3)}{2\rho_3^2 \; r_{A3}^7} \left\{ 2\rho_A s\theta \, c2\phi_1 - 3(\rho_3 + \rho_A c\theta) s2\phi_1 \right\}$$

$$+ \; Gm_B \left\{ \frac{r_{B3}^2 \rho_B s\theta + 3\rho_B \rho_3 s\theta (\rho_3 - \rho_B c\theta)}{2\rho_3^2 \; r_{B3}^5} \right\} (2 + \frac{3Q}{m_3 r_{B3}^2})$$

$$- 3Gm_B \frac{\rho_B\rho_3 s\theta}{2\rho_3^2 r_{B3}^7}\left\{3S_1\rho_B s\theta\, s2\phi_2 - \frac{2Q}{m_3}(\rho_3 - \rho_B c\theta)\right\}$$

$$- 3Gm_B S_1 \frac{s2\phi_2}{2\rho_3^2 r_{B3}^7}\left\{r_{B3}^2\rho_B c\theta + 2\rho_3(\rho_3 - \rho_B c\theta)(\rho_B c\theta - \rho_3)\right\}$$

$$- 3Gm_B S_1 \frac{\rho_3(\rho_B c\theta - \rho_3)}{2\rho_3^2\ r_{B3}^7}\left\{2\rho_B s\theta\, c2\phi_2 + 3(\rho_3 - \rho_B c\theta)s2\phi_2\right\} \tag{4.17}$$

$$a_{\Theta\phi} = \frac{3Gm_A S_1}{2\rho_3^2 r_{A3}^5}\left\{3\rho_A\rho_3 s\theta\, s2\phi_1 + 2\rho_3(\rho_A c\theta + \rho_3)c2\phi_1\right\}$$

$$- \frac{3Gm_B S_1}{2\rho_3^2\ r_{B3}^5}\left\{3\rho_B\rho_3 s\theta\, s2\phi_2 + 2\rho_3(\rho_B c\theta - \rho_3)c2\phi_2\right\} \tag{4.18}$$

$$a_{R_3\Theta} = 2\rho_3(\dot{\psi} + \Theta) \tag{4.19}$$

$$a_{R_3R_3} = a_{R_3\Phi} = 0 \tag{4.20}$$

$$a_{R_3\theta} = Gm_A\rho_A s\theta\left\{\frac{r_{A3}^2 - 3\rho_3(\rho_3 + \rho_A c\theta)}{2r_{A3}^5}\right\}\left(2 + \frac{3P}{m_3 r_{A3}^2}\right)$$

$$+ \frac{3}{2}\ Gm_A\rho_3\frac{\rho_3 + \rho_A c\theta}{r_{A3}^7}\left\{S_1(\rho_3 + \rho_A c\theta)s2\phi_1 - \frac{2P}{m_3}\rho_A s\theta\right\}$$

$$- \frac{3}{2}\ Gm_A\rho_A S_1\left\{\frac{r_{A3}^2 c\theta + 2\rho_A\rho_3 s^2\theta}{r_{A3}^7}\right\} s2\phi_1$$

$$- \frac{3}{2}\ Gm_A\rho_A\rho_3 S_1 s\theta\left\{\frac{2(\rho_3 + \rho_A c\theta)c2\phi_1 - 3\rho_A s\theta\, s2\phi_1}{r_{A3}^7}\right\}$$

$$
\begin{aligned}
&-Gm_B\rho_B s\theta\left\{\frac{r_{B3}^2 - 3\rho_3(\rho_3 - \rho_B c\theta)}{2r_{B3}^5}\right\}(2 + \frac{3Q}{m_3 r_{B3}^2}) \\
&+ \frac{3}{2} Gm_B\rho_3 \frac{\rho_3 - \rho_B c\theta}{r_{B3}^7}\left\{3S_1(\rho_3 - \rho_B c\theta)s2\phi_2 + \frac{2Q}{m_3}\rho_B s\theta\right\} \\
&+ \frac{3}{2} Gm_B\rho_B S_1\left\{\frac{r_{B3}^2 c\theta - 2\rho_B\rho_3 s^2\theta}{r_{B3}^7}\right\} s2\phi_2 \\
&+ \frac{3}{2} Gm_B\rho_B\rho_3 S_1 s\theta\left\{\frac{2(\rho_3 - \rho_B c\theta)c2\phi_2 - 3\rho_B s\theta s2\phi_2}{r_{B3}^7}\right\}
\end{aligned}
\tag{4.21}
$$

$$
\begin{aligned}
a_{R_3\rho_3} = (\dot\psi + \Theta)^2 &- Gm_A\left\{\frac{r_{A3}^2 - 3(\rho_3 + \rho_A c\theta)^2}{2r_{A3}^5}\right\}(2 + \frac{3P}{m_3 r_{A3}^2}) \\
&+ \frac{3}{2} Gm_A \frac{\rho_3 + \rho_A c\theta}{r_{A3}^7}\left\{3S_1\rho_A s\theta s2\phi_1 + \frac{2P}{m_3}(\rho_3 + \rho_A c\theta)\right\} \\
&+ 3Gm_A S_1\left\{\frac{\rho_A s\theta(\rho_3 + \rho_A c\theta)s2\phi_1}{r_{A3}^7}\right\} \\
&+ \frac{3}{2} Gm_A S_1 \frac{\rho_A s\theta}{r_{A3}^7}\left\{3(\rho_3 + \rho_A c\theta)s2\phi_1 - 2\rho_A s\theta c2\phi_1\right\} \\
&- Gm_B\left\{\frac{r_{B3}^2 - 3(\rho_3 - \rho_B c\theta)^2}{2r_{B3}^5}\right\}(2 + \frac{3Q}{m_3 r_{B3}^2}) \\
&+ \frac{3}{2} Gm_B \frac{\rho_3 - \rho_B c\theta}{r_{B3}^7}\left\{-3S_1\rho_B s\theta s2\phi_2 + \frac{2Q}{m_3}(\rho_3 - \rho_B c\theta)\right\} \\
&- 3Gm_B S_1\left\{\frac{\rho_B s\theta(\rho_3 - \rho_B c\theta)s2\phi_2}{r_{B3}^7}\right\}
\end{aligned}
$$

$$- \frac{3}{2} \; Gm_B S_1 \frac{\rho_B s\theta}{r_{B3}^7} \left\{ 3(\rho_3 - \rho_B c\theta) s2\phi_2 + 2\rho_B s\theta\, c2\phi_2 \right\} \tag{4.22}$$

$$a_{R_3\phi} = - \frac{3Gm_A S_1}{2r_{A3}^5} \left\{ 3(\rho_3 + \rho_A c\theta) s2\phi_1 - 2\rho_A s\theta\, c2\phi_1 \right\}$$

$$- \frac{3Gm_B S_1}{2r_{B3}^5} \; 3(\rho_3 - \rho_B c\theta) s2\phi_2 + 2\rho_B s\theta\, c2\phi_2 \tag{4.23}$$

$$a_{\Phi\Theta} = 0 \tag{4.24}$$

$$a_{\Phi R_3} = 0 \tag{4.25}$$

$$a_{\Phi\Phi} = 0 \tag{4.26}$$

$$a_{\Phi\theta} = - 3GS_2 \frac{m_A \rho_3}{r_{A3}^5} \left\{ (3/2)\rho_A s\theta\, s2\phi_1 - (\rho_3 + \rho_A c\theta) c2\phi_1 \right\}$$

$$- 3GS_2 \frac{m_B \rho_3}{r_{B3}^5} \left\{ (3/2)\rho_B s\theta\, s2\phi_2 - (\rho_3 - \rho_B c\theta) c2\phi_2 \right\} \tag{4.27}$$

$$a_{\Phi\rho_3} = - 3GS_2 \frac{m_A}{r_{A3}^5} \left\{ (3/2)(\rho_3 + \rho_A c\theta) s2\phi_1 - \rho_A s\theta\, c2\phi_1 \right\}$$

$$- 3GS_2 \frac{m_B}{r_{B3}^5} \left\{ (3/2)(\rho_3 - \rho_B c\theta) s2\phi_2 + \rho_B s\theta\, c2\phi_2 \right\} \tag{4.28}$$

$$a_{\Phi\phi} = - 3GS_2 \left\{ \frac{m_A}{r_{A3}^3} c2\phi_1 + \frac{m_B}{r_{B3}^3} c2\phi_2 \right\} \tag{4.29}$$

$$a_{\theta\Theta} = a_{\rho_3 R_3} = a_{\phi\Phi} = 1 \tag{4.30}$$

$$a_{\theta R_3} = a_{\theta\Phi} = a_{\theta\theta} = a_{\theta\rho_3} = a_{\theta\phi} = 0 \tag{4.31}$$

$$a_{\rho_3\Theta} = a_{\rho_3\Phi} = a_{\rho_3\theta} = a_{\rho_3\rho_3} = a_{\rho_3\phi} = 0 \tag{4.32}$$

$$a_{\phi\Theta} = a_{\phi R_3} = a_{\phi\theta} = a_{\phi\rho_3} = a_{\phi\phi} = 0 \tag{4.33}$$

where r_{A_3} and r_{B_3} are defined by equations (3.2) and (3.3) and ϕ_1, ϕ_2, P, Q, S_1, S_2 are defined by equations (3.10) through (3.21).

