A UNIVERSAL TIME OF FLIGHT EQUATION FOR SPACE MECHANICS

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(ABSTRACT)

A universal time of flight equation for any orbit is developed as a function of the initial and final radius, the change in true anomaly and the initial flight path angle. Lambert's theorem, a new corollary to this theorem, a trigonometric variable substitution and a continuing fraction expression are used in this development. The resulting equation is not explicitly dependent upon eccentricity and is determinate for $-2\pi <$ (change in true anomaly) $< 2\pi$. A method to make the continuing fraction converge rapidly is evaluated using a top down algorithm. Finally, the accuracy of the universal time of flight equation is examined for a representative set of orbits including near parabolic and near rectilinear orbits.

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SYMBOLS

B - decimal accuracy of the normalized difference for the universal time of flight equation

B_1, B_2, B_3 - maximum, average, minimum value of B

CC - convergence criterion for the continuing fraction

D - decimal accuracy of the convergence criterion

D_1', D_2' - parabolic eccentric anomalies at P_1 and P_2 of the original orbit

E - eccentric anomaly, eccentric anomaly at P_2 of the transformed orbit

E_0', E_1', E_2' - elliptic eccentric anomalies at P_0, P_1 and P_2 of the original orbit

F - decimal accuracy of the normalized difference for the continuing fraction substitute, inverse tangent and inverse hyperbolic tangent functions

F_1', F_2' - hyperbolic eccentric anomalies at P_1 and P_2 of the original orbit

G - continuing fraction expressions of the universal time of flight equation

N - multiple for π, an index for $x^2G(x^2)$, $\zeta^2\xi(\zeta^2)$ and $\omega^2\xi(\omega^2)$, the number of terms required to converge the continuing fraction

N_{max} - largest value of N for a parametric study case

ND_{CF} - normalized difference for the continuing fraction substitute

ND_{TE} - normalized difference for the universal time of flight equation

P_0, P_1, P_2 - the normal point, the initial point and the final point of the original or transformed orbit

TCF - type of continuing fraction used

V'_0, V'_1 - total velocity at P_0 and P_1 of the original orbit

$\bar{V}'_0, \bar{V}'_1$ - total velocity vector at P_0 and P_1 of the original orbit

W $= \left(z^2 + x^2\right)\left(1 + x^2\right)$

a - semi-major axis of an orbit, semi-major axis of the original or transformed orbit

c - half the distance between the foci of an orbit

c_h - chord between P_1 and P_2

e - eccentricity of an orbit, eccentricity of the transformed orbit

$e', \bar{e}'$ - eccentricity and eccentricity vector of the original orbit

f - true anomaly in the transformed orbit

h' - angular momentum of the original orbit

$\bar{h}'$ - angular momentum vector of the original orbit

p' - parameter, or semi-latus rectum, of the original orbit

p_p - "parameter of the parabolic"

r - radius from the occupied focus to a point on the orbit

r_o, r_1, r_2 - radius to P_o, P_1 and P_2

$\dot{r}, \dot{r}_o, \dot{r}_1$ - time derivatives of r, r_o and r_1

$\bar{r}_o, \bar{r}_1, \bar{r}_2$ - radius vectors to P_o, P_1 and P_2

$x^2 \quad = \tan^2\left(\frac{E}{2}\right)$ - variable substitution

y - argument of the continuing fraction expressions for the inverse tangent or inverse hyperbolic tangent

$z^2 \quad = \tan^2\left(\frac{f}{2}\right)$ - variable substitution

Δt - time of flight from P_1 to P_2 in the original or transformed orbit

Δt_k - time of flight from P_1 to P_2 using one of the three classical time of flight equations

$\Delta \nu$ - true anomaly increment in the original orbit

$\Delta \eta$ - change in true anomaly increment in the original orbit

γ_o', γ_1' - compliment of the flight path angle at P_o and P_1 of the original orbit

$\delta \quad = \nu_o - \nu_1$

$\zeta^2 \quad = \dfrac{x^2}{\left(\sqrt{x^2 + 1} + 1\right)^2}$ - variable substitution

$\eta \quad = \nu_2 - \nu_1$ - change in true anomaly in the original orbit

μ - gravitational parameter (=1 in canonical units)

ν - true anomaly

ν_o, ν_1, ν_2 - true anomaly at P_o, P_1 and P_2 in the original orbit

ε' - specific mechanical energy of the original orbit

τ - time of periapsis passage

ϕ_1 - flight path angle at P_1 of the original orbit

ω^2 $= \dfrac{\zeta^2}{\left(\sqrt{\zeta^2 + 1} + 1\right)^2}$ - variable substitution

FIGURES

<u>TABLES</u>

CHAPTER 1 INTRODUCTION

Under suitable assumptions the motions of satellites in space are described by the equations of motion associated with the so-called two-body problem. The solutions to these equations are obtained by replacing time with the true anomaly as the independent variable. The resulting solution is called the orbit equation since it describes the position as a function of the angle or equivalently, the orbit in space. Depending upon the energy of the orbit, it can be either an ellipse, parabola or hyperbola. However, in order to carry out typical orbital operations such as intercept, rendezvous and navigation, knowing the position in orbit at a given past, present or future time is necessary.

In order to introduce time into the problem a relationship between time and true anomaly must be determined. The classical solution to this problem consists of three different equations, one for each type of orbit. These equations determine time as a function of the independent variable, the true anomaly. Consequently the problem of determining the true anomaly given the time turns out to be transcendental and must be solved by iteration. In any case, the classical time of flight equations for the elliptic and hyperbolic orbits become very sensitive at values of eccentricity near one.

Additional concerns may arise if these time equations and trajectory equations are coupled with some non-linear programming algorithms for the purposes of reducing fuel consumption or time of flight operations. Algorithms which use gradient techniques require smooth derivatives in order to be successful. For operations which require near parabolic orbits, the switching among three types of equations when calculating the time can lead to possible numerical discontinuities in time and its derivatives because of the inaccuracies of the classical time of flight equations for near parabolic orbits ($e \approx 1.0$).

In order to determine the position in any orbit at a given past, present or future time and to overcome any numerical problems a single universal time of flight equation, as a function of the change in true anomaly, will be derived. As expected this equation turns out to be transcendental in the change in true anomaly. Therefore with this universal time of flight equation and the starting true anomaly the position in any orbit as a function of time can be found using one equation. The use of one equation eliminates discontinuities in time and its derivative which arise from switching among the three classical time of flight equations.

The universal time of flight equation between two points in any orbit will be developed using Lambert's theorem, a new corollary to this theorem, a trigonometric substitution and a continuing fraction expression. In the development of this equation an inverse tangent term will appear. For this equation to be continuous the argument of this inverse tangent term must be bounded by $\pm \frac{\pi}{2}$. Lambert's theorem and a trigonometric substitution will be used to accomplish this. Then a continuing fraction expression will be substituted for the inverse tangent term resulting in one time of flight equation for all three orbits. Finally, the new corollary to Lambert's theorem will be used to make this universal time of flight equation a function of parameters of two arbitrary points in any orbit. The motivation for this development is provided by an AIAA paper entitled "A New Transformation Invariant in the Orbital Boundary-Value Problem", written by Richard H. Battin, Thomas J. Fill and Stanley W. Shepperd (Ref. 1).

The resulting universal time of flight equation will be shown to be determinate for the principle range of the orbit parameters. The accuracy of the continuing fraction will be examined as a function of the convergence criterion and as a function of a method used to make the continuing fraction converge faster. Finally, the accuracy of the universal time of flight equation will be examined for near parabolic and near rectilinear orbits.

CHAPTER 2 THE UNIVERSAL TIME OF FLIGHT EQUATION

To begin the development of the universal time of flight equation, Lambert's theorem is used to transform an orbit containing two arbitrary, fixed points P_1 and P_2 to an orbit where P_1 and P_2 are symmetric about the semi-major axis. According to Lambert's theorem if the sum of the two radius vectors, the semi-major axis and the chord between the two fixed points remain unchanged in the transformation then the time of flight between P_1 and P_2 of both the original orbit and the transformed orbit are the same. If the focus of the original orbit is moved on an ellipse such that the sum of the radius vectors is a constant the focus of the transformed orbit is defined by the point where the radius vectors of the transformed orbit are equal. This results in both of the radius vectors of the transformed orbit being equal to half the sum of the radius vectors of the original orbit. Refer to Fig. 1 and Fig. 2.

Because of the symmetry of P_1 and P_2 about the semi-major axis of the transformed orbit the times from P_1 to P_0 and P_0 to P_2 are the same. Kepler's equation for the elliptical orbit is used to express the time of flight from P_1 to P_2, Δt, in the transformed orbit.

$$\Delta t = 2\sqrt{\frac{a^3}{\mu}}\left(E - e \sin E\right) \tag{1}$$

Where E is the eccentric anomaly of P_2 in the transformed orbit.

First the eccentricity of the transformed orbit e, in eq.(1), is expressed as a function of r_1, r_2, η and r_0 where r_0 is the radius to the normal point. The normal point, P_0, is the point on an orbit where the velocity vector is parallel to the chord between P_1 and P_2, refer to Fig. 3.

A fundamental property of the normal point is that its eccentric anomaly is the arithmetic mean of the eccentric anomalies of the two termini, P_1 and P_2, which define the normal point of the original orbit Fig. 1, $E_0' = (E_2' + E_1')/2$. The new corollary to Lambert's theorem (Ref. 1) states that the radius to the normal point is invariant in the orbit transformation. Therefore r_0 for both the original and transformed orbit is expressed as follows;

$$r_0 = a\left[1 - e'\cos\left(\frac{E_2' + E_1'}{2}\right)\right] \tag{2}$$

where e' is the eccentricity of the original orbit. The sum of the two radius vectors is also expressed as a function of a, e', E_1' and E_2'.

$$r_1 + r_2 = a\left[2 - e'\left(\cos E_2' + \cos E_1'\right)\right]$$

Using the function-sum trigonometric relation and substituting $\cos\phi$ for $e'\cos\left(\frac{E_2' + E_1'}{2}\right)$ and ψ for $\frac{E_2' - E_1'}{2}$ the sum of the radius vectors of the original orbit becomes

$$r_1 + r_2 = 2\,a\left(1 - \cos\phi\cos\psi\right) \tag{3}$$

Using the definition of eccentricity

$$e = \frac{c}{a} = \frac{a - r_0}{a} ,$$

where c is half the distance between the foci, an expression for the product of the eccentricity of the transformed orbit and the semi-major axis is obtained

$$ea = a - r_0 \tag{4}$$

Subtracting the expression $\left(a \cos \phi \cos \psi\right)$ from each side of eq.(4), substituting in eq.(2) and eq.(3) and using the following relationship between e and e'

$$e = \frac{a - r_0}{a} = e'\cos\left(\frac{E_2' + E_1'}{2}\right)$$

an expression for the eccentricity of the transformed orbit as a function of r_1, r_2, r_0, E_2', E_1' and the semi-major axis, is developed. Note that e can be less than zero.

$$e = \frac{\left(\frac{r_1 + r_2}{2}\right) - r_0}{a\left[1 - \cos\left(\frac{E_2' - E_1'}{2}\right)\right]} \tag{5}$$

Using the following equations for r_1, r_2 and η,

$$r_1 = a\left(1 - e'\cos E_1'\right)$$

$$r_2 = a\left(1 - e'\cos E_2'\right)$$

$$\eta = \nu_2 - \nu_1$$

the following trigonometric identities and equations relating true anomaly to eccentric anomaly

$$\frac{1 + \cos \eta}{2} = \cos^2\left(\frac{\eta}{2}\right)$$

$$\frac{e' - \cos E'}{e'\cos E' - 1} = \cos \nu$$

$$\cos^2\left(\frac{E_2' + E_1'}{2}\right) = \frac{1}{2}\left[1 + \cos\left(E_2' + E_1'\right)\right]$$

$$\cos^2\left(\frac{E_2' - E_1'}{2}\right) = \frac{1}{2}\left[1 + \cos\left(E_2' - E_1'\right)\right]$$

the following equation is generated

$$\sqrt{r_1 r_2}\cos\left(\frac{\eta}{2}\right) = a\left[\cos\left(\frac{E_2' - E_1'}{2}\right) - e'\cos\left(\frac{E_2' + E_1'}{2}\right)\right] \tag{6}$$

Using eq.(6) in eq.(5) the eccentricity of the transformed orbit becomes a function of r_1, r_2, r_0 and η.

$$e = \frac{\left(\frac{r_1 + r_2}{2}\right) - r_0}{r_0 - \sqrt{r_1 r_2}\cos\left(\frac{\eta}{2}\right)} \tag{7}$$

