Decentralized Control Of Large Space Structures: An Overview
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(ABSTRACT)

This thesis examines several techniques for the design of decentralized control strategies for the active control of vibrational damping in large space structures. A brief description of the finite element method is presented to explain the derivation of mathematical models of flexible structures represented by systems of linear second-order ordinary differential equations. The fundamental ideas of modal analysis are introduced to explain the concepts of vibrational modes and mode shapes, and derive the modal coordinate state space representation of flexible structures.

The decentralized fixed modes of a system are defined, and several important characterizations of decentralized fixed modes are presented. Alternate characterizations of fixed modes yield additional insight into the nature of fixed modes and often provide new methods for calculating the fixed modes of a system.

The use of collocated rate feedback for robust vibrational damping control is described. It is shown that the robustness of collocated rate feedback is due to the positivity of large space structures, an extension of the mathematical concept of positive real functions to dynamic systems.

Another strategy for the control of vibrational damping in large space structures, known as uniform damping control, is also described. It is shown that compared to collocated rate feedback, uniform damping control achieves increased performance at the price of decreased robustness at low frequencies.
The application of decomposition techniques to the design of decentralized control laws is described, and a special type of decomposition known as an overlapping decomposition is introduced. It is shown how overlapping decompositions can be used to design control laws for systems for which the more familiar disjoint decomposition techniques often fail to yield satisfactory results.

Finally, these decentralized control techniques are illustrated using a model of a proposed large space structure, the NASA COFS mast.
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Finally, I would like to dedicated this work to the memory of my great-uncle Theodore Taylor.
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1.0 Introduction

The next generation of spacecraft proposed by the National Aeronautics and Space Administration (NASA) and its counterparts, is characterized by large orbiting space structures constructed from lightweight composite materials. Many of the proposed spacecraft are significantly larger than any structures previously placed in orbit. The present generation of earth orbiting spacecraft consists primarily of satellites carried into orbit as a single unit by the space shuttle or a traditional rocket booster. Even the United States' Skylab module and the Soviet Union's Mir space station were launched as single rocket payloads.

By employing new composite materials with high strength-to-weight ratios, engineers can design much larger structures without exceeding launch vehicle payload weight constraints. Due to space limitations, however, these large structures must be designed to fit into small containers that can be carried by the space shuttle or conventional rocket boosters. Otherwise, the structures will have to be launched in sections, then assembled in space.

Examples of proposed large space structures include space-based antennas ranging in size from 30m to 20km, and a permanent manned space station. Other NASA
projects which involve the study of large space structure control include the Spacecraft Control Laboratory Experiment (SCOLE), and the Control Of Flexible Structures (COFS) experiment.

As a result of the use of lightweight composite materials in their construction, large space structures often have vibrational modes with low natural frequencies and very little damping. This characteristic lack of inherent structural damping presents a particular problem when the structure is designed to function as a satellite or a platform for sensitive instrumentation and experiments. Consequently, the active suppression of unwanted structural vibrations is expected to be one of the primary concerns in the control of large space structures.

The primary objective of active vibration suppression for large space structure is to add damping to the structure. The implementation of vibration suppression control strategies for large space structures typically requires large numbers of sensors and actuators located throughout the structure. The sensors measure the spacecraft motion and provide feedback information for the control system, and the actuators apply the control forces to the structure.

Due to the spatial distribution of the sensors and actuators on a large space structure, minimization of the exchange of information between the controller and the sensors and actuators is desirable. Assuming there is a correlation between system reliability and the complexity of the feedback interconnections between the controller and the sensors and actuators, minimizing these interconnections should increase system reliability, which is a primary concern in space based systems. In addition, depending on the function of the structure, the on board computing capabilities available to the controller may be limited; hence, a reduction in the control law complexity is also desirable.
The majority of the existing techniques for analyzing dynamic systems and designing active control strategies rely on the assumption that the system is centralized; that is, all of the information about the system is available at one central location. In centralized control strategies, one central controller processes all of the available sensor information and generates the necessary actuator inputs; the control input to any actuator may be a function of all of the sensor outputs. Unfortunately, many of the centralized analysis and design techniques fail when the system is decentralized.

When the control strategy for a system consists of a number of controllers with no communication between them, we say that the control strategy is decentralized. Each of the individual controllers is usually designed to control some subsystem of the overall system; hence, each of the individual controllers should require less computational ability than a single centralized controller for the overall system. Therefore, the reductions in control law complexity and information flow suggest that decentralized control is a natural choice for vibration suppression in large space structures.

In the control of large space structures, we are interested in decentralized control strategies in which the input to each actuator only depends upon measurements from sensors located within some spatial neighborhood of that actuator; thus, the vibration suppression control law for the structure consists of a collection of local feedback control laws for the individual actuators. The geographic neighborhoods may also be defined so that several actuators act together to suppress vibrations in a larger region of the structure. The most restrictive form of decentralization corresponds to the situation in which sensors and actuators are mounted on the structure in pairs and feedback is only permitted between the collocated pairs of sensors and actuators.

In this thesis we examine techniques for the design of decentralized control strategies for large space structures. We are primarily interested in the active control
of structural vibrations, however, several of the techniques presented in this thesis also apply to more general classes of decentralized control problems.

There are many important issues associated with the design of decentralized control systems for large space structures. An investigation of all of these issues, however, is beyond the scope of this thesis. The primary issues that we address are large space structure modelling, decentralized stabilizability and controllability, robust controller design, decentralized feedback controller design, and the application of decomposition techniques to decentralized control.

Previous surveys of decentralized control techniques for large scale systems concentrated on identifying the important issues associated with the decentralized control of large scale systems, and discussed only one or two actual techniques for decentralized control design [1,2]. Instead of repeating this type of study we have chosen to present a detailed examination of several decentralized techniques, each of which is representative of an important issue in the decentralized control of large space structures.

Mathematical models for large space structures are often determined using the finite element method. The mathematical models for flexible structures obtained through finite element analysis consist of a system of linear, second-order, ordinary differential equations. The coordinate system in which the mathematical model for a structure is represented is very important to the control law design. For any real structure, there exists a coordinate transformation that completely decouples the system of ordinary differential equations that describe the mathematical model [4,5]. This process, known as modal analysis, identifies the vibrational modes and mode shapes of the structure. In modal coordinates, the mathematical model for the structure consists of a system of independent second-order ordinary differential equations - one for each of the modelled vibrational modes.
Another primary concern in the design of decentralized control strategies is decentralized stabilizability - the ability of decentralized control to stabilize a system. One of the most important results concerning the stabilization of linear time invariant systems is that there exists a linear time invariant dynamic controller such that the closed loop poles of the system can be placed arbitrarily if and only if the system is controllable and observable. This result fails, however, when the controller is not centralized. The existence of a stabilizing decentralized control law for a system depends upon the properties of a set of numbers for the system known as "decentralized fixed modes" [7].

The question of how many and which vibrational modes to include in the mathematical model of a flexible structure is a difficult problem that is still being explored [3]. In addition, the numerical model parameters determined by finite element analysis are often in error by as much as 10%. As a result, it is very important that any decentralized control design for a large space structure be robust. One approach to robust feedback design for large space structures is the use of colocated rate feedback.

Colocated rate feedback employs direct feedback connections between colocated pairs of rate sensors and force actuators. When implemented using perfect sensors and actuators, colocated rate feedback controllers provide closed-loop stability regardless of the number of modes included in the design model and model parameter errors [25,26]. The robustness of colocated rate feedback for large space structures is due to the positivity of large space structures [23], an extension of the mathematical notion of positive real analytic functions to dynamic systems. Although colocated rate feedback is robust, large feedback gains are usually required to achieve the desired closed-loop damping.

Uniform damping control is an example of an output feedback control strategy designed specifically for the suppression of structural vibrations in large space structures. The uniform damping control law represents the discrete implementation of a distributed
optimal control law for vibration suppression in flexible space structures [31]. The uniform damping control law employs a linear combination of position and rate feedback, and the feedback gains are chosen so that all of the closed-loop poles of the system are stable and have the same real part. This is accomplished, however, without altering the system's natural modes or natural frequencies. Because all of the closed-loop poles have the same (negative) real part, the motion due to each of the system's vibrational modes decays at one uniform exponential rate, hence the name uniform damping control. One drawback of uniform damping control is that the control law may not be decentralized. In many applications, however, we can approximate the uniform damping control law with a decentralized control law with little change in the closed loop system poles.

Decomposition plays an important role in the analysis and control of large scale systems. It is often beneficial, if not absolutely necessary, to decompose a large scale system into a number of interconnected subsystems. Each subsystem is considered independently, and the individual solutions are then combined to obtain a solution for the original system. Thus, decomposition reduces the task of solving a given problem for a large scale system to solving the problem for a number of smaller systems, and in the process, reducing the computational burden. The decomposition of the system not only serves to reduce the computational burden of solving a large scale system problem, but may also provide valuable insight into the effect of the subsystem interconnections on the behavior of the combined system. The notion of connective stability relates changes in the subsystem interconnections to stability of the overall system [33].

Decentralized control is a natural extension of the concept of system decomposition to control system design. After a decomposition is available for a given system, control laws are designed for each of the subsystems, then implemented locally. Typically, little attention, if any, is given to the subsystem interconnections in the
development of the decentralized control laws. Therefore, the ability of the decentralized control law to stabilize the system or achieve a certain closed-loop system behavior may depend upon the subsystem interconnections - which were ignored in the design process.

Many large scale systems often contain subsystems which are strongly connected. These so called strong subsystem interconnections may arise through the sharing of certain system dynamics between subsystems, and the subsystems are said to overlap. Typically, when a system contains strongly connected subsystems, disjoint decompositions fail to produce useful results. When the strong subsystem interconnections arise from overlapping subsystems, another type of decomposition, known as an overlapping decomposition, may prove more useful. In an overlapping decomposition, the portion of the state vector representing the system dynamics shared by the strongly connected subsystems is duplicated in the subsystems, so that the sum of the subsystem dimensions is greater than the dimension of the overall system [36].

The state space - and possibly the input and output spaces - of the dynamic system are "expanded", via singular transformations, into corresponding state (and input or output) spaces of higher dimension such that the overlapping subsystems appear disjoint [34,35].

Although we have tried to present a comprehensive representation of decentralized techniques for large space structures, there are several important issues that we have not addressed. One such issue is model simplification and reduction. Because of the size of the mathematical models for flexible structures, model reduction is a very important problem.

Another issue that we have not discussed in detail is the stability of decentralized systems - how is the stability of the composite system related to the stability of the subsystems? A related topic is the issue of connective stability, that is, how is the
stability of the system affected by eliminating one or more of the subsystems. We have also not discussed decentralized stochastic control, decentralized state estimation.

One class of techniques for the design of decentralized control strategies that we have failed to mention, but which has received a lot of attention in the past, uses parameter optimization to determine the decentralized feedback gains for the system. The decentralized control problem is posed as a constrained optimization problem and solved numerically.

In Chapter 2 we discuss the mathematical modelling of flexible structures. A brief description of finite element analysis is provided as background for the discussion of flexible structure modelling, since the finite element method is frequently used to generate mathematical models of flexible structures. The mathematical models determined using the finite element method consist of systems of linear, second-order, ordinary differential equations. The basic concepts of modal analysis are introduced to decouple the system ordinary differential that comprise the mathematical model and to define the ideas of structural modes, and mode shapes. Finally, a state variable model for large space structures is derived from the system of independent second-order ordinary differential equations identified using modal analysis.

Chapter 3 is concerned with decentralized fixed modes. Although fixed modes may be defined with respect to any arbitrary feedback scheme, our primary interest is in fixed modes of systems for which the controllers are decentralized. In this chapter, we present necessary and sufficient conditions for decentralized stabilizability and pole placement in terms of the decentralized fixed modes of a system. Several characterizations of fixed modes are presented which provide insight into the relationship between fixed modes and the system structure, and also yield additional computational tests for the identification of fixed modes.
In Chapter 4 we describe the application of positivity concepts to control system design, and the design of robust controllers for vibration supression in large space structures, using collocated rate feedback. The robustness properties of collocated rate feedback depend upon the plant and controller both satisfying certain positivity concepts. We describe the mathematical concept of positive real analytic functions and show the extension of positivity concepts to dynamic systems. The application of positivity concepts to control system design guarantees closed-loop stability of systems in which the plant and controller both satisfy certain positivity requirements. We also show that large space structures with collocated rate sensors and force actuators represent positive real systems, and that negative definite collocated rate feedback guarantees stability of the closed loop system. Although this result holds only for perfect sensors and actuators, we show that these conditions can be relaxed without affecting the stability of the closed-loop system.

Chapter 5 describes uniform damping control; a feedback control law designed to provide the same amount of damping in all of the structures modelled vibrations modes, without affecting the natural frequencies or mode shapes. In addition to describing the uniform damping control law, we compare uniform damping control and collocated rate feedback. Using root locus analysis, we study the effects of both uniform damping control and collocated rate feedback on single input single output models for flexible structures.

In Chapter 6 we examine the use of overlapping decompositions in system analysis and control design. The familiar notions of aggregation and restriction are redefined in the framework of the Inclusion Principle. The notions of expansions and contractions of dynamic systems are introduced, and their application to control system design is explained. Finally, the notion of inclusion is extended to include not only dynamic
system but also cost functional to improve measures of suboptimality in overlapping control designs.

Chapter 7 provides an example of decentralized control design for large space structures. The decentralized control techniques described in this thesis are applied to a mathematical model to a proposed large space structure, the NASA COFS mast. This example serves to illustrate some of the advantages and disadvantages of the techniques presented in this thesis.

Chapter 8 contains conclusions and explores some possible topics for future research.
2.0 Flexible space structure dynamics

One of the first issues that must be addressed in the analysis and design of control systems for any system is the development or identification of a mathematical model for the system. The importance of the mathematical model cannot be underestimated; almost all analysis and design is based on the mathematical model of the system, not the physical system.

Most mathematical models for flexible structures are described by either partial or ordinary differential equations. There are advantages and disadvantages to both types model representations, however, and the question of which provides a more accurate description of flexible structures remains an active topic of debate [3]. Currently, the most popular approach to flexible structure modelling may be the finite element method. The finite element method is a numerical method for determining the numerical parameters for mathematical models of flexible structures consisting of systems of simultaneous ordinary differential equations.

The solution of a system of ordinary differential equations describing the mathematical model of a flexible structure is greatly simplified when the equations are decoupled. Even when the equations are not decoupled, however, there exists a linear
transformation which completely decouples the system of ordinary differential equations. The transformation to the new coordinate system, known as modal coordinates [4], identifies the structure's vibrational modes and permits description of structural vibrations as a linear combination of vibrations due to each of these modes. The use of modal coordinates leads to a state space representation of the system that we shall use throughout this thesis.

2.1 Mathematical modeling

There are several approaches to the modelling of flexible structures, but the two most common are the use of partial and ordinary differential equations. Partial differential equations typically describe mathematical models of distributed parameter systems; the model parameters are considered distributed throughout the domain of the system. Ordinary differential equations, on the other hand, represent discrete or lumped-parameter models for flexible structures. In lumped-parameter models, the attributes of entire regions of a structure are lumped together and the structure is modelled as a finite collection of discrete elements. Lumped-parameter models for structures represent systems having a finite number of degrees of freedom. Systems described by distributed parameter models are said to have an infinite number of degrees of freedom.

Distributed parameter models for flexible structures are usually described by partial differential equations of the form

\[
\rho(x) \frac{\partial^2}{\partial t^2} q(x,t) = - Lq(x,t) + f(x,t)
\]  \hspace{1cm} (2.1.1)
where \( q(x,t) \) denotes the inertial displacement at a point \( x \) in the domain of the structure, at time \( t \). The mass density of the structure at the point \( x \) is \( \rho(x) \), and \( f(x,t) \) represents the combined effects of any external forces acting on the structure, including any control forces and force disturbances. The differential operator \( L \) is known as the stiffness operator, and the term \( Lq(x,t) \) in (2.1.1) represents the spatial derivative of the structural displacement \( q(x,t) \). It is typically assumed that the unforced motion of the structure is undamped and linear, in which case the differential stiffness operator \( L \) is positive semidefinite, self-adjoint, and admits a discrete spectrum [31].

One popular approach to the solution of partial differential equations such as (2.1.1) is the separation of variables method. In the separation of variables method, we assume that the spacecraft displacement \( q(x,t) \) can be expressed as an infinite sum of displacements, each of which is the product of a spatially dependent component and a time dependent component. The resulting solution to the partial differential equation has the form:

\[
q(x,t) = \sum_{i=1}^{\infty} \phi_i(x)\eta_i(t). \tag{2.1.2}
\]

The spatially dependent components of the displacement, \( \phi_i(x) \), are called mode shapes, and the time dependent components, \( \eta_i(t) \), are known as vibrational modes. The vibrational modes \( \eta_i(t) \) in (2.1.2) represent an infinite dimensional basis for describing the structural displacement \( q(x,t) \), and the mode shapes \( \phi_i(x) \) denote the displacement at a point \( x \) relative to this basis.

Substitution of (2.1.2) into the partial differential equation (2.1.1) results in a separation of the partial differential equation into two infinite dimensional systems of ordinary differential equations - one for the mode shapes, and one for the vibrational
modes. Consequently it is often said that flexible structures are described by infinite dimensional mathematical models.

Lumped-parameter, or discrete models for flexible structures are described by systems of ordinary differential equations. Some opponents of the use of discrete modelling techniques for flexible structures argue that the lumped-parameter models represent only an approximation of the actual structure [3]. The use of discrete models for flexible structures also represents one approach to solving the partial differential equations governing the motion of a structure; mathematical models designed to approximate distributed parameter systems are often derived as the limiting cases of finite-dimensional discrete system models as the number of discrete mass elements approaches infinity and the respective masses become infinitesimal. This type of approximation contributes to the notion that flexible structures must be represented by infinite-dimensional models.

In the modelling of flexible structures, it is usually assumed that the generalized displacement of the structure from equilibrium is small enough so that the force-displacement and velocity-displacement relationships governing the motion of the structure are linear. This assumption, known as the "small motions assumption", results in descriptions of lumped-parameter models for flexible structures by systems of linear, second-order, ordinary differential equations. Similar assumptions resulted in restrictions on the stiffness operator, \( L \), in the partial differential equation (2.1.1).

The general form of a system of ordinary differential equations describing the motion of a flexible structure is

\[
M\ddot{q}(t) + C\dot{q}(t) + Kq(t) = Ef(t)
\]  

(2.1.3)

where \( q(t) \) is the displacement of the structure in generalized coordinates. Each element of the vector \( q(t) \) represents the displacement of a point on the structure. The vector
$f(t)$ represents control forces applied to the structure. We assume that we can model any other forces which may affect the system as disturbances in the control force. The matrix $E$, known as the input influence matrix, describes the effect of each input $f(t)$ on the system.

The matrices $M$, $C$, and $K$ in (2.1.3) are known respectively as the mass, damping, and stiffness matrices, and are all dimensioned appropriately. Many modelling techniques are unable to identify structural damping, however, and the damping matrix $C$ is often not included in the model (2.1.3).

For any real structure, the mass, damping, and stiffness matrices in (2.1.3) are all symmetric, and the mass and stiffness matrices are positive definite, and is positive semidefinite respectively. The positivity requirements on the mass and stiffness matrices are the result of similar conditions on the kinetic and potential energy of the structure [4,5].

Although the mass matrix is always positive definite, the stiffness matrix may be either positive definite or positive semidefinite depending on the potential energy of the system. When the potential energy of the system (2.1.3) is positive definite, the unforced system may exhibit undamped free vibration, but no rigid-body motion; the stiffness matrix, in this case, is positive definite and the model is said to include only vibrational modes, or that the rigid-body modes have been removed from the model. When the model contains both vibrational and rigid-body modes, the structure may exhibit both undamped free vibration and rigid-body motion; therefore, the potential energy of the system, and the stiffness matrix, are positive semidefinite.
2.2 Finite element model

One of the most popular methods of determining the numerical parameters for mathematical models of flexible structures is the use of finite element analysis, also known as the finite element method. The finite element method regards a complex structure as a collection of finite elements, each of which is part of a continuous structural member. Certain points shared by several elements are known as nodes. At each node, the displacements of the adjacent elements are required to be compatible and the internal forces in balance. When these boundary conditions at the nodes are satisfied, the collection of elements, and hence the entire structure, acts as one entity.

Although the finite element method considers continuous elements, it is essentially a discretization process; the displacement at any point on the structure is expressed in terms of a finite number of nodal displacements multiplied by corresponding interpolation functions [4]. Because of this interpretation, it has been suggested that the finite element method should be considered as a numerical approach to the solution of the partial differential equation model describing a structure [3].

Mathematical models of flexible structures determined using finite elements have the form

\[ M\ddot{q}(t) + Kq(t) = u(t) \]  \hspace{1cm} (2.2.1)

where \( q(t) \) is the displacement of the structure in generalized coordinates, and \( M \) and \( K \) are the mass and stiffness matrices respectively, as in (2.1.3). The vector \( u(t) \) denotes control, and disturbance, inputs to the structure. When the effects of individual actuators are included in the model the input vector is given by
\[ u(t) = E f(t), \]  

(2.2.2)

where \( f(t) \) denotes the force generated by the actuators, as in (2.1.3).

One of the distinguishing characteristics of flexible structures is little inherent structural damping, and most finite element methods assume no structural damping. Consequently, the damping in (2.2.1) is identically equal to zero. Although the amount may be very small, however, all real structures have some damping; therefore, when the mathematical model for a structure is determined using finite element analysis, the structural damping, represented by the matrix \( C \) in (2.1.3), must be determined separately. In most cases, any structural damping included in the model is either determined experimentally, or simply assumed based on previous experience with similar structures.

### 2.3 Modal analysis

The mathematical model for a flexible structure, such as (2.2.1), determined via the finite element method consists of a system simultaneous linear second-order ordinary differential equations with constant coefficients. The solution of this system of equations is greatly simplified when the mass and stiffness matrices, \( M \) and \( K \), are both diagonal.

Although \( M \) and \( K \), are both symmetric, neither is usually diagonal. When the mass matrix is diagonal, the system is said to be inertially decoupled, and when the stiffness matrix is diagonal, the system is elastically decoupled. When both \( M \) and \( K \) are simultaneously diagonal the system of equations is completely decoupled and the
problem of solving the system of differential equations for the displacement of the structure is reduced to the problem of solving a system of independent linear second-order ordinary differential equations.

Despite the fact that neither the mass nor the stiffness matrix determined by the finite element method is usually diagonal, we can often find a transformation matrix which will either inertially or elastically decouple the system. Let $T$ be an appropriately dimensioned transformation matrix, such that

$$ q(t) = T\eta(t). \quad (2.3.1) $$

Substituting (2.3.1) into (2.2.1), and multiplying both sides of the equation by the transpose of $T$, we have

$$ T^TMT\dot{\eta}(t) + T^TKT\dot{\eta}(t) = T^Tu(t). \quad (2.3.2) $$

If $T$ is chosen such that $T^TMT$ is diagonal, the system of equations is inertially decoupled with respect to the new coordinates $\eta_i(t)$; if $T^TKT$ is diagonal, the equations of motion are elastically decoupled in the new coordinate system. Ideally, we would like to find a transformation such that the equations of motion are completely decoupled in the new coordinate system.

A linear transformation which simultaneously diagonalizes the mass and stiffness matrices does exist, since for any real structure, $M$ and $K$ are both symmetric and $M$ is positive definite. For any positive definite, symmetric matrix $M$ and symmetric $K$, there exist $n$ real eigenvalues $\lambda_k$ and corresponding linearly independent eigenvectors $z_k$ such that

$$ Kz_k = \lambda_kMz_k, \quad k = 1,2, \ldots, n \quad (2.3.3) $$
where $M, K \in \mathbb{R}^{nn}$ [5]. We can choose the characteristic vectors, $z_k$, such that

$$z_j^T M z_k = \delta_{jk} \quad (2.3.4)$$

where $\delta_{jk}$ is the Kronecker delta function; therefore, multiplying both sides of (2.3.3) by $z_j^T$, yields

$$z_j^T K z_k = \lambda_k \delta_{jk}. \quad (2.3.5)$$

When the characteristic vectors are chosen as in (2.3.4) we say that they are orthonormal with respect to the mass matrix, $M$.

Let the matrix $\Phi$ be defined by

$$\Phi \triangleq [ z_1 | z_2 | ... | z_n ], \quad (2.3.6)$$

where the columns $z_j$ are the characteristic vectors in (2.3.3). Then

$$\Phi^T M \Phi = I_n, \quad (2.3.7)$$

and

$$\Phi^T K \Phi = \Omega, \quad (2.3.8)$$

where $I_n$ is the identity matrix in $\mathbb{R}^n$ and $\Omega$ is a diagonal matrix with diagonal elements $\lambda_i$. Since $K$ is positive semidefinite, the characteristic values are all nonnegative; hence, we can write

$$\Omega = \text{diag}(\omega_1^2, \omega_2^2, ..., \omega_n^2), \quad (2.3.9)$$

where $\omega_i^2 = \lambda_i$.  

Chapter 2
The transformation matrix, \( \Phi \), which simultaneously diagonalizes \( M \) and \( K \), is known as the modal matrix, and the columns of \( \Phi \), which we shall denote by \( \phi_i \), are called modal vectors. The transformed system coordinates, \( \eta_i(t) \) are called natural or modal coordinates, and the characteristic values \( \omega_i^2 \) are referred to as the modal or natural frequencies. This procedure of solving a system of simultaneous differential equations by transformation to a system of independent equations via the modal matrix is known as modal analysis [4].

### 2.4 Modal coordinate model

Recall from Section 2.3 that the modal vectors \( \phi_i \), the columns of the modal matrix \( \Phi \), are linearly independent [5]; therefore, they form a basis for \( \mathbb{R}^n \) and any \( n \)-dimensional vector can be expressed as a linear combination of the modal vectors. Physically, this means that we can regard any structural motion as a superposition of the natural modes, represented by the corresponding modal vector, multiplied by appropriate constants. These constants represent the position or displacement of the structure in modal coordinates.

Denoting the position of the structure in modal coordinates by the \( n \)-dimensional vector \( \eta(t) \), the displacement in generalized coordinates is given by

\[
q(t) = \Phi \eta(t).
\]  

Substituting the expression for \( q(t) \) in (2.4.1) into the system of second-order ordinary differential equations (2.1.3), we have
\[ M \Phi \ddot{\eta}(t) + C \Phi \dot{\eta}(t) + K \Phi \eta(t) = u(t). \quad (2.4.2) \]

Premultiplying both sides of (2.4.2) by \( \Phi^T \), yields

\[ \Phi^T M \Phi \ddot{\eta}(t) + \Phi^T C \Phi \dot{\eta}(t) + \Phi^T K \Phi \eta(t) = \Phi^T u(t), \quad (2.4.3) \]

which is equivalent to

\[ \ddot{\eta}(t) + \Phi^T C \Phi \dot{\eta}(t) + \Omega \eta(t) = \Phi^T u(t), \quad (2.4.4) \]

by (2.3.7) and (2.4.8).

The orthogonality of the modal vectors with respect to the mass and stiffness matrices leads to simultaneous diagonalization of \( M \) and \( K \) under the transformation to modal coordinates, but \( \Phi^T C \Phi \), although symmetric, generally is not a diagonal matrix. Hence, the system of equations (2.4.4) may still be coupled through the damping matrix.

