Nonlinear Oscillations Under
Multifrequency Parametric Excitation

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(ABSTRACT)

A second-order system of differential equations containing a multifrequency parametric excitation and weak quadratic and cubic nonlinearities is investigated. The method of multiple scales is used to carry out a general analysis, and three resonance conditions are considered in detail. First, the case in which the sum of two excitation frequencies is near two times a natural frequency, $\lambda_s + \lambda_t \approx 2\omega_q$, is examined. Second, the influence of an internal resonance, $\omega_q \approx 3\omega_q$, on the previous case is studied. Finally, the effect of the internal resonance $\omega_q \approx 3\omega_q$ on the resonance $\lambda_s + \lambda_t \approx 2\omega_q$ is investigated. Results are presented as plots of response amplitudes as functions of a detuning parameter, excitation amplitude, and, for the first case, a measure of the relative values of $\lambda_s$ and $\lambda_t$.

When the parametric resonance, $\lambda_s + \lambda_t \approx 2\omega_q$, is the only resonance present, no more than one stable nontrivial solution is possible for the parameters studied. Addition of the internal resonance, $\omega_q \approx 3\omega_q$, to the parametric resonance produces at most one new stable nontrivial solution. Therefore, including the stable nontrivial solutions from the parametric resonance only, up to three stable steady-state solutions are present for certain parameter ranges. The internal resonance, $\omega_q \approx 3\omega_q$, eliminates the solutions for $\lambda_s + \lambda_t \approx 2\omega_q$ only, and again no more than one stable
nontrivial solution is present for any given set of parameters. In this case, there are regions where no stable solutions are possible.
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Chapter 1

Introduction

In this thesis the behavior of a multidegree-of-freedom system subjected to a parametric excitation having multiple harmonic components and containing quadratic and cubic nonlinearities is studied. A parametric excitation appears as a time-varying coefficient in the governing equations of motion. Many natural phenomena such as waves, earthquakes, and wind produce forces which may appear as parametric excitations in the equations governing structures on which they act. Rotating machinery and pumps are examples of man-made devices which may produce the same effect. Parametric excitations act on missiles, rotors and fluid-conveying pipes as well as numerous other mechanical and structural systems.

The system of governing equations examined here,
\[
\ddot{u}_n + 2\varepsilon c_n(\varepsilon)\dot{u}_n + \omega_n^2 u_n + 2\varepsilon \sum_{m=1}^{M} \cos(\lambda_m t + \tau_m) \sum_{j=1}^{\infty} Q_{jn}^m(\varepsilon) u_j
\]

\[+ \varepsilon \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \Lambda_{jkn} u_j u_k + \varepsilon^2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \Gamma_{jkl} u_j u_k u_l = 0; \quad n = 1, 2, ...,\]

(1.1)

is typical of those encountered when using discretizing methods such as Rayleigh-Ritz or finite elements. Another discretizing method, modal analysis, reduces governing partial differential equations in space and time to ordinary differential equations and often results in an infinite set similar to (1.1).

In (1.1), the \(u_n\) are generalized coordinates, the dots denote derivatives with respect to time, the \(c_n\) are damping coefficients, and the \(\omega_n\) are the linear natural frequencies. The \(Q_{jn}^m, \lambda_m,\) and \(\tau_m\) are the amplitudes, frequencies and phases of the excitation, respectively, and the amplitudes, frequencies, and phases of the excitations are independent. The \(\Lambda_{jkn}\) and \(\Gamma_{jkl}\) are constant coefficients of the nonlinear terms, and \(\varepsilon\) is a small dimensionless parameter.

Using the method of multiple scales (MMS), we will determine the equations governing the steady-state amplitudes and phases of the response when certain resonance conditions are met. MMS provides a means of obtaining approximate analytical solutions of systems such as (1.1), which are valid for small but finite values of \(\varepsilon\). The transient solution of the nonlinear problem does not necessarily decay as it does for the linear case, and we will see that MMS predicts multivalued nontrivial solutions unlike linear approximations.
Attention is focused here on works involving systems whose only excitation is parametric with multiple harmonic components. In many cases, a single frequency cannot satisfactorily describe a dynamic force, and in these situations a multifrequency excitation provides a more accurate model. A general multifrequency excitation can be written in the form

\[ \sum_{m=1}^{M} P_m \cos(\lambda_m t + \phi_m) \]  

where the \( P_m, \lambda_m, \) and \( \phi_m \) are the amplitudes, frequencies, and phases, respectively. Excitations involving \( \cos \lambda t \) and \( \sin \lambda t \) can be written in the form of (1.2) by rewriting \( \sin \lambda t = \cos(\lambda - \pi/2) \). Similarly, coefficients such as \((a + b \cos \lambda t)^2\) can be represented by (1.2) in part by writing \( \cos^2 \lambda t = (1/2 + 1/2 \cos 2\lambda t) \).

In this light, [1]-[6] discuss systems with essentially a two-frequency parametric excitation. To identify the regions of instability of a beam subjected to a sinusoidal axial force, Elmaraghy and Tabarrok [1] used a discretizing method and approximated the beam motion as having a finite number of modes. In addition to an excitation acting on the linear term as in (1.1), they considered the case with a time-varying coefficient in the damping. Mallik, Kulkarni and Ram [2] investigated a linear system using a Ritz-Galerkin procedure and investigated both one- and two-degree-of-freedom models. Using Galerkin's method, Noah and Hopkins [3] considered the effect of support flexibility on the motion of pipes transporting a pulsating fluid. Their linear equation of motion contained a parametric excitation on the velocity as well as the two-frequency excitation previously mentioned. Noah and
Hopkins [4] examined the stability of the trivial solution for a problem very similar to the one in [3] using a generalized Hill's analysis. Bajaj [5] studied the nonlinear planar motions of articulated tubes with periodic flow through them, and his system involved a cubic nonlinearity as well as the parametric excitation. Barr and McWhannel [6] examined the lateral motions of structures undergoing vertical ground motion. Their single-degree-of-freedom model was linear, contained no damping, and involved a two-frequency parametric excitation with the form of (1.2).

Other references with $M > 2$ in (1.2) are [7]-[9], but in each of these studies certain limitations were placed on the $\lambda_m$. In their book, Schmidt and Tondl [7] discussed a very general nonlinear single-degree-of-freedom system. Their parametric excitation contained multiple harmonic components; however, all frequencies were integral multiples of a single frequency, and an external excitation was included in addition to the parametric excitation. Only certain special cases were covered in detail. Watt and Barr [8] studied the stability boundary of a single-degree-of-freedom linear system containing a multifrequency parametric excitation. Their pseudo-random excitation was composed of only frequencies near two times a linear natural frequency of the system. More generally, yet still somewhat restricted, Bogdanoff and Citron [9] presented results from experiments on an inverted pendulum subjected to a multifrequency excitation and compared their results to theoretical predictions. The theoretical predictions were based on a linear ordinary differential equation of motion with constant coefficients. The excitation amplitudes were small, while the frequencies were large, and there were large differences between excitation frequencies.

Most relevant to the present work are [10]-[12]. In [10], MMS was used by Nayfeh to study a two-degree-of-freedom linear system with the identical excitation as in (1.1). He examined three different cases with two simultaneous resonances in each
case. In each resonance, the excitation frequency was either a sum or difference of the two natural frequencies of the system, or one or two times a natural frequency. Nayfeh and Jebril [11] extended Nayfeh's previous study by adding quadratic and cubic nonlinearities to the problem. Once again, four possible cases were examined: (1) principal parametric resonance, (2) the sum of the two natural frequencies near an excitation frequency, (3) the difference of the two natural frequencies near an excitation frequency, and (4) simultaneous principal parametric resonances. Finally, Plaut [12] investigated an air-inflated cylindrical membrane whose internal pressure fluctuated slightly. The fluctuation appeared as a parametric term in the governing equation similar to the excitation in (1.1). Using MMS, he analyzed the single-degree-of-freedom nonlinear system for five resonances: an excitation frequency near one, two, or three times a natural frequency, and the sum or difference of two excitation frequencies near two times a natural frequency.

All of the studies discussed above lack the generality in their mathematical models that is found in (1.1). Often, all nonlinear effects are ignored [1]-[4], [6], [8]-[10]. All of the previous examples consider a finite number of degrees of freedom [1]-[12]. Limitations are frequently placed on the number of excitation frequencies [2]-[7] or on the relationships between the $\lambda_n$ [6]-[8].

This thesis will partially fill the void in the literature above by providing a general MMS analysis of (1.1), which is an infinite-degree-of-freedom system containing a multifrequency parametric excitation with independent amplitudes, frequencies, and phases. Three resonance conditions will be considered in detail. First, we will study the case when the sum of two excitation frequencies is near twice a natural frequency of the system, $\lambda_n + \lambda_\gamma \approx 2\omega_q$. Next, the influence of an internal resonance, $\omega_q \approx 3\omega_q$, on the previous case will be examined. Finally, the effect of the internal resonance $\omega_q \approx 3\omega_q$ on $\lambda_n + \lambda_\gamma \approx 2\omega_q$ will be investigated. In each case, plots of response...
amplitudes as functions of a detuning parameter and excitation amplitude will be presented. In the first case, the variation of the response amplitude will also be presented as a function of the relative magnitudes of $\lambda_s$ and $\lambda_r$. 