The matrix [a] in equation (4.6) has to be evaluated at the equilateral triangular point and as such the elements a_{ij} are now evaluated at the equilateral triangle point.

$$a_{\Theta\Theta}\Big|_o = 0 \tag{4.34}$$

$$a_{\Theta R_3}\Big|_o = -2\dot{\psi}/\rho_{3o} \tag{4.35}$$

$$a_{\Theta\Phi}\Big|_o = 0 \tag{4.36}$$

$$a_{\Theta\theta}\Big|_o = Gm_A\left\{\frac{(\rho_A+\rho_B)^2\rho_A\rho_{3o}c\theta_o+3(\rho_A\rho_{3o}s\theta_o)^2}{2\rho_{3o}^2(\rho_A+\rho_B)^5}\right\}\left\{2+3\,\frac{\{S_3+3(B_3-A_3)c^2(\pi/3-\phi_o)\}}{m_3(\rho_A+\rho_B)^2}\right\}$$

$$-\ 3Gm_A\left\{\frac{\rho_A s\theta}{2(\rho_A+\rho_B)^7}\right\}\left\{3S_1(\rho_{3o}+\rho_A c\theta)s2(\pi/3-\phi_o) - \frac{2\rho_A s\theta_o}{m_3}\{S_3+3(B_3-A_3)c^2(\pi/3-\phi_o)\}\right\}$$

$$+\ 3Gm_A S_1\left\{\frac{\rho_A s\theta_o s2(\pi/3-\phi_o)}{2\rho_{3o}(\rho_A+\rho_B)^7}\right\}\left\{(\rho_A+\rho_B)^7 - 2(\rho_{3o}+\rho_A c\theta)\right\}$$

$$- 3Gm_A S_1 \left\{\frac{\rho_{3o}+\rho_A c\theta_o}{2(\rho_A+\rho_B)^7}\right\} \left\{2(\rho_{3o}+\rho_A c\theta_o)c2(\pi/3-\phi_o) + 3\rho_A s\theta_o s2(\pi/3-\phi_o)\right\}$$

$$- Gm_B\left\{\frac{(\rho_A+\rho_B)^2 \rho_B\rho_{3o}c\theta_o-3(\rho_B\rho_{3o}s\theta_o)^2}{2\rho_{3o}^2 \ (\rho_A+\rho_B)^5}\right\}\left\{2+3 \ \frac{3\{S_3+3(B_3-A_3)c^2(2\pi/3-\phi_o)}{m_3(\rho_A+\rho_B)^2}\right\}$$

$$+ 3Gm_B \left\{\frac{\rho_B s\theta_o}{2(\rho_A+\rho_B)^7}\right\} \left\{3S_1(\rho_{3o}-\rho_B c\theta_o)s2(2\pi/3 \ -\phi_o)- \frac{2\rho_B s\theta_o}{m_3} \ \{S_3+3(B_3-A_3)c^2(2\pi/3-\phi_o)\}\right\}$$

$$- 3Gm_B S_1 \left\{\frac{\rho_B s\theta_o s2(2\pi/3-\phi_o)}{2\rho_{3o}(\rho_A+\rho_B)^7}\right\} \left\{(\rho_A+\rho_B)^2 + 2(\rho_{3o}-\rho_B c\theta_o)\right\} \tag{4.37}$$

$$+ 3Gm_B S_1 \left\{\frac{\rho_{3o}-\rho_B c\theta_o}{2(\rho_A+\rho_B)^7}\right\} \left\{2(\rho_{3o}-\rho_B c\theta_o)c2(2\pi/3-\phi_o) + 3\rho_B s\theta_o s2(2\pi/3-\phi_o)\right\}$$

$$a_{\Theta\rho_3}\Big|_o = -Gm_A\rho_A\left\{\frac{(\rho_A+\rho_B)^2 s\theta_o+3\rho_{3o}s\theta_o(\rho_{30}+\rho_A c\theta_o)}{2\rho_{3o}^2(\rho_A+\rho_B)^5}\right\}\left\{2+3 \ \frac{\{S_3+3(B_3-A_3)c^2(\pi/3-\phi_o)\}}{m_3(\rho_A+\rho_B)^2}\right\}$$

$$- 3Gm_A \left\{\frac{\rho_A\rho_{3o}s\theta_o}{2\rho_{3o}^2(\rho_A+\rho_B)^7}\right\}\left\{3S_1\rho_A s\theta_o s2(\pi/3-\phi_o)+ \frac{2(\rho_{3o}+\rho_A c\theta_o)}{m_3} \ \{S_3+3(B_3-A_3)c^2(\pi/3-\phi_o)\}\right\}$$

$$+3Gm_A S_1\left\{\frac{s2(\pi/3-\phi_o)}{2\rho_{3o}^2(\rho_A+\rho_B)^7}\right\} \left\{(\rho_A+\rho_B)^2 \rho_A c\theta_o+2\rho_{3o}(\rho_{3o}+\rho_A c\theta_o)^2\right\}$$

$$-3Gm_A S_1\left\{\frac{\rho_{3o}(\rho_{3o}+\rho_A c\theta_o)}{2\rho_{3o}^2(\rho_A+\rho_B)^7}\right\}\left\{2\rho_A s\theta_o c2(\pi/3-\phi_o)-3(\rho_{3o}+\rho_A c\theta_o)s2(\pi/3-\phi_o)\right\}$$

$$+ Gm_B\rho_B\left\{\frac{(\rho_A+\rho_B)^2 s\theta_o+3\rho_{3o}s\theta_o(\rho_{30}-\rho_B c\theta_o)}{2\rho_{3o}^2(\rho_A+\rho_B)^5}\right\}\left\{2+ \ \frac{3\{S_3+3(B_3-A_3)c^2(2\pi/3-\phi_o)}{m_3(\rho_A+\rho_B)^2}\right\}$$

$$-3Gm_B\left\{\frac{\rho_B\rho_{3o}s\theta_o}{2\rho_{3o}^2(\rho_A+\rho_B)^7}\right\}\left[3S_1\rho_B s\theta_o s2(2\pi/3-\phi_o)-\frac{2(\rho_{3o}-\rho_B c\theta_o)}{m^3}\{S_3+3(B_3-A_3)c^2(2\pi/3-\phi_o)\}\right]$$

$$-3Gm_B S_1\left\{\frac{s2(2\pi/3-\phi_o)}{2\rho_{3o}^2(\rho_A+\rho_B)^7}\right\}\left\{(\rho_A+\rho_B)^2\ \rho_B c\theta_o-2\rho_{3o}(\rho_{3o}-\rho_B c\theta_o)^2\right\}$$

$$+\ 3Gm_B S_1\left\{\frac{\rho_{3o}(\rho_{3o}-\rho_B c\theta_o)}{2\rho_{3o}^2(\rho_A+\rho_B)^7}\right\}\left\{2\rho_B s\theta_o c2(2\pi/3-\phi_o)+3(\rho_{3o}-\rho_B c\theta_o)s2(2\pi/3-\phi_o)\right\} \qquad (4.38)$$

$$a_{\Theta\phi}\Big|_o = \frac{3Gm_A S_1}{2\rho_{3o}^2(\rho_A+\rho_B)^5}\left\{3\rho_A s\theta_o s2(\pi/3-\phi_o)\ +2(\rho_{3o}+\rho_A c\theta_o)c2(2\pi/3-\phi_o)\right\}$$

$$-\frac{3Gm_B S_1}{2\rho_{3o}^2(\rho_A+\rho_B)^5}\left\{3\rho_B s\theta_o s2(2\pi/3-\phi_o)\ -\ 2(\rho_{3o}-\rho_B c\theta_o)c2(2\pi/3-\phi_o)\right\} \qquad (4.39)$$

$$a_{R_3\Theta}\Big|_o = 2\rho_{3o}\ \dot{\psi} \qquad (4.40)$$

$$a_{R_3R_3}\Big|_o = 0 \qquad (4.41)$$

$$a_{R_3\Phi}\Big|_o = 0 \qquad (4.42)$$

$$a_{R_3\theta}\Big|_o = Gm_A\rho_A s\theta_o\left\{\frac{(\rho_A+\rho_B)^2-3\rho_{3o}(\rho_{3o}+\rho_A c\theta_o)}{2(\rho_A+\rho_B)^5}\right\}\left\{2+\frac{3\{S_3+3(B_3-A_3)c^2(\pi/3-\phi_o)\}}{m_3(\rho_A+\rho_B)^2}\right\}$$

