FREQUENCY DEPENDENT ACOUSTIC TRANSMISSION IN NONUNIFORM MATERIALS

by

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(ABSTRACT)

A one dimensional normalized model for the frequency response of the acoustic power transmitted through nonuniform materials is developed. Using the ideal mixture model to relate acoustic velocity and impedance, this normalized model demonstrates that the power transmission characteristics are completely determined using only a composition profile and the parameters defining percent variation in acoustic velocity and impedance. For purposes of comparison, an analytically exact solution for exponential tapers is obtained.

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A special thanks goes to my family and friends who have sent love, support, and encouragement across the miles. I am especially grateful to my parents for helping me through the rough times and giving me the opportunity to attend college.

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1. Introduction

1.1 Overview

The purpose of this thesis was to develop a normalized one dimensional model for the frequency response of the acoustic power transmitted through nonuniform materials. Using the ideal mixture model which relates acoustic velocity and impedance, this normalized model demonstrates that the power transmission characteristics are completely determined using only a composition profile and the parameters defining percent variation in velocity and impedance. The process of normalization alleviates the necessity of specifying in the algorithm the actual design lengths and specific acoustic frequencies, thereby, generalizing the numerically generated results to a broader class of problems.

The format of the thesis is described in the following paragraphs. Chapter one is the introduction and includes an overview which highlights the main topics. The next section of chapter one is a noncomprehensive literature review. Realizing the difficulty of presenting an exhaustive review, section 1.2 focuses on those references which have been most supportive to this thesis.

Chapter two is a review of transmission line formalism as it applies to acoustics. This chapter has been included for the convenience of the reader and also serves as a reference for the more specialized discussions to follow. Section 2.1 contains basic equations and describes their physical significance. Section 2.2, which is a continuation of 2.1, presents results for the special case of quarterwave matching.

In chapter three, the analysis for the frequency dependent normalized model is developed using the ideal mixture model. Section 3.1 explains the ideal mixture model. A physical description of the multilayer acoustic transformer is given in section 3.2. In section 3.3, the initial normalization of frequency is developed. Section 3.4 covers the normalization for the acoustic impedance and wave velocity. Section 3.5 continues with the final normalization of the frequency. Section 3.6 demonstrates the normalization process for an exponential impedance profile. The integrated model for power transmission calculations is summarized as a flowchart in section 3.7.

In chapter four, analytical results are presented which will be used to demonstrate the validity of the normalized formalism. Section 4.1 includes analytical calculations for bandwidth measurements under quarterwave matching conditions. Section 4.2 discusses an equation for one dimensional plane wave propagation which is analytically exact even for nonuniform materials. An exact solution to this equation is derived for one dimensional acoustic transformers exhibiting an exponential impedance profile and a uniform velocity profile. This solution is presented in section 4.3. Finally, section 4.4 develops a frequency normalization for the solution to the case s=1.

Chapter five is the conclusion. The validity of the ideal mixture model is demonstrated in section 5.1 by comparing its predictions with experimental data. Section 5.2 contains results of the quarterwave bandwidth analysis. Similarly, section 5.3 contains results of the analysis of the exact exponential impedance profile solution. Lastly, section 5.4 contains final remarks.

Additional information is provided in the appendices. A superposition of plane waves analysis is found in Appendix A. Appendix B provides an analogy between electromagnetic fields and acoustics. Appendix C discusses the applicability of the propagation equation (found in Chapter 4) to acoustics and provides mathematical validity for the equation. The physical significance of this equation is demonstrated by three examples found in Appendix D. Finally, Appendix E discusses the limitations of the ideal mixture model.

1.2 Literature Review

Throughout this thesis, references to [1] by V. Ristic appear. In particular, Ristic was referred to in section 2.1 where the application of transmission line formalism to acoustics is discussed, in section 3.1 where the ideal mixture model is described, and in appendix B where an analogy between electromagnetics and acoustics is derived. To summarize, Ristic's book was used to provide basic equations and fundamental theories.

The ideal mixture model is discussed in section 3.1 where [2] is referenced. This paper presents measurements of the acoustic impedance and velocity for various types of tungsten plastic composites. Specifically, a plot of velocity and impedance versus percent volume of tungsten is given for a tungsten-vinyl composite. Results from the ideal mixture model are compared with this data in a later section.

R. E. Collin derives in [3] an equation for the bandwidth of a quarterwave transformer. We have shown that the results predicted by his formula agree with numerical calculations generated with the normalized formalism. This work can be found in sections 4.1 and 5.2.

N. Wax and L. R. Walker's paper [4] was used primarily to establish the validity of our formalism in the *non*quarterwave case. They presented an equation describing the one dimensional variation of the reflection coefficient on a nonuniform transmission line. The equation is not only applicable to transverse electromagnetic waves [6] but also applicable to one dimensional acoustic plane wave propagation in isotropic nonhomogeneous media. See sections 4.2 - 4.4, 5.3, and Appendices C and D for work involving their equation.

Using the transmission line model a bandwidth optimizing impedance profile for the acoustic transformer layer used between an acoustic transducer and water has been proposed by R. F. Vogel [5]. However, this modeling ignored the expected variation in acoustic wave velocity across the matching layer.

Of course the literature contains articles which deal with multilayer acoustic transformers that are not referenced in the body of this thesis. An example of such a paper is [6] by A. M. Khilla. Khilla discussed the use of a computer to design four different continuous transmission line tapers, one of which is the exponential. His analysis differs from ours in that he makes the assumption that the reflection coefficient is much smaller than one and thus is able to neglect a term that we include.

I. P. Dunn and W. A. Davern have also done research in the area of multilayer acoustic transformers. In their article, [7], they also used a microcomputer to calculate the normal acoustic impedance for a multilayer system. Their program called upon a database to provide the wave impedance and the propagation coefficient , both as functions of frequency, for each material. Having established such a database, their program could evaluate potential layer combinations much quicker than experimental analysis in an impedance tube. Dunn and Davern applied their program to the designing of a flat-walled anechoic lining. Their results showed close agreement between the computer generated reflection coefficients and those obtained by direct measurement in an impedance tube.

R. D. Corsaro and J. D. Klunder have tested the ideal mixture model in a laboratory and shown it to be accurate. In [8], they describe how they constructed and acoustically characterized a materials system that can be used to prepare spray-deposited thin coatings of materials with continuously variable acoustic properties. During the development of their model, Corsaro and Klunder show that their experimental data is in close agreement with predictions of the ideal mixture model.

2. Review of Transmission Line Formalism Applied to Acoustics

2.1 Basic Equations

It is possible to develop an analogy between transverse electromagnetic fields and acoustics [1]. The origin of this analogy is further clarified in Appendices B and C. The results of such a comparison are

2.1.1 $E \rightarrow -T$ where: E = electric field
T = stress

2.1.2 $H \rightarrow v_p$ where: H = magnetic field
v_p = particle velocity

2.1.3 $\mu \rightarrow p_m$ where: μ = permeability
p_m = mass density

2.1.4 $\varepsilon \rightarrow \frac{1}{c}$ where: ε = permitivitty
c = elastic stiffness constant

The intrinsic impedance also has a counterpart in acoustics,

2.1.5 $$\sqrt{\frac{\mu}{\varepsilon}} \rightarrow \sqrt{p_m c}\,.$$

The time dependent electromagnetic wave equation can be written as,

2.1.6 $$\frac{\partial^2}{\partial x^2}\begin{Bmatrix} E \\ H \end{Bmatrix} - \varepsilon\mu \frac{\partial^2}{\partial t^2}\begin{Bmatrix} E \\ H \end{Bmatrix} = 0$$

where x is the space coordinate. Applying the analogy above, the acoustic time dependent wave equation is,

2.1.7 $$\frac{\partial^2}{\partial x^2}\begin{Bmatrix} T \\ v_p \end{Bmatrix} - \frac{p_m}{c}\frac{\partial^2}{\partial t^2}\begin{Bmatrix} T \\ v_p \end{Bmatrix} = 0.$$

The acoustic wave velocity, v_w, is therefore given by,

2.1.8 $$v_w \equiv \sqrt{\frac{c}{p_m}}\,.$$

For harmonic sources of radian frequency ω, the corresponding acoustic phasors are defined using the engineering convention,

2.1.9a $\quad T(x,t) = \mathrm{Re}\left(T(x)e^{j\omega t}\right)$

and

2.1.9b $\quad v_p(x,t) = \mathrm{Re}\left(v_p(x)e^{j\omega t}\right).$

Solutions are then obtained from the one dimensional Hemholtz equation,

2.1.10 $\quad \dfrac{d^2}{dx^2}\begin{Bmatrix} T \\ v_p \end{Bmatrix} + \dfrac{p_m \omega^2}{c}\begin{Bmatrix} T \\ v_p \end{Bmatrix} = 0.$

Solutions to (2.1.10) take the form of a superposition of plane waves,

2.1.11 $\quad \begin{Bmatrix} T(x) \\ v_p(x) \end{Bmatrix} = \begin{Bmatrix} T^+ \\ v_p{}^+ \end{Bmatrix} e^{-j\beta x} + \begin{Bmatrix} T^- \\ v_p{}^- \end{Bmatrix} e^{+j\beta x}$

where the superscript +/- is indicative of waves propagating in the +/- direction. In order for (2.1.11) to be a solution to (2.1.10) it is required that the phase constant β satisfy,

2.1.12 $\quad \beta = \dfrac{\omega}{\sqrt{\dfrac{c}{p_m}}} \equiv \dfrac{\omega}{v_w} \equiv \dfrac{2\pi}{\lambda}$

where λ is the acoustic wavelength. The acoustic wave propagation can now be treated using electrical transmission line theory by noting the analogy that exists between transverse electromagnetic fields and transmission lines [9]. Specifically, the transmission line voltage (V) and transmission line current (I) are related to the corresponding electromagnetic fields as,

2.1.13 $\quad E \rightarrow V$

2.1.14 $\quad H \rightarrow I$

thus permitting the direct application of results taken from transmission line theory. These results are derived in most undergraduate engineering texts on electromagnetic theory. See for example [10].

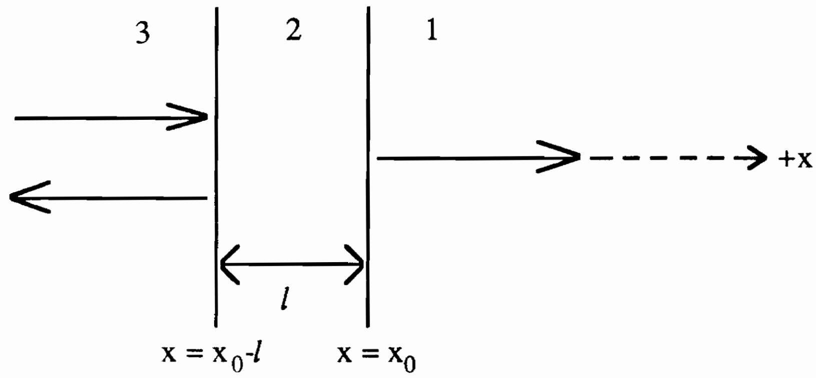


Figure 1 - Acoustic Waves at an Interface

For purposes of illustration, Figure 1 shows a slab of material of length l sandwiched between two semi-infinite volumes. The slab is used for matching purposes and is called an acoustic transformer. In general, all three regions are acoustically distinct. The boundary conditions require that stress and velocity are continuous across boundaries [1]. The intrinsic impedance of material j (j = 1, 2, 3) can be defined as,

2.1.15 $$Z_0^{(j)} = \sqrt{p_m c},$$

where the density and stiffness constant are evaluated in the layer j. The acoustic impedance, looking in the +x direction, is defined as,

2.1.16 $$Z = \frac{-T(x)}{v_p(x)}$$

which is strictly continuous even across boundaries. The reflection coefficient can be defined for points within region j according to,

2.1.17 $$r(x) = \frac{Z(x) - Z_0^{(j)}}{Z(x) + Z_0^{(j)}}.$$

As a special case of (2.1.17) the reflection coefficient evaluated at a point just to the left of $x = x_0$ takes the form,

2.1.18 $$r(x_0) = \frac{Z_0^{(1)} - Z_0^{(2)}}{Z_0^{(1)} + Z_0^{(2)}}$$

where the acoustic impedance for all points to the right of $x = x_0$ reduces to the intrinsic impedance of the material $Z_0^{(1)}$. The reflection coefficient for an arbitrary point, x, within the slab [x: $x_0 - l \leq x \leq x_0$] is related to the reflection coefficient at x_0 according to,

2.1.19 $\quad r(x_0 + x) = r(x_0)\, e^{-j2\beta(x - x_0)}.$

Consistent with transmission line formalism the acoustic impedance for point x within region j can also be expressed as,

2.1.20 $\quad Z(x) = Z_0^{(j)}\left(\frac{1 + r(x)}{1 - r(x)}\right)$

The power transmission coefficient, Γ, for arbitrary x is simply expressed as,

2.1.21 $\quad \Gamma(x) = 1 - |r(x)|^2.$

2.2 Quarterwave Matching Conditions

In order for an acoustic transformer to be considered quarterwave, two conditions must be met. These conditions are obtained by setting the impedance (2.1.20) evaluated at $x=x_0-l$ equal to the characteristic impedance of region 3. Thus,

2.2.1 $\quad Z_0^{(3)} = Z(x_0 - l) = Z_0^{(2)}\left(\frac{1 + r(x_0 - l)}{1 - r(x_0 - l)}\right)$

This guarantees that the reflection coefficient in region three is zero. The two conditions which result for this process are,

2.2.2a $\quad l = \frac{m\lambda^{(2)}}{4}$ where: $\lambda^{(2)}$ is the acoustic wavelength in region 2 and m is an odd integer

and

2.2.2b $\quad Z_0^{(2)} = \sqrt{Z_0^{(3)} Z_0^{(1)}}$

as defined in Figure 1. This last condition (2.2.2b) can be expressed in terms of a condition on single interface reflection coefficients defined below.

2.2.3a $$r_1 = \frac{\sqrt{Z_0^{(3)} Z_0^{(1)}} - Z_0^{(3)}}{\sqrt{Z_0^{(3)} Z_0^{(1)}} + Z_0^{(3)}}$$

and

2.2.3b $$r_2 = \frac{Z_0^{(1)} - \sqrt{Z_0^{(3)} Z_0^{(1)}}}{\sqrt{Z_0^{(3)} Z_0^{(1)}} + Z_0^{(1)}}.$$

The physical significance of which is shown in Figure 2.

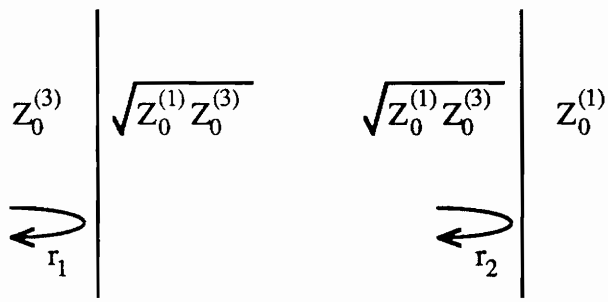


Figure 2 - Reflection Coefficients at an Interface

It is easily shown that condition (2.2.2b) is equivalent to

2.2.4 $r_1 = r_2$.

In Appendix A this result is derived using the method of plane wave superposition for a uniform impedance acoustic transformer. An impedance profile for bandwidth optimization with an acoustic transformer has been proposed in the current literature and for reference it is shown in Figure 3.

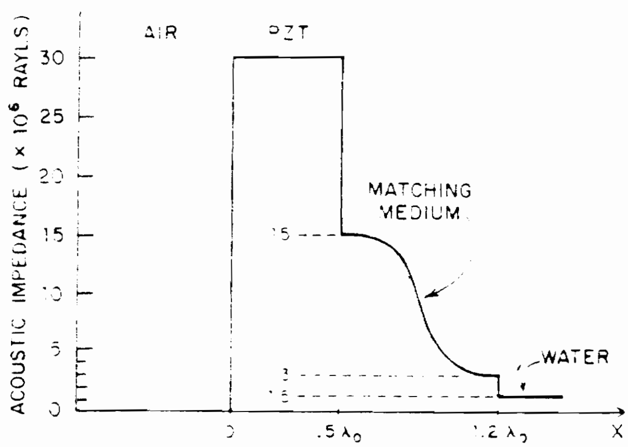


Figure 3 - Impedance vs. distance, d, for an Airbacked Transducer. After [5].

Calculating the single interface reflection coefficients for this design shows that equation (2.2.4) is in good agreement with $r \equiv \frac{1}{3}$. As shown in Appendix A, condition (2.2.4) is valid as long as reflections internal to the acoustic transformer can be ignored. Such an assumption would be applicable if the impedance profile were slowly varying between interfaces at x=0 and x=-l.

3. A Normalized Formalism for Acoustic Wave Calculations

Computer simulation is a commonly used tool for analyzing nonuniform transmission lines [5,6,7]. Investigations such as these have often been posed in a partially normalized form. In several cases the frequency has been normalized by a fixed resonant frequency determined from physical considerations unrelated to the transforming layer [5,11]. The purpose of this chapter is to develop a fully normalized formalism which under the assumption of ideal mixing requires the absolute minimum acoustic information in the computer implementation of the algorithm. Such an approach is intended to provide a broad scope of applicability for data generated from the formalism. In addition, it also lends insight into the fundamentally independent combination of acoustic parameters effecting acoustic power transmission.

Despite the positive points for the formalism discussed above there are several disadvantages which deserve mentioning. First, by assuming the applicability of the ideal mixture model, generality has been sacrificed. A model providing the relationship between velocity and impedance which covers nonaggregate mixing has, to the best of the author's knowledge, not appeared in the literature. Therefore, the alternative to employing the ideal mixture model is to assume the availability of measured data for acoustic wave velocity and impedance for the range of composites being tested. Although this approach has not been pursued directly in this thesis, the potential exists for the development of a normalized formalism under these constraints. Second, as with most normalization processes the resultant parameters are dimensionless. Because of this, some sacrifice in the convenience of physical interpretation is inevitable.

3.1 Ideal Mixture Model

Because the ideal mixture model is an integral part of the analytical development to follow, a brief derivation [8] of the ideal mixture equations has been reproduced here. The ideal mixture model is based on the initial assumption that in a composite, the volumes of the individual components are additive. In other words, when the components are mixed, each one retains its original volume; there is no shrinkage or expansion. Therefore, the total volume of the mixture, τ_m, can be expressed as,

3.1.1 $$\tau_m = \sum_i \tau_i$$

where τ_i is the volume of the i^{th} component. The sum is performed over the constituents present in the mixture. The fractional volume of the i^{th} component, V_i, is defined as,

3.1.2 $$V_i = \frac{\tau_i}{\tau_m},$$

where $\sum_i V_i = 1$.

Similarly, the total mass, w_m, can be defined as,

3.1.3 $$w_m = \sum_i w_i,$$

where w_i is the mass of the i^{th} component. The mass fraction of the i^{th} component, W_i, is defined as,

3.1.4 $$W_i = \frac{w_i}{w_m},$$

where $\sum_i W_i = 1$. Let p_m be the density of the final mix. Then, it follows that,

3.1.5 $$p_m = \frac{w_m}{\tau_m}.$$

Similarly, the density of the i^{th} component is given by,

3.1.6 $$p_i = \frac{w_i}{\tau_i}.$$

Substituting equation (3.1.3) into equation (3.1.5), with reference to (3.1.6) leads to, 3.1.7 $\quad p = \sum_i \frac{w_i}{\tau_m} = \sum_i \frac{p_i \tau_i}{\tau_m}$

It also follows from substituting equation (3.1.2) into equation (3.1.7) that,

3.1.8 $\quad p_m = \sum_i p_i V_i,$

which is the first of the ideal mixture equations.