In the special case where the damping matrix is assumed to be a linear combination of the mass and stiffness matrices, the system of equations (2.4.4) is completely decoupled. Suppose

\[ C = \alpha M + \beta K, \quad (2.4.5) \]

where \( \alpha \) and \( \beta \) are scalar constants. After the transformation to modal coordinates, the damping matrix becomes

\[ \Phi^T C \Phi = \alpha I_n + \beta \Omega, \quad (2.4.6) \]

reducing (2.4.4) to a system of independent equations. This special case is known as proportional damping. We typically write

\[ \Phi^T C \Phi = D \quad (2.4.7) \]
where $D$ is a diagonal matrix with diagonal elements

$$d_{ii} = 2\zeta_i\omega_i.$$  \hspace{1cm} (2.4.8)

$\zeta_i$ is called the damping factor or damping ratio, and is chosen so that

$$2\zeta_i\omega_i = \alpha + \beta\omega_i^2.$$ \hspace{1cm} (2.4.9)

Assuming proportional damping, the motion of the structure in modal coordinates is described by a system of independent equations of the form

$$\ddot{\eta}_i(t) + 2\zeta_i\omega_i\dot{\eta}_i(t) + \omega_i^2\eta_i(t) = \phi_i^T u(t),$$ \hspace{1cm} (2.4.10)

or by the matrix differential equation

$$\ddot{\eta}(t) + D\dot{\eta}(t) + \Omega\eta(t) = \Phi^T u(t).$$ \hspace{1cm} (2.4.11)

### 2.5 The state space model

Consider the following finite-dimensional state space model for a large space structure

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$ \hspace{1cm} (2.5.1)

where $x \in \mathbb{R}^{2n}$, $u \in \mathbb{R}^n$, and $y \in \mathbb{R}^r$. The dimension of the state vector is equal to twice the number of structural modes included in the model. The dimensions of the control input,
$u(t)$, and the measured output, $y(t)$, correspond to the number of actuators and sensors, respectively, included in the model.

With the mathematical model for a flexible structure expressed in modal coordinates, the motion due to each of the structure's flexible modes is described by a second-order ordinary differential equation of the form (2.4.10). Choosing modal position and velocity, $\eta_i$ and $\dot{\eta}_i$, as state variables, the motion due to each mode is described by a state equation of the form

$$
\begin{bmatrix}
\dot{\eta}_i \\
\ddot{\eta}_i
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\omega_i^2 & -2\zeta_i \omega_i
\end{bmatrix}
\begin{bmatrix}
\eta_i \\
\dot{\eta}_i
\end{bmatrix}
+ \sum_{j=1}^{m}
\phi_i(x_{aj})u_j,
$$

(2.5.2)

where $\omega_i$ is the modal frequency of the $i$-th mode, $\zeta_i$ is the corresponding damping ratio, and $\phi_i(x_{aj})$ is the $j$-th element of the $i$-th modal vector $\phi_i^T$ in (2.4.10). In practice, $\phi_i(x_{aj})$ denotes the mode shape of the $i$-th mode evaluated at the point $x = x_{aj}$, which corresponds to the location of the $j$-th actuator.

With the model expressed in modal coordinates, we can define the composite state vector for the entire structure, $x(t)$, by

$$
x \triangleq \begin{bmatrix}
\eta_1, \dot{\eta}_1, \eta_2, \dot{\eta}_2, \ldots, \eta_N, \dot{\eta}_N
\end{bmatrix}^T.
$$

(2.5.3)

With the state vector (2.5.3), the $A$ matrix for the composite system model, (2.5.1), has the form

$$
A = \text{block diag}(A_1, A_2, \ldots, A_N),
$$

(2.5.4)

where
\[ A_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2z_i^\prime \omega_i \end{bmatrix}. \] (2.5.5)

Each diagonal block, \( A_i \), represents one of the structural modes included in the model.

The \( n \times m \) input matrix for the structure is given by

\[
B = \begin{bmatrix}
0 \\
b_1 \\
\vdots \\
0 \\
b_2 \\
\vdots \\
0 \\
b_N
\end{bmatrix}
\] (2.5.6)

where the nonzero rows, \( b_i \), are \( 1 \times m \) matrices of the form

\[ b_i = [ \phi_1(z_{a1}), \phi_1(z_{a2}), \ldots, \phi_1(z_{am}) ], \] (2.5.7)

as in (2.5.2). The column dimension of the input matrix corresponds to the number of actuators included in the model.

We shall denote the output matrices for \( y(t) \) corresponding to measured position and velocity by \( C_x \) and \( C_y \), respectively. For \( y(t) \) equal to the measured position, the \( r \times n \) output matrix has the form

\[
C_p = \begin{bmatrix}
c_1 \\ 0
\end{bmatrix} \begin{bmatrix}
c_2 \\ 0 \\ \vdots \\ c_N \\ 0
\end{bmatrix}. \] (2.5.8)

The nonzero columns of the output matrix \( C_x \) are given by

\[
c_i^T = [ \phi_i(z_{s1}), \phi_i(z_{s2}), \ldots, \phi_i(z_{sr}) ] \] (2.5.9)
where \( \phi_i \) is the mode shape corresponding to the \( i \)-th vibrational mode and \( z_{n1}, z_{n2}, ..., z_{nr} \) denote the locations of \( r \) position sensors. When \( y(t) \) denotes measured velocity, the output matrix is

\[
C_y = \begin{bmatrix}
0 & c_1 & | & 0 & c_2 & | & ... & | & 0 & c_N
\end{bmatrix},
\]  

(2.5.10)

where the nonzero columns \( c_i^T \) are as in (2.5.9). If the rate and position sensors are collocated then the nonzero columns of \( C_n \) are identical to the nonzero columns of \( C_r \); if the sensors are not collocated, then for \( C_n \), the modes are evaluated at the locations of the rate sensors. When the output consists of a linear combination position and velocity measurements, the matrix \( C \) is modified accordingly.

2.6 Summary

Mathematical models for large space structures may be described by either partial or ordinary differential equations. One of the most popular methods of large space structure modelling is the use of finite element analysis, which yields a mathematical model consisting of a system of linear, second-order, ordinary differential equations. Flexible structures characteristically have little inherent structural damping, and finite element methods typically do not model what little structural damping does exist. Consequently, any structural damping included in mathematical models for flexible structures, generated using the finite element method, is either determined experimentally, or assumed by the modeller based on prior experience.

The ordinary differential equations describing the mathematical model for a structure are decoupled by transformation to modal coordinates. When the model is
expressed in modal coordinates, each differential equation describes the displacement associated with one of the structure’s flexible modes. The overall motion of the structure is a linear combination of the motions associated with each mode; thus, modelled modes form a finite dimensional basis for approximating the overall motion of the structure. Choosing modal position and velocity as state variables results in a convenient state space representation of the structure.
3.0 Decentralized Fixed Modes

One of the primary purposes of almost any control design is to insure system stability. The system stabilization problem, however, is a special case of the more general pole placement problem - given a specified region of the complex plane, does there exist a linear time-invariant dynamic controller such that all of the closed loop poles of the system lie within the region? When the proposed control law is decentralized, the existence of a solutions to both the pole placement and system stabilization problems depends on the properties of a set of numbers for the system known as "decentralized fixed modes" [7].

The study of decentralized fixed modes is not the only approach to the problem of determining when a system may be stabilized using decentralized control. Much of the research in the area of large-scale system stability has concentrated on the use of Lyapunov stability criteria and functional analysis techniques to providing conditions for both overall and connective stability of the systems in question. The application of these results to decentralized stabilizability, however, has been limited primarily to systems with specific internal structures (i.e. triangular, tridiagonal, etc.).
Decentralized fixed modes are defined as those modes of the system that are invariant to decentralized output feedback. Thus, we see the direct connection between the fixed modes of the system and the existence of a controller which solves the pole placement or stabilization problem. In fact, direct application of the definition of fixed modes provides necessary and sufficient conditions for the existence of a solution to the decentralized stabilizability problem [7].

Much of the research in the area of decentralized fixed modes has centered on finding different characterizations of fixed modes. These alternate characterizations often provide insight into the relationship between the existence of fixed modes and the system structure [8,15].

One of the most useful results regarding the nature of decentralized fixed modes is a recursive characterization, which reduces the problem of studying the fixed modes of a system with $N$ local feedback controllers to the study of fixed modes in a system with only two local controllers [8]. An algebraic characterization of decentralized fixed modes leads to rank tests for fixed modes that are applicable to systems described by either state equations or matrix fraction decompositions [9]. Interpretation of the results of these algebraic tests implies a characterization of decentralized fixed modes in terms of the transmission zeros of the system [11].

Many systems to which we wish to apply decentralized control techniques arise from the interconnection of several independent subsystems. There are several characterizations of decentralized fixed modes specifically for systems comprised of interconnected subsystems [16,13,14]. The study of interconnected systems provides additional insight into the relationship between fixed modes and system structure.
3.1 Output feedback characterization

Consider the linear time-invariant system described by the following set of state equations:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{align*}
\]  

(3.1.1)

where \(x \in \mathbb{R}^n\) is the state, \(u \in \mathbb{R}^m\) and \(y \in \mathbb{R}^r\) are the system's inputs and measurable outputs respectively, and the matrices \(A\), \(B\), and \(C\) are all appropriately dimensioned. We are interested in controlling the system (3.1.1) using output feedback; therefore, we are interested in control laws of the form

\[
u(t) = -Ky(t) + r(t)
\]  

(3.1.2)

where \(K \in \mathbb{R}^{m \times n}\) is a matrix of feedback gains, and \(r \in \mathbb{R}^n\) is a vector of reference inputs.

The flow of information through the controller (3.1.2) is constrained by the structure of the feedback matrix \(K\). The relative positions of the zero and nonzero elements in \(K\) determine the flow of feedback information between the outputs and inputs.

We denote the elements of the feedback gain matrix \(K\) in by \(k_{ij}\). Suppose \(\overline{K}\) is dimensionally equivalent to \(K\), and the elements \(\overline{k}_{ij}\) of \(\overline{K}\) are defined such that \(\overline{k}_{ij} = 0\) if \(k_{ij} = 0\) and \(\overline{k}_{ij} = 1\) if \(k_{ij} \neq 0\). The matrices \(K\) and \(\overline{K}\) share the same structure with respect to their zero and nonzero elements; hence, we say that they define the same feedback information flow constraint.
Definition 3.1.1 [6]: A structured matrix $\overline{M}$ is a matrix which has a number of fixed zero elements, and unity entries elsewhere.

With each structured matrix $\overline{M}$, we associate a parameter space $R^*$, where $v$ denotes the number of nonzero elements in $\overline{M}$. Every point $d \in R^*$ defines a matrix, $M = \overline{M}(d)$, which is obtained by replacing the nonzero entries of $\overline{M}$ by the corresponding elements of the parameter vector $d$. Conversely, for any matrix $M$ with $v$ nonzero elements, there exists a corresponding structured matrix $\overline{M}$ and parameter space $R^*$ such that $M = \overline{M}(d)$ for some $d \in R^*$.

Definition 3.1.2 [6]: Two matrices, $M_1$ and $M_2$, are structurally equivalent if there is a one-to-one correspondence between their zero and nonzero entries, that is, if both matrices have the same corresponding structured matrix $\overline{M}$.

Suppose $\overline{K} \in R_v^{v \times v}$ has matrix elements $\overline{k}_{ij}$, such that $\overline{k}_{ij} = 1$ if there exists a feedback interconnection between the $j$-th output $y_j(t)$ and the $i$-th input $u_i(t)$ in (3.1.1), and $\overline{k}_{ij} = 0$ otherwise. The structured matrix $\overline{K}$ represents the feedback information flow constraint for the system (3.1.1).

The admissible feedback gain matrices for the controller (3.1.2) subject to a feedback information flow constraint represented by a structured matrix $\overline{K}$, belong to the set of matrices that are structurally equivalent to $\overline{K}$. We denote this set of allowable feedback gain matrices by

$$\{\overline{K}\} = \{K \mid K = \overline{K}(d), d \in R^*\}, \quad (3.1.3)$$

where $R^*$ denotes the parameter space associated with the structured matrix $\overline{K}$.
The fixed modes of a dynamic system such as (3.1.1), with the feedback controller (3.1.2), are defined with respect to a feedback information flow constraint represented by a structured matrix $\overline{K}$, or the class of allowable feedback matrices $\{\overline{K}\}$, in (3.1.3), which it defines.

**Definition 3.1.3** [7]: Suppose the controller (3.1.2) is applied to the system (3.1.1). The fixed modes of the system with respect to the class of feedback matrices $\{\overline{K}\}$ defined by the controller information flow constraint represented by the structured matrix $\overline{K}$ are given by

$$\Lambda(A, B, \overline{K}, C) = \bigcap_{K \in (\overline{K})} \lambda(A + BK C)$$  

(3.1.4)

where $\lambda(G)$ denotes the set of eigenvalues of the square matrix $(G)$.

Definition 3.1.3 characterizes fixed modes as those eigenvalues of the open loop system that are invariant under a specified nondynamic output feedback structure. It can be shown [7], however, that the fixed modes of the system, $\Lambda(A, B, \overline{K}, C)$, are also invariant with respect to dynamic output feedback controllers subject to the same constraints on the feedback structure.

Suppose the structure of the feedback information flow is unconstrained, then the set of admissible feedback gain matrices is all of $\mathbb{R}^{m \times m}$. The corresponding structured matrix is $\overline{K} \in \mathbb{R}^{m \times m}$ with $\overline{k}_{ij} = 1$ for all $i$ and $j$. We denoted the set of allowable feedback gain matrices for full output feedback by

$$\{\overline{K}_e\} = \{K \mid K = \overline{K}(d), d \in \mathbb{R}^m\} \quad (3.1.5)$$

$$= \{K \mid K \in \mathbb{R}^{\times m}\}.$$
When there are no constraints on the structure of the feedback gain matrix, the control law (3.1.2) corresponds to full, or centralized, output feedback for the system (3.1.1). We refer to the fixed modes of the system with respect to full output feedback as centralized fixed modes. The characterization of fixed modes implied by Definition (3.1.3) suggests a correspondence between centralized fixed modes and the familiar concepts of controllability and observability.

**Theorem 3.1.1:** The centralized fixed modes, \( \Lambda(A, B, \overline{K}, C) \), of a system such as (3.1.1), are those modes of the system that are not both controllable and observable.

The fixed modes of a system are defined relative to the output feedback structure imposed on the system by the controller information flow constraint. Centralized fixed modes correspond to the most general class of feedback gain matrices (3.1.5). We may also define fixed modes with respect to other types of feedback information flow constraints, each of which corresponds to a specific feedback gain matrix structure. Examples of other types of fixed modes that may be of interest include those defined by triangular, tridiagonal, diagonal, and block diagonal feedback gain matrix structures.

Consider again the dynamic system (3.1.1), and the output feedback controller (3.1.2). Suppose that the inputs \( u_i(t) \) and outputs \( y_j(t) \) of the system have been reordered in such a way that there exists a feedback path from the first \( r_1 \) outputs to the first \( m_1 \) inputs, from the next \( r_2 \) outputs to the next \( m_2 \) inputs, ... from the next \( r_{N-1} \) outputs to the next \( m_{N-1} \) inputs, and no feedback path from the remaining \( r_N \) outputs to the remaining \( m_N \) inputs.

Partitioning the input and output vectors, \( u(t) \) and \( y(t) \), in (3.1.1) in a manner compatible with the allowable feedback interconnections yields
\[ u \triangleq \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix} \]  \hfill (3.1.6)

and

\[ y \triangleq \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix} \]  \hfill (3.1.7)

where \( u_i \in \mathbb{R}^n \) and \( y_i \in \mathbb{R}^r \) for \( i = 1, 2, \ldots, N \). With the system inputs and outputs partitioned as in (3.1.6) and (3.1.7), the control law (3.1.2) has the form

\[ u_i = -K_i y_i + v_i, \quad i = 1, 2, \ldots, N - 1. \]  \hfill (3.1.8)

We say that the feedback control law (3.1.8) and that the feedback controller information flow constraint for the system are decentralized.

**Definition 3.1.4**: The feedback information flow constraint for the output feedback controller (3.1.2) is *decentralized* if there exists a suitable permutation of the inputs and outputs such that

\[ K \in \{ \overline{K}_d \} \triangleq \{ K \mid K = \text{block diag } (K_1, K_2, \ldots, K_N) ; \]
\[ K_i \in \mathbb{R}^{r_i \times m_i}, \quad i = 1, 2, \ldots, N \} \]  \hfill (3.1.9.a)

where \( m_i \) and \( r_i \) are the dimensions of the \( N \) sets of local inputs and measurable outputs respectively.
The partition of the input and output vectors in (3.1.6) and (3.1.7) implied by the decentralized feedback information flow constraint also implies a partitioning of the corresponding input and output matrices in (3.1.1); that is,

$$B \triangleq \begin{bmatrix} B_1, B_2, \ldots, B_N \end{bmatrix}$$ \hspace{1cm} (3.1.10a)

$$C \triangleq \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}$$ \hspace{1cm} (3.1.10b)

where $B_i \in \mathbb{R}^{n \times m_i}$ and $C_i \in \mathbb{R}^{n \times n}$. Representation of the system (3.1.1) with respect to the partitioning induced by the decentralized information flow constraint yields the following set of state equations

$$\dot{x} = Ax + \sum_{i=1}^{N} B_i u_i$$ \hspace{1cm} (3.1.11)

$$y_i = C_i x, \hspace{1cm} i = 1, 2, \ldots, N$$

We refer to the system (3.1.11) subject to the decentralized control law (3.1.8) as a decentralized $N$-control agent system. The fixed modes of the system are called decentralized fixed modes.

**Definition 3.1.5**: Assume that the controller (3.1.8) is applied to the system (3.1.11); then, the system is said to have a *decentralized fixed mode*, $\lambda$, if

$$\lambda \in \Lambda(A,B, \overline{K}_d,C) = \bigcap_{\kappa \in (\overline{K}_d)} \lambda(A + BK_C)$$ \hspace{1cm} (3.1.12)
where \( \{\overline{K}_d\} \) denotes the set of allowable feedback gain matrices defined by the decentralized information flow constraint (3.1.9).

The following algorithm, derived directly from Definition 3.1.5, illustrates the characterization of decentralized fixed modes as those eigenvalues of the open loop system that are invariant to local output feedback and provides a method of calculating the decentralized fixed modes of a system.

Algorithm 3.1.1 [7]:

1. Compute the eigenvalues of \( A \)
2. Select 'arbitrary' local feedback gain matrices \( K_i, i = 1,2, ..., N \) (using, for example, a random number generator)
3. Compute the eigenvalues of
   \[
   A + \sum_{i=1}^{N} B_i K_i C_i
   \]  
   (3.1.13)
4. The decentralized fixed modes are contained in those eigenvalues of the system in (3.1.13) which are also eigenvalues of \( A \). This is true for almost all \( K_i, i = 1,2, ..., N \) chosen.

The following theorem illustrates the relationship between decentralized fixed modes, centralized fixed modes, and the open loop poles of the system.
Theorem 3.1.2: Let $\lambda(A)$ denote the open-loop poles of the system (3.1.1), and $\Lambda(A, B, \bar{K}_c, C)$ and $\Lambda(A, B, \bar{K}_d, C)$ the centralized and decentralized fixed modes of the system respectively. Then

$$\Lambda(A, B, \bar{K}_d, C) \subset \Lambda(A, B, \bar{K}_c, C) \subset \lambda(A). \quad (3.1.14)$$

Proof: From the definitions of $\{\bar{K}_c\}$ and $\{\bar{K}_d\}$,

$$\emptyset \in \{\bar{K}_d\} \subset \{\bar{K}_c\}. \quad (3.1.15)$$

Hence,

$$\bigcap_{K \in \{\bar{K}_d\}} \lambda(A + BKC) \subset \bigcap_{K \in \{\bar{K}_c\}} \lambda(A + BKC) \subset \bigcap_{K \neq \emptyset} \lambda(A + BKC), \quad (3.1.16)$$

which is equivalent to (3.1.14).

The following corollary follows immediately from Theorems 3.1.1 and 3.1.2.

Corollary 3.1.3: The decentralized fixed modes of a system include any modes of the system that are either uncontrollable, unobservable, or both.

The necessary and sufficient conditions for arbitrary placement of the closed loop poles of a system via centralized output feedback may be expressed in terms of the controllability and observability of the system. These same necessary and sufficient conditions may also be expressed in terms of the centralized fixed modes of the system, since the centralized fixed modes correspond to those modes of the system that are not both controllable and observable. This suggests that necessary and sufficient conditions
for decentralized pole placement may be expressed in terms of decentralized fixed modes. The following theorem provides such a set of necessary and sufficient conditions.

**Theorem 3.1.4 [7]**: Let $S$ be any nonempty symmetric open subset of the complex plane. There exists a linear, time-invariant, decentralized controller for the system (3.1.11) such that all of the poles of the resulting closed loop system are contained in $S$, if and only if the decentralized fixed modes of the system (if they exist) are contained in $S$.

The problem of stabilizing a system using decentralized output feedback is actually a special case of the decentralized pole placement problem. The following corollary follows directly from Theorem 3.1.4 with $S$ equal to the open left-half of the complex plane.

**Corollary 3.1.5 [7]**: There exists a solution to the decentralized stabilization problem for the system (3.1.11), if and only if the decentralized fixed modes of the system (if any exist) are contained in the open left-half of the complex plane.

### 3.2 Recursive characterization

Consider the $N$-agent control system (3.1.11), and suppose that we combine the local inputs and outputs of subsystems $i$ and $(i + 1)$. This results in a representation of the $N$ channel system (3.1.11) as an $(N - 1)$ control agent system. We shall represent this $(N - 1)$-control agent representation of the system by
\[
\begin{bmatrix}
C_1 \\
\vdots \\
C_i \\
C_{i+1} \\
\vdots \\
C_N
\end{bmatrix}
, A
, 
\begin{bmatrix}
B_1 \\
\vdots \\
B_i \\
B_{i+1} \\
\vdots \\
B_N
\end{bmatrix}
\]

(3.2.1)

In studying decentralized fixed modes, we would like to be able to reduce the size of the systems with which we must work, without a loss of generality. The recursive characterization of decentralized fixed modes shows that the existence of fixed modes in an \(N\)-agent control system, such as (3.1.11), reduces to the existence of fixed modes in a \((N - 1)\)-agent control system.

**Theorem 3.2.1** [8]: Given the \(N\)-agent decentralized control system (3.1.11) with \(N \geq 3\), \(\lambda \in \lambda(A)\) is not a decentralized fixed mode of the system, if and only if \(\lambda\) is not a decentralized fixed mode of any of the following \((N - 1)\)-control agent systems:

\[
\begin{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
\vdots \\
C_N
\end{bmatrix}
, A
, 
\begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
\vdots \\
B_N
\end{bmatrix}
\end{bmatrix}
\]

(1)

\[
\begin{bmatrix}
\begin{bmatrix}
C_1 \\
C_2 \\
C_3 \\
\vdots \\
C_N
\end{bmatrix}
, A
, 
\begin{bmatrix}
B_1 \\
B_2 \\
B_3 \\
\vdots \\
B_N
\end{bmatrix}
\end{bmatrix}
\]

(2)
\[
\begin{align*}
(N-2) & \left[ \begin{bmatrix} C_1 \\ \vdots \\ C_{N-2} \\ C_{N-1} \\ C_N \end{bmatrix}, A, [B_1, \ldots, [B_{N-2}, B_{N-1}], B_N] \right] \\
(N-1) & \left[ \begin{bmatrix} C_1 \\ \vdots \\ C_{N-2} \\ C_{N-1} \\ C_N \end{bmatrix}, A, [B_1, \ldots, B_{N-2}, [B_{N-1}, B_N]] \right]
\end{align*}
\]

Successive application of Theorem 3.2.1 leads to the following corollary.

**Corollary 3.2.2** [8]: Given the \(N\)-agent decentralized control system (3.1.11) with \(N \geq 3\); then \(\lambda \in \lambda(A)\) is not a decentralized fixed mode of the system, if and only if \(\lambda\) is not a decentralized fixed mode of any of the following 2-control agent systems

\[
(1) \left[ \begin{bmatrix} C_1 \\ \vdots \\ C_2 \\ \vdots \\ C_N \end{bmatrix}, A, [B_1, [B_2, \ldots, B_N]] \right]
\]
Corollary 3.2.2 reduces the study of decentralized fixed modes in $N$-agent control systems to the study of fixed modes in 2-agent control systems. This allows us to generalize characterizations of decentralized fixed modes in systems with 2-control agents to systems with an arbitrary number of control agents.
3.3 Structurally fixed modes

Fixed modes may originate from two distinct sources. One type of fixed mode, which we shall call unstructured, arises from perfect matching between system parameters; therefore, the presence of an unstructured fixed mode is affected by changes in the system parameters. Structured fixed modes are the result of the structure of the system. Structurally fixed modes remain fixed independent of any perturbation of the system parameters.

The object of this section is to characterize structurally fixed modes. As with the general notion of fixed modes, structurally fixed modes are defined with respect to system constraints on the feedback information flow.

We can extend the notion of parametrizing matrices in terms of a structurally fixed matrix $\bar{M}$ and a point $\bar{d}$ in a parameter space $R^r$ to dynamic systems. Let the triple $(A, B, C)$ denote a system $S$ as in (3.1.1). For each system $S$ there exists a structurally fixed system $\bar{S}$, which we shall denote by the triple $(\bar{A}, \bar{B}, \bar{C})$ and a parameter space $E$ such that $S = \bar{S}(e)$ for some $e \in E$. The following definitions extend the concepts of structured matrices and structurally equivalent matrices to dynamic systems.

**Definition 3.3.1** [6]: A structured system $\bar{S}$, which we shall denote by the matrix triple $(\bar{A}, \bar{B}, \bar{C})$, represents the state space description of a system, such as (3.1.1), in which the matrices $A$, $B$, and $C$ are all structured matrices. When $\bar{S}$ denotes the structured system corresponding to a dynamic system $S$, the structured matrices in the matrix triple $(\bar{A}, \bar{B}, \bar{C})$ denote the structured matrices corresponding to the matrices $A$, $B$, and $C$ for the system $S$. □
Definition 3.3.2 [6]: Two systems, \( S_1 \) and \( S_2 \), are \textit{structurally equivalent} if the corresponding matrices in the matrix triples, \((A_1, B_1, C_1)\) and \((A_2, B_2, C_2)\), associated with each system are structurally equivalent.

If we let \( v \) equal the total number of nonzero parameters in the triple \((A, B, C)\), and let \( R^v \) denote the associated parameter space, then Definition 3.3.2 implies that every dynamic system that is structurally equivalent to a given dynamic system \( S \) is a member of the set of dynamic systems

\[
\{\tilde{S}\} = \{S \mid S = \tilde{S}(d), \ d \in R^v\}.
\] (3.3.1)

All of the dynamic systems \( S \in \{\tilde{S}\} \) are structurally equivalent to the structured system \( \tilde{S} \), with matrix triple \((\tilde{A}, \tilde{B}, \tilde{C})\), corresponding to the given system \( S \).

We can now turn our attention to the characterization of structured decentralized fixed modes.

Definition 3.3.3 [6]: A system \( S \) has \textit{structurally fixed modes} with respect to a specified feedback information flow constraint, if every system \( \tilde{S} \) that is structurally equivalent to \( S \) has fixed modes with respect to the same information flow constraint.

Suppose \( \lambda \) is a fixed mode of the system \( S \). It follows from Definition 3.3.3 that if there exists \( d \in R^v \) such that \( \lambda \) is not a fixed mode of \( \tilde{S}(d) \), then \( \lambda \) is not a structurally fixed mode of \( S \). If the system \( S \) has no structurally fixed modes, it can be shown [6] that almost all systems that are structurally equivalent to \( S \) also have no fixed modes.
Clearly, not all systems that are structurally equivalent will even have the same open loop eigenvalues, let alone the same fixed modes. There are some system representations in which the existence of structured or unstructured fixed modes depends completely on the input and output matrices.