$$+\ \frac{3}{2}\ Gm_A\rho_{3o}\left\{\frac{\rho_{3o}+\rho_A c\theta_o}{(\rho_A+\rho_B)^7}\right\}\left\{3S_1(\rho_{3o}+\rho_A c\theta_o)s2(\pi/3-\phi_o)-\frac{2\rho_A s\theta_o}{m_3}\{S_3+3(B_3-A_3)c^2(\pi/3-\phi_o)\}\right\}$$

$$- \frac{3}{2}\, Gm_A \rho_A S_1 \left\{ \frac{(\rho_A+\rho_B)^2 c\theta_o + 2\rho_A \rho_{3o} s^2\theta_o}{(\rho_A+\rho_B)^7} \right\} s2(\pi/3-\phi_o)$$

$$- \frac{3}{2}\, Gm_A \rho_A \rho_{3o} S_1\, s\theta_o \left\{ \frac{2(\rho_{3o}+\rho_A c\theta_o)c2(\pi/3-\phi_o) - 3\rho_A s\theta_o s2(\pi/3-\phi_o)}{(\rho_A+\rho_B)^7} \right\}$$

$$- Gm_B \rho_B s\theta_o \left\{ \frac{(\rho_A+\rho_B)^2 - 3\rho_{3o}(\rho_{3o}-\rho_B c\theta_o)}{2(\rho_A+\rho_B)^5} \right\} \left\{ 2 + \frac{3\{S_3 + 3(B_3-A_3)c^2(2\pi/3-\phi_o)\}}{m_3(\rho_A+\rho_B)^2} \right\}$$

$$+ \frac{3}{2}\, Gm_B \rho_{3o} \left\{ \frac{\rho_{3o}-\rho_B c\theta_o}{(\rho_A+\rho_B)^7} \right\} \left\{ 3S_1(\rho_{3o}-\rho_B c\theta_o)s2(2\pi/3-\phi_o) + \frac{2\rho_B s\theta_o}{m_3} \{S_3+3(B_3-A_3)c^2(2\pi/3-\phi_o)\right.$$

$$+ \frac{3}{2}\, Gm_B \rho_B S_1 \left\{ \frac{(\rho_A+\rho_B)^2 c\theta_o - 2\rho_B \rho_{3o} s^2\theta_o}{(\rho_A+\rho_B)^7} \right\} s2\,(2\pi/3-\phi_o)$$

$$+ \frac{3}{2}\, Gm_B \rho_B \rho_{3o} S_1 s\theta_o \left\{ \frac{2(\rho_{3o}-\rho_B c\theta_o)c2(2\pi/3-\phi_o) - 3\rho_B s\theta_o s2(2\pi/3-\phi_o)}{(\rho_A+\rho_B)^7} \right\} \tag{4.43}$$

$$a_{R_3\rho_3}\big|_o = \dot{\psi}^2 - Gm_A \left\{ \frac{(\rho_A+\rho_B)^2 - 3(\rho_3 + \rho_A c\theta_o)^2}{2(\rho_A+\rho_B)^5} \right\} \left\{ 2 + \frac{3\{S_3 + 3(B_3 - A_3)c^2(\pi/3-\phi_o)\}}{m_3(\rho_A+\rho_B)^2} \right\}$$

$$+ \frac{3}{2}\, Gm_A \left\{ \frac{\rho_{3o}+\rho_A c\theta_o}{(\rho_A+\rho_B)^7} \right\} \left\{ 3S_1 \rho_A s\theta_o s2(\pi/3-\phi_o) + \frac{2(\rho_{3o}+\rho_A c\theta)}{m_3} \{S_3 + 3(B_3-A_3)c^2(\pi/3-\phi)\} \right\}$$

$$+ 3Gm_A S_1 \left\{ \frac{\rho_A s\theta_o(\rho_{3o}+\rho_A c\theta_o)s2(\pi/3-\phi_o)}{(\rho_A+\rho_B)^7} \right\}$$

$$+ \frac{3}{2}\, Gm_A S_1 \left\{ \frac{\rho_A s\theta_o}{(\rho_A+\rho_B)^7} \right\} \left\{ 3(\rho_{3o}+\rho_A c\theta_o)s2(\pi/3-\phi_o) - 2\rho_A s\theta_o c2(\pi/3-\phi_o) \right\}$$

$$- Gm_B \left\{ \frac{(\rho_A + \rho_B)^2 - 3(\rho_{3o} - \rho_B c\theta_o)^2}{2(\rho_A + \rho_B)^5} \right\} \left\{ 2 + \frac{3\{S_3 + 3(B_3 - A_3)c^2(2\pi/3 - \phi_o)\}}{m_3(\rho_A + \rho_B)^2} \right\}$$

$$+ \frac{3}{2} Gm_B \left\{ \frac{\rho_3 - \rho_B c\theta_o}{(\rho_A + \rho_B)^7} \right\} \left\{ -3S_1 \rho_B s\theta_o s2(2\pi/3 - \phi_o) + \frac{(2\rho_{3o} - \rho_B c\theta_o)}{m_3} \{S_3 + 3(B_3 - A_3)c^2(2\pi - \phi_o)\} \right\}$$

$$- 3Gm_B S_1 \left\{ \frac{\rho_B s\theta_o (\rho_{3o} - \rho_B c\theta_o) s2(2\pi/3 - \phi_o)}{(\rho_A + \rho_B)^7} \right\}$$

$$- \frac{3}{2} Gm_B S_1 \left\{ \frac{\rho_B s\theta_o}{(\rho_A + \rho)^7} \right\} \left\{ 3(\rho_{3o} - \rho_B c\theta_o) s2(2\pi/3 - \phi_o) + 2\rho_B s\theta_o c2(2\pi/3 - \phi_o) \right\} \tag{4.44}$$

$$a_{R_3\phi}\Big|_o = - \frac{3Gm_A S_1}{2(\rho_A + \rho_B)^5} \left\{ 3(\rho_{3o} + \rho_A c\theta_o) s2(\pi/3 - \phi_o) - 2\rho_A s\theta_o c2(\pi/3 - \phi_o) \right\}$$

$$- \frac{3Gm_B S_1}{2(\rho_A + \rho_B)^5} \left\{ 3(\rho_{3o} - \rho_B c\theta_o) s2(2\pi/3 - \phi_o) + 2\rho_B s\theta_o c2(2\pi/3 - \phi_o) \right\} \tag{4.45}$$

$$a_{\Phi\Theta}\Big| = 0 \tag{4.46}$$

$$a_{\Phi R_3}\Big|_o = 0 \tag{4.47}$$

$$a_{\Phi\Phi}\Big|_o = 0 \tag{4.48}$$

$$a_{\Phi\theta}\Big|_o = -3GS_2 \frac{m_A \rho_{3o}}{(\rho_A + \rho_B)^5} \left\{ \left(\frac{3}{2}\right) \rho_A s\theta_o s2(\pi/3 - \phi_o) - (\rho_{3o} + \rho_A c\theta) c2(\pi/3 - \phi_o) \right\}$$

$$- 3GS_2 \frac{m_B \rho_{3o}}{(\rho_A + \rho_B)^5} \left\{ \left(\frac{3}{2}\right) \rho_B s\theta_o s2(2\pi/3 - \phi_o) - (\rho_{3o} - \rho_B c\theta_o) c2(2\pi/3 - \phi_o) \right\} \tag{4.50}$$

$$a_{\Phi\phi}\Big|_o = -\frac{3GS_2}{(\rho_A+\rho_B)^3}\left\{m_A c2(\pi/3-\phi_o) + m_B c2(2\pi/3-\phi_o)\right\} \tag{4.51}$$

$$a_{\theta\Theta}\Big|_o = a_{\rho_3 R_3}\Big|_o = a_{\phi\Phi}\Big|_o = 1 \tag{4.52}$$

$$a_{\theta R_3}\Big|_o = a_{\theta\Phi}\Big|_o = a_{\theta\theta}\Big|_o = a_{\theta\rho_3}\Big|_o = a_{\theta\phi}\Big|_o = 1 \tag{4.53}$$

$$a_{\rho_3\Theta}\Big|_o = a_{\rho_3\Phi}\Big|_o = a_{\rho_3\theta}\Big|_o = a_{\rho_3\rho_3}\Big|_o = a_{\rho_3\phi}\Big|_o = 0 \tag{4.54}$$

$$a_{\phi\Theta}\Big|_o = a_{\phi R_3}\Big|_o = a_{\phi\theta}\Big|_o = a_{\phi\rho_3}\Big|_o = a_{\phi\phi}\Big|_o = 0 \tag{4.55}$$

where S_1 and S_2 are given by (3.20) and (3.21) and where

$$S_3 = (A_3 + B_3 + C_3) - 3B_3 \tag{4.56}$$

ρ_{3o} and θ_o are the values of ρ_3 and θ at the equilateral triangle position and ϕ_o measures the libration of the third body when placed at the equilateral triangle point.