The pressure derivative taken at constant temperature is $\frac{\partial}{\partial p}$. The composite compressibility of the mixture, k_m, can be defined as,

3.1.9 $\quad k_m \equiv - \frac{1}{\tau_m} \frac{\partial}{\partial p} \tau_m.$

Similarly, the compressibility of the i^{th} component, k_i, can be defined as,

3.1.10 $\quad k_i = - \frac{1}{\tau_i} \frac{\partial}{\partial p} \tau_i.$

Substituting equation 3.1.1 into equation 3.1.9 leads to,

3.1.11 $\quad k_m = - \frac{1}{\tau_m} \frac{\partial}{\partial p} \sum_i \tau_i,$

which after multiplying by $\frac{\tau_i}{\tau_i}$ shows that,

3.1.12 $\quad k_m = \sum_i \frac{\tau_i}{\tau_m} \frac{1}{\tau_i} \left(\frac{\partial}{\partial p} \tau_i \right)$

Substituting equations 3.1.2 and 3.1.10 into equation 3.1.12 produces,

3.1.13 $\quad k_m = \sum_i V_i k_i,$

which is the second of the ideal mixture equations. From equations 3.1.8 and 3.1.13, the composite density and compressibility can be calculated. Now that these quantities are known, the usual equations for acoustic impedance (Z_0) and velocity (v) are determined from [1].

3.1.14 $\quad Z_0 = p_m v_m$

3.1.15 $\quad v_m = \frac{1}{\sqrt{p_m k_m}}$

The above equations (3.1.14) and (3.1.15) have, subject only to assumptions stated (i.e. ideal mixing), been proposed as part of the ideal mixture model [1],[8]. However, comparison of equations (2.1.8) and (2.1.15) to equations (3.1.15) and (3.1.14), respectively, indicates that unless the stiffness constant (c) and the incompressibility $\left(\frac{1}{k}\right)$ are equal an additional source of error has been introduced in the ideal mixture model. An examination of the representation of these parameters in terms of the stiffness Lamé constants suggests that there is an inconsistency. These relations are

$$\frac{1}{k} \equiv \lambda + \frac{2}{3}\mu$$

and

$$c = \lambda + 2\mu, \text{ (longitudinal wave)}$$

where λ is the first Lamé constant and μ is the shear modulus [1]. In the application of this model in this thesis, the following assumption is being made regarding the behavior of the mixture stiffness constant in terms of fractional components.

3.1.16 $$\frac{1}{c_m} = \sum_i \frac{V_i}{c_i}$$

Note that this corresponds to replacing k by $\left(\frac{1}{c}\right)$ in equation (3.1.13). It then follows that the corresponding mixture equation (3.1.15) changes to,

3.1.17 $$v_m = \sqrt{\frac{c_m}{p_m}}.$$

It can be concluded that the error in the model, as applied here, is related to the assumption (3.1.16) rather than to the difference between incompressibility and stiffness. In Appendix E, this assumption is examined analytically. The analysis shows that for tungsten-vinyl composites the error introduced with assumption (3.1.16) can be as high as 30%.

In order to avoid ambiguity in notation the composite density and compressibility will henceforth be denoted without the subscript m. They are represented here assuming the mixture has only two constituents.

3.1.18 $p = \sum_{i=1}^{2} p_i V_i$

and

3.1.19 $\frac{1}{c} = \sum_{i=1}^{2} \frac{1}{c_i} V_i.$

The acoustic wave velocity and intrinsic impedance are defined respectively as,

3.1.20 $v = \sqrt{\frac{c}{p}}$

and

3.1.21 $Z_0 = pv.$

Similarly, the acoustic wave velocity and intrinsic impedance of the undiluted components satisfy respectively,

3.1.22 $v_i = \sqrt{\frac{c_i}{p_i}} \quad i = 1, 2$

and

3.1.23 $(Z_0)_i = p_i v_i \quad i = 1, 2.$

Thus, the ideal mixture model defines the acoustic impedance and velocity of composites. Note that this model is not valid if there is significant mixing on the molecular level. This restriction usually applies to liquids and solutions. Aggregates, however, can be good candidates for the ideal mixture model. An analysis of the error introduced with assumption (3.1.16) appears in appendix E. A comparison of actual measured data for tungsten-vinyl composites with a theoretical prediction based on the ideal mixture model is presented in section 5.1.

3.2 Physical Description of the Multilayer Acoustic Transformer

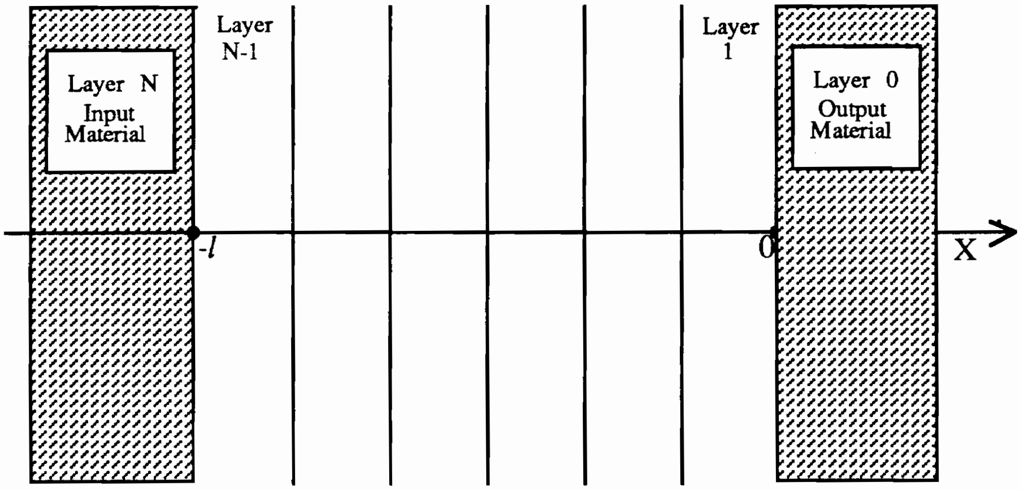


Figure 4 - Multilayer Transformer Model

Figure 4 represents a multilayer acoustic transformer of length l. Acoustic power is assumed to be incident from the left. There are shown two semi-infinite acoustically distinct slabs characterized by the labels "input" on the left and "output" on the right. Including the outer regions there exist N+1 homogeneous regions. Layers 1 through N-1 are composites formed from the outer regions. The conventions defined below will be followed in the subsequent analysis.

1 Subscripts are indicative of quantities related to the materials used in the outer regions. Here the subscript 1 refers to the rightmost layer (output) while subscript 2 refers to the leftmost layer (input).

2 Superscripts identify a layer number.

It follows from the stated conventions that:

$V_1 \equiv V^{(0)}$

and

$V_2 \equiv V^{(N)}$.

The ideal mixture model can be applied to the multilayer system to predict the composition dependent acoustic wave velocity and impedance. By application of equations (3.1.18 - 3.1.23) the layer dependent acoustic velocity and impedance can be shown to satisfy, respectively,

3.2.1 $$v^{(j)} = \frac{1}{\sqrt{\sum_{i=1}^{2} V_i^{(j)} \rho_i}} \frac{1}{\sqrt{\sum_{i=1}^{2} \frac{V_i^{(j)}}{\rho_i v_i^2}}}$$ where: $j = 0,1, \ldots\ldots N$,

and

3.2.2 $$Z_0^{(j)} = \frac{\sqrt{\sum_{i=1}^{2} V_i^{(j)} \rho_i}}{\sqrt{\sum_{i=1}^{2} \frac{V_i^{(j)}}{\rho_i v_i^2}}}$$ where: $j = 0,1, \ldots\ldots N$.

3.3 Initial Frequency Normalization

The problem of calculating the power transmission coefficient as a function of frequency is to be posed in a normalized form which will be applicable to "real" problems once the normalization transformation is reversed. We begin with an initial normalization of the frequency.

From Figure 4, the length of layer j, $l^{(j)}$, is defined as, 3.3.1 $\quad -l^{(j)} \equiv x^{(j)} - x^{(j-1)}$,

where x is the space coordinate. Also from Figure 4, the composite length, l, is defined as,

3.3.2 $\quad l \equiv \sum_{i=1}^{N-1} l^{(j)}.$

The normalized length of layer j, $\overline{l}^{(j)}$, is defined by,

3.3.3 $\quad \overline{l}^{(j)} \equiv \frac{l^{(j)}}{l}.$

Similarly the normalized space coordinate of layer j, $\overline{x}^{(j)}$, is defined by,

3.3.4 $\quad \overline{x}^{(j)} \equiv \frac{x_i}{l}.$

Substituting equations (3.3.3) and (3.3.4) into equation (3.3.2) leads to,

3.3.5 $\quad \overline{x}^{(j)} - \overline{x}^{(j-1)} = -\overline{l}^{(j)}.$

The propagation constant of layer j, $\beta^{(j)}$, is defined by,

3.3.6 $\quad \beta^{(j)} \equiv \frac{2\pi}{\lambda^{(j)}} = \frac{2\pi f}{v^{(j)}} = \frac{\omega}{v^{(j)}},$

where f is the Hertz frequency of the wave. Let r(x) be the reflection coefficient at x, then from (2.1.19),

3.3.7 $\quad r(x^{(j)}) = r^{(j),(j-1)}\, e^{-(2j\beta^{(j)}(x^{(j-1)} - x))},$

where $r(x^{(j)})$ is the reflection coefficient at the interface of the layers j, j-1. Multiplying and dividing the argument of the exponent in (3.3.7) by l and applying equation (3.3.4) leads to,

3.3.8 $\quad r(x^{(j)}) = r^{(j),(j-1)}\, e^{+(2j\beta^{(j)} l(\overline{x} - \overline{x}^{(j-1)}))}.$

Evaluating r(x) at $x = \overline{x}^{(j)}$ and applying equation (3.3.5) shows that,

3.3.9 $\quad r(\overline{x}^{(j)}) = r^{(j),(j-1)}\, e^{-(2j\beta^{(j)} l\,(\overline{l}^{(j)}))}.$

The quarterwave center Hertz frequency is denoted by f_c. Similarly, the quarterwave center radian frequency is denoted by ω_c. Motivated by the relation to the allowed quarterwave transformer frequencies, (2.2.2a), the center wavelength, λ_c, is defined by, 3.3.10 $\lambda_c \equiv 4l = \frac{v_g}{f_c} = \frac{2\pi v_g}{2\pi f_c} = \frac{2\pi v_g}{\omega_c}$,

where $v_g \equiv \sqrt{v_1 v_2}$ and the subscript g denotes the geometric mean of the input and output materials. Note that the quantity ω_c is being defined in the normalization process as a reference frequency for a multilayer transformer. In addition, ω_c does not necessarily correspond to a frequency for peak power transmission for the multilayer system. Nonetheless, it serves reasonably well as a physically justifiable reference. Isolating l in equation (3.3.10) shows that,

3.3.11 $l = \frac{\pi v_g}{2\omega_c}$.

Multiplying equation (3.3.6) by l and substituting in equation (3.3.11) results in,

3.3.12 $\beta^{(j)} l = \frac{\pi v_g}{2\omega_c} \frac{\omega}{v^{(j)}}$.

The normalized acoustic velocity of layer j, $\bar{v}^{(j)}$, and the normalized angular frequency, ϖ, are defined respectively by,

3.3.13 $\bar{v}^{(j)} \equiv \frac{v^{(j)}}{v_g}$

and

3.3.14 $\varpi \equiv \frac{\omega}{\omega_c}$.

Equation (3.3.14) is plugged into equation (3.3.12) yielding,

3.3.15 $\beta^{(j)} l = \frac{\pi}{2} \frac{\varpi}{\bar{v}^{(j)}}$.

Similarly, equation (3.3.15) is plugged into equation (3.3.9) giving,

3.3.16 $r(\bar{x}^{(j)}) = r^{(j),(j-1)} \exp\left(- \pi j \frac{\varpi}{\bar{v}^{(j)}} \bar{l}^{(j)} \right)$

This result indicates that it is possible to define the reflection coefficient at the normalized space coordinate of layer j, $\bar{x}^{(j)}$, in terms of the normalized velocity (3.3.13) and the normalized frequency (3.3.14).

3.4 Normalization for Impedance and Velocity

The ideal mixture model will be used to relate the characteristic impedance and velocity profiles. These quantities will then be normalized.

Dividing both sides of equation (3.2.1) by v_g yields,

$$3.4.1 \quad \frac{v^{(j)}}{v_g} = \frac{1}{\sqrt{v_g^2}} \frac{1}{\sqrt{\sum_{i=1}^{2} v_i^{(j)} p_i}} \frac{1}{\sqrt{\sum_{i=1}^{2} \frac{v_i^{(j)}}{p_i v_i^2}}}.$$

Applying equation (3.3.13) and simplifying leads to,

$$3.4.2 \quad \bar{v}^{(j)} = \frac{1}{\sqrt{\sum_{i=1}^{2} v_i^{(j)} p_i}} \frac{1}{\sqrt{\sum_{i=1}^{2} \frac{v_i^{(j)}}{p_i \bar{v}_i^2}}}.$$

The geometrical mean of the composite density, p_g, and the normalized density of material i, $\bar{p}_i$, are defined respectively by,

$$3.4.3 \quad p_g \equiv \sqrt{p_1 p_2}$$

and

$$3.4.4 \quad \bar{p}_i \equiv \frac{p_i}{p_g}.$$

Multiplying the right hand side of equation (3.4.2) by $\frac{\sqrt{p_g}}{\sqrt{p_g}}$ yields,

$$3.4.5 \quad \bar{v}^{(j)} = \frac{1}{\sqrt{\sum_{i=1}^{2} v_i^{(j)} \bar{p}_i}} \frac{1}{\sqrt{\sum_{i=1}^{2} \frac{v_i^{(j)}}{\bar{p}_i \bar{v}_i^2}}}.$$

The normalized intrinsic impedance of material i, $(Z_0)_i$, and the normalized impedance of material i, $(Z)_i$, are defined respectively as, 3.4.6a $$(\overline{Z_0})_i \equiv \frac{(Z_0)_i}{\sqrt{(Z_0)_1 (Z_0)_2}} \equiv \frac{(Z_0)_i}{(Z_0)_g}$$

and

3.4.6b $$(\overline{Z})_i \equiv \frac{(Z)_i}{\sqrt{(Z_0)_1 (Z_0)_2}} \equiv \frac{(Z)_i}{(Z_0)_g}.$$

Substituting equation (3.1.21) into equation (3.4.6) leads to,

3.4.7 $$(\overline{Z_0})_i = \frac{p_i v_i}{\sqrt{p_1 v_1 p_2 v_2}} = \frac{p_i}{\sqrt{p_1 p_2}} \frac{v_i}{\sqrt{v_1 v_2}} = \overline{p}_i \overline{v}_i.$$

Isolating $\overline{p}_i$ shows that,

3.4.8 $$\overline{p}_i = \frac{(\overline{Z_0})_i}{\overline{v}_i}.$$

Plugging equation (3.4.8) into equation (3.4.5) yields,

3.4.9 $$\overline{v}^{(j)} = \frac{1}{\sqrt{\sum_{i=1}^{2} \frac{V_i^{(j)} (\overline{Z_0})_i}{\overline{v}_i}} \sqrt{\sum_{i=1}^{2} \frac{V_i^{(j)}}{(\overline{Z_0})_i \overline{v}_i}}}.$$

In this equation only the fractional volumes $V_i^{(j)}$ are layer dependent. The other variables are associated with either the input or the output and are assumed to be known. Continuing the normalization process, the normalized characteristic impedance of layer j, $(\overline{Z_0})^{(j)}$, is defined by,

3.4.10 $$(\overline{Z_0})^{(j)} \equiv \frac{(Z_0)^{(j)}}{\sqrt{(Z_0)_1 (Z_0)_2}} \equiv \frac{(Z_0)^{(j)}}{(Z_0)_g} = \frac{(Z_0)^{(j)}}{\sqrt{p_g v_g}}.$$

Plugging equation (3.1.16) into equation (3.1.19) and using normalization and geometric mean definitions leads to,

3.4.11 $$(Z_0)^{(j)} = \sum_{i=1}^{2} V_i^{(j)} p_i v^{(j)} = \sum_{i=1}^{2} V_i^{(j)} \overline{p}_i (p_g v_g) \overline{v}^{(j)}.$$

Substituting equations (3.4.10) into equation (3.4.11) yields,

$$3.4.12 \quad (Z_0)^{(j)} = \bar{v}^{(j)} \sum_{i=1}^{2} V_i^{(j)} \bar{p}_i .$$

Plugging equation (3.4.8) into equation (3.4.12) shows that,

$$3.4.13 \quad (Z_0)^{(j)} = \bar{v}^{(j)} \sum_{i=1}^{2} V_i^{(j)} \frac{(Z_0)_i}{\bar{v}_i} .$$

Now, if we plug equation (3.4.9) into (3.4.13), once again the only unknown layer dependent variable is the fractional volume.

Observations

1) A profile is specified by defining $V_1^{(j)}$ or $V_2^{(j)}$ for j = 1,2,......N-1. This follows from the fact that the sum of fractional volumes adds to 1 in every layer. Note that $V_1^{(0)} \equiv 1$ and $V_1^{(N)} \equiv 0$, by definition of the formalism.

2) The additional information that is required in equations (3.4.9) and (3.4.13) is related to the input and output materials. Specifically, the variables $(Z_0)_1$, $(Z_0)_2$, $\bar{v}_1$, and $\bar{v}_2$ will completely define these equations.

3) Equations (3.3.13) and (3.4.10) are applicable to materials not necessarily satisfying the ideal mixture model. These equations would serve as a basis of a normalization independent of the ideal mixture model.

In the interest of reducing the required number of variables, an extended normalization will be posed in terms of parameters defining a percent variation in impedance and velocity. The ratio of acoustic impedances, q, and the ratio of acoustic velocities, s, are defined respectively as,

$$3.4.14 \quad q \equiv \frac{(Z_0)_2}{(Z_0)_1}$$

and

3.4.15 $\quad s \equiv \frac{\bar{v}_2}{\bar{v}_1}$,

thereby reducing the required number of variables by 2. In the extended normalization process all variables are referenced to the output or right hand side as shown in Figure 4. The extended normalized intrinsic impedance, $\gamma_0^{(j)}$, and the extended normalized impedance, $\gamma(\bar{x}^{(j)})$, are defined respectively as,

3.4.16 $\quad \gamma_0^{(j)} \equiv \frac{(Z_0)^{(j)}}{(Z_1)_0} = \frac{(Z_0)^{(j)}}{(Z_0)^{(0)}}$ (Note: $\gamma_0^{(0)} = 1,\ \gamma_0^{(N)} = q$)

and

3.4.17 $\quad \gamma(\bar{x}) \equiv \frac{Z(\bar{x})}{(Z_0)_1}$.