Consider the system

\[ \dot{x}(t) = Ax(t) + \sum_{i=0}^{N} B_i u_i \]
\[ y_i(t) = C_i x(t), \quad i = 1, 2, ..., N \]

where \( x \in \mathbb{R}^n \), \( u_i \in \mathbb{R}^{n_i} \), and \( y_i \in \mathbb{R}^{n_i} \). Let the matrix triplet \((A, B, C)\) denote the system (3.3.2), and let

\[ B \triangleq \begin{bmatrix} B_1, B_2, ..., B_N \end{bmatrix} \]
\[ C \triangleq \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix} \]

(3.3.3)

denote the composite input and output matrices. We shall assume the system \((A, B, C)\) is both completely controllable and completely observable, and the matrix \( A \) has \( n \) distinct eigenvalues \( \{\lambda_i\}_{i=1}^n \).

Suppose the system \((A, B, C)\) is transformed to Jordan canonical form, then (3.3.2) becomes

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\[
\dot{x}(t) = \Lambda \bar{x}(t) + \sum_{i=0}^{N} \tilde{B}_i u_i(t) \\
y_i(t) = \tilde{C}_i \bar{x}(t), \quad i = 1, 2, \ldots, N,
\]

where \(\Lambda\) is the diagonal matrix

\[
\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}.
\]

**Definition 3.3.4 [19]:** Let \(\lambda_i\) be a fixed mode of (3.3.2) with respect to a given feedback information flow constraint; then, \(\lambda_i\) is a **structurally fixed mode** if \(\lambda_i\) is a fixed mode of (3.3.4) for all nonzero elements of \(\tilde{B}_i\) and \(\tilde{C}_i\), \(i = 1, 2, \ldots, N\). Otherwise, \(\lambda_i\) is an **unstructured fixed mode**.

The condition in Definition 3.3.4 is equivalent to considering all systems that are structurally equivalent to the system (3.3.2) and have the same open loop eigenvalues \(\{\lambda_1, \lambda_2, \ldots, \lambda_n\}\).

Suppose the system (3.3.2) is a 2-agent control system, then the Jordan form of the system becomes

\[
\dot{x}(t) = \Lambda \bar{x}(t) + \begin{bmatrix}
b_1^1 \\
b_2^1 \\
\vdots \\
b_n^1
\end{bmatrix} u_1(t) + \begin{bmatrix}
b_1^2 \\
b_2^2 \\
\vdots \\
b_n^2
\end{bmatrix} u_2(t)
\]

\[
y_1(t) = \begin{bmatrix}
c_1^1, c_2^1, \ldots, c_n^1
\end{bmatrix} \bar{x}(t)
\]

\[
y_2(t) = \begin{bmatrix}
c_1^2, c_2^2, \ldots, c_n^2
\end{bmatrix} \bar{x}(t).
\]
The test for decentralized fixed modes is simplified when the system is expressed in Jordan form, as illustrated in the following theorem.

**Theorem 3.3.1 [19]:** Given the system (3.3.6), \( \lambda_i \) is a structured decentralized fixed mode if and only if one of the following conditions holds:

\[
b_i^1 = c_i^2 = 0 \quad \text{and} \quad b_j^1 c_j^2 = 0, \quad \forall \ j \in [1,n]
\]  
(3.3.7a)

or

\[
b_i^2 = c_i^1 = 0 \quad \text{and} \quad b_j^2 c_j^1 = 0, \quad \forall \ j \in [1,n].
\]  
(3.3.7b)

\[\square\]

**Theorem 3.3.2 [19]:** Given the system (3.3.6), \( \lambda_i \) is an unstructured decentralized fixed mode if and only if one of the following conditions holds:

\[
b_i^1 = c_i^2 = 0 \quad \text{and} \quad \sum_{j=1, j \neq i}^{n} \frac{b_j^1 c_j^2}{(\lambda_i - \lambda_j)} = 0,
\]  
(3.3.8a)

where \( b_i^1 c_i^2 \neq 0 \) for at least one \( i \neq j \), or

\[
b_i^2 = c_i^1 = 0 \quad \text{and} \quad \sum_{j=1, j \neq i}^{n} \frac{b_j^2 c_j^1}{(\lambda_i - \lambda_j)} = 0,
\]  
(3.3.8b)
where $b_\ell c_j \neq 0$ for at least one $i \neq j$.

The extension of the characterizations of structurally fixed and unstructured decentralized fixed modes presented in Theorems 3.3.1 and 3.3.2 respectively for 2-agent control systems to systems with $N$-control agents, follows from the recursive characterization of decentralized fixed modes presented in Section 3.2.

Before proceeding, we introduce the idea of the generic rank of a matrix. This provides a means of testing whether two matrices are structurally equivalent.

**Definition 3.3.5** [6]: Let $\rho(M)$ denote the rank of the matrix $M$. The *generic rank* of $M$, is

$$\bar{\rho}(M) = \max_{deR^r} (\rho[\overline{M}(d)])$$

(3.3.9)

where $\overline{M}$ and $R^r$ are the structured matrix and parameter space associated with $M$.

The set $\{deR^r|\rho[\overline{M}(d)] < \bar{\rho}(M)\}$ has Lebesgue measure zero [6]. This implies the following lemma.

**Lemma 3.3.3** [6]: Almost all matrices that are structurally equivalent to a matrix $M$ have rank $\bar{\rho}(M)$.

The following theorem provides necessary and sufficient conditions for the existence of structured fixed modes with respect to decentralized feedback. First, however, we must introduce some necessary notation. Let

$$\mathcal{F} \equiv \{i_1, i_2, \ldots, i_k\} \subset \{1, 2, \ldots, N\}$$

(3.3.10)
and let

\[ \mathcal{I}^c = \{1, 2, \ldots, N\} \setminus \mathcal{I} = \{i_{k+1}, i_{k+2}, \ldots, i_N\}, \tag{3.3.11} \]

where \( N \) is the number of control agents implied by the partitioning of the system inputs and outputs. With input and output matrices

\[
B = [B_1, B_2, \ldots, B_N], \quad C = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix}, \tag{3.3.12}
\]

we can define the following partitions with respect to the subset \( \mathcal{I} \) of the set \( \{1, 2, \ldots, N\} \)

\[
B^\mathcal{I} = [B_{i_1}, B_{i_2}, \ldots, B_{i_N}], \quad B^\mathcal{I}^c = [B_{i_{k+1}}, B_{i_{k+2}}, \ldots, B_{i_N}], \tag{3.3.13}
\]

and

\[
C^\mathcal{I} = \begin{bmatrix} C_{i_1} \\ C_{i_2} \\ \vdots \\ C_{i_N} \end{bmatrix}, \quad C^\mathcal{I}^c = \begin{bmatrix} C_{i_{k+1}} \\ C_{i_{k+2}} \\ \vdots \\ C_{i_N} \end{bmatrix}. \tag{3.3.14}
\]

**Theorem 3.3.4 [6]:** The system \( S = (A, B, C) \) has structurally fixed modes with respect to the decentralized feedback information flow if and only if either of the following conditions holds:

1. There exists \( \mathcal{I} \subset \{1, 2, \ldots, N\} \) and a permutation matrix \( P \) such that
\[ P^T A P = \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}. \] (3.3.15)

\[ P^T B^* = \begin{bmatrix} B_1^* \\ B_2^* \\ B_3^* \end{bmatrix}, \quad P^T B^* = \begin{bmatrix} 0 \\ 0 \\ B_2^* \end{bmatrix}. \] (3.3.16)

and

\[ C^* P = \begin{bmatrix} C_1^* & 0 & 0 \end{bmatrix}, \quad C^* P = \begin{bmatrix} C_1^* \\ C_2^* \\ C_3^* \end{bmatrix}. \] (3.3.17)

2. There exists \( S \subseteq \{1, 2, ..., N\} \) such that

\[ \bar{\rho} \begin{bmatrix} A & B^* \\ C^* & 0 \end{bmatrix} < n. \] (3.3.18)

Condition two in Theorem 3.3.4 corresponds to the algebraic test for decentralized fixed modes presented in Theorem 3.4.3. Theorem 3.3.4 illustrates that structurally fixed modes are the result of the system structure (i.e. a lack of information flow between the control channels). The structural characterization of fixed modes provided by the theorem also suggests a method for determining minimal decentralized realizations for systems [6].
3.4 Algebraic characterization

In this section we present algebraic tests for the existence of decentralized fixed modes. This algebraic characterization of decentralized fixed modes follows from a basic result of linear algebra. Using this result we may discuss fixed modes using either a matrix fraction, or state-variable description of a system [9].

The output feedback characterization of fixed modes presented in Section 3.1 lead to Algorithm 3.1.1 for computing the fixed modes of a system, but provided little insight into the origins of fixed modes. Furthermore, the output feedback characterization of decentralized fixed modes requires finite dimensionality of the system; the algebraic characterization, however, applies infinite as well as finite dimensional systems.

The following theorem provides the basic linear algebra result which is the basis for the algebraic characterization of fixed modes.

**Theorem 3.4.1** [9]: Let $A_1, A_2, ..., A_N$ be $N$ matrices with $A_i \in R^{n \times n}$, and let $B_1, B_2, ..., B_N$ be $N$ matrices with $B_i \in R^{m \times m}$.

$$\text{rank} \left[ \begin{array}{cccc} A_1 + B_1 K_1 & A_2 + B_2 K_2 & \cdots & A_N + B_N K_N \end{array} \right]$$

$$< \min(n - \delta, \sum \gamma_i - \varepsilon)$$  \hspace{1cm} (3.4.1)

$\forall K_i \in R^{n \times n} (i = 1, 2, \ldots, N)$ and some $\delta \geq 0$ and $\varepsilon \geq 0$, if and only if there exists a nonempty subset $\mathcal{S} = \{i_1, i_2, \ldots, i_j\}$ of $\{1, 2, \ldots, N\}$ for which
\[
\text{rank } \begin{bmatrix} A_i & B_i & \ldots & A_j & B_j \end{bmatrix} < \min(n - \delta - \sum_{i \in \mathcal{I}} \gamma_i, \sum_{i \in \mathcal{I}} \gamma_i - \varepsilon). \tag{3.4.2}
\]

The integers \(\delta\) and \(\varepsilon\) in Theorem 3.4.1 are used to keep track of rank deficiencies. The matrix on the left-hand side of (3.4.1) has less than full column and row rank if and only if (3.4.2) is satisfied.

The following corollary is a special case of Theorem 3.4.1 with \(B_N \equiv 0\).

**Corollary 3.4.2 [9]:** Let \(A_1, A_2, \ldots, A_N\) and \(B_1, B_2, \ldots, B_{N-1}\) be as in Theorem 3.4.1.

\[
\begin{align*}
\text{rank } \begin{bmatrix} A_1 + B_1 & K_1 & \ldots & A_{N-1} + B_{N-1} & K_{N-1} & A_N \end{bmatrix} \\
&< \min(n - \delta, \sum \gamma_i - \varepsilon)
\end{align*}
\tag{3.4.3}
\]

\(\forall K_i \in \mathbb{R}^{n \times n} (i = 1, 2, \ldots, N)\) and some \(\delta \geq 0\) and \(\varepsilon \geq 0\), if and only if there exists a nonempty subset \(\mathcal{I} = \{i_1, i_2, \ldots, i_J\}\) of \(\{1, 2, \ldots, N\}\) for which

\[
\begin{align*}
\text{rank } \begin{bmatrix} A_{i_1} & B_{i_1} & \ldots & A_{i_J} & B_{i_J} \end{bmatrix} < \min(n - \delta - \sum_{i \in \mathcal{I}} \gamma_i, \sum_{i \in \mathcal{I}} \gamma_i - \varepsilon) \\
\end{align*}
\tag{3.4.4}
\]

except that if \(N \in \mathcal{I}\), \(B_N\) is omitted in the matrix appearing in (3.4.4). \(\square\)

Consider a linear, time-invariant, system with \(m\) inputs and \(r\) outputs. By reordering the inputs and outputs if necessary, assume there exists a feedback path from
the first \( y_1 \) outputs to the first \( m_1 \) inputs, from the next \( y_2 \) outputs to the next \( m_2 \) inputs, 
... from the next \( y_{N-1} \) outputs to the next \( m_{N-1} \) inputs, and no feedback path from the 
remaining \( y_N \) outputs to the remaining \( m_N \) inputs.

Let \( u(t) \) and \( y(t) \) denote the composite input and output vectors defined by the 
allowable feedback paths, then

\[
\begin{bmatrix}
  u_1 \\
  u_2 \\
  \vdots \\
  u_N
\end{bmatrix} \triangleq u \tag{3.4.5}
\]

and

\[
\begin{bmatrix}
  y_1 \\
  y_2 \\
  \vdots \\
  y_N
\end{bmatrix} \triangleq y \tag{3.4.6}
\]

where \( u_i \in \mathbb{R}^{m_i} \) and \( y_i \in \mathbb{R}^{n_i} \) for \( i = 1, 2, \ldots, N \).

While we require that \( y_i \) and \( m_i \) all be positive for \( i = 1, 2, \ldots, N - 1 \), either \( y_N \) or 
\( m_N \), or both, can be zero. In addition, there is no restriction requiring that the entries 
of \( y_i \) be different from the entries of \( y_j \) so that one or more of the \( N \) outputs may be the 
same. As a result, different inputs could depend upon some of the same outputs.

The mathematical model for the system is described by the set of state equations

\[
\begin{align*}
  \dot{x} &= Ax + Bu \\
  y &= Cx
\end{align*} \tag{3.4.7}
\]

where \( x \in \mathbb{R}^n \) represents the system state, and \( u \) and \( y \) are the system inputs and 
measurable outputs defined in (3.4.5) and (3.4.6). We shall assume that the following feedback controller is applied to the system

\[ \text{Chapter 3} \]
\[ u_i = -K_i y_i + r_i, \quad i = 1, 2, \ldots, N - 1. \] (3.4.8)

The decentralized feedback control law (3.4.8) is compatible with the partitioning of the system input and output vectors.

We can use the algebraic test described in Theorem 3.4.1 to characterize the decentralized fixed modes of a system described by state equations.

**Theorem 3.4.3 [9]:** Consider the system (3.1.11). A necessary and sufficient condition for \( \lambda \in \lambda(A) \) to be a decentralized fixed mode of the system is that for some partition of the set \( \{1, 2, \ldots, N\} \) into disjoint subsets \( \{i_1, i_2, \ldots, i_s\} \) and \( \{i_{s+1}, i_{s+2}, \ldots, i_N\} \) there exists

\[
\begin{bmatrix}
    A - \lambda I & B_{i_1} & B_{i_2} & \cdots & B_{i_s} \\
    C_{i_1} & 0 & 0 & \cdots & 0 \\
    C_{i_2} & 0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    C_{i_N} & 0 & 0 & \cdots & 0
\end{bmatrix}
\]

\[
\text{rank} \quad \begin{bmatrix}
    A - \lambda I & B_{i_1} & B_{i_2} & \cdots & B_{i_s} \\
    C_{i_1} & 0 & 0 & \cdots & 0 \\
    C_{i_2} & 0 & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    C_{i_N} & 0 & 0 & \cdots & 0
\end{bmatrix} < n \quad (3.4.9)
\]

The algebraic characterization of decentralized fixed modes presented in Theorem 3.4.3 is not particularly useful for computing the fixed modes of a system, however, it does provide valuable insight into the nature of fixed modes; it has led to other alternate characterizations of fixed modes and has proven to be a valuable tool in proving new characterizations. Interpretation of the condition (3.4.9) in the theorem implies a characterization of decentralized fixed modes in terms of the transmission zeros of the system.

Using the algebraic characterization of decentralized fixed modes, we can prove the assertion of Corollary 3.1.3; that is, that a necessary condition for a system to have no
decentralized fixed modes is that the system be both controllable and observable. When
\( k = 0 \) in Theorem 3.4.3 (3.4.9) becomes
\[
\text{rank} \begin{bmatrix} A - \lambda I \\ C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix} < n, \tag{3.4.10}
\]
which is precisely the Popov, Belevitch, and Hautus (PBH) rank test for observability [10]. Similarly, when \( k = N \), (3.4.9) becomes
\[
\text{rank} \begin{bmatrix} A - \lambda I & B_1 & B_2 & \ldots & B_N \end{bmatrix} < n, \tag{3.4.11}
\]
the corresponding PBH test for controllability. Hence, Theorem 3.4.3 shows that a
necessary condition for the system to have no decentralized fixed modes is that the
system be completely controllable and observable (in a centralized sense). Since we have
considered only two of the many possible partitions of the set \( \{1,2,\ldots,N\} \) in Theorem
3.4.3., controllability and observability of the centralized system are necessary but not
sufficient conditions for decentralized controllability.

To further illustrate the algrebraic characterization of fixed modes, consider the
2-agent control system
\[
\begin{align*}
\dot{x} &= Ax + B_1 u_1 + B_2 u_2 \\
 y_i &= C_i x, \quad i = 1,2. \tag{3.4.12}
\end{align*}
\]
The eigenvalue \( \lambda \in \lambda(A) \) is not a decentralized fixed mode of (3.4.12) if and only if the
following conditions - corresponding to the possible partitions of the set \( \{1,2\} \) described
in Theorem 3.4.3 - are all satisfied:
1. \[ \text{rank}\left[\begin{array}{c} A - \lambda I \\ C_1 \\ C_2 \end{array}\right] = n \] (3.4.13)

2. \[ \text{rank}\left[\begin{array}{c} A - \lambda I \\ C_1 \\ C_2 \end{array}\right] = n \] (3.4.14)

3. \[ \text{rank}\left[\begin{array}{c} A - \lambda I \\ B_1 \\ C_2 \end{array}\right] \geq n \] (3.4.15)

4. \[ \text{rank}\left[\begin{array}{c} A - \lambda I \\ B_2 \\ C_1 \end{array}\right] \geq n \] (3.4.16)

The first two conditions (3.4.13) and (3.4.14) correspond to the necessary condition that the system be completely controllable and observable with respect to centralized feedback. Equation (3.4.15) corresponds to following set of simultaneous requirements:

1. \( \lambda \) is controllable from control input 1

2. \( \lambda \) is observable from output 2

3. \( \lambda \) is not a transmission zero of certain subsystems of the system - in particular \( \lambda \) cannot be a zero of the transfer function from input 1 to output 2

The fourth condition (3.4.16) corresponds to a similar set of requirements concerning the second input and the first output. If \( \lambda \) is not a transmission zero of the interconnection subsystem from system 2 to system 1, then information about that mode is transmitted to system 1, even if it is unobservable from the output.

Instead of state equations we can also describe the mathematical model for a large space structure in the frequency domain. Suppose that the structure has a real rational
transfer function matrix $H(s)$, with left matrix fraction description $A^{-1}(s)B(s)$. The Laplace transform of the system, with respect to the left matrix fraction description, is

$$A(s)Y(s) = B(s)U(s).$$  \hspace{1cm} (3.4.17)

The partitioning of the system inputs and outputs in (3.4.5) and (3.4.6) implies a corresponding partitioning of the matrices $A(s)$ and $B(s)$:

$$A(s) = \begin{bmatrix} A_1(s) & \cdots & A_N(s) \end{bmatrix}$$
$$B(s) = \begin{bmatrix} B_1(s) & \cdots & B_N(s) \end{bmatrix}$$  \hspace{1cm} (3.4.18)

where $A_i(s)$ and $B_i(s)$ possess $\gamma_i$ and $m_i$ columns respectively. We can then express (3.4.17) as

$$\sum_{i=1}^{N} A_i(s)Y_i(s) = \sum_{i=1}^{N} B_i(s)u_i(s).$$  \hspace{1cm} (3.4.19)

Using (3.4.19) we can show that the closed-loop system, with the feedback control law (3.4.8), has a left matrix fraction description $\overline{A}^{-1}(s)\overline{B}(s)$ where

$$\overline{A}(s) = \begin{bmatrix} A_1(s) + B_1(s)K_1 & \cdots & A_{N-1}(s) + B_{N-1}(s)K_{N-1} & A_N(s) \end{bmatrix}.$$  \hspace{1cm} (3.4.20)

The following theorem characterizes the decentralized fixed modes of a system in terms of its matrix fraction description.

**Theorem 3.4.4 [9]:** Given a system with matrix fraction description $A^{-1}(s)B(s)$, and controller information flow constraint of the form (3.4.8) defined by the positive integers $\gamma_1, \ldots, \gamma_{N-1}$ and $m_1, \ldots, m_{N-1}$, and non-negative $\gamma_N$ and $m_N$, the system has a fixed mode
with respect to this controller information flow at \( s = s_0 \) if \( \overline{A}(s_0) \) is singular for all \( K_i \), \( i = 1, 2, \ldots, N - 1 \).

**Proof:** The characteristic equation of the closed-loop system is given by the determinant of \( \overline{A}(s) \); hence \( \overline{A}(s_0) \) is singular if and only if

\[
\det \overline{A}(s_0) = 0
\]  

(3.4.21)

which implies \( s_0 \) is an eigenvalue of the system. Then, by Definition 3.1.3, \( s = s_0 \) is a fixed mode if \( \overline{A}(s_0) \) is singular for all \( K_i \).

Corollary 3.4.2 provides a means of expressing the condition for the existence of a fixed mode at \( s = s_0 \) independent of the feedback matrices \( K_i \). Taking \( \delta = \varepsilon = 0 \), \( p = \sum \gamma_i \), and setting \( A_i = A_i(s_0) \) and \( B_i = B_i(s_0) \) yields the following theorem.

**Theorem 3.4.5** [9]: A system with matrix fraction description \( A^{-1}(s)B(s) \), controller information flow constraint (3.4.8), the positive integers \( \gamma_1, \ldots, \gamma_{N-1} \) and \( m_1, \ldots, m_{N-1} \), and non-negative \( \gamma_N \) and \( m_N \), has a fixed mode at \( s = s_0 \) if and only if there exists a nonempty subset \( \mathcal{F} \equiv \{ i_1, i_2, \ldots, i_j \} \subset \{ 1, 2, \ldots, N \} \) for which

\[
\text{rank} \left[ A_{i_1}(s_0) \ B_{i_1}(s_0) \ \ldots \ A_{i_j}(s_0) \ B_{i_j}(s_0) \right] < \sum_{i \in \mathcal{F}} \gamma_i.
\]  

(3.4.22)

except that if \( N \in \mathcal{F} \), \( B_N(s_0) \) is omitted in the matrix.
Suppose the controller information flow constraint for a system permits full output feedback. Full output feedback corresponds to \( N = 1 \) in Theorem 3.4.5, and \( s = s_0 \) is a fixed mode of the system if and only if

\[
\text{rank } \begin{bmatrix} A(s_0) & B(s_0) \end{bmatrix} < p. \tag{3.4.23}
\]

Equation (3.4.23), however, is satisfied if and only if \( A(s) \) and \( B(s) \) are not coprime; therefore, the system has centralized fixed modes if and only if it is not completely controllable and observable. Theorem 3.1.1 contains this exact result; thus, we can derive the same conclusions using different characterizations of fixed modes.

### 3.5 Fixed modes and transmission zeros

Application of the algebraic tests for decentralized fixed modes associated with Theorem 3.5.3 results in a set of conditions which must be satisfied if the system is to be free of fixed modes. One interpretation of these conditions suggests a correlation between fixed modes and the transmission zeros of the system [11]. The resulting characterization provides further insight into the nature of fixed modes in decentralized systems, and suggests additional computational methods for their identification.

Consider the system (3.1.1), and suppose the input and output matrices, \( B \in \mathbb{R}^n \) and \( C \in \mathbb{R}^r \), are partitioned so that

\[
B \triangleq \begin{bmatrix} b_1, b_2, \ldots, b_m \end{bmatrix}. \tag{3.5.1a}
\]

and
We shall assume that the partition (3.5.1) corresponds to a decentralized feedback information flow in which there is a path from the \( i \)-th output \( y_i = c_i x \), to the \( i \)-th input \( u_i \).

The following theorem provides a characterization of fixed modes in terms of transmission zeros of the certain subsystems of the composite system.

**Theorem 3.5.1** [11]: Suppose the controller (3.1.2) is applied to the system (3.1.1), subject to the feedback information flow constraint (3.1.3). The eigenvalue \( \lambda \in \lambda(A) \) is a fixed mode of the system with respect to the class of feedback matrices \( \{K\} \) defined by the controller information flow constraint, if and only if \( \lambda \) is a transmission zero of all of the following subsystems:

\[
\begin{bmatrix}
    [c_{j_{k_1}}] \\
    [c_{j_{k_2}}] \\
    \vdots \\
    [c_{j_{k_r}}]
\end{bmatrix}
\Lambda,
[\begin{bmatrix} b_{i_{k_1}} & b_{i_{k_2}} & \ldots & b_{i_{k_r}} \end{bmatrix}]
\]

\( k_1 = 1, 2, \ldots, s + 1 - r; \)
\( k_2 = k_1 + 1, k_1 + 2, \ldots, s + 2 - r; \)
\( \vdots \)
\( k_t = k_{t-1} + 1, k_{t-1} + 2, \ldots, s; \)

for \( t = 1, 2, \ldots, \min(m,r) \).
The subsystems generated in (3.5.2) correspond to the subsystems associated with all nonsingular $1 \times 1$, $2 \times 2$, ..., $\min(m,r) \times \min(m,r)$ submatrices of the composite feedback gain matrix $K$. If the feedback information flow constraint (3.1.3) defines a decentralized information flow such as (3.1.9), Theorem 3.5.1 identifies decentralized fixed modes.

Corollary 3.5.2 presents several results which follow directly from Theorem 3.5.1.

**Corollary 3.5.2 [11]:**

1. If any of the subsystems in (3.5.2) have transmission zeros that are disjoint from the eigenvalues of $A$, then the system has no decentralized fixed modes with respect to $(\bar{K}_d)$.

2. If any of the subsystems in (3.5.2) are minimum phase, then any fixed modes of the system with respect to $(\bar{K}_d)$ are stable.

3. Suppose $m = r$. If the composite feedback gain matrix $K \in (\bar{K}_d)$ is nonsingular, then $\lambda \in \lambda(A)$ is a decentralized fixed mode with respect to $(\bar{K}_d)$ only if $\lambda$ is a transmission zero of the composite system (3.1.1). □

When a system has distinct eigenvalues, the characterization of decentralized fixed modes in terms of the transmission zeros of the system leads to a characterization in terms of the zeros of the matrix transfer function for the system.

Consider an $N$-agent control system with the matrix transfer function description...
\[
\begin{bmatrix}
Y_1(s) \\
Y_2(s) \\
\vdots \\
Y_N(s)
\end{bmatrix}
= W(s)
\begin{bmatrix}
u_1(s) \\
u_2(s) \\
\vdots \\
u_N(s)
\end{bmatrix}
\] (3.5.3)

in which the \(i\)-th control agent has \(m_i\) inputs and \(r_i\) outputs. The matrix transfer function \(W(s)\) has the form

\[
W(s) \triangleq \begin{bmatrix}
W_{11}(s) & W_{12}(s) & \ldots & W_{1N}(s) \\
W_{21}(s) & W_{22}(s) & \ldots & W_{2N}(s) \\
\vdots & \vdots & \ddots & \vdots \\
W_{N1}(s) & W_{N2}(s) & \ldots & W_{NN}(s)
\end{bmatrix}
\] (3.5.4)

where \(W_{ij}\) is the \(r_i \times m_j\) matrix transfer function from the \(j\)-th input to the \(i\)-output

\[
Y_i(s) = W_{ij}(s)U_j(s).
\] (3.5.5)

Suppose that the poles of the system are distinct; then, we may factor \(W(s)\) as

\[
W(s) = \frac{A_1}{s - \lambda_1} + \sum_{i=2}^{n} \frac{A_i}{s - \lambda_i},
\] (3.5.6)

where \(A_1 \neq 0\) and \(\lambda_i \neq \lambda_1\), for \(i = 2, 3, \ldots, n\). The following results, although presented for 2 and 3-agent control systems, may be extended to systems with \(N\)-control agents as a result of Corollary 3.2.2.