Chapter 1
Chapter 2

Analysis

To approximate the solution of

\[
\ddot{u}_n + 2\varepsilon c_n(\varepsilon)\dot{u}_n + \omega_n^2 u_n + 2\varepsilon \sum_{m=1}^{M} \cos(\lambda_m t + \tau_m) \sum_{j=1}^{\infty} Q_{jn}^m(\varepsilon) u_j \\
+ \varepsilon \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \Lambda_{ijk}^1 u_i u_k + \varepsilon^2 \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \Gamma_{jkl}^1 u_i u_k u_l = 0; \quad n = 1, 2, \ldots, \tag{1.1}
\]

we introduce the series

\[
c_n(\varepsilon) = c_{n1} + \varepsilon c_{n2} + \ldots \quad \text{and} \quad Q_{jn}^m(\varepsilon) = Q_{jn1}^m + \varepsilon Q_{jn2}^m + \ldots \tag{2.1}
\]

for the damping coefficients and parametric excitation amplitudes, respectively, where \(c_n \geq 0\) for all \(n\). Following the method of multiple scales [13], we assume that the solution of (1.1) can be expressed as
for \( n = 1, 2, \ldots \), where \( t \) has been replaced by "multiple time scales", i.e.,

\[
T_n = \varepsilon^n t. 
\]

Derivatives with respect to time become sums of partial derivatives with respect to the \( T_n \):

\[
\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 + \ldots 
\]

\[
\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (2D_0 D_2 + D_1^2) + \ldots 
\]

where

\[
D_n = \frac{\partial}{\partial T_n}. 
\]

We substitute (2.1), (2.2), (2.4), and (2.5) into (1.1). Because each \( u_{nj} \) is independent of \( \varepsilon \), the coefficient of each power of \( \varepsilon \) can be set to zero independently to obtain the governing equations for the \( u_{nj} \). Considering only terms of order \( \varepsilon^3 \) or less, we obtain

\[
O(\varepsilon^0): \quad D_0^2 u_{n0} + \omega_n^2 u_{n0} = 0 
\]

\[
O(\varepsilon^1): \quad D_0^2 u_{n1} + \omega_n^2 u_{n1} = -2(D_0 D_1 + c_n D_0) u_{n0} 
\]

\[
- 2 \sum_{m=1}^{M} \cos(\lambda_m T_0 + \tau_m) \sum_{j=1}^{\infty} Q_{j0}^n u_{j0} - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} A_{jkn} u_{j0} u_{k0} 
\]
\[ O(\varepsilon^2): \quad \begin{align*}
D_0^2 u_{n2} + \omega_n^2 u_{n2} &= -2(D_0D_1 + c_{n1}D_0)u_{n1} \\
&= -(2D_0D_2 + D_1^2 + 2c_{n2}D_0 + 2c_{n1}D_1)u_{n0} \\
&\quad - 2\sum_{m=1}^{M} \cos(\lambda_m T_0 + \tau_m)\sum_{j=1}^{\infty} (Q_{j0}^m u_{j0} + Q_{j1}^m u_{j1}) \\
&\quad - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \Lambda_{jkn}^* u_{j0} u_{k0} - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \Gamma_{jkn}^* u_{j0} u_{k0}
\end{align*} \]

where

\[ \Lambda_{jkn}^* = \Lambda_{jkn} + \Lambda_{kjn}^*. \] (2.10)

The general solution of (2.7) can be written as

\[ u_{n0} = A_n(T_1, T_2, \ldots) \exp(i\omega_n T_0) + \text{cc} \] (2.11)

where cc denotes the complex conjugate of the preceding terms on the right-hand side of the equation. Substituting (2.11) into (2.8) results in

\[ \begin{align*}
D_0^2 u_{n1} + \omega_n^2 u_{n1} &= -2i\omega_n(D_1A_n + c_{n1}A_n) \exp(i\omega_n T_0) \\
&\quad - \sum_{j=1}^{\infty} \sum_{m=1}^{M} Q_{n1}^m [A_j \exp[i(\lambda_j + \omega_j) T_0 + \tau_j]] + \bar{A}_j \exp[i(\lambda_j - \omega_j) T_0 + \tau_j]] \\
&\quad - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \Lambda_{jkn} A_k \exp[i(\omega_j + \omega_k) T_0] + \bar{A}_k \exp[i(\omega_j - \omega_k) T_0] + \text{cc}
\end{align*} \] (2.12)

where overbars denote complex conjugates. Small divisors will occur in the solution for \( u_{n1} \) if one or more of the following conditions is met (for any \( j, k, m, \) and \( n \):)

\[ \omega_n \approx |\omega_j \pm \omega_k|, \quad \text{including } \omega_n \approx 2\omega_j; \] (2.13)

\[ \omega_n \approx |\lambda_m \pm \omega_j|, \quad \text{including } 2\omega_n \approx \lambda_m. \] (2.14)
We consider the case where none of the resonance conditions in (2.13) and (2.14) is met and eliminate the secular terms from \( u_n \) by setting

\[ c_{n1} = 0 \quad \text{and} \quad D_1 A_n = 0. \]  

(2.15)

For convenience we also set

\[ c_{n2} = \mu_n, \]  

(2.16)

and it follows from (2.15) that

\[ A_n = A_n(T_2). \]  

(2.17)

As in the linear case, the approximate solution is the sum of homogeneous and particular parts; however, as explained by Nayfeh and Mook [13], it is unnecessary to include the homogeneous solution in any terms of the expansion except the first. Therefore, the solution of (2.12) is given by

\[
\begin{align*}
    u_{n1} &= - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \Lambda_{jk} A_j R_{njk} A_k \exp[i(\omega_j + \omega_k)T_0] + S_{njk} \overline{A_k} \exp[i(\omega_j - \omega_k)T_0] \\
    &\quad - \sum_{j=1}^{\infty} \sum_{m=1}^{M} Q_{jm}^m R_{njm} A_j \exp[i(\lambda_m + \omega_j)T_0 + \tau_m] \\
    &\quad + S_{njm}^m A_j \exp[i(\lambda_m - \omega_j)T_0 + \tau_m]] + \text{cc}
\end{align*}
\]  

(2.18)

where
\[ \mathbf{R}_{njk} = \frac{1}{[\omega_n^2 - (\omega_j + \omega_k)^2]}, \quad \mathbf{S}_{njk} = \frac{1}{[\omega_n^2 - (\omega_j - \omega_k)^2]}, \]

(2.19)

\[ \mathbf{R}^m_{n,jl} = \frac{1}{[\omega_n^2 - (\lambda_m + \omega_j)^2]}, \quad \mathbf{S}^m_{n,jl} = \frac{1}{[\omega_n^2 - (\lambda_m - \omega_j)^2]}. \]

Using (2.11) and (2.18) in (2.9), we obtain

\[
\begin{align*}
D_0^2 u_{n2} + \omega_n^2 u_{n2} &= -2\mathbf{w}_n(D_2 \mathbf{A}_n + \mu_n \mathbf{A}_n) \exp(i\omega_n T_0) \\
&\quad + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \Lambda^{ij}_{kl} \phi_y \psi_y \exp[i(v_y T_0 + \eta_y)] \\
&\quad + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \Lambda^{ij}_{klm} \phi_y \psi_y \exp[i(v_y T_0 + \eta_y)] \\
&\quad - \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{v=1}^{\infty} \sum_{y=1}^{\infty} Q_{jlmn}^{+} \phi_y \psi_y \exp[i(v_y T_0 + \eta_y)] \\
&\quad + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{v=1}^{\infty} \sum_{y=1}^{\infty} Q_{jlmn}^{+} \phi_y \psi_y \exp[i(v_y T_0 + \eta_y)] \\
&\quad + \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{v=1}^{\infty} \sum_{y=1}^{\infty} Q_{jlmn}^{+} \phi_y \psi_y \exp[i(v_y T_0 + \eta_y)] + \text{cc}
\end{align*}
\]

(2.20)

where \( \phi_y, \psi_y, v_y, \text{ and } \eta_y \) are given in Appendix A.

If \( Q_{jlm}^+ \neq 0 \) and \( Q_{jlm}^- = 0 \), small divisors will occur in the solution for \( u_{n2} \) if one or more of the following conditions are met (for any values of \( j,k,l,m,v \) and \( n \)):

\[ \omega_n \approx |\omega_j \pm \omega_k \pm \omega_l|, \quad \text{including } \omega_n \approx |\omega_j \pm 2\omega_k|, \quad \omega_n \approx 3\omega_j, \quad \omega_n \approx \omega_j; \]

(2.21)

\[ \omega_n \approx |\lambda_m \pm \omega_j \pm \omega_k|, \quad \text{including } \omega_n \approx |\lambda_m \pm 2\omega_j|, \quad \omega_n \approx \lambda_m, \quad 3\omega_n \approx \lambda_m; \]

(2.22)
If $Q_{m1} = 0$ and $Q_{m2} \neq 0$, small divisors will occur if (2.14) or (2.21) is satisfied. Because (2.14) has been ruled out, and (2.21) produces small divisors for $Q_{m1} \neq 0$, we will consider the case in which $Q_{m2} = 0$.