In order to relate the extended normalized impedance to the reflection coefficient, equation (2.1.20) is repeated here. Note that the reflection coefficient is given by (3.3.16) where it is defined in terms of normalized variables.

3.4.18 $\quad Z(\bar{x})^{(j)} = Z_0{}^{(j)}\frac{1 + r(\bar{x})^{(j)}}{1 - r(\bar{x})^{(j)}}$.

Dividing both sides of equation (3.4.18) by $(Z_0)_g$ and referring to (3.4.6) leads to,

3.4.19 $\quad Z(\bar{x})^{(j)} = Z_0{}^{(j)}\frac{1 + r(\bar{x})^{(j)}}{1 - r(\bar{x})^{(j)}}$.

Similarly, dividing both sides of equation (3.4.19) by $(Z_0)_1$ and referring to equations (3.4.16) and (3.4.17) leads to,

3.4.20 $\quad \gamma(\bar{x}^{(j)}) = \gamma_0^{(j)}\frac{1 + r(\bar{x})^{(j)}}{1 - r(\bar{x})^{(j)}}$.

In the calculations resulting from the normalized formalism, equation (3.4.20) will be evaluated at the left hand side of the layer j. In order to complete the calculation, the reflection coefficient, r, has to be evaluated at the right hand side of layer j+1. Following (2.1.17), 3.4.21 $$r^{(j)} = \frac{Z^{(j-1)} - Z_0^{(j)}}{Z^{(j-1)} + Z_0^{(j)}} .$$

Multiplying the right hand side of equation (3.4.21) by $\frac{Z_g}{Z_g}\frac{(Z_0)_1}{(Z_0)_1}$ and using equation (3.4.17) yields,

3.4.22 $$r^{(j)} = \frac{\gamma(\bar{x}^{(j)}) - \gamma_0^{(j)}}{\gamma(\bar{x}^{(j)}) + \gamma_0^{(j)}} .$$

The reflection coefficient is now expressed in terms of the extended normalized impedances.

The extended normalized velocity of layer j, $\alpha^{(j)}$, is defined as,

3.4.23 $$\alpha^{(j)} \equiv \frac{\bar{v}^{(j)}}{\bar{v}_1} \equiv \frac{v^{(j)}}{v_1} .$$

Substituting equation (3.4.13) into equation (3.4.16) leads to,

3.4.24 $$\gamma_0^{(j)} = \frac{(Z_0)^{(j)}}{(Z_0)_1} = \bar{v}^{(j)} \sum_{i=1}^{2} V_i^{(j)} \frac{(Z_0)_i}{(Z_0)_1} \frac{1}{\bar{v}_i} .$$

By expanding the summation and using equation (3.4.14), it can be shown that,

3.4.25 $$\gamma_0^{(j)} = \bar{v}^{(j)} \left(V_1^{(j)} \frac{1}{\bar{v}_1} + V_2^{(j)} \frac{q}{\bar{v}_2} \right)$$

Factoring out $\bar{v}_1$ and using equations (3.4.23) and (3.4.15) yields,

3.4.26 $$\gamma_0^{(j)} = \alpha^{(j)} \left(V_1^{(j)} + V_2^{(j)} \frac{q}{s} \right)$$

Substituting equation (3.4.9) into equation (3.4.26) gives,

3.4.27 $$\alpha^{(j)} \equiv \frac{1}{\sqrt{\bar{v}_1^2}} \frac{1}{\sqrt{\displaystyle\sum_{i=1}^{2} \frac{V_i^{(j)} (Z_0)_i}{\bar{v}_i}}} \frac{1}{\sqrt{\displaystyle\sum_{i=1}^{2} \frac{V_i^{(j)}}{(Z_0)_i \bar{v}_i}}} .$$

Expanding the sums, substituting in equations (3.4.14) and (3.4.15), and multiplying by $\frac{(Z_0)_1}{(Z,^-)_{01}}$ yields,

3.4.28 $$\alpha^{(j)} = \frac{1}{\sqrt{V_1^{(j)} + V_2^{(j)}\frac{q}{s}}}\,\frac{1}{\sqrt{V_1^{(j)} + \frac{V_2^{(j)}}{sq}}}.$$

Substituting equation (3.4.28) into equation (3.4.26) and simplifying leads to,

3.4.29 $$\gamma_0^{(j)} = \frac{\sqrt{V_1^{(j)} + V_2^{(j)}\frac{q}{s}}}{\sqrt{V_1^{(j)} + \frac{V_2^{(j)}}{sq}}}.$$

Using $V_1^{(j)} + V_2^{(j)} \equiv 1$, squaring both sides and cross multiplying produces,

3.4.30 $$\left(\gamma_0^{(j)}\right)^2\left(V_1^{(j)}\left(1 - \frac{1}{sq}\right) + \frac{1}{sq}\right) = \left(V_1^{(j)}\left(1 - \frac{q}{s}\right) + \frac{q}{s}\right).$$

Isolating $V_1^{(j)}$ and multiplying by $\frac{-1}{-1}$ gives,

3.4.31 $$V_1^{(j)} = \frac{\frac{(\gamma_0^{(j)})^2}{qs} - \frac{q}{s}}{\left(1 - \frac{q}{s}\right) - (\gamma_0^{(j)})^2\left(1 - \frac{1}{qs}\right)}.$$

By factoring out $\frac{1}{qs}$ and regrouping the denominator, it can be shown that,

3.4.32 $$V_1^{(j)} = \frac{\frac{1}{qs}\left((\gamma_0^{(j)})^2 - q^2\right)}{\left(1 - (\gamma_0^{(j)})^2\right) + \frac{1}{qs}\left((\gamma_0^{(j)})^2 - q^2\right)}.$$

In the special case that s = q = 1, referring to equation (3.4.29) leads to,

3.4.33 $$(\gamma_0^{(j)})^2 \equiv 1.$$

This demonstrates the fact that in the case that the input and output materials are acoustically equivalent the fractional volume plays no role in the acoustic description of the interface.

3.5 Final Frequency Normalization

Having established an extended normalization process applicable to impedance and velocity, it is advantageous at this point to return to the frequency dependent equations in order to pose the frequency dependence in a compatible form. From equations (3.3.10) and (3.3.11) it can be shown that,

3.5.1 $$\omega_c = \sqrt{\frac{\pi v_1}{2l}}\sqrt{\frac{\pi v_2}{2l}} \equiv \sqrt{\omega_1\omega_2}$$

where ω_1 and ω_2 are defined by the second half of equation (3.5.1). ω_1 and ω_2 correspond physically to the peak transmission frequencies obtained by replacing the acoustic transformer with a quarterwave transformer having the same velocity as the output and input materials respectively.

In the endeavor to complete the extended normalization for the reflection coefficient (3.3.16), it is necessary to examine the term $\frac{\varpi}{\overline{v}^{(j)}}$ more carefully. Specifically, applying equations (3.3.1), (3.3.13), and (3.3.14) leads to,

3.5.2 $$\frac{\varpi}{\overline{v}^{(j)}} = \frac{\omega}{\omega_c}\frac{v_g}{v^{(j)}} = \frac{\omega}{\left(\frac{\pi v_g}{2l}\right)}\frac{v_g}{v^{(j)}} = \frac{\omega}{\left(\frac{\pi}{2l}\right)}\frac{1}{v^{(j)}}.$$

Multiplying and dividing by $\sqrt{v^{(1)}v^{(N-1)}}\,\frac{1}{v_1}$ gives,

3.5.3 $$\frac{\varpi}{\overline{v}^{(j)}} = \frac{\omega}{\left(\frac{\pi}{2l}\right)\sqrt{v^{(1)}v^{(N-1)}}}\;\frac{\sqrt{v^{(1)}v^{(N-1)}}}{v^{(j)}}\;\frac{\frac{1}{v_1}}{\frac{1}{v_1}}.$$

Applying equation (3.4.24) yields,

3.5.4 $$\frac{\varpi}{\overline{v}^{(j)}} = W\,\frac{\sqrt{\alpha^{(1)}\alpha^{(N-1)}}}{\alpha^{(j)}}$$

where $W = \frac{\omega}{\left(\frac{\pi}{2l}\right)\sqrt{v^{(1)}v^{(N-1)}}}$ and W is the extended normalized frequency. Substituting equation (3.5.4) into equation (3.3.16) shows that,

3.5.5 $$r(\overline{x}^{(j)}) = r^{(j),(j-1)}\exp\left(-\pi j W\,\frac{\sqrt{\alpha^{(1)}\alpha^{(N-1)}}}{\alpha^{(j)}}\,T^{(j)}\right),$$

which is the reflection coefficient posed completely in terms of extended normalized variables. Assuming that all N - 1 layers are of equal length and that the total length l is defined to be 1 leads to,

3.5.6 $$l^{(j)} = \frac{1}{N - 1}.$$

Plugging equation (3.5.6) into equation (3.5.5) gives,

3.5.7 $$r(\bar{x}^{(j)}) = r^{(j),(j-1)} \exp\left(-\pi j W \frac{\sqrt{\alpha^{(1)}\alpha^{(N-1)}}}{\alpha^{(j)}(N-1)} \right).$$

Equation (3.5.7) is the form of the equation for the reflection coefficient used in the program.

Having defined the extended normalized frequency, W, the values of W for which peak power transmission will occur in a quarterwave transformer are now derived. W is defined in equation (3.5.4) as,

3.5.8 $$W = \frac{\omega}{\left(\frac{\pi}{2l}\right)\sqrt{v^{(1)}v^{(N-1)}}}.$$

Under quarterwave matching conditions, the number of layers, N, is defined by,

3.5.9 $$N \equiv 2.$$

Plugging equation (3.5.9) into equation (3.5.8) results in,

3.5.10 $$W = \frac{\omega}{\left(\frac{\pi v^{(1)}}{2l}\right)}.$$

After referencing equation (2.2.2a), the term in the denominator of equation (3.5.10) satisfies the fundamental quarterwave resonant frequency. Therefore, for all quarterwave transformers, W will have peak power transmission at odd integer values.

The values of W at which peak power transmission occur for the case that the velocity profile is continuous across the interfaces at x=0, x=-l, are derived below. The definition of W is repeated for convenience.

3.5.11 $$W = \frac{\omega}{\left(\frac{\pi}{2l}\right)\sqrt{v^{(1)}v^{(N-1)}}}$$

If the velocity profile is continuous across the left and right interfaces this requires $N \rightarrow \infty$. The following approximation holds,

3.5.12 $$\alpha^{(1)} \approx \alpha^{(0)} \text{ and } \alpha^{(N-1)} \approx \alpha^{(N)}.$$

Substituting equation (3.5.12) into equation (3.5.11) gives,

3.5.13 $$\frac{\varpi}{v^{(j)}} = \frac{\omega}{\left(\frac{\pi}{2l}\right)\sqrt{v^{(0)}v^{(N)}}} = \frac{\omega}{\left(\frac{\pi}{2l}\right)\sqrt{v_1 v_2}}$$

Substituting equations (3.3.10), (3.3.11), and (3.3.14) into equation (3.5.13) leads to,

3.5.14 $$W = \frac{\omega}{\left(\frac{\pi v_g}{2l}\right)} = \frac{\omega}{\omega_c} = \varpi.$$

Thus, in the limit that $N \rightarrow \infty$, W reduces to ϖ as defined by (3.3.14). This reduction is significant because ϖ has a physical description as presented by equation (3.5.1).

It would be convenient to be able to reverse the normalization process for all values of W and provide a physical interpretation. Along these lines, a frequency factor, ff, which converts W to ϖ is derived. Referring to equation (3.5.4), substituting in equations (3.3.13) and (3.4.23), and simplifying leads to,

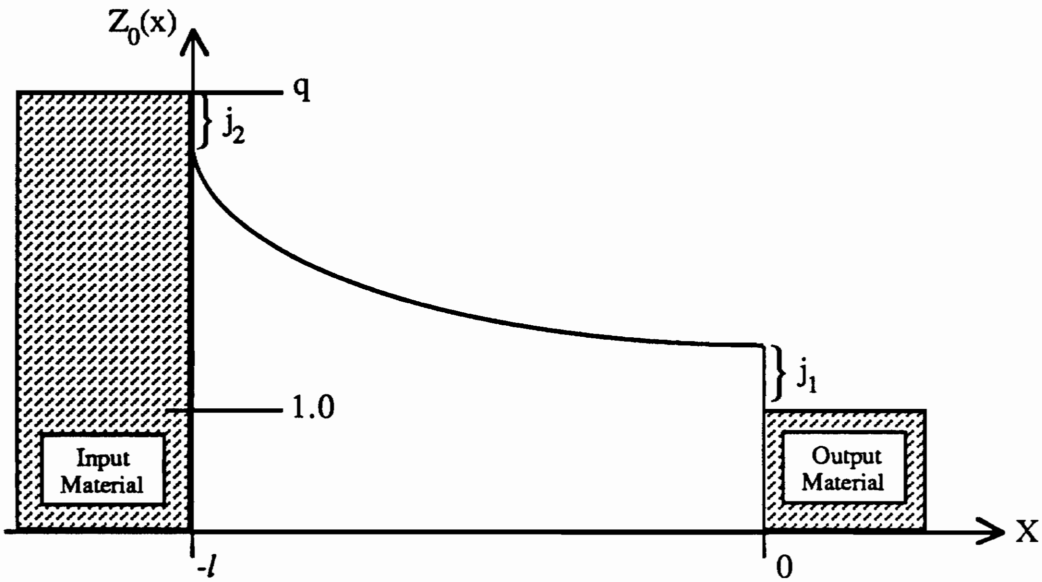
3.5.15 $$\varpi = W\sqrt{\alpha^{(1)}\alpha^{(N-1)}}\,\frac{1}{\sqrt{s}}.$$

It follows that,

3.5.16 $$ff \equiv \sqrt{\alpha^{(1)}\alpha^{(N-1)}}\,\frac{1}{\sqrt{s}}.$$

3.6 Development of an Exponential Impedance Profile

The above formalism assumed prior knowledge of an impedance profile. As an illustration, an impedance profile based on an exponential taper will be defined. See Figure 5. The profile will account for discontinuous jumps in the characteristic impedance of the output and the first layer in the matching media, as well as between the input and the N-1 matching layer.



The region to the left of $x = -l$ is the input
The region to the right of $x = 0$ is the output
The region in between the input and output consists of the matching layers.
j_1 is the percentage of change in the magnitude of $Z_0(x)$ at $x = 0$.
j_2 is the percentage of change in the magnitude of $Z_0(x)$ at $x = -l$.

Figure 5 - Continuous Impedance Profile Transformer Model

In order to form an exponential taper, let,

3.6.1 $\quad \gamma_0(x) = c_2 e^{c_1 x}$

where c_1 and c_2 are arbitrary constants. From Figure 5, it is obvious that,

3.6.2 $\quad \gamma_0(x = 0) = 1 + j_1.$

Equation (3.5.1) demonstrates that,

3.6.3 $\quad \gamma_0(x = 0) = c_2.$

Setting equations (3.6.2) and (3.6.3) equal, solving for c_2, and substituting into equation (3.6.1) results in,

3.6.4 $\gamma_0(x) = (1 + j_1)e^{c_1 x}$.

From Figure 5, it is obvious that,

3.6.5 $\gamma_0(x = -l) = q - j_2$.

Evaluating equation (3.6.4) at $x = -l$ leads to,

3.6.6 $\gamma_0(x = -l) = (1 + j_1)e^{c_1(-l)}$.

Setting equations (3.6.5) and (3.6.6) equal, and solving for c_1 gives,

3.6.7 $c_1 = \frac{-1}{l}\ln\left(\frac{q - j_2}{1 + j_1}\right)$

Substituting equation (3.6.7) into equation (3.6.4) yields,

3.6.8 $\gamma_0(x) = (1 + j_1)\exp\left(-\ln\left(\frac{q - j_2}{1 + j_1}\right)\left(\frac{x}{l}\right)\right)$

Assuming each of the N-1 layers are of equal length results in,

3.6.9 $\frac{x}{l} = \frac{-j}{N - 1}$.

Substituting equation (3.6.9) into equation (3.6.8) shows that,

3.6.10 $\gamma_0^{(j)} = (1 + j_1)\exp\left(+\ln\left(\frac{q - j_2}{1 + j_1}\right)\left(\frac{j}{N - 1}\right)\right)$ for $j = 1, N-1$

By definition,

3.6.11 $\gamma_0^{(0)} \equiv 1.0$ and $\gamma_0^{(N)} \equiv q$.

Equations (3.6.10) and (3.6.11) completely define an exponential impedance profile.

3.7 Flow Chart for Power Transmission Calculations

The flow chart presented below is intended to summarize the process for calculating the power transmission coefficient as a function of frequency for a multilayer transformer.

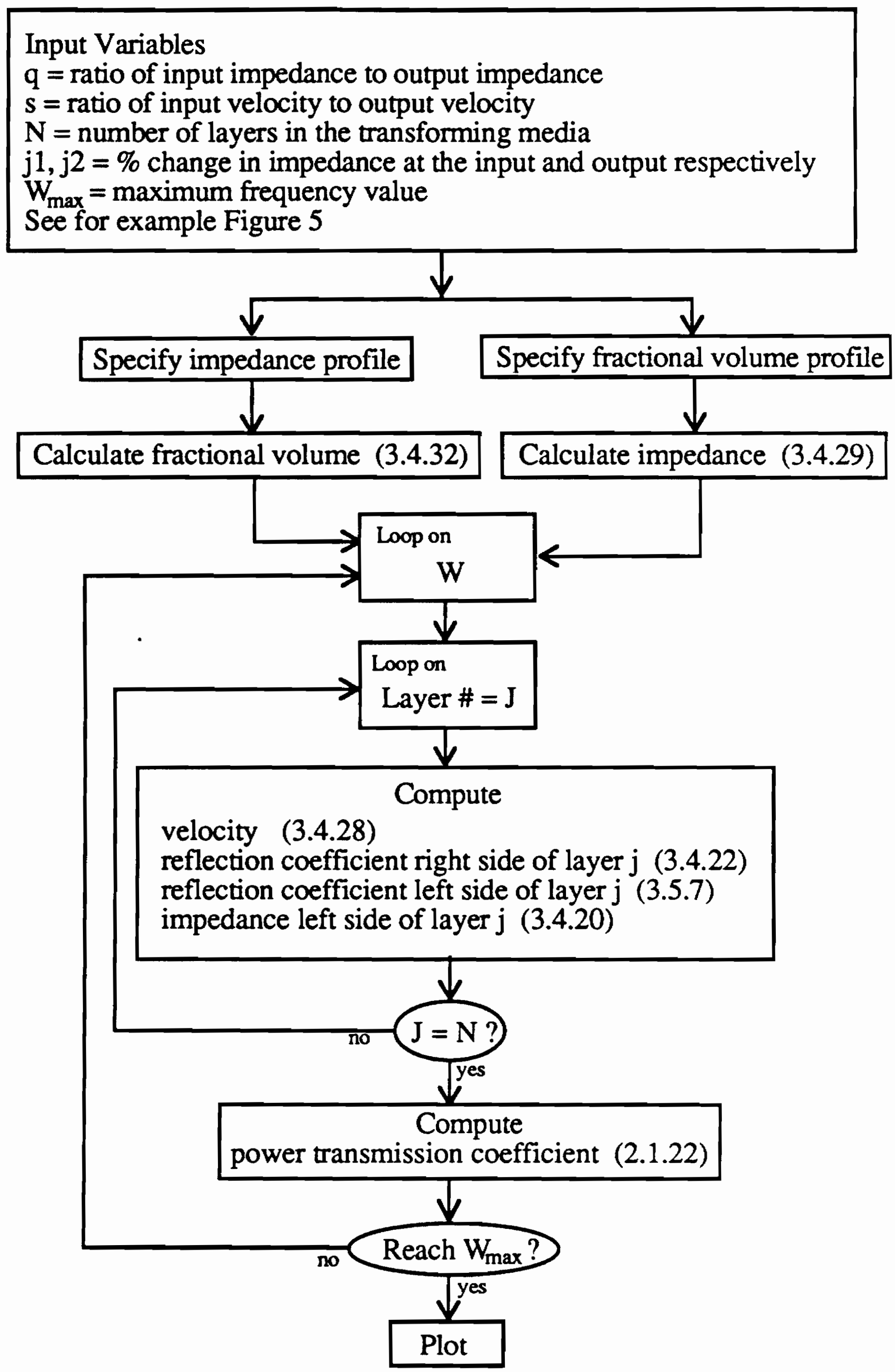


Figure 6 - Flowchart

4. Analytical Results for two Special Cases

In chapter four, analytical results are presented which will be used to demonstrate the validity of the normalized formalism. Section 4.1 includes analytical calculations for bandwidth measurements under quarterwave matching conditions. Section 4.2 discusses an equation for one dimensional plane wave propagation. An exact solution to this equation is derived in section 4.3 for acoustic transformers exhibiting an exponential impedance profile. Finally, section 4.4 develops a frequency normalization for the solution to the case s=1.