**Theorem 3.5.3** [8]: \(\lambda_1\) is not a decentralized fixed mode of the system (3.5.3), if and only if none of the following conditions occur with respect to the matrices \(A_i\) and
or their respective transposes, $A_1^T$ and

$$
\left[ W(s) - \frac{A_1}{s - \lambda_1} \right] \bigg|_{s=\lambda_1} \quad (3.5.8)
$$

Case 1: ($N = 2$)

$$
A_1 = \begin{bmatrix} 0 & X \\ \hline 0 & 0 \end{bmatrix} \quad \text{and} \quad \left[ W(s) - \frac{A_1}{s - \lambda_1} \right] \bigg|_{s=\lambda_1} = \begin{bmatrix} X & X \\ \hline 0 & X \end{bmatrix} \quad (3.5.9)
$$

Case 2: ($N = 3$)

$$
A_1 = \begin{bmatrix} 0 & 0 & X \\ \hline 0 & 0 & X \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \left[ W(s) - \frac{A_1}{s - \lambda_1} \right] \bigg|_{s=\lambda_1} = \begin{bmatrix} X & X & X \\ \hline X & X & X \\ 0 & 0 & X \end{bmatrix} \quad (3.5.10a)
$$

$$
A_1 = \begin{bmatrix} 0 & X & X \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \left[ W(s) - \frac{A_1}{s - \lambda_1} \right] \bigg|_{s=\lambda_1} = \begin{bmatrix} X & X & X \\ \hline 0 & X & X \\ 0 & X & X \end{bmatrix} \quad (3.5.10b)
$$

$$
A_1 = \begin{bmatrix} 0 & X & 0 \\ \hline 0 & 0 & 0 \\ 0 & X & 0 \end{bmatrix} \quad \text{and} \quad \left[ W(s) - \frac{A_1}{s - \lambda_1} \right] \bigg|_{s=\lambda_1} = \begin{bmatrix} X & X & X \\ \hline 0 & X & 0 \\ X & X & X \end{bmatrix} \quad (3.5.10c)
$$

where $X$ denotes elements whose values are not necessarily zero.
The condition (3.5.9), for the case $N = 2$, requires that, after cancellation, $W_{11}(s)$, $W_{21}(s)$, and $W_{22}(s)$ have no elements with a pole at $\lambda_i$ and that all elements of $W_{21}(s)$ have a zero (or zeros) at $s = \lambda_i$. Similar interpretations are also possible for the conditions (3.5.10) for a system with three control agents.

### 3.6 Characterization for interconnected systems

Many decentralized control problems involve systems comprised of interconnected subsystems. Consequently, it is useful to study the nature of decentralized fixed modes in this particular class of systems.

Consider the following system comprised of interconnected subsystems:

\begin{align}
\dot{x}_i &= A_i x_i + G_i v_i + B_i u_i \quad (3.6.1a) \\
z_i &= H_i x_i \quad (3.6.1b) \\
y_i &= C_i x_i \quad (3.6.1c) \\
v_i &= \sum_{j \neq i} L_{ij} z_j \quad (3.6.1d)
\end{align}

where $x_i \in \mathbb{R}^n$, $u$, $v_i \in \mathbb{R}^n$, and $y_i, z_i \in \mathbb{R}^n$, for $i = 1, 2, ..., N$. The matrices $A_i$, $B_i$, $C_i$, $G_i$, $H_i$, and $L_{ij}$ all have appropriate dimension. Assume that the following control law is applied to the system.
\[ u_i = K_{ij} z_j + K_i x_i + r_i \] (3.6.2)

where \( K_{ij} \neq 0 \) only if \( L_{ij} \neq 0 \). The decentralized control depends not only on the local states, but may also depend on local interconnection variables. If we desire, we may impose further restrictions on the control law.

The internal subsystems, defined by the triples \((A_i, G_i, H_i)\), characterize the internal interaction between the decoupled subsystems

\[ \dot{x}_i = A_i x_i \] (3.6.3)

through the interconnection structure (3.6.1d). The external subsystem, defined by the input-output pair \((u_i, y_i)\), characterizes the interaction between the i-th decoupled subsystem and the environment.

If \( B_i = G_i \) and \( C_i = H_i \) for all \( i \), we obtain the component connection model (CCM) for the system \([12]\). In this case, the subsystems interact internally the same way they interact with the environment.

The composite interconnected system has the form

\[
\begin{bmatrix}
A_1 & G_1 L_{12} H_2 & \cdots & G_1 L_{1N} H_N \\
G_2 L_{21} H_1 & A_2 & \cdots & G_2 L_{2N} H_N \\
\vdots & \vdots & \ddots & \vdots \\
G_N L_{N1} H_1 & G_N L_{N2} H_2 & \cdots & A_N
\end{bmatrix}
+ \sum_{i=1}^{N} B_i \begin{bmatrix} u_i \\ 0 \end{bmatrix}
\]

\[ y_i = \begin{bmatrix} 0 & \cdots & 0 & C_i & 0 & \cdots & 0 \end{bmatrix}, \quad i = 1, 2, \ldots, N \]
where \( x \in \mathbb{R}^n \) and \( n = \sum_{i=1}^{N} n_i \). The partitioning of the input and output matrices is compatible with the partitioning of the state vector, with \( B_i \) and \( C_i \) occupying the \( i \)-th blocks of their respective matrices. The composite feedback control law is

\[
\mathbf{u} = \begin{bmatrix}
K_1 & K_{12} H_2 & \cdots & K_{1N} H_N \\
K_{21} H_1 & K_2 & \cdots & K_{2N} H_N \\
\vdots & \vdots & \ddots & \vdots \\
K_{N1} H_1 & K_{N2} H_2 & \cdots & K_N
\end{bmatrix} \mathbf{x} + \mathbf{r} \tag{3.6.5}
\]

where \( u \in \mathbb{R}^m \) and \( m = \sum_{i=1}^{N} m_i \).

We refer to the subsystem model (3.6.1) as a decomposition of the composite system model (3.6.4). Note that the interconnection variables \((v_i, z_i)\) appear only in the decomposition and not the composite model.

When the internal subsystem model corresponds to a Component Connection Model, the existence decentralized fixed modes depends upon the controllability and observability of the internal subsystems.

**Theorem 3.6.1** [13,14]: Given the interconnected system (3.6.1), with \( B_i = G_i \) and \( C_i = H_i \), then necessary and sufficient conditions for the system to have no decentralized fixed modes are that the internal subsystems be both controllable and observable. \( \square \)

Theorem 3.6.1 provides necessary and sufficient conditions for fixed modes in systems with input-output interconnection \((B_i = G_i \text{ and } C_i = H_i)\). Somewhat weaker results also exist for the more general case where \( G_i \) and \( H_i \) are not necessarily equal to \( B_i \) and \( C_i \) respectively.
Theorem 3.6.2 [15]: Given the system (3.6.1), assume that \((C_i, A_i, B_i), i = 1, 2, ..., N\) are all controllable and observable; then the system has no decentralized fixed modes for almost all interconnection gains \(L_{ij}\) (the class of interconnection gain matrices \(\{L\}\) for which the system has decentralized fixed modes is either empty or lies on a subset of a hypersurface in the parameter space of \(L\)).

Suppose that we restrict the class of interconnected systems in (3.6.1) to include only those systems with scalar interconnections between subsystems. The interconnection structure in (3.6.1d) becomes

\[
v_i = \sum_{i \neq j} \ell_{ij} z_j
\]  

(3.6.6)

where \(v_i, z_i \in \mathbb{R}\), for \(i = 1, 2, ..., N\), and \(\ell_{ij}\) are scalar interconnection gains.

We are primarily interested here in the poles of the composite system, which we may describe in terms of the subsystem decomposition. Considering the internal interconnection structure (3.6.1) with scalar subsystem interconnections, \(L_{ij} = \ell_{ij}\), we may view the internal subsystem model as a signal flow graph whose nodes are the internal interconnection variables \(v_i\) and \(z_i\). Consequently, we can express the relationship between \(v_i\) and \(z_i\) as the transfer function of the internal subsystem

\[
S_i(s) = \frac{\gamma z_i(s)}{p_i(s)} = H_i(sI - A_i)^{-1} G_i
\]  

(3.6.7)

where \(z_i(s)\) is monic. The pole polynomial for the composite system may now be computed from the determinant of the signal flow graph using a method such as Mason's Gain Rule.
When the pole polynomial is computed using this method it has the following special structure:

\[ p(s) = p_a(s) + z_a(s) \]  
(3.6.8)

where

\[ p_a(s) = p_1(s) p_2(s) \ldots p_n(s) \]  
(3.6.9)

and

\[ z_a(s) = p(s) - p_a(s) \]  
(3.6.10)

The roots of the zero polynomial, \( z_a(s) \), are determined by the zeros of the internal transfer functions (3.6.7) and the interconnection structure (3.6.6).

This representation of the composite system pole polynomial leads to a characterization of fixed modes in interconnected systems with scalar interconnection gains. Let \( \{ K \} \) represent the set of all admissible control laws in (3.6.2), where \( K_i \neq 0 \) only if \( \epsilon_i \neq 0 \).

**Theorem 3.6.3** [16]: The complex number \( \lambda_i \) is a decentralized fixed mode of the interconnected system (3.6.1) with scalar interconnection gains, if \( (s - \lambda_i) \) is a common factor of both the numerator and denominator of the transfer function of the internal subsystem, \( S_i \), for all admissible controls in \( \{ K \} \).

\[ \square \]
3.7 Sampling and decentralized fixed modes

In Section 3.1 we showed that a continuous time linear system is stabilizable using decentralized control if and only if there are no unstable decentralized fixed modes [7]. In fact, one characterization of decentralized fixed modes is precisely those open loop eigenvalues of the system which are unaffected by decentralized output feedback. Despite the presence of decentralized fixed modes, however, it is possible to drive the system states to zero using time-varying controllers [17,18].

Although there are many characterizations, all decentralized fixed modes may be classified as either structured or unstructured. Structurally fixed, or structured, modes generally arise due to a physical lack of interaction between certain parts of a system and the system inputs and outputs. Unstructured decentralized fixed modes typically arise due to pole-zero cancellations.

In sampled data systems with zero-order-holds at each input, for almost all sampling intervals, the continuous-time unstructured fixed modes are not present in the discrete-time system model [19]. Hence, a continuous-time system with unstable decentralized fixed modes is stabilizable with decentralized digital controllers for almost all sampling rates, as long as the fixed modes are not structurally fixed.

Consider the system

\[
\dot{x}(t) = Ax(t) + \sum_{i=1}^{N} B_i u_i \\
y_i(t) = C_i x(t), \quad i = 1, 2, \ldots, N
\]

(3.7.1)

where \( x \in \mathbb{R}^n \), \( u_i \in \mathbb{R}^{m_i} \), and \( y_i \in \mathbb{R}^{n_i} \). Let the matrix triplet \((A,B,C)\) denote the system (3.7.1), and let
\[ B \overset{\Delta}{=} \begin{bmatrix} B_1, B_2, \ldots, B_N \end{bmatrix} \]
\[ C \overset{\Delta}{=} \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_N \end{bmatrix} \tag{3.7.2} \]

denote the composite input and output matrices. We shall assume the system \((A,B,C)\) is both completely controllable and completely observable, and that matrix \(A\) has \(n\) distinct eigenvalues which we shall denote \(\{\lambda_i\}_{i=1}^n\).

Suppose that the system \((A,B,C)\) is transformed to Jordan canonical form, then (3.7.1) becomes

\[
\dot{x}(t) = \Lambda \tilde{x}(t) + \sum_{i=1}^{N} \widetilde{B}_i u_i(t) \tag{3.7.3}
\]
\[ y_i(t) = C_i \tilde{x}(t), \quad i = 1, 2, \ldots, N, \]

where \(\Lambda\) is the diagonal matrix

\[ \Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}. \tag{3.7.4} \]

Recall the recursive characterization of decentralized fixed modes presented in Section 3.2. The characterizations of structurally fixed and unstructured decentralized fixed modes presented in Theorems 3.7.4 and 3.7.2 respectively for a 2-agent control system can be extended to more general systems with \(N\)-control agents.
Suppose that zero-order-holds are placed on the control inputs and a digital controller with sampling interval $h$ is used to control the system. The sampled-data representation of the Jordan form (3.7.3) of the system $(A,B,C)$ in (3.7.1) is described by

$$\tilde{x}(k+1) = e^{\Delta h} \tilde{x}(k) + \sum_{i=1}^{N} \Gamma B_i u_i$$

$$y_i(k) = C_i \tilde{x}(k), \quad i = 1, 2, \ldots, N,$$

where the time between sampling instances is $h$, and

$$\Gamma \triangleq \text{diag}\{\Gamma_1, \Gamma_2, \ldots, \Gamma_n\}$$

where

$$\Gamma_i \triangleq \begin{cases} e^{h\lambda_i} - 1 \over \lambda_i, & \text{if } \lambda_i \neq 0, \\ h, & \text{if } \lambda_i = 0. \end{cases}$$

**Theorem 3.7.1 [19]:** Suppose that the system (3.7.1), has $p_u \geq 0$ unstructured decentralized fixed modes and $p_s \geq 0$ structured decentralized fixed modes $\lambda_i$, $i = 1, 2, \ldots, p_s$; then the sampled system (3.7.5) has $p_s$ structured decentralized fixed modes $\{\lambda^d_i\}$ given by

$$\lambda_i^d = e^{h\lambda_i}, \quad i = 1, 2, \ldots, p_s, \quad \forall \ h > 0,$$

and no unstructured decentralized fixed modes for almost all $h > 0$. \hfill \square
This theorem states that even if a plant has decentralized fixed modes, if all of the fixed modes are unstructured, then a digital controller can be used to stabilize the plant (or for pole placement, etc.) for almost all sampling intervals. The disappearance of the unstructured decentralized fixed modes in the sampled system is due to the fact that they arise in the continuous-time system because of pole-zero cancelations. Sampling effects poles and zeros differently so that the pole-zero cancellations that created unstructured decentralized fixed modes in the continuous-time model are no longer present in the sampled system.

3.8 Summary

Let $S$ denote an arbitrary nonempty open subset of the complex plane. Given any linear, time-invariant, dynamic system, there exists a linear, time-invariant, decentralized output feedback controller for the system such that the closed-loop poles of the system are contained in $S$, if and only if any decentralized fixed modes of the system are contained in $S$. Thus, the role of decentralized fixed modes in decentralized pole placement is similar to that of the uncontrollable and unobservable modes for systems with centralized feedback.

The fixed modes of a system are defined as those open-loop eigenvalues of the system that are invariant with respect to a specified feedback information flow constraint that defines which outputs may provide feedback to which inputs in the system. Decentralized fixed modes are defined with respect to decentralized output feedback, however, fixed modes may be defined with respect to any arbitrary feedback information flow constraint. Centralized fixed modes, modes that are fixed with respect to full or
centralized output feedback, are precisely those system modes that are either uncontrollable or unobservable.

There has been much research into the identification of alternate characterizations of decentralized fixed modes. These alternate characterizations provide insight into the relationships between system structure and the presence of decentralized fixed modes as well as alternate methods of determining the decentralized fixed modes of a system.

Although there are many characterizations of fixed modes, all decentralized fixed modes can be classified as either structured or unstructured. Structured, or structurally fixed, modes arise due to the lack of physical interconnections within a system. Unstructured fixed modes, on the other hand, are the result of perfect matching between system parameters; hence, if the system parameters are perturbed in some way, the unstructured fixed modes are no longer present. Any system modes that are unstructured fixed modes with respect to a continuous-time controller, are not fixed with respect to a sampled date, or digital, controller subject to the same feedback information flow constraints. Structurally fixed modes, however, remain invariant to both continuous and discrete feedback control.
4.0 Velocity feedback control

Collocated control, the use of collocated sensors and actuators, represents an attractive strategy for robust attitude control and vibration suppression in large space structures. Collocated attitude controllers are designed to control rigid-body attitude (i.e. the spacecraft's rigid body modes) as well as structural vibrations. Collocated direct velocity feedback controllers, also known as collocated rate feedback controllers, are designed to enhance structural damping, without affecting the rigid body modes.

The performance requirements for vibration suppression control systems are relatively low: the primary requirement is typically good relative stability. When the plant and controller both satisfy certain frequency domain conditions, collocated controllers guarantee stability even in the presence of plant uncertainties [25,26]. When implemented using perfect sensors and actuators, such controllers provide closed-loop stability regardless of the number of modes included in the design model and model parameter errors. The sensors and actuators available in practice, however, often have nonlinearities and phase shifts. Therefore, in order to be useful in practical applications, the controller should be tolerant to such common imperfections as saturation, relays, dead zones, actuator dynamics, and computational delays [30].
The stability and robustness of colocated rate feedback, as well as other classes of colocated controllers, depends upon both system and controller satisfying certain frequency domain conditions. These conditions are satisfied by systems and controllers that are positive real, an extension of the mathematical concept of positive real functions to system theory [22,24,20]. Positivity concepts are applicable not only to velocity feedback, however, but also to more general controller designs [23].

4.1 Positive real systems

The theory of positive dynamic systems is an extension of the mathematical concept of positivity, or positive real functions, to dynamic systems. In control theory, the concept of positive dynamic systems has appeared explicitly or implicitly in many problems such as the stability of linear and nonlinear feedback systems and optimal control. The work done in this area has played an important role in establishing a link between control theoretic results in the time and frequency domains.

The theory of positive systems also plays an important role in network theory. When the transfer function matrix describing a system is strictly positive real, the system is energy dissipating, and if the system represents an electrical network, the network is realizable using only passive circuit components.

In this section we shall present the basic mathematical definitions of positivity and positive real functions, then show the extension of these results to positive dynamic systems.

A positive real function of a complex variable has the following formal definition:
**Definition 4.1.1** [20]: A rational function $h(s)$ of the complex variable $s = \sigma + j\omega$ is **positive real** if

1. $h(s)$ is real for all real $s$.
2. $\text{Re}[h(s)] \geq 0$ for all $\text{Re}[s] > 0$.

The examination of $\text{Re}[h(s)]$ over the entire open right-half plane required for direct application of Definition 4.1.1 is a tedious operation. The following lemma provides a more useful method of determining if a function is positive real.

**Lemma 4.1.2** [20]: A rational function $h(s)$ of the complex variable $s = \sigma + j\omega$ is **positive real** if

1. $h(s)$ is real for all real $s$.
2. $h(s)$ has no poles in the open right-half plane, $\text{Re}[s] > 0$.
3. Any poles of $h(s)$ that lie on the imaginary axis, $\text{Re}[s] = 0$ (i.e. $s = j\omega$), are distinct, and the residues associated with these poles are real and positive (or zero).
4. $\text{Re}[h(j\omega)] \geq 0$ for all real $\omega$ for which $s = j\omega$ is not a pole of $h(s)$.

The following definition describes sufficient conditions for a rational function $h(s)$ to be strictly positive real.

**Definition 4.1.3** [20]: A rational function $h(s)$ of the complex variable $s = \sigma + j\omega$ is **strictly positive real** if

1. $h(s)$ is real for all real $s$.
2. $h(s)$ has no poles in the closed right-half plane, $\text{Re}[s] \geq 0$.
3. \( \text{Re}[h(j\omega)] > 0, \ -\infty < \omega < +\infty \). 

From the second condition in Lemma 4.1.2 we see that if a positive real function represents the transfer function for a dynamic system, the system may be unstable, since \( h(s) \) may have poles on the imaginary axis. The definition of strictly positive real functions, however, requires that the poles of \( h(s) \) all lie in the open left-half plane; hence, dynamic systems represented by strictly positive real transfer functions are stable.

Further comparison of Lemma 4.1.2 and Definition 4.1.3 shows that for a strictly positive real function \( h(s) \), \( \text{Re}[h(j\omega)] > 0 \) for all real \( \omega \), but if \( h(s) \) is only positive real the condition is relaxed somewhat and \( \text{Re}[h(j\omega)] \geq 0 \). The interpretation of this difference in terms of dynamic systems is that systems with positive real transfer functions have phase margins greater than or equal to 90 degrees, while systems with strictly positive real transfer functions have phase margin greater than 90 degrees.

**Theorem 4.1.1:** Suppose that the transfer function for a dynamic system is strictly positive real, then the corresponding Nyquist plot of the transfer function lies entirely in the open right half plane, and the system has more than 90° phase margin. 

**Corollary 4.1.2:** The Nyquist plot of a positive real transfer function lies entirely in the closed right half plane, and has at least 90° of phase margin.

When a positive real function \( h(s) \) has the form

\[
h(s) = \frac{n(s)}{d(s)} \tag{4.1.1}
\]
where \( n(s) \) and \( d(s) \) are relatively prime polynomials of the complex variable \( s \), the function and the numerator and denominator polynomials have several special properties.

\textbf{Theorem 4.1.3} [20]: If \( h(s) \) is a positive real function of the form (4.1.1), then

1. \( n(s) \) and \( d(s) \) have real coefficients.
2. \( 1/h(s) \) is also a positive real function.
3. \( n(s) \) and \( d(s) \) are Hurwitz polynomials (they verify the Hurwitz criterion, and their zeros have negative real parts).
4. The order of \( d(s) \) does not differ from the order of \( n(s) \) by more than \( \pm 1 \). \( \square \)

We also have the following theorem which imposes limits on the behavior of the function as \( \omega \) approaches infinity.

\textbf{Theorem 4.1.4} [21]: Let \( n^* \) be the relative degree of the polynomial fraction \( h(s) \) in (4.1.1) (i.e. \( n^* = \text{deg}[n(s)] - \text{deg}[d(s)] \)), and assume \( h(s) \) is not identically zero for all \( s \). Then \( h(s) \) is strictly positive real if and only if

1. \( h(s) \) is analytic in \( \text{Re}[s] \geq 0 \)
2. \( \text{Re}[h(j\omega)] > 0 \quad \forall \omega \in (-\infty, +\infty) \)
3. If \( n^* = 1 \), then
   \[
   \lim_{\omega \to -\infty} \omega^2 \text{Re}[h(j\omega)] > 0 \quad (4.1.2)
   \]
4. If \( n^* = -1 \), then
\[
\lim_{\omega \to \infty} \text{Re}[h(j\omega)] > 0 \tag{4.1.3}
\]

and

\[
\lim_{|\omega| \to \infty} \frac{h(j\omega)}{j\omega} > 0. \tag{4.1.4}
\]

The concept of positive real systems is an extension of the theory of positive real functions to dynamic systems. Consider the following frequency domain description of a dynamic system

\[ Y(s) = H(s)U(s) \tag{4.1.5} \]

where \( U(s), (m \times 1), \) represents the system inputs, \( Y(s), (r \times 1), \) the output of the system, and \( H(s) \) is the \((r \times m)\) system transfer function matrix.

**Definition 4.1.4** [22]: A square transfer function matrix \( H(s) \) is **positive real** if

1. \( H(s) \) has real elements for all real \( s \).
2. All elements of \( H(s) \) are analytic in the open right-half plane, \( \text{Re}[s] > 0 \).
3. \( H^*(s) + H(s) \) is nonnegative definite for all \( s \) with \( \text{Re}[s] > 0 \), where \( H^*(s) \) is the complex conjugate transpose of \( H(s) \).

We say that the dynamic system described by (4.1.5) is positive real if the transfer function matrix, \( H(s) \), for the system is positive real.
A dynamic system is said to be strictly positive real if the transfer function matrix, $H(s)$, which describes the system is strictly positive real.

**Definition 4.1.5** [22]: A square transfer function matrix $H(s)$ is strictly positive real if

1. $H(s)$ has real elements for all real $s$.
2. All elements of $H(s)$ are analytic in the closed right-half plane, $\text{Re}[s] \geq 0$.
3. $H^*(j\omega) + H(j\omega)$ is positive definite for all real $\omega$.

Comparing the conditions in Definitions 4.1.4 and 4.1.5, the difference between positive real and strictly positive real systems is essentially the same as the difference between positive real and strictly positive real functions. A positive real system may have poles along the $j\omega$-axis, while a strictly positive real system must have all system poles in the open left half plane, and $H^*(j\omega) + H(j\omega)$ must be positive semidefinite for a positive real system, but must be positive definite for a system to be strictly positive real.

For single-input/single-output (SISO) systems, the transfer function matrix, $H(s)$, is simply a scalar transfer function, and Definitions 4.1.4 and 4.1.5 simplify to the definitions for positive and strictly positive real functions. A SISO system is positive real if its scalar transfer function $h(s)$ is a positive real function, and the system is strictly positive real if $h(s)$ is strictly positive real.

Direct application of Definitions 4.1.4 and 4.1.5 to multi-input/multi-output (MIMO) systems is typically a nontrivial exercise. The following theorem, however, provides a state space characterization of positive dynamic systems; it incorporates a system's state space and transfer function matrix representations and is equally well adapted for SISO and MIMO systems.
Theorem 4.1.5 [22]: Let \( H(s) \) be a square matrix of rational transfer functions such that \( H(\infty) \) is finite and \( H(s) \) has poles which either lie in \( Re[s] < 0 \) or are simple on \( Re[s] = 0 \). Let \( \{A, B, C, H(\infty)\} \) be a minimal state-space realization of \( H(s) \). Then \( H(s) \) is positive real if and only if there exists a symmetric, positive definite matrix \( P \) and matrices \( W_0 \) and \( L \) such that

\[
PA + A^T P = -LL^T, \tag{4.1.6}
\]

\[
W_0^T W_0 = H(\infty) + H(\infty)^T, \tag{4.1.7}
\]

\[
C^T = PB + LW_0. \tag{4.1.8}
\]

\( \square \)

Theorem 4.1.5 allows only simple (nonrepeated) poles on the \( j\omega \)-axis; however, any real structure has some damping, although it may be small, so that repeated modal frequencies are removed from the \( j\omega \)-axis by this limiting argument. Furthermore, examination of the proof of Theorem 4.1.5 reveals that it would allow for completely undamped modal frequencies along the \( j\omega \)-axis with multiplicity greater than one because the normal coordinate formulation of the structure allows for strict diagonalization even with repeated eigenvalues [23].

The primary difference between positive and strictly positive real systems is in the nature of the transfer function matrix \( H(s) \) as \( s \) approaches the \( j\omega \)-axis in the left-half plane. This leads to the following characterization of strictly positive real transfer function matrices.

Theorem 4.1.6 [24]: The transfer function matrix \( H(s) \) is strictly positive real if there exists \( \sigma > 0 \) such that
\[ \hat{H}(s) \triangleq \mathcal{H}(s - \sigma) \]  

is positive real.

The result of Theorem 4.1.6 corresponds to modifying (4.1.6) in Theorem 4.1.5 so that

\[ PA + A^T P = -Q \]  

(4.1.10)

where \( Q = LL^T \) must be positive definite. This leads to the following corollary.

**Corollary 4.1.7**: Let \( H(s) \) be a square matrix of rational transfer functions such that \( H(\infty) \) is finite and \( H(s) \) has poles which either lie in \( \text{Re}[s] < 0 \) or are simple on \( \text{Re}[s] = 0 \). Let \( \{A, B, C, H(\infty)\} \) be a minimal state-space realization of \( H(s) \). Then \( H(s) \) is strictly positive real if and only if there exists a symmetric positive definite matrix \( P \) and matrices \( W_0 \) and \( L \) such that

\[ PA + A^T P = -Q, \]  

(4.1.11)

\[ W_0^T W_0 = H(\infty) + H(\infty)^T, \]  

(4.1.12)

\[ C^T = PB + LW_0, \]  

(4.1.13)

where \( Q = LL^T \) is symmetric and positive definite.