Using terms 2-4, 17, and 18 from Appendix A, we eliminate secular terms from $u_{n2}$ by setting

$$-2\omega_n \left\{ i(D_2 A_n + \mu_n A_n) + e_n A_n - 4A_n \sum_{j=1}^{\infty} \alpha_{jn} A_j \right\} + \text{Int} + \text{Para} = 0 \quad (2.24)$$

where Int and Para represent terms resulting from internal resonances (involving only $\omega$'s) and parametric resonances (involving $\lambda$'s and $\omega$'s), respectively, and

$$\alpha_{jn} = \frac{1}{8\omega_n (1 + \delta_{jn})} \left\{ - \Gamma_{jnn}^* + \sum_{k=1}^{\infty} \left[ \Lambda_{knn}^* \Lambda_{jkk}^* S_{knn} + \Lambda_{jknn}^* \Lambda_{jnn}^* (R_{kjn} + S_{kjn}) \right] \right\} \quad (2.25)$$

and

$$e_n = -\frac{1}{2\omega_n} \sum_{j=1}^{\infty} \sum_{m=1}^{M} (R_{jn}^m + S_{jn}^m) Q_{m1}^m Q_{m2}^m \quad (2.26)$$

In (2.25), $\delta_{jn}$ is the Kronecker delta, i.e.,

$$\delta_{jn} = \begin{cases} 1 & j = n \\ 0 & j \neq n \end{cases} \quad (2.27)$$

and

$$\Gamma_{jnn}^* = \Gamma_{jnn} + \Gamma_{knn} + \Gamma_{jknn} + \Gamma_{kljn} + \Gamma_{lkn} + \Gamma_{kjnn} \quad (2.28)$$
Chapter 3

The Case When $\lambda_s + \lambda_t \simeq 2\omega_q$

We will first consider the case when $\lambda_s + \lambda_t \simeq 2\omega_q$ and no other resonances exist. A detuning parameter, $\sigma$, is introduced to provide a quantitative measure of the relationship between the frequencies and is defined by

$$\lambda_s + \lambda_t = 2\omega_q + \varepsilon^2 \sigma. \quad (3.1)$$

Using (3.1) with term 16 of Appendix A, we find that secular terms can be eliminated from the solution for $u_n$ if

$$i(D_2 A_n + \mu_n A_n) + e_n A_n - 4A_n \sum_{j=1}^{\infty} \alpha_{jn} \bar{A}_j + \delta_{nq} n_q \bar{A}_q \exp[i(\sigma T_2 + \tau_s + \tau_t)] = 0 \quad (3.2)$$

where $\alpha_{jn}$ can be found from (2.25), $e_n$ from (2.26), and
Using the form

\[ A_n(T_2) = \frac{1}{2} a_n(T_2) \exp[i\beta_n(T_2)] \]  

in (3.2), and separating real and imaginary parts, we obtain for \( n \neq q \):

\[ a_n' + \mu_n a_n = 0, \]  

(3.5)

\[ a_n(\beta_n' - e_n + \sum_{j=1}^{\infty} \alpha_j a_j^2) = 0, \]  

(3.6)

and for \( n = q \):

\[ a_q' + \mu_q a_q + h_q a_q \sin \gamma = 0, \]  

(3.7)

\[ a_q[\frac{1}{2} (\sigma - \gamma') - e_q + \sum_{j=1}^{\infty} \alpha_j a_j^2 - h_q \cos \gamma] = 0 \]  

(3.8)

where

\[ \gamma = \sigma T_2 + \tau_s + \tau_t - 2\beta_q \]  

(3.9)

has been defined to produce an autonomous set of equations for the \( a_n \). The solution of (3.5) shows that all \( a_n \) for \( n \neq q \) decay exponentially with time.
To determine the steady-state response, we put \(a_n = 0\) for \(n \neq q\), \(a_q' = 0\) and \(y' = 0\). Then, it follows from (3.7) and (3.8) that

\[
a_q (\mu_q + h_q \sin \gamma) = 0
\]

(3.10)

\[
a_q \left( \frac{1}{2} \sigma - e_q + \alpha_{qq} a_q^2 - h_q \cos \gamma \right) = 0.
\]

(3.11)

Obviously, a trivial solution, \(a_q = 0\), is possible. Nontrivial solutions are also possible with \(a_q\) and \(y\) given by

\[
a_q^2 = \frac{1}{\alpha_{qq}} (e_q - \frac{1}{2} \sigma \pm \sqrt{h_q^2 - \mu_q^2})
\]

(3.12)

\[
sin \gamma = -\frac{\mu_q}{h_q}
\]

(3.13)

where (3.13) was obtained by solving (3.10) for \(\sin \gamma\), and (3.12) was obtained by solving (3.11) for \(\cos \gamma\) and squaring and adding the result and (3.13) to eliminate \(\gamma\).

Stability of the nontrivial steady-state solutions is determined by "perturbing" the steady-state amplitudes and phases and investigating the behavior of the "perturbations." We let

\[
a_q = \tilde{a}_q + \delta a_q
\]

(3.14)

\[
y = \tilde{\gamma} + \delta \gamma
\]

(3.15)

where \(\tilde{a}_q\) is the steady-state amplitude, \(\tilde{\gamma}\) is the steady-state phase, and \(\delta a_q\) and \(\delta \gamma\) are perturbations of the amplitude and phase, respectively. Substituting (3.14) and (3.15) into (3.7) and (3.8) and using the fact that \(\tilde{a}_q\) and \(\tilde{\gamma}\) satisfy (3.10) and (3.11) result in the matrix equation.
\[
\begin{align*}
\begin{bmatrix} \delta a_q' \\ \delta y' \end{bmatrix} &= [M] \begin{bmatrix} \delta a_q \\ \delta y \end{bmatrix} \\
[M] &= \begin{bmatrix} 0 & -h_q \bar{a}_q \cos \gamma \\ 2\alpha_{qq} \bar{a}_q & -\mu_a \end{bmatrix}.
\end{align*}
\] (3.16)

where

Solutions for $\delta a_q$ and $\delta y$ are of the form

\[
\begin{align*}
\delta a_q &= C_1 e^{\lambda_1 T_2} + C_2 e^{\lambda_2 T_2} \\
\delta y &= C_3 e^{\lambda_1 T_2} + C_4 e^{\lambda_2 T_2}
\end{align*}
\] (3.18) (3.19)

where $\lambda_1$ and $\lambda_2$ are the eigenvalues of $[M]$, the $C_n$ are constants of integration, and

\[
\lambda_1, \lambda_2 = \frac{-\mu_a \pm \sqrt{4\alpha_{qq} h_q \bar{a}_q^2 \cos \gamma}}{2}.
\] (3.20)

If either $\lambda_1$ or $\lambda_2$ has a positive real part, the perturbations grow, and the solution is unstable. On the other hand, if both $\lambda_1$ and $\lambda_2$ have negative real parts, the perturbations decay to zero, and the solution is stable.

Stability of the trivial solution may also be determined by substituting (3.14) and (3.15) into (3.7) and (3.8) and then letting $\bar{a}_q = 0$. This results in the governing equations

\[
\begin{align*}
\delta a_q' &= -(\mu_a + h_q \sin \gamma)\delta a_q \\
(\frac{1}{2} \sigma - e_q - h_q \cos \gamma)\delta a_q &= 0.
\end{align*}
\] (3.21) (3.22)
For a nonzero perturbation, \( \delta a_q \neq 0 \), \( \bar{y} \) can be found from (3.22), and this value is used in (3.21). If \( (\mu_q + h_q \sin \bar{y}) > 0 \), the trivial solution is stable, and if \( (\mu_q + h_q \sin \bar{y}) < 0 \), the trivial solution is unstable. Numerical studies have shown that \( (\mu_q + h_q \sin \bar{y}) \) is greater than zero for small excitation amplitudes. At bifurcation points, \( (\mu_q + h_q \sin \bar{y}) \) equals zero, and the stability property of the trivial solution changes at these points.

Alternatively, the stability of the trivial solution may be obtained by letting

\[
A_q = \tilde{A}_q + \delta A_q
\]

(3.23)

and substituting into (3.2). For the trivial solution, \( \tilde{A}_q = 0 \) and we write

\[
\delta A_q = (p + iq) \exp \left[ i \frac{1}{2} \left( \sigma T_2 + \tau_s + \tau_i \right) \right].
\]

(3.24)

Using (3.24) in (3.2) and separating real and imaginary parts gives the governing equations for \( p \) and \( q \):

\[
\begin{bmatrix} p' \\ q' \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix}
\]

(3.25)

where

\[
[T] = \begin{bmatrix} -\mu_q & h_q - e_q + \frac{1}{2} \sigma \\ h_q + e_q - \frac{1}{2} \sigma & -\mu_q \end{bmatrix}.
\]

(3.26)

As for the nontrivial solutions, the eigenvalues of \( [T] \) determine the stability properties of the trivial solution. The eigenvalues of \( [T] \) are

\[
\lambda_1, \lambda_2 = -\mu_q \pm \sqrt{h_q^2 - (e_q - \frac{1}{2} \sigma)^2}.
\]

(3.27)
Stability of the trivial solution changes at the points where $(h^2 - \mu^2)^2 = (e_q - \alpha/2)^2$, and these are also the points where the nontrivial solutions bifurcate from the trivial solution. The two methods described for finding the stability of the trivial solution agree completely, and both indicate that the stability of the trivial solution changes at the bifurcation points.

Frequency-response curves (plots of response amplitude as a function of detuning parameter) can have either of the forms shown in Figure 1, hardening behavior when $\alpha_{qq} < 0$ as shown in part (a) or softening behavior when $\alpha_{qq} > 0$ as shown in part (b). The maximum response does not occur when the resonance is perfectly tuned, i.e., $\sigma = 0$, as would be expected in the linear approximation. For $\alpha_{qq} > 0$, the maximum response occurs when $\lambda_s + \lambda_i$ is less than $2\omega_q$, and for $\alpha_{qq} < 0$, the maximum response occurs for $\lambda_s + \lambda_i$ greater than $2\omega_q$. As damping increases, the curves approach each other, and the values on the abscissa are $\sigma = 2(e_q \pm \sqrt{h^2 - \mu^2})$. The curves will not close as $\sigma$ is increased in part (a) or decreased in part (b) because the MMS analysis has only been carried to order $\varepsilon^2$, and thus this analysis is only valid for a small range of $\sigma$ near perfect tuning. Solid curves in the figures indicate stable steady-state solutions, while dashed curves represent unstable solutions.