4.1 Bandwidth Measurements Under Quarterwave Matching Conditions

In order to demonstrate the validity of our formalism, we have compared our results to a *quarterwave* bandwidth analysis found in [3] by Collin. When given the same inputs, Collin's formalism and ours produce the same results.

After pointing out that quarterwave transformers will only provide maximum power transfer over a narrow band of frequencies, Collin suggests the use of multilayer transformers to increase bandwidth. He proceeds to derive a formula which will calculate the bandwidth of a quarterwave transformer in order to show its limited applicability. The formula is,

4.1.1 $$\frac{\Delta f}{f} = 2 - \frac{4}{\pi}\cos^{-1}\left(\frac{2p_m\sqrt{Z_2 Z_1}}{(Z_2 - Z_1)\sqrt{1-p_m^2}}\right)$$

where: Z_1 = characteristic impedance of the output,

Z_2 = input impedance at distance l (must be purely resistive),

p_m = maximum value of reflection coefficient that can be tolerated,

Δf = bandwidth,

and

f = fundamental quarterwave resonant frequency.

Rewriting equation (4.1.1), it can be shown that,

4.1.2 $$\frac{\Delta f}{f} = 2 - \frac{4}{\pi}\cos^{-1}\left(\frac{\dfrac{2p_m}{\sqrt{1-p_m^2}}}{\dfrac{(Z_2 - Z_1)}{\sqrt{Z_2 Z_1}}}\right).$$

The normalized impedance of the input and output are defined respectively as,

4.1.3 $$\mathcal{Z}_2 \equiv \frac{Z_2}{\sqrt{Z_2 Z_1}} \quad \text{and} \quad \mathcal{Z}_1 \equiv \frac{Z_1}{\sqrt{Z_2 Z_1}}.$$

Substituting equation (4.1.3) into equation (4.1.2) leads to,

4.1.4 $$\frac{\Delta f}{f} = 2 - \frac{4}{\pi}\cos^{-1}\left(\frac{\dfrac{2p_m}{\sqrt{1-p_m^2}}}{\mathcal{Z}_2 - \mathcal{Z}_1}\right).$$

The ratio of the impedances, q, is defined in (3.4.14) as,

4.1.5 $$q \equiv \frac{Z_2}{Z_1} = \frac{\mathcal{Z}_2}{\mathcal{Z}_1}.$$

Substituting equation (4.1.5) into equation (4.1.4) and factoring results in,

4.1.6 $$\frac{\Delta f}{f} = 2 - \frac{4}{\pi}\cos^{-1}\left(\frac{\dfrac{2p_m}{\sqrt{1-p_m^2}}}{\mathcal{Z}_1\,(q - 1)}\right).$$

$\mathcal{Z}_1$ can be represented in terms of q as follows,

4.1.7 $$\mathcal{Z}_1 = \frac{Z_1}{\sqrt{Z_2 Z_1}} = \sqrt{\frac{Z_1}{Z_2}} = \frac{1}{\sqrt{q}}.$$

Plugging equation (4.1.7) into equation (4.1.6) and simplifying gives,

4.1.8 $$\frac{\Delta f}{f} = 2 - \frac{4}{\pi}\cos^{-1}\left(\frac{2p_m\sqrt{q}}{\sqrt{1-p_m^2}\,(q - 1)}\right).$$

Thus, Collin's bandwidth formula depends only on q and p_m. In order to compare Collin's bandwidth formula to our formalism, it must be shown that a change in the extended normalized frequency, ΔW, is equal to Collin's $\frac{\Delta f}{f}$. The left hand side of equation (4.1.8) can be expressed as,

4.1.9 $$\frac{\Delta f}{f_0} = \frac{\Delta \omega}{\omega_0},$$

where ω_0 is the radian frequency for which

4.1.10 $$\beta l = \frac{\pi}{2}$$

and β is the phase constant. Substituting equations (2.2.2a) and (3.3.6) into equation (4.1.9) and letting j=1 for quarterwave matching leads to,

4.1.11 $$\frac{\omega_0 l}{v^{(1)}} = \frac{\pi}{2}.$$

Isolating ω_0 and plugging equation (4.1.11) into equation (4.1.9),

4.1.12 $$\frac{\Delta \omega}{\omega_0} = \frac{\Delta \omega}{\frac{\pi}{2}\frac{v^{(1)}}{l}}.$$

At this point, focus is shifted to our extended normalized frequency definition. Equation (3.5.4) is repeated here for convenience.

4.1.13 $$W \equiv \frac{\omega}{\left(\frac{\pi}{2l}\right)\sqrt{v^{(1)}v^{(N-1)}}}$$

Since N = 2 for quarterwave matching, it follows that,

4.1.14 $$\Delta W \equiv \frac{\Delta \omega}{\left(\frac{\pi v^{(1)}}{2l}\right)}.$$

Comparing equations (4.1.12) and (4.1.14) one sees that they are identical. Thus, our definition of bandwidth is indeed equivalent to Collin's definition. Having established the equivalency in the two definitions for frequency, the bandwidths predicted by Collin's formalism and the normalized formalism can be compared. This comparison is done in section 5.2 where several examples show that the formulas provide the equivalent bandwidths when given the same inputs. This equivalency demonstrates the validity of the normalized formalism in the special case of quarterwave transformers.

4.2 An Exact Solution for an Exponential Taper Using Wax and Walker's Equation for Propagation

In [4], a very brief sketch of a proof for the differential equation describing the evolution of the voltage reflection coefficient in nonuniform materials has been given. The details of this analysis appear in Appendix C. Also included in Appendix C is a discussion covering the analysis which mathematically links transmission parameters to parallel acoustic parameters. The remainder of this chapter will apply this equation to an acoustic transformer with a continuous, exponential taper in order to develop a program which will calculate the amount of power transmitted through the transformer. The results of this program will be compared to the results of our normalized formalism and shown to be equivalent and thus, demonstrate the validity of our formalism in a nonquarterwave case.

In this section, a *general* solution for the reflection coefficient associated with an exponential impedance taper is derived. As a check, the general solution is analyzed under quarterwave matching conditions and yields results which agree with acoustic theory. The acoustically transformed version of Walker and Wax's equation, (C.1.4), is derived in Appendix C and repeated here for convenience.

4.2.1 $$\frac{\partial R}{\partial x} - 2R\Gamma(x) + 0.5\,\frac{\partial(\ln Z_0(x))}{\partial x}(1 - R^2) = 0$$

where R = the reflection coefficient, Γ = the propagation factor, and Z_0 = the characteristic impedance. R can be defined as,

4.2.2 $$R = e^{j\phi(x)}.$$

Plugging equation (4.2.2) into equation (4.2.1) and simplifying yields, 4.2.3 $\quad j\frac{\partial\phi}{\partial x}e^{j\phi(x)} - 2\Gamma(x)e^{j\phi(x)} + 0.5\frac{\partial(\ln Z_0(x))}{\partial x}\left(1 - e^{j2\phi(x)}\right) = 0.$

Multiplying by $\frac{1}{j}e^{-j\phi(x)}$ leads to,

4.2.4 $\quad \frac{\partial\phi}{\partial x} + j2\Gamma(x) + \frac{\partial(\ln Z_0(x))}{\partial x}\left(\frac{e^{-j\phi(x)} - e^{j\phi(x)}}{2j}\right) = 0.$

Recalling the definition of the sin function [12], it follows that,

4.2.5 $\quad \frac{\partial\phi}{\partial x} + j2\Gamma(x) - \frac{\partial(\ln Z_0(x))}{\partial x}\sin\phi(x) = 0.$

For nonlossy materials the propagation constant can be represented by $j\beta$ where the phase constant β is strictly real. The phase constant is also assumed to be independent of x. This corresponds physically to a uniform velocity profile. The second assumption is necessary in order to obtain a mathematically tractable problem. Plugging $\Gamma(x) = j\beta$ into equation (4.2.6), isolating the first term, and simplifying results in,

4.2.6 $\quad \frac{\partial\phi}{\partial x} = 2\beta + \left(\frac{1}{Z_0(x)}\frac{\partial Z_0(x)}{\partial x}\right)\sin\phi(x).$

Under the special case of an exponential taper,

4.2.7 $\quad Z_0(x) = c_2 e^{c_1 x},$

where c_1 and c_2 are arbitrary constants. Plugging equation (4.2.7) into equation (4.2.6), simplifying, and integrating from $x = 0$ to $x = x$ yields,

4.2.8 $\quad \int_{\phi(x=0)}^{\phi(x)} \frac{\partial\phi}{2\beta + c_1\sin\phi(x)} = \int_0^x \partial x,$

where the direction of x is defined as in Figure 5. Performing a change of variables from $\phi(x)$ to R, noting that $R = e^{j\phi(x)}$ leads to,

4.2.9 $\quad \int_{R(x=0)}^{R(x)} \frac{\partial R}{jR\left(2\beta + \frac{c_1}{2j}\left(R - \frac{1}{R}\right)\right)} = \int_0^x \partial x.$

After multiplying jR through the denominator and multiplying by $1 = \frac{2}{c_1}\frac{c_1}{2}$, it follows that,

$$4.2.10 \quad \frac{2}{c_1} \int_{R(x=0)}^{R(x)} \frac{\partial R}{\frac{4ßRj}{c_1} + R^2 - 1} = \int_0^x \partial x.$$

Use the quadratic formula to solve for the roots of R in the denominator of equation (4.2.10). R_P denotes the positive root, R_N denotes the negative root, and R_{PN} denotes both roots.

$$4.2.11 \quad R_{PN} = \frac{\frac{-4ßj}{c_1} \pm \sqrt{\left(\frac{4ßj}{c_1}\right)^2 + 4}}{2}$$

Simplifying the expression gives,

$$4.2.12 \quad R_{PN} = \frac{-2ßj}{c_1} \pm \sqrt{1 - \left(\frac{2ß}{c_1}\right)^2}.$$

Performing a partial fraction expansion of the term $\frac{1}{R^2 + \frac{4ßRj}{c_1} - 1}$ found in equation (4.2.10) results in,

$$4.2.13 \quad \frac{1}{R^2 + \frac{4ßRj}{c_1} - 1} = \frac{d_1}{R - R_P} + \frac{d_2}{R - R_N}$$

where $d_1 = \frac{1}{R_P - R_N}$ and $d_2 = -d_1$.

Plugging equation (4.2.13) into equation (4.2.10) leads to,

$$4.2.14 \quad \frac{2d_1}{c_1}\left(\int_{R(x=0)}^{R(x)} \frac{\partial R}{R - R_P} - \int_{R(x=0)}^{R(x)} \frac{\partial R}{R - R_N} \right) = \int_0^x \partial x.$$

Performing the integrations, applying the properties of natural logarithms, and simplifying yields,

$$4.2.15 \quad \frac{R(x) - R_P}{R(x) - R_N} = \frac{R(x=0) - R_P}{R(x=0) - R_N} e^{(c_1 x / 2d_1)}.$$

A quantity, f(x), is defined for purposes of simplifying the expression.

$$4.2.16 \quad f(x) = \frac{R(x=0) - R_P}{R(x=0) - R_N} e^{(c_1 x / 2d_1)}$$

After plugging equation (4.2.16) into equation (4.2.15) and isolating R(x), it follows that, 4.2.17 $\quad R(x) = \dfrac{R_P - f(x)R_N}{1 - f(x)}$.

Evaluating d_1 via equation (4.2.13) and using equation (4.2.12) produces,

4.2.18 $\quad d_1 = \dfrac{1}{R_P - R_N} = \dfrac{1}{2\sqrt{1 - \left(\frac{2\beta}{c_1}\right)^2}}$.

Plugging equation (4.2.16) into equation (4.2.17) and simplifying gives,

4.2.19 $\quad R(x) = \dfrac{R_P[R(x=0) - R_N] - R_N[R(x=0) - R_P]\, e^{(c_1 x / 2d_1)}}{(R(x=0) - R_N) - (R(x=0) - R_P)\, e^{(c_1 x / 2d_1)}}$.

As a check of equation (4.2.19), the equation is analyzed under quarterwave conditions to find its value at $x = -l^+$. The superscript + is indicative of the right side of the interface at -l. A quarterwave transformer has only one matching layer, thus $Z_0(x)$ is constant. Since $Z_0(x) = c_2 e^{c_1 x}$, it is obvious that c_1 must equal zero for $Z_0(x)$ to be constant. In the limit as c_1 goes to zero, equation (4.2.18) provides,

4.2.20 $\quad \lim c_1 \to 0, \quad \dfrac{c_1}{2d_1} = \sqrt{c_1^{\,2} - (2\beta)^2} = j2\beta$.

Referring to equation (4.2.12) and employing the binomial expansion results in,

4.2.21 $\quad \lim c_1 \to 0, \quad R_P = 0$

and

4.2.22 $\quad \lim c_1 \to 0, \quad R_N = \infty$.

Substituting equations (4.2.20), (4.2.21), and (4.2.22) into equation (4.2.19) and noting that $R_N >> R(x=0)$ leads to,

4.2.23 $\quad \lim c_1 \to 0, \quad R(x) = R(x=0)e^{j2\beta x}$.

After evaluating at $x = -l^+$, it follows that,

4.2.24 $\quad R(x = -l^+) = R(x=0)e^{j2\beta(-l+)}$.

As expected, equation (4.2.24) is in exact agreement with acoustic theory (2.1.19).

4.3 The Normalized Exact Solution for the Exponential Impedance Profile

This section is devoted to the development of equations to be used in the program which will calculate the amount of power transmitted in an acoustic transformer with an exponential taper. These equations are summarized at the end of the chapter. Normalization of some of the variables was performed in order to minimize the number of input variables required by the program.

To begin the derivation of the necessary equations, focus is shifted to the establishment of an exponential impedance profile for the matching medium.

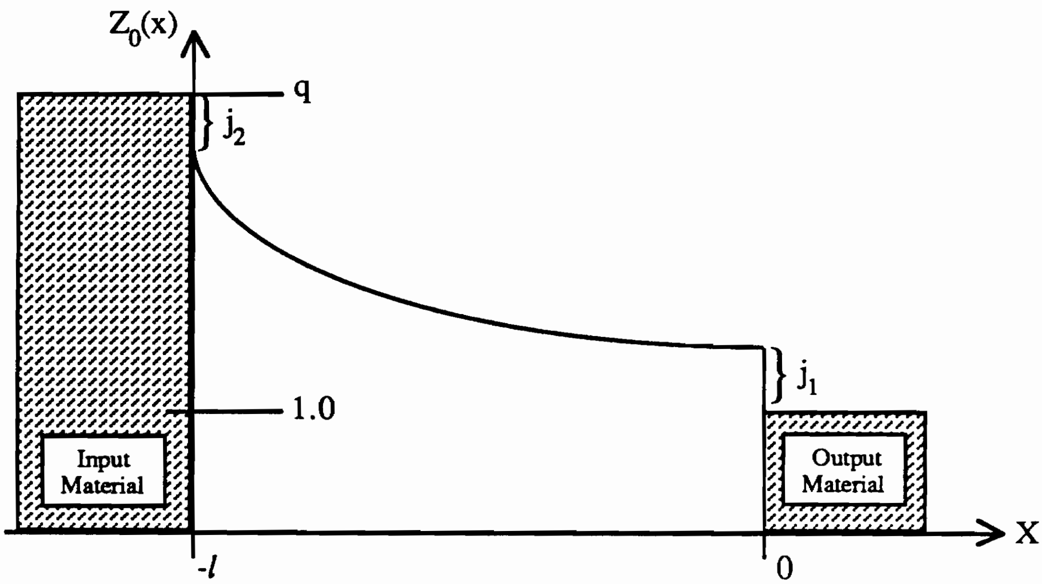


Figure 7 - Exponential Impedance Profile

Referring to Figure 7, since $\gamma_0(x = 0) = 1$, the parameter q is not only the value of $\gamma_0(x)$ at $x = -l$, but also the ratio of the impedances input to output. Using equation (4.2.7) and evaluating $\gamma_0(x)$ at $x = -l^+$ and at $x = 0$ produces, 4.3.1 $\quad \gamma_0(x = -l^+) = c_2 e^{-c_1 l^+}$

and

4.3.2 $\quad \gamma_0(x = 0^-) = c_2.$

From Figure 7, one can see that,

4.3.3 $\quad \gamma_0(x = -l^+) = q - j_2$

and

4.3.4 $\quad \gamma_0(x = 0^-) = 1 + j_1.$

Setting equation (4.3.1) equal to equation (4.3.3) gives,

4.3.5 $\quad c_2 e^{-c_1 l^+} = q - j_2.$

Similarly, setting equation (4.3.2) equal to equation (4.3.4) gives,

4.3.6 $\quad c_2 = 1 + j_1.$

Substituting equation (4.3.6) into equation (4.3.5) and isolating c_1 results in,

4.3.7 $\quad c_1 = \frac{-1}{l^+} \ln\left(\frac{q - j_2}{1 + j_1}\right)$

Having defined an impedance profile, the reflection coefficient at $x = 0^-$ is now calculated. Equation (2.1.18) demonstrates that,

4.3.8 $\quad R(x = 0^-) = \frac{\gamma_0(x = 0^+) - \gamma_0(x = 0^-)}{\gamma_0(x = 0^+) + \gamma_0(x = 0^-)}.$

From Figure 7, one can see that,

4.3.9 $\quad \gamma_0(x = 0^+) = 1.$

Plugging equations (4.3.9) and (4.3.4) into equation (4.3.8) leads to,

4.3.10 $\quad R(x = 0^-) = \frac{-j_1}{2 + j_1}.$

$R(x = -l^+)$ is given by equation (4.2.19) evaluated at $x = -l^+$. The derivation for $R(x = -l^-)$ follows directly from the acoustically transformed version of Wax and Walker's equation, (C.1.4), and is quite lengthy. To avoid an unnecessary diversion, the derivation for this equation is not given here but can be found in Appendix D. Equation (D.2.9) is repeated here for convenience.