Now an expression relating the true anomaly (f) of P_2 of the transformed orbit, Fig. 2, to the change in true anomaly of the original orbit (η), Fig. 1, is needed. The law of cosines is used to equate expressions for the chord (c_h), between P_1 and P_2, in terms of the original and transformed orbits.

$$c_h{}^2 = r_1^2 + r_2^2 - 2r_1r_2\cos\eta = 2\left(\frac{r_1 + r_2}{2}\right)^2 - 2\left(\frac{r_1 + r_2}{2}\right)^2\cos\left(2f\right)$$

The trigonometric identities

$$2\sin^2 f = 1 - \cos\left(2f\right)$$

and

$$\cos^2\left(\frac{\eta}{2}\right) = \frac{1 + \cos\eta}{2}$$

are used to obtain

$$\left(\frac{r_1 + r_2}{2}\right)\cos f = \sqrt{r_1 r_2}\cos\left(\frac{\eta}{2}\right) \tag{8}$$

Now substituting eq.(8) into eq.(7) and rearranging yields

$$\frac{2r_o}{r_1 + r_2} = \frac{1 + e \cos f}{1 + e}$$

Using the classical relationship between the true anomaly f and the eccentric anomaly E (Ref. 2),

$$\cos f = \frac{e - \cos E}{e \cos E - 1}$$

and the trigonometric substitutions

$$\cos^2\left(\frac{f}{2}\right) = \frac{1 + \cos f}{2}$$

$$\cos^2\left(\frac{E}{2}\right) = \frac{1 + \cos E}{2}$$

the following expression results.

$$\frac{2r_o}{r_1 + r_2} = \frac{\left[1 + \tan^2\left(\frac{E}{2}\right)\right]}{\left[1 + \tan^2\left(\frac{f}{2}\right)\right]} \qquad (9)$$

Because of the symmetry property of Lambert's theorem the arguments of the trigonometric substitutions of eq.(9) are bounded by $\pm\frac{\pi}{2}$. Therefore the orbit transformation, using Lambert's theorem, and the trigonometric substitution allows the change in true anomaly of the original orbit to vary between $\pm 2\pi$ while keeping the tangent function within its principle range $\pm\infty$.

Now the variable substitution $x^2 = \tan^2\left(\frac{E}{2}\right)$ and $z^2 = \tan^2\left(\frac{f}{2}\right)$ change eq.(9) into

$$\frac{2r_o}{r_1 + r_2} = \frac{1 + x^2}{1 + z^2} \tag{10}$$

The equation for the eccentricity of the transformed orbit, eq.(7), is multiplied by $1 = \left(\frac{2}{r_1 + r_2}\right)\left(\frac{r_1 + r_2}{2}\right)$, then eq.(8) is substituted for $\sqrt{r_1 r_2}\cos\left(\frac{\eta}{2}\right)$. Now the trigonometric identity

$$\cos^2\left(\frac{f}{2}\right) - \sin^2\left(\frac{f}{2}\right) = \cos f$$

and eq.(10) are used to obtain an expression for the eccentricity of the transformed orbit, eq.(1), in terms of x^2 and z^2

$$e = \frac{z^2 - x^2}{z^2 + x^2} \tag{11}$$

Now the invariant semi-major axis of both orbits is derived as a function of x^2, z^2 and p_p, where p_p is the "parameter of the parabolic" (Ref. 1). Beginning with the double angle relation of the true anomaly of the transformed orbit

$$1 + \cos f = 2\cos^2\left(\frac{f}{2}\right)$$

the sum of x^2 and z^2 is expressed as follows

$$z^2 + x^2 = \left(\frac{1}{1 - e}\right)\left[\frac{4x^2}{\left(1 + z^2\right)\left(1 + \cos f\right)}\right]$$

Using eq.(4) and eq.(10) the sum becomes

$$z^2 + x^2 = \frac{8ax^2}{\left(r_1 + r_2\right)\left(1 + x^2\right)\left(1 + \cos f\right)} \tag{12}$$

Next the "parameter of the parabolic" is defined. Setting e of eq.(7) equal to one and using the relationship between η and f of eq.(8) the radius to the normal point r_0 becomes

$$r_0\left(e=1\right) = \left(\frac{r_1 + r_2}{4}\right)\left(1 + \cos f\right) \tag{13}$$

Using the orbit equation of the transformed orbit

$$r = \frac{p}{1 + e \cos f}$$

with e=1 and f=0° a second equation for r_0 results

$$r\left(e=1, f=0^\circ\right) = r_0\left(e=1\right) = \frac{p_p}{2} \tag{14}$$

Since e equals one, the parameter p of the orbit equation is unique. It is called the "parameter of the parabolic" (p_p) (Ref. 1). With eq.(13) equal to eq.(14) the following definition is provided

$$p_p = \left(\frac{r_1 + r_2}{2}\right)\left(1 + \cos f\right) \tag{15}$$

Substituting p_p into eq.(12) and setting

$$w = \left(z^2 + x^2\right)\left(1 + x^2\right)$$

results in an expression for the semi-major axis in terms of w, x^2 and p_p.

$$a = \frac{w\ p_p}{4x^2} \tag{16}$$

The "parameter of the parabolic" is used to make the development tractable. Later in the development p_p will be expressed as a function of the original orbit parameters.

With the eccentricity and semi-major axis of the transformed orbit expressed as a function of z^2, x^2 and p_p the eccentric anomaly of P_2 of the transformed orbit E, and sin E, of eq.(1) need to be expressed as a function of x and x^2. Using the definition of x^2

$$x^2 = \tan^2\left(\frac{E}{2}\right)$$

E is resolved

$$E = 2 \arctan \left(\sqrt{x^2}\right) \tag{17}$$

Now using the trigonometric identity

$$\sin E = 2 \sin \left(\frac{E}{2}\right) \cos \left(\frac{E}{2}\right)$$

the sine of the eccentric anomaly of the transformed orbit is expressed as a function of x and x^2

$$\sin E = \frac{2x}{1 + x^2} \tag{18}$$

Substituting equations (11), (16), (17) and (18) into eq.(1) results in a time of flight equation as a function of w, x, x^2, z^2 and p_p.

$$\Delta t = \frac{1}{2x^2} \sqrt{\frac{w^3 \, p_p^3}{\mu}} \left[\frac{\arctan x}{x} - \left(\frac{z^2 - x^2}{z^2 + x^2}\right)\left(\frac{1}{1 + x^2}\right)\right] \tag{19}$$

Note that in the development of this equation the arguments of the tangent expressions $x^2 = \tan^2\left(\frac{E}{2}\right)$ and $z^2 = \tan^2\left(\frac{f}{2}\right)$ were constrained to $\pm\frac{\pi}{2}$ which results in $0 \leq x^2 \leq \infty$ and $0 \leq z^2 \leq \infty$.

Later x^2 will be expressed as a function of r_1, r_2, η and ϕ_1 of the original orbit, and consequently, can be less than zero. The trigonometric identity

$$-i \operatorname{arctanh} \left(i\sqrt{x^2}\,\right) = \arctan \left(\sqrt{x^2}\,\right) \;, \text{ for } x^2 < 0$$

indicates that the inverse tangent function must be replaced by the inverse hyperbolic tangent function for $x^2 < 0$.

Note that the quotient

$$\frac{i \operatorname{arctanh}\left(\sqrt{|x^2|}\right)}{i\sqrt{|x^2|}} \quad , \text{ for } x^2 < 0$$

is still a real number. This replacement results in essentially two time of flight equations.

To turn eq.(19) into a universal time of flight equation a continuing fraction is substituted for the quotient

$$\frac{\arctan x}{x} \quad , \quad \text{(Ref. 3)}$$

Using the following continuing fraction expressions for the inverse tangent and inverse hyperbolic tangent

$$\arctan y = \cfrac{y}{1 + \cfrac{y^2}{3 + \cfrac{2^2 y^2}{5 + \cfrac{3^2 y^2}{7 + \cfrac{4^2 y^2}{9 + \cdots}}}}} \qquad -\frac{\pi}{2} \leq \arctan y \leq \frac{\pi}{2}, \quad -\infty \leq y \leq \infty$$

$$\operatorname{arctanh} y = \cfrac{y}{1 - \cfrac{y^2}{3 - \cfrac{2^2 y^2}{5 - \cfrac{3^2 y^2}{7 - \cfrac{4^2 y^2}{9 - \cdots}}}}} \qquad -\infty \leq \operatorname{arctanh} y \leq \infty, \quad -1 \leq y \leq 1$$

the continuing fraction substitute, $\dfrac{1}{1 + x^2G}$ (Ref. 1), is derived as follows

$$\frac{1}{1 + x^2G} = \frac{\arctan x}{x} = \frac{i \,\operatorname{arctanh}\left(\sqrt{|x^2|}\right)}{i\sqrt{|x^2|}}, \text{ for } -1 \leq x^2 \leq \infty$$

$$\text{where } G = G\left(x^2\right) = \cfrac{1}{3 + \cfrac{2^2x^2}{5 + \cfrac{3^2x^2}{7 + \cfrac{4^2x^2}{9 + \cdots}}}}$$

Using

$$i \,\operatorname{arctanh}\left(\sqrt{|x^2|}\right) = \frac{i}{2}\log_e\left[\frac{1 + \sqrt{|x^2|}}{1 - \sqrt{|x^2|}}\right]$$

a three dimensional plot of the functions arctan x and $i \,\operatorname{arctanh}\left(\sqrt{|x^2|}\right)$ as a function of the independent variable x^2 is presented in Fig. 4. The continuing fraction substitute is plotted as a function of x^2 in Fig. 5. Note that the substitute is continuous at $x^2 = 0$.

Now substituting

$$\frac{1}{1 + x^2G} = \frac{\arctan x}{x} \tag{20}$$

and replacing w with $\left(z^2 + x^2\right)\left(1 + x^2\right)$, eq.(19) becomes

$$\Delta t = \frac{1}{2}\sqrt{\frac{p_p^3}{\mu}}\sqrt{\left(z^2 + x^2\right)\left(1 + x^2\right)}\left\{2 + z^2 + x^2 - G\left[\frac{\left(z^2 + x^2\right)\left(1 + x^2\right)}{1 + x^2 G}\right]\right\} \tag{21}$$

for $-1 \le x^2 \le \infty$. The convergence of the continuing fraction $G\left(x^2\right)$ will be examined in a subsequent section. Equation (21) is a universal time of flight equation as a function of x^2, z^2 and p_p.

Now x^2, z^2 and p_p are expressed as functions of r_1, r_2, η and ϕ_1 of the original orbit. This provides a universal time of flight equation for any type of orbit as a function of the parameters of the original orbit. Note that the eccentricity of the original orbit is not needed.

First z^2 is derived as a function of r_1, r_2 and η. Using the relationship between η and f, eq.(8), the cosine of the true anomaly of the transformed orbit is

$$\cos f = \left(\frac{2\sqrt{r_1 r_2}}{r_1 + r_2}\right)\cos\left(\frac{\eta}{2}\right) \tag{22}$$

Now the variable z^2 is expressed as a function of cos f using

$$\frac{1 - \cos f}{1 + \cos f} = \tan^2\left(\frac{f}{2}\right) = z^2$$

Substituting eq.(22) for cos f yields

$$z^2 = \frac{r_1 + r_2 - 2\sqrt{r_1 r_2}\cos\left(\frac{\eta}{2}\right)}{r_1 + r_2 + 2\sqrt{r_1 r_2}\cos\left(\frac{\eta}{2}\right)} \tag{23}$$

To find x^2 as a function of r_1, r_2, η and ϕ_1, where ϕ_1 is the flight path angle at P_1 of the original orbit, the radius to the normal point must be expressed as a function of these variables.