Jumps can be seen in both Figures 1(a) and (b). Consider initial conditions which are near the trivial solution for $\sigma$ sufficiently positive in part (a). The trivial solution is stable in this region, and, as $\sigma$ is decreased, the response will stay near $a_q = 0$. When $\sigma$ reaches the bifurcation point, however, the trivial solution is no longer stable, and the response amplitude will "jump" up to the stable nontrivial solution. As $\sigma$ is decreased further, the response will follow the stable solution curve until the response becomes trivial again. A similar response can be seen in part (b) by choosing initial conditions near the trivial solution, beginning with a sufficiently
negative value of $\sigma$ and slowly increasing $\sigma$ until the bifurcation point where the
response amplitude will jump to the stable nontrivial solution as $\sigma$ is increased
further. More increase in $\sigma$ will result in a smooth decrease in the response along
the stable solution curve to the trivial solution. These jumps will only be seen for the
conditions described above. If initial conditions are chosen in part (a) for a large $\sigma$
such that the response begins on the nontrivial solution, $a_q$ will smoothly decrease
along the nontrivial solution curve to the trivial solution. The same steady response
would be seen for large initial conditions and small $\sigma$ as $\sigma$ is increased in part (b).
Any initial conditions for sufficiently negative $\sigma$ in part (a) would decay to the trivial
solution, and as $\sigma$ is increased, the response would follow the stable solution curve.
Similarly, any initial conditions for $\sigma$ very large in part (b) would decrease to the trivial
solution and the response would follow the continuous stable solution curve as $\sigma$ is
decreased.

In order to illustrate the variation of the response amplitude, $a_q$, as a function of
the excitation amplitudes, we consider the case when all excitation amplitudes are
zero except $Q_{q21}$ and $Q_{q21}'$, which are equal and denoted by $Q$, i.e.,

$$Q = Q_{q21}^5 = Q_{q21}^1. \quad (3.28)$$

Then, using (3.1) and (3.28) in (2.26) and (3.3), we can write (to order $\varepsilon^4$):

$$e_q = -k_1Q^2, \quad h_q = -k_2Q^2 \quad (3.29)$$

where

$$k_1 = \frac{2(\lambda_s^2 + 3\lambda_s\lambda_t + \lambda_t^2)}{\lambda_s\lambda_t(\lambda_s + \lambda_t)(\lambda_s + 2\lambda_t)(\lambda_s + \lambda_t)} \quad (3.30)$$

and

Chapter 3
\[ k_2 = \frac{2}{\lambda_s \lambda_t (\lambda_s + \lambda_t)} \]
with \( k_2 > k_1 > 0 \). Also, we define

\[ k_3 = k_2 - k_1 > 0 \]

and

\[ k_4 = k_1 + k_2 > 0. \] (3.31)

We will first consider the undamped case, \( \mu_q = 0 \), while all other \( \mu_n \) remain greater than 0. Then (3.12) with (3.28)-(3.31) yields the solutions

\[
\begin{align*}
\alpha_q^2 &= \frac{1}{\alpha_{qq}} (k_3 Q^2 - \frac{1}{2} \sigma), \\
\beta_q^2 &= -\frac{1}{\alpha_{qq}} (k_4 Q^2 + \frac{1}{2} \sigma).
\end{align*}
\] (3.32)

If \( \sigma = 0 \), \( a_q \) is directly proportional to \( |Q| \) as shown in Figure 2(a). Figure 2(b) is applicable if \( \sigma > 0 \) and \( \alpha_{qq} > 0 \) or \( \sigma < 0 \) and \( \alpha_{qq} < 0 \). The value of \( Q \) at the bifurcation point is \( Q = \sqrt{\sigma/(2k_3)} \) in the former case and \( Q = \sqrt{-\sigma/(2k_4)} \) in the latter. The behavior of \( a_q \) as a function of \( Q \) when \( \sigma \) and \( \alpha_{qq} \) are of opposite signs can be seen in Figure 2(c). For the case when \( \alpha_{qq} > 0 \) and \( \sigma < 0 \), the value of \( Q \) at the bifurcation point is \( Q = \sqrt{-\sigma/(2k_4)} \). When \( \alpha_{qq} < 0 \) and \( \sigma > 0 \), the value of \( Q \) at the bifurcation point is \( Q = \sqrt{\sigma/(2k_3)} \). For both cases, the nontrivial value of \( a_q \) where \( Q = 0 \) is \( a_q = \sqrt{-\sigma/(2\alpha_{qq})} \). In all the cases shown, the stable solutions grow as \( Q \) is increased.

In the damped case, \( \mu_q > 0 \), (3.12) becomes
with the use of (3.28)-(3.30). Figure 2(d) shows the variation of \( a_q \) with \( Q \) when \( Q_a \), the value of \( Q \) at the turning point, is less than \( Q_b \), the value of \( Q \) at the bifurcation point, and \( a^*_q \), the value of \( a_q^2 \) at the turning point, is positive, where

\[
a_q^2 = \frac{1}{\alpha_{qq}} \left( -k_1 Q^2 - \frac{1}{2} \sigma \pm \sqrt{k_2^2 Q^4 - \mu_q^2} \right)
\]  

(3.33)

The transition from stable to unstable solutions occurs at the point of vertical tangency. Figure 2(e) illustrates the behavior of \( a_q \) as a function of \( Q \) when \( a^*_q \) is negative and only one solution is possible.

A jump in the response amplitude can be seen in Figure 2(c). Initial conditions near the trivial solution for small \( Q \) will decay to the trivial solution, and as \( Q \) is increased past the bifurcation point, the response will jump to the stable nontrivial solution. Jumps can be observed for both increasing and decreasing \( Q \) in part (d). For small \( Q \) and small initial conditions, the response will continue to be trivial as \( Q \) is increased up to the bifurcation. At this point, the response will jump to the nontrivial solution. The response for any initial conditions at a large \( Q \) will converge to the stable nontrivial solution and continue along this curve to the vertical tangent.

\[
Q_a = \sqrt{\frac{\mu_q}{k_2}}, \quad Q_b^2 = \frac{k_2 \sigma + \sqrt{k_2^2 \sigma^2 + 4 \mu_q^2 (k_2^2 - k_1^2)}}{2(k_2^2 - k_1^2)}
\]  

(3.34)

\[
a^2_a = -\frac{1}{2k_2 \alpha_{qq}} (2k_1 \mu_q + \sigma k_2).
\]
where \( a_q \) will jump down to the trivial solution. No jumps would be seen in the response for parts (a), (b), and (e).

The response amplitude, \( a_q \), also depends upon the relative magnitudes of the excitation frequencies, \( \lambda_s \) and \( \lambda_r \). We define

\[
\Gamma = \frac{\lambda_s}{2\omega_q}
\]

(3.35)

and look at the variation of \( a_q \) with \( \Gamma \) to illustrate this dependence. Substituting (3.35) into (3.33) gives

\[
a_q^2 = \frac{1}{\sigma_{qa}} \left[ \frac{(2\Gamma - 2\Gamma - 1)Q^2}{4\omega_{q}\Gamma(2 - \Gamma)(1 - \Gamma^2)} - \frac{1}{2} \sigma \pm \sqrt{\frac{Q^4}{16\omega_{q}^3(1 - \Gamma^2)} - \mu_q^2} \right].
\]

(3.36)

Near \( \Gamma = 0, 0.5, \) or 1, (3.36) is not valid because other resonances exist. Figure 3 shows some possibilities of the behavior of \( a_q \) with \( \Gamma \), with \( \mu_q = 0, \alpha_q < 0 \) in part (a), \( \mu_q > 0, \alpha_q > 0 \) in (b) and \( \mu_q > 0, \alpha_q < 0 \) in (c). Jumps would be observed in the response in part (c), while none would occur in parts (a) and (b).

Next, we will continue to assume that \( Q_{q1}^s \) and \( Q_{q1}^t \) are the only nonzero excitation amplitudes; however, we will fix one and vary the other. Let

\[
Q_s = Q_{q1}^s, \quad Q_t = Q_{q1}^t,
\]

(3.37)

and assume \( Q_t \) is fixed. Now (2.26) and (3.3) can be written as

\[
e_q = -k_5 - k_5Q_s^2, \quad h_q = -k_7Q_s
\]

(3.38)

where
\[ k_5 = \frac{2Q_t^2}{(\lambda_s + \lambda_t)(\lambda_s + 2\lambda_t)\lambda_s} , \quad k_6 = \frac{2}{(\lambda_s + \lambda_t)(\lambda_t + 2\lambda_s)\lambda_t} \]

and

\[ k_7 = \frac{2Q_t}{(\lambda_s + \lambda_t)\lambda_s \lambda_t} . \]

Using (3.36) and (3.37), (3.12) can be written as

\[ a_q^2 = -\frac{1}{2\alpha_{qq}} (2k_5 + 2k_6 Q_s^2 + \sigma \pm 2\sqrt{k_7^2 Q_s^2 - \mu_q^2}) . \quad (3.40) \]

Figure 4 shows some examples of the variation of \( a_q \) with \( Q \), where \( Q = Q_s \). In Figure 4(a), one stable nontrivial solution is present for small values of \( Q \). A second nontrivial solution is possible for larger values of \( Q \); however, it is always unstable. The trivial solution is unstable in the region where only one solution is present but becomes stable when the second nontrivial solution appears. In Figure 4(b), no nontrivial solutions are possible for very small values of \( Q \), and two solutions are present near the first bifurcation point. A lower nontrivial solution appears for \( Q \) sufficiently large. As in Figure 4(a), the lower nontrivial solution is always unstable, and the trivial solution is stable when no nontrivial solutions or two nontrivial solutions are present. The trivial solution is unstable in the region where only one nontrivial solution is possible. A similar case to Figure 4(b) is seen in Figure 4(c); however, there is only one bifurcation point, and the area where two nontrivial solutions exist, near the bifurcation, has expanded. The stability of the trivial solution is the same as that in Figure 4(b). In Figure 4(d), one nontrivial solution is possible for a limited range of \( Q \), and it is stable. Outside this range, only the trivial solution
is possible. Jumps can be observed in the response in parts (a)-(c) with only smooth transitions in part (d).