4.3.11 $$R(x=-l^-) = \frac{(1+R(x=-l^+)) - \dfrac{\gamma_0(x=-l^-)}{\gamma_0(x=-l^+)}(1-R(x=-l^+))}{(1+R(x=-l^+)) + \dfrac{\gamma_0(x=-l^-)}{\gamma_0(x=-l^+)}(1-R(x=-l^+))}$$

From Figure 7, it is obvious that,

4.3.12 $\gamma_0(x=-l^-) = q.$

Plugging equations (4.3.3) and (4.3.12) into equation (4.3.11) yields,

4.3.13 $$R(x=-l^-) = \frac{(1+R(x=-l^+)) - \dfrac{q}{q-j_2}(1-R(x=-l^+))}{(1+R(x=-l^+)) + \dfrac{q}{q-j_2}(1-R(x=-l^+))}.$$

Having determined the reflection coefficient, an extended normalized frequency, F, applicable to Walker and Wax's equation is now defined. Equation (3.3.15) is repeated here for convenience.

4.3.14 $$\frac{\varpi}{\bar{v}} = \frac{2l\beta}{\pi}.$$

The extended normalized frequency, F, is defined by,

4.3.15 $$F \equiv \frac{\varpi}{\bar{v}},$$

where ϖ and $\bar{v}$ are defined by equations (3.3.14) and (3.3.13) respectively.

Plugging equation (4.3.15) into equation (4.3.14) and isolating β leads to,

4.3.16 $$\beta = \frac{\pi F}{2l}.$$

Using equation (4.3.7) and simplifying results in,

4.3.17 $$\frac{\beta}{c_1} = \frac{\pi F}{2}\frac{1}{\ln\left(\dfrac{1+j_1}{q-j_2}\right)}.$$

The equations which were used in the program are listed below. The goal of calculating the power transmission coefficient is realized by equation 9. Note that this equation depends on equation 8 which in turn depends on equation 7, etc. In other words, these equations are coupled and are all necessary to compute the power transmission coefficient. While it is possible to uncouple the equations, it is also inadvisable due to the extremely complex nature of the equation which would result. Note that all of the equations can be expressed in terms of j_1, j_2, q, W, and l. To simplify the equations, l is defined to be 1 in the program.

1 $$c_1 = \frac{-1}{l} \ln\left(\frac{q - j_2}{1 + j_1}\right)$$

2 $$\frac{\beta}{c_1} = \frac{\pi F}{2} \frac{1}{\ln\left(\frac{1 + j_1}{q - j_2}\right)}$$

3 $$R_{PN} = \frac{-2\beta j}{c_1} \pm \sqrt{1 - \left(\frac{2\beta}{c_1}\right)^2}$$

Let x = the distance into the matching layer,

4 $$\frac{c_1 x}{2d_1} = \frac{c_1}{2d_1} l \frac{x}{l} = c_1 l \sqrt{1 - \left(\frac{2\beta}{c_1}\right)^2} \frac{x}{l}$$

5 $$\left.\frac{c_1 x}{2d_1}\right|_{x = -l+} = c_1 l \sqrt{1 - \left(\frac{2\beta}{c_1}\right)^2} (-1)$$

6 $$R(x = 0^-) = \frac{-j_1}{2 + j_1}$$

7 $$R(x = -l^+) = \left.\frac{R_P\,(R(x = 0^-) - R_N) - R_N\,(R(x = 0^-) - R_P)e^{(c_1 x / 2d_1)}}{(R(x = 0^-) - R_N) - (R(x = 0^-) - R_P)\,e^{(c_1 x / 2d_1)}}\right|_{x = -l^+}$$

8 $$R(x = -l^-) = \frac{(1 + R(x = -l^+)) - \frac{q}{q - j_2}(1 - R(x = -l^+))}{(1 + R(x = -l^+)) + \frac{q}{q - j_2}(1 - R(x = -l^+))}$$

9 $$\text{Power Transmitted} = 1 - |R(x = -l^-)|^2$$

4.4 Frequency Normalization for the Case s=1

In order to compare the normalized formalism with Wax and Walker's, our definition for frequency must be modified so that it equals theirs. Beginning with Wax and Walker's definition for frequency, equation (4.3.15) is repeated here,

4.4.1 $$\frac{\varpi}{\bar{v}} = F.$$

Similarly, our definition for frequency, (3.5.4), is also repeated here,

4.4.2 $$\frac{\varpi}{\bar{v}^{(j)}} = W\frac{\sqrt{\alpha^{(1)}\alpha^{(n-1)}}}{\alpha^{(j)}}.$$

To show equivalency of these two definitions, one need only show that, F=W or equivalently,

4.4.3 $$\frac{\sqrt{\alpha^{(1)}\alpha^{(n-1)}}}{\alpha^{(j)}} = 1.$$

Equation (3.4.29) is repeated here for convenience.

4.4.4 $$\alpha^{(j)} = \frac{\sqrt{V_1^{(j)} + V_2^{(j)}\frac{s}{q}}}{\sqrt{V_1^{(j)} + \frac{V_2^{(j)}}{sq}}}$$

It is obvious that if s=1, $\alpha^{(j)} = 1$ for all j. Thus, equation (4.4.2) becomes,

4.4.5 $$\frac{\varpi}{\bar{v}} = W = F.$$

This implies that if s is set equal to 1, the normalized formalism and Walker and Wax's equation should provide equivalent graphs of power transmission versus frequency. Examples demonstrating this fact can be found in section 5.3.

5. Conclusion

5.1 Testing the Ideal Mixture Model

In order to demonstrate the validity of the ideal mixture model, a comparison of its predictions and actual experimental data is presented. In [2], a graph of acoustic velocity and impedance versus percent volume of tungsten is given for tungsten-vinyl composites. The data provided for the graph is experimental and represents measurements of various compositions between pure vinyl and pure tungsten. The graph is reproduced below.

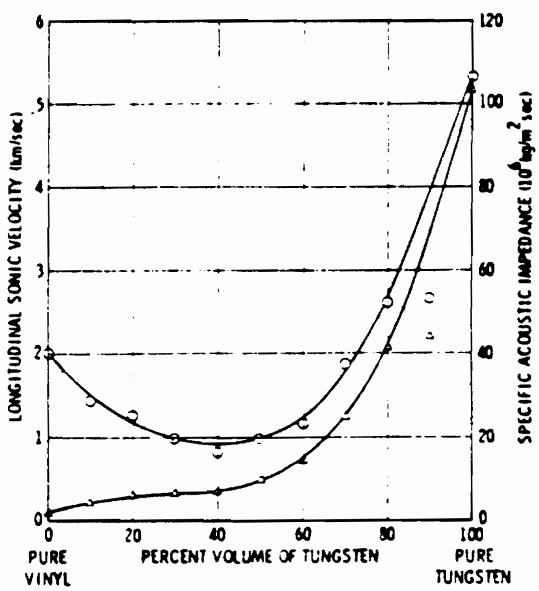


Figure 8 - Acoustic Velocity and Impedance vs. Percent Volume of Tungsten. After [2].

By allowing the fractional volume to range from 0 to 1 and using the equations for acoustic velocity and impedance derived in the normalized formalism, we produced similar graphs shown below.

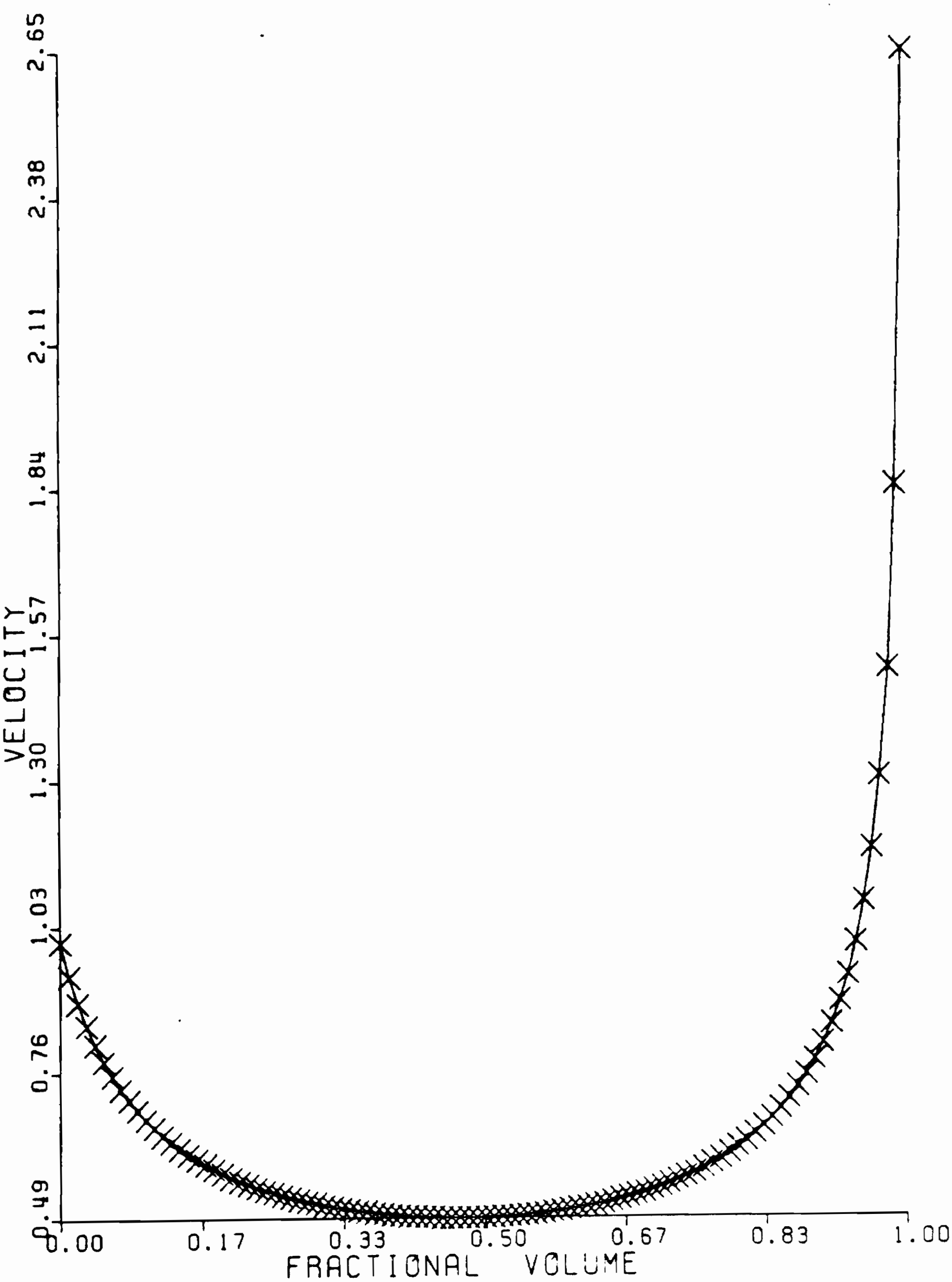


Figure 9 - Acoustic Velocity vs. Fractional Volume

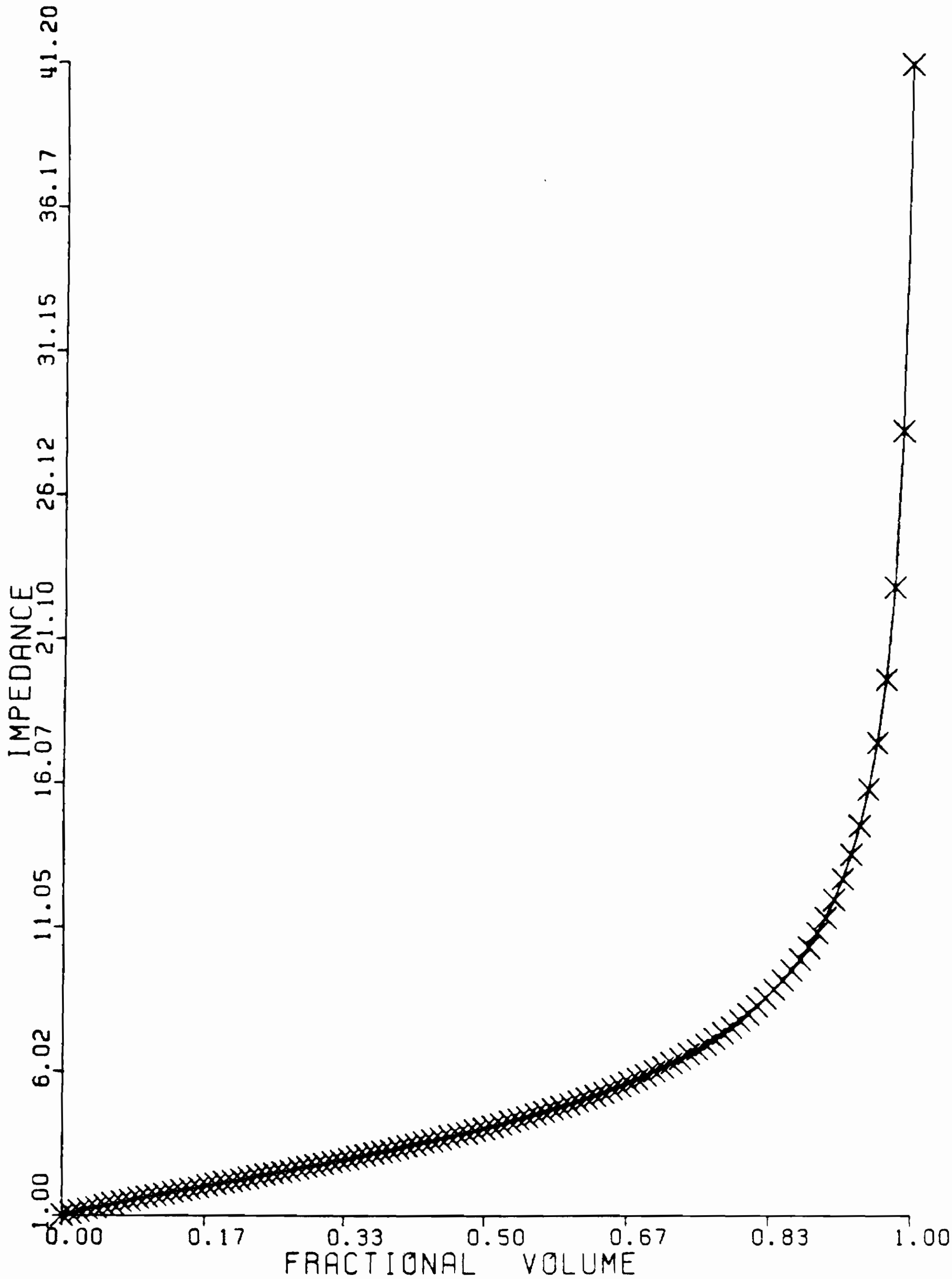


Figure 10 - Acoustic Impedance vs. Fractional Volume

In order to compare our graphs which were based on the ideal mixture model with the graphs from [2] which were based on experimentation, the manipulation of equations which follows was necessary. Equations (3.3.42) and (3.3.50) are repeated here for convenience.

5.1.1 $\quad s \equiv \frac{\bar{v}_2}{\bar{v}_1} = \frac{v_2}{v_1}$

5.1.2 $\quad \alpha^{(j)} \equiv \frac{\bar{v}^{(j)}}{\bar{v}_1} \equiv \frac{v^{(j)}}{v_1}$

Letting $v_1 = 2.0$ and isolating $v^{(j)}$ leads to,

5.1.3 $\quad v^{(j)} = (2.0)\alpha^{(j)}$.

Equations (3.3.41) and (3.3.43) are also repeated here,

5.1.4 $\quad q \equiv \frac{(Z_0)_2}{(Z_0)_1} = \frac{(Z_0)_2}{(Z_0)_1}$

5.1.5 $\quad \gamma_0^{(j)} \equiv \frac{(Z_0)^{(j)}}{(Z_0)_1} = \frac{(Z_0)^{(j)}}{(Z_0)_1}$

Letting $(Z_0)_1 = 2.5$ and isolating $(Z_0)^{(j)}$ leads to,

5.1.6 $\quad (Z_0)^{(j)} = (2.5)\gamma_0^{(j)}$.

Equations (5.1.3) and (5.1.6) were used to convert the data provided by the normalized formalism and shown in Figures 9 and 10 into the form used in [2] and shown in Figure 8. A comparison of the data generated by the normalized formalism and the data given by [2] is presented in the following two tables.

Table 1 - Velocity

% Tungsten	α	Predicted Velocity	Experimental Velocity	% error
20	0.565	1.13	1.3	13
40	0.493	0.986	0.82	20
60	0.503	1.006	1.2	17
80	0.618	1.235	2.7	54

Table 2 - Impedance

% Tungsten	γ_0	Predicted Impedance	Experimental Impedance	% error
20	2.208	5.520	6.4	14
40	3.361	8.403	7.1	18
60	4.898	12.245	14.6	16
80	7.807	19.518	41.6	53

The error shown in the tables above is due to the fact that the volumes of tungsten and vinyl are not additive as the ideal mixture model assumes. Additional error for this case not covered by the model's standard assumptions is described in Appendix E. Nonetheless, the ideal mixture model is in qualitative agreement with the data.

5.2 Results of Quarterwave Bandwidth Analysis

In [3], a formula is developed by Collin for calculating the bandwidth of a quarterwave transformer. Section 4.1 proves that Collin's definition for bandwidth is equivalent to a change in the extended normalized frequency, ΔW. The purpose of this section is to provide examples which demonstrate that Collin's formula and the normalized formalism provide the same results when given the same inputs and thus demonstrate the validity of the normalized formalism in the special case of quarterwave transformers.

Equation (4.1.8) expresses Collin's bandwidth formula in terms of the ratio of acoustic impedances, q, and the maximum value of reflection coefficient that can be tolerated, p_m. Therefore, given q and p_m, Collin's formula provides bandwidth. Determining the corresponding bandwidth predicted by the normalized formalism requires the following procedure. First, the amount of power transferred, T, must be calculated using p_m and equation (2.1.21). For example, if $p_m = \sqrt{0.1}$ then T = 0.9 or 90% power transmission. Second the normalized formalism is invoked to calculate the amount of power transferred as a function of the extended normalized frequency W (see section 3.7). Third, the values of W where the percent power transmitted, T, is equal to the value designated by p_m are recorded. For example, in Figure 11 these values would be 1.26 and 0.76 when T = 0.9.