To begin the derivation of r_0 the derivative of the orbit equation of the original orbit is generated

$$\frac{dr}{dt} = \dot{r} = \frac{h'e' \sin \nu}{p'}$$

where ν is the true anomaly measured in the original orbit. The trigonometric identity

$$\cot\left(\frac{\nu_0 - \nu_1}{2}\right)\left(\cos \nu_1 - \cos \nu_0\right) = \sin \nu_1 + \sin \nu_0$$

is used to generate the sum of the derivatives of r_1 and r_0 of the original orbit

$$\dot{r}_1 + \dot{r}_0 = \frac{h'e'}{p'}\left(\cos \nu_1 - \cos \nu_0\right) \cot\left(\frac{\delta}{2}\right) \qquad (24)$$

where $\delta = \nu_0 - \nu_1$

Now the orbit equation is used to provide the following expressions

$$e' \cos \nu_1 = \frac{p'}{r_1} - 1$$

$$e' \cos \nu_0 = \frac{p'}{r_0} - 1$$

which when substituted into eq.(24) results in

$$\dot{r}_1 + \dot{r}_0 = h'\left(\frac{1}{r_1} - \frac{1}{r_0}\right)\cos\left(\frac{\delta}{2}\right)$$

Substituting $\sqrt{p'\mu} = h'$ and multiplying through by the product $(r_1 r_0)$ yields

$$r_0\left(\frac{\bar{r}_1 \cdot \bar{V}_1'}{\sqrt{\mu}}\right) + r_1\left(\frac{r_0 V_0' \cos \gamma_0'}{\sqrt{\mu}}\right) = \sqrt{p'}\left(r_0 - r_1\right)\cot\left(\frac{\delta}{2}\right) \qquad (25)$$

where $\bar{r}_1 \cdot \bar{V}_1' = r_1 V_1' \cos \gamma_1' = r_1 \dot{r}_1$

$r_0 V_0' \cos \gamma_0' = r_0 \dot{r}_0$

The parameters V_0', γ_0', V_1', and γ_1' are of the original orbit, Fig. 1 and Fig. 3. Now an expression for the compliment of the flight path angle at the normal point of the original orbit (γ_0') as a function of r_1, r_2 and η is needed.

To begin the derivation of γ_0' the orbit equation of the original orbit

$$r = \frac{p'}{1 + e'\cos \nu}$$

is used to generate the following expression

$$\bar{e}' \cdot \left(\bar{r}_1 - \bar{r}_2\right) = r_2 - r_1 \qquad (26)$$

Using the definition of the eccentricity vector as a function of normal point parameters

$$\bar{e}' = \frac{1}{\mu}\left(\bar{V}_0' \times \bar{h}'\right) - \frac{\bar{r}_0}{r_0}$$

eq.(26) becomes

$$\left[\frac{1}{\mu}\left(\bar{V}_0' \times \bar{h}'\right) - \frac{\bar{r}_0}{r_0}\right]\cdot\left(\bar{r}_1 - \bar{r}_2\right) = r_2 - r_1 \qquad (27)$$

Referring to Fig. 6 the vector $\bar{V}_0' \times \bar{h}'$, where $\bar{h}'$ is normal to the page, is perpendicular to the vector $\left(\bar{r}_2 - \bar{r}_1\right)$ and

$$\left(\bar{V}_0' \times \bar{h}'\right)\cdot\left(\bar{r}_1 - \bar{r}_2\right) = 0$$

Therefore eq.(27) reduces to

$$\bar{r}_0\cdot\left(\bar{r}_2 - \bar{r}_1\right) = r_0\left(r_2 - r_1\right) \qquad (28)$$

But from the geometry of Fig. 3

$$\bar{r}_0\cdot\left(\bar{r}_2 - \bar{r}_1\right) = r_0 \, c_h \cos \gamma_0'$$

where $c_h = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\eta}$

Equating these expressions

$$\cos \gamma_0' = \frac{r_2 - r_1}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos \eta}}$$

Substituting this expression for $\cos \gamma_0'$ and the following equality

$$r_1V_1' \sin \phi_1 = \bar{r}_1 \cdot \bar{V}_1'$$

into eq.(25) and solving for r_0 yields

$$r_0 = r_1 \left(\frac{\sqrt{p'} + \left(\dfrac{r_0V_0'\left(r_2 - r_1\right)}{\sqrt{\mu}\ c_h}\right) \tan\left(\dfrac{\delta}{2}\right)}{\sqrt{p'} - \left(\dfrac{r_1V_1' \sin \phi_1}{\sqrt{\mu}}\right) \tan\left(\dfrac{\delta}{2}\right)} \right)$$

Now substituting

$$h' = r_0V_0' \sin \gamma_0'$$

$$\frac{\sqrt{c_h^2 - \left(r_2 - r_1\right)^2}}{c_h} = \sqrt{1 - \cos^2\gamma_0'} = \sin \gamma_0'$$

and

$$\sqrt{p'} = \frac{h'}{\sqrt{\mu}} = \frac{r_1V_1' \cos \phi_1}{\sqrt{\mu}}$$

into this equation for r_0 yields

$$r_0 = r_1 \left[\frac{1 + \left(\dfrac{r_2 - r_1}{\sqrt{c_h^2 - \left(r_2 - r_1\right)^2}} \right) \tan \left(\dfrac{\delta}{2} \right)}{1 - \tan \phi_1 \tan \left(\dfrac{\delta}{2} \right)} \right] \tag{29}$$

This expression gives the radius to the normal point as a function of r_1, r_2, η, ϕ_1 and δ. An expression relating η and δ is needed.

To find an expression relating η $\left(= \nu_2 - \nu_1 \right)$ and δ $\left(= \nu_0 - \nu_1 \right)$ eq.(28) is expanded

$$\bar{r}_0 \cdot \bar{r}_2 - \bar{r}_0 \cdot \bar{r}_1 = r_0 \left(r_2 - r_1 \right)$$

Substituting

$$r_0 r_2 \cos\left(\eta - \delta\right) = r_0 r_2 \cos\left(\nu_2 - \nu_1 - \nu_0 + \nu_1\right) = \bar{r}_0 \cdot \bar{r}_2$$

and

$$r_0 r_1 \cos \delta = r_0 r_1 \cos\left(\nu_0 - \nu_1\right) = \bar{r}_0 \cdot \bar{r}_1$$

into this equation yields

$$\frac{r_1}{r_2}\left(1 - \cos \delta\right) = 1 - \cos \eta \cos \delta - \sin \eta \sin \delta \tag{30}$$

Adding $0 = \frac{1}{2}\left(\cos \delta - \cos \delta - \cos \eta + \cos \eta\right)$ to the right hand side of eq.(30) results in

$$\frac{r_1}{r_2}\left(\frac{1 - \cos \delta}{1 + \cos \delta}\right) = \frac{1 - \cos \eta}{2} - \frac{\sin \eta \sin \delta}{1 + \cos \delta} + \frac{1 + \cos \eta}{2}\left(\frac{1 - \cos \delta}{1 + \cos \delta}\right)$$

Using the following trigonometric identities

$$\tan^2\left(\frac{\delta}{2}\right) = \frac{1 - \cos\delta}{1 + \cos\delta}$$

$$\sin^2\left(\frac{\eta}{2}\right) = \frac{1 - \cos\eta}{2}$$

$$\cos^2\left(\frac{\eta}{2}\right) = \frac{1 + \cos\eta}{2}$$

in this equation yields

$$\left(\frac{r_1}{r_2}\right)\tan^2\left(\frac{\delta}{2}\right) = \sin^2\left(\frac{\eta}{2}\right) - \frac{\sin\eta\,\sin\delta}{1 + \cos\delta} + \cos^2\left(\frac{\eta}{2}\right)\tan^2\left(\frac{\delta}{2}\right)$$

Using two more trigonometric identities

$$\tan\left(\frac{\delta}{2}\right) = \frac{\sin\delta}{1 + \cos\delta}$$

$$2\sin\left(\frac{\eta}{2}\right)\cos\left(\frac{\eta}{2}\right) = \sin\eta$$

results in an expression for η as follows

$$\tan\left(\frac{\delta}{2}\right) = \frac{\sin\left(\frac{\eta}{2}\right)}{\sqrt{\frac{r_1}{r_2}} + \cos\left(\frac{\eta}{2}\right)}$$

This expression is substituted into eq.(29) to generate

$$r_o = r_1 \left(\frac{1 + \left(\dfrac{r_2 - r_1}{\sqrt{c_h^2 - \left(r_2 - r_1\right)^2}} \right) \left(\dfrac{\sin\left(\frac{\eta}{2}\right)}{\sqrt{\dfrac{r_1}{r_2}} + \cos\left(\frac{\eta}{2}\right)} \right)}{1 - \tan \phi_1 \left(\dfrac{\sin\left(\frac{\eta}{2}\right)}{\sqrt{\dfrac{r_1}{r_2}} + \cos\left(\frac{\eta}{2}\right)} \right)} \right)$$

Finally, the trigonometric identity

$$\sin\left(\frac{\eta}{2}\right) = \sqrt{\frac{1 - \cos \eta}{2}}$$

is used to obtain an expression for the radius to the normal point as a function of r_1, r_2, η and ϕ_1 as follows

$$r_o = r_1 \left(\frac{\sqrt{\dfrac{r_1}{r_2}} + \cos\left(\frac{\eta}{2}\right) + \left(\dfrac{r_2 - r_1}{2\sqrt{r_1 r_2}} \right)}{\sqrt{\dfrac{r_1}{r_2}} + \cos\left(\frac{\eta}{2}\right) - \tan \phi_1 \sin\left(\frac{\eta}{2}\right)} \right) \tag{31}$$

Now that r_o has been expressed as a function of r_1, r_2, η, and ϕ_1, x^2 is expressed as a function of these variables. Equation (10) is solved for x^2, then eq.(31) and eq.(23) are substituted for r_o and z^2, and the result reduces to

$$x^2 = \frac{\sqrt{\dfrac{r_1}{r_2}} - \cos\left(\dfrac{\eta}{2}\right) + \tan\phi_1 \sin\left(\dfrac{\eta}{2}\right)}{\sqrt{\dfrac{r_1}{r_2}} + \cos\left(\dfrac{\eta}{2}\right) - \tan\phi_1 \sin\left(\dfrac{\eta}{2}\right)} \tag{32}$$

With z^2 and x^2 expressed as functions of r_1, r_2, η and ϕ_1, the sums $\left(z^2 + x^2\right)$ and $\left(1 + x^2\right)$ are needed for substitution into eq.(21). Summing equations (23) and (32) produces

$$z^2 + x^2 = \frac{2\left(r_1 + r_2\right)\sqrt{\dfrac{r_1}{r_2}} - 4\sqrt{r_1 r_2}\cos\left(\dfrac{\eta}{2}\right)\left[\cos\left(\dfrac{\eta}{2}\right) - \tan\phi_1 \sin\left(\dfrac{\eta}{2}\right)\right]}{\left[r_1 + r_2 + 2\sqrt{r_1 r_2}\cos\left(\dfrac{\eta}{2}\right)\right]\left[\sqrt{\dfrac{r_1}{r_2}} + \cos\left(\dfrac{\eta}{2}\right) - \tan\phi_1 \sin\left(\dfrac{\eta}{2}\right)\right]} \tag{33}$$

and using eq.(32)

$$1 + x^2 = \frac{2\sqrt{\dfrac{r_1}{r_2}}}{\sqrt{\dfrac{r_1}{r_2}} + \cos\left(\dfrac{\eta}{2}\right) - \tan\phi_1 \sin\left(\dfrac{\eta}{2}\right)} \tag{34}$$

Finally substituting eq.(22) for cos f into eq.(15) yields the "parameter of the parabolic" as a function of r_1, r_2 and η.

$$p_p = \frac{r_1 + r_2}{2}\left[1 + \left(\frac{2\sqrt{r_1 r_2}}{r_1 + r_2}\right)\cos\left(\frac{\eta}{2}\right)\right] \tag{35}$$

Now substitute eq.(35), eq.(34), eq.(33), eq.(32) and eq.(23) into eq.(21). After a considerable amount of algebra the universal time of flight equation reduces to a function of r_1 r_2, η and ϕ_1 as follows

$$\Delta t = \sqrt{\frac{r_1}{8\mu r_2}} \left(\frac{\sqrt{r_1 + r_2 - 2\, r_2 \cos\left(\frac{\eta}{2}\right)\left[\cos\left(\frac{\eta}{2}\right) - \tan\phi_1 \sin\left(\frac{\eta}{2}\right)\right]}}{\sqrt{\frac{r_1}{r_2}}\left(1 + G\right) + \left[\cos\left(\frac{\eta}{2}\right) - \tan\phi_1 \sin\left(\frac{\eta}{2}\right)\right]\left(1 - G\right)} \right)$$

$$\left(\frac{\left(1-G\right)2\left[r_2\cos\left(\frac{\eta}{2}\right)-\left(r_1+r_2\right)\tan\phi_1 \sin\left(\frac{\eta}{2}\right)\right]+\left(6+2G\right)r_1\cos\left(\frac{\eta}{2}\right)+4\left(r_1+r_2\right)\sqrt{\frac{r_1}{r_2}}}{\sqrt{\frac{r_1}{r_2}} + \cos\left(\frac{\eta}{2}\right) - \tan\phi_1 \sin\left(\frac{\eta}{2}\right)} \right) \tag{36}$$

$$\text{where } G = G\left(x^2\right) = \cfrac{1}{3 + \cfrac{2^2x^2}{5 + \cfrac{3^2x^2}{7 + \cfrac{4^2x^2}{9 + \cdots}}}}$$

$$\text{and} \quad x^2 = \frac{\sqrt{\frac{r_1}{r_2}} - \cos\left(\frac{\eta}{2}\right) + \tan\phi_1 \sin\left(\frac{\eta}{2}\right)}{\sqrt{\frac{r_1}{r_2}} + \cos\left(\frac{\eta}{2}\right) - \tan\phi_1 \sin\left(\frac{\eta}{2}\right)}$$

"The Universal Time of Flight Equation"

Equation 36 is a universal time of flight equation for any orbit as a function of the original orbit parameters r_1, r_2, η and ϕ_1 and is not explicitly a function of the eccentricity of the original orbit. This equation has been developed with the constraint $-2\pi \leq \eta \leq 2\pi$. Next eq.(36) will be checked for indeterminate points, conditions where the numerator and denominator go to zero. The check will reveal that eq.(36) is determinate for $-2\pi < \eta < 2\pi$.

