OPTIMAL STATE ESTIMATION
FOR THE
OPTIMAL CONTROL OF FAR-FIELD ACOUSTIC RADIATION
PRESSURE FROM SUBMERGED PLATES

by

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ABSTRACT

Sound pressure radiating from vibrating structures submerged in fluid, as in the case of a vibrating panel in a submarine hull, is usually undesirable. An optimal control methodology for the suppression of far-field acoustic radiation pressure from submerged structures has been developed by Meirovitch (ref. 1). The linear modal state feedback control law developed implies that the full state (displacements and velocities) is available, perhaps through measurements. However, in practice it is not always feasible to measure the full modal state vector for feedback. To permit practical implementation of the feedback control law, an optimal stochastic state estimator, or Kalman-Bucy filter, has been developed here for incorporation into the control system design. The development has been specialized to a uniform simply supported rectangular plate.

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NOMENCLATURE

x,y,z	Cartesian coordinates with origin on plate surface
R,θ,ϕ	Spherical coordinates with origin on the plate surface
$P = P(x,y)$	Typical point on plate surface
$P_f = P_f(R,\theta,\phi)$	Typical point in the fluid medium
Ω	Domain of the structure
$\partial\Omega$	Boundary of the structure
Ω_f	Region in the fluid over which the far-field pressure is to be minimized
$m(P)$	Mass density of the plate material
L	Homogeneous self-adjoint differential operator
$2a$	Length of plate along x coordinate
$2b$	Length of plate along y coordinate
E	Young's modulus of plate material
υ	Poisson's ratio of plate material
B_i	Homogeneous boundary differential operators
$f_c(x,y,t)$	Control force density
$f_d(x,y,t)$	Harmonic persistent disturbance force density
$f_p(x,y,t)$	Pressure exerted by fluid on the plate surface
$w(P,t)$	Displacement of plate surface in z-direction
$p(P_f,t)$	Acoustic pressure in fluid
c	Speed of sound in fluid
η	Outward normal to plate surface
ρ	Mass density of the fluid
k	Acoustic wavenumber
$w(P)$	Displacement amplitude
$\ddot{w}(P)$	Acceleration amplitude
ω	Frequency of harmonic oscillation
$p(P_f)$	Acoustic pressure amplitude
γ_x, γ_y	Fourier transform variables

$\tilde{f}(\gamma_x, \gamma_y)$	Double Fourier transform of f(x,y)
W_{rs}, W_r	Orthogonal *in vacuo* modes of plate vibration
ω_{rs}, ω_r	*In vacuo* natural frequencies of plate vibration
$q_{rs}(t)$, $q_r(t)$	Modal displacements
$f_{jrs}(t)$	Modal forces (j = c,d,p)
$\underset{\sim}{q}(t)$	Modal displacement vector
Λ	Diagonal matrix of eigenvalues
C	Fluid-loading matrix
$\underset{\sim}{q}_C(t)$	Vector of controlled modal displacements
$\underset{\sim}{q}_R(t)$	Vector of uncontrolled, or residual, modal displacements
C_C	Controlled partition of fluid-loading matrix
C_R	Residual partition of fluid-loading matrix
Λ_C	Controlled partition of eigenvalue matrix
Λ_R	Residual partition of eigenvalue matrix
I	Identity matrix
n	Number of controlled modes
N	Number of actuators
q	Number of sensors
B_{cC}', B_{cR}'	Modal participation matrices
$h_1(P)$, $h_2(P)$	Weighting functions in performance measure for optimal control
a(P), b(R,θ,φ)	Weighting functions in performance measure for optimal control
$\underset{\sim}{x}(t)$	Modal state vector
A,B,D,C_o	Coefficient matrices for standard state notation
H,Q*,R*	Weighting matrices for discretized performance measure
Q,R,S,T,U,V	Weighting matrices for discretized performance measure
$\underset{\sim}{p}(t)$	Costate Vector
$\underset{\sim}{F}_c(t)$	Feedback control force
$\underset{\sim}{\hat{x}}(t)$	Estimated state vector
$\underset{\sim}{e}(t)$	Observation error vector
G(t)	Matrix of feedback control gains