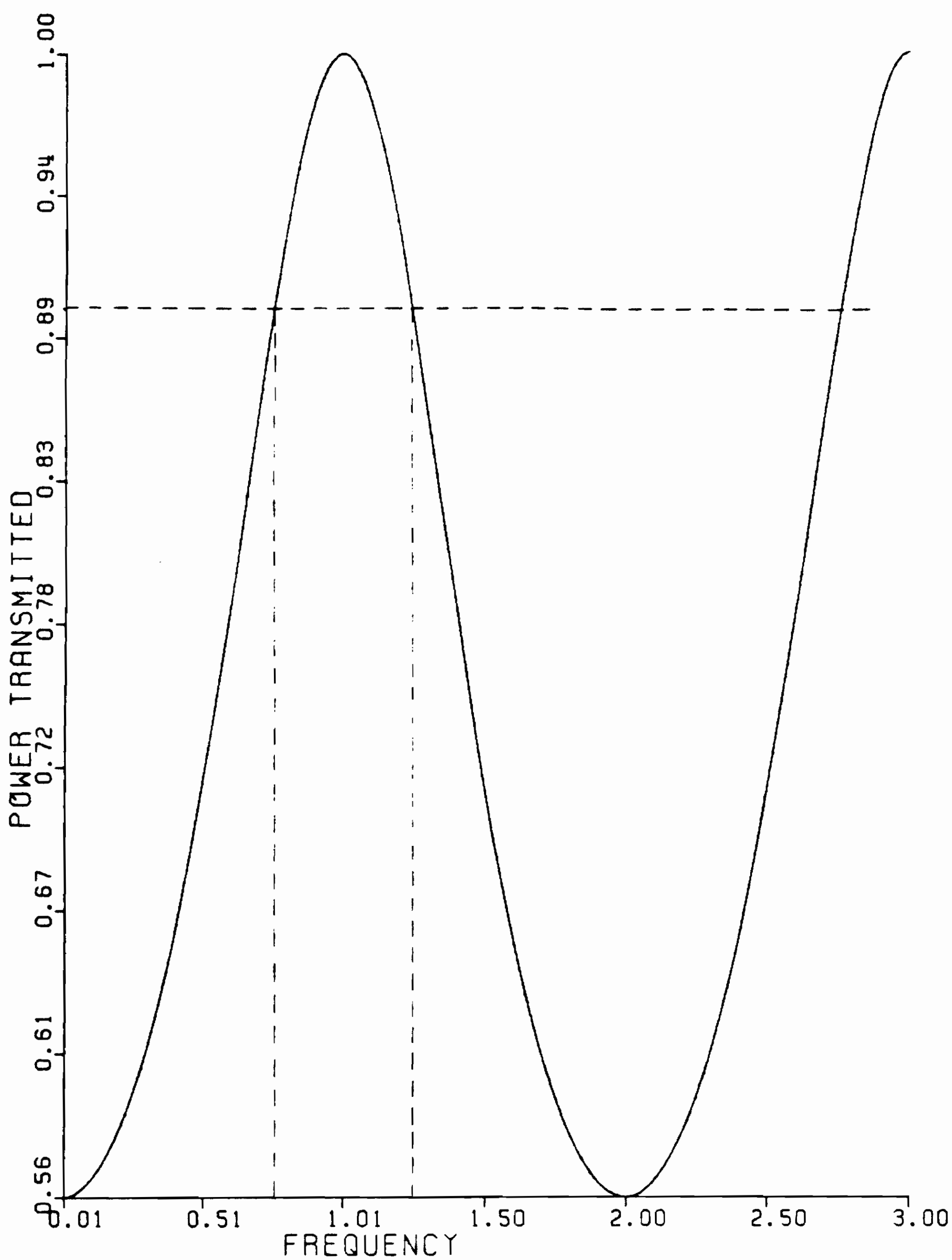


Figure 11 - Normalized Formalism with q=5

Finally, the bandwidth is equal to the absolute value of the algebraic difference of these frequencies. Therefore, in Figure 11 the bandwidth is $| 0.76 - 1.26 |$ or 0.5. Note that this value is approximate and a more exact value is obtained for bandwidth when the computer is programmed to determine the frequencies corresponding to T and to calculate their difference. The graphs are provided as a means of visual endorsement.

A second example for calculating bandwidth using the normalized formalism is presented in Figure 12.

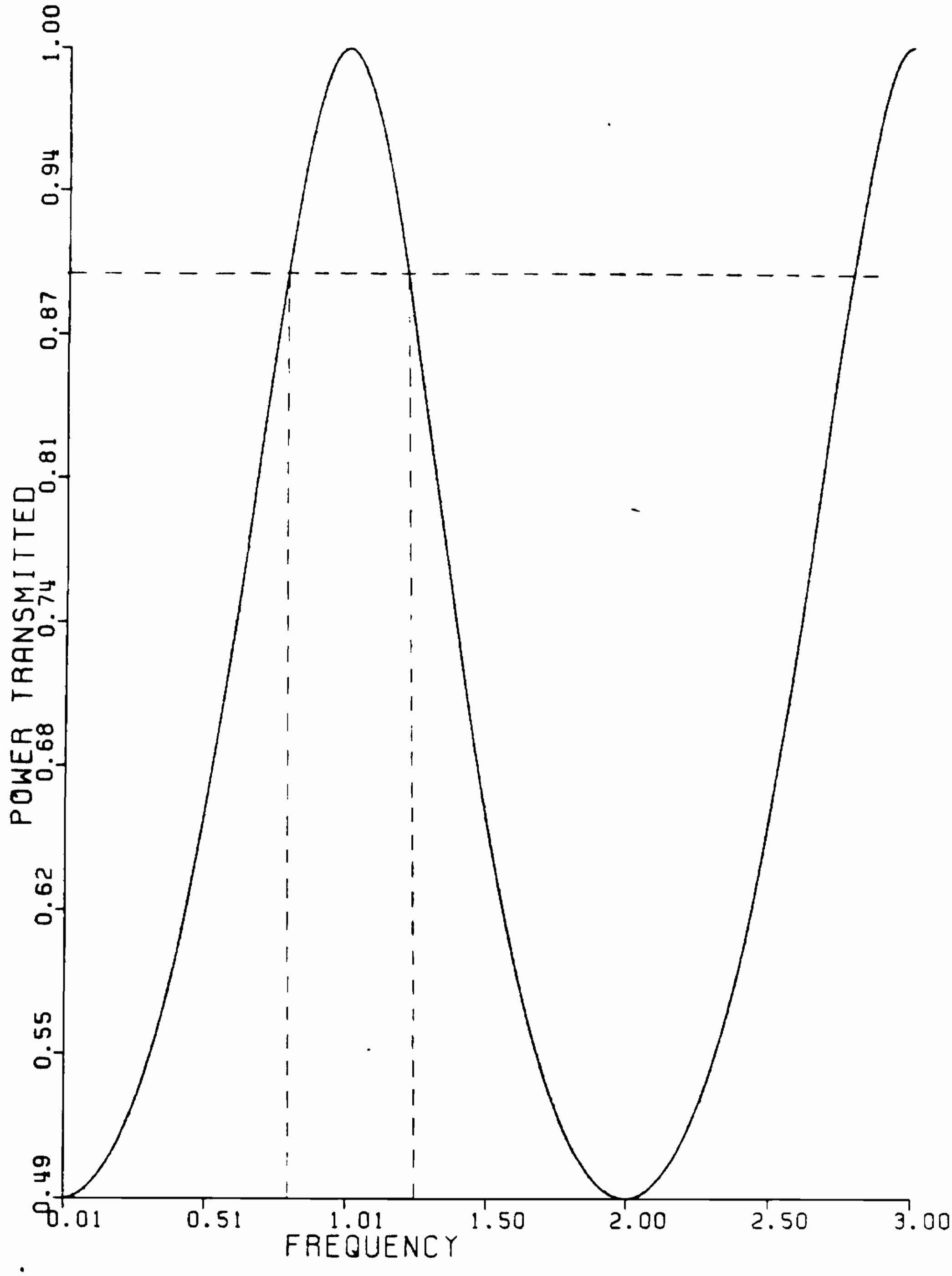


Figure 12 - Normalized Formalism with q=6

Once again $p_m = \sqrt{0.1}$ and thus T = 0.9. The values of W corresponding to T=90% are 0.81 and 1.25 resulting in a bandwidth of 0.44. The table below compares the values of bandwidth predicted by Collin's formula, the computer, and where appropriate the graph. Note that Figure 11 corresponds to q=5 and Figure 12 corresponds to q=6. The percent error calculated is between Collin's formula and the computer's prediction.

Table 3 - Bandwidth Comparisons

q	$(p_m)^2$	Collin's Formula	Computer	Graph	% Error
5	0.1	0.486	0.488	0.50	.411
5	0.3	1.045	1.047	N/A	.095
6	0.1	0.424	0.424	0.44	.007
6	0.3	0.887	0.887	N/A	.045

The extremely small percent errors indicate the validity of the normalized formalism in the special case of quarterwave transformers.

5.3 Results of Non Quarterwave Power Transmission Analysis

Using [4], an exact solution for an acoustic transformer exhibiting an exponential impedance profile is developed in section 4.3. Section 4.4 demonstrates that under the special condition that the ratio of acoustic velocities, s, is equal to 1, the normalized formalism can be directly compared to the exact solution and thus its validity can be tested. The graph of power transmission vs. frequency found in Figure 13 was formed using the exact solution.

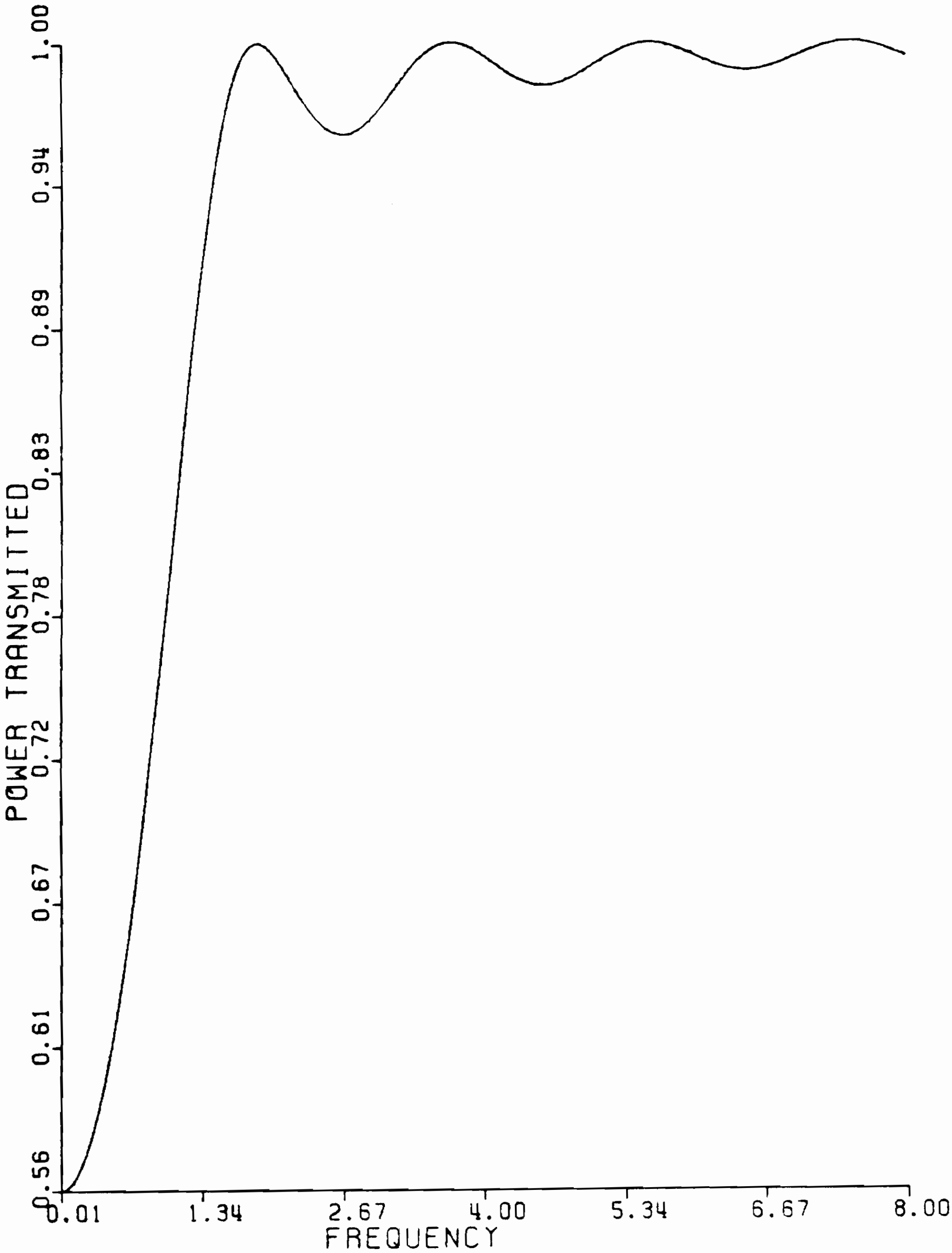


Figure 13 - Exact Solution

The required inputs as described in section 4.3 were the percent change in impedance at the input and output, j_2 and j_1, and the ratio of acoustic impedances, q. These variables were given the values 51%, 5.2%, and 5 respectively. For purposes of comparison, the normalized formalism was applied using the same inputs and s=1. As indicated by the flow chart in Figure 6, a value for the number of layers, N, was also needed. N was chosen to be 20 as this number was sufficiently large enough to ensure a continuous profile solution. The resulting graph of power transmission vs. frequency is shown in Figure 14.

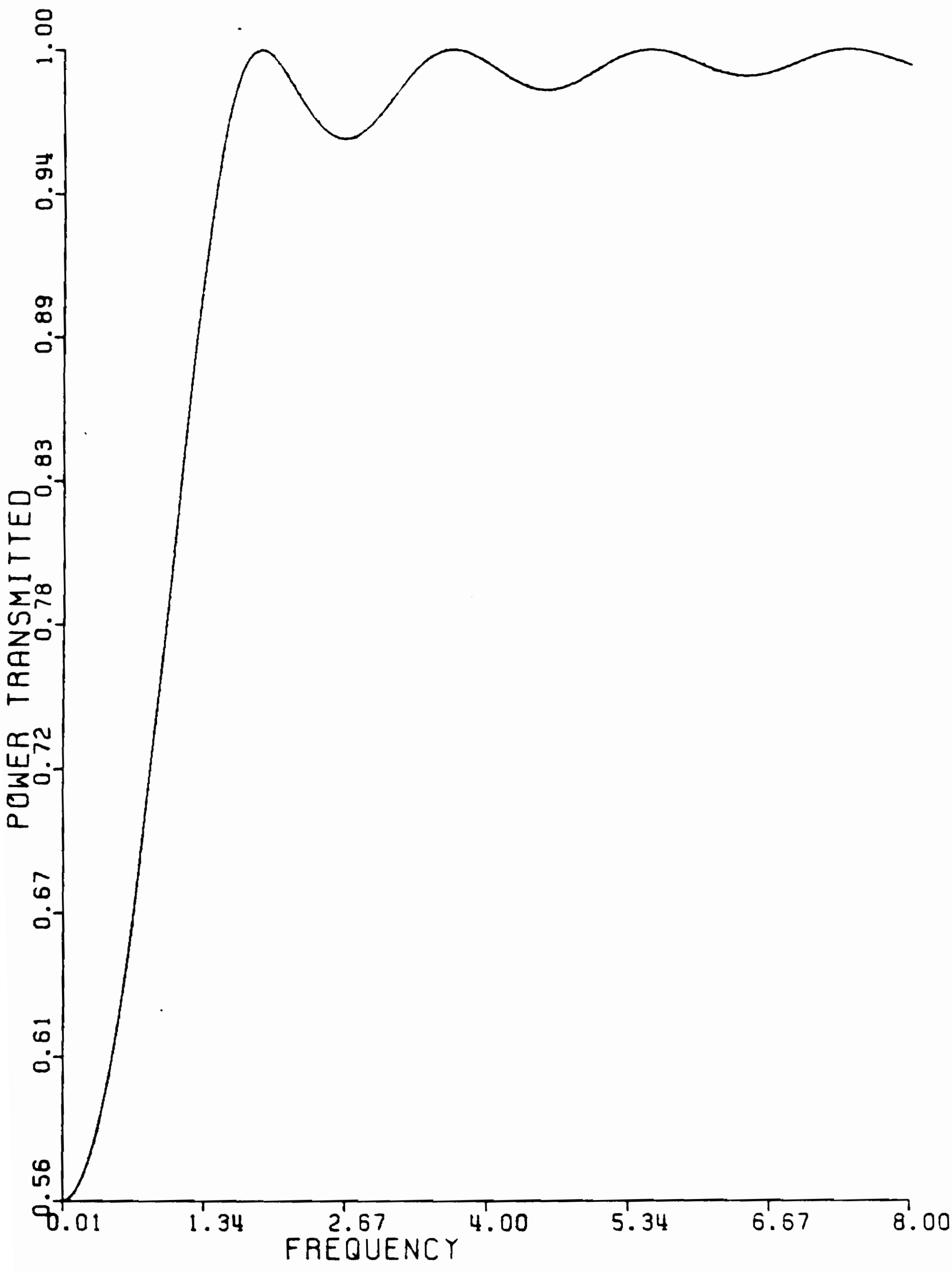


Figure 14 - Normalized Formalism

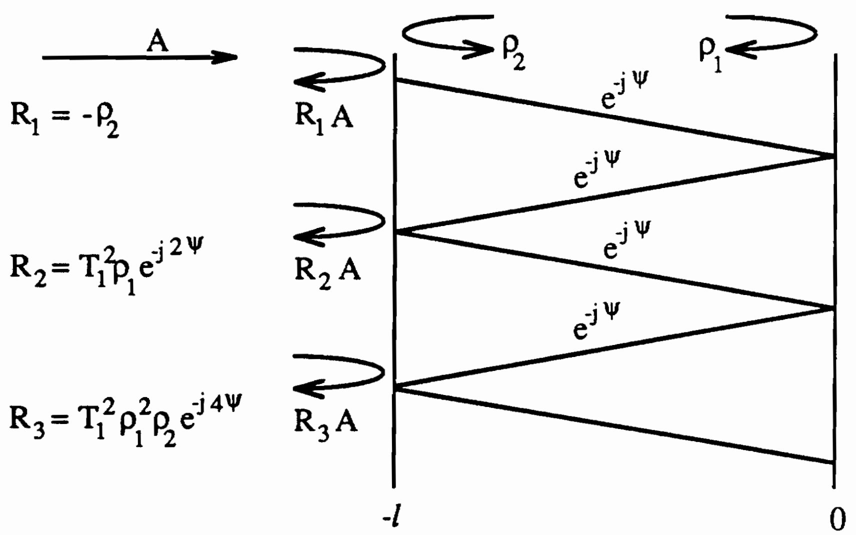
On comparison of Figures 13 and 14 , it is obvious that the two graphs are virtually identical. In fact, the numerical data for power transmission in the graphs always agreed to at least the second decimal place. Thus, the validity of the normalized formalism in a non quarterwave case has been demonstrated.

5.4 Final Remarks

A one dimensional normalized model for the frequency response of the acoustic power transmitted through nonuniform materials has been developed. Using the ideal mixture model, the normalized model demonstrated that the power transmission characteristics are completely determined using only a composition profile and the parameters defining percent variation in acoustic velocity and impedance. The inherent limitations of the ideal mixing model as proposed in the literature was discussed. Note that the two cases which show very close agreement with the normalized formalism do not take full advantage of the ideal mixture model. In the first case, the quarterwave transformer, the transformer acoustic velocity fell out of the formalism. In the second case, the exponential profile, the s parameter was set to one. As expected, the ideal mixing model forced the velocity variation across the transformer to be zero. It is undeniable that these two special cases could have been implemented without a physical model relating velocity and impedance. In spite of the limitations of the ideal mixture model, the inclusion of the model in the normalized formalism demonstrated the impact a more refined mixture model could have on practical problems. The normalized model also has potential for interfacing with a normalized version of the Mason model. This extended model could then be adapted to include the piezoelectric effect. The kind of applications where these considerations are important fall into the area of ultrasonic transducers.