To examine the determinacy of the universal time of flight equation the following constraints are applied.

$$0 < r_1 \leq \infty$$

$$0 < r_2 \leq \infty$$

$$-\frac{\pi}{2} \leq \phi_1 \leq \frac{\pi}{2}$$

Then the range of η for a determinate equation is established. Remember that the universal time of flight equation was developed with the constraint $-2\pi \leq \eta \leq 2\pi$.

First the values of η which drive the denominator of the second factor of eq.(36) to zero are determined.

$$\sqrt{\frac{r_1}{r_2}} + \cos\left(\frac{\eta}{2}\right) - \tan \phi_1 \sin\left(\frac{\eta}{2}\right) = 0 \qquad (37)$$

Substituting the orbit equation for r_1 and r_2 of the original orbit, using the definition $\eta = \nu_2 - \nu_1$ and squaring both sides, eq.(37) becomes

$$\frac{1 + e'\cos\left(\nu_1 + \eta\right)}{1 + e'\cos \nu_1} = \cos^2\left(\frac{\eta}{2}\right) - 2 \tan \phi_1 \cos\left(\frac{\eta}{2}\right) \sin\left(\frac{\eta}{2}\right) + \tan^2\phi_1 \sin^2\left(\frac{\eta}{2}\right) \qquad (38)$$

The tangent of the flight path angle at P_1 is expressed as a function of e' and ν_1 by using the time derivative of the orbit equation.

$$\tan \phi_1 = \frac{e'\sin \nu_1}{1 + e'\sin \nu_1}$$

Substituting for $\tan \phi_1$ in eq.(38) and multiplying the result by $\left(1 + e'\cos \nu_1\right)^2$ yields

$$\left[1 + e'\cos\left(\nu_1 + \eta\right)\right]\left(1 + e'\cos \nu_1\right) = \cos^2\left(\frac{\eta}{2}\right)\left(1 + e'\cos \nu_1\right)^2$$
$$-2e'\sin \nu_1 \cos\left(\frac{\eta}{2}\right)\sin\left(\frac{\eta}{2}\right)\left(1 + e'\cos \nu_1\right) + \left(e'\sin \nu_1\right)^2\sin^2\left(\frac{\eta}{2}\right)$$

Using the following trigonometric identities

$$\frac{1 + \cos \eta}{2} = \cos^2\left(\frac{\eta}{2}\right)$$

$$\frac{1 - \cos \eta}{2} = \sin^2\left(\frac{\eta}{2}\right)$$

$$\sin \eta = 2 \sin\left(\frac{\eta}{2}\right)\cos\left(\frac{\eta}{2}\right)$$

and reducing the resulting equation yields the following expression.

$$\frac{1}{2} + \left(\frac{(e')^2}{2}\right)\cos^2\nu_1 \cos \eta = \left(\frac{(e')^2}{2}\right) + \left(\frac{\cos \eta}{2}\right) - \left(\frac{(e')^2}{2}\right)\sin^2\nu_1 \cos \eta$$

Multiplying through by 2 and reducing further results in

$$\eta = \arccos (1)$$

Next the denominator

$$\sqrt{\frac{r_1}{r_2}} + \cos\left(\frac{\eta}{2}\right) - \tan\,\phi_1\,\sin\left(\frac{\eta}{2}\right)$$

is evaluated for $\eta = N\pi$ since this equation was squared to produce this result. Noting that $r_1 = r_2$ for $\eta = N\pi$, where $N = \pm 0,2,4,6,8,10,\cdots$, the denominator is equal to 2 for $N = \pm 0,4,8,\cdots$ and is equal to 0 for $N = \pm 2,6,10,\cdots$. Therefore the denominator of the second factor of eq.(36) is zero for $\eta = N\pi$, where $N = \pm 2,6,10,\cdots$. Now check the denominator of the first factor of eq.(36).

The denominator of the first factor is expanded and set equal to zero as follows

$$\sqrt{\frac{r_1}{r_2}} + \cos\left(\frac{\eta}{2}\right) - \tan\,\phi_1\,\sin\left(\frac{\eta}{2}\right) + G\left[\sqrt{\frac{r_1}{r_2}} - \cos\left(\frac{\eta}{2}\right) + \tan\,\phi_1\,\sin\left(\frac{\eta}{2}\right)\right] = 0$$

Using eq.(20) G becomes

$$G = \frac{x - \arctan x}{x^2 \arctan x}$$

After substituting G into the expanded denominator of the first factor and reducing, the following expression results

$$\sqrt{\frac{r_1}{r_2}} - \cos\left(\frac{\eta}{2}\right) + \tan\,\phi_1\,\sin\left(\frac{\eta}{2}\right) = 0 \tag{39}$$

Evaluating this expression for $\eta = N\pi$, where $N = \pm 0,2,4,6,8,10,\cdots$, and $r_1 = r_2$ results in this expression going to zero for $\eta = N\pi$, where $N = \pm 0,4,8,\cdots$. When $\eta = 0$, $\Delta t = 0$, which is considered a a trivial case. Therefore the denominator of the first factor of eq.(36) is considered never to go to zero for $-2\pi \leq \eta \leq 2\pi$.

With the denominators evaluated it is only necessary to find one of the two numerators equal to zero for $\eta = \pm 2\pi$. Therefore the numerator of the first factor of eq.(36)

$$\sqrt{r_1 + r_2 - 2\, r_2 \cos\left(\frac{\eta}{2}\right)\left[\cos\left(\frac{\eta}{2}\right) - \tan\phi_1 \sin\left(\frac{\eta}{2}\right)\right]}$$

is evaluated. By inspection this numerator equals zero for $\eta = N\pi$, where $N = \pm 0,2,4,6,8,\cdots$. Consequently eq.(36) is indeterminate for $\eta = N\pi$, where $N = \pm 2,6,10,\cdots$.

L'Hospital's rule could be used to determine if in the limit, as η approaches $\pm 2\pi$, eq.(36) has a definite value. But of prime interest is the range of orbital parameters for which eq.(36) provides an answer. Therefore the universal time of flight equation is considered determinate for $-2\pi < \eta < 2\pi$.

The accuracy of the universal time of flight equation is dependent upon the convergence of the continuing fraction, G, which results when the continuing fraction expression for the inverse tangent function is used to provide one time of flight equation for any orbit.

$$\frac{1}{1 + x^2G} = \frac{\arctan x}{x} = \frac{i \operatorname{arctanh}\left(\sqrt{|x^2|}\right)}{i\sqrt{|x^2|}} \quad , \text{ for } -1 \le x^2 \le \infty$$

$$\text{where } G = G\left(x^2\right) = \cfrac{1}{3 + \cfrac{2^2x^2}{5 + \cfrac{3^2x^2}{7 + \cfrac{4^2x^2}{9 + \cdots}}}}$$

Note that $G(x^2=0) = \frac{1}{3}$ and the continuing fraction substitute $\frac{1}{1 + x^2G}$, at $x^2=0$, equals 1.

The following "top down algorithm" (Ref. 1) is used to evaluate $G(x^2)$.

```
      N = 0
      AK = X²
      AK2 = 2*2*X²
      BK = 3
      BK2 = 5
      UK = 1
      VK = AK/BK
      WK = VK
1     UK = 1/(1+AK2/(BK*BK2)*UK)
```

```
      VK = VK*(UK-1)
      WK = WK+VK
      IF (ABS(VK).LT.CC)GOTO 2
      N = N+1
      AK2 = (N+2)*(N+2)*X2
      BK = 2*N+3
      BK2 = 2*(N+1)+3
      GOTO 1
    2 G = WK/X2
```

where CC ≡ convergence criterion

Using this algorithm the product $x^2G(x^2)$ is calculated and checked for convergence to obtain $G(x^2)$. The difference between the values of $x^2G\left(x^2,N=i+1\right)$ and $x^2G\left(x^2,N=i\right)$ is compared to the convergence criterion as follows

for cycle 1 (N=0), is $\left|x^2G\left(x^2,N=0\right) - x^2G\left(x^2,N=-1\right)\right| < CC$

$$\left| \frac{x^2}{3 + \frac{2^2x^2}{5}} - \frac{x^2}{3} \right| < CC$$

for cycle 2 (N=1), is $\left|x^2G\left(x^2,N=1\right) - x^2G\left(x^2,N=0\right)\right| < CC$

$$\left| \frac{x^2}{3 + \frac{2^2x^2}{5 + \frac{3^2x^2}{7}}} - \frac{x^2}{3 + \frac{2^2x^2}{5}} \right| < CC$$

The algorithm is repeated until a difference less than CC is obtained resulting in a converged value for $x^2G\left(x^2,N=i+1\right)$. This value is divided by x^2 to obtain $G(x^2)$.

The accuracy of the converged continuing fraction $G(x^2)$ is evaluated by tabulating the decimal accuracy (F) of the following normalized difference.

$$ND_{CF} \times 10^F = \frac{\left(\frac{\arctan x}{x}\right) - \left(\frac{1}{1 + x^2 G}\right)}{\frac{\arctan x}{x}} \quad , \text{ for } x^2 > 0$$

$$ND_{CF} \times 10^F = \frac{\left(\frac{i \operatorname{arctanh}\left(\sqrt{|x^2|}\right)}{i \sqrt{|x^2|}}\right) - \left(\frac{1}{1 + x^2 G}\right)}{\left(\frac{i \operatorname{arctanh}\left(\sqrt{|x^2|}\right)}{i \sqrt{|x^2|}}\right)} \quad , \text{ for } x^2 < 0$$

The decimal accuracy, F, of the normalized difference and the number of terms required for convergence of the continuing fraction (N) are tabulated as a function of x^2 and the decimal accuracy (D) of the convergence criterion.

where $CC = 1 \times 10^D$

Table I presents the data for $G(x^2)$ and two continuing fraction expressions, $G(x^2,\zeta^2)$ and $G(x^2,\zeta^2,\omega^2)$, which will be developed later.

The data of Table I for $G(x^2)$ indicates that as the convergence criterion is reduced the accuracy of the continuing fraction is increased but the number of terms needed for convergence also increases. The limiting value appearing in the data, F=-16, is the decimal accuracy of the computer. If the normalized difference happens to be zero then $F=-\infty$.