$L(t)$	Matrix of observer gains
$E\{\ \}$	Expected value or mean value of stochastic vector
C_v	Covariance matrix of stochastic vector
R_v	Correlation matrix of stochastic vector
R_{vz}	Cross-correlation matrix of two stochastic vectors
$V_1(t)$, $V_2(t)$	Intensities of white noise processes
$\underset{\sim}{w}_1(t)$	Random state excitation vector
$\underset{\sim}{w}_2(t)$	Random observation noise vector
$\tilde{Q}(t)$, $Q(t)$	Variance matrices of the observation error vector
$L^*(t)$	Matrix of optimal observer gains
ε_1, ε_2	Coordinate system with origin at corner of plate
k_r, k_s, k_m, k_n	Modal wavenumbers
K_r, K_s, K_m, K_n	Nondimensional modal wavenumbers
G_x, G_y	Nondimensional Fourier transform variables
K	Nondimensional acoustic wavenumber
θ, X, Y	Polar coordinates in G_x, G_y plane

I. Introduction

When a plate submerged in fluid undergoes vibration, energy is radiated from the surface of the structure in the form of sound pressure. There are many applications in which this sound radiation pressure is an undesirable effect, such as the case of sound radiating from the hull of a submarine or noise generated by vibration of panels on an aircraft wing. Therefore, developing methods of reducing or eliminating sound radiation pressure from submerged plates has been a topic of great interest in recent years (ref. 1).

The presence of the surrounding fluid has three specific effects on the vibration of the plate (ref. 2). First, the plate must push the fluid and thus appears to have an added inertia when compared to a plate vibrating *in vacuo*. This added inertia tends to lower the natural frequencies of vibration of the plate. Secondly, the vibrating plate radiates energy into the fluid. This loss of energy causes the amplitude of a freely vibrating plate to decay over time in a fashion similar to the effects of viscous damping. Finally, the fluid serves as a coupling medium for the transfer of energy between modes of vibration of the plate. Motion in one mode causes acoustic pressures which excite other modes. This transfer of energy causes the mode shapes of a plate vibrating in fluid to differ from those of the plate vibrating *in vacuo*. All of these fluid-structure interaction effects increase the complexity of the problem and add to the difficulty in formulating a control methodology for the suppression of the radiated acoustic pressure.

The fluid-structure interaction problem involves calculation of acoustic pressures at the plate surface and determination of the structural response. When a plate is vibrating in a low density fluid such as air, the radiation loading at the plate surface is generally negligible (ref. 3). Consequently, the plate dynamic response may be solved independently of the acoustic pressure field generated by the vibrating plate. On the other hand, for a dense fluid such as seawater the dynamic response of the plate depends on the fluid-loading, while the fluid-loading depends on the acceleration of the plate surface. Thus the two problems are integrally coupled and must be solved simultaneously. The complex nature of the problem almost inevitably leads to the use of numerical techniques.

One possible method of controlling sound radiation pressure is by reducing the vibration of the plate passively, through the addition of external damping materials (ref. 1). This approach may be able to reduce to some extent the pressure contributed by all modes of vibration of the structure. However, damping materials may not produce the desired overall sound pressure reduction. It has been shown that only certain modes contribute significantly to the far-field pressure (ref. 3). Thus, controlling all of the plate modes represents overdesign which can lead to undesirable excess structural weight. A more efficient method of controlling sound radiation pressure is to suppress only the modes of vibration which contribute to the far-field pressure. Since suppressing selective modes is not possible using passive means, this suggests the use of an active control system (ref. 4).

The use of active controls to suppress sound radiation dates back as early as 1936, when Paul Lueg received a patent for a control system design based on the destructive interference of two sound waves 180 degrees out of phase. This type of open-loop approach works well only for simple sources such as point sources or uniformly vibrating spheres. For the case of a vibrating plate, the surface does not move in uniform phase, and therefore many sound waves of varying phase are radiated. Attenuation of the far-field pressure would require a large number of secondary sources (ref. 5).

Another active approach to suppressing far-field radiation is to control the vibration of the structure via a closed-loop approach known as state feedback (ref. 4). This method of control involves the placement of discrete sensors and actuators on the plate surface and the calculation of feedback forces so as to suppress the modes contributing to the far-field radiation. Using modern control theory, an optimal control system can be designed to minimize a system performance measure involving such quantities as the far-field acoustic pressure, the state of the system, and the amount of control effort (ref. 4). This technique permits automatic selection of feedback gains so as to control the necessary modes of vibration.