Appendix A - Plane Wave Superposition Analysis

A mathematical analysis of $\rho_2 = -\rho_1$ (ρ_2 = reflection coefficient at the input, ρ_1 = reflection coefficient at the output) follows. This condition is also discussed in section 2.2 where ρ_2 is analogous to r_1 and ρ_1 is analogous to minus r_2, as is evident by equation (2.2.4). Note that other than at the interfaces at z=0 and z=-l, the variation in intrinsic impedance is ignored.



R_i = Power Reflection Coefficient $\quad$ ρ_1 = reflection coefficient at z = 0

T_i = Transmission Coefficient $\quad$ ρ_2 = reflection coefficient at z = $-l$

$\Psi = \int_0^l \beta \Delta l$ where ß is defined by (2.1.12)

A = Amplitude of a normally incident plane wave

Figure 15 - Superposition of a Plane Wave

From Figure 15, R_i can be generalized in the form,

A.1 $\qquad R_i = T_1{}^2(\rho_1\rho_2)^{(i-2)} * e^{-j(i-1)2\Psi} \qquad$ for $i \geq 2$.

Representing the total power reflection coefficient, R, discretely leads to,

A.2 $$R = \sum_{i=1}^{\infty} R_i .$$

Substituting equation (A.1) into equation (A.2) yields,

A.3 $$R = R_1 + \left(\sum_{i=2}^{\infty} (\rho_1\rho_2)^{i-2} e^{-j(i-1)2\Psi} \right) (T_1{}^2\rho_1).$$

Taking non i dependent quantities out of the summation gives,

A.4 $$R = R_1 + \frac{e^{j2\Psi} T_1{}^2 \rho_1}{(\rho_1\rho_2)^2} \sum_{i=2}^{\infty} (\rho_1\rho_2 e^{-j2\Psi})^i .$$

A geometric series can be approximated by,

A.5 $$\sum_{i=2}^{\infty} r^i \approx \frac{r^2}{1-r} .$$

Letting $r = \rho_1\rho_2 e^{-j2\Psi}$ and substituting equation (A.5) into equation (A.4) results in,

A.6 $$R = R_1 + \frac{e^{-j2\Psi}\, T_1{}^2 \rho_1}{1 - \rho_1\rho_2 e^{-j2\Psi}} .$$

Applying the condition for maximum power transmission, that is zero reflection leads to,

A.7 $$0 = R_1 + \frac{e^{-j2\Psi}\, T_1{}^2 \rho_1}{1 - \rho_1\rho_2 e^{-j2\Psi}} .$$

Noting that $R_1 = -\rho_2$ (see Figure 15) and simplifying produces,

A.8 $$\rho_1\rho_2 + T_1{}^2 \frac{\rho_1}{\rho_2} = e^{j2\Psi}.$$

After multiplying by $\frac{\rho_2}{\rho_1}$ and considering only magnitudes, it follows that,

(Note: the magnitude of $e^{j2\Psi}$ must be -1 because the left hand side of the equation is negative)

A.9 $$R_1{}^2 + T_1{}^2 = (-1)\frac{\rho_2}{\rho_1} .$$

From [1],

A.10 $\quad R_1^2 + T_1^2 = 1.$

Substituting equation (A.10) into equation (A.9) results in,

A.11 $\quad \rho_1 = -\rho_2.$

or

A.12 $\quad r_1 = r_2$

due to the correspondence of r's and ρ's discussed at the beginning of this appendix. Thus setting ρ_1 equal to $-\rho_2$ is conducive to 100% power transmission when the condition that $e^{j2\Psi} = -1$ is satisfied. Note that the conditions obtained here include equations (2.2.2a) and (2.2.2b) as a special case.

Appendix B - Field-Acoustics Analogy

This appendix contains a derivation of acoustic equations for longitudinal waves which are analogous to the transmission line electrical equations.

The strain, S, is defined by,

B.1 $$\frac{\partial u}{\partial x} \equiv S(x,t)$$

where u is the particle displacement and x is the space coordinate. The particle velocity, v_p, is defined by ,

B.2 $$\frac{\partial u}{\partial t} \equiv v_p(x,t)$$

where t is the time.

Taking the time derivative of equation (B.1) yields,

B.3 $$\frac{\partial S(x,t)}{\partial t} = \frac{\partial^2 u}{\partial x \partial t}.$$

Similarly, taking the space derivative of equation (B.2) yields,

B.4 $$\frac{\partial v_p(x,t)}{\partial x} = \frac{\partial^2 u}{\partial x \partial t}.$$

From the equivalency of equations (B.3) and (B.4), it follows that,

B.5 $$\frac{\partial S(x,t)}{\partial t} = \frac{\partial v_p(x,t)}{\partial x}.$$

The stress, T, is defined by,

B.6 $$T(x,t) = cS(x,t)$$

where c is the elastic stiffness constant.

Taking the time derivative of equation (B.6) gives,

B.7 $$\frac{\partial T(x,t)}{\partial t}\frac{1}{c} = \frac{\partial S(x,t)}{\partial t}.$$

Substituting equation (B.5) into equation (B.7) leads to,

B.8 $$\frac{\partial T(x,t)}{\partial t}\frac{1}{c} = \frac{\partial v_p(x,t)}{\partial x}.$$

The stress can also be represented as,

B.9 $$T = \frac{F}{A}$$

where F is the force and A is the area.

From equation (B.9) the net force follows as,

B.10 $$F_{net} = [T(x) - T(x + \Delta L)]A$$

where ΔL is the change in length.

From Newton's Law, it follows that,

B.11 $$F_{net} = ma = p_m(A\Delta L)\frac{\partial^2 u}{\partial t^2}$$

where m is the mass, a is the acceleration, and p_m is the mass per unit volume.

Substituting equation (B.11) into equation (B.10) and simplifying results in,

B.12 $$\frac{[T(x) - T(x + \Delta L)]}{\Delta L} = p_m\frac{\partial^2 u}{\partial t^2}.$$

Applying the definition of derivative leads to,

B.13 $$\frac{\partial T(x,t)}{\partial x} = p_m\frac{\partial^2 u}{\partial t^2}.$$

Taking the time derivative of equation (B.2) yields,

B.14 $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial v_p(x,t)}{\partial t}.$$

After plugging equation (B.14) into equation (B.13), it can be shown that,

B.15 $$\frac{\partial T(x,t)}{\partial x} = p_m\frac{\partial v_p(x,t)}{\partial t}.$$

Letting $\frac{\partial}{\partial t} \rightarrow j\omega$ and rewriting equations (B.8) and (B.15) as phasors gives,

B.16 $$\frac{\partial v_p}{\partial x} = -j\omega\frac{1}{c}(-T)$$

and

B.17 $$\frac{\partial(-T)}{\partial x} = -p_m j\omega v_p.$$

Finally, equations (B.16) and (B.17) are analogous to

B.18 $$\frac{\partial V}{\partial x} = -Z(x)I$$

and

B.19 $$\frac{\partial I}{\partial x} = -Y(x)V,$$

taken from [4]. The analysis in this section confirms the analogy applied in section 2.1.

Appendix C - Proof of Wax and Walker's Equation [4]

In section 4.2, the differential equation describing the evolution of the reflection coefficient in nonuniform materials is applied to acoustic wave propagation. This equation was originally proposed within the context of electrical transmission line analysis. In this appendix, the corresponding acoustic version is shown to be applicable. The mathematical validity of this equation is also established.

C.1 Applicability of the Wax and Walker Equation to Acoustic Propagation

The development of the required correspondence between electrical transmission line waves and acoustic waves begins with the voltage current relations,

C.1.1a $\frac{\partial V}{\partial x} = -Z(x)I$

and

C.1.1b $\frac{\partial I}{\partial x} = -Y(x)V$

written in terms of the corresponding series impedance Z(x) and shunt admittance Y(x). These can be shown to be related to the electrical characteristic impedance, $Z_0(x)$

C.1.2a $Z_0(x) = \sqrt{\frac{Z(x)}{Y(x)}}.$

Similarly, the electrical propagation constant is given by,

C.1.2b $\Gamma = \sqrt{Z(x)Y(x)}.$

Consistent with the discussion in section 2.1, the voltage reflection coefficient is defined as,

C.1.3 $$R(x) = \frac{\frac{V}{I} - Z_0(x)}{\frac{V}{I} + Z_0(x)}$$

Making use of the above equations it is possible to show that R(x) satisfies the first order differential equation

C.1.4 $$\frac{\partial R}{\partial x} - 2R(x)\Gamma(x) + 0.5\left(\frac{\partial(\ln Z_0(x))}{\partial x}\right)(1 - R^2(x)) = 0.$$

The demonstration of this statement is the subject of the next section.

In order to complete the correspondence equations (B.17) and (B.16) have been repeated here for reference.

C.1.5 $$\frac{\partial(-T)}{\partial x} = -f_1(x)v_p \qquad \text{where } f_1(x) = j\omega p_m(x)$$

and

C.1.6 $$\frac{\partial v_p}{\partial x} = -f_2(x)(-T) \qquad \text{where } f_2(x) = j\omega\frac{1}{c(x)}.$$

The definition for the characteristic impedance (Z_0) and the nominal propagation factor ($\Gamma(x)$) are given by equations (C.1.2a) and (C.1.2b). The analogous acoustic equations are,

C.1.7 $$Z_0(x) \equiv \sqrt{\frac{f_1(x)}{f_2(x)}}$$

and

C.1.8 $$\Gamma(x) = \sqrt{f_1(x)f_2(x)}\,.$$

Plugging in $f_1(x)$ and $f_2(x)$ yields,

C.1.9 $$Z_0(x) \equiv \sqrt{p_m c(x)}$$

and

C.1.10 $$\Gamma(x) \equiv j\omega\sqrt{\frac{p_m(x)}{c(x)}}\,.$$

Similarly, the definition for the reflection coefficient (R) is given by equation (C.1.3). The acoustic impedance (Z) is defined in equation (2.1.16) and is repeated here for convenience,

C.1.11 $$Z(x) \equiv \frac{-T}{v_p}\,.$$

The analogous acoustic equation for the reflection coefficient is,

C.1.12 $$R(x) = \frac{Z(x) - Z_0(x)}{Z(x) + Z_0(x)}\,.$$

The acoustic analogy for equation (C.1.4) is now established using acoustic parameters defined by equations (C.1.9), (C.1.10), and (C.1.12).

C.2 Mathematical Validity

The validity of equation (C.1.4) shall be proven now. Each term in the equation will be simplified and the sum of the simplified terms will be shown to be zero.

To simplify the first term, we use equation (C.1.8) and take the derivative of R with respect to x.

$$\text{C.2.1} \quad \frac{\partial R}{\partial x} = \frac{2Z_0(x)\frac{\partial Z}{\partial x} - 2Z(x)\frac{\partial Z_0}{\partial x}}{(Z(x) + Z_0(x))^2}$$

Plugging equation (C.2.1) into equation (C.1.4) results in,

$$\text{C.2.2} \quad \frac{2Z_0(x)\frac{\partial Z}{\partial x} - 2Z(x)\frac{\partial Z_0}{\partial x}}{(Z(x) + Z_0(x))^2} - 2R(x)\Gamma(x) + 0.5\left(\frac{\partial(\ln Z_0(x))}{\partial x}\right)(1 - R^2(x)) = 0.$$

To simplify the second term, equations (C.1.8) and (C.1.6) are applied.

$$\text{C.2.3} \quad R(x)\Gamma(x) = \frac{Z(x) - Z_0(x)}{Z(x) + Z_0(x)}\sqrt{f_1(x)f_2(x)}$$

Multiplying by $1 = \frac{Z(x) + Z_0(x)}{Z(x) + Z_0(x)}$ and plugging equation (C.1.3) into equation (C.2.3) yields,

$$\text{C.2.4} \quad R(x)\Gamma(x) = \frac{Z^2(x)\sqrt{f_1(x)f_2(x)} - f_1(x)Z_0(x)}{(Z(x) + Z_0(x))^2}.$$

Plugging equation (C.2.4) into equation (C.2.2) gives,

$$\text{C.2.5} \quad \frac{2Z_0(x)\frac{\partial Z}{\partial x} - 2Z(x)\frac{\partial Z_0}{\partial x}}{(Z(x) + Z_0(x))^2} - 2\left(\frac{Z^2(x)\sqrt{f_1(x)f_2(x)} - f_1(x)Z_0(x)}{(Z(x) + Z_0(x))^2}\right) + 0.5\left(\frac{\partial(\ln Z_0(x))}{\partial x}\right)(1 - R^2(x)) = 0.$$

The first step in simplifying the third term is to take the derivative. Thus,

$$\text{C.2.6} \quad 0.5\left(\frac{\partial(\ln Z_0(x))}{\partial x}\right)(1 - R^2(x)) = \frac{1}{2Z_0(x)}\frac{\partial Z_0(x)}{\partial x}(1 - R^2(x)).$$

Plugging equation (C.1.8) into equation (C.2.6) and simplifying leads to,

$$\text{C.2.7} \quad 0.5\left(\frac{\partial(\ln Z_0(x))}{\partial x}\right)(1 - R^2(x)) = \frac{\partial Z_0(x)}{\partial x}\frac{2Z(x)}{(Z(x) + Z_0(x))^2}.$$

Plugging equation (C.2.7) into equation (C.2.5) shows that,

C.2.8 \f(2Z_0(x)\f(∂Z,∂x) - 2Z\f(∂Z_0,∂x),(Z(x) + Z_0(x))^2) - 2\b(\f(Z^2(x) \R(f_1(x)f_2(x)) - f_1(x)Z_0(x),(Z(x) + Z_0(x))^2))+ $\frac{\partial Z_0(x)}{\partial x}\frac{2Z(x)}{(Z(x) + Z_0(x))^2}$.

Dividing by 2 and multiplying by $(Z(x) + Z_0(x))^2$ yields,

$$\text{C.2.9} \quad Z_0(x)\frac{\partial Z}{\partial x} - Z^2(x)\sqrt{f_1(x)f_2(x)} + f_1(x)Z_0(x) = 0.$$

It becomes apparent that the first term needs to be simplified by taking the derivative of Z.

Using equation (C.1.7), it follows that,

$$\text{C.2.10} \quad Z_0(x)\frac{\partial Z}{\partial x} = Z_0(x)\frac{\partial}{\partial x}\left(\frac{-T}{v}\right) = Z_0(x)\left(\frac{T}{v^2}\frac{\partial v}{\partial x} - \frac{1}{v}\frac{\partial T}{\partial x}\right).$$

Plugging equations (C.1.1) and (C.1.2) into equation (C.2.10) results in,

$$\text{C.2.11} \quad Z_0(x)\frac{\partial Z}{\partial x} = Z_0(x)\left(\frac{T^2}{v^2}f_2(x) - f_1(x)\right).$$

Plugging equation (C.1.7) into equation (C.2.11) leads to,

$$\text{C.2.12} \quad Z_0(x)\frac{\partial Z}{\partial x} = Z_0(x)\, Z^2(x)\, f_2(x) - Z_0(x)\, f_1(x).$$

Plugging equation (C.1.3) into equation (C.2.12) gives,

$$\text{C.2.13} \quad Z_0(x)\frac{\partial Z}{\partial x} = \sqrt{f_1(x)f_2(x)}\, Z^2(x) - Z_0(x)\, f_1(x).$$

Plugging equation (C.2.13) back into equation (C.2.9) yields,

$$\text{C.2.14} \quad \sqrt{f_1(x)f_2(x)}\, Z^2(x) - Z_0(x)\, f_1(x) - Z^2(x)\sqrt{f_1(x)f_2(x)} + f_1(x)Z_0(x) = 0.$$

Brief examination of the left side of equation (C.2.14) indicates that it is identically equal to zero. Thus, by direct substitution, equation (C.1.4) has been confirmed.

Appendix D - Background Examples of Wax and Walker's Equation [4]

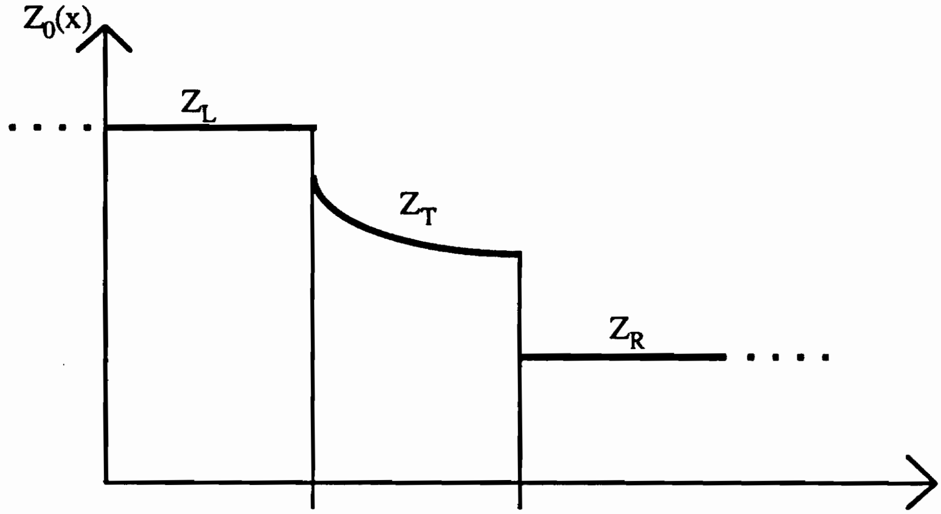


Figure 16 - Basic Transformer Model

Figure 16 represents an acoustic transformer. Z_L is the input, Z_R is the output, and Z_T represents the matching layer(s) which may contain 1 to N layers where N is an arbitrary integer. Figure 16 will be used to provide physical significance to equation (C.1.4) in three examples. The first example deals with reflection at the x=0 interface and is found in section D.1. Similarly, section D.2 deals with reflection at the $x=-l$ interface. Finally, section D.3 shows that under the requirements of a single layer transformer and zero reflection coefficient, equation (C.1.4) leads to quarterwave matching conditions as expected.

D.1 Reflection at x=0.

Acoustic theory predicts (2.1.18) that,

D.1.1 At $x > 0^+$, $R = 0$,

and

D.1.2 At $x = 0^-$, $R = \frac{Z_R - Z_T}{Z_R + Z_T}$.

Using Walker and Wax's equation, the value of R at $x=0^-$ will be calculated and shown to be in agreement with equation (D.1.2) and thus with acoustic theory (2.1.18).

The acoustic equation analogous to Wax and Walker's equation is given by equation (C.1.4) and repeated here.