In this chapter we have examined the resonance that occurs when the sum of two parametric excitation frequencies is near two times a natural frequency of the system, i.e., $\lambda_1 + \lambda_2 \approx 2\omega_s$. In each of the cases shown, all stability changes occur at bifurcation points or points of vertical tangency. At most, two nontrivial solutions are possible for the cases shown here. In the regions where two stable solutions are present, initial conditions determine which response will occur.
Figure 1. Frequency-response curves: In (a), \( \mu_q = 0.1, \ x_q = -1, \ e_q = -5, \ h_q = 3. \) In (b), \( \mu_q = 0.1, \ x_{qq} = 1, \ e_q = 5, \ h_q = 3. \)
Figure 2. Response amplitude as a function of excitation amplitude: In (a), \( \mu_a = 0, \alpha_{ee} = 1, \sigma = 0, k_3 = 3 \). In (b), \( \mu_a = 0, \alpha_{ee} = 1, \sigma = 10, k_3 = 3 \). In (c), \( \mu_a = 0, \alpha_{ee} = 1, \sigma = -10, k_3 = 3 \). In (d), \( \mu_a = 0.5, \alpha_{ee} = 1, \sigma = 5, k_1 = 2, k_2 = 3 \). In (e), \( \mu_a = 0.1, \alpha_{ee} = 1, \sigma = 0.1333, k_1 = 2, k_2 = 3 \).
Figure 3. Response amplitude as a function of \( r \): in (a), \( \mu_q = 0, \sigma_q = -1, \sigma = 0, Q = 4, \omega_q = 1 \). In (b), \( \mu_q = 1.25, \sigma_q = 1, \sigma = 0, Q = 1, \omega_q = 1 \). In (c), \( \mu_q = 1.25, \sigma_q = -1, \sigma = 0, Q = 1, \omega_q = 1 \).
Figure 4. Response amplitude as a function of excitation amplitude: In (a), $\mu_q = 0$, $\sigma_q = -1$, $\sigma = -2.1333$, $Q_i = 2$, $\lambda_a = 1.5$, $\lambda_t = 0.5$. In (b), $\mu_q = 0.5$, $\sigma_q = -1$, $\sigma = 0$, $Q_i = 1$, $\lambda_a = 0.5$, $\lambda_t = 1.5$. In (c), $\mu_q = 0.5$, $\sigma_q = -1$, $\sigma = 3$, $Q_i = 4$, $\lambda_a = 1.5$, $\lambda_t = 0.5$. In (d), $\mu_q = 0$, $\sigma_q = 1$, $\sigma = 0$, $Q_i = 2$, $\lambda_a = 1.5$, $\lambda_t = 0.5$. 
The Case When

\[ \lambda_s + \lambda_t \approx 2\omega_q \quad \text{and} \quad \omega_q \approx 3\omega_r \]

We now examine the effect of an internal resonance, \( \omega_q \approx 3\omega_r \), on the previous case where \( \lambda_s + \lambda_t \approx 2\omega_r \). Again we assume no other resonances exist. Detuning parameters \( \sigma_1 \) and \( \sigma_2 \) are defined by

\[ \lambda_s + \lambda_t = 2\omega_q + \varepsilon^2 \sigma_1 \] \hspace{1cm} (4.1)

and

\[ \omega_q = 3\omega_r + \varepsilon^2 \sigma_2. \] \hspace{1cm} (4.2)

Using (4.1), (4.2) and terms 1-4 and 16 of Appendix A, (2.24) can be written as
\( i(D_2A_n + \mu_n A_n) + e_n A_n - 4A_n \sum_{j=1}^{\infty} \alpha_{njn} A_j \bar{A}_j + \delta_{nq} h_q A_q \exp[i(\sigma_1 T_2 + \tau_s + \tau_f)] \)

\[ - 4\delta_{nq} W_{rqr} A_q^3 \exp(-i\sigma_2 T_2) - 4\delta_{nr} V_{qr} A_q A_r^2 \exp(i\sigma_2 T_2) = 0 \]

where \( \alpha_{nj} \) is given by (2.25), \( \epsilon_n \) by (2.26), \( h_q \) by (3.3),

\[ W_{rq} = \frac{1}{8\omega_q} \left[ -\Gamma_{rrq} + \sum_{j=1}^{\infty} \Lambda_{jrq}^{*} \Lambda_{jrr} R_{jrr} \right] \]

and

\[ V_{qr} = \frac{1}{8\omega_r} \left[ -\frac{1}{2} \Gamma_{qrr}^{*} + \sum_{j=1}^{\infty} (\Lambda_{jrr}^{*} \Lambda_{jqr} S_{jqr} + \Lambda_{jq}^{*} \Lambda_{jr}^{*} R_{jrr}) \right] \]

where \( \Gamma_{jk\alpha} \) is defined by (2.28) and \( \Lambda_{jk}^{*} \) by (2.10).

Substituting (3.4) into (4.3) results in (3.5) and (3.6) for \( n \neq q \) or \( r \), and

\[ a_q' + \mu_q a_q + h_q a_q \sin \gamma_1 + W_{rqa_r^3} \sin \gamma_2 = 0, \]

\[ a_q \left( \beta_q' - e_q + \sum_{j=1}^{\infty} \alpha_{jq} a_q^2 h_q \cos \gamma_1 \right) + W_{rqa_r^3} \cos \gamma_2 = 0. \]

\[ a_r' + \mu_r a_r - V_{qr} a_q a_r^2 \sin \gamma_2 = 0, \]

\[ a_r \left( \beta_r' - e_r + \sum_{j=1}^{\infty} \alpha_{jr} a_r^2 + V_{qr} a_q a_r \cos \gamma_2 \right) = 0 \]
for \( n = q \) and \( r \) where

\[
\gamma_1 = \sigma_1 T_2 + \tau_s + \tau_t - 2\beta_q
\]  

(4.10)

and

\[
\gamma_2 = \sigma_2 T_2 + \beta_q - 3\beta_r.
\]  

(4.11)

In the steady state, \( a_n' = 0, \gamma_1' = 0, \) and \( \gamma_2' = 0 \). As before, all \( a_n \) for \( n \neq q \) or \( r \) are zero, and the nontrivial steady-state solutions for \( a_q, a_r, \gamma_1, \) and \( \gamma_2 \) are governed by

\[
a_q(\mu_q + h_q \sin \gamma_1) + W_{rq} a_r^3 \sin \gamma_2 = 0,
\]  

(4.12)

\[
a_q(\xi_1 - h_q \cos \gamma_1) + W_{rq} a_r^3 \cos \gamma_2 = 0,
\]  

(4.13)

\[
a_r(\mu_r - V_{qr} a_q a_r \sin \gamma_2) = 0,
\]  

(4.14)

\[
a_r(\xi_2 + \xi_3 + V_{qr} a_q a_r \cos \gamma_2) = 0,
\]  

(4.15)

where

\[
\xi_1 = \frac{1}{2} \sigma_1 - e_q + \alpha_{qq} a_q^2 + \alpha_{rq} a_r^2,
\]  

(4.16)

\[
\xi_2 = \alpha_{qr} a_q^2 + \alpha_{rr} a_r^2,
\]  

(4.17)

\[
\xi_3 = \frac{1}{6} (\sigma_1 + 2\sigma_2) - e_r.
\]  

(4.18)

A trivial solution, \( a_q = a_r = 0 \), exists along with nontrivial solutions. One possibility is \( a_r = 0 \) and \( a_q \neq 0 \) which reduces to the case studied in the previous section where no
internal resonance was present. Nontrivial solutions with \(a_r \neq 0\) and \(a_q \neq 0\) also exist for certain parameter ranges.

The stability of the steady-state solutions is determined by the procedure described in the previous chapter. We let

\[
a_q = \tilde{a}_q + \delta a_q \tag{4.19}
\]

\[
a_r = \tilde{a}_r + \delta a_r \tag{4.20}
\]

\[
\gamma_1 = \tilde{\gamma}_1 + \delta \gamma_1 \tag{4.21}
\]

\[
\gamma_2 = \tilde{\gamma}_2 + \delta \gamma_2 \tag{4.22}
\]

Using (4.19)-(4.22) in (4.6)-(4.9) and taking advantage of (4.12)-(4.15) gives

\[
\begin{bmatrix}
\delta a_q' \\
\delta a_r' \\
\delta \gamma_1' \\
\delta \gamma_2'
\end{bmatrix} = [M]
\begin{bmatrix}
\delta a_q \\
\delta a_r \\
\delta \gamma_1 \\
\delta \gamma_2
\end{bmatrix} \tag{4.23}
\]

where the elements of \([M]\) are

\[
M_{11} = - (\mu_q + h_q \sin \tilde{\gamma}_1)
\]

\[
M_{12} = - 3W_{r q} \tilde{a}_r^2 \sin \tilde{\gamma}_2
\]

\[
M_{13} = - h_q \tilde{a}_q \cos \tilde{\gamma}_1
\]

\[
M_{14} = - W_{r q} \tilde{a}_r^3 \cos \tilde{\gamma}_2
\]
\[ M_{21} = V_{qr} \tilde{a}_r \sin \tilde{y}_2 \]

\[ M_{22} = 2V_{qr} \tilde{a}_q \tilde{a}_r \sin \tilde{y}_2 - \mu_r \]

\[ M_{23} = 0 \]

\[ M_{24} = V_{qr} \tilde{a}_q \tilde{a}_r \cos \tilde{y}_2 \]

\[ M_{31} = 2 \left( 2x_{aq} \tilde{a}_q - \frac{W_{rq} \tilde{a}_r^3}{\tilde{a}_q^2} \cos \tilde{y}_2 \right) \]

\[ M_{32} = 2 \left( \frac{3W_{rq} \tilde{a}_r^2}{\tilde{a}_q} \cos \tilde{y}_2 + 2x_{rq} \tilde{a}_r \right) \]

\[ M_{33} = 2h_q \sin \tilde{y}_1 \]

\[ M_{34} = - \frac{W_{rq} \tilde{a}_r^3}{\tilde{a}_q} \sin \tilde{y}_2 \]

\[ M_{41} = \frac{W_{rq} \tilde{a}_r^3}{\tilde{a}_q^2} \cos \tilde{y}_2 + 2(3x_{qr} - \alpha_{aq}) \tilde{a}_q + 3V_{qr} \tilde{a}_r \cos \tilde{y}_2 \]

\[ M_{42} = - \frac{3W_{rq} \tilde{a}_r^2}{\tilde{a}_q} \cos \tilde{y}_2 + 2(3\alpha_{rr} - \alpha_{rq}) \tilde{a}_r + 3V_{qr} \tilde{a}_q \cos \tilde{y}_2 \]

\[ M_{43} = - h_q \sin \tilde{y}_1 \]

\[ M_{44} = - 3V_{qr} \tilde{a}_q \tilde{a}_r \sin \tilde{y}_2 + \frac{W_{rq} \tilde{a}_r^3}{\tilde{a}_q} \sin \tilde{y}_2. \]
The solutions for $\delta a_q$, $\delta a_r$, $\delta \gamma_1$, and $\delta \gamma_2$ can be written in the form

\[ \delta a_q = \sum_{j=1}^{4} C_{1j} e^{\lambda_j T_z} \]  

\[ \delta a_r = \sum_{j=1}^{4} C_{2j} e^{\lambda_j T_z} \]  

\[ \delta \gamma_1 = \sum_{j=1}^{4} C_{3j} e^{\lambda_j T_z} \]  

\[ \delta \gamma_2 = \sum_{j=1}^{4} C_{4j} e^{\lambda_j T_z} \]

where the $\lambda_j$ are eigenvalues of $[M]$, and the $C_{ij}$ are constants of integration. As before, any eigenvalue with a positive real part indicates an unstable solution, and if the real parts of all the $\lambda_j$ are less than zero, the solution is stable.