When dealing with complex systems, it is not always feasible to measure all of the state variables (modal displacements and velocities) for feedback. In this situation, it becomes necessary to generate an estimate of the full state based on the output from a finite number of discrete sensors. The estimated state vector is then substituted into the feedback control law in place of the actual state vector. State estimation is carried out by means of an observer, a dynamic system which has as input the inputs and measured outputs of the original system (ref. 6). When the sensor and actuator noise to signal ratio is small, the observer is deterministic; it is commonly known as a Luenberger observer. When the sensor noise to signal ratio is high, the observer is stochastic; it is referred to as a Kalman-Bucy filter. The dynamics and initial conditions of the Luenberger observer are relatively free to be determined by the designer through the choice of appropriate observer gains. In general, the gains should be chosen so that the dynamics of the observer are faster than the dynamics of the original system. This corresponds to placing the poles of the observer to the left of the plant poles in the complex plane (ref. 7). The Kalman-Bucy filter has the added advantage that the observer gains and initial conditions are optimal in the sense that they minimize the observation error (ref. 8).

This thesis covers the development of a Kalman-Bucy filter to estimate the state for the optimal feedback control of the far-field acoustic pressure radiation from a plate submerged in fluid. The development of the plate model and control system design, assuming complete measurement of the state for feedback, has been carried out by Meirovitch (ref. 1) and is discussed in Chapter 2. The Kalman-Bucy filter design including the optimal selection of observer gains and initial conditions is derived in Chapter 3. Some of the numerical solution procedures are discussed in Chapter 4. Chapter 5 contains conclusions and suggestions for future work.

II. Plate Modelling and Control System Design

2.1 Plate Model

The plate model used in this application is shown in Figure 2.1. The plate is of length 2a in the x-direction and 2b in the y-direction and has thickness h. The plate is assumed to be bounded on one side by air and on the other by fluid. A typical point in the fluid medium is denoted by $P_f(R,\theta,\phi)$, where R, θ and ϕ are spherical coordinates. External excitation is applied to the dry side of the plate in the form of a harmonic persistent disturbing force.

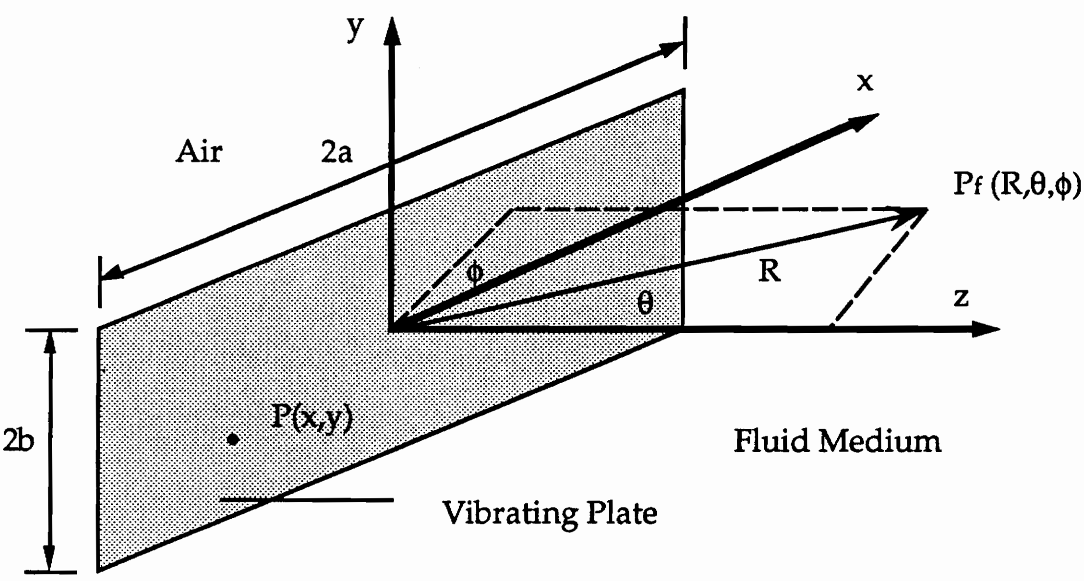

Figure 2.1 Vibrating Plate Coordinate Systems

As stated earlier, the feedback control system design has been completed by Meirovitch (ref. 1) assuming perfect sensing of all state variables for feedback. This chapter represents a review of the developments in reference 1.