D.1.3 $$\frac{\partial R}{\partial x} - 2R\Gamma(x) + 0.5\frac{\partial(\ln Z_0(x))}{\partial x}(1 - R^2) = 0.$$

Dividing by $(1 - R^2)$ and isolating the first term gives,

D.1.4 $$\frac{1}{1 - R^2}\frac{\partial R}{\partial x} = \frac{2R}{1 - R^2}\Gamma(x) - 0.5\frac{\partial(\ln Z_0(x))}{\partial x}.$$

Multiplying both sides of equation (D.1.4) by dx and integrating from $x=0^-$ to $x=0^+$ leads to,

D.1.5 $$\int_{R(x = 0^-)}^{R(x = 0^+)} \frac{1}{1 - R^2}\partial R = \int_{0^-}^{0^+}\left(\frac{2R}{1 - R^2}\right)\Gamma(x)\partial x - 0.5\int_{0^-}^{0^+}\frac{\partial(\ln Z_0(x))}{\partial x}\partial x.$$

Integrating the third term and simplifying with the properties of natural logs shows that,

D.1.6 $$\int_{R(x = 0^-)}^{R(x = 0^+)} \frac{1}{1 - R^2}\partial R = \int_{0^-}^{0^+}\left(\frac{2R}{1 - R^2}\right)\Gamma(x)\partial x - 0.5\ln\left(\frac{Z_0(x = 0+)}{Z_0(x = 0^-)}\right)$$

From Figure 16, $Z_0(x = 0^+) = Z_R$ and $Z_0(x = 0^-) = Z_T$. Substituting into equation (D.1.6) results in,

D.1.7 $$\int_{R(x = 0^-)}^{R(x = 0^+)} \frac{1}{1 - R^2}\partial R = \int_{0^-}^{0^+}\left(\frac{2R}{1 - R^2}\right)\Gamma(x)\partial x - 0.5\ln\left(\frac{Z_R}{Z_T}\right)$$

From [12], it can be shown that,

D.1.8 for $x^2 < a^2$, $\int \frac{\partial x}{a^2 - x^2} = \frac{1}{2a} \ln\left(\frac{a + x}{a - x}\right)$.

Noting that R is always less than 1, equation (D.1.8) is applied to equation (D.1.7) with a = 1 , x = R.

D.1.9 $$0.5 \ln\frac{1 + R}{1 - R}\Bigg|_{R(x = 0^-)}^{R(x = 0^+)} = \int_{0^-}^{0^+} \left(\frac{2R}{1 - R^2}\right)\Gamma(x)\partial x - 0.5 \ln\left(\frac{Z_R}{Z_T}\right).$$

But, $R(x = 0^+) = 0$ since there are no reflections once the output region is reached. Substituting into equation (D.1.9) gives,

D.1.10 $$-0.5 \ln\left(\frac{1 + R(x = 0^-)}{1 - R(x = 0^-)}\right) = \int_{0^-}^{0^+} \left(\frac{2R}{1 - R^2}\right)\Gamma(x)\partial x - 0.5 \ln\left(\frac{Z_R}{Z_T}\right).$$

Plugging equation (C.1.10) into equation (D.1.10) yields,

D.1.11 $$-0.5 \ln\left(\frac{1 + R(x = 0^-)}{1 - R(x = 0^-)}\right) = j\omega \int_{0^-}^{0^+} \left(\frac{2R}{1 - R^2}\right)\sqrt{\frac{p_m(x)}{c(x)}}\partial x - 0.5 \ln\left(\frac{Z_R}{Z_T}\right).$$

The mass per unit volume (p_m) and the elastic stiffness constant (c) remain bounded for all x. Similarly, the quantity in brackets in the integral of equation (D.1.11) is also bounded. Therefore,

D.1.12 $$\int_{0^-}^{0^+} \sqrt{\frac{p_m(x)}{c(x)}}\partial x = 0.$$

Plugging equation (D.1.12) into equation (D.1.11), simplifying, and isolating $R(x = 0^-)$ leads to,

D.1.13 $$R(x = 0^-) = \frac{Z_R - Z_T}{Z_R + Z_T}.$$

Thus, equation (D.1.13) which was derived from an acoustically transformed version of Walker and Wax's equation, (C.1.4), is in exact agreement with equation (D.1.2) and thus with acoustic theory (2.1.18).

D.2 Reflection at x=-*l*

As a second proof of the physical significance of Walker and Wax's equation the reflection coefficient at $x = -l^-$ is considered. One can see from Figure 16 that if $R(x = -l^+) = 0$, or equivalently there exists one interface at $x=-l$, acoustic theory (2.1.18) predicts $R(x = -l^-)$ to be,

D.2.1 $$R(x = -l^-) = \frac{Z_T - Z_L}{Z_T + Z_L}.$$

The following calculations show that Walker and Wax's equation yields the same result. The acoustically transformed version of this equation, (C.1.4), is repeated here,

D.2.2 $$\frac{\partial R}{\partial x} - 2R\Gamma(x) + 0.5\,\frac{\partial(\ln Z_0(x))}{\partial x}(1 - R^2) = 0.$$

Dividing by $(1 - R^2)$ and isolating the first term gives,

D.2.3 $$\frac{1}{1 - R^2}\frac{\partial R}{\partial x} = \frac{2R}{1 - R^2}\Gamma(x) - 0.5\,\frac{\partial(\ln Z_0(x))}{\partial x}.$$

Multiplying both sides of equation (D.2.3) by ∂x and integrating from $x=-l^-$ to $x=-l^+$ leads to,

D.2.4 $$\int_{R(x = -l^-)}^{R(x = -l^+)} \frac{1}{1 - R^2}\partial R = \int_{-l^-}^{-l^+}\left(\frac{2R}{1 - R^2}\right)\Gamma(x)\partial x - 0.5\int_{-l^-}^{-l^+}\frac{\partial(\ln Z_0(x))}{\partial x}\partial x.$$

Integrating the third term and simplifying with the properties of natural logs shows that,

D.2.5 $$\int_{R(x = -l^-)}^{R(x = -l^+)} \frac{1}{1 - R^2}\partial R = \int_{-l^-}^{-l^+}\left(\frac{2R}{1 - R^2}\right)\Gamma(x)\partial x - 0.5\ln\left(\frac{Z_0(x = -l^+)}{Z_0(x = -l^-)}\right).$$

From Figure 16, $Z_0(x = -l^-) = Z_L$ and $Z_0(x = -l^+) = Z_T$. Substituting into equation (D.2.5) and simplifying results in, D.2.6 $$\int_{R(x=-l^-)}^{R(x=-l^+)} \frac{1}{1-R^2}\partial R = \int_{-l^-}^{-l^+} \left(\frac{2R}{1-R^2}\right)\Gamma(x)\partial x + 0.5\ln\left(\frac{Z_L}{Z_T}\right).$$

Noting that R is always less than 1, and thus $R^2 < 1^2$, equation (D.1.8) is applied to equation (D.2.6) with a = 1 , x = R. Simplifying with the properties of natural logs leads to,

D.2.7 $$0.5\ln\left(\frac{1+R(x=-l^+)}{1-R(x=-l^+)}\,\frac{1-R(x=-l^-)}{1+R(x=-l^-)}\right) = \int_{-l^-}^{-l^+} \left(\frac{2R}{1-R^2}\right)\Gamma(x)\partial x + 0.5\ln\left(\frac{Z_L}{Z_T}\right).$$

Following the same argument that lead to equation (D.2.5) and noting that at $x=-l$ R is less than 1 leads to,

D.2.8 $$\int_{-l^-}^{-l^+} \Gamma(x)\partial x = 0.$$

Plugging equation (D.2.8) into equation (D.2.7), simplifying, and isolating $R(x=-l^-)$ leads to,

D.2.9 $$R(x=-l^-) = \frac{1 - \dfrac{Z_L}{Z_T}\,\dfrac{1-R(x=-l^+)}{1+R(x=-l^+)}}{1 + \dfrac{Z_L}{Z_T}\,\dfrac{1-R(x=-l^+)}{1+R(x=-l^+)}}.$$

The above equation describes the reflection coefficient at $x=-l^+$ for the general case represented in Figure 16. This equation in a modified form was applied in section 4.3 for the analysis of an acoustic transformer with an exponential intrinsic impedance profile. Under the special condition that $R(x=-l^-) = 0$, equation (D.2.9) becomes,

D.2.10 $$R(x=-l^-) = \frac{Z_T - Z_L}{Z_T + Z_L}.$$

This equation is in exact agreement with (D.2.1) and thus, with acoustic theory (2.1.18).

D.3 Quarterwave Analysis

As a final demonstration of the physical significance of Walker and Wax's equation, the requirement of zero reflection for a single layer transformer is shown to be equivalent to quarterwave matching conditions. The acoustic equation analogous to equation Wax and Walker's equation, (C.1.4), is repeated here,

D.3.1 $$\frac{\partial R}{\partial x} - 2R\Gamma(x) + 0.5\frac{\partial(\ln Z_0(x))}{\partial x}(1 - R^2) = 0.$$

As can be seen in Figure 16, $Z_0(x) = Z_T$ in the case of a quarterwave transformer. Since Z_T is a constant, its derivative is zero and the third term in equation (D.3.1) is zero. Thus,

D.3.2 $$\frac{\partial R}{\partial x} - 2R\Gamma(x) = 0.$$

Rearranging and integrating from $x = 0^-$ to x=x shows that,

D.3.3 $$\int_{R(x=0^-)}^{R(x)} \frac{\partial R}{R} = 2\int_{0^-}^{x} \Gamma(x)\partial x.$$

Integrating the first term and applying the properties of natural logarithms leads to,

D.3.4 $$R(x) = R(x = 0^-)\exp\left(2\int_{0^-}^{x}\Gamma(x)\partial x\right).$$

From equation (C.1.6), $\Gamma(x) \equiv j\omega\sqrt{\frac{p_m(x)}{c(x)}}$. Assuming that the mass per unit volume (p_m) and the elastic stiffness constant (c) do not vary with x within the single matching medium of the acoustic transformer, then $\omega\sqrt{\frac{p_m(x)}{c(x)}}$ can be replaced by the constant ß. Therefore, $\Gamma(x) = j\text{ß}$. Plugging into equation (D.3.4) gives,

D.3.5 $$R(x) = R(x = 0^-)\exp\left(2\int_{0^-}^{x} j\text{ß}\partial x\right).$$

Integrating the exponent and evaluating R(x) at the point $x = -l^+$ yields,

D.3.6 $$R(x = -l^+) = R(x = 0^-)e^{2j\text{ß}(-l^+)}.$$

This equation for $R(x = -l^+)$ was found under the condition of a single layer transformer. A second equation for $R(x = -l^+)$ will now be derived under the condition for zero reflection in a general transformer. Next, these two equations will be set equal to each other and the condition for l to satisfy both equations will be solved for. As expected, these conditions are the quarterwave matching conditions given by equations (2.2.2a) and (2.2.2b). After simplifying equation (D.2.9), it can be shown that,

D.3.7 $$R(x = -l^-) = \frac{Z_T(1 + R(x = -l^+)) - Z_L(1 - R(x = -l^+))}{Z_T(1 + R(x = -l^+)) + Z_L(1 - R(x = -l^+))}.$$

For total transmission, $R(x = -l^-)$ must be zero. Using this fact, rearranging, and isolating $R(x = -l^+)$, results in,

D.3.8 $$R(x = -l^+) = \frac{Z_L - Z_T}{Z_L + Z_T}.$$

Setting equations (D.3.6) and (D.3.8) equal gives,

D.3.9 $$R(x = 0^-)e^{2j\beta(-l^+)} = \frac{Z_L - Z_T}{Z_L + Z_T}.$$

It is apparent that the right side of the equation is a real quantity. The expression for $R(x = 0^-)$, which is given by equation (D.1.13), is also real. Thus, $e^{2j\beta(-l^+)}$ must also be real. Expanding with Euler's formula leads to,

D.3.10 $$e^{2j\beta(-l^+)} = \cos(2\beta l) - j\sin(2\beta l).$$

For $e^{2j\beta(-l^+)}$ to be real, it must equal either ± 1. If $e^{2j\beta(-l^+)} = +1$, then,

D.3.11 $$2\beta l = 2n\pi$$

where n is an integer.

Dividing by 2β and plugging equation (3.3.6) into equation (D.3.11) yields,

D.3.12 $$l = \frac{n\lambda}{2}.$$

If $e^{2j\beta(-l^+)} = -1$, then,

D.3.13 $$2\beta l = m\pi,$$

where m is an **odd** integer.

Dividing by β and using equation (3.3.6) yields, D.3.14 $l = \frac{m\lambda}{4}$.

Equations (D.3.12) and (D.3.14) specify two conditions for l. Further analysis is required.

Let $e^{2j\beta(-l+)} = +1$. Plugging this condition into equation (D.3.9) results in,

D.3.15 $R(x = 0^-) = \frac{Z_L - Z_T}{Z_L + Z_T}$.

Plugging into equation (D.1.13) leads to,

D.3.16 $\frac{Z_R - Z_T}{Z_R + Z_T} = \frac{Z_L - Z_T}{Z_L + Z_T}$.

Equation (D.3.16) implies that $Z_R = Z_L$. If this were true, there would be no matching layer and hence no quarterwave transformer. Thus the condition that $l = \frac{n\lambda}{2}$ (D.3.12) is not useful. Performing an analysis similar to the one above with $e^{2j\beta - l+} = -1$ yields,

D.3.17 $Z_T = \sqrt{Z_R Z_L}$.

Or, equivalently, the impedance of the matching layer is equal to the geometric mean of the input and output materials. This condition for Z_T and the corresponding initial condition for l, $l = \frac{m\lambda}{4}$, where m is an odd integer, are in agreement with acoustic theory (2.2.2.a, 2.2.2b).

Appendix E - Limitations of the Ideal Mixture Model

As noted in the text, the last two steps, (3.1.14) and (3.1.15), which related the incompressibility to the acoustic impedance and velocity are inaccurate. In order to more clearly account for this defect, the ideal mixing assumption was applied directly to the stiffness constant of the mixture (3.1.16). The modified mixture equations (3.1.22) and (3.1.23) are then completely accurate in the saturation limits of the mixing; a feature not shared by relations (3.1.14) and (3.1.15). To access the error inherent in the assumption for the stiffness of the mixture,

E.1 $$\frac{1}{c_m} = \sum_{i=1}^{N} \frac{V_i}{c_i},$$

it is necessary to make use of the established relation for the compressibility of the mixture,

E.2 $$k_m = \sum_{i=1}^{N} k_i V_i.$$

In order to keep this appendix self contained, the discussion begins with the fundamental tensor relations defining stress and strain. For nonpiezoelectric materials, stress T and strain S can be related, in full tensor notation, using the stiffness c, according to,

E.3 $$T_{kl} = c_{ijkl} S_{ij}.$$

Similarly, a second equivalent form using compliances s is given below.

E.4 $$S_{kl} = s_{ijkl} T_{ij}.$$

Due to inherent symmetries present in materials an abbreviated tensor notation has been introduced [1], [13]. Using the abbreviated notation stress and strain become tensors of rank 1 with 6 components and both the compliance and stiffness reduce to a tensor of rank 2. For materials with a level of symmetry higher than cubic, both the compliance and stiffness (6x6) matrices take the following form:

E.5
$$\begin{pmatrix} x_{11} & x_{12} & x_{12} & 0 & 0 & 0 \\ x_{12} & x_{11} & x_{12} & 0 & 0 & 0 \\ x_{12} & x_{12} & x_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & x_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & x_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & x_{44} \end{pmatrix}$$

where [13]

E.6a $$s_{11} = \frac{c_{11} + c_{12}}{(c_{11} - c_{12})\,(c_{11} + 2c_{12})},$$

E.6b $$s_{12} = \frac{-\,c_{12}}{(c_{11} - c_{12})\,(c_{11} + 2c_{12})},$$

E.6c $$s_{44} = \frac{1}{c_{44}}.$$

Because of the symmetry in form or by direct inversion relations equations (E.6) are also correct upon interchange of s and c. For isotropic materials, the following condition should also be included,

E.7 $$\frac{c_{11} - c_{12}}{2} = c_{44}.$$

Consistent with (E.7), the standard Lamé coefficients, introduced in section 3.1, satisfy,

E.8a $c_{11} = \lambda + 2\mu$

E.8b $c_{12} = \lambda$

E.8c $c_{44} = \mu$.

Similarly, for isotropic materials the compliance coefficients can be shown to satisfy [13]

E.9 $2(s_{11} - s_{12}) = s_{44}$.

Consistent with equation (E.9) the compliance version of the Lamé coefficients are defined here.

E.10a $s_{11} = \eta + \frac{U}{2}$

E.10b $s_{12} = \eta$

E.10c $s_{44} = U$

The stress-strain relation for the case of nonshear stresses can be written [1],

E.11 $T_i = \lambda(S_1 + S_2 + S_3) + 2\mu S \quad i = 1, 2, 3$

where the subscript refers to the direction in space (x, y, z $\rightarrow$ 1,2,3). The corresponding version in terms of compliance Lamé constants would take the form,

E.12 $$S_i = \eta(T_1 + T_2 + T_3) + \frac{U}{2}T_i \quad i = 1, 2, 3.$$

The compressibility can be defined as,

E.13 $$k = \frac{\Delta}{T}$$

where Δ is the volume dilation [1] under conditions of an ambient stress T. Under these conditions,

E.14a $$\Delta \equiv S_1 + S_2 + S_3 = 3S_1$$

$$T = - T_1 = - T_2 = - T_3$$

After combining equations (E.12-E.14), it follows that,

E.15 $$k = 9\eta + \frac{3}{2} U.$$

Equation (E.15) would be applicable to mixtures if η_m and U_m were known. Specifically, required equality of (E.2) and (E.15) suggests that the independent coefficients η and U are necessarily linear for the fractional volumes (V_i) and therefore,

E.16a $$\eta_m = \sum_i \eta_i V_i$$

E.16b $$U_m = \sum_i U_i V_i.$$

In order to access the error in (E.1) the linearity in fractional volumes of the reciprocal stiffness c_{11} can be checked. After expressing $\frac{1}{c_{11}}$ using relation (E.6) and writing in terms of η and U (E.10) the following relation is obtained,

E.17a $$\frac{1}{c_{11}} = \frac{1}{2}U\left(1 + \frac{\eta}{2\eta + \frac{1}{2}U}\right)$$

which using equations (E.6) and (E.8), can be written as,

E.17b $$\frac{1}{c_{11}} = \frac{1}{2}U\left(1 + \frac{- \lambda}{\lambda + 2\mu}\right)$$

It is noted that the leading term U according to equation (E.16b) is linear in the fractional volume. The error E inherent in the assumption (E.1) is due to the variation across the mixture defined by,

E.18a $$E = \delta\left(\frac{-\lambda}{\lambda + 2\mu}\right).$$

Using (E.8) this can be expressed as

E.18b $$E = \delta\left(\frac{c_{11} - 2c_{44}}{c_{11}}\right).$$

As an example, the tungsten-vinyl case discussed in section 5.1 can be examined. The following data taken from Table 3 in [1] is needed in the calculation.

Tungsten	$c_{11} = 58.1$	$c_{44} = 13.4$
Polyethylene (plastic)	$c_{11} = 0.34$	$c_{44} = 0.026$

where the stiffness constants have been normalized by $10^{10}\,\frac{\mathrm{Nt}}{\mathrm{m}^2}$. By direct calculation,

$$\frac{58.1 - 26.8}{58.1} = 0.53$$

$$\frac{0.34 - 0.052}{0.34} = 0.84.$$

Since (0.84 - 0.53) = 0.31, the error introduced by the ideal mixture model even under ideal conditions can be significant.

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Vita

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