Perhaps the most important contribution of Theorem 4.1.5 (and similarly Corollary 4.1.7), is that it permits us to determine the output matrix \( C \) of a system so that its transfer function matrix.
\[ H(s) = C(zI - A)^{-1}B \]  \hspace{1cm} (4.1.14)

is positive real. When \( H(\infty) = 0 \), the conditions (4.1.7) and (4.1.12) disappear. We can then choose \( L \), solve the Lyapunov equation (4.1.6), or (4.1.11), for \( P \), and compute the output matrix \( C \) from \( C^T = PB \). The structure of the input matrix, \( B \) is usually fixed; however, since \( L \) is arbitrary, there is, in general, no unique \( C \) for which the system is positive real (or strictly positive real). This permits a certain amount of design flexibility while keeping the system positive (or strictly positive) real.

### 4.2 Control design using positivity concepts

Suppose the block diagram shown in Figure 1 represents a feedback control system for active vibration suppression in large space structures. The square transfer function matrix \( G(s) \) represents the structure, and the square transfer function matrix \( H(z) \) represents a controller, which may include a reduced order observer. The vector \( u(t) \) denotes the control forces applied to the structure by the actuators, and \( y(t) \) represents the outputs of sensors located on the structure. The vector-valued function \( r(t) \) represents the reference input to the closed loop system; of course, for active vibration damping, the reference input is identically equal to zero. The difference between the sensor measurements \( y(t) \) and the reference input \( r(t) \) generate error signals which are represented by the vector \( e(t) \).

The transfer function matrices \( G(z) \) and \( H(s) \) in Figure 1 are both square; hence, the plant and the controller must both have the same number of inputs and outputs. Although a square transfer function matrix typically implies that the plant has an equal
Figure 1. Feedback control system with controller in forward path
number of sensors and actuators, the system could have more sensors than actuators; some outputs, $y_i$, may be linear combinations of sensor measurements. We should also remember that while the dimensions of the matrices $G(s)$ and $H(s)$ reflect the number of system inputs and outputs, they indicate neither the order of the system model nor the order of the controller.

The following theorem provides sufficient conditions for asymptotic stability of the closed loop system in Figure 1.

**Theorem 4.2.1** [23]: Let $S$ be a system with the block diagram description shown in Figure 1, where $G(s)$ and $H(s)$ are square transfer function matrices. If at least one of the transfer function matrices $G(s)$ or $H(s)$ is strictly positive real and the other is positive real, then the system is asymptotically stable.

Let $T(s)$ denote the transfer function matrix from the reference input, $r(t)$, to the measured output, $y(t)$ of the system in Figure 1, then

$$T(s) = (sI + G(s)H(s))^{-1} G(s)H(s), \quad (4.2.1)$$

and the closed-loop poles of the system are the zeros of the characteristic equation

$$\det(sI + G(s)H(s)) = 0. \quad (4.2.2)$$

Now consider the feedback control system in Figure 2, where $H(s)$ and $G(s)$ represent the same plant and controller as in Figure 1, but now the controller is located in the feedback path instead of the feedforward path. The corresponding transfer function matrix description of this system is

$$W(s) = (sI + G(s)H(s))^{-1} G(s). \quad (4.2.3)$$
Both system configurations have the same characteristic equation, (4.2.2), and hence, the same closed loop poles. Consequently, we would expect Theorem 4.2.1 to also hold when the controller is located in the feedback path, as in Figure 2.

**Corollary 4.2.2** [23]: Let $S$ be a system with the block diagram description shown in Figure 2, where $G(s)$ and $H(s)$ are square transfer function matrices. If at least one of the transfer function matrices $G(s)$ or $H(s)$ is strictly positive real and the other is positive real, then the system is asymptotically stable.

Theorem 4.2.1 and Corollary 4.2.2 both employ transfer function representations of their respective systems. The transfer function matrix, however, only describes that part of a system which is completely controllable and observable. Therefore, Theorem 4.2.1 and Corollary 4.2.2 only guarantee asymptotic stability for those states of the closed loop system that are both controllable and observable.

Application of Theorem 4.2.1 requires that the plant $G$ be either positive real or strictly positive real. This requirement may initially seem too restrictive for the theorem to be very useful. A technique called "embedding", however, may be applied when this condition is not satisfied, which permits application of the theorem.

Embedding is essentially a block diagram transformation of the original system into an equivalent system, in the input/output sense, that embeds the original transfer function matrices (or operators) in new and different transfer function matrices which satisfy the positivity conditions required for stability. Figure 3 illustrates the embedding process. The transfer function matrices (or operators) are

$$
\tilde{H} = (I - F^{-1}HD)^{-1}F^{-1}H
$$

(4.2.4)

and
Figure 2. Feedback control system with controller in the feedback path
Figure 3. Block diagram transformation for embedding
\[ \tilde{G} = GF + D. \quad (4.2.5) \]

The operators F and D in Figure 3 are selected to make \( \tilde{G} \) (strictly) positive real, even though \( G \) is not. The controller \( H \), however, must be very stable and positive for the new operator \( \tilde{H} \) to remain (strictly) positive real, since the embedding process often results in wrapping it, \( H \), in a positive feedback loop.

Although the embedding technique illustrated in Figure 3 applies specifically to systems such as the one in Figure 1, a similar block diagram transformation exists for systems with the controller in the feedback path. Embedding techniques are important when we wish to include sensor and actuator dynamics in our system model, since their addition often results in a loss of positivity.

### 4.3 Positivity of large space structures

Consider the state space model

\[
\begin{align*}
x(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
\end{align*}
\quad (4.3.1)
\]

for a large space structure with state vector

\[
x \triangleq [\eta_1, \dot{\eta}_1, \eta_2, \dot{\eta}_2, \ldots, \eta_N, \dot{\eta}_N]^T.
\quad (4.3.2)
\]

This is the modal coordinate model described in Section 2.5. The state vector has dimension \( 2N \), where \( N \) is equal to the number of structural modes included in the
model. The dimensions of the control input, \( u(t) \), and the measured output, \( y(t) \), correspond to the number of actuators and sensors included in the mathematical model.

With the state vector (4.3.2), the system matrix, \( A \), is block diagonal with diagonal blocks \( A_i \in \mathbb{R}^{2 \times 2} \) as in (2.5.5). The input matrix \( B \) has alternating zero and nonzero rows, with the nonzero rows comprised of mode shape coefficients as described in equations (2.5.6) and (2.5.7).

An important contribution of Theorem 4.2.5 is that it provides conditions on the output matrix \( C \) in (4.3.1) so that the matrix transfer function

\[
G(s) = C(sI - A)^{-1}B
\]  

(4.3.3)

for the structure is positive real. Corollary 4.2.7 provides similar conditions for determining output matrices \( C \) for which the system is strictly positive real.

Let

\[
P = \text{block diag}(P_1, P_2, \ldots, P_N),
\]

(4.3.4)

with

\[
P_i = \begin{bmatrix}
\omega_i^2 & 0 \\
0 & 1
\end{bmatrix}.
\]

(4.3.5)

Clearly, \( P \) is symmetric, and \( P \) is positive definite provided none of the modal frequencies, \( \omega_i \), are identically equal to zero. This condition is satisfied when the mathematical model (4.3.1) contains only flexible modes; rigid body modes have \( \omega_i = 0 \), but \( \omega_i > 0 \) for all flexible modes. Because the \( A \) matrix in (2.5.1) has the same block diagonal structure as \( P \) (i.e. both are composed of \( 2 \times 2 \) diagonal blocks), it is easy to see that \( P \) satisfies the Lyapunov equation (4.2.6) with

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\[ L = \text{block diag} (L_1, L_2, \ldots, L_N), \]  

where

\[
L_i = \begin{bmatrix}
0 & 0 \\
0 & 2\xi_i \omega_i \nu_i
\end{bmatrix}. 
\]

By Theorem 4.2.5, the transfer function matrix \( G(s) \) for the system is positive real for

\[ C^T = PB. \]  \hspace{1cm} (4.3.8)

Due to the structure of the input matrix \( B \) and our choice for the matrix \( P \),

\[ PB = B. \]  \hspace{1cm} (4.3.9)

Taking the transpose of both sides of (4.3.8), the output matrix

\[ C = B^T \]  \hspace{1cm} (4.3.10)

makes the transfer function matrix for the structure positive real.

Equation (4.3.10) implies that the nonzero column vectors, \( c_j \), of the output matrix, \( C \), are identical to the nonzero rows, \( b_i \), of the input matrix, \( B \). The locations of the zero and nonzero columns in the output matrix \( C = B^T \) imply that output \( y(t) = Cx(t) \) represents measured velocity. The nonzero rows of \( B \) contain the values of the mode shapes evaluated at the actuator locations, and the nonzero columns of the output matrices contain the values of the mode shapes evaluated at the sensor locations; therefore, (4.3.10) implies that the actuators and rate sensors are collocated.
Due to the block diagonal structure of the $A$ matrix and the matrices $P$ and $L$, the derivation of the output matrix (4.3.10) which resulted in the system being positive real was independent of the number of modes included in the model, and the numerical values of the mode shapes and modal frequencies. The following theorem summarizes these results.

**Theorem 4.3.1** [23]: The transfer function matrix of a large space structure from force input to velocity output is positive real if the structure employs perfect, collocated rate sensors and force actuators. Furthermore, this result is independent of the number of modes included in the model, the numerical values of the modal frequencies and the mode shapes of the structure, and the amount of structural damping assumed for each mode.

The output matrix $C = B^T$ makes the system positive real, but not strictly positive real. For $L$ as in (4.3.6) and (4.3.7), $LL^T$ is only positive semidefinite, not positive definite; Corollary 4.2.7 requires that $LL^T$ be positive definite for the system to be strictly positive real.

It is also important to note that our choice of $L$, and therefore the corresponding solution of the Lyapunov equation (4.3.6), is not unique. Other choices for $L$, and consequently $C$, are possible for which the system is positive real.
4.4 Robustness of collocated rate feedback

Theorem 4.2.1 provides the basis for the application of positivity concepts to control system design; a closed-loop system consisting of a positive real plant and a strictly positive real controller, or a strictly positive real plant and positive real controller, is asymptotically stable. By Theorem 4.3.1 large space structures with collocated actuators and rate sensors, are positive real. These two results, when combined, account for the stability and robustness properties of collocated rate feedback control for large space structures.

Consider again, the state space model

\[
\dot{x}(t) = Ax(t) + Bu(t) \\
y(t) = Cx(t) \tag{4.4.1}
\]

for a large space structure with state vector

\[
x \triangleq [\eta_1, \dot{\eta}_1, \eta_2, \dot{\eta}_2, \ldots, \eta_N, \dot{\eta}_N]^T. \tag{4.4.2}
\]

Suppose that the structure employs collocated pairs of force actuators and rate sensors; then, \(y(t)\) represents measured velocity, and

\[
C = B^T. \tag{4.4.3}
\]

We are interested in output feedback control laws of the form

\[
u(t) = -Ky(t) + r(t) \tag{4.4.4}
\]
where \( y(t) \) is the velocity output in (4.4.1), and \( r(t) \) is some reference input. Because of the use of collocated actuators and rate sensors, the feedback scheme in (4.4.4) is referred to as collocated rate feedback, or direct velocity feedback.

The following theorem provides necessary conditions for guaranteed stability and robustness using rate feedback for large space structures.

**Theorem 4.4.1** [25,26]: Direct velocity feedback control of large space structures guarantees stability regardless of the number of modes included in the mathematical model of the structure and uncertainties in the model parameters, provided the following conditions are satisfied:

1. The structure employs perfect, collocated rate sensors and force actuators.

2. The actuators do not excite - or at least maintain constant energy in - the rigid-body modes of the structure.

3. The feedback matrix \( K \) in (4.4.4) is positive definite.

In Section 4.3 we showed that the a large space structure with collocated actuators and rate sensors is positive real; hence, condition one in Theorem 4.4.1 represents a sufficient condition for positivity of the structure. We assumed in our development in Section 4.3, however, that the structure's rigid-body modes were excluded from the model. In addition, the rigid-body modes are unobservable from the rate sensors; therefore, we should avoid exciting any rigid body motion.

If the structure is positive real, application of the results of Theorem 4.2.1 requires that the controller be strictly positive real. For static output feedback, this implies that
the feedback matrix to be positive definite, as required by condition number three in Theorem 4.4.1.

A variety of design methods exist which capitalize on the stability and robustness of collocated rate feedback [27,28]. Several of these methods include output feedback modal decoupling, pole assignment, and optimal and suboptimal control.

One of drawback of collocated rate feedback control is relatively low performance; although it is possible to achieve acceptable performance using direct velocity feedback [29], the corresponding controller gains and control inputs are often unacceptably large.

Another disadvantage of collocated rate feedback is that the stability and robustness properties depend upon implementation of the control law with perfect sensors and actuators. Unfortunately, the sensors and actuators available in practice are not perfect and may contain nonlinearities and phase lags. In order to be useful in practice, controller designs must tolerate nonlinearities such as saturation, relays, and dead zones, and phase shifts resulting from computational delays or actuator dynamics.

Collocated rate feedback can tolerate certain types of sensor and actuator of nonlinearities and dynamics. Theorem 4.4.2 provides sufficient conditions under which collocated rate feedback controllers provide stability for large space structures, even with nonideal sensors and actuators. First, however, we need the following definition:

**Definition 4.4.1** A function $\psi(v)$ is said to belong to the $(0, \infty)$ sector if $\psi(0) = 0$ and $v\psi(v) > 0$ for $v \neq 0$; $\psi$ is said to belong to the $[0, \infty)$ sector if $v\psi(v) \geq 0$ for $v \neq 0$. 

**Theorem 4.4.2** [30]: Collocated rate feedback controllers guarantee stability when the following conditions are all satisfied:

1. The controller is strictly positive real.
2. Any time-varying sensor or actuator nonlinearities belong to the \([0, \infty)\) sector.

3. The controller consists of stable first order dynamics followed by time-invariant nonlinearities belonging to the \([0, \infty)\) sector.

Theorem 4.4.1 states that collocated rate feedback control guarantees closed-loop asymptotic stability, regardless of the number of modes included in the design model and the numerical values of the model parameters, when the structure employs perfect sensors and actuators. Theorem 4.4.2, however, shows that collocated rate feedback preserves asymptotic stability under much weaker conditions.

Collocated direct velocity feedback controllers have a 90 deg phase margin and are tolerant to time-varying sensor and actuator nonlinearities belonging to the \([0, \infty)\) sector. Furthermore, collocated rate feedback controllers can also tolerate \([0, \infty)\) nonlinearities and first-order dynamics in the loop simultaneously.

One practical implication of these results is that if the actuators, or sensors, used in practice are not perfect, but can be approximated by first-order dynamics, then collocated rate feedback still stabilizes the system. Even when the sensors and actuators are not perfect, these stability results are still valid regardless of the number of modes in the model and the numerical values of the model parameters.

4.5 Summary

In this chapter we introduced the mathematical definition of positive real and strictly positive real rational functions, and the extension of these concepts to matrices
rational functions. These positivity concepts were extended to include dynamic systems by considering the input-output representation of the system. A system is positive real or strictly positive real if the transfer function, or transfer function matrix, for the system is positive or strictly positive real.

One application of positivity concepts to control system design concerns the closed-loop stability of systems with output feedback control. If a system is positive real then the closed-loop system is stable for any negative definite output feedback gain matrix. Similarly, given any strictly positive real system and any negative semidefinite output feedback gain matrix, the closed-loop system is stable.

In this chapter, we also showed that the transfer function matrix, from actuator input to sensor output, for a large space structure is positive real if the structure employs perfect, collocated rate sensors and force actuators. This result was independent of the numerical values of the parameters in the mathematical model, and the model order. Therefore, negative definite collocated rate feedback guarantees robust stability of the closed-loop system. Collocated rate feedback was also shown to guarantee stability even when the sensors and actuators are not perfect, that is sensors and actuators having first order dynamics, or nonlinearities that satisfy certain sign conditions.
5.0 Uniform damping control

Uniform damping control is a discrete implementation of a distributed optimal control law for vibration suppression in flexible space structures. The distributed optimal control law minimizes a quadratic cost functional for the spacecraft which includes measures of the total control energy expenditure and the total energy of the structure [31].

The uniform damping control law employs a linear combination of position and rate feedback. The feedback gains are chosen so that all of the closed-loop poles of the system are stable and have the same real part. This is accomplished, however, without altering the system's natural modes or natural frequencies. Because all of the closed-loop poles have the same (negative) real part, the motion due to each of the system's vibrational modes decays at one uniform exponential rate, hence the name uniform damping control.

One drawback of uniform damping control is that the control law generally is not decentralized. In many applications, however, we can approximate the uniform damping control law with a decentralized control law with little change in the closed-loop pole locations of the system. Another disadvantage of uniform damping control is that the
number of modes included in the mathematical model for the structure must equal the number of sensors and actuators included in the model. This arises due to the calculation of the uniform damping control feedback gain matrix from the modal matrix for the system.

Using root locus analysis, we can compare the effects of uniform damping control and collocated rate feedback on the closed-loop poles and zeros of a system [32]. Although uniform damping control employs a combination of both velocity and position feedback, in many applications - particularly when there are limits on the maximum control force which can be applied to the structure - the control input consists primarily of velocity feedback.

5.1 Performance specifications

The uniform damping control law is designed to meet the following dynamic performance specifications for the closed-loop system:

1. The motion due to each mode decays at a single exponential rate.

2. The closed-loop frequencies of oscillation are identical to the uncontrolled natural frequencies.

3. The closed-loop vibrational modes are identical to the uncontrolled natural modes of vibration.
As a result of the first performance specification, all vibrations in the spacecraft will decay at the same exponential rate. The other two closed-loop system specifications are designed to minimize the unnecessary expenditure of additional control effort.

The lumped-parameter model for a large space structure with negligible inherent structural damping is given by

\[ \ddot{\eta}(t) + \Omega \eta(t) = \Phi^T u(t) , \]  

where \( \eta(t) \) is the spacecraft position in modal coordinates, \( \Omega \) is a diagonal matrix with diagonal elements \( \omega_j^2 \) corresponding to the natural frequencies of the vibrational modes included in the model, \( u(t) \) represents control forces input to the system, and \( \Phi \) is the modal matrix. Because the state of the spacecraft can be completely described by the position and velocity of the motion due to each of the vibrational modes, we are interested in control laws of the form

\[ u(t) = Gq(t) + H\dot{q}(t) , \]  

where \( q(t) \) and \( \dot{q}(t) \) are the measured position and velocity of the spacecraft in generalized coordinates.

The position (and velocity) of the spacecraft in modal coordinates are related to the measured position (and velocity) by the modal matrix \( \Phi \). The spacecraft displacement, \( q(t) \) can be expressed as a linear combination of the displacements due to each of the vibrational modes,

\[ q(t) = \Phi \eta(t) = \sum_{r=1}^{n} C_r \phi_r e^{i\omega_r t} \]  

(5.1.3)
where \( \lambda = \alpha + j\beta \), is the closed-loop eigenvalue corresponding to the \( r \)-th vibrational mode, \( \phi \), is the \( r \)-th modal vector, and \( C \) is a complex constant which depends on the system's initial conditions. The real part \( \alpha \), of the eigenvalue \( \lambda \), represents the exponential rate of decay (\( \alpha < 0 \)) or growth (\( \alpha > 0 \)) of the motion due to the \( r \)-th vibrational mode; the imaginary part, \( \beta \), denotes the corresponding frequency of vibration.

The primary criteria in the design of vibrational control systems for large space structures are that the motion due to each of the structures vibrational modes be stable, and that the magnitude of all vibrations approach zero at some rate \( \alpha \), greater than some minimum rate \( \alpha \). Hence, we require

\[
\|q(t)\| \leq C_0 e^{-\alpha t}
\]  

(5.1.4)

where \( \alpha > 0 \), \( \|\| \) is some applicable vector norm, and \( C_0 = \|q(t_0)\| \).

The motion at any point on the structure is a linear combination of the motion due to each of the vibrational modes; therefore, assuming that the structure is stable, the exponential rate of decay of the motion at any point is bounded below by the rate of decay of the mode with the smallest damping factor \( \alpha \). One of the most notable characteristics of the uniform damping control law is that

\[
\alpha_r = \alpha
\]  

(5.1.5)

for each of the closed-loop eigenvalues \( \lambda = \alpha + j\beta \). The name uniform damping control derives from this characteristic of the closed-loop system - the motion due to each of the vibrational modes decays at the same uniform rate.

An additional important question in specifying the desired performance criteria for our system is whether there is any benefit in altering the shapes of the natural modes or the natural frequencies - velocity feedback alone changes both the real and imaginary
parts of the open-loop eigenvalues, thus changing the natural frequencies. Appreciable changes in either the shape of the natural modes or their frequency of vibration, however, typically requires large control forces; therefore, we should avoid attempting to alter them if we wish to minimize the expenditure of control effort [31].

5.2 The uniform damping control law

With the feedback control law (5.2.2), the modal coordinate model for the system, (5.2.1), becomes

\[ \ddot{\eta}(t) + \Omega \eta(t) = \Phi^T H \Phi \dot{\eta}(t) + \Phi^T G \Phi \eta(t) \]  

(5.2.1)

The left side of equation (5.2.1) represents a system of independent equations; therefore, it is convenient to choose the uniform damping control feedback gain matrices \( G \) and \( H \) such that

\[ \Phi^T G \Phi = \text{diag} \{ g_1, g_2, ..., g_n \}, \]
\[ \Phi^T H \Phi = \text{diag} \{ h_1, h_2, ..., h_n \}. \]  

(5.2.2)

When the feedback gain matrices satisfy the conditions in (5.2.2), the closed-loop system model in modal coordinates becomes a system of independent second-order differential equations of the form

\[ \ddot{\eta}_r(t) - h_r \dot{\eta}_r(t) + (\omega_r^2 - g_r)\eta_r(t) = 0, \quad r = 1, 2, ..., n. \]  

(5.2.3)

The closed-loop eigenvalues of the system are the zeros of the characteristic equations
\[ \lambda_r^2 - h_r \lambda_r + (\omega_r^2 - g_r) \lambda_r = 0, \quad r = 1, 2, \ldots, n. \]  
(5.2.4)

Hence, the closed-loop eigenvalues are

\[ \lambda_r = \frac{h_r}{2} \pm \frac{1}{2} \sqrt{h_r^2 - 4(\omega_r^2 - g_r)}, \quad r = 1, 2, \ldots, n. \]  
(5.2.5)

The desired closed-loop eigenvalue locations are given by \( \lambda_r = -\alpha \pm j\omega_r \), where \( \alpha > 0 \) and \( \omega_r \) is the \( r \)-th natural frequency. Equating the real and imaginary parts of \( \lambda_r \) and \( \lambda_r^* \)

\[
\begin{align*}
h_r &= -2\alpha \\
g_r &= -\alpha^2, \quad r = 1, 2, \ldots, n. 
\end{align*}
\]  
(5.2.6)

With the feedback gains \( g_r \) and \( h_r \) defined as in equation (5.2.6), the solutions to the differential equations for each closed-loop mode are

\[ \eta_r(t) = C_r e^{-\alpha t} e^{j\omega_r t}, \quad r = 1, 2, \ldots, n. \]  
(5.2.7)

Then

\[ |\eta_r(t)| = |C_r| e^{-\alpha t} \leq C e^{-\alpha t}, \quad r = 1, 2, \ldots, n, \]  
(5.2.8)

where

\[ C = \max_{1 \leq r \leq n} |C_r| \]  
(5.2.9)

Therefore, the choices \( g_r = -\alpha^2 \) and \( h_r = -2\alpha \) for the feedback gains in modal coordinates satisfy the design criteria of the previous section.

The combined control input in modal coordinates is given by

\[ \Phi^T u(t) = -\alpha^2 \eta(t) - 2\alpha \dot{\eta}(t). \]  
(5.2.10)
Multiplying both sides of equation (5.2.10) by the modal matrix $\Phi$, and substituting $q(t) = \Phi \eta(t)$ yields

$$u(t) = -\alpha^2(\Phi \Phi^T)^{-1} q(t) - 2\alpha(\Phi \Phi^T)^{-1} \dot{q}(t).$$  \hfill (5.2.11)

Recalling that the modal vectors are orthonormal with respect to the mass, or inertia, matrix $M$, the uniform damping control law in generalized coordinates is

$$u(t) = -\alpha^2 M q(t) - 2\alpha M \dot{q}(t).$$  \hfill (5.2.12)

The uniform damping control law (5.2.10), expressed in modal coordinates, is decentralized, but the corresponding control law expressed in generalized coordinates (5.2.12) is decentralized only if $M$ is a diagonal matrix. Unfortunately, the mass matrix generally is not diagonal; although, it is often diagonally dominant.

We can approximate the uniform damping control law (5.2.12) using

$$u(t) = -\alpha^2 \hat{M} q(t) - 2\alpha \hat{M} \dot{q}(t).$$  \hfill (5.2.13)

where $\hat{M}$ is a diagonal matrix, so that the approximate control law is decentralized. The elements of $\hat{M}$ are chosen to approximate the local mass of the structure in the region surrounding each corresponding sensor/actuator pair. When the mass matrix is diagonally dominant, $\hat{M}$ may be chosen as the main diagonal of $M$. Therefore, even when the uniform damping control law is not decentralized, we can use it to generate a set of feedback gains for a decentralized controller which should approximate its performance.
5.3 Uniform damping control vs rate feedback

We can illustrate the relationship between direct velocity feedback and uniform damping control by considering a system model with one vibrational mode and one collocated sensor/actuator pair:

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u(t) \\
\begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} &= \begin{bmatrix} \beta & 0 \\ 0 & \beta \end{bmatrix} x(t),
\end{align*}
\]

where \( q(t) \) and \( \dot{q}(t) \) are the measured position and velocity of the system, \( \omega \) is the natural frequency of the modeled vibrational mode, and \( \beta \) is the mode shape at the actuator/sensor location. The velocity feedback control law for the system is

\[
u(t) = -k \dot{q}(t),
\]

and the corresponding transfer function is

\[
\frac{Q_r(s)}{U(s)} = \frac{\beta^2 s}{s^2 + \omega^2},
\]

where \( Q_r(s) \) and \( U(s) \) represent the Laplace transforms of the measured velocity and control input respectively. The root locus of the closed loop poles of the system with rate feedback is shown in Figure 4.

The uniform damping control law for the system, in generalized coordinates, is

\[
u(t) = -\alpha^2 Mq(t) - 2\alpha M\dot{q}(t),
\]
Figure 4. Root locus of one mode system with rate feedback.
where in this case \( M \) is a scalar constant which represents the mass of the system. We can rewrite the uniform damping control law as

\[
u(t) = -M\begin{bmatrix} \alpha^2 & 2\alpha \\ \dot{q}(t) & \dot{q}(t) \end{bmatrix}.
\] (5.3.5)

Considering the form of the uniform damping control law, define a new system output

\[
z(t) = \begin{bmatrix} \alpha^2 & 2\alpha \\ \dot{q}(t) & \dot{q}(t) \end{bmatrix} \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}
= \beta\begin{bmatrix} \alpha^2 & 2\alpha \end{bmatrix} x(t).
\] (5.3.6)

The transfer function from the control input to the new output \( z(t) \) is

\[
\frac{Z(s)}{U(s)} = \frac{2\beta^2 \alpha(s + \frac{\alpha}{2})}{s^2 + \omega^2},
\] (5.3.7)

where \( Z(s) \) represents the Laplace transform of the output \( z(t) \). Figure 5 shows the root locus for the single mode system with the control law

\[
u(t) = -kz(t).
\] (5.3.8)

For the proper choice of the gain \( k \), (5.3.8) is equivalent to the uniform damping control law. From equations (5.2.12) and (5.3.6), the necessary gain for uniform damping control is

\[
k = \frac{1}{\beta^2}.
\] (5.3.9)
Figure 5. Root locus for one mode with uniform damping control.
which is a measure of the relative mass density of the structure in a region surrounding the sensor/actuator pair, and represents a scalar version of the mass matrix $M$. The closed-loop pole locations corresponding to this feedback gain are marked on the root locus plot in Figure 5.