2.2 Boundary-Value Problem for a Vibrating Submerged Plate

The vibration of an elastic structure is governed by the partial differential equation (ref. 9)

$$\mathcal{L}\, w(P,t) + m(P)\, \ddot{w}(P,t) = f(P,t), \quad P \in \Omega \tag{2.1}$$

where w(P,t) is the displacement at time t of a point P(x,y) located in the domain Ω of the structure, $\mathcal{L}$ is a homogeneous self-adjoint differential operator of order 2p, in which p is an integer, $m(x,y)$ is the mass density, and f(x,y,t) is a force distributed over the surface of the structure. For a vibrating plate p = 2, and the differential operator takes the form

$$\mathcal{L} = \frac{Eh^3}{12(1-\upsilon^2)}\left(\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}\right) \tag{2.2}$$

where E is the material modulus of elasticity, h is the plate thickness and υ is the Poisson's ratio. The solution w(x,y,t) of Eq. (2.1) must satisfy the problem boundary conditions, which can be written in the following form:

$$B_i w(P,t) = 0, \quad P \in \partial\Omega, \quad i=1,2,\ldots,p \tag{2.3}$$

in which the B_i are homogeneous boundary differential operators of order ranging from zero to 2p-1 and $\partial\Omega$ is the boundary of Ω. For a simply supported rectangular plate, as shown in Figure 2.1, the B_i take the following values

$$B_1 = 1, \qquad B_2 = \frac{\partial^2}{\partial x^2}, \qquad x = -a, a$$

$$B_1 = 1, \qquad B_2 = \frac{\partial^2}{\partial y^2}, \qquad y = -b, b \tag{2.4}$$

The force density f(x,y,t) arises from three sources and it can be written in the form

$$f(x,y,t) = f_p(x,y,t) + f_c(x,y,t) + f_d(x,y,t) \tag{2.5}$$

where $f_c(x,y,t)$ is the control force, $f_d(x,y,t)$ is the persistent disturbing force and $f_p(x,y,t)$ is the pressure exerted by the fluid on the plate surface.

2.3 Boundary-Value Problem for the Fluid Motion

Acoustic pressure is defined here as the time-dependent local fluctuation about the ambient hydrostatic pressure. The propagation of this pressure disturbance throughout the fluid medium is governed by the three-dimensional wave equation. For small amplitude pressure fluctuations in an ideal homogeneous compressible fluid, the wave equation has the form

$$\nabla^2 p(P_f,t) = \frac{\ddot{p}(P_f,t)}{c^2} \tag{2.6}$$

where ∇^2 is the three-dimensional Laplace operator and c is the speed of sound in the fluid. At the plate surface, the acceleration of the fluid and the plate surface coincide. Using Newton's Law, the normal component of the fluid pressure at the plate surface must equal the normal acceleration times the mass density of the fluid. This relation can be expressed in the following boundary condition:

$$\rho \ddot{w}(P,t) = -\frac{\partial p(P_f,t)}{\partial \eta}\Big|_{P_f=P} \tag{2.7}$$

where η is the outward normal to the structure, ρ is the mass density of the fluid and $P_f = P$ indicates that the boundary condition is imposed on the fluid-structure interface. The pressure of the fluid exerted on the plate surface can be written as

$$f_p(P,t) = -p(P_f,t)\big|_{P_f=P} \tag{2.8}$$

where the negative sign is included because the pressure acts in the -z direction.

2.4 Relations Between Plate Vibrations and Fluid Pressure

Suppression of the far-field acoustic pressure radiating from a vibrating submerged plate is carried out by controlling certain characteristics of the plate's vibration. Specifically, the sound radiated from a vibrating plate depends on the acceleration of the plate surface. Therefore, before the control design can be carried out, one must derive a relationship between the vibration of the plate and the resulting radiated acoustic pressure.

Equation (2.6) governs the propagation of acoustic pressure and its solution is subject to the boundary condition given by Eq. (2.7), which contains the plate acceleration. In the case of transient vibration of a plate, the solution of Eqs. (2.6) and (2.7) involves a convolution integral in the time domain (ref. 1). This type of relationship between pressure and plate acceleration involves integration over the entire time domain and is not suitable for use in control system design.

For the steady-state case, such as that of a plate vibrating harmonically, it is possible to eliminate the time dependence, thus reducing the complexity of the problem. For harmonic time variation, we can write the pressure and acceleration as

$$p(P_f,t) = p(P_f)e^{-i\omega t}, \qquad \ddot{w}(P,t) = \ddot{w}(P)e^{-i\omega t} \tag{2.9a,b}$$

where $p(P_f)$ is the pressure amplitude, $\ddot{w}(P)$ is the acceleration amplitude and ω is the frequency of the harmonic oscillation, so that, inserting Eq. (2.9a) into the 3-dimensional wave equation, Eq. (2.6), we obtain the Helmholtz equation

$$(\nabla^2 + k^2)p(P_f) = 0 \tag{2.10}$$

where

$$k = \omega/c \tag{2.11}$$

is defined as the acoustic wavenumber. Similarly, substitution of Eqs. (2.9) into the boundary condition, Eq. (2.7), results in

$$\rho \ddot{w}(P) = -\frac{\partial p(P_f)}{\partial z}\Big|_{P_f = P} \tag{2.12}$$