ρ_{3o} and θ_o can be calculated from the geometry of the figure and are given by

$$\rho_{3o} = \frac{\{(\rho_B-\rho_A)^2 + 3(\rho_B+\rho_A)^2\}^{\frac{1}{2}}}{2} \tag{4.57}$$

$$\theta_o = c^{-1}\frac{\rho_B - \rho_A}{\{(\rho_B-\rho_A)^2 + 3(\rho_B+\rho_A)^2\}^{\frac{1}{2}}} \tag{4.58}$$

ϕ_o can be obtained by substituting (4.57) and (4.58) into (3.15) and setting $\ddot{\phi}$ equal to zero, and is given by

$$\phi_o = \tfrac{1}{2}\, t^{-1}\left\{\frac{m_B s(\pi/3)}{m_A - m_B c(\pi/3)}\right\} + (\pi/3) \tag{4.59}$$

or

$$= \tfrac{1}{2}\, t^{-1}\left\{\frac{m_B s(\pi/3)}{m_A - m_B c(\pi/3)}\right\} - (\pi/6) \tag{4.60}$$

Knowing the earth moon distance, and the mass of the earth and the moon, the values of θ_o, ρ_{3o}, and ϕ_o can be calculated. These are given by

$$\theta_o = 1.0578 \text{ radians} \tag{4.61}$$

$$\rho_{3o} = 3.329 \times 10^5 \text{ km.} \tag{4.62}$$

$$\phi_o = 1.0578 \text{ and } (1.0578 - \pi/2) \text{ radians} \tag{4.63}$$

It may be noted that the value of ϕ_o obtained (Fig. 5) agrees exactly with the value ($60^\circ 36'$) given by Robinson (Ref. 8) for a dumb-bell shaped station. The position $\phi_o = (1.0578 - \pi/2)$ will later be shown to be unstable and hence for the immediate analysis we use the stable position

$$\phi_o = 1.0578 \text{ radians} \tag{4.64}$$

At this stage, the differences between the true equilibrium position and the equilateral position can be found by using equation 4.6 after 4.61, 4.62, and 4.64 have been substituted in equations 4.34 through 4.55. These differences were calculated for a dumb-bell shaped station for which the differences can be expected to be a maximum (being zero for a symmetric station). These were found to be negligibly small and hence we can take the equilibrium position to be the equilateral point.

The equations of motion (4.1) can now be expanded (linearized) in a Taylor's series about an equilibrium point denoted by the subscript 'oo' which, in general, is different from the equilateral point denoted by the subscript 'o'. The linearized equations of motion are given by

$$\{\dot{y}\} = [a]_{oo}\{y\} \tag{4.65}$$

where

$$\{y\} = [\delta\Theta \; \delta R_3 \; \delta\Phi \; \delta\theta \; \delta\rho_3 \; \delta\phi]^T$$

and

$$[a] = \begin{bmatrix} a_{\Theta\Theta} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{\Theta\phi} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{\phi\Theta} & \cdot & \cdot & \cdot & \cdot & \cdot & a_{\phi\phi} \end{bmatrix} \tag{4.66}$$

The elements of [a] are given by equations 4.13 through 4.33. Since we have shown that the equilibrium position is very close to the equilateral point, we can replace the 'oo' in 4.65 by 'o'. Hence the linearized equations are given by

$$\{\dot{y}\} = [a]_o\{y\} \tag{4.67}$$

where the elements of $[a]_o$ are given by equations 4.36 through 4.55.

It is worth noting that the effects of the non-symmetric nature of the third body are given by those terms which contain the size and shape factors S_2 and S_2. If these are set equal to zero, we would be left with the equations of motion for a spherical body in a restricted three body problem.

5. NON-DIMENSIONALISATION OF THE EQUATIONS OF MOTION

The linearized equations of motion as given by equation 4.67 are in a dimensional form. In order to make the results more tractable, it will be advantageous to non-dimensionalise the equations of motion. The non-dimensionalising scheme is based on a "Newtonian formulation of Kepler's third law".

$$\dot{\psi}^2 = \frac{G(m_A + m_B)}{(\rho_A + \rho_B)^3} \tag{5.1}$$

The non-dimensional equations of motion are given by

$$\{\dot{y}\} = [TT]^{-1}[T]\ [a]\ [T]^{-1}\ \{y\} \tag{5.2}$$

It should be noted that the dot above the y in equation 5.2 and in 5.5 (which follows) denotes the differentiation of y with respect to non-dimensional time.

[TT] is a 2nx2n diagonal matrix given by

$$[TT] = \begin{bmatrix} \ddots & & 0 \\ & \dot{\psi} & \\ 0 & & \ddots \end{bmatrix} \tag{5.3}$$

and [T] is another diagonal matrix given by

$$[T] = \begin{bmatrix} 1/\dot{\psi} & & & & & 0 \\ & 1/(\rho_A + \rho_B)\dot{\psi} & & & & \\ & & 1/\dot{\psi} & & & \\ & & & 1 & & \\ & & & & 1/(\rho_A + \rho_B) & \\ 0 & & & & & 1 \end{bmatrix} \tag{5.4}$$

Equation 5.2 can now be written as

$$\{\dot{y}\} = [a]_o\{y\} \tag{5.5}$$

where the states $\{y\}$ and consequently $[a]_o$ are non-dimensional.

6. STABILITY, RESONANCE, AND THE SOLUTION TO THE EQUATIONS OF MOTION

In accordance with classical results which can be found in a good textbook on space mechanics (example, Ref. 10), we can expect ρ_3 and θ to vary such that the center of mass describes an elliptic orbit about the equilateral point. The semiminor axis of the resulting orbit will point towards the barycenter of the system (Fig. 1). These results are valid for a point mass. In this problem, we are including the possible effect of the libration motion (of a non-symmetric station) on the motion of the center of mass. Though this effect is very small, it is conceivable that if the libration motion is periodic (which depends on the sign of the shape factor S_2), and furthermore has the same period as that of the center of mass, resonance will occur.

Before investigating the possibility of the resonant condition, it would be fruitful to find out the condition under which the libration motion is periodic and hence not unstable. In general, $-1 \leq S_2 \leq 1$ (6.1) The libration motion can be examined for periodic motion by solving the linearized libration equation. It will be observed that the case $-1 \leq S_2 < 0$ is equivalent to that of a spring mass system with the spring having a negative striffness. If $0 < S_2 \leq 1$, the solution is periodic and will be shown to be given by equation 6.5. Hence we can state the first stability rule

$$B_3 > A_3 \text{ for stability} \tag{6.2}$$

Now in order to look for resonant behavior, the solution to the non-dimensional and linearized equations have to be found. The eigenvalues [a] are found (for any $(m_B/(m_A+m_B)) < 0.0385$) to be three complex conjugate pairs with no real parts ($\mathrm{Im}\,\lambda_1 = -\mathrm{Im}\,\lambda_2 > 0$, $\mathrm{Im}\,\lambda_3 = -\mathrm{Im}\,\lambda_4 > 0$, $\mathrm{Im}\,\lambda_5 = -\mathrm{Im}\,\lambda_6 > 0$). However, from previous knowledge, we know that for no libration motion (for a spherical station) the position of the center of mass (i.e., $\delta\theta$ and $\delta\rho_3$) varies harmonically. Also if the station remains at L_4, its libration motion (i.e., $\delta\phi$) is harmonic for $B_3 > A_3$. However for our case, coupling between the motions exist due to the $a_{\Theta\phi}$, $a_{R_3\phi}$, $a_{\Phi\theta}$, and $a_{\Phi\rho_3}$ terms. Hence each of the solutions (for $\delta\theta$, $\delta\rho_3$, and $\delta\phi$) can be written out in terms of a harmonic solution (with two constants of integration) for the homogeneous part and a term (say Q_k, $k = 1, 2, 3$) due to the coupling or forcing term. Since these coupling terms are themselves harmonic, all the Q_k will contain a $(\lambda_i^2 - \lambda_j^2)$ term in the denominator where $i \neq j$; $i, j = 1, 3, 5$. If $\lambda_i = \lambda_j$ a resonant condition occurs. Based on the above arguments the solutions can be shown to be given by

$$\delta\theta = \delta\theta^o \cos\lambda_s t + \frac{\delta\dot{\theta}^o}{\lambda_s}\sin\lambda_s t + \frac{a_{\Theta\phi}}{\lambda_s^2 - \lambda_\phi^2}\left(\delta\phi^o \cos\lambda_\phi t + \frac{\delta\dot{\phi}^o}{\lambda_\phi}\sin\lambda_\phi t\right) \quad (6.3)$$

$$\delta\rho_3 = \delta\rho_3^o \cos\lambda_s t + \frac{\delta\dot{\rho}_3^o}{\lambda_s}\sin\lambda_s t + \frac{a_{R_3\phi}}{\lambda_s^2 - \lambda_\phi^2}\left(\delta\phi^o \cos\lambda_\phi t + \frac{\delta\dot{\phi}^o}{\lambda_\phi}\sin\lambda_\phi t\right) \quad (6.4)$$

$$\delta\phi = \delta\phi^o \cos\lambda_\phi t + \frac{\delta\dot{\phi}^o}{\lambda_\phi}\sin\lambda_\phi t + \frac{a_{\Phi\theta}}{\lambda_\phi^2 - \lambda_s^2}\left(\delta\theta^o \cos\lambda_s t + \frac{\delta\dot{\theta}^o}{\lambda_s}\sin\lambda_s t\right)$$

$$+ \frac{a_{\Phi\rho_3}}{\lambda_\phi^2 - \lambda_s^2}\left(\delta\rho_3^o \cos\lambda_s t + \frac{\delta\dot{\rho}_3^o}{\lambda_s}\sin\lambda_s t\right) \quad (6.5)$$