Stability of the trivial solution may also be determined by substituting (4.19)-(4.22) into (4.6)-(4.9) and using (4.12)-(4.15); however, $\tilde{a}_s = \tilde{a}_r = 0$, and we can no longer divide by $\tilde{a}_q$ and $\tilde{a}$, to get (4.24). In this case, the governing equations for $\delta a_q$ and $\delta a_r$ are

\[ \delta a_q' = - (\mu_q + h_q \sin \tilde{\gamma}_1) \delta a_q \]  

\[ \delta a_r' = - \mu_r \delta a_r \]
where $\gamma_1$ is arbitrary for the trivial solution. Equation (4.29) is identical to (3.21) and indicates the same stability properties for $\tilde{a}_q = 0$ as in Chapter 3. When $(\mu_q + h_q \sin \tilde{\gamma}_1) = 0$, $a_q$ bifurcates from the trivial solution, and the trivial solution changes its stability character as in Chapter 3. It is assumed that $\mu > 0$, and the solution of (4.30) shows that $\delta a_q$ always decays and $\tilde{a}_r = 0$ is stable. No nontrivial $a_q$ solutions bifurcate from $a_r = 0$. For certain ranges of excitation amplitude, initial conditions near $a_q = a_r = 0$ will result in the growth of the $q$th mode while the $r$th mode remains small.

The alternative procedure described in Chapter 3 may also be used to determine the stability of the trivial solution. For $n = q$, (3.25) is obtained with $[T]$ again given by (3.26). The stability of the $A_q$ trivial solution is then the same as that of the trivial solution in Chapter 3. For $n = r$, let

$$A_r = \tilde{A}_r + \delta A_r,$$

(4.31)

substitute (4.31) into (4.3), let $\tilde{A}_r = 0$, and write

$$\delta A_r = (p_r + iq_r) \exp \left[ i \frac{1}{2} (\sigma_i T_2 + \tau_s + \tau_i) \right].$$

(4.32)

Using (4.32) in (4.3) and separating real and imaginary parts results in

$$\begin{bmatrix} p' \cr q' \cr \end{bmatrix} = [T_r] \begin{bmatrix} p_r \cr q_r \cr \end{bmatrix}$$

(4.33)

where

$$[T_r] = \begin{bmatrix} -\mu_r & -\epsilon_r \\ \epsilon_r & -\mu_r \end{bmatrix}.$$

(4.34)
The eigenvalues of \([ T_r, T_r]\) are

\[
\lambda_1, \lambda_2 = -\mu_r \pm i\epsilon_r
\]  

(4.35)

indicating that the \(A_r\) trivial solution is always stable, because the real parts of the eigenvalues of \([ T_r, T_r]\) are always negative. As expected, the two methods for determining the stability of the trivial solution produce the same results and are equivalent.

When \(a_q \neq 0, a_r \neq 0,\) and \(\sin \gamma_2 \neq 0,\) we can solve (4.14) to obtain

\[
a_q = \frac{\mu_r}{a_r V_{qr} \sin \gamma_2}.
\]  

(4.36)

Using (4.36) in (4.15) leads to

\[
a_r^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}
\]  

(4.37)

where

\[
A = \alpha_{rr} V_{qr}^2 \sin^2 \gamma_2, \quad B = (\mu_r \cot \gamma_2 + \xi_3) V_{qr}^2 \sin^2 \gamma_2,
\]

and

\[
C = \alpha_{qr} \mu_r.
\]  

(4.38)

Solving (4.12) for \(\sin \gamma,\) and (4.13) for \(\cos \gamma\), and then squaring and adding the results eliminates \(\gamma,\) and yields

\[
a_q^2 h_q^2 = (a_q \mu_q + W_{rq} a_r^3 \sin \gamma_2)^2 + (a_q \xi_4 + W_{rq} a_r^3 \cos \gamma_2)^2.
\]  

(4.39)
Again, we will consider the case (3.28) and assume all other excitation amplitudes are zero. Using (3.29) in (4.39) and solving for \( Q^2 \) results in

\[
Q^2 = -\frac{E \pm \sqrt{E^2 - 4DF}}{2D}
\]

(4.40)

where

\[
D = k_2^2 - k_1^2, \quad E = -2k_1G - \frac{2W_{rq}a_r^3k_1 \cos \gamma_2}{a_q},
\]

(4.41)

and

\[
F = -\mu^2 - G^2 - \frac{W_{rq}a_r^6}{a_q^2} - \frac{2W_{rq}a_r^3}{a_q} (\mu \sin \gamma_2 + G \cos \gamma_2).
\]

In (4.41),

\[
G = \frac{1}{2} \sigma_1 + \alpha_{qq}a_q^2 + \alpha_{rr}a_r^2.
\]

(4.42)

To determine the behavior of the response amplitudes, \( a_q \) and \( a_r \), as functions of \( Q \), we specify \( \sigma_1 \) and \( \sigma_2 \), and vary \( \gamma_2 \) from 0 to \( 2\pi \). For each value of \( \gamma_2 \), \( a_r \) is found from (4.37), \( a_q \) from (4.36), and \( Q \) from (4.40). Frequency-response curves can be obtained by fixing \( Q \) and \( \sigma_2 \), varying \( \sigma_1 \), and solving (4.37), (4.36), and (4.40) numerically for \( \gamma_2 \), \( a_r \), and \( a_q \).

Numerical results are presented in Figures 5 and 6, where \( a_q \) and \( a_r \) are shown as functions of \( \sigma_1 \) and \( Q \), respectively. In both cases, there are a trivial solution \( (a_r = a_q = 0) \), nontrivial solutions where \( a_q \neq 0 \) and \( a_r = 0 \), and finally, nontrivial
solutions where both \( a_q \) and \( a_r \) are nonzero (\( a_q \neq 0, a_r \neq 0 \)). For the case when \( a_q \neq 0 \) and \( a_r = 0 \), \( a_q \) bifurcates from the trivial solution, resulting in regions of instability of the \( a_q \) trivial solution; however, no nontrivial solutions bifurcate from the trivial solution \( a_r = 0 \), and small perturbations of \( a_r \) from \( a_r = 0 \) do not grow large.

Consider the solutions with nonzero \( a_r \) and \( a_q \). In this case, no solutions bifurcate from the trivial solution, and in Figures 5(a) and (b), there are two branches. One is unstable, while the other has a stable portion for \( \sigma \) sufficiently negative. In Figures 5(c) and (d) the response is shown for a larger value of \( \Omega \), and a third branch appears on the right which has a stable portion. As \( \Omega \) is increased further, the right branch seems to meet the middle one and the gap between them disappears. As shown in Figures 5(e) and (f), there are three solutions again, but now two are unstable, while the third contains a region of instability in the center and stable solutions on each end. For clarity, Figure 5 is repeated in Figure 6 with unstable solutions removed and corresponding solutions marked.

For the cases of nonzero \( a_r \) and \( a_q \) which are presented as functions of \( \Omega \) in Figure 7, the solution has only one branch. Stable solutions exist in each case for a limited range of \( \Omega \). One end of the stable portion is located at a point of vertical tangency.

The effect of an internal resonance \( \omega_q = 3\omega \), on the resonance \( \lambda_s + \lambda_r \approx 2\omega_q \) was illustrated in the previous examples. For a fixed set of parameters, including \( \sigma_1, \sigma_2 \) and \( \Omega \), up to four new steady-state solutions may exist. However, in the cases shown in Figures 5-7, not more than one of the new solutions is stable. Initial conditions determine which solution will occur when more than one stable solution is possible for a set of parameters. Also, the boundary of the stable portion of a solution branch is not always at a point of vertical tangency as it is when the internal resonance is not present or active. Moreover, numerical studies have revealed that at times, when
the initial conditions are near one of the unstable branches, the motion never reaches a steady state. Though stable steady states exist for the set of parameters, the modal response amplitudes may oscillate in a limit cycle around the unstable branch.
Figure 5. Frequency-response curves: In (a)-(f), $\mu_q = \mu_r = 0.1$, $V_{fr} = W_{fr} = \gamma_{eq} = \gamma_{r} = \gamma_{m} = 1$, $a_r = -1$, $\sigma_2 = 0$, $\lambda_q = 1.5$, $\lambda_r = 0.5$. In (a) and (b), $Q = 0.388$. In (c) and (d), $Q = 1.494$. In (e) and (f), $Q = 2.028$. 
Figure 6. Frequency-response curves: Repeat of Figure 5 with unstable solutions removed and corresponding solutions marked.
Figure 7. Response amplitudes as functions of excitation amplitude: In (a)-(f), $\mu_q = \mu_r = 0.1$, $\nu_q = \nu_r = \sigma_q = \sigma_r = 1$, $x_r = -1$, $\sigma_1 = 0$, $\lambda_q = 1.5$, $\lambda_r = 0.5$. In (a) and (b), $\sigma_1 = -25$. In (c) and (d), $\sigma_1 = 0$. In (e) and (f), $\sigma_1 = 25$. 