Equations (2.10) and (2.12) represent a time-independent boundary-value problem in terms of the fluid pressure and plate acceleration amplitudes. The solution of this problem is conveniently treated using double Fourier transform techniques (ref. 3). For any function f(x,y), the Fourier transform pair is defined as

$$\tilde{f}(\gamma_x,\gamma_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-i(\gamma_x x + \gamma_y y)}\, dxdy \tag{2.13a}$$

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{f}(\gamma_x, \gamma_y)\, e^{i(\gamma_x x + \gamma_y y)}\, d\gamma_x d\gamma_y \tag{2.13b}$$

The double Fourier transform of the pressure $\tilde{p}(\gamma_x,\gamma_y,z)$ must satisfy the transform of the Helmholtz equation

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) p(x,y,z)\, e^{-i(\gamma_x x + \gamma_y y)}\, dxdy = 0 \tag{2.14}$$

subject to the transformed boundary condition

$$\rho \tilde{\ddot{w}}(\gamma_x, \gamma_y) = -\frac{\partial \tilde{p}(\gamma_x, \gamma_y, z)}{\partial z}\Big|_{z=0} \tag{2.15}$$

Carrying out the integrations by parts in Eq. (2.14), we obtain

$$\left(k^2 - \gamma_x^2 - \gamma_y^2 + \frac{\partial^2}{\partial z^2}\right)\tilde{p}(\gamma_x, \gamma_y, z) = 0 \tag{2.16}$$

which has the solution

$$\tilde{p}(\gamma_x, \gamma_y, z) = A \exp[i(k^2 - \gamma_x^2 - \gamma_y^2)^{1/2} z] \tag{2.17}$$

The constant A in equation (2.17) is determined by applying the boundary condition, Eq. (2.15); the result is an expression for the double Fourier transform of the pressure amplitude. The pressure amplitude is then found by using the inverse transform, Eq. (2.13b), resulting in

$$p(x,y,z) = \frac{i\rho}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\ddot{\tilde{w}}(\gamma_x, \gamma_y)\exp[\, i(\gamma_x x + \gamma_y y) + i(k^2 - \gamma_x^2 - \gamma_y^2)^{1/2} z\,]}{(k^2 - \gamma_x^2 - \gamma_y^2)^{1/2}} d\gamma_x d\gamma_y \tag{2.18}$$

The amplitude of the pressure exerted by the fluid on the plate is given by letting z = 0 in Eq. (2.18). The resulting expression is given by

$$f_p(x,y) = -\,p(x,y,0) = -\frac{i\rho}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\ddot{\tilde{w}}(\gamma_x, \gamma_y)\exp[\, i(\gamma_x x + \gamma_y y)]}{(k^2 - \gamma_x^2 - \gamma_y^2)^{1/2}} d\gamma_x d\gamma_y \tag{2.19}$$

For control purposes, we must have the relation between the plate vibration and the far-field radiation pressure, in addition to the pressure at the plate surface. The far-field of a radiator is defined as the region in the fluid in which the relationships between pressure, fluid velocity and sound intensity approach those for a plane wave (ref. 3). The spatial dependence of the far-field acoustic pressure is simpler than that in the region close to the plate surface. The far-field acoustic radiation pressure for a rectangular plate is given by Rayleigh's formula (ref. 3)

$$p(R,\theta,\phi) = \frac{\rho e^{ikR}}{2\pi R} \int_{-b}^{b} \int_{-a}^{a} \ddot{w}(x,y) e^{-ik \sin\theta \, (x \cos\phi + y \sin\phi)} \, dxdy \tag{2.20}$$

where R, θ and ϕ are spherical coordinates in the fluid medium.

2.5 The Free Vibration, *In Vacuo* Eigenvalue Problem

At the present time, modern control theory does not permit solutions to control problems in terms of partial differential equations. Since the motion of the plate in this case is governed by a partial differential equation, some form of spatial discretization is necessary before the solution to the control problem can be carried out. The natural approach to discretization is to expand the solution in terms of the *in vacuo* modes of vibration of the plate. The free vibration, *in vacuo* modes of vibration are found by letting $f(P,t) \equiv 0$ in Eq. (2.1) and using the method of separation of variables to eliminate the time dependence (ref. 9). The resulting modes are not the actual modes for a plate vibrating in fluid. However, the free vibration modes do represent a complete set of admissible functions (ref. 9) which can be used to reduce the governing partial differential equation to a set of ordinary differential equations.