To converge the continuing fraction to a smaller value of CC requiring fewer terms for convergence, a method described by Battin and Vaughan (Ref. 4) is employed. The objective of the method is to reduce by half the argument, x, of the inverse tangent function being evaluated by the continuing fraction expression $G(x^2)$.

$$\frac{E}{2} = 2\left(\frac{E}{4}\right)$$

$$\frac{E}{2}\left(\frac{\tan\left(\frac{E}{2}\right)}{\tan\left(\frac{E}{2}\right)}\right) = 2\left(\frac{E}{4}\right)\left(\frac{\tan\left(\frac{E}{4}\right)}{\tan\left(\frac{E}{4}\right)}\right) \tag{40}$$

Using $x^2 = \tan^2\left(\frac{E}{2}\right)$ and $\zeta^2 = \tan^2\left(\frac{E}{4}\right)$, the continuing fraction expression for $\arctan\left(\zeta\right)$

$$\arctan \zeta = \cfrac{\zeta}{1 + \cfrac{\zeta^2}{3 + \cfrac{2^2\zeta^2}{5 + \cfrac{3^2\zeta^2}{7 + \cfrac{4^2\zeta^2}{9 + \cdots}}}}}$$

produces $\dfrac{\arctan \zeta}{\zeta} = \dfrac{1}{1 + \zeta^2 G(\zeta^2)}$

where $$G(\zeta^2) = \cfrac{\zeta}{3 + \cfrac{2^2\zeta^2}{5 + \cfrac{3^2\zeta^2}{7 + \cfrac{4^2\zeta^2}{9 + \cdots}}}}$$

and $\dfrac{1}{1 + \zeta^2 G(\zeta^2)} = \dfrac{\arctan \zeta}{\zeta} = \dfrac{\left(\frac{E}{4}\right)}{\tan\left(\frac{E}{4}\right)}$

Recall that $\dfrac{1}{1 + x^2G(x^2)} = \dfrac{\arctan x}{x} = \dfrac{\left(\frac{E}{2}\right)}{\tan\left(\frac{E}{2}\right)}$

After substituting, eq.(40) becomes

$$\frac{1}{1 + x^2G(x^2)} = \left(\frac{2\tan\left(\frac{E}{4}\right)}{\tan\left(\frac{E}{2}\right)}\right)\frac{1}{1 + \zeta^2G(\zeta^2)} \tag{41}$$

$$\text{where} \quad G(x^2) = \cfrac{1}{3 + \cfrac{2^2x^2}{5 + \cfrac{3^2x^2}{7 + \cfrac{4^2x^2}{9 + \cdots}}}}$$

Now using the trigonometric identities

$$\frac{1 - \cos\left(\frac{E}{2}\right)}{2} = \sin^2\left(\frac{E}{4}\right)$$

$$\frac{1 + \cos\left(\frac{E}{2}\right)}{2} = \cos^2\left(\frac{E}{4}\right)$$

then $\tan^2\left(\frac{E}{4}\right) = \dfrac{\sec\left(\frac{E}{2}\right) - 1}{\sec\left(\frac{E}{2}\right) + 1}$

With $x^2 = \tan^2\left(\frac{E}{2}\right)$, then ζ^2 becomes

$$\frac{\sqrt{x^2 + 1} - 1}{\sqrt{x^2 + 1} + 1} = \zeta^2 = \tan^2\left(\frac{E}{4}\right) \tag{42}$$

Substituting eq.(42) into eq.(41) yields

$$\frac{1}{1 + x^2G(x^2)} = \left(\frac{2}{\sqrt{x^2+1}+1}\right)\frac{1}{1+\zeta^2G(\zeta^2)}$$

Now solve for $G(x^2)$

$$G(x^2) = \frac{1 + G(\zeta^2)}{2\left(\sqrt{x^2+1}+1\right)}$$

Substituting for $G(x^2)$ and $G(\zeta^2)$ into this equation yields

$$\frac{1}{3 + \left[\dfrac{4x^2}{5 + x^2\xi(x^2)}\right]} = \frac{1}{2\left(\sqrt{x^2+1}+1\right)}\left(1 + \left(\frac{1}{3 + \left[\dfrac{4\zeta^2}{5+\zeta^2\xi(\zeta^2)}\right]}\right)\right) \tag{43}$$

where $\xi(x^2) = \cfrac{3^2}{7 + \cfrac{4^2x^2}{9 + \cdots}}$

where $\xi(\zeta^2) = \cfrac{3^2}{7 + \cfrac{4^2\zeta^2}{9 + \cdots}}$

Note $$G(x^2) = \frac{1}{3 + \left[\dfrac{4x^2}{5 + x^2\xi(x^2)}\right]} \tag{44}$$

Now eq.(43) is solved for $x^2\xi(x^2)$

$$x^2\xi(x^2) = \frac{8\left(\sqrt{x^2+1}+1\right)}{3 + \left[\dfrac{1}{\zeta^2 + 5 + \zeta^2\xi(\zeta^2)}\right]} - 5$$

and substituting into eq.(44) to produce

$$G(x^2,\zeta^2) = \cfrac{1}{3 + \cfrac{\cfrac{4x^2}{8\left(\sqrt{x^2+1}+1\right)}}{3 + \cfrac{1}{\zeta^2 + 5 + \cfrac{3^2\zeta^2}{7 + \cfrac{4^2\zeta^2}{9 + \cdots}}}}} \tag{45}$$

$$\text{where} \quad \zeta^2 = \frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}$$

The algorithm used to evaluate $G(x^2,\zeta^2)$ is a modified version of the algorithm used to evaluate $G(x^2)$.

Using $\zeta^2 = \dfrac{x^2}{\left(\sqrt{x^2+1}+1\right)^2} = \dfrac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}$

the following algorithm results

```
      ZTA = SQRT(X2+1)+1
      ZTA2 = X2/(ZTA*ZTA)
      N = 2
      AK = 3*3*ZTA2
      AK2 = 4*4*ZTA2
      BK = 7
      BK2 = 9
      UK = 1
      VK = AK/BK
      WK = VK
   1  UK = 1/(1+AK2/(BK*BK2)*UK)
      VK = VK*(UK-1)
      WK = WK+VK
      IF (ABS(VK).LT.CC)GOTO 2
      N = N+1
      AK2 = (N+2)*(N+2)*ZTA2
      BK = 2*N+3
      BK2 = 2*(N+1)+3
      GOTO 1
   2  G = 5+ZTA2+WK
      G = 3+1/G
      G = 8*(SQRT(X2+1)+1)/G
      G = 2*2*X2/G
      G = 1/(3+G)
```

Using this algorithm the product $\zeta^2\xi(\zeta^2)$ is calculated and checked for convergence to obtain $G(x^2,\zeta^2)$. The difference between the values of $\zeta^2\xi\left(\zeta^2, N=i+1\right)$ and $\zeta^2\xi\left(\zeta^2, N=i\right)$ is compared to the convergence criterion as follows

for cycle 1 (N=2), is $\left|\zeta^2\xi\left(\zeta^2, N=2\right) - \zeta^2\xi\left(\zeta^2, N=1\right)\right| < CC$

$$\left| \frac{3^2\zeta^2}{7 + \frac{4^2\zeta^2}{9}} - \frac{3^2\zeta^2}{7} \right| < CC$$

for cycle 2 (N=3), is $\left|\zeta^2\xi\left(\zeta^2, N=3\right) - \zeta^2\xi\left(\zeta^2, N=2\right)\right| < CC$

$$\left| \frac{3^2\zeta^2}{7 + \frac{4^2\zeta^2}{9 + \frac{5^2\zeta^2}{11}}} - \frac{3^2\zeta^2}{7 + \frac{4^2\zeta^2}{9}} \right| < CC$$

As before, this algorithm is repeated until a difference less than the convergence criterion is obtained resulting in a converged value for $\zeta^2\xi\left(\zeta^2, N=i+1\right)$. The following expression is used to obtain $G(x^2,\zeta^2)$

$$G(x^2,\zeta^2) = \frac{1}{3 + \left[\frac{\frac{4x^2}{8\left(\sqrt{x^2+1} + 1\right)}}{3 + \left[\frac{1}{\zeta^2 + 5 + \zeta^2\xi(\zeta^2)}\right]} \right]} \tag{46}$$

The value of $G(x^2,\zeta^2)$ is substituted for G in eq.(36). Again $G(x^2=0,\zeta^2=0) = \frac{1}{3}$.

The decimal accuracy, F, of the normalized difference, where $G(x^2,\zeta^2)$ is substituted for G, and the number of terms for convergence, N, are tabulated as a function of x^2 and the decimal accuracy, D, of the convergence criterion for $G(x^2,\zeta^2)$ in Table I.

As before, the decimal accuracy of the normalized difference is limited by the decimal accuracy of the computer. This time the improvement in the increase in accuracy of the continuing fraction $G(x^2,\zeta^2)$, as D is decreased from -1 to -15, ranges from a factor of 2.5 to .5. But the decrease in the number of terms required for convergence ranges from a factor of 200 to 0 as D goes from -1 to -15. This demonstrates the advantage of using this method of halving the argument of the inverse tangent function to produce a machine accurate answer to $\dfrac{1}{1 + x^2 G}$ with fewer terms required for convergence.

If this method is applied again to halve the argument, ζ, of the inverse tangent function which is evaluated by the continuing fraction expression $G(x^2,\zeta^2)$, will the number of terms required for convergence be reduced further? The same procedure for halving the argument is followed as before.

$$\left(\frac{E}{4}\right) = 2\left(\frac{E}{8}\right)$$

with $\omega^2 = \tan^2\left(\frac{E}{8}\right) = \dfrac{\sqrt{\zeta^2 + 1} - 1}{\sqrt{\zeta^2 + 1} + 1}$

$$\frac{\arctan \omega}{\omega} = \frac{1}{1 + \omega^2 G(\omega^2)}$$

$$\text{where} \quad G(\omega^2) = \cfrac{1}{3 + \cfrac{2^2\omega^2}{5 + \cfrac{3^2\omega^2}{7 + \cfrac{4^2\omega^2}{9 + \cdots}}}}$$

$$\frac{1}{1 + \zeta^2 G(\zeta^2)} = \left(\frac{2 \tan\left(\frac{E}{8}\right)}{\tan\left(\frac{E}{4}\right)} \right) \frac{1}{1 + \omega^2 G(\omega^2)}$$

Using the trigonometric identities

$$\frac{1 - \cos\left(\frac{E}{4}\right)}{2} = \sin^2\left(\frac{E}{8}\right)$$

$$\frac{1 + \cos\left(\frac{E}{4}\right)}{2} = \cos^2\left(\frac{E}{8}\right)$$

and substituting $\frac{\sqrt{\zeta^2 + 1} - 1}{\sqrt{\zeta^2 + 1} + 1}$ for ω^2 yields

$$\frac{1}{1 + \zeta^2 G(\zeta^2)} = \left(\frac{2}{\sqrt{\zeta^2 + 1} + 1} \right) \frac{1}{1 + \omega^2 G(\omega^2)}$$

$$G(\zeta^2) = \frac{1 + G(\omega^2)}{2\left(\sqrt{\zeta^2 + 1} + 1\right)}$$

$$\frac{1}{3 + \left[\dfrac{4\zeta^2}{5 + \zeta^2 \xi(\zeta^2)}\right]} = \frac{1}{2\left(\sqrt{\zeta^2 + 1} + 1\right)} \left(1 + \left(\frac{1}{3 + \left[\dfrac{4\omega^2}{5 + \omega^2 \xi(\omega^2)}\right]}\right)\right)$$

$$\text{where} \quad \xi(\omega^2) = \cfrac{3^2}{7 + \cfrac{4^2\omega^2}{9 + \cdots}}$$

Note $G(\zeta^2) = \cfrac{1}{3 + \left[\cfrac{4\zeta^2}{5 + \zeta^2\xi(\zeta^2)}\right]}$

and
$$\zeta^2\xi(\zeta^2) = \cfrac{8\left(\sqrt{\zeta^2 + 1} + 1\right)}{3 + \left[\cfrac{1}{\omega^2 + 5 + \omega^2\xi(\omega^2)}\right]} - 5 \tag{47}$$

Finally

$$G(x^2,\zeta^2,\omega^2) = \cfrac{1}{3 + \left[\cfrac{\cfrac{4x^2}{8\left(\sqrt{x^2+1}+1\right)}}{3 + \left[\cfrac{1}{\zeta^2 + \left[\cfrac{8\left(\sqrt{\zeta^2+1}+1\right)}{3 + \left[\cfrac{1}{\omega^2 + 5 + \left[\cfrac{3^2\omega^2}{7 + \cfrac{4^2\omega^2}{9 + \cdots}}\right]}\right]}\right]}\right]}\right]} \tag{48}$$

where $\omega^2 = \dfrac{\sqrt{\zeta^2 + 1} - 1}{\sqrt{\zeta^2 + 1} + 1}$

Again the algorithm to evaluate $G(x^2,\zeta^2,\omega^2)$ is a modified version of the algorithm to evaluate $G(x^2)$.