Chapter 4
Chapter 5

The Case When

\[ \lambda_s + \lambda_t \approx 2\omega_q \quad \text{and} \quad 3\omega_q \approx \omega_r \]

In this section, we suppose that \( \lambda_s + \lambda_t \approx 2\omega_q \) and \( 3\omega_q \approx \omega_r \), and assume no other resonances exist. Detuning parameters, \( \sigma_1 \) and \( \sigma_2 \), are defined by

\[ \lambda_s + \lambda_t = 2\omega_q + \varepsilon^2 \sigma_1 \quad (5.1) \]

\[ 3\omega_q = \omega_r + \varepsilon^2 \sigma_2. \quad (5.2) \]

As a result, we also have

\[ \lambda_s + \lambda_t = \omega_r - \omega_q + \varepsilon^2 (\sigma_1 + \sigma_2). \quad (5.3) \]

To eliminate secular terms in the solution for \( u_\phi \), we use (5.1)-(5.3) and terms 1-4, 15, and 16 of Appendix A, and see that (2.24) becomes
\[ i(D_{2}A_{n} + \mu_{n}A_{n}) + e_{n}A_{n} - 4A_{n}\sum_{j=1}^{\infty} \alpha_{j}A_{j} + \delta_{nq}h_{q}A_{q} \exp\{i(\sigma_{1}T_{2} + \tau_{s} + \tau_{t})\} \\
- 4\delta_{nq}V_{rq}A_{q}^{2}A_{r} \exp(-i\sigma_{2}T_{2}) - 4\delta_{nr}W_{qr}A_{r}^{3} \exp(i\sigma_{2}T_{2}) \\
- \delta_{nq}g_{tr}A_{r} \exp\{i[(\sigma_{1} + \sigma_{2})T_{2} + \tau_{s} + \tau_{t}]\} \\
- \delta_{nr}p_{qr}A_{q} \exp\{i[(\sigma_{1} + \sigma_{2})T_{2} + \tau_{s} + \tau_{t}]\} = 0 \tag{5.4} \]

where \(\alpha_{jn}\) can be found from (2.25), \(e_{n}\) from (2.26), \(h_{q}\) from (3.3), \(W_{qr}\) from (4.4), \(V_{rq}\) from (4.5),

\[ g_{rq} = \frac{1}{2\omega_{q}} \sum_{j=1}^{\infty} (R_{1j}^{s}Q_{q1j1}^{s} + R_{j1}^{s}Q_{q1j1}^{s}) \tag{5.5} \]

and

\[ p_{qr} = \frac{1}{2\omega_{r}} \sum_{j=1}^{\infty} (S_{1j}^{s}Q_{r1j1}^{s} + S_{j1}^{s}Q_{r1j1}^{s}) \tag{5.6} \]

Using the form (3.4) in (5.4), we obtain (3.5) and (3.6) for \(n \neq q \) or \(r\), and

\[ a_{q}' + \mu_{q}a_{q} + h_{q}a_{q} \sin \gamma_{1} + V_{rq}a_{q}^{2}a_{r} \sin \gamma_{2} + g_{rq}a_{r} \sin(\gamma_{1} + \gamma_{2}) = 0 \tag{5.7} \]

\[ a_{q}\left(\beta_{q}' - e_{q} + \sum_{j=1}^{\infty} \alpha_{jq}a_{j}^{2} - h_{q} \cos \gamma_{1} + V_{rq}a_{q}a_{r} \cos \gamma_{2}\right) + g_{rq}a_{r} \cos(\gamma_{1} + \gamma_{2}) = 0 \tag{5.8} \]

\[ a_{r}' + \mu_{r}a_{r} - W_{qr}a_{q}^{3} \sin \gamma_{2} - p_{qr}a_{q} \sin(\gamma_{1} + \gamma_{2}) = 0 \tag{5.9} \]
for \( n = q \) and \( r \) where

\[
\gamma_1 = \sigma_1 T_2 + \tau_s + \tau_t - 2\beta_q
\]  \hspace{1cm} (5.11)

\[
\gamma_2 = \sigma_2 T_2 + 3\beta_q - \beta_r.
\]  \hspace{1cm} (5.12)

In the steady state \( a_n' = 0, \gamma_n' = 0, \) and \( \gamma_n' = 0. \) For \( n \neq q \) or \( r, \) \( a_n = 0 \) is the only steady-state solution, and for \( n = q \) and \( r, \) \( a_n = 0 \) is governed by

\[
a_q (\mu_q + h_q \sin \gamma_1 + V_{rq} a_q a_r \sin \gamma_2) + g_{rq} a_r \sin (\gamma_1 + \gamma_2) = 0
\]  \hspace{1cm} (5.13)

\[
a_q (\zeta_1 - h_q \cos \gamma_1 + V_{rqr} a_q a_r \cos \gamma_2) + g_{rq} a_r \cos (\gamma_1 + \gamma_2) = 0
\]  \hspace{1cm} (5.14)

\[
\mu_r a_r - W_{qr} a_q^3 \sin \gamma_2 - p_{qr} a_q \sin (\gamma_1 + \gamma_2) = 0
\]  \hspace{1cm} (5.15)

\[
a_r \zeta_2 + W_{qr} a_q^3 \cos \gamma_2 + p_{qr} a_q \cos (\gamma_1 + \gamma_2) = 0
\]  \hspace{1cm} (5.16)

where

\[
\zeta_1 = \frac{1}{2} \sigma_1 - e_q + \alpha_{qq} a_q^2 + \alpha_{rq} a_r^2
\]  \hspace{1cm} (5.17)

\[
\zeta_2 = \frac{3}{2} \sigma_1 + \sigma_2 - e_r + \alpha_{qr} a_q^2 + \alpha_{rr} a_r^2.
\]  \hspace{1cm} (5.18)

We consider the case (3.28), and see that (5.5) and (5.6) give

\[
g_{rq} = 0, \quad p_{qr} = 0.
\]  \hspace{1cm} (5.19)
A trivial solution, \( a_r = a_q = 0 \), exists. Nontrivial solutions with both \( a_r \neq 0 \) and \( a_r \neq 0 \) are also possible, and these solutions are governed by

\[
\mu_q + h_q \sin \gamma_1 + V_{rq}a_qa_r \sin \gamma_2 = 0 \tag{5.20}
\]

\[
\zeta_1 - h_q \cos \gamma_1 + V_{rq}a_qa_r \cos \gamma_2 = 0 \tag{5.21}
\]

\[
\mu_r a_r - W_{qr}a_q^3 \sin \gamma_2 = 0 \tag{5.22}
\]

\[
a_r \zeta_2 + W_{qr}a_q^3 \cos \gamma_2 = 0. \tag{5.23}
\]

Stability properties of the nontrivial steady-state solutions are determined by substituting (4.19)-(4.22) into (5.7)-(5.10) and using (5.13)-(5.16) to obtain (4.23) with

\[
M_{11} = -(\mu_q + h_q \sin \tilde{\gamma}_1 + 2V_{rq}\tilde{a}_q\tilde{a}_r \sin \tilde{\gamma}_2)
\]

\[
M_{12} = -(V_{rq}\tilde{a}_q^2 \sin \tilde{\gamma}_2)
\]

\[
M_{13} = -h_q\tilde{a}_q \cos \tilde{\gamma}_1
\]

\[
M_{14} = -V_{rq}\tilde{a}_q^2 \tilde{a}_r \cos \tilde{\gamma}_2
\]

\[
M_{21} = 3W_{qr}\tilde{a}_q^2 \sin \tilde{\gamma}_2
\]

\[
M_{22} = -\mu_r
\]

\[
M_{23} = 0
\]

\[
M_{24} = W_{qr}\tilde{a}_q^3 \cos \tilde{\gamma}_2 \tag{5.24}
\]
Solutions for \( t_5a, t_5r, t_5y \) and \( t_5Y^2 \) can be written as (4.25)-(4.28). Once again, any \( \lambda_j \) with a positive real part indicates a perturbation which grows as \( t \to \infty \) and an unstable solution. All eigenvalues with real parts less than zero indicate a decaying perturbation and a stable solution.