The *in vacuo* modes are obtained by assuming that in free vibration, $f(x,y,t) = 0$, all points on the plate undergo synchronous motion, which implies that the ratio of the displacement amplitudes of any two points does not change with time during motion (ref. 9). This is equivalent to the assumption that the solution is separable in space and time, or that

$$w(x,y,t) = W(x,y)F(t) \tag{2.21}$$

Setting $f(x,y,t) = 0$ in Eq. (2.1) and using Eq. (2.21), we conclude that the time dependent part $F(t)$ must satisfy

$$\ddot{F}(t) + \omega^2 F(t) = 0 \tag{2.22}$$

so that it is harmonic, and that the spatial dependent part $W(x,y)$ must satisfy the partial differential equation

$$\mathcal{L}W(x,y) = \lambda m(x,y)W(x,y), \quad \lambda=\omega^2, \quad -a<x<a, \quad -b<y<b \tag{2.23}$$

Moreover, eliminating the time dependence from Eq. (2.3), the boundary conditions take the form

$$B_i W(x,y) = 0, \; i=1,2; \; -a<x<a, \; y = \pm b \text{ or } -b<y<b, \; x = \pm a \tag{2.24}$$

The problem defined by Eqs. (2.23) and (2.24) represents a differential eigenvalue problem.

The differential operator $\mathcal{L}$ given by Eq. (2.2) can be shown to be self-adjoint and positive definite. The fact that $\mathcal{L}$ is self-adjoint guarantees that the solution of the eigenvalue problem consists of a denumerably infinite set of real eigenvalues λ_{mn} and orthogonal eigenfunctions $W_{mn}(x,y)$ (m,n=1,2,3,...) (ref. 9). Furthermore, because $\mathcal{L}$ is positive definite, all the eigenvalues $\lambda_{mn} = \omega_{mn}^2$ are positive, where ω_{mn} are the natural frequencies of the plate. The eigenfunctions are orthogonal both with respect to m(x,y) and $\mathcal{L}$ and can be normalized so as to satisfy

$$\int_{-b}^{b}\int_{-a}^{a} m(x,y)W_{mn}(x,y)W_{rs}(x,y)dxdy = \delta_{mnrs}, \quad r,s=1,2,... \tag{2.25a}$$

$$\int_{-b}^{b}\int_{-a}^{a} W_{mn}(x,y)\mathcal{L}W_{rs}(x,y)dxdy = \omega_{rs}^2\delta_{mnrs}, \quad r,s=1,2,... \tag{2.25b}$$

where the Kronecker delta, δ_{mnrs}, is nonzero only when m = r and n = s. The conditions given by Eqs. (2.25) are known as the orthonormality conditions (ref. 9). For the simply supported uniform plate shown in Figure 2.1, the mode shapes are given by

$$W_{rs}(x,y) = \frac{1}{\sqrt{m\,ab}}\sin\frac{r\pi(x+a)}{2a}\sin\frac{s\pi(y+b)}{2b} \tag{2.26}$$

with the corresponding natural frequencies by

$$\omega_{rs} = \frac{\pi^2}{4}\sqrt{\frac{Eh^3}{12m\,(1-\upsilon^2)}\left[\left(\frac{r}{a}\right)^2+\left(\frac{s}{b}\right)^2\right]} \tag{2.27}$$

2.6 Problem Discretization

Discretization in space is accomplished by expressing the solution as a linear combination of the *in vacuo* modes. Indeed, according to the expansion theorem (ref. 9), the plate displacement can be written in the form

$$w(x,y,t) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} W_{rs}(x,y)q_{rs}(t) \tag{2.28}$$

where $W_{rs}(x,y)$ are the *in vacuo* modes and $q_{rs}(t)$ are time-dependent generalized coordinates known as "modal" coordinates. Modal analysis is then used to reduce the partial differential equation, Eq. (2.1), to a set of ordinary differential equations in terms of the modal coordinates. This reduction is possible in view of the orthogonality of the *in vacuo* modes. Substituting Eq. (2.28) into Eq. (2.1), multiplying by $W_{mn}(x,y)$ and integrating over the entire plate, we obtain

$$\begin{aligned}&\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\left[\int_{-a}^{a}\int_{-b}^{b} W_{mn}(x,y)\mathcal{L}W_{rs}(x,y)q_{rs}(t)dydx\right] + \\ &\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\left[\int_{-a}^{a}\int_{-b}^{b} W_{mn}(x,y)m(x,y)W_{rs}(x,y)\ddot{q}_{rs}(t)dydx\right] \\ &\qquad = \int_{-a}^{a}\int_{-b}^{b} W_{mn}(x,y)f(x,y,t)dydx\end{aligned} \tag{2.29}$$