Using $\zeta^2 = \dfrac{x^2}{\left(\sqrt{x^2 + 1} + 1\right)^2}$

and $\omega^2 = \dfrac{\zeta^2}{\left(\sqrt{\zeta^2 + 1} + 1\right)^2} = \dfrac{\sqrt{\zeta^2 + 1} - 1}{\sqrt{\zeta^2 + 1} + 1}$

the following algorithm results

```
ZTA = SQRT(X2+1)+1
ZTA2 = X2/(ZTA*ZTA)
OMG = SQRT(ZTA2+1)+1
OMG2 = ZTA2/(OMG*OMG)
```

```
      N = 2
      AK = 3*3*OMG2
      AK2 = 4*4*OMG2
      BK = 7
      BK2 = 9
      UK = 1
      VK = AK/BK
      WK = VK
    1 UK = 1/(1+AK2/(BK*BK2)*UK)
      VK = VK*(UK-1)
      WK = WK+VK
      IF (ABS(VK).LT.CC)GOTO 2
      N = N+1
      AK2 = (N+2)*(N+2)*OMG2
      BK = 2*N+3
      BK2 = 2*(N+1)+3
      GOTO 1
    2 G = 5+OMG2+WK
      G = 3+1/G
      G = 8*(SQRT(ZTA2+1)+1)/G
      G = 1/(ZTA2+G)
      G = 3+G
      G = 8*(SQRT(X2+1)+1)/G
      G = 2*2*X2/G
      G = 1/(3+G)
```

This algorithm is used to calculate the continuing fraction $\omega^2\xi(\omega^2)$ to obtain $G(x^2,\zeta^2,\omega^2)$. The difference between $\omega^2\xi\left(\omega^2,N=i+1\right)$ and $\omega^2\left(\omega^2,N=i\right)$ is compared to the convergence criterion as follows

for cycle 1 (N=2), is $\left|\omega^2\xi\left(\omega^2,N=2\right) - \omega^2\xi\left(\omega^2,N=1\right)\right| < CC$

$$\left|\frac{3^2\omega^2}{7+\dfrac{4^2\omega^2}{9}} - \frac{3^2\omega^2}{7}\right| < CC$$

for cycle 2 (N=3), is $\left|\omega^2\xi\left(\omega^2,N=3\right) - \omega^2\xi\left(\omega^2,N=2\right)\right| < CC$

$$\left|\frac{3^2\omega^2}{7+\dfrac{4^2\omega^2}{9+\dfrac{5^2\omega^2}{11}}} - \frac{3^2\omega^2}{7+\dfrac{4^2\omega^2}{9}}\right| < CC$$

The algorithm is repeated to obtain a converged value for $\omega^2\xi(\omega^2, N=i+1)$. The following expression is used to obtain $G(x^2,\zeta^2,\omega^2)$.

$$G(x^2,\zeta^2,\omega^2) = \cfrac{1}{3 + \left[\cfrac{\cfrac{4x^2}{8\left(\sqrt{x^2+1}+1\right)}}{3 + \left[\cfrac{1}{\zeta^2 + \left[\cfrac{8\left(\sqrt{\zeta^2+1}+1\right)}{3 + \left[\cfrac{1}{\omega^2 + 5 + \omega^2\xi(\omega^2)}\right]}\right]}\right]}\right]}$$

The value of $G(x^2,\zeta^2,\omega^2)$ is substituted for G in eq.(36). Again $G(x^2=0,\zeta^2=0,\omega^2=0) = \frac{1}{3}$.

The decimal accuracy, F, of the normalized difference, where $G(x^2,\zeta^2,\omega^2)$ is substituted for G, and the number of terms for convergence, N, are tabulated as a function of x^2 and the decimal accuracy, D, of the convergence criterion for $G(x^2,\zeta^2,\omega^2)$ in Table I.

Again the decimal accuracy of the normalized difference is limited by the decimal accuracy of the computer. The improvement in the increase in accuracy of the continuing fraction $G(x^2,\zeta^2,\omega^2)$, as D is decreased from −1 to −15, ranges from a factor of .6 to 0 when compared to F for $G(x^2,\zeta^2)$. This time the decrease in the number of terms required for convergence, N, ranges from a factor of 4 to 0.

Now $\omega^2\xi(\omega^2)$ is converged for $x = i0.99999$, $x^2 = -.9999800001$, to obtain $G(x^2,\zeta^2,\omega^2)$. The continuing fraction converges to a criterion of 1×10^{-1} at the third cycle (N=4) with a normalized difference of less than 2.5×10^{-3}. Therefore the convergence of the continuing fraction $G(x^2,\zeta^2,\omega^2)$ of the universal time of flight equation, eq.(36), has been examined for $-.9999800001 \leq x^2 \leq 3249.0$, corresponding to

$$i6.103033823 \text{ hyperbolic radians} \leq i \operatorname{arctanh}\left(\sqrt{|x^2|}\right) \text{ and}$$

$$\arctan x \leq \left(\frac{\pi}{2.00039196}\right) \text{ circular radians,}$$

and found to be machine accurate with an acceptable number of terms required for convergence.

It is not surprising that the decrease in the number of terms required for convergence between $G(x^2,\zeta^2)$ and $G(x^2,\zeta^2,\omega^2)$ is much smaller than the decrease between $G(x^2)$ and $G(x^2,\zeta^2)$. Table II presents the number of terms required to converge the continuing fractions associated with x^2, ζ^2 and ω^2 as E varies from 0° to 177.99° for a convergence criterion of 1×10^{-15} . The first application of the method of halving the argument of the inverse tangent function reduces the number of terms required for convergence significantly for large values of E. A second application of this method does not reduce the number of terms required for convergence as much. The reason for this is evident from the values of the arguments of the continuing fractions, x^2, ζ^2 and ω^2. With the first application of the method the argument of the resulting continuing fraction, ζ^2, is made less than one for the largest value of E, x and x^2. With $\zeta^2 < 1$ the continuing fraction can converge much faster. The second application of the method does not reduce the argument of the associated continuing fraction, ω^2, nearly as much. Therefore the continuing fraction of ω^2 does not converge much faster that the continuing fraction of ζ^2.

The universal time of flight equation has been found to be determinate for $-2\pi < \eta < 2\pi$ and the convergence of the continuing fraction G has been examined. Now the accuracy of the universal time of flight equation, eq.(36), is examined using the following normalized difference

$$\frac{\Delta t_k - \Delta t}{\Delta t_k}$$

Where Δt_k is one of three classical time of flight equations depending upon the type of orbit: elliptic, parabolic or hyperbolic. Remember, the universal time of flight equation is valid for any type of orbit. The normalized difference will provide a measure of accuracy relative to the classical time of flight equations.

The following classical time of flight equation for the elliptic orbit is know as "Kepler's Equation" (Ref. 2).

$$\sqrt{\frac{\mu}{a^3}}\left(t - \tau\right) = E - e \sin E$$

From this equation the following equation for the elliptic orbit is derived

$$\Delta t_k\left(0 \leq e' < 1\right) = \sqrt{\frac{a^3}{\mu}}\left[\left(E_2' - e'\sin E_2'\right) - \left(E_1' - e'\sin E_1'\right)\right]$$

$$\text{where} \quad \cos E_1' = \frac{e' + \cos \nu_1}{e'\cos \nu_1 + 1}$$

$$\cos E_2' = \frac{e' + \cos \nu_2}{e'\cos \nu_2 + 1}$$

$$\text{and} \quad \nu_2 = \nu_1 + \eta$$

Note that this equation is a function of the eccentric anomaly of the original orbit (E'). The classical time of flight equations for the parabolic and hyperbolic orbits can be derived using the concepts of Kepler's equation (Ref. 2). These equations are

$$\Delta t_k\left(e' = 1\right) = \frac{1}{2\sqrt{\mu}}\left[\left(p'D_2' + \frac{\left(D_2'\right)^3}{3}\right) - \left(p'D_1' + \frac{\left(D_1'\right)^3}{3}\right)\right]$$

$$\text{where} \quad D_1 = \sqrt{p'} \tan\left(\frac{\nu_1}{2}\right)$$

$$D_2 = \sqrt{p'} \tan\left(\frac{\nu_2}{2}\right)$$

for the parabolic orbit and

$$\Delta t_k\left(1 < e'\right) = \sqrt{\frac{(-a)^3}{\mu}}\left[\left(e'\sinh F_2' - F_2'\right) - \left(e'\sinh F_1' - F_1'\right)\right]$$

$$\text{where} \quad \cosh F_1' = \frac{e' + \cos \nu_1}{e'\cos \nu_1 + 1}$$

$$\cosh F_2' = \frac{e' + \cos \nu_2}{e'\cos \nu_2 + 1}$$

and $$F_1' = \log_e\left(\cosh F_1' + \sqrt{\cosh^2 F_1' - 1}\right)$$

$$F_2' = \log_e\left(\cosh F_2' + \sqrt{\cosh^2 F_2' - 1}\right)$$

for the hyperbolic orbit. These equations are functions of the semi-major axis, a, and the semi-latus rectum (parameter), p', of the original orbit.

These three classical equations and the universal time of flight equation are mechanized on a computer in canonical units to generate the normalized difference $\frac{\Delta t_k - \Delta t}{\Delta t_k}$. These mechanized time of flight equations are incorporated into a parametric study to examine the overall accuracy of the universal time of flight equation.

The first parametric study, P.S.1.1, is conducted as a function of r_1, e', CC and the type of continuing fraction used (TCF) for an array of ν_1 $\left(0^\circ \leq \nu_1 < 360^\circ \ , \ \Delta\nu = 40^\circ\right)$ and η $\left(-360^\circ \leq \eta \leq 360^\circ \ , \ \Delta\eta = 20^\circ\right)$. This results in 324 calculations of the normalized difference for each r_1, e', CC and TCF combination (parametric study case). The combinations for the five cases of P.S.1.1 are as follows

<u>P.S.1.1</u>

For r_1 = 1.2 , e' = .0, .5, .9
for each e' , TCF = $G(x^2)$, $G(x^2,\zeta^2)$, $G(x^2,\zeta^2,\omega^2)$
for each TCF , CC = 1×10^{-1}, 1×10^{-5}, 1×10^{-15}

For r_1 = 1.2 , e' = .99, .9999, .999999
for each e' , TCF = $G(x^2,\zeta^2,\omega^2)$
for each TCF , CC = 1×10^{-1}, 1×10^{-5}, 1×10^{-15}

For r_1 = 1.2 , e' = 1.0
for each e' , TCF = $G(x^2)$, $G(x^2,\zeta^2)$, $G(x^2,\zeta^2,\omega^2)$
for each TCF , CC = 1×10^{-1}, 1×10^{-5}, 1×10^{-15}

For r_1 = 1.2 , e' = 1.0000001, 1.00001, 1.001
for each e' , TCF = $G(x^2,\zeta^2,\omega^2)$
for each TCF , CC = 1×10^{-1}, 1×10^{-5}, 1×10^{-15}

For r_1 = 1.2 , e' = 1.1, 2.5
for each e' , TCF = $G(x^2)$, $G(x^2,\zeta^2)$, $G(x^2,\zeta^2,\omega^2)$
for each TCF , CC = 1×10^{-1}, 1×10^{-5}, 1×10^{-15}

The inputs CC and TCF pertain to the convergence of and type of continuing fraction used in the universal time of flight equation. The parametric study inputs ν_1 and η are chosen to ensure that representative combinations of starting true anomaly and change in true anomaly are considered for each orbit specified by r_1 and e'. The parametric study input e' is chosen to ensure that all three types of orbits are included in the study.

But the universal time of flight equation is a funciton of r_1, r_2, η and ϕ_1 where the inputs of the parametric study are r_1, e', ν_1 and η. These input parameters provide control over the orbital cases being studied since the variables of the universal time of flight equation do not permit this direct control. The variables of the universal time of flight equation are generated from the parameteric study inputs as follows

$$h' = \sqrt{r_1\mu\left(1 + e'\cos\nu_1\right)}$$

$$\varepsilon' = \frac{\mu^2\left((e')^2 - 1\right)}{2(h')^2} \equiv \text{ specific mechanical energy of the original orbit}$$

$$V_1' = \sqrt{2\left(\varepsilon' + \frac{\mu}{r_1}\right)}$$

$$\phi_1 = \begin{cases} -\arccos\left[h'/\left(r_1 V_1'\right)\right] & , \text{ if } \nu_1 > \frac{\pi}{2} \\ \arccos\left[h'/\left(r_1 V_1'\right)\right] & , \text{ if } \nu_1 \leq \frac{\pi}{2} \end{cases}$$

$$p' = \frac{(h')^2}{\mu}$$

$$r_2 = \frac{p'}{1 + e'\cos\left(\nu_1 + \eta\right)}$$

where $\mu=1$ (in canonical units)

The semi-major axis of the original orbit is generated as follows.

$$a = \frac{(h')^2}{\mu\left(1 - (e')^2\right)}$$

Because of the large variation in the normalized differences of each parametric study case (r_1, e', CC, and TCF combination), the decimal accuracy, B, of the normalized difference is more significant than the value itself.

$$ND_{TE} \times 10^B = \frac{\Delta t_k - \Delta t}{\Delta t_k}$$

Because of the large number of normalized differences for each parametric study case, a maximum (B_1), average (if any) (B_2), and minimum value (B_3) of the decimal accuracy B are tabulated as a function of r_1, e', TCF and D. The variable D is the decimal accuracy of the convergence criterion, CC,

$CC = 1 \times 10^D$

for the type of continuing fraction used, TCF, and

$B_1 \equiv$ a maximum decimal accuracy from the 324 normalized differences of the parametric study case

$B_2 \equiv$ an average decimal accuracy (if any) from the 324 normalized differences of the parametric study case

$B_3 \equiv$ a minimum decimal accuracy from the 324 normalized differences of the parametric study case

The value of B is limited by the decimal accuracy of the computer (B_{max}=−16) used to do the parametric study. If the normalized difference is exactly zero then $B_3 = -\infty$. The values of B are presented in Table III.