Using (4.19)-(4.22) in (5.7)-(5.10), taking advantage of (5.13)-(5.16), and letting \( \tilde{a}_q = \tilde{a}_r = 0 \) results in (4.29) and (4.30) as the equations governing perturbations from the trivial solution. As in the previous two chapters, the stability of the \( a_q \) trivial solution changes when \( (\mu + h_q \sin \tilde{\gamma}_1) = 0 \); however, the stable nontrivial solution for \( a_q \) which bifurcates from this point no longer corresponds to a stable trivial solution.

\[
M_{31} = 2(2x_{qq}\tilde{a}_q + V_{rq}\tilde{a}_r \cos \tilde{\gamma}_2)
\]
\[
M_{32} = 2(2x_{rr}\tilde{a}_r + V_{rq}\tilde{a}_q \cos \tilde{\gamma}_2)
\]
\[
M_{33} = 2h_q \sin \tilde{\gamma}_1
\]
\[
M_{34} = -2V_{rq}\tilde{a}_q\tilde{a}_r \sin \tilde{\gamma}_2
\]
\[
M_{41} = -3(2x_{qq}\tilde{a}_q + V_{rq}\tilde{a}_r \cos \tilde{\gamma}_2) + 2x_{rr}\tilde{a}_q + \frac{3W_{rqrq}a_q^2}{a_r} \cos \tilde{\gamma}_2
\]
\[
M_{42} = -3(2x_{rr}\tilde{a}_r + V_{rq}\tilde{a}_q \cos \tilde{\gamma}_2) + 3x_{rr}\tilde{a}_r + \frac{3}{2} \frac{\sigma_1 + \sigma_2}{a_q} \tilde{a}_r + \frac{\alpha_{qr}\tilde{a}_q}{a_r}
\]
\[
M_{43} = -3h_q \sin \tilde{\gamma}_1
\]
\[
M_{44} = 3V_{rq}\tilde{a}_q\tilde{a}_r \sin \tilde{\gamma}_2 - \frac{W_{rqrq}a_q^3}{a_r} \sin \tilde{\gamma}_2.
\]
for $a_r$. A trivial $a_q$ indicates a trivial $a_r$ and vice-versa. Hence, when a perturbation from $\tilde{a}_q = 0$ can grow, we say that the solution $(\tilde{a}_q, \tilde{a}_r) = (0, 0)$ is unstable, and both trivial solutions are then represented by dashed lines in the figures. Equivalently, (3.25) with (3.26) is obtained for $n = q$, and (4.33) with (4.34) is obtained for $n = r$ when the alternative procedure is used to determine the stability properties of the trivial solution. As in the previous two chapters, the two methods for determining the stability of the trivial solution agree completely.

Solving (5.22) for $a_r$ results in

$$a_r = \frac{W_{qr}a_q^3 \sin \gamma_2}{\mu_r}. \quad (5.25)$$

Using (5.25) in (5.23) gives a cubic equation governing $a_r^2$:

$$(a_q^2)^3 + \frac{\alpha_{qr} \mu_r^2}{\alpha_{rr} W_{qr}^2 \sin^2 \gamma_2} (a_q^2)^2 + \frac{\mu_r^2}{\alpha_{rr} W_{qr}^2 \sin^2 \gamma_2} \left( \frac{3}{2} \sigma_1 + \sigma_2 + \frac{\mu_r \cos \gamma_2}{\sin \gamma_2} \right) = 0. \quad (5.26)$$

Solving (5.20) for $\sin \gamma_1$, (5.21) for $\cos \gamma_1$, and squaring and adding the results gives

$$h_q^2 = (\mu_q + V_q a_q a_r \sin \gamma_2)^2 + (\xi_1 + V_{rq} a_q a_r \cos \gamma_2)^2. \quad (5.27)$$

Making use of (3.21) and (3.22) in (5.27) and solving for $Q^2$ yields

$$Q^2 = \frac{-1 \pm \sqrt{1^2 - 4HJ}}{2H} \quad (5.28)$$

where
\[ H = k_2^2 - k_1^2, \quad I = -2k_1(V_{rq}a_qa_r \cos \gamma_2 + K) \]

and

\[ J = -\left[ \mu_q^2 + V_{rq}a_qa_r(2\mu_q \sin \gamma_2 + V_{rq}a_qa_r + 2K \cos \gamma_2) + K^2 \right]. \]

In (5.29),

\[ K = \frac{1}{2} \sigma_1 + \alpha_{qq}a_q^2 + \alpha_{rr}a_r^2. \]  

Attempts to repeat the solution procedure of the previous section for frequency-response curves (fixing \( Q \), incrementing \( \sigma_1 \), and solving (5.25), (5.26), and (5.28) numerically for the corresponding \( \gamma_2 \)) failed due to the nature of the problem. The solutions are very sensitive for \( \gamma_2 \) near 0 and \( \pi \). For this reason the iterative procedure which attempted to find \( \gamma_2 \) did not converge on all possible solutions. Therefore, an alternative procedure was used, and \( \gamma_2 \) was removed from the problem.

By squaring and adding (5.22) and (5.23), \( \gamma_2 \) is eliminated, and solving the result for \( \sigma_1 \) gives

\[ \sigma_1 = \frac{2}{3} \left( -\sigma_2 - \alpha_{qr}a_q^2 - \alpha_{rr}a_r^2 \pm \sqrt{\frac{W_{qr}a_q^6}{a_r^2} - \mu_r^2} \right). \]  

Solving (5.20) for \( \sin \gamma_1 \), (5.21) for \( \cos \gamma_1 \), squaring and adding the results, and using values for \( \sin \gamma_2 \) and \( \cos \gamma_2 \) from (5.22) and (5.23) respectively gives

\[ -k_2^2Q^2 + \left( \mu_q + \frac{V_{rq}a_r^2\mu_r}{W_{qr}a_q^2} \right)^2 + \left( \xi_1 - \frac{\xi_2V_{rq}a_r^2}{W_{qr}a_q^2} \right)^2 = 0 \]  

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where $\xi_1$ is defined by (5.17) and $\xi_2$ by (5.18). Frequency-response curves can now be obtained by fixing $\Omega$ and $\sigma_2$, varying $a_n$, substituting (5.31) into (5.32), and solving the resulting equation numerically for $a_q$.

In Figure 8 the variation of the response amplitudes, $a_q$ and $a_r$, as functions of a detuning parameter, $\sigma_\nu$, are shown for the same values of $\Omega$ as in the preceding section. For the lowest value of $\Omega$, shown in Figures 8(a) and (b), two solutions are possible for most negative values of $\sigma_\nu$; however, only the upper solution is stable. Small irregularities seen in Figures 8(a) and (b) grow with $\Omega$ in Figures 8(c)-(f). Once again, no more than two branches of the solutions are present for any given value of $\sigma_\nu$, and the lower solution is always unstable. The upper solution is no longer always stable, though. Stable solutions exist for $\sigma_\nu$ very negative, and the stability properties alternate for values of $\sigma_\nu$ sufficiently large, ending in a stable region near the bifurcation point.

The variation of the response amplitudes, $a_q$ and $a_r$, as a function of $\Omega$ was determined by specifying $\sigma_1$ and $\sigma_2$ and varying $\gamma_2$ from 0 to $2\pi$. For each value of $\gamma_2$, $a_q$ is found from (5.26), $a_r$ from (5.25) and $Q$ from (5.28).

In Figure 9, the behaviors of $a_q$ and $a_r$ with $Q$ are shown. There is a trivial solution, $a_q = a_r = 0$, and nontrivial solutions with both $a_q$ and $a_r$ nonzero. For a given $\sigma_1$ and $\sigma_2$, all nontrivial response amplitudes bifurcate from the trivial solution at the same value of $Q$.

Consider solutions with both $a_q$ and $a_r$ nontrivial. As seen in Figure 9, the solution has only one branch. Stable solutions exist for negative $\sigma_1$ and $Q$ sufficiently large, as seen in Figures 9(a) and (b). Figures 9(c) and (d) show that as $\sigma_1$ increases, there are two stable regions with an unstable portion in between. Finally, as $\sigma_1$ becomes more positive, stable solutions occur for a limited range of $Q$ near the bifurcation point, as shown in Figures 9(e) and (f).
Figure 10 examines the region in Figures 9(c) and (d) where no stable steady-state solutions exist. The variation of $a_r$ with $a_q$ is periodic for the parameters studied, indicating limit-cycle behavior in this region. Limit cycles for values of $Q$ just inside the unstable area of the nontrivial solution are shown in Figures 10(a) and (c), while a limit cycle for a value of $Q$ near the center of the region where no stable steady states exist can be seen in part (b).

The influence of an internal resonance, $\omega_2 \approx 3\omega_1$, on the parametric resonance, $\lambda_r + \lambda_2 \approx 2\omega_1$, has been examined in this chapter. For each case studied, there are no more than two nontrivial solutions possible, and the lower of these solutions is always unstable. In the previous chapter where $\lambda_r + \lambda_2 \approx 2\omega_1$ and $\omega_2 \approx 3\omega_1$, a stable steady state exists for any given set of parameters and the cases shown; however, in the present chapter, there are regions where no stable steady-state solutions exist. In these regions, limit cycle behavior of the response amplitude has been observed in numerical studies. As before, initial conditions determine which response will occur when more than one stable steady state is present for a fixed set of parameters.
Figure 8. Frequency-response curves: In (a)-(f), $\mu_1 = \mu_2 = 0.1$, $V_0 = W_0 = \alpha_\sigma = \alpha_\tau = \alpha_{\phi} = 1$, $\sigma_\mu = -1$, $\sigma_\nu = 0$, $\lambda_\sigma = 1.5$, $\lambda_\tau = 0.5$. In (a) and (b), $Q = 0.398$. In (c) and (d), $Q = 1.484$. In (e) and (f), $Q = 2.028$. 

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Figure 9. Response amplitudes as functions of excitation amplitude: In (a)-(f), \( \mu_q = \mu_r = 0.1 \), \( V_{eq} = W_{eq} = \alpha_{eq} = \alpha_{eq} = 1 \), \( x_r = 1 \), \( \sigma_q = 0 \). \( \lambda_s = 1.5 \), \( \lambda_t = 0.5 \). In (a) and (b), \( \sigma_s = -25 \). In (c) and (d), \( \sigma_s = 0 \). In (e) and (f), \( \sigma_s = 25 \).
Figure 10. Limit-cycle behavior: Plots of $a_r$ as a function of $a_q$ in a region where no stable steady-state solutions exist. In (a)-(c), $\sigma_1 = 0$. In (a), $Q = 0.904$. In (b), $Q = 1.100$. In (c), $Q = 1.385$. 

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Bibliography


Appendix A

Coefficients and Arguments in Equation (2.20)

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<td>( A_j A_k A_l )</td>
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<tr>
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<tr>
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</tbody>
</table>
Jeanette J. "Tina" Gentry was born November 11, 1964 in Nicholas County, West Virginia. After graduating from Charleston High School in Charleston, West Virginia she attended Virginia Polytechnic Institute and State University, where she acquired her Bachelor of Science degree in Engineering Science and Mechanics in 1987. She then began working for a Master's Degree in Engineering Mechanics.