In the above expressions, the superscript 'o' stands for initial conditions which have been chosen so that the long period mode is eliminated. λ_s and λ_ϕ are two of the eigenvalues of [a]. λ_s is the frequency of the short period mode and λ_ϕ is the libration frequency which depends on the shape of the station and hence on the shape factor S_2.

For a particular value of S_2, the libration frequency can be equal to the frequency of the short period mode. Resonance occurs and the amplitudes of motion may become large. This resonant condition can be obtained by equating the frequency of the short period mode to the libration frequency.

$$\lambda_s^2 = \lambda_\phi^2 = S_2\lambda_{\phi D}^2 \tag{6.6}$$

$\lambda_{\phi D}$ is the libration frequency of a dumb-bell shaped station which is one of the eigenvalues of [a] when A_3 is equal to zero.

Substituting the value of S_2 from equation 3.21 into 6.6, we obtain

$$\frac{B_3}{A_3} = \frac{\lambda_{\phi D}^2 + \lambda_s^2}{\lambda_{\phi D}^2 - \lambda_s^2} \tag{6.7}$$

Taking the value of μ for the earth moon system to be 0.01235, we obtain

$$B_3/A_3 = 1.9201 \text{ for resonance} \tag{6.8}$$

The stability results obtained can be conveniently plotted in an inertia domain by drawing stability boundaries (Fig. 6). The motion is stable in the space where $B_3 > A_3$ except along the resonance line. In practice, any points close to the line will correspond to station inertias which will cause large amplitudes of motion. Hence the station inertias should be chosen such that they are away (to any desired degree) from the line, but in the stability region defined by equation 6.2.

It can be shown that for any value of S_2, $a_{\Theta\phi}$ and $a_{R_3\phi}$ are many orders of magnitude smaller than $a_{\Phi\theta}$ and $a_{\Phi\rho_3}$. In fact $a_{\Theta\phi}$ are much smaller than any of the other terms in equations 3.3 through 3.5. As such, the coupling terms in equations 3.3 and 3.4 can be neglected as long as the station inertias are away from the resonant condition. This shall be done in the sequel. Hence we can conclude that if the libration motion is kept finite, the motion of the center of mass will also be finite. In other words, stability criteria are more restrictive for libration motion than for motion of the center of mass.

7. INITIAL CONDITIONS AND POINTING ERRORS

The selection of initial conditions can hardly be overemphasized, for their appropriate selection is crucial for the ensuing motion. The four constants of motion in equations 6.3 and 6.4 are not arbitrary. They have been chosen to completely eliminate the long period mode, and the relationship between them can be obtained by using equations 6.3 and 6.4 with equation 4.67. If the initial position for the center of mass of the station is chosen, the velocities that must be imparted to it can be shown to be given by the following relations

$$\delta\dot{\theta}^o = \frac{a_{R_3\theta}\,\delta\theta^o + (a_{R_3\rho_3} + \lambda_s^2)\,\delta\rho_3^o}{a_{R_3\Theta}} \tag{7.1}$$

$$\delta\dot{\rho}_3^o = -\frac{(a_{\Theta\theta} + \lambda_s^2)\ \delta\theta^o + a_{\Theta\rho_3}\,\delta\rho_3^o}{a_{\Theta R_3}} \tag{7.2}$$

The libration motion (given by equation 6.5) is a superposition of two periodic motions one having the period of the short period made and one with the period of libration. The initial libration position and velocity are selected to be zero, i.e. $\delta\phi^o = \delta\dot{\phi}^o = 0$. These correspond to an initial configuration such that the center of mass is on a semiminor axis (Fig 1) and one initial orientation is such that the station points towards the barycenter of the earth-moon system.

We are now in a position to analyze the pointing errors of a trancommunication station.

As soon as the station moves away from its initial orbit position, pointing errors of two kinds will occur. One of them is due to the effect of gravity gradients which will cause the station to rotate about its center of mass. This error is given by the value of $\delta\phi(t)$. Even if this error was not present, another kind of pointing error would occur due to the finite size of the orbit. This error is given by $[-\delta\theta(t)]$. Hence, at any position in the orbit, the total pointing error is given by

$$\in \delta\phi(t) = \delta\phi(t) - \delta\theta(t) \tag{7.3}$$

where $\in$ stands for an error in the variable which follows it. From equations 6.4 and 6.5 (remembering the remarks made at the end of last chapter and that $\delta\phi^{o} = \delta\phi^{o} = 0$), the total pointing error can be shown to be given by

$$\begin{aligned}
\in \delta\phi(t) = {} & \frac{a_{\Phi\theta}}{\lambda_\phi^2 - \lambda_s^2} \left[\delta\theta^{o} \cos \lambda_s t + \frac{\delta\dot{\theta}^{o}}{\lambda_s} \sin \lambda_s t \right] \\
& + \frac{a_{\Phi\rho_3}}{\lambda_\phi^2 - \lambda_s^2} \left[\delta\rho_3^{o} \cos \lambda_s t + \frac{\delta\dot{\rho}_3^{o}}{\lambda_s} \sin \lambda_s t \right] \\
& - \left(\delta\theta^{o} \cos \lambda_s t + \frac{\delta\dot{\theta}^{o}}{\lambda_s} \sin \lambda_s t \right)
\end{aligned} \tag{7.4}$$

Knowing the nature and magnitude of the pointing error, a suitable control system can be designed to minimize the error.

8. DUMB-BELL SHAPED STATION

The analytical results obtained are now checked for their validity by studying the dynamics of a dumb-bell shaped station. This is done because some of the results obtained for a dumb-bell shaped station have been obtained by Robinson (Ref. 8). Furthermore, the gravity gradient effect is a maximum for a dumb-bell shaped station.

The following station parameters were used for the study.

$$m_3 = 1.0 \times 10^3 \text{ Kg} \tag{8.1}$$

$$A_3 = 0 \tag{8.2}$$

$$B_3 = 625 \times 10^3 \text{ Kg. m}^2 \tag{8.3}$$

$$C_3 = 625 \times 10^3 \text{ Kg. m}^2 \tag{8.4}$$

The following non-dimensional results are obtained

$$[a] = \begin{bmatrix} 0 & -2.012 & 0 & 0.2744 \times 10^{-1} & -0.1555 \times 10^{-1} & 0.3274 \times 10^{-19} \\ 1.988 & 0 & 0 & -0.1536 \times 10^{-1} & 2.973 & 0.9926 \times 10^{-16} \\ 0 & 0 & 0 & 2.923 & 0.2347 \times 10^{-1} & -2.999 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{8.5}$$

It should be noted that since $a_{\Theta\phi}$ and $a_{R_3\phi}$ are much smaller than

$a_{\Theta\theta}$, $a_{\Theta\rho_3}$, $a_{R_3\theta}$, $a_{R_3\rho_3}$, the libration motion does not effect the motion of the center of mass. Since $a_{\Phi\theta}$ and $a_{\Phi\rho_3}$ are of comparable magnitude to $a_{\Phi\phi}$, the motion of the center of mass does effect the libration motion. Furthermore, since $a_{\Theta\phi}$ and $a_{R_3\phi}$ are much smaller than $a_{\Phi\theta}$ and $a_{\Phi\rho_3}$ (all of which are the coupling terms which can cause resonant behavior), stability criteria are more restrictive for libration motion than for motion of the center of mass.