For each parametric study case 324 continuing fractions were converged. Table IV contains N_{max}, the largest value of N (where N is the number of terms required to converge the continuing fraction) for each parametric study case, as a function of r_1, e', TCF and D.

Referring to Table III and Table IV the expected decrease in the normalized difference, B, as D decreases is apparent. As TCF varies from $G(x^2)$ to $G(x^2,\zeta^2)$ there is a small decrease in B but a significantly large decrease in N_{max}. This is why the method of halving the argument of the inverse tangent function was used. As TCF is changed from $G(x^2,\zeta^2)$ to $G(x^2,\zeta^2,\omega^2)$ there is essentially no change in B, and N_{max} is cut in half. The decrease in the decimal accuracy of the normalized difference appears to be independent of eccentricity with the exception of the region just below and above e'=1. This may be due to the error generated in computing ϕ_1 and r_2 from r_1, e', ν_1

and η. But N_{max} does not appear to be affected. From the parametric study, the variable x^2 is found to be a very large number for e' just less than one and a very small number for e' equal to one. This would explain the apparent independence of B and N_{max} from TCF and D at e'=1. It is interesting to note that the decimal accuracy of the normalized difference for e' $\leq$ .9 and e' $\geq$ 1.1 is about the same whereas N_{max} increases with e'. This is probably due to the nature of the inverse tangent and inverse hyperbolic tangent functions.

The second part of the first parametric study P.S.1.2 examines B and N_{max} for a larger initial radius as follows

P.S.1.2

For r_1 = 5.0 , e' = .0, .5, .9, 1.0, 1.1, 2.5
for each e' , TCF = $G(x^2,\zeta^2,\omega^2)$
for each TCF , CC = 1×10^{-15}

The data of Table V indicates that the decimal accuracy of the normalized difference and N_{max} are independent of r_1.

The universal time of flight equation appears to be consistently accurate when compared to the classical time of flight equations and the accuracy is improved by the method of halving the argument of the inverse tangent function and decreasing the convergence criterion.

The second parametric study examines the accuracy of the universal time of flight equation for a small, constant specific mechanical energy as the eccentricity approaches one from above and below e'=1. This study is to examine the performance of the universal time of flight equation for near parabolic and near rectilinear orbits. The study is conducted as a function of the specific mechanical energy of the original orbit (ε'), e', TCF and CC for any array of ν_1 $\left(0^\circ \leq \nu_1 < 360^\circ\ ,\ \Delta\nu = 40^\circ\right)$ and η $\left(-360^\circ \leq \eta \leq 360^\circ\ ,\ \Delta\eta = 20^\circ\right)$. The combinations of ε', e', TCF and CC for the four cases of P.S.2 are as follows

<u>P.S.2</u>

For $\varepsilon' = -0.1$, $e' = .9999, .999999$
for each e' , TCF $= G(x^2, \zeta^2, \omega^2)$
for TCF , CC $= 1\times10^{-15}$

For $\varepsilon' = -0.0000001$, $e' = .9999, .999999$
for each e' , TCF $= G(x^2, \zeta^2, \omega^2)$
for TCF , CC $= 1\times10^{-15}$

For $\varepsilon' = 0.0000001$, $e' = 1.0000001, 1.00001$
for each e' , TCF $= G(x^2, \zeta^2, \omega^2)$
for TCF , CC $= 1\times10^{-15}$

For $\varepsilon' = 0.1$, $e' = 1.0000001, 1.00001$
for each e' , TCF $= G(x^2, \zeta^2, \omega^2)$
for TCF , CC $= 1\times10^{-15}$

Again the universal time of flight equation is a function of r_1, r_2, η and ϕ_1. These variables are generated from the parametric study inputs ε' and e' as follows

$$r_1 = \frac{\left((e')^2 - 1\right)\mu}{2\ \varepsilon'\left(1 + e'\cos \nu_1\right)}$$

$$h' = \sqrt{r_1\ \mu\left(1 + e'\cos \nu_1\right)}$$

The rest of the variables are generated as in P.S.1.

Table VI contains the decimal accuracy of the normalized differences, B, and N_{max} as a function of ϵ', e', TCF and D for P.S.2. This data indicates that the accuracy is similar to the first parametric study. The decimal accuracy of the normalized difference is larger for a value of e' closer to one. Again this may be due to the error generated in computing r_1 using the term $\left((e')^2 - 1\right)$. The universal time of flight equation appears to be accurate for a small, constant value of specific mechanical energy for near parabolic and near rectilinear orbits.

CHAPTER 6 SUMMARY

A universal time of flight equation for flight between two arbitrary points in any orbit was developed as a function of r_1, r_2, η and ϕ_1. The symmetry property of Lambert's theorem was used to transform the original orbit into an orbit symmetric about an invariant semi-major axis. A trigonometric substitution was used to keep the argument of an inverse tangent function within the principle range of the function. The quotient, $\frac{\arctan x}{x}$, was replaced by a continuing fraction expression which resulted in one time of flight equation for any orbit. The new corollary to Lambert's theorem was used to relate parameters of the transformed orbit to the parameters of the original orbit, r_1, r_2, η and ϕ_1. The resulting universal time of flight equation is not explicitly a function of eccentricity. This universal time of flight equation was found to be determinate for $-2\pi < \eta < 2\pi$.

A method of halving the argument of the inverse tangent function to make the continuing fraction converge faster was examined using a top down algorithm to evaluate the resulting continuing fraction. The method does make the continuing fraction converge faster to the decimal accuracy of the computer used.

The universal time of flight equation was compared to the classical time of flight equations. For all types of orbits, including near parabolic orbits, the universal time of flight equation was found to be machine accurate with an acceptable number of terms required to converge the continuing fraction. This was also found to be true for near parabolic and near rectilinear orbits with small values of specific mechanical energy.

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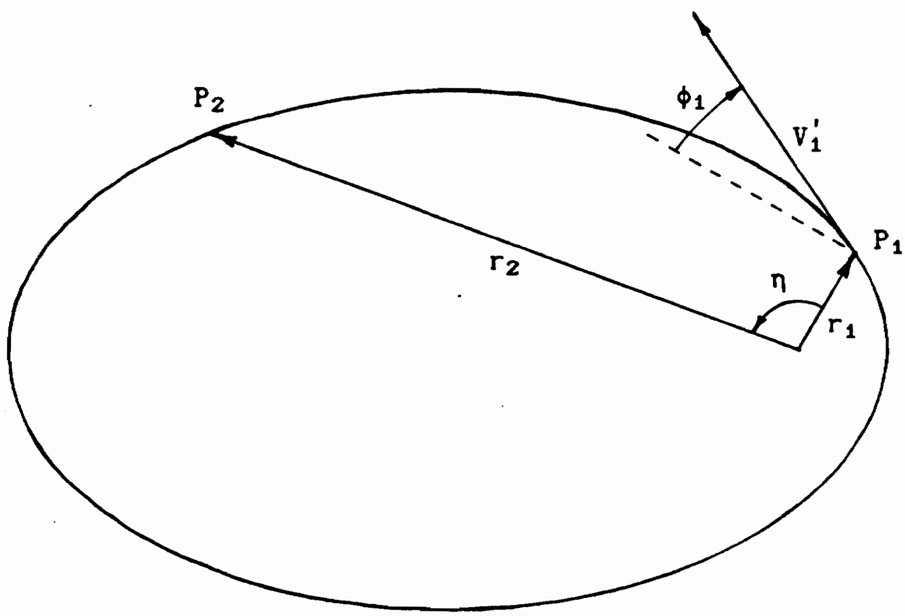