Then, applying the orthonormality conditions, Eqs. (2.25), Eq. (2.29) reduces to the "modal" equations

$$\ddot{q}_{mn}(t) + \omega_{mn}^2 q_{mn}(t) = f_{cmn}(t) + f_{dmn}(t) + f_{pmn}(t), \quad m,n = 1,2,\ldots \tag{2.30}$$

in which

$$f_{jmn}(t) = \int_{-a}^{a}\int_{-b}^{b} W_{mn}(x,y)f_j(x,y,t)dydx, \quad j=c,d,p; \quad m,n = 1,2,\ldots \tag{2.31}$$

represent the "modal" forces. Equations (2.30) are coupled through the terms $f_{crs}(t)$ and $f_{prs}(t)$, which generally depend on all of the modal coordinates.

The modal forces due to fluid pressure depend on the acceleration of the plate surface. In order to derive an explicit expression for these modal forces, we begin by taking the double Fourier transform of Eq. (2.28), so that

$$\tilde{w}(\gamma_x,\gamma_y,t) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \tilde{W}_{rs}(\gamma_x,\gamma_y)q_{rs}(t) \tag{2.32}$$

where, according to Eq. (2.13a),

$$\tilde{W}_{rs}(\gamma_x,\gamma_y) = \int_{-b}^{b}\int_{-a}^{a} W_{rs}(x,y)e^{-i[\gamma_x x + \gamma_y y]}dxdy, \quad r,s=1,2,\ldots \tag{2.33}$$

in which we recalled that -a<x<a, -b<y<b. Introducing Eqs. (2.32) and (2.33) into Eq. (2.19), we obtain

$$f_p(x,y,t) = -\frac{i\rho}{4\pi^2}\sum_{r,s=1}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\tilde{W}_{rs}(\gamma_x,\gamma_y)e^{i[\gamma_x x + \gamma_y y]}}{(k^2-\gamma_x^2-\gamma_y^2)^{1/2}}d\gamma_x d\gamma_y\right]\ddot{q}_{rs}(t) \tag{2.34}$$

Inserting Eq. (2.34) into Eq. (2.31) with j=p, we obtain an expression for the modal surface pressure in the form

$$f_{pmn}(t) = -\frac{i\rho}{4\pi^2}\sum_{r,s=1}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\tilde{W}_{rs}(\gamma_x,\gamma_y)\tilde{W}^*_{mn}(\gamma_x,\gamma_y)}{(k^2-\gamma_x^2-\gamma_y^2)^{1/2}}d\gamma_x d\gamma_y\right]\ddot{q}_{rs}(t) \quad (2.35)$$

where we have defined

$$\tilde{W}^*_{mn}(\gamma_x,\gamma_y) = \int_{-b}^{b}\int_{-a}^{a}W_{mn}(x,y)e^{i[\gamma_x x+\gamma_y y]}dxdy, \quad m,n=1,2,\ldots \quad (2.36)$$

for convenience.

It should be noted that the double integral in Eq. (2.35) cannot be evaluated in closed form. It follows that we must resort to numerical integration to evaluate the integral. A discussion of the numerical procedure chosen for this task is presented in Chapter 4.

When computations are to be carried out on a computer, it is advantageous to formulate the problem in terms of matrix notation. In this case, the modal equations can be written in matrix form

$$\ddot{\underset{\sim}{q}}(t) + \Lambda\underset{\sim}{q}(t) = \underset{\sim}{f}_c(t) + \underset{\sim}{f}_d(t) + \underset{\sim}{f}_p(t) \quad (2.37)$$

in which $\Lambda = \mathrm{diag}[\omega_1^2\ \omega_2^2\ \ldots]$ and the tilde under a symbol indicates a vector quantity. In practice, the modal displacements and accelerations are expressed in vector form by using a single subscript and arranging the components according to increasing magnitude of the corresponding natural frequencies, with $\omega_1 \le \omega_2 \le \ldots \le \omega_\infty$. Following this numbering scheme, we can rewrite Eq. (2.35) as

$$f_{pr}(t) = -\frac{i\rho}{4\pi^2}\sum_{s=1}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\tilde{W}_s(\gamma_x,\gamma_y)\tilde{W}_r^*(\gamma_x,\gamma_y)}{(k^2-\gamma_x^2-\gamma_y^2)^{1/2}}d\gamma_x d\gamma_y\right]\ddot{q}_s(t)$$

$$= -\sum_{s=1}^{\infty} c_{rs}\ddot{q}_s(t), \quad r = 1,2,\ldots,\infty \tag{2.38}$$