The eigenvalues of [a] are

$$\lambda_{1,2} = \pm\ 1.7319i \text{ where } i = \sqrt{-1} \tag{8.6}$$

$$\lambda_{3,4} = \pm\ 0.95429i \tag{8.9}$$

$$\lambda_{5,6} = \pm\ 0.29877i \tag{8.10}$$

Classical results (Ref. 10) show that λ_3 and λ_5 correspond to the frequency of short and long period modes. Hence λ_1 must be the frequency of libration of a dumb-bell shaped station. Hence we can write

$$\lambda_{\phi D} = \lambda_1 = 1.7319i \tag{8.11}$$

$$\lambda_s = \lambda_3 = 0.95429i \tag{8.12}$$

$$\lambda_\ell = \lambda_5 = 0.29877i \tag{8.13}$$

where λ_ℓ is the frequency of the long period mode which is eliminated by appropriately selecting the initial conditions. The value of $\lambda_{\phi D}$ corresponds to a libration period of 15.8 days which is of the "order of 16 days" which was given by Robinson (Ref. 8). It should be noted that the libration frequency of any other station can be obtained from the relation

$$\lambda_{\phi} = 1.7319 \sqrt{S_2} \tag{8.14}$$

The initial conditions chosen are given

$$\delta\theta^{o} = 0 \tag{8.15}$$

$$\delta\rho_3^{o} = 0.001 \tag{8.16}$$

$$\delta\phi^{o} = 0 \tag{8.17}$$

$$\delta\dot{\phi}^{o} = 0 \tag{8.18}$$

The calcuated initial conditions (velocities that should be imparted to the center of mass) are given by

$$\delta\dot{\theta}^{o} = 1.9535 \times 10^{-3} \tag{8.19}$$

$$\delta\dot{\rho}_3^{o} = -0.7727 \times 10^{-5} \tag{8.20}$$

The pointing errors for various orbit positions are tabulated in Table I.

9. CONCLUSIONS

Based on this work, the following conclusions can be drawn:

EQUILIBRIUM POSITION

Equilibrium position is negligibly affected by the non-symmetric shape. Stations of all shapes point to the barycenter of the system.

EXTENT OF COUPLING BETWEEN THE EQUATIONS OF MOTION

Motion of the center of mass is negligibly affected by libration motion provided the station inertias are significantly different away from the resonant condition. The libration motion is affected by center of mass motion.

INITIAL CONDITIONS

Proper selection of initial conditions enables:

Center of Mass to move in a short period mode.

Libration motion of any station to have the same period as that of the short period mode.

STABILITY

For stability, $B_3 > A_3$.

Statbility criteria are more restrictive for the libration motion than for the motion of the center of mass.

POINTING ERRORS

Pointing errors due to the center of mass not being at the equilateral point act in opposition (and hence reduce) the pointing errors due to the finite size of the orbit.

The initial conditions were chosen so that the above two pointing errors are of the same period - that of the short period mode.

TABLE I

POINTING ERRORS

Location in orbit	Due to gravity gradient effects = $\delta\phi$ (Radians)	Due to finite size of orbit = $-\delta\theta$ (Radians)	Total $\epsilon\delta\phi$ (Radians)
$\frac{1}{4}$	-2.8645×10^{-3}	2.0471×10^{-3}	-0.8174×10^{-3}
$\frac{1}{2}$	-1.1236×10^{-5}	0.0	-1.1236×10^{-5}
$\frac{3}{4}$	2.8645×10^{-3}	-2.0471×10^{-3}	0.8174×10^{-3}
1	1.1236×10^{-5}	0.0	1.1236×10^{-5}