Fig. 1 - The Original Orbit

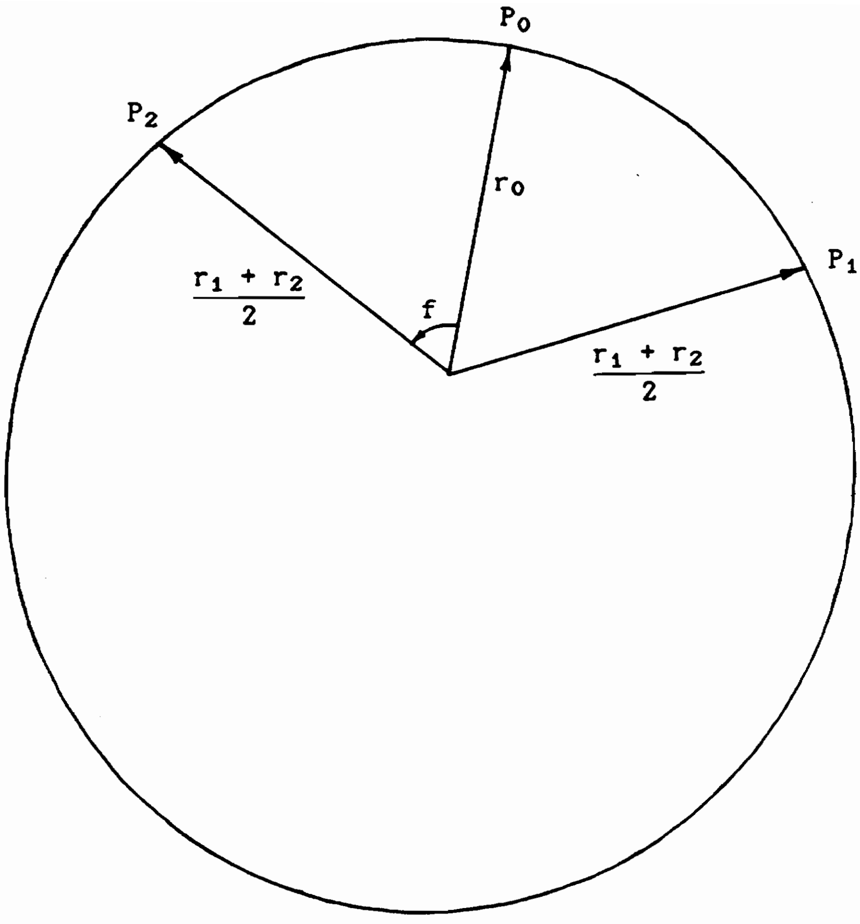

Fig. 2 - The Transformed Orbit

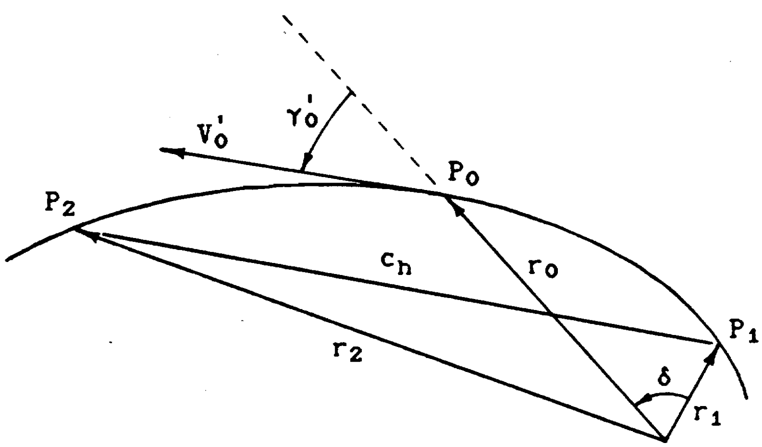

Fig. 3 - The Normal Point

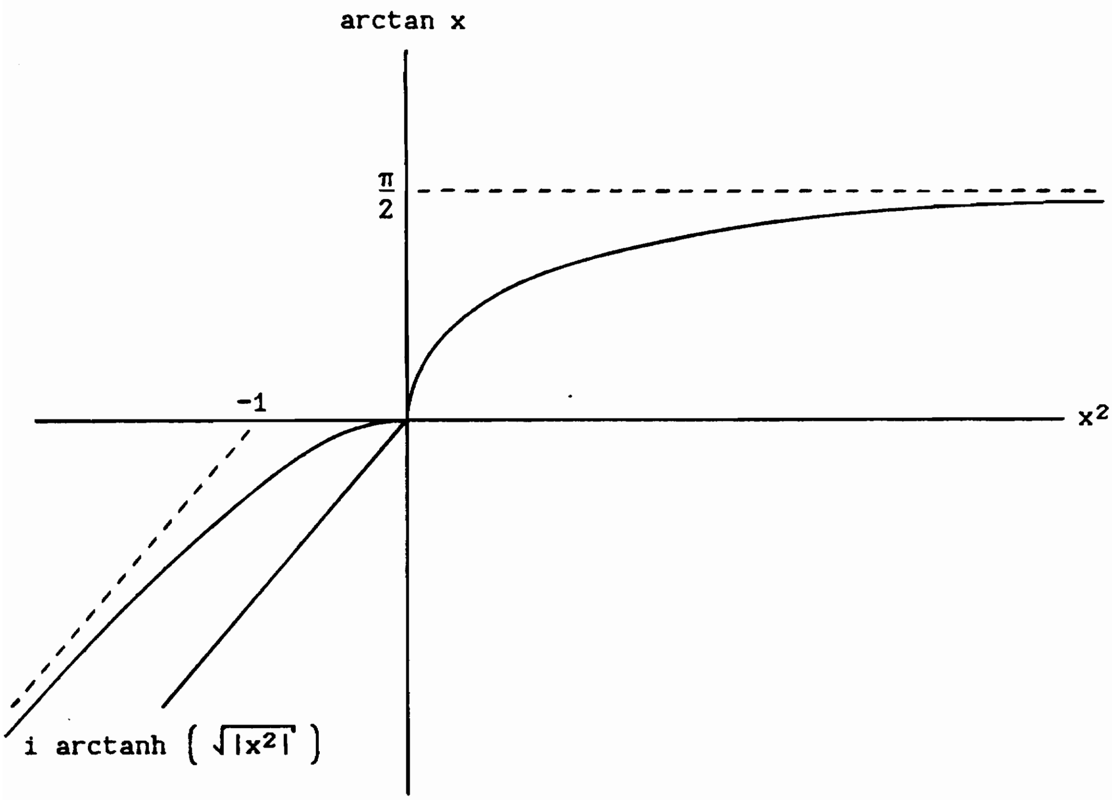

Fig. 4 - The Arctangent and Hyperbolic Arctangent Functions

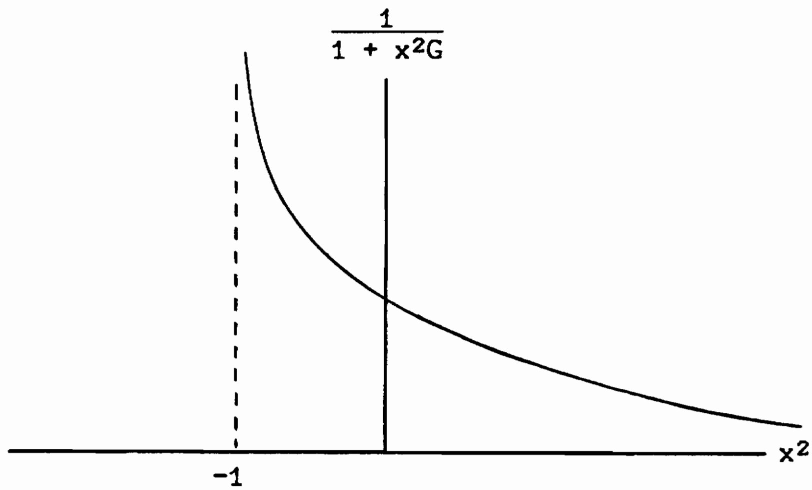

Fig. 5 - The Continuing Fraction Substitution

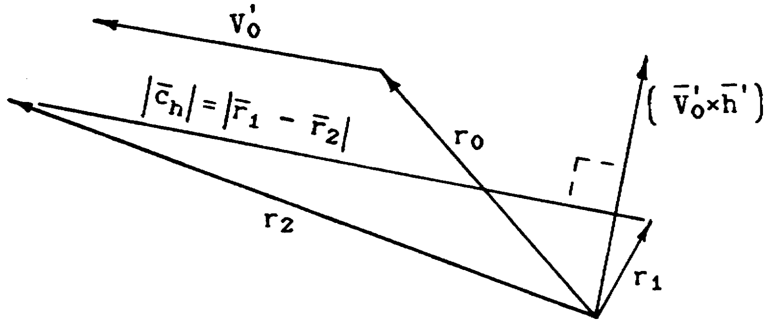

Fig. 6 - $\bar{V}_0' \times \bar{h}'$

TABLE I ACCURACY OF THE CONTINUING FRACTIONS

F
N

x^2	$G(x^2)$			$G(x^2,\zeta^2)$			$G(x^2,\zeta^2,\omega^2)$		
	D=-1	-5	-15	-1	-5	-15	-1	-5	-15
-.81	F=-2	-6	-16	-5	-9	-∞	-8	-11	-14
	N=0	10	35	0	5	16	0	2	9
-.25	-4	-7	-∞	-9	-12	-∞	-12	-13	-∞
	0	3	12	0	4	8	0	1	5
-.01	-8	-8	-16	-16	-16	-16	-16	-16	-16
	0	0	4	0	0	3	0	0	3
.01	-8	-8	-15	-15	-15	-∞	-∞	-∞	-∞
	0	0	4	0	0	3	0	0	3
.25	-4	-8	-16	-10	-13	-16	-13	-15	-16
	0	3	11	0	2	7	0	1	5
1.0	-3	-7	-16	-7	-12	-16	-10	-12	-16
	0	6	19	0	3	10	0	1	6
100.0	-3	-7	-13	-5	-9	-16	-8	-11	-∞
	26	72	187	1	6	18	0	2	9
225.0	-3	-7	-16	-5	-9	-∞	-8	-12	-∞
	43	112	285	1	6	18	0	3	9
625.0	-3	-7	-10	-5	-9	-∞	-7	-12	-15
	79	194	482	1	6	19	0	3	10
1225.0	-3	-7	-15	-5	-9	-∞	-7	-12	-∞
	117	278	681	1	6	19	0	3	10
1600.0	-3	-7	-16	-5	-9	-16	-7	-12	-11
	137	321	781	1	6	19	0	3	10
2500.0	-3	-7	-16	-5	-9	-15	-7	-12	-16
	177	407	983	1	6	19	0	3	10
3249.0	-3	-7	-10	-5	-9	-16	-7	-12	-11
	206	468	1124	1	6	19	0	3	10

$ND_{CF} \times 10^F$ = Normalized Difference of the Continuing Fraction
N = Number of terms required to converge the continuing fraction
$CC = 1 \times 10^D$ = Convergence Criterion

TABLE II CONVERGENCE OF THE CONTINUING FRACTIONS

E	x	x^2	ζ^2	ω^2
0.0°	0.0	0.0	0.0	0.0
11.42°	0.1	0.01	0.002486	0.0006211
90.00°	1.0	1.0	0.1716	0.03957
168.58°	10.0	100.0	0.8190	0.1485
176.73°	35.0	1225.0	0.9445	0.1647
177.99°	57.0	3249.0	0.9655	0.1674

	N for CC = 1×10^{-15}		
x	$N(x^2)$	$N(\zeta^2)$	$N(\omega^2)$
0.0	0	0	0
0.1	4	3	3
1.0	19	10	6
10.0	187	18	9
35.0	681	19	10
57.0	1124	19	10

x^2, ζ^2, ω^2 = Arguments of the continuing fractions

N = Number of terms required to converge the continuing fraction

TABLE III ACCURACY OF THE UNIVERSAL TIME OF FLIGHT EQUATION

$r_1=1.2$ B_1 B_2 B_3

e'	$G(x^2)$			$G(x^2,\zeta^2)$			$G(x^2,\zeta^2,\omega^2)$		
	D=-1	-5	-15	-1	-5	-15	-1	-5	-15
.0	B_1=-2 B_2=* B_3=-8	-6 * -9	-8 -16 -∞	-5 * -16	-8 * -16	-8 -16 -∞	-8 * -16	-8 * -16	-8 -16 -∞
.5	-2 * -10	-6 * -11	-8 -16 -∞	-5 * -16	-8 * -16	-8 -16 -∞	-8 * -16	-8 * -16	-8 -16 -∞
.9	-2 * -10	-6 * -11	-8 * -16	-5 * -16	-7 * -16	-8 * -16	-7 * -16	-8 * -16	-8 * -∞
.99							-5 * -16	-5 -11 -16	-8 * -16
.9999							-7 * -14	-7 * -14	-5 * -14
.999999							-3 -7 -12	-5 * -12	-4 * -12
1.0	-10 -16 -∞	-10 -16 -∞	-10 * -16	-10 * -16	-10 * -16	-10 * -16	-10 * -16	-10 * -16	-10 * -16
1.0000001							-4 * -10	-4 * -10	-4 * -11
1.00001							-5 * -13	-5 * -13	-5 * -13
1.001							-7 * -14	-7 * -14	-7 * -14
1.1	-4 * -11	-7 * -10	-7 * -16	-7 * -15	-7 * -16	-7 * -16	-7 * -16	-7 * -16	-7 * -16
2.5	-3 * -10	-8 * -10	-8 -16 -∞	-8 * -15	-8 * -15	-8 -16 -∞	-8 * -15	-8 * -16	-8 -16 -∞

$ND_{TE} \times 10^{Bi}$ = Normalized Difference of the universal Time of flight Equation where i=1,2,3

$CC = 1 \times 10^{D}$ = Convergence Criterion

* There is no discernable average value for B.

TABLE IV ACCURACY OF THE UNIVERSAL TIME OF FLIGHT EQUATION

$r_1=1.2$ — N_{max}

e'	$G(x^2)$ D=-1	$G(x^2)$ -5	$G(x^2)$ -15	$G(x^2,\zeta^2)$ -1	$G(x^2,\zeta^2)$ -5	$G(x^2,\zeta^2)$ -15	$G(x^2,\zeta^2,\omega^2)$ -1	$G(x^2,\zeta^2,\omega^2)$ -5	$G(x^2,\zeta^2,\omega^2)$ -15
.0	31	83	215	1	6	18	0	2	9
.5	59	150	377	1	6	18	0	3	10
.9	175	402	972	1	6	19	0	3	10
.99							0	3	10
.9999							0	3	10
.999999							0	3	10
1.0	0	0	0	0	0	0	0	0	0
1.0000001							0	0	0
1.00001							0	0	1
1.001							0	0	3
1.1	0	4	14	0	2	9	0	1	6
2.5	0	6	22	0	3	12	0	2	7

N_{max} = largest value of N in a study case, where N is the number of terms required to converge a continuing fraction

CC = 1×10^D = Convergence Criterion

TABLE V ACCURACY OF THE UNIVERSAL TIME OF FLIGHT EQUATION

$r_1 = 5.0$

e'	$G(x^2, \zeta^2, \omega^2)$, D=-15 B_1 B_2 B_3	N_{max}
.0	-8 -16 $-\infty$	11
.5	-7 -16 $-\infty$	10
.9	-5 -16 $-\infty$	10
1.0	-8 -16 $-\infty$	0
1.1	-5 * -16	6
2.5	-7 * -16	7

$ND_{TE} \times 10^{Bi}$ = Normalized Difference of the universal Time of flight Equation where i=1,2,3

N_{max} = largest value of N in a study case, where N is the number of terms required to converge a continuing fraction

$CC = 1 \times 10^D$ = Convergence Criterion

* There is no discernable average value for B.

TABLE VI ACCURACY OF THE UNIVERSAL TIME OF FLIGHT EQUATION

$r_1=1.2$ Constant Specific Mechanical Energy

ε'	e'	$G(x^2,\zeta^2,\omega^2)$, D=-15 B_1	B_2	B_3	N_{max}
-.1	.9999	-6	*	-14	10
-.1	.999999	-3	*	-12	10
-.0000001	.9999	-3	*	-14	10
-.0000001	.999999	-2	*	-12	10
.0000001	1.0000001	-4	*	-11	0
.0000001	1.00001	-5	*	-13	1
.1	1.0000001	-3	*	-11	0
.1	1.00001	-5	*	-13	1

$ND_{TE} \times 10^{Bi}$ = Normalized Difference of the universal Time of flight Equation where i=1,2,3

N_{max} = largest value of N in a study case, where N is the number of terms required to converge a continuing fraction

CC = 1×10^D = Convergence Criterion

* There is no discernable average value for B.

VITA

The author was born on the 14th of January 1953 at the Presidio, San Francisco, CA. He served in the U.S. Air Force from September 1972 to December 1976 as an Air Cargo Specialist. The author graduated from Virginia Polytechnic Institute and State University with a Bachelor of Science degree in Aerospace and Ocean Engineering, June of 1981. In September of 1981 he was employed as an Aerospace Engineer at the Naval Surface Weapons Center, Dahlgren, VA where he enrolled in the Virginia Tech Graduate Program in 1982. This thesis fulfills the requirements for a Master of Science degree in Aerospace Engineering from Virginia Polytechnic Institute and State University.

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