A comparison of Figure 4 and Figure 5 illustrates the difference between pure rate feedback and uniform damping control. The position component of the uniform damping control acts to move the zero of the transfer function from the origin into the left half-plane; this allows the plant poles to be moved further into the left half-plane. The root loci show that for small $\alpha$, uniform damping control behaves much like pure rate feedback.

We can rewrite the transfer function equation (5.3.7) as

$$\frac{Z(s)}{U(s)} = 2a(s + \frac{\alpha}{2}) \frac{Q(s)}{U(s)},$$

(5.3.10)

where $Q(s)/U(s)$ is the plant transfer function from the control input to the measured position output. This shows that, in general, uniform damping control cascades a pure lead compensator with a zero at $-\alpha/2$ in a position feedback loop. Indexing the modal subsystems by $i$, the corresponding transfer function for an $n$-th order system model is given by
\[
\frac{Z(s)}{U(s)} = \sum_{i=1}^{n} \frac{2\beta_i^2 (s + \frac{\alpha}{2})}{s^2 + \omega_i^2} \\
= 2\alpha (s + \frac{\alpha}{2}) \sum_{i=1}^{n} \frac{\beta_i^2}{s^2 + \omega_i^2} \\
= 2\alpha (s + \frac{\alpha}{2}) \frac{Q(s)}{U(s)} .
\]

(5.3.11)

The transfer function corresponding to velocity feedback is

\[
\frac{Q_v(s)}{U(s)} = \frac{sQ(s)}{U(s)} .
\]

(5.3.12)

A second comparison of uniform damping control and rate feedback is possible from equations (5.3.11) and (5.3.12). Note that the phase of transfer function \(Q(s) / U(s)\) is always between \(0^\circ\) and \(-180^\circ\). Therefore, using velocity feedback, the phase of the transfer function in (5.3.12) is always between \(\pm 90^\circ\), the Nyquist plot is always in the right half plane and the system is stable. From equation (5.3.11), however, we see that uniform damping control guarantees stability only at frequencies above \(\alpha / 2\). Hence, we see that uniform damping control obtains increased performance at the price of decreased robustness at low frequencies [32].
5.4 Summary

Uniform damping control is an output feedback control law which employs both position and velocity feedback. The uniform damping control law is designed to provide the same amount of damping for each of the structure's vibrational modes; the motion due to each mode of the closed-loop system decays at the same exponential rate. To minimize the unnecessary expenditure of additional control energy, the closed-loop frequencies of oscillation and vibrational modes are identical to the frequencies and mode shapes.

There are, several disadvantages to uniform damping control, however. The uniform damping control law requires that the number of modes included in the mathematical model of the structure, and the number of sensors and actuators on the structure, all be equal. Another disadvantage of uniform damping control is it may not be decentralized. The feedback gain matrix in the uniform damping control law is proportional to the mass matrix for the system; hence, the control law is decentralized only if the mass matrix is diagonal. In some cases, such as when the mass matrix is diagonally dominant, we can approximate the uniform damping control law by replacing the mass matrix for the system with a diagonal matrix with diagonal elements which approximate the structural mass in the vicinity of the sensors and actuators.
6.0 Overlapping decompositions and control

Decomposition plays an important role in the analysis and control of large scale systems. It is often beneficial, if not absolutely necessary, to decompose a given large scale system into a number of interconnected subsystems. For the purpose of analysis or control design, each subsystem is considered independently; the solutions to the problem for the individual subsystems are combined to obtain a solution for the composite system. Decompositions reduce the computational burden associated with large scale system problems by reducing the task of solving a problem for a large scale system to solving the problem for a number of smaller systems.

Many large scale systems arise naturally from the interconnection of individual subsystems, in which case decomposition of the system not only reduces the computational burden of solving a large scale system problem, but may also provide valuable insight into the effect of the subsystem interconnections on the behavior of the combined system. The notion of connective stability relates changes in the subsystem interconnections to stability of the overall system [33].

Decentralized control is a natural extension of the concept of system decomposition to control system design. After a decomposing a system into...
interconnected subsystems, local control laws are designed for each subsystems. The control laws for the subsystems are implemented locally as if the subsystems were disjoint. The collection of local feedback control laws comprise a decentralized controller for the composite system.

In general, little attention, if any, is given to the effect of subsystem interconnections in the development of decentralized control laws. The ability of a decentralized controller to stabilize the system or achieve a certain closed-loop system behavior, however, may depend upon the subsystem interconnections ignored in the design process.

Decompositions in which the subsystem states represent a disjoint partition of the complete system state are known as disjoint decompositions. This type of decomposition is most useful when the subsystems are weakly connected, that is, when there is little or no exchange of information between subsystems. Many large scale systems, however, often contain subsystems that are strongly connected. Strong subsystem interconnections may arise due to many different factors, including the sharing of certain system dynamics between subsystems. Subsystems that are interconnected through shared subsystem dynamics are said to overlap, and are referred to as overlapping subsystems.

Typically, when a system contains strongly connected subsystems, disjoint decompositions fail to produce useful results. When the strong subsystem interconnections arise because of overlapping subsystems, another type of decomposition, known as overlapping decomposition, may prove more useful. In an overlapping decomposition, the state space - and possibly the input and output spaces - of the dynamic system are "expanded", via singular transformations, into corresponding state (and input or output) spaces of higher dimension such that the overlapping subsystems appear disjoint [34,35]. The portion of the state vector representing the
system dynamics shared by the strongly connected subsystems is duplicated in each of the expanded subsystems, so that the sum of the subsystem dimensions in the expanded system is greater than the dimension of the original system [36].

The general mathematical framework for overlapping decompositions and control is the Inclusion Principle [37]. Decentralized control laws are designed in the expanded space, using standard techniques for systems with disjoint subsystems. The control laws designed for the expanded system are then contracted, or transformed, for implementation on the original system [38,34,35]. The design of optimal control laws using expansions and contractions also requires inclusion of the cost functions [39,38], and although the contracted optimal control law may not be optimal for the original system we are able to calculate a measure of suboptimality in some cases.

6.1 Overlapping decomposition

Consider the following linear time-invariant system:

\[ S: \dot{x}(t) = Ax(t) + Bu(t) \]  

(6.1.1)

where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^n \) are the system state and input respectively. Suppose the state and input vectors are partitioned as

\[
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix}
\]

(6.1.2)

and
\[ u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \]  

where \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \). Suppose also, that the matrices \( A \) and \( B \) in (6.1.1) are partitioned in a manner compatible the state and input vectors, (6.1.2) and (6.1.3), and that they have the following forms:

\[
A = \begin{bmatrix}
A_{11} & A_{12} & 0 \\
A_{21} & A_{22} & A_{23} \\
0 & A_{32} & A_{33}
\end{bmatrix}
\]  

(6.1.4)

and

\[
B = \begin{bmatrix}
B_{11} & 0 \\
B_{21} & B_{22} \\
0 & B_{32}
\end{bmatrix}
\]  

(6.1.5)

Consider the state transformation

\[
\tilde{x} = T x = \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
\]  

(6.1.6)

where \( \tilde{x} \in \mathbb{R}^\tilde{n} \), \( \tilde{n} = n + n_2 \), and let \( T^\dagger \) denote an inverse transformation such that

\[
T^\dagger \tilde{x} = x, \quad T^\dagger T = I_n
\]

(6.1.7)
where \( I_n \) is the identity matrix in \( \mathbb{R}^n \). The transformation (6.1.6) is essentially an expansion of the system’s state space. Applying the transformation (6.1.6), to the system \( S \) in (6.1.1) yields the system

\[
\tilde{S}: \quad \dot{x}(t) = \tilde{A} \tilde{x}(t) + \tilde{B}u(t)
\]

where

\[
\tilde{A} = \begin{bmatrix}
A_{11} & A_{12} & 0 & 0 \\
A_{21} & A_{22} & 0 & A_{23} \\
A_{21} & 0 & A_{22} & A_{23} \\
0 & 0 & A_{32} & A_{33}
\end{bmatrix}
\]

(6.1.9)

and

\[
\tilde{B} = \begin{bmatrix}
B_{11} & 0 \\
B_{21} & B_{22} \\
B_{21} & B_{22} \\
0 & B_{32}
\end{bmatrix}.
\]

(6.1.10)

We say that the system \( \tilde{S} \) is an expansion of the system \( S \).

We can represent the expansion \( \tilde{S} \) of \( S \) as two interconnected subsystems

\[
\tilde{S}: \quad \begin{align*}
\dot{x}_1(t) &= \tilde{A}_1 \tilde{x}_1(t) + \tilde{B}_1 u_1(t) + \tilde{A}_{12} \tilde{x}_2(t) + \tilde{B}_{12} u_2(t) \\
\dot{x}_2(t) &= \tilde{A}_2 \tilde{x}_2(t) + \tilde{B}_2 u_2(t) + \tilde{A}_{21} \tilde{x}_1(t) + \tilde{B}_{21} u_1(t)
\end{align*}
\]

(6.1.11)

where
\[ \tilde{A}_1 = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} \]
\[ \tilde{A}_2 = \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{33} \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} B_{22} \\ B_{12} \end{bmatrix} \]

(6.1.12)

are the matrices associated with the decoupled subsystems

\[ \tilde{S}_1: \dot{\tilde{x}}_1(t) = \tilde{A}_1\tilde{x}_1(t) + \tilde{B}_1u_1(t) \]
\[ \tilde{S}_2: \dot{\tilde{x}}_2(t) = \tilde{A}_2\tilde{x}_2(t) + \tilde{B}_2u_2(t) \]

(6.1.13)

and

\[ \tilde{A}_{12} = \begin{bmatrix} 0 & 0 \\ 0 & A_{22} \end{bmatrix}, \quad \tilde{B}_1 = \begin{bmatrix} 0 \\ B_{22} \end{bmatrix} \]
\[ \tilde{A}_{21} = \begin{bmatrix} A_{21} & 0 \\ 0 & 0 \end{bmatrix}, \quad \tilde{B}_2 = \begin{bmatrix} B_{21} \\ 0 \end{bmatrix} \]

(6.1.14)

are interconnection matrices. We say that the decoupled subsystems in (6.1.13) represent an overlapping decomposition of the system \( S \).

The overlapping decomposition (6.1.13) separates the dynamics of the overlapping subsystems and can be used to design overlapping control laws for (6.1.1). Given the decoupled subsystems (6.1.13) we can design a decentralized control law

\[ \tilde{u}(t) = \begin{bmatrix} \tilde{K}_1 & 0 \\ 0 & \tilde{K}_2 \end{bmatrix} \dot{\tilde{x}}(t) + \tilde{v}(t) \]

(6.1.15)

for the expanded system \( \tilde{S} \). The control law for the (6.1.15) is then contracted to a control law for the original system \( S \) via the inverse transformation.
\[ x = T' \tilde{x}. \] (6.1.16)

Necessary and sufficient conditions for the application of overlapping decompositions and control design are posed in the framework of the Inclusion Principle. When the expansion \( \tilde{S} \) of \( S \) is chosen properly, stability of the closed-loop system in the expanded space guarantees stability of the original system subject to the contracted control law.

### 6.2 The Inclusion Principle

Consider two linear time-invariant systems

\[
\begin{align*}
\dot{x} &= Ax + Bu \\
S: \quad y &= Cx
\end{align*}
\] (6.2.1)

and

\[
\begin{align*}
\dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}\tilde{u} \\
\tilde{S}: \quad \tilde{y} &= \tilde{C}\tilde{x}
\end{align*}
\] (6.2.2)

where \( x \in \mathbb{R}^n, \tilde{x} \in \mathbb{R}^{\tilde{n}}, u \in \mathbb{R}^m, \tilde{u} \in \mathbb{R}^{\tilde{m}}, y \in \mathbb{R}^r, \) and \( \tilde{y} \in \mathbb{R}^{\tilde{r}} \) are the states, inputs and measurable outputs of the systems \( S \), and \( \tilde{S} \) respectively. We denote the state and corresponding output of the system \( S \) at time \( t \), for a fixed input \( u(t) \) and the initial state \( x(t_0) \), by \( x(t; x_0, u) \) and \( y[x(t)] \) respectively. Similarly, \( \tilde{x}(t; \tilde{x}_0, \tilde{u}) \) and \( \tilde{y}[\tilde{x}(t)] \) denote the state and output of the system \( \tilde{S} \).
We shall assume that the dimensions of the state, input and output spaces of the system $S$ are less than or equal to the dimensions of the corresponding state, input, and output spaces of the system $\tilde{S}$; that is,

$$\tilde{n} \geq n, \quad \tilde{m} \geq m, \quad \tilde{r} \geq r. \quad (6.2.3)$$

We shall also assume that the state, input, and output spaces of the systems $S$ and $\tilde{S}$ are related by the following linear, possibly nonsingular, transformations

$$\tilde{x} = Vx, \quad x = V^T\tilde{x}, \quad V^TV = I_n, \quad (6.2.4a)$$

$$\tilde{u} = Uu, \quad u = U^T\tilde{u}, \quad U^TU = I_m, \quad (6.2.4b)$$

$$\tilde{y} = Ty, \quad y = T^T\tilde{y}, \quad T^TT = I_r, \quad (6.2.4c)$$

where $V$, $U$, and $T$ are constant matrices with appropriate dimension and full column rank, $V^T$, $U^T$, and $T^T$ are constant matrices with appropriate dimension and full row rank, and $I_r$ is the identity matrix in $R^r$.

With these transformation definitions, the Inclusion Principle has the following formulation:

**Definition 6.2.1 [34,35]:** The system $\tilde{S}$ *includes* the system $S$, or equivalently, $S$ is included by $\tilde{S}$, if there exist transformation pairs $(V',V)$ and $(U',U)$ such that, for any initial state $x_0$ and any fixed input $u(t)$ of $S$, the choice

$$\tilde{x}_0 = Vx_0$$

$$\tilde{u}(t) = Uu(t) \quad \forall \ t \geq 0 \quad (6.2.5)$$

of the initial state $\tilde{x}_0$ and input $\tilde{u}(t)$ for $\tilde{S}$, implies
\[ x(t; x_0, u) = \nu^t \bar{x}(t; \bar{x}_0, \bar{u}) \]
\[ y[x(t)] = T^i \bar{y}[\bar{x}(t)] \quad \forall \ t \geq 0. \tag{6.2.6} \]

The condition (6.2.6) in the definition of inclusion implies that the system \( \tilde{S} \) contains all of the necessary information about the behavior of the system \( S \). For example, if the system \( \tilde{S} \) includes the system \( S \), then since the transformation matrices are constant, (6.2.6) implies that the system \( S \) is stable if and only if the system \( \tilde{S} \) is stable. This is the underlying idea behind the Inclusion Principle [36,38].

If the system \( \tilde{S} \) includes the system \( S \), we say that \( \tilde{S} \) is an expansion of the system \( S \), or similarly, that \( S \) is a contraction of \( \tilde{S} \). This terminology reflects our assumption that the dimensions of the state, input, and output spaces of the system \( \tilde{S} \) are greater than or equal to the dimensions of the corresponding state, input, and output spaces of the system \( S \).

The following theorem provides necessary and sufficient conditions for inclusion of the system \( S \) by the system \( \tilde{S} \).

**Theorem 6.2.1 [34,35]:** The system \( \tilde{S} \) includes the system \( S \) if and only if there exist linear transformations represented by the matrices \( V^t, V, U, \) and \( T^t \), such that

\[ A^t = V^t \tilde{A}^t V, \quad A^t B = V^t \tilde{A}^t \tilde{B} U \]
\[ C A^t = T^t \tilde{C} A^t V, \quad C A^t B = T^t \tilde{C} A^t \tilde{B} U, \quad \forall \ i = 1, 2, ... \tag{6.2.7} \]
We are often interested in generating an expansion $\tilde{S}$ of a given system $S$. Given the linear transformations (6.2.4), between the state, input and output spaces of $S$ and $\tilde{S}$, the matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ of the system $\tilde{S}$ can be expressed as

$$\tilde{A} = VAV^T + M, \quad \tilde{B} = VBU^T + N, \quad \tilde{C} = TCV^T + L$$  \hfill (6.2.8)

where $M$, $N$, and $L$ are complementary matrices of appropriate dimensions. The matrices $M$, $N$, and $L$ must be chosen properly for $\tilde{S}$ to be an expansion of $S$, as the following theorem illustrates.

**Theorem 6.2.2 [34,35]:** The system $\tilde{S}$ is an expansion of the system $S$, if and only if

$$V^TM^TV = 0, \quad V^TM^{i-1}NU = 0$$

$$T^LM^{i-1}V = 0, \quad T^LM^{i-1}NU = 0, \quad \forall \ i = 1, 2, \ldots, \tilde{n}.$$  \hfill (6.2.9)

There are several special cases of expansion and contraction which occur often in applications. Restrictions and aggregations are two such types of contractions. A special type of expansion that is very useful for the design of control laws using overlapping decompositions, is an extension.

Restrictions are so named because the initial state and control input for the expanded system $\tilde{S}$ are restricted to subsets of the expanded state and input spaces spanned by the columns of the transformation matrices $V$ and $U$.

**Definition 6.2.2 [34,35]:** The system $S$ is a restriction of $\tilde{S}$, or equivalently, $\tilde{S}$ is an unrestricted of $S$, if there exist transformation matrices $V$, $U$, and $T$ such that, for any initial state $x_0$ and any fixed input $u(t)$ of $S$, the choice
\[ \tilde{x}_0 = Vx_0 \]
\[ \tilde{u}(t) = Uu(t), \quad \forall \ t \geq 0 \] (6.2.10)

of the initial state and control input for the expanded system \( \tilde{S} \), implies

\[ \tilde{x}(t; \tilde{x}_0, \tilde{u}) = Vx(t; x_0, u) \]
\[ \tilde{y}[\tilde{x}(t)] = Ty[x(t)], \quad \forall \ t \geq 0. \] (6.2.11)

\[ \square \]

Aggregation, another special type of contraction, is the dual of restriction.

Definition 6.2.3 [34,35]: The system \( S \) is an aggregation of \( \tilde{S} \), or equivalently, \( \tilde{S} \) is a disaggregation of \( S \), if there exist transformation matrices \( V' \), \( U' \), and \( T' \) such that, for any initial state \( \tilde{x}_0 \) and any fixed input \( \tilde{u}(t) \) of \( \tilde{S} \), the choice

\[ x_0 = V'\tilde{x}_0 \]
\[ u(t) = U'\tilde{u}(t), \quad \forall \ t \geq 0 \] (6.2.12)

of the initial state and input for the system \( S \), implies

\[ x(t; x_0, u) = V'x(t; \tilde{x}_0, \tilde{u}) \]
\[ y[x(t)] = T'y[\tilde{x}(t)], \quad \forall \ t \geq 0. \] (6.2.13)

\[ \square \]

Extensions are a special type expansion defined for an arbitrary input \( \tilde{u} \) in the input space of the expanded system \( \tilde{S} \); the input for the original space is obtained from the contraction \( u(t) = U'\tilde{u}(t) \).
Definition 6.2.4 [39]: The system \( \tilde{S} \) is an \textit{extension} of the system \( S \), and \( S \) is a \textit{disextension} of \( \tilde{S} \), if there exist transformation matrices \( V \) and \( U^t \) such that for any initial state \( x_0 \) of \( S \) and fixed input \( \tilde{u}(t) \) of \( \tilde{S} \), the choice

\[
\tilde{x}_0 = Vx_0 \\
u(t) = U^t\tilde{u}(t), \quad \forall \ t \geq 0
\]

(6.2.14)

for the initial state of \( \tilde{S} \) and input of \( S \), implies

\[
\tilde{x}(t; \tilde{x}_0, \tilde{u}) = Vx(t; x_0, u), \quad \forall \ t \geq 0.
\]

(6.2.15)

Although they may appear similar, there are important differences between extensions and unrestrictions [34,35]. In Definition 6.2.2, the unrestriction is defined for an arbitrary input \( u(t) \) in the original input space, and the input in the expanded space is obtained from the transform \( \tilde{u}(t) = Uu(t) \). If \( \tilde{S} \) is an unrestriction of \( S \), the inputs \( \tilde{u}(t) \) are contained in an \( m \)-dimensional subspace of \( \mathbb{R}^\tilde{n} \) spanned by the column vectors of the transformation matrix \( U \). Any control law designed in the expanded space must remain in this \( m \)-dimensional subspace of the expanded input space. To see that this is true, suppose that \( \tilde{u} \) that is not an element of the subspace spanned by the columns of the transformation matrix \( U \), then there exists no control input \( u \in \mathbb{R}^n \) for the system \( S \) such that \( \tilde{u} = Uu \).

Extensions, are defined for arbitrary inputs \( \tilde{u}(t) \) in the expanded input space, and the input for the system \( S \) is obtained from the contraction \( u(t) = U^t\tilde{u}(t) \). The set of allowable inputs for the expanded system is all of \( \mathbb{R}^\tilde{n} \); therefore, if \( \tilde{S} \) is an extension of
$S$, any feedback control law designed in the expanded space $\tilde{S}$ can be contracted to a control law for the original space $S$.

As with the definition of inclusion, there exist theorems which provide necessary and sufficient conditions for contractions and expansions to be restrictions, aggregations and extensions. The following theorem gives necessary and sufficient conditions for the system $S$ to be a restriction of the system $\tilde{S}$.

**Theorem 6.2.3 [34,35]:** The system $S$ is a restriction of $\tilde{S}$, if and only if there exists transformation matrices $V$, $U$, and $T$ such that

$$\tilde{A}V = VA, \quad \tilde{B}U = VB, \quad \tilde{C}V = TC. \quad (6.2.16)$$

Restriction, like inclusion, also has a characterization in terms of the complementary matrices $M$, $N$, and $L$ in (6.2.8).

**Theorem 6.2.4 [34,35]:** The system $S$ is a restriction of the system $\tilde{S}$, if and only if

$$MV = 0, \quad NU = 0, \quad LV = 0 \quad (6.2.17)$$

The dual version of Theorem 6.2.3 provides necessary and sufficient conditions for the system $S$ to be a aggregation of the system $\tilde{S}$.

**Theorem 6.2.5 [34,35]:** The system $S$ is an aggregation of $\tilde{S}$, if and only if there exists transformation matrices $V^t$, $U^t$, and $T^t$ such that
\[ V^t\tilde{A} = AV^t , \quad V^t\tilde{B} = BU^t , \quad T^t\tilde{C} = CV^t . \] (6.2.18)

\[ \square \]

In terms of the complementary matrices \( M \), \( N \), and \( L \), aggregation has the following characterization:

**Theorem 6.2.6 [34,35]:** The system \( S \) is an aggregation of the system \( \tilde{S} \), if and only if

\[ V^tM = 0 , \quad V^tN = 0 , \quad T^tL = 0 . \] (6.2.19)

\[ \square \]

As with restriction and aggregation, there are certain necessary and sufficient conditions for one system to be an extension of another.

**Theorem 6.2.7 [39]:** The system \( \tilde{S} \) is an extension of the system \( S \) if and only if there exist transformation matrices \( V \) and \( U^t \) such that

\[ VA = \tilde{A}V , \quad VBU^t = \tilde{B} \] (6.2.20)

\[ \square \]

The following theorem provides necessary and sufficient conditions in terms of the complementary matrices \( M \) and \( N \) such that \( \tilde{S} \) is an extension of the system \( S \).

**Theorem 6.2.8 [39]:** The system \( \tilde{S} \) is an extension of the system \( S \) if and only if
\[ MV = 0 \quad , \quad N = 0. \quad \quad (6.2.21) \]

### 6.3 Control law contractability

One of the primary applications of the inclusion principle is the design of decentralized control laws for systems with overlapping subsystems. Given a system \( S \) with overlapping subsystems, and a corresponding expanded system \( \tilde{S} \), we design a decentralized control law for the system \( \tilde{S} \), then contract the control law for implementation on the original system \( S \).

Consider the linear state feedback control laws

\[ u(t) = Kx(t) + v(t) \quad \quad (6.3.1) \]

for the system \( S \), and

\[ \tilde{u}(t) = \tilde{K}\tilde{x}(t) + \tilde{v}(t) \quad \quad (6.3.2) \]

for the system \( \tilde{S} \), where the matrices \( K \) and \( \tilde{K} \) are constant and have appropriate dimension, and \( v \in R^m \), \( \tilde{v} \in R^{\tilde{m}} \) are reference inputs for the closed-loop systems. One of the key questions in the control law design, however, is whether the control law (6.3.2) can be contracted to a realizable control law (6.3.1) for the original system \( S \).
**Definition 6.3.1** [39]: The control law (6.3.2) is contractable to the control law (6.3.1), if there exist transformation matrices $V$ and $U'$, as in (6.2.4), such that

$$Kx(t; x_0, U'\tilde{u}) = U'\tilde{K}x(t; Vx_0, \tilde{u}), \quad \forall \ i \geq 0$$  \hspace{1cm} (6.3.3)

for any initial state $x_0 \in \mathbb{R}^n$ and any fixed input $\tilde{u}(t) \in \mathbb{R}^m$, $0 \leq t < \infty$.

The control law (6.3.2) for $\tilde{S}$ is contractable to the control law (6.3.1) for the system $S$ if the conditions

$$\begin{align*}
\tilde{x}_0 &= Vx_0 \\
\tilde{u}(t) &= Uu(t), \quad \forall \ t \geq 0
\end{align*} \hspace{1cm} (6.3.4)$$

imply

$$\tilde{K}x(t; \tilde{x}_0, \tilde{u}) = UKx(t; x_0, u), \quad \forall \ t \geq 0$$  \hspace{1cm} (6.3.5)

for any initial state $x_0$ and any fixed input $u(t)$ of $S$. Contractability of the control law (6.3.2) to the control law (6.3.1) implies that the closed loop system

$${\begin{align*} \dot{x} &= (A + BK)x + Bu \\
S_c: \quad y &= Cx \end{align*}} \hspace{1cm} (6.3.6)$$

is a contraction of the closed loop system

$$\begin{align*}
\tilde{S}_c: \quad \dot{\tilde{x}} &= (\tilde{A} + \tilde{BK})\tilde{x} + \tilde{B}\tilde{u} \\
\tilde{y} &= \tilde{C}\tilde{x}
\end{align*} \hspace{1cm} (6.3.7)$$

The feedback gain matrices in (6.3.6) and (6.3.7) are related by the transformation

$$\tilde{K} = UKV' + F$$  \hspace{1cm} (6.3.8)
where $F$ is a complementary matrix which must satisfy certain necessary and sufficient conditions for contractability.

**Theorem 6.3.1 [34,35]:** The control law (6.3.2) for $\tilde{S}$ is contractable to the control law (6.3.1) for $S$ if and only if

$$ U K A^T = \tilde{K} \tilde{A}^T V, $$
$$ U K A^T B = \tilde{K} \tilde{A}^T \tilde{B} U, \quad \forall \ i = 0, 1, 2, ... \quad (6.3.9) $$

or, equivalently,

$$ F M^{i-1} V = 0 $$
$$ F M^{i-1} N U = 0, \quad \forall \ i = 0, 1, 2, ... \quad (6.3.10) $$

The following corollary follows directly from Theorem 6.3.1.