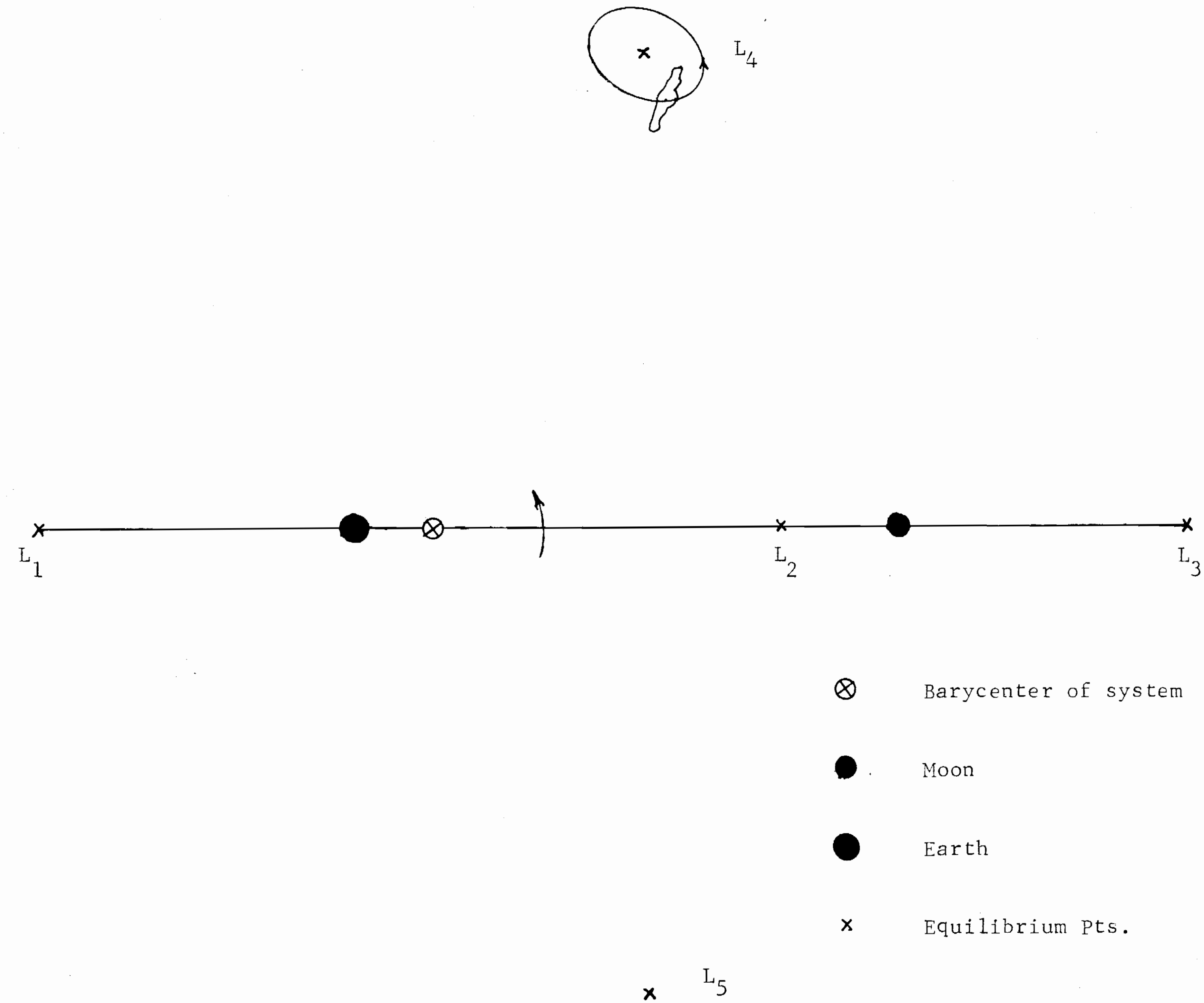

Fig. 1 EQUILIBRIUM POINTS OF THE EARTH-MOON SYSTEM

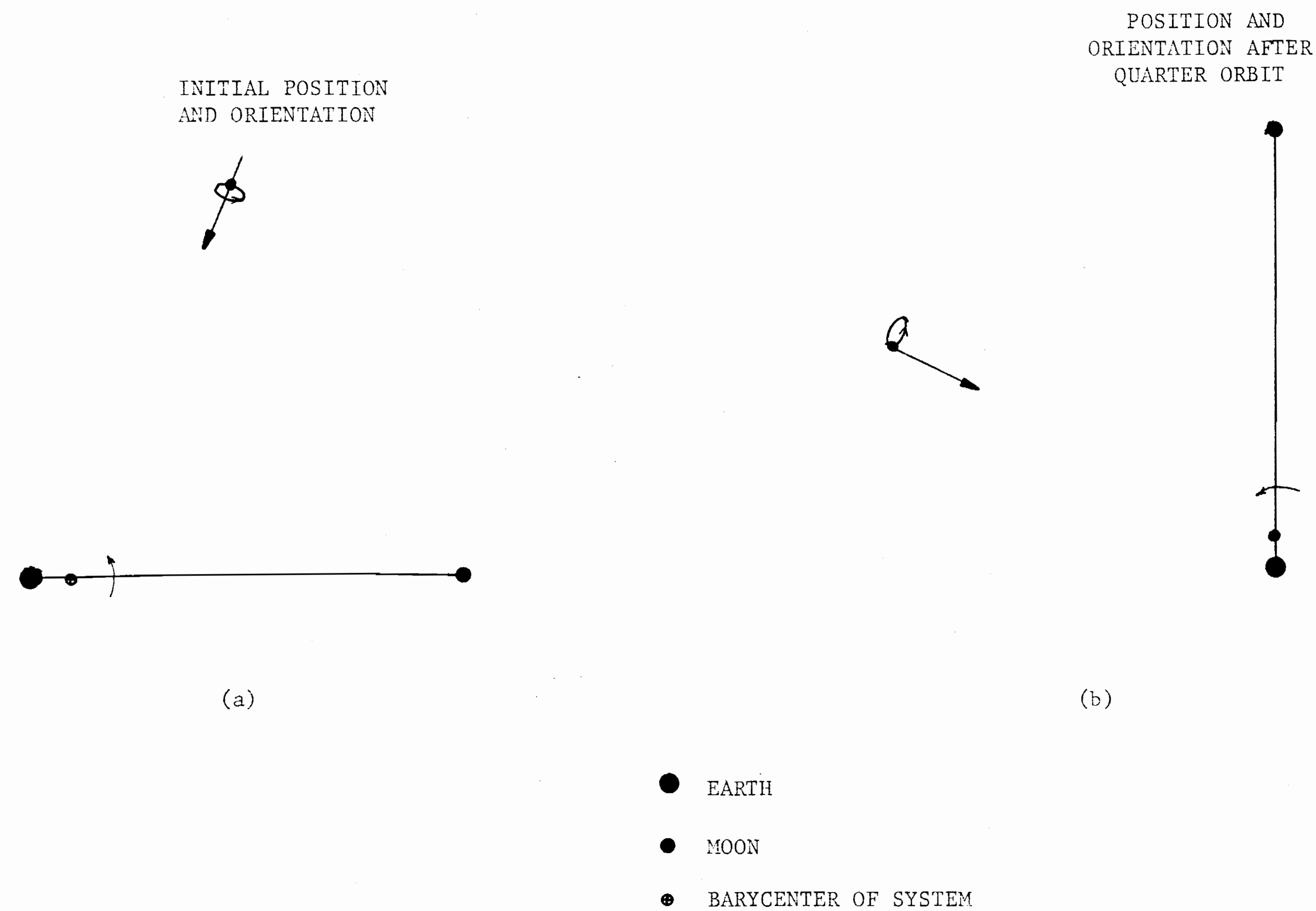

FIG. 2. TRANSCOMMUNICATION STATION

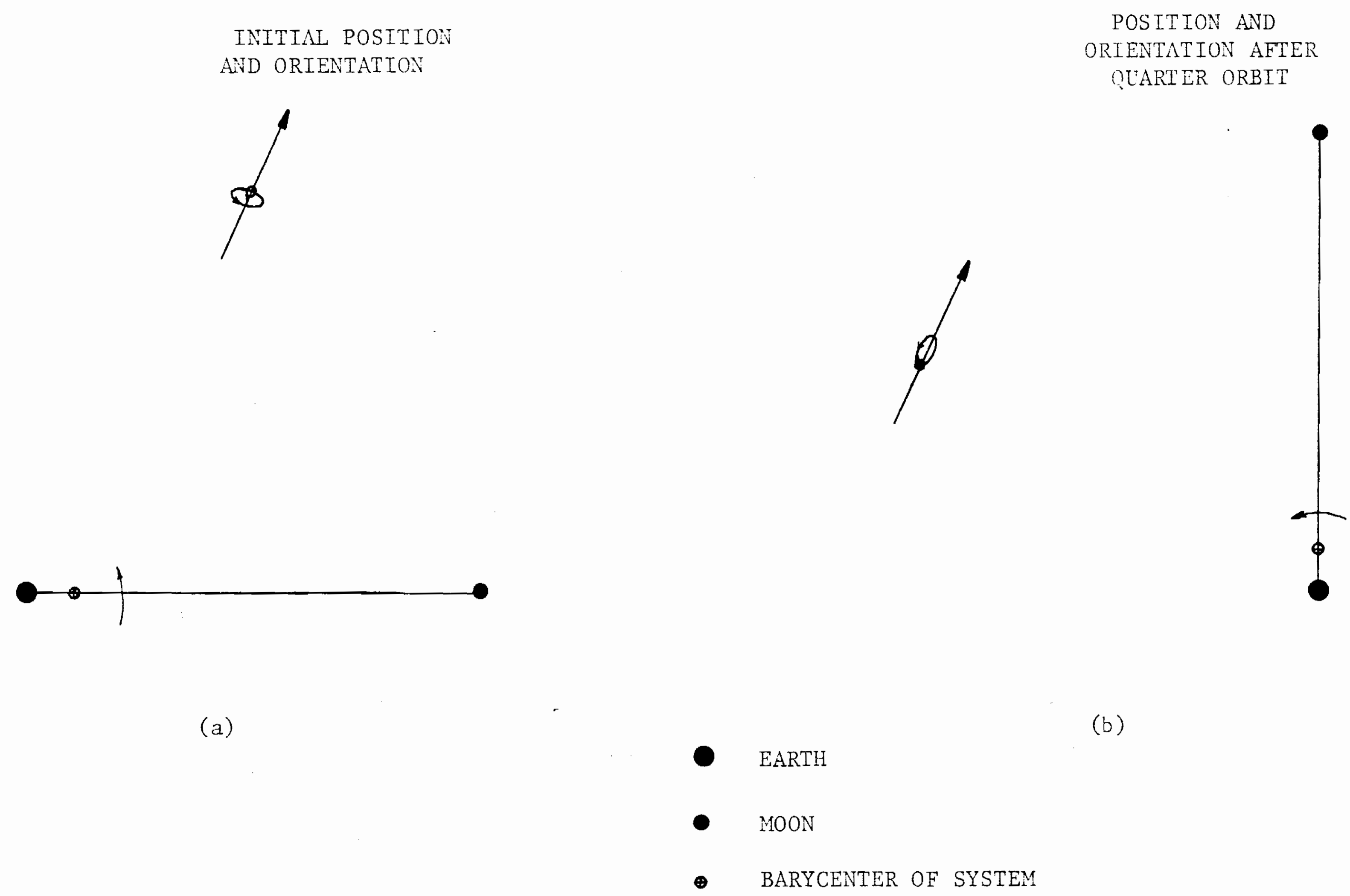

FIG. 3. ASTRONOMICAL STATION

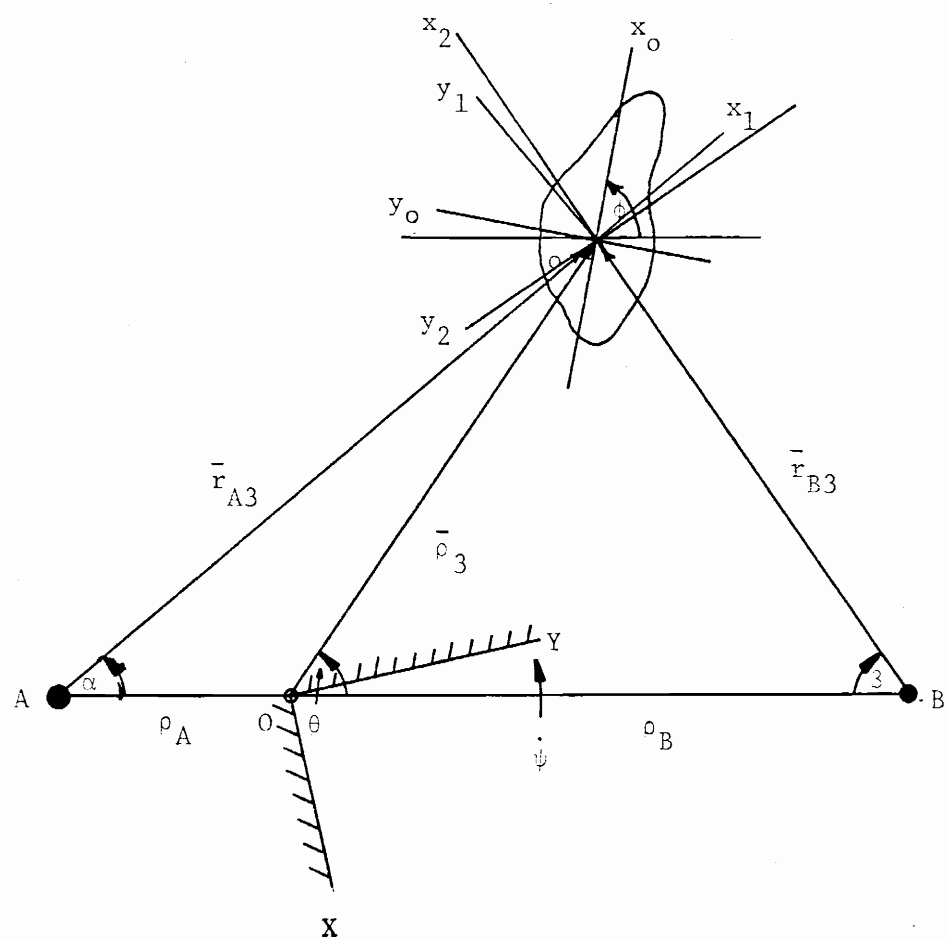

FIG. 4. GEOMETRY OF THE PROBLEM

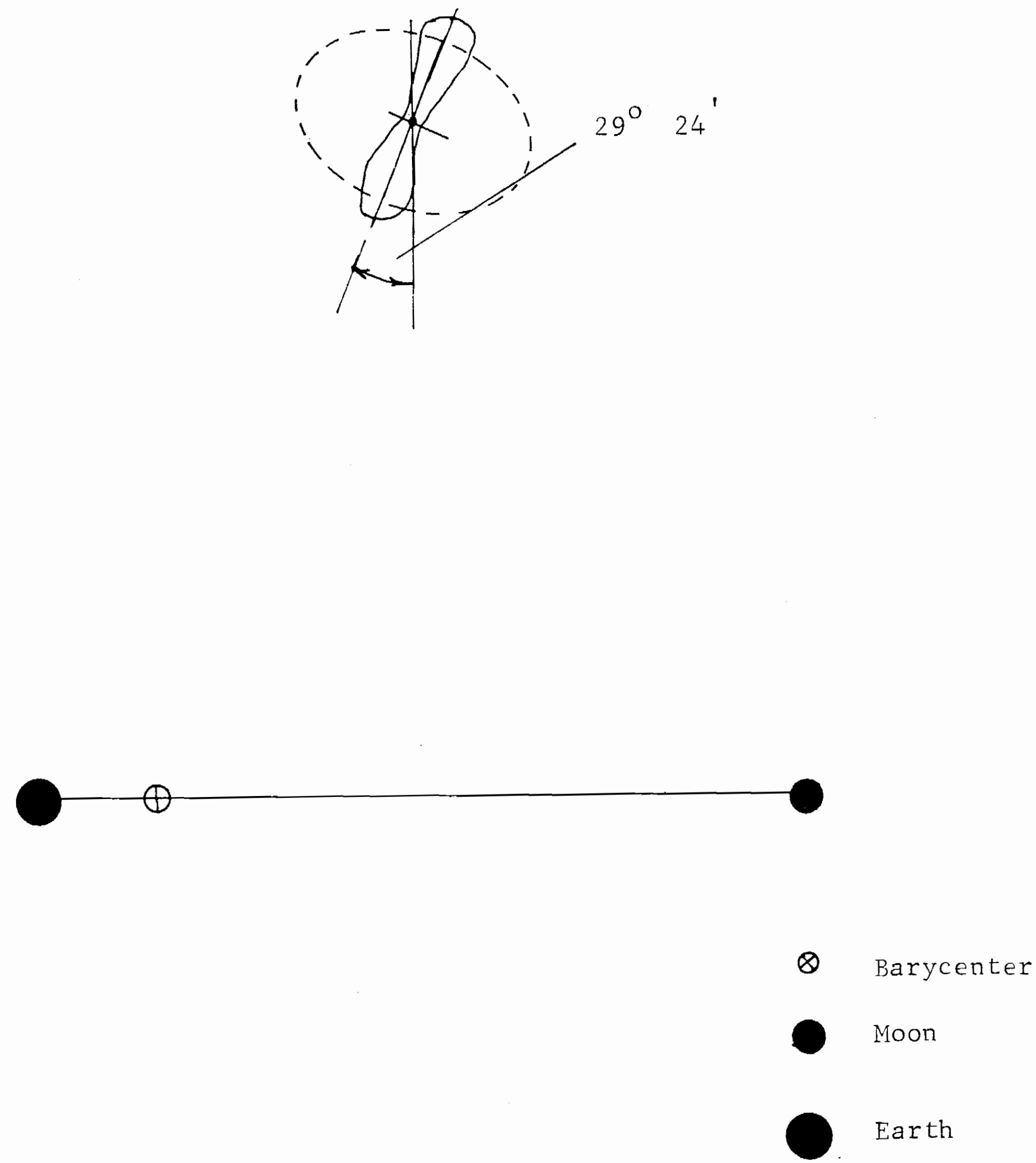

Fig. 5 EQUILIBRIUM ATTITUDE FOR AN ARBITRARILY SHAPED STATION

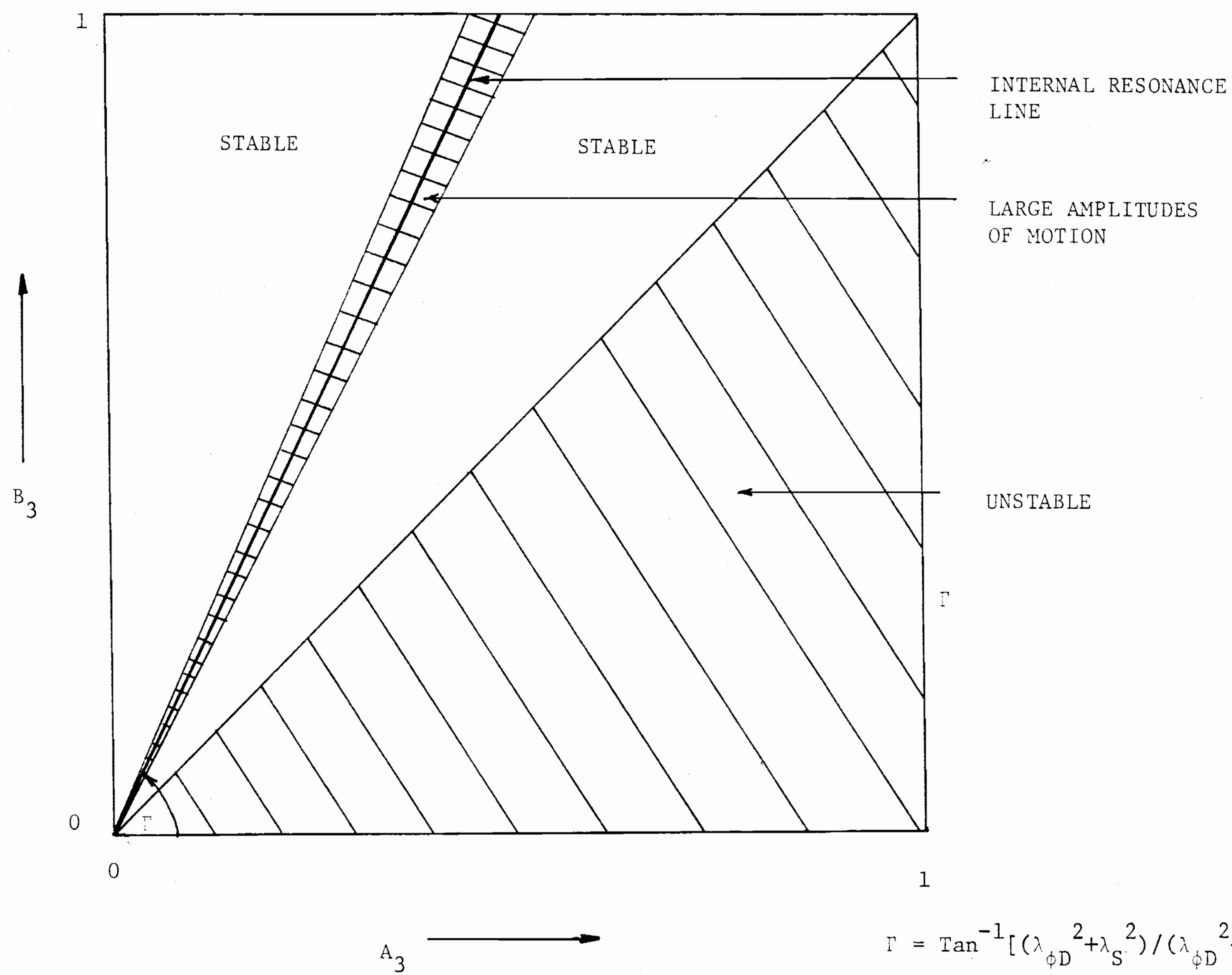

Fig. 6. STABILITY BOUNDARIES

$$\Gamma = \mathrm{Tan}^{-1}[(\lambda_{\phi D}{}^2+\lambda_S{}^2)/(\lambda_{\phi D}{}^2-\lambda_S{}^2)]$$

$$= 62°29'$$

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AN ARBITRARILY SHAPED SATELLITE IN THE RESTRICTED PROBLEM OF THREE BODIES

Ashok Kumar Saxena

Abstract

It is proposed to place a space station in a short period orbit about the L_4 triangular point in the earth moon system for the purpose of communication and possibly astronomical observations. Within limits of a linear analysis, an exhaustive and general approach to the dynamical problem is presented. The equations of motion for a finite sized body of arbitrary shape and internal mass distribution are derived. The phenomenon of resonance and stability are discussed. Stability boundaries for such a space station are established. Initial conditions and pointing errors are analyized.