**Corollary 6.3.2 [34,35]:** If the control law (6.3.2), designed in the expanded space $\tilde{S}$, is contractable to a control law of the form (6.3.1), for the system $S$, then the contracted gain matrix $K$ is given by

$$ K = U^T \tilde{K} V. \quad (6.3.11) $$

Corollary 6.3.2 implies that if the control law in the expanded space is contractable, the only possible candidate for contraction is the matrix $K$ in (6.3.11). Substituting (6.3.11) into (6.3.8), the complementary matrix $F$, must be given by
\[ F = \tilde{K} - UU^T \tilde{K} V V^T, \] 

if the control law is contractable. Hence, to determine the contractability of the control law \((6.3.2)\), we only need to apply Theorem 6.3.1 to the complementary matrix \(F\) in \((6.3.12)\).

The conditions for control law contractability are greatly simplified when \(\tilde{S}\) is an extension of \(S\).

**Corollary 6.3.3 [39]:** If \(\tilde{S}\) is an extension of \(S\), then any control law of the form \((6.3.2)\) is contractable to a control law of the form \((6.3.1)\), and the contracted gain matrix \(K\) is given by \((6.3.11)\).

\[
\begin{aligned}
\text{By Corollary 6.3.3, control law contractability is guaranteed when } \tilde{S} \text{ is an extension of } S.
\end{aligned}
\]

\section{6.4 Overlapping optimal control}

The design of optimal control laws using expansions and contractions requires not only state, input, and output inclusion, but also inclusion of the cost functions \([39,38]\). Once again, we wish to design the control law, this time an optimal control law, for an expanded system, and contract the control law for implementation on the original system. If \(\tilde{S}\) is an extension of a system \(S\), then by Corollary 6.3.3, any feedback control law of the form \((6.2.2)\) is contractable to a feedback control law of the form \((6.2.1)\), and the contracted feedback gain matrix is given by \((6.4.11)\). The contraction of a control
law that is optimal for the expanded system $\tilde{S}$, however, may not be optimal for the system $S$; although, in some cases, we can calculate a measure of suboptimality for the contracted control law.

Consider the cost functions

$$J = \int_0^{\infty} (x^TQx + u^TRu)dt$$  \hspace{1cm} (6.4.1)$$

and

$$\tilde{J} = \int_0^{\infty} (\tilde{x}^T\tilde{Q}\tilde{x} + \tilde{u}^T\tilde{R}\tilde{u})dt$$  \hspace{1cm} (6.4.2)$$

for the systems $S$ and $\tilde{S}$ respectively. We can extend the original definition of cost function inclusion [38], that is, for the case of state inclusion only, to state and input inclusion.

**Definition 6.4.1** [39]: The pair $(\tilde{S}, \tilde{J})$ includes the pair $(S,J)$ if there exists a transformation (6.2.4), such that $\tilde{S}$ includes $S$, and

$$J(x_0, u(t)) = \tilde{J}(\tilde{V}x_0, \tilde{u})$$  \hspace{1cm} (6.4.3)$$

for any initial state $x_0 \in \mathbb{R}^n$ and any fixed input $\tilde{u} \in \mathbb{R}^{n\tilde{n}}$, and $0 \leq t < \infty$. 

From the condition (6.4.3) in Definition 6.4.1, cost function inclusion requires
where \( \tilde{R} \) is the control weighting matrix in (6.4.2). For a symmetric \( \tilde{R} \), (6.4.4) implies \( \tilde{R} \) is nonpositive definite when \( \tilde{m} > m \). The control weighting matrix in the quadratic cost functional (6.4.2), however, is required to be positive definite; therefore, a direct extension of the concept of state, input, and output inclusion to quadratic cost functions as in Definition 6.4.1 is unacceptable. We can overcome this difficulty by introducing a new concept of inclusion - inclusion with respect to the optimal control.

Definition 6.4.2 [39]: Suppose that there exists a bounded control of the form (6.2.2) such that the cost function \( \tilde{J}(\tilde{x}, \tilde{u}) \) is minimized for all \( \tilde{x}_0 \in \tilde{R}^c \); then, the pair \( (\tilde{S}, \tilde{J}) \) includes the pair \( (S, J) \) with respect to the optimal, if \( \tilde{S} \) includes \( S \) and

\[
\min \tilde{J}(x_0, u) = \min \tilde{J}(Vx_0, \tilde{u})
\]  \hspace{1cm} (6.4.5)

for any initial state \( x_0 \in R^n \).

Under the transformations (6.2.4) the cost weighting matrices in (6.4.1) and (6.4.2) are related by

\[
\tilde{Q} = (V')^TQV' + M_Q \hspace{1cm} (6.4.6)
\]

and

\[
\tilde{R} = (U')^TRU' + N_R \hspace{1cm} (6.4.7)
\]

where \( M_Q \) and \( N_R \) are constant complementary matrices. If the system \( \tilde{S} \) is an extension of the system \( S \), then there exist sufficient conditions in terms of the complementary
matrices $M_\phi$ and $N_\kappa$, such that the pair $(\tilde{S}, \tilde{J})$ includes the pair $(S, J)$ with respect to the optimal control.

**Theorem 6.4.1 [39]:** If $\tilde{S}$ is an extension of $S$ and there exists a bounded control of the form (6.2.2) such that the cost function $\tilde{J}(\tilde{x}_0, \tilde{u})$ is minimized for all $\tilde{x}_0 \in \tilde{\mathcal{R}}$, then the pair $(\tilde{S}, \tilde{J})$ includes the pair $(S, J)$ with respect to the optimal if

$$V^TM_\phi V = 0 \quad (6.4.8)$$

and

$$U^TN_\kappa U = 0. \quad (6.4.9)$$

Given positive definite and positive semi-definite weighting matrices $R$ and $Q$ for the system $S$, it is always possible to construct symmetric complementary matrices $M_\phi$ and $N_\kappa$ satisfying Theorem 6.4.1 such that the weighting matrices $\tilde{R}$ and $\tilde{Q}$ for the system $\tilde{S}$ are also positive definite and positive semi-definite respectively. An alternate approach would be to begin by specifying the weighting matrices for the expanded system, $\tilde{Q}$ and $\tilde{R}$, to insure that they are positive semi-definite and positive definite respectively, then construct the corresponding weighting matrices $Q$ and $R$ for the system $S$ using the transformations

$$Q = V^T\tilde{Q}V \quad (6.4.10)$$

and

$$R = Q^T\tilde{R}Q. \quad (6.4.11)$$
We can calculate the optimal solutions to minimization problems for expanded systems and then calculate the cost function for the original system that is minimized by the contracted optimal control law.

6.5 Summary

In this chapter, we have introduced the concepts of overlapping decomposition and overlapping control. These techniques are most applicable to systems composed of strongly connected subsystems. When strong subsystem interconnections arise from overlapping subsystems, subsystems that share certain system dynamics, overlapping decomposition and control techniques often yield better results than standard disjoint decomposition techniques.

In an overlapping decomposition, the state space - and possibly the input and output spaces - of the system are "expanded", via linear, and possibly singular, transformations into corresponding state (and input or output) spaces of higher dimension such that the overlapping subsystems appear disjoint. The portions of the state vector representing the system dynamics shared by strongly connected subsystems are duplicated in the subsystems of the expanded system; hence, the sum of the dimensions of the expanded subsystems is greater than the dimension of the original system. We design control laws for the expanded system using standard disjoint decentralized control design techniques. The control laws for the expanded system are then "contracted", via an inverse transformation to suitable control laws for the original system.
The mathematical framework for overlapping decomposition and control design is the Inclusion Principle. When the expanded system includes the original system, it contains the necessary information about the original system to conclude such things as stability of the original system from stability of the expanded system.

There are several special types of expansions and contractions that occur frequently in applications. Restrictions and aggregations are examples of contractions. Extensions and unrestricteds are examples of special types of expansions. The difference between restrictions and aggregations depends on where the initial state and control input are defined; for aggregations, the initial state and input are defined in the expanded spaces, whereas the initial state and input in restrictions are defined in the original spaces. The difference between unrestricteds and extensions is more subtle; the initial states for both expansions are defined in the original state space, but the control inputs in the expanded input space may differ. In an unrestricted, the control input in the expanded space is confined to a subspace of the expanded input space with dimension equal to the dimension of the original input space. Extensions, however, are defined so that the control input for the expanded system can take on any value in the whole expanded input space.

An important question in overlapping decomposition and control design is the question of control law contractability - when can control laws designed for the expanded system be contracted to control laws for the original system? When the expanded system is an extension of the original system, control laws designed in the expanded is contractable to a control law for the original system. When we wish to design optimal control laws using overlapping decomposition and control techniques, special care must be taken to insure that the contracted control law is still optimal for the original system.
7.0 Decentralized control example

In this chapter we present an example of decentralized control design for large space structures. Several of the techniques described in the preceding chapters are employed in the design of a decentralized control system to enhance structural damping in a proposed large space structure. The structure examined in this example is known as the COFS mast, and has been proposed as part of the NASA Control Of Flexible Structures program.

Using the mathematical model for the COFS mast, we demonstrate the design of decentralized control laws using direct velocity feedback, uniform damping control, and overlapping decompositions. Using structural damping in the closed loop system as a measure of performance, we compare the centralized uniform damping control law with two control laws designed to approximate uniform damping control - one completely decentralized, and another that is decentralized except for coupled feedback between the sensors and actuators at the tip. We also examine the application of overlapping decompositions to decentralized control design for this structure; we show that the success of decomposition techniques depends on the coordinate system in which the structural model is represented.
7.1 The COFS Mast

The COFS mast is an extendable truss with a triangular cross section. The mast is designed to collapse into a canister which can be carried into orbit in the space shuttle's cargo bay. Once in orbit, the mast can be extended through the open shuttle bay doors and used as a testbed for a variety of control and identification experiments.

The extended mast is 60 meters long, and has a one meter triangular cross section. Figure 6 and Figure 7 show side and top views, respectively, of the space shuttle and the extended mast. The orientation of the coordinate system employed in Figure 6 and Figure 7 is used throughout this example. The $x$-axis is parallel to the shuttle body with the positive direction pointing toward the shuttle's tail section; the $z$-axis runs parallel to the extended mast with the positive direction away from the shuttle bay, and the orientation of the $y$-axis follows the right-hand rule with the positive direction away from the shuttle.

Sensors, actuators, and other instrumentation may be mounted on specially designed bays located between each of the mast's collapsible sections. We have numbered the bays 1 through 54 as shown in Figure 6.

There are ten sensors, capable of measuring position and velocity, and ten force actuators attached to the mast. The sensors and actuators are collocated; there are four sensor/actuator pairs located at the tip of the mast and two pairs at each of the bays 46, 28, and 10. The sensors and actuators are mounted on the structure so that each pair senses motion and applies force only in one plane of motion - either the $x$-$z$ plane or the $y$-$z$ plane. We have numbered the actuators from one to ten starting at the tip; odd numbered actuator/sensor pairs are aligned parallel to the $x$-$z$ plane and even numbered pairs lie in the $y$-$z$ plane. Table 1 lists the location and orientation of each
Figure 6. COFS mast extended from space shuttle, side view
Figure 7. COFS mast extended from space shuttle, top view
Table 1. Actuator numbers, locations, and orientations.

<table>
<thead>
<tr>
<th>Number</th>
<th>Location</th>
<th>Direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>tip</td>
<td>x</td>
</tr>
<tr>
<td>2</td>
<td>tip</td>
<td>y</td>
</tr>
<tr>
<td>3</td>
<td>tip</td>
<td>x</td>
</tr>
<tr>
<td>4</td>
<td>up</td>
<td>y</td>
</tr>
<tr>
<td>5</td>
<td>Bay 46</td>
<td>x</td>
</tr>
<tr>
<td>6</td>
<td>Bay 46</td>
<td>y</td>
</tr>
<tr>
<td>7</td>
<td>Bay 28</td>
<td>x</td>
</tr>
<tr>
<td>8</td>
<td>Bay 28</td>
<td>y</td>
</tr>
<tr>
<td>9</td>
<td>Bay 10</td>
<td>x</td>
</tr>
<tr>
<td>10</td>
<td>Bay 10</td>
<td>y</td>
</tr>
</tbody>
</table>
sensor/actuator pair on the mast: the $x$ direction denotes sensing and actuation in the $x$-$z$ plane and the $y$ direction denotes sensing and actuation in the $y$-$z$ plane. The sensor/actuator pairs located at bays 10, 28, and 46 are mounted perpendicular to one another in the center of the bay; the tip sensor/actuator pairs are mounted in a square configuration as shown in Figure 8. This configuration permits sensing and actuation of torsional motion in the structure.

7.2 The mast model

The numerical data for our mathematical model of the COFS mast was supplied by NASA. The modal frequencies and mode shape coefficients were determined from a finite element analysis of the structure; the structural damping was selected by NASA as typical for a structure of this nature.

The examples presented in this chapter are based on a ten mode model which includes four bending modes in the $x$-direction, four $y$-bending modes, and two torsion modes. The ten modes chosen for the model are the first ten flexible modes of the structure, selected in order of increasing natural frequency. The model size reflects our interest in controlling structural vibrations with frequencies below 50 rad/sec. The modal frequencies and the assumed damping for each of the modeled modes are listed in Table 2. The mode shape coefficients for each of mode, listed in Table 3, are the values of the corresponding mode shapes evaluated at each of the sensor/actuator locations.

The state space model for the structure, in modal coordinates, has the form.
Figure 8. COFS mast tip sensor/actuator configuration
Table 2. Modal frequencies and damping.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency (rad/s)</th>
<th>Assumed Damping (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 x</td>
<td>1.175</td>
<td>0.2</td>
</tr>
<tr>
<td>1 y</td>
<td>1.5205</td>
<td>0.2</td>
</tr>
<tr>
<td>2 y</td>
<td>8.168</td>
<td>0.3</td>
</tr>
<tr>
<td>2 x</td>
<td>8.545</td>
<td>0.3</td>
</tr>
<tr>
<td>1 t</td>
<td>9.1106</td>
<td>0.5</td>
</tr>
<tr>
<td>3 y</td>
<td>23.688</td>
<td>0.5</td>
</tr>
<tr>
<td>3 x</td>
<td>24.693</td>
<td>0.5</td>
</tr>
<tr>
<td>2 t</td>
<td>32.107</td>
<td>0.5</td>
</tr>
<tr>
<td>4 y</td>
<td>41.469</td>
<td>0.5</td>
</tr>
<tr>
<td>4 x</td>
<td>42.977</td>
<td>0.5</td>
</tr>
</tbody>
</table>
Table 3. Mode shape coefficients.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Mode Shape Coefficient</th>
<th>Bay 54</th>
<th>Bay 46</th>
<th>Bay 28</th>
<th>Bay 10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 x</td>
<td></td>
<td>.0486</td>
<td>.0373</td>
<td>.0149</td>
<td>.000874</td>
</tr>
<tr>
<td>1 y</td>
<td></td>
<td>.0375</td>
<td>.0263</td>
<td>.0048</td>
<td>-.0048</td>
</tr>
<tr>
<td>2 y</td>
<td></td>
<td>-.0240</td>
<td>.0193</td>
<td>.0639</td>
<td>.0206</td>
</tr>
<tr>
<td>2 x</td>
<td></td>
<td>-.0243</td>
<td>.0194</td>
<td>.0647</td>
<td>.0213</td>
</tr>
<tr>
<td>1 t</td>
<td></td>
<td>.0887</td>
<td>.0795</td>
<td>.0461</td>
<td>.0194</td>
</tr>
<tr>
<td>3 y</td>
<td></td>
<td>.0164</td>
<td>-.0487</td>
<td>.0117</td>
<td>.0489</td>
</tr>
<tr>
<td>3 x</td>
<td></td>
<td>.0167</td>
<td>-.0493</td>
<td>.0111</td>
<td>.0505</td>
</tr>
<tr>
<td>2 t</td>
<td></td>
<td>-.0496</td>
<td>.0122</td>
<td>.1161</td>
<td>.0739</td>
</tr>
<tr>
<td>4 y</td>
<td></td>
<td>-.0103</td>
<td>.0516</td>
<td>-.0553</td>
<td>.0665</td>
</tr>
<tr>
<td>4 x</td>
<td></td>
<td>-.0107</td>
<td>.0530</td>
<td>-.0567</td>
<td>.0683</td>
</tr>
</tbody>
</table>
\[ \dot{x}(t) = Ax(t) + Bu(t) \]
\[ p(t) = C_p x(t) \]
\[ v(t) = C_v x(t). \]  \hspace{1cm} (7.2.1)

The dimensions of the control input, \( u(t) \), and the measured outputs, \( p(t) \) and \( v(t) \), correspond to the number of actuators and sensors on the structure respectively. The outputs \( p_j \) and \( v_j \) the position and velocity at the \( j \)-th sensor/actuator pair. The control input \( u(t) \) is given by

\[ u(t) = Ef(t) \]  \hspace{1cm} (7.2.2)

where \( f(t) \) denotes the control force generated by the actuators mounted on the structure. Because we have the same number of actuators as modelled modes, however, \( E = I_N \), and the \( j \)-th control input \( u_j \) represents the output of the \( j \)-th actuator.

The state vector \( x(t) \) in (7.2.1) is given by

\[ x \triangleq [\eta_1, \dot{\eta}_1, \eta_2, \dot{\eta}_2, ..., \eta_N, \dot{\eta}_N]^T, \]  \hspace{1cm} (7.2.3)

where \( \eta_i \) and \( \dot{\eta}_i \) are the position and velocity of the motion associated with the \( i \)-th flexible mode. The state vector has dimension \( 2N \), where \( N \) is equal to the number of modes included in the model. With the model expressed in modal coordinates, and the state vector (7.2.3), the matrix \( A \) is block diagonal with diagonal blocks of the form (2.5.5). Because the sensors and actuators are collocated,

\[ C_v = B^T, \]  \hspace{1cm} (7.2.4)

where the input matrix, \( B \), has the form (2.5.6), with nonzero rows (2.5.7). Due to our choice for the state vector, the position output matrix \( C_p \) is just a column shifted version of \( C_v \), as are as described in Section 2.5.
7.3 **Decentralized stabilizability**

Although we have assumed some structural damping in each of the flexible modes included in our mathematical model of the mast, design criteria for most structures typically specify structural damping of 5% or more for each mode. Our objective is to design a decentralized feedback control system to increase the damping in each mode.

The existence of a solution to the decentralized pole placement problem, depends on the existence and location of any decentralized fixed modes of the system. Theorem 3.1.4 states, in effect, that as long as the system has no decentralized fixed modes we can design a dynamic decentralized output feedback controller to achieve any desired set of closed loop poles. Therefore, if our system has no decentralized fixed modes, we can design a decentralized controller to achieve any amount of structural damping.

Before designing a decentralized control system for the proposed large space structure, we must determine the location of any decentralized fixed modes. If decentralized fixed modes are present, we must then determine whether they are structured or unstructured. As the discussion in section 3.3 indicates, unstructured fixed modes may be controlled using a sampled data or digital control, but structurally fixed modes will remain invariant to decentralized feedback. As a result, we will be unable to add damping to any flexible modes that correspond to structurally fixed modes.

Algorithm 3.1.1 provides a convenient computational method of identifying the fixed modes of a system with respect to a given feedback information flow constraint. The decentralized fixed modes of the system lie in the intersection, taken over all admissible feedback gain matrices, of the closed loop poles of the system.
We are interested in a decentralized feedback control law in which feedback is only permitted between collocated sensors and actuators; thus, all admissible feedback gain matrices belong to the set of matrices that are structurally equivalent to

$$\bar{K}_d = I_n$$  \hspace{1cm} (7.3.1)

where $I_n$ is the identity matrix in $R^n$.

Figure 9 shows a scatter plot of the closed loop poles of the system with decentralized velocity feedback, for several random feedback gain matrices $K \in \{\bar{K}_d\}$. The large X's denote open-loop system poles, and the *'s denote poles of the closed-loop system.

$$\dot{x}(t) = (A - BK_{C_0})x(t).$$  \hspace{1cm} (7.3.2)

Figure 10 shows the closed loop poles of the system with decentralized position feedback; that is, the closed-loop system:

$$\dot{x}(t) = (A - BK_{C_p})x(t).$$  \hspace{1cm} (7.3.3)

The X's denote eigenvalues of the open-loop system, and the *'s denote closed-loop poles of the system for several random decentralized feedback gain matrices $K \in \{\bar{K}_d\}$.

Figure 9 shows that none of the modelled modes are fixed with respect to decentralized velocity feedback. While velocity feedback primarily affects structural damping, position feedback affects the modal frequencies. Consequently, very large position feedback gains are required to produce noticeable changes in the higher frequency modes. In order to better illustrate the effects of decentralized position feedback, Figure 10 shows only the five flexible modes. The effects of position feedback on the lower frequency modes shown in Figure 10, extend to all of the modelled modes.
Figure 9. Closed loop poles with random decentralized velocity feedback
Figure 10. Closed loop poles with random decentralized position feedback
however, and that none of the modes are fixed with respect to decentralized position feedback.

Table 4 contains a list of the eigenvalues of the open loop system, and the eigenvalues of the closed loop system for random decentralized velocity and decentralized position feedback. Comparing the three sets of closed-loop eigenvalues in Table 4, we see that none of the open-loop eigenvalues are fixed with respect to either decentralized velocity or decentralized position feedback; therefore, we can design decentralized control laws to achieve any desired structural damping for each mode.

7.4 Mast model positivity

Consider the mathematical model for the COFS mast described in Section 7.2. The matrix transfer function for the system which relates the control input to the measured velocity output is

\[ T(s) = C_v(sI - A)^{-1}B. \]  \hspace{1cm} (7.4.1)

By Theorem 4.3.1, the transfer function matrix \( T(s) \), is positive real since the actuators and rate sensors on the mast are collocated. This result is independent of the number of modes included in the model, the amount of structural damping, and the numerical values of the model parameters; therefore, the model will remain positive real even if we change the number of modes in the model and the numerical values of the modal frequencies or the mode shapes.

Suppose the following feedback control law is used to control the the mast
Table 4. Closed loop poles with position and velocity feedback.

<table>
<thead>
<tr>
<th>Open loop</th>
<th>Closed loop $(A - BK_c)$</th>
<th>Closed loop $(A - BK_c)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2149 ± j42.9765</td>
<td>-0.21488 ± j42.989</td>
<td>-0.26947 ± j42.976</td>
</tr>
<tr>
<td>-0.2073 ± j41.4685</td>
<td>-0.20735 ± j41.481</td>
<td>-0.25912 ± j41.468</td>
</tr>
<tr>
<td>-0.1605 ± j32.1066</td>
<td>-0.16053 ± j32.181</td>
<td>-0.40063 ± j32.104</td>
</tr>
<tr>
<td>-0.1235 ± j24.6927</td>
<td>-0.12346 ± j24.704</td>
<td>-0.15177 ± j24.692</td>
</tr>
<tr>
<td>-0.1184 ± j23.6877</td>
<td>-0.11844 ± j23.699</td>
<td>-0.14563 ± j23.688</td>
</tr>
<tr>
<td>-0.0456 ± j9.1105</td>
<td>-0.04551 ± j9.3771</td>
<td>-0.29160 ± j9.1035</td>
</tr>
<tr>
<td>-0.0256 ± j8.5450</td>
<td>-0.02566 ± j8.5803</td>
<td>-0.05627 ± j8.5459</td>
</tr>
<tr>
<td>-0.0245 ± j8.1680</td>
<td>-0.02452 ± j8.2043</td>
<td>-0.05454 ± j8.1686</td>
</tr>
<tr>
<td>-0.0030 ± j1.5205</td>
<td>-0.00304 ± j1.6323</td>
<td>-0.02079 ± j1.5197</td>
</tr>
<tr>
<td>-0.0024 ± j1.1750</td>
<td>-0.00235 ± j1.4185</td>
<td>-0.03404 ± j1.1743</td>
</tr>
</tbody>
</table>
\[ u(t) = Ky(t), \]  

where \( y(t) \) is measured velocity from the collocated rate sensors. By Theorem 4.4.1, the closed loop system

\[ \dot{x}(t) = (A - BKCy)x(t) \]  

is stable for all negative definite feedback gain matrices \( K \). When the control law is decentralized, \( K \) must be diagonal; therefore, stability of the closed loop system is guaranteed for all decentralized feedback control laws with diagonal elements \( k_y < 0 \).

Figure 11 is a plot of the closed loop system poles for random, negative definite, decentralized velocity feedback gain matrices. The closed loop poles in Figure 11 are all contained in the open left half plane, indicating that the system remains stable. We can compare this result to the closed loop poles plotted in Figure 9 where the decentralized velocity feedback gain matrices were not necessarily negative definite, and some of the closed loop poles lie in the right half plane, indicating that the system is unstable for some nonnegative definite feedback matrices \( K \).

Collocated rate feedback guarantees stability of the closed-loop system despite uncertainty in either the numerical parameters of the system or the number of flexible modes included in the mathematical model. One disadvantage of collocated rate feedback, however, is relatively low performance. The feedback gains required to achieve significant changes in the structural damping of the system are usually undesirably large. Large feedback gains make the system sensitive to disturbances such as sensor noise, and are usually unacceptable. Therefore, the system performance that we can achieve using pure rate feedback is typically limited by restrictions on the magnitudes of the feedback gains.
Figure 11. Closed loop poles with negative definite velocity feedback
7.5 Uniform damping control

The mathematical model for the mast, in modal coordinates, is

\[ \begin{align*}
\dot{x}(t) &= Ax(t) + Bu(t) \\
\dot{v}(t) &= B^T x(t) \\
p(t) &= C_p x(t)
\end{align*} \quad (7.5.1) \]

and the state vector \( x(t) \) is given by

\[ x \triangleq [\eta_1, \eta_2, \eta_2, \ldots, \eta_N, \dot{\eta}_N]^T. \quad (7.5.2) \]

We can rewrite the state equations as a set of second-order linear ordinary differential equations of the form

\[ \ddot{\eta}(t) + D\dot{\eta}(t) + \Omega \eta(t) = \Phi^T u(t), \quad (7.5.3) \]

where

\[ \eta(t) = [\eta_1, \eta_2, \ldots, \eta_N]^T \quad (7.5.4) \]

is the position of the structure in modal coordinates. The matrices \( D \) and \( \Omega \) are given by

\[ D = \text{diag}\{\zeta_1 \omega_1, 2\zeta_2 \omega_2, \ldots, 2\zeta_N \omega_N\}. \quad (7.5.5) \]

\[ \Omega = \text{diag}\{\omega_1^2, \omega_2^2, \ldots, \omega_N^2\}, \quad (7.5.6) \]

where \( \omega_i \) is the natural frequency of the \( i \)-th mode, and \( \zeta_i \) is the corresponding damping ratio. The matrix \( \Phi \) in (7.5.2) is the modal matrix described in Chapter 2.
The uniform damping control law, introduced in Chapter 5, is given by

\[ u(t) = -\alpha^2 M q(t) - 2\alpha M \dot{q}(t), \quad (7.5.7) \]

where \( q(t) \) and \( \dot{q}(t) \) are the position and velocity of the structure in generalized coordinates and \( M \) is the mass matrix for the system. The measured position \( q(t) \) is given by

\[ q(t) = \Phi \eta(t). \quad (7.5.8) \]

Recall from the discussion in Chapter 2 that the columns, \( \phi_j \), of the modal matrix \( \Phi \) are orthonormal with respect to the mass matrix \( M \); therefore,

\[ \Phi^T M \Phi = I. \quad (7.5.9) \]

The modal vectors are also linearly independent; hence, the modal matrix is invertible, and given \( \Phi \), we can determine the mass matrix \( M \) from

\[ M = (\Phi \Phi^T)^{-1}. \quad (7.5.10) \]

Comparing the two model representations, (7.5.1) and (7.5.2), the modal vectors \( \phi_j \) correspond to the nonzero rows of the input matrix \( B \). The input matrix \( B \) for the COFS mast, however, does not have full column rank, regardless of the number of modelled modes; therefore, \( \Phi \) will always be nonsingular and we are unable to calculate the corresponding mass matrix \( M \).

Although the mast is equipped with 10 actuators, the actuators yield only nine force inputs; hence, the input matrix, and subsequently \( \Phi^T \), have column rank equal to nine instead of 10, and \( \Phi \) is nonsingular. The redundancy is due to the configuration of the four tip actuators shown in Figure 8. Actuators 1 and 3 generate force in the...
x-direction, actuators 2 and 4 generate force in the y-direction, and together the four tip actuators are capable of generating a torsional force. Thus, the four tip actuators are effectively capable of generating three force inputs.

We can remove the redundancy in the tip actuators by applying a geometric transformation on the input matrix which effectively replaces the four tip actuators with three actuators generating the same effective forces. Suppose the control input for the mast is given by

\[ u = [u_1, u_2, u_3, u_4, u_5, \ldots, u_{10}]^T \]  \hspace{1cm} (7.5.11)

where \( u_j \) is the force generated by the \( j \)-th actuator. Assume there exist three equivalent actuators located at the center of the mast tip which generate the same effective forces as the four original tip actuators. If we denote the outputs of the effective actuators by \( u_{ae} \), \( u_{ay} \), and \( u_{et} \), then the new input vector for the system is

\[ u_a = [u_{ae}, u_{ay}, u_{et}, u_5, \ldots, u_{10}]^T. \]  \hspace{1cm} (7.5.12)

This process is essentially aggregation of the system inputs, and we can write

\[ u(t) = Eu_a(t), \]  \hspace{1cm} (7.5.13)

where

\[ E = \begin{bmatrix} E_a & 0 \\ 0 & I \end{bmatrix}, \]  \hspace{1cm} (7.5.14)

and

Chapter 7
\[
E_a = \begin{bmatrix}
1 & 0 & .5 \\
0 & 1 & .5 \\
1 & 0 & -.5 \\
0 & 1 & -.5 \\
\end{bmatrix},
\] (7.5.15)

Substituting (7.5.13) into (7.5.1) the model becomes
\[
\dot{x} = Ax + BEu_a = Ax + B_a u_a.
\] (7.5.16)

The aggregate input matrix \(B_a\) now has full column rank. Each of the columns corresponds to one of the actuators represented in \(u_a\). Since the modal matrix must be nonsingular, the number of actuators must equal the number of modes in the model; therefore, we are forced to reduce the number of modes included in the model to agree with the number of effective actuators. The second torsional mode is removed so the number of bending modes in the \(x\) and \(y\)-directions remains equal.

The matrix \(\Phi^T\) is formed from the aggregate input matrix \(B_a\) for the model containing nine flexible modes and nine effective actuators. The mass matrix for the nine mode/nine actuator model \(M\) is determined according to (7.5.10).

The uniform damping control law (7.5.7) is decentralized only if the mass matrix, \(M\), is diagonal. Although the mass matrix is seldom diagonal, it may be diagonally dominant. When \(M\) is diagonally dominant, we can approximate uniform damping control with a decentralized control law. The decentralized approximation to the uniform damping control law consists of a set of local output feedback control laws of the form
\[
u_j = -M_j (\alpha^2 q_j + 2\alpha \dot{q}_j),
\] (7.5.17)
where $M_s$ represents an approximation of the mass of the structure in the vicinity of the sensor/actuator pair.

The composite decentralized control law for the aggregate system is

$$u_a(t) = -\alpha^2 M_a q(t) - 2\alpha M_a \dot{q}(t), \quad (7.5.18)$$

where $M_s$ is a diagonal matrix formed from the main diagonal of $M$. The control law (7.5.18) is a decentralized approximation to the uniform damping control law for the aggregate system (7.5.16) but is not completely decentralized with respect to the original system (7.5.1) due to coupling through the effective torsion actuator located at the tip.

The aggregation of the actuators, also applies to the colocated sensors; therefore, the velocity $q(t)$ in the aggregate system is

$$\dot{q}(t) = B_d^T x(t)$$
$$= E^T B_d^T x(t)$$
$$= E^T v(t), \quad (7.5.19)$$

where $v(t)$ is the velocity measured by the actuators mounted on the structure. Similarly, the measured position in the aggregate system is

$$q(i) = E^T p(i). \quad (7.5.20)$$

where $p(i)$ is the position measured by the actuators mounted on the structure.

Substituting the uniform damping control law (7.5.7) into the aggregate system model (7.5.16), we have

$$\dot{x} = Ax - B_d(\alpha^2 Mq + 2\alpha M\dot{q}). \quad (7.5.21)$$

Substituting the expressions for the aggregate position and velocity into (7.5.21) we have
\[ \dot{x} = A \dot{x} - B_d(\alpha^2 ME^T \dot{p} + 2\alpha ME^T \dot{v}) \]
\[ = A \dot{x} - B E(\alpha^2 ME^T \dot{p} + 2\alpha ME^T \dot{v}) \]
\[ = A \dot{x} - B E ME^T (\alpha^2 p + 2\alpha \dot{v}). \] (7.5.22)

Therefore, the uniform damping control law, in terms of the measured position and velocity, is given by

\[ u(t) = -K[\alpha^2 \dot{p}(t) + 2\alpha \dot{v}(t)], \] (7.5.23)

where

\[ K = E ME^T. \] (7.5.24)

The approximation (7.5.18) of the uniform damping control law corresponds to

\[ K = K_\alpha = EM_a E^T, \] (7.5.25)
in (7.5.23). With \( K = K_\alpha \), the control law is decentralized with respect to the aggregate sensor/actuator pairs, but not the real sensors and actuators; the four tip actuators are coupled through the aggregate torsion actuator. Since the coupled actuators are all located at the tip, however, \( K_\alpha \) may represent an acceptable control design even though it is not completely decentralized, since it is decentralized with respect to the bays in which the sensor/actuator pairs are mounted.

The uniform damping control law (7.5.23) is completely decentralized, only when the feedback gain matrix \( K \) is diagonal matrix. To get an approximation of the uniform damping control law that is completely decentralized we approximate the gain matrix \( K \) by a diagonal matrix \( K_\alpha \) with diagonal elements equal to the diagonal elements of \( K \).

Figure 12 shows the locus of the closed-loop poles of the system with uniform damping control. The large X's denote the open-loop pole locations and the asterisks
denote the closed-loop pole locations as \( \alpha \) varies from 0.05 to 1.0. Figure 13 shows the eigenvalues of the closed-loop system when the uniform damping control gain matrix \( K \) is replaced with the decentralized approximation \( K_p \).

Table 5 contains a listing of the closed-loop poles of the structure for each of the three designs: uniform damping control with \( \alpha = 0.1 \), a decentralized approximation of the uniform damping control law with coupling between the tip actuators, and a completely decentralized approximation of the uniform damping control law. Table 6 shows the corresponding damping ratios for the closed-loop poles in Table 5.

We see from Table 5 and Table 6 that the uniform damping control law for a given \( \alpha \) results in the closed-loop poles of the system having negative real parts less than or equal to \( \alpha \). In fact, if the open-loop system were completely undamped, all of the closed-loop poles would have the same negative real part with uniform damping control. The real parts of the closed-loop poles for the two approximations to the uniform damping control law do not have same uniformity as the closed-loop poles with full uniform damping control, but they provide good approximations. While there are differences between the closed-loop poles of the system with the approximate control laws and the closed-loop poles with uniform damping control, the closed-loop poles for the two approximations are very similar.

### 7.6 Decompositions and control design

Decomposition techniques play an important role in reducing the computational burden of analyzing large scale systems. Large scale systems are often represented as a collection of interconnected subsystems of smaller dimension. The subsystems are
Figure 12. Closed loop poles for uniform damping control
Figure 13. Closed loop poles for decentralized uniform damping control
Table 5. Uniform damping and decentralized approx., closed loop poles.

<table>
<thead>
<tr>
<th>Open loop</th>
<th>Uniform damping control ($x = 0.5$)</th>
<th>Coupled tip control</th>
<th>Decentralized control</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.215 ± 42.976</td>
<td>-0.315 ± 42.976</td>
<td>-0.375 ± 42.975</td>
<td>-0.374 ± 42.975</td>
</tr>
<tr>
<td>-0.207 ± 41.469</td>
<td>-0.307 ± 41.468</td>
<td>-0.384 ± 41.467</td>
<td>-0.382 ± 41.467</td>
</tr>
<tr>
<td>-0.123 ± 24.693</td>
<td>-0.223 ± 24.692</td>
<td>-0.208 ± 24.692</td>
<td>-0.206 ± 24.692</td>
</tr>
<tr>
<td>-0.118 ± 23.688</td>
<td>-0.218 ± 23.687</td>
<td>-0.217 ± 23.687</td>
<td>-0.212 ± 23.687</td>
</tr>
<tr>
<td>-0.046 ± 9.110</td>
<td>-0.146 ± 9.109</td>
<td>-0.818 ± 9.067</td>
<td>-0.778 ± 9.068</td>
</tr>
<tr>
<td>-0.026 ± 8.545</td>
<td>-0.126 ± 8.545</td>
<td>-0.111 ± 8.549</td>
<td>-0.106 ± 8.549</td>
</tr>
<tr>
<td>-0.025 ± 8.168</td>
<td>-0.125 ± 8.168</td>
<td>-0.131 ± 8.173</td>
<td>-0.124 ± 8.175</td>
</tr>
<tr>
<td>-0.003 ± 1.520</td>
<td>-0.103 ± 1.520</td>
<td>-0.085 ± 1.520</td>
<td>-0.061 ± 1.521</td>
</tr>
<tr>
<td>-0.002 ± 1.175</td>
<td>-0.102 ± 1.175</td>
<td>-0.098 ± 1.175</td>
<td>-0.080 ± 1.175</td>
</tr>
</tbody>
</table>
Table 6. Uniform damping and decentralized approx., closed loop damping.

<table>
<thead>
<tr>
<th>Open loop (%)</th>
<th>Uniform damping control (%)</th>
<th>Coupled tip control (%)</th>
<th>Decentralized control (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>0.73</td>
<td>0.87</td>
<td>0.87</td>
</tr>
<tr>
<td>0.2</td>
<td>0.74</td>
<td>0.93</td>
<td>0.92</td>
</tr>
<tr>
<td>0.3</td>
<td>0.90</td>
<td>0.84</td>
<td>0.83</td>
</tr>
<tr>
<td>0.3</td>
<td>0.92</td>
<td>0.92</td>
<td>0.90</td>
</tr>
<tr>
<td>0.5</td>
<td>1.60</td>
<td>8.98</td>
<td>8.54</td>
</tr>
<tr>
<td>0.5</td>
<td>1.47</td>
<td>1.30</td>
<td>1.24</td>
</tr>
<tr>
<td>0.5</td>
<td>1.52</td>
<td>1.61</td>
<td>1.48</td>
</tr>
<tr>
<td>0.5</td>
<td>6.76</td>
<td>5.56</td>
<td>4.04</td>
</tr>
<tr>
<td>0.5</td>
<td>8.68</td>
<td>8.33</td>
<td>6.80</td>
</tr>
</tbody>
</table>
considered isolated from each other and analyzed independently. The separate solutions for the isolated subsystems are later combined to provide a solution for the large scale system.

Decentralized control is a natural extension of decomposition techniques to control system design for large scale systems such as space structures. Decentralized control is constrained by the allowable feedback information structure associated with a system; hence, the decentralized control scheme corresponds to a set of local feedback control laws for the subsystems identified by a decomposition of the structure compatible with the information structure of the system.

The application of modal analysis to the mathematical models of flexible structures results in a representation of any structural displacement in terms of a series of modal displacements. Representation of the state space model for flexible structures in modal coordinates, as discussed in Section 2.5, leads to a disjoint decomposition of the state space, illustrated by the blocked diagonal structure of the A matrix in (2.5.1).

In the COFS mast, the configuration of the four sensor/actuator pairs located at the tip of the mast introduces coupling between the flexible modes in the x and y directions through the torsional modes. This coupling results in an overlap between certain subsystems in the modal coordinate model; therefore, an overlapping decomposition of the system is suggested as another approach to the design of a decentralized control system for the mast.

The nature of the finite element method, used in the modelling of many flexible structures, results in overlapping subsystems in the structural dynamics. The finite element method regards a complex structure as a collection of finite elements, each of which is part of a continuous structural member. Certain points shared by several elements are known as nodes. At each node, the displacements of the adjacent elements are required to be compatible and the internal forces in balance; therefore, the structural
dynamics at the nodes are shared by the adjacent elements. This suggests an overlapping decomposition of the structure. In the overlapping decomposition, the structural dynamics shared by the adjacent structural elements are duplicated so that the adjacent elements appear as disjoint.

With the state space model for the mast described in modal coordinates, each flexible mode is modelled independently by a state equation of the form

\[
\dot{\eta}_i = \begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta_i\omega_i \end{bmatrix} \eta_i + \sum_{j=1}^{m} \alpha_{ij} \eta_j, \tag{7.6.1}
\]

where \( m \) is the number of sensor/actuator pairs mounted on the structure. Modal analysis provides a disjoint decomposition of the state space into \( N \) subsystems corresponding to the \( N \) flexible modes included in the model. The subsystems are interconnected, however, by the sensor/actuator pairs. We see from the mode shape coefficients listed in Table 3, that although each sensor/actuator pair influences some flexible modes more than others, each mode is affected by more than one sensor/actuator pair, resulting in subsystem coupling.

The issue of sensor and actuator placement is very important in the control of flexible structures. If the sensor/actuator locations can be chosen so that each sensor/actuator pair affects only one mode, then the model is completely decoupled in modal coordinates. Although it may not be possible to achieve total decoupling, the sensor and actuator locations can often be chosen in such a way that the coupling is minimized. This is not the case for the COFS mast, however, and the disjoint decomposition does not lead to an acceptable decentralized control law.
In the state space model (2.5.1), the vibrational modes were represented in the state vector in order of increasing natural frequency, and the control inputs \( u(t) \) were arranged by increasing actuator number. There exists a similarity transform, so that the state space model, given in modal coordinates, has the form

\[
\begin{bmatrix}
\dot{q}_x \\
\dot{q}_t \\
\dot{q}_y
\end{bmatrix} =
\begin{bmatrix}
A_x & 0 & 0 \\
0 & A_t & 0 \\
0 & 0 & A_y
\end{bmatrix}
\begin{bmatrix}
q_x \\
q_t \\
q_y
\end{bmatrix} +
\begin{bmatrix}
B_{xx} & 0 \\
B_{tx} & B_{ty} \\
0 & B_{yy}
\end{bmatrix}
\begin{bmatrix}
u_x \\
u_t \\
u_y
\end{bmatrix},
\]  

(7.6.2)

where \( q_x, q_t, \) and \( q_y \) are the system states corresponding to the \( x \)-bending, torsion, and \( y \)-bending modes respectively. The force inputs \( u_x \) and \( u_y \) represent the actuators in the \( x \) and \( y \) directions respectively. The state space in (7.6.2) is decoupled, but the subsystems are not disjoint because of the overlapping influence of the two sets of actuators on the torsional modes.

Consider the following pairs of transformation matrices

\[
V = \begin{bmatrix}
I_{n_x} & 0 & 0 \\
0 & I_{n_t} & 0 \\
0 & 0 & I_{n_y}
\end{bmatrix}, \quad V' = \begin{bmatrix}
I_{n_x} & 0 & 0 & 0 \\
0 & \frac{1}{2} I_{n_t} & \frac{1}{2} I_{n_t} & 0 \\
0 & 0 & 0 & I_{n_y}
\end{bmatrix},
\]  

(7.6.3)

and

\[ U = U' = I_m, \]

(7.6.4)

where \( n_x, n_t, \) and \( n_y \) denote the dimensions of \( q_x, q_t, \) and \( q_y \) respectively, \( I_m \) is the identity matrix in \( R^m \). Clearly
\[ V'V = I_n \]  
(7.6.3)

and

\[ U'U = I_m \]  
(7.6.6)

where \( n \) is the total number of states; hence, the transform pairs \( V, V' \), and \( U, U' \), satisfy the conditions (6.2.4).

We are interested in generating an expansion of the system (7.6.2). With the complementary matrices

\[
M = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & \frac{1}{2} A_t & -\frac{1}{2} A_t & 0 \\
0 & -\frac{1}{2} A_t & \frac{1}{2} A_t & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]  
(7.6.7)

and

\[ N = 0 \]  
(7.6.8)

the necessary and sufficient conditions in Theorem 6.2.2 are satisfied, and the transformation given in (6.2.8) defines an expansion of the system (7.6.2). The state space model for the expanded system, given the transformation matrices \( V, V', U, \) and \( U' \), and the complementary matrices \( M \), and \( N \), is
\[
\begin{bmatrix}
\dot{q}_x \\
\dot{q}_t \\
\dot{q}_r \\
\dot{q}_y
\end{bmatrix} = \begin{bmatrix}
A_x & 0 & 0 & 0 \\
0 & A_t & 0 & 0 \\
0 & 0 & A_t & 0 \\
0 & 0 & 0 & A_y
\end{bmatrix} \begin{bmatrix}
q_x \\
q_t \\
q_r \\
q_y
\end{bmatrix} + \begin{bmatrix}
B_{xx} & 0 \\
B_{lx} & B_{ry} \\
B_{tx} & B_{sy} \\
0 & B_{yy}
\end{bmatrix} \begin{bmatrix}
u_x \\
u_y
\end{bmatrix}.
\] (7.6.9)

A decomposition of the expanded system (7.6.9) yields the following set of disjoint subsystems

\[
\tilde{S}_x : \begin{bmatrix}
\dot{q}_x \\
\dot{q}_t
\end{bmatrix} = \begin{bmatrix}
A_x & 0 \\
0 & A_t
\end{bmatrix} \begin{bmatrix}
q_x \\
q_t
\end{bmatrix} + \begin{bmatrix}
B_{xx} \\
B_{lx}
\end{bmatrix} u_x
\] (7.6.10)

\[
\tilde{S}_y : \begin{bmatrix}
\dot{q}_t \\
\dot{q}_y
\end{bmatrix} = \begin{bmatrix}
A_t & 0 \\
0 & A_y
\end{bmatrix} \begin{bmatrix}
q_t \\
q_y
\end{bmatrix} + \begin{bmatrix}
B_{sy} \\
B_{yy}
\end{bmatrix} u_y.
\] (7.6.11)

The subsystem \( \tilde{S}_x \) represents that part of the system affected by the \( x \)-actuators, and \( \tilde{S}_y \) that part of the system affected by the \( y \)-actuators. We say that \( \tilde{S}_x \) and \( \tilde{S}_y \) represent an overlapping decomposition of the system (7.6.2).

Using the overlapping decomposition (7.6.10) and (7.6.11) of the modal coordinate description of the system (7.6.2), permits us to design a set of two local control laws for the system - one control law for the \( x \)-actuators, and one control law for the \( y \)-actuators. While this does reduce the order of the control design problem, the overlapping decomposition does not yield a control law that is decentralized with respect to the physical composition of the structure.

The failure of the overlapping decomposition to provide a physically meaningful decentralized control law can be attributed to the fact that we started with the modal coordinate model for the system (7.6.2). An alternate representation of the mast model is given by
\[ M\ddot{z}(t) + C\dot{z}(t) + Kz(t) = u(t) \]  

(7.6.12)

where \( z \) is the displacement of the structure in generalized coordinates, \( u \) is the force input to the mast by the actuators, and \( M, C, \) and \( K \) are the mass, damping, and stiffness matrices respectively as in (2.1.3). Multiplying both sides of (7.6.12) by the inverse of the mass matrix \( M \), and defining the state vector

\[ x(t) \triangleq \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix} \]  

(7.6.13)

we have the following of the state space model for the mast in generalized coordinates:

\[ \dot{x}(t) = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} u(t). \]  

(7.6.14)

The Inclusion Principle (6.2.1), and the associated theory of overlapping decompositions and control is defined in the context of state space representations for systems. From a physical standpoint, we would expect that the any overlapping of structural dynamics between adjacent structural elements would appear in the matrices of the second order matrix differential equation (7.6.12). Because we are restricted by the formulation of the Inclusion Principle to work with the state space description of the system, however, we must base our overlapping decomposition on the matrix \( M^{-1}K \) in (7.6.14) [40]. Unfortunately, the matrix \( M^{-1}K \) calculated for the COFS mast, using the available data, did not exhibit any type of overlapping structure; therefore we were unable to apply an overlapping decomposition to the mast model (7.6.14).
7.7 Summary

In this chapter, we have considered the application of decentralized control techniques to the mathematical model of a proposed large space structure, the NASA COFS mast. Using the techniques presented in Chapter 3, we showed that the structure has no fixed modes with respect to either decentralized velocity or position feedback; hence, there exist decentralized output feedback controllers capable of achieving any desired set of closed-loop poles for the system.

The use of collocated rate sensors and force actuators makes the mathematical model of the COFS mast positive real; therefore, negative definite collocated rate feedback guarantees stability of the closed-loop system. After removing the redundancy caused by the configuration of the tip actuators, we were able to calculate the mass matrix for the structure, and generate the feedback gains for uniform damping control. The mass matrix, however, was not diagonal; hence the uniform damping control law for the mast was not decentralized. Two decentralized approximations to the uniform damping control law were examined - one completely decentralized, and one employing coupled tip sensors and actuators - to and the respective closed-loop poles were compared.

Finally, we attempted to employ overlapping decompositions to design an overlapping control law for the mast. Starting with the modal coordinate model of the structure resulted in two overlapping control laws - one for the x-modes and another for the modes in the y-direction. Although this reduces the computational complexity of the control law design by approximately a factor of two, what we really desire is a spatially decentralized control system; that is, a decentralized control law which permits feedback only between collocated pairs of sensors and actuators. Attempts to design a
decentralized control law based on an overlapping decomposition of the generalized coordinate model of the system also failed, due to the lack of an identifiable overlapping structure in the mass matrix for the system.
8.0 Conclusion

This thesis has examined several decentralized techniques for the analysis and control of large space structures. These techniques were then applied in the analysis and design of decentralized control laws for a proposed flexible structure, the NASA COFS mast.

Modal analysis identifies a basis for describing the spacecraft motion. The basis consists of a set of so-called natural motions associated with undamped free vibration of the structure. In addition, the terminology associated with modal analysis is widely used in the structural control community, and this discussion served to explain and define many of these terms.

The role of fixed modes in decentralized control design and the analysis of large scale systems, is comparable to controllability and observability for centralized systems. Necessary and sufficient conditions for closed loop pole placement, and therefore system stabilization, are posed in terms of the locations of any fixed modes of the system. There are many characterizations of decentralized fixed modes. The most important characterization of fixed modes, in terms of further research into their nature, is probably the recursive characterization, which reduces the problem of studying fixed

Conclusion.
modes in a system with $N$ local feedback controllers to the study of systems with only two local controllers. From an applications standpoint, the most important result concerning fixed modes may be the relationship between unstructured fixed modes and sampled data control; those modes of a system that are not structurally fixed are not fixed with respect to sampled-data control.

Extension of the mathematical concept of positive real functions to dynamic systems provides a method for designing robust controllers for large space structures. We showed that large space structures with colocated rate sensors and force actuators are positive real; therefore, negative definite direct velocity feedback guarantees stability of the closed loop system regardless of the model order and changes in the numerical values of the model parameters. In addition, these results can be extended to systems with certain types of sensor/actuator and plant nonlinearities and first order dynamics.

Uniform damping control is a decentralized approximation to a distributed optimal control problem which uniformly damps all of the flexible modes; all of the closed loop poles have the same real part. A comparison of uniform damping control and colocated rate feedback shows that uniform damping control law achieves increased performance at the cost of decreased robustness.

Decomposition techniques have two applications to large space structures; they are commonly used to reduce the computational burden associated with large scale system, and they often lead naturally to decentralized control designs. There exist many systems for which a decomposition of the system into disjoint subsystems is not very useful because certain system dynamics are shared by adjacent subsystems. Overlapping decompositions in effect duplicate those portions of the system dynamics shared by adjacent subsystems to form an expanded system in which the previously overlapping subsystems are disjoint. A disjoint decomposition of the expanded system is used to design decentralized control laws in the usual manner; the control laws designed for the
expanded system are contracted via a pseudo inverse transformation to control laws for the original system. When this decomposition satisfies certain criteria, the contractibility of the control laws is guaranteed.

Finally, we applied the decentralized techniques to a mathematical model of the COFS mast, a large space structure proposed by NASA. The model for the mast was given in modal coordinates, and the mast was determined to have no decentralized fixed modes. The proposed structure employs collocated rate sensors and force actuators satisfying the necessary conditions for positivity, and we showed that the structure was always stable for negative definite collocated rate feedback.

We also applied uniform damping control to the structure. The uniform damping control, however, was not decentralized because the mass matrix associated with the mast model was not diagonal. Two approximations to uniform damping control laws were examined. One approximation consisted of a control law that was completely decentralized; the other was decentralized except for coupling mast’s tip sensors and actuators. A comparison of the three control designs showed that the decentralized approximations did not exhibit the uniform distribution of the closed loop poles associated with uniform damping control but still resulted in acceptable closed loop damping.

The mast model used in this analysis did not lend itself well to the application of overlapping decompositions. Starting with the model in modal coordinates, overlapping decomposition of the model results in an expansion with two disjoint subsystems - one containing those flexible modes influenced by actuators and sensors aligned in the x-z plane of the mast and another containing those modes influenced by the sensors and actuators aligned in the y-z plane. While this reduces the computational burden of the control design, it does not result in a decentralized design. We also tried applying
overlapping decompositions using a representation of the model in generalized coordinates, but the model did not exhibit any type of overlapping structure.

In this thesis, we have described several techniques for the decentralized control of vibrational damping in large space structures. Based on the information presented in this thesis, we can identify several directions for future research. As we already mentioned in the discussion of flexible structure modelling, the question of how many, and which, vibrational modes to include in the mathematical model of a structure remains open. Approaches that have been suggested include the use of some type of modal energy criterion [3], or model reduction techniques based on the Generalized Hessenberg Representation [41].

Another area that deserves further investigation is the positivity of large space structures. We showed that the input-output model for a large space structures is positive real when the structure employs collocated rate sensors and force actuators. We would like to know what other types of sensor and actuator arrangements also result in the model being positive real; in particular, can the positivity results be extended to systems with distributed sensors and actuators.

Finally, there remain several open questions concerning overlapping decompositions and the design of overlapping control laws. One of the most important questions concerns the necessary system representation for application of these techniques to structural control - is there one particular system representation that identifies the overlapping nature of the structural dynamics and lends itself to the application of these techniques. Another question is whether these techniques can be applied to the matrix second-order ordinary differential equation description of the system instead of the first-order state variable representation.
References


References

Vita

Mr. Reichard was born in Baltimore, MD in 1962. In 1966, his family moved from Baltimore to Forest Hill, MD. He graduated from Bel Air Senior High School in 1980.

Mr. Reichard graduated Summa Cum Laude from Virginia Polytechnic Institute and State University, in 1985, with a degree of Bachelor of Science in Electrical Engineering. While at VPI&SU, he received a T. Marshall-Hahn Engineering Merit Scholarship, the Department of Electrical Engineering’s H.L. Kraus award, and was selected to participate in the College of Engineering’s summer study abroad program. Mr. Reichard also participated in the cooperative education program with the U.S. Army at Aberdeen Proving Grounds, MD.

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