Equation (2.38) can be expressed in the matrix form

$$\underset{\sim}{f}_p(t) = -[C]\ddot{\underset{\sim}{q}}(t) \tag{2.39}$$

where C is a matrix with entries given by

$$c_{rs} = \frac{i\rho}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\tilde{W}_s(\gamma_x,\gamma_y)\tilde{W}_r^*(\gamma_x,\gamma_y)}{(k^2-\gamma_x^2-\gamma_y^2)^{1/2}}d\gamma_x d\gamma_y, \qquad r,s = 1,2,\ldots \tag{2.40}$$

Using Eq. (2.39), the modal equations can be rewritten in the compact form

$$(I + C)\ddot{\underset{\sim}{q}}(t) + \Lambda\underset{\sim}{q}(t) = \underset{\sim}{f}_c(t) + \underset{\sim}{f}_d(t) \tag{2.41}$$

2.7 Truncated Equations for Control

Equation (2.41) expresses the motion of a vibrating plate in terms of an infinite number of modes and modal coordinates. Designing a feedback control system with an infinite number of state variables is not feasible. Fortunately, in practice only a finite number of modes are excited and are in need of control, which permits truncation of the system to a manageable dimension. We propose to control n degrees of freedom and write the modal displacement vector in the form

$$\underset{\sim}{q}(t) = [\underset{\sim}{q}_C^T(t)\ \underset{\sim}{q}_R^T(t)]^T \tag{2.42}$$

where $\underset{\sim}{q}_C(t)$ is an n-dimensional vector of controlled modal displacements and $\underset{\sim}{q}_R(t)$ is an infinite-dimensional vector of uncontrolled, or residual modal displacements. The presence of the fluid causes vibration in one mode to excite vibration in other modes. This coupling effect is manifested in the fluid-loading C matrix. In the case of harmonic excitation of a submerged plate, as in the case at hand, the fluid coupling of the modes tends to become negligible as the mode number increases (ref. 3). In view of this, we can write

$$C = \begin{bmatrix} C_C & 0 \\ 0 & C_R \end{bmatrix} \tag{2.43}$$

where C_C and C_R are the fluid-loading matrices for the controlled and residual modes, respectively. Then, if we partition the diagonal matrix of eigenvalues as follows

$$\Lambda = \begin{bmatrix} \Lambda_C & 0 \\ 0 & \Lambda_R \end{bmatrix} \tag{2.44}$$

Eq. (2.41) can be separated into two uncoupled equations, one for the controlled modes and one for the residual modes, or

$$[I + C_C]\ddot{\underset{\sim}{q}}_C(t) + \Lambda_C \underset{\sim}{q}_C(t) = \underset{\sim}{f}_{cC}(t) + \underset{\sim}{f}_{dC}(t) \tag{2.45a}$$

$$[I + C_R]\ddot{\underset{\sim}{q}}_R(t) + \Lambda_R \underset{\sim}{q}_R(t) = \underset{\sim}{f}_{cR}(t) + \underset{\sim}{f}_{dR}(t) \tag{2.45b}$$

We propose to carry out the control using actuators at a finite number of points on the plate surface. Mathematically, point actuators can be treated as distributed using spatial Dirac delta functions. The control force can be written as

$$f_c(x,y,t) = \sum_{k=1}^{N} F_{ck}(t)\delta(x - x_k, y - y_k) \tag{2.46}$$

where $F_{ck}(t)$ are control force amplitudes and $\delta(x - x_k, y - y_k)$ are spatial Dirac delta functions, in which $x = x_k$, $y = y_k$ are the locations of the actuators, and N is the total number of actuators. Inserting Eq. (2.46) into Eq. (2.31) with j = c, performing the integration and separating the controlled and residual modal forces, we obtain

$$\underset{\sim}{f}_{cC}(t) = B'_{cC}\,\underset{\sim}{F}_c(t)\,, \quad \underset{\sim}{f}_{cR}(t) = B'_{cR}\,\underset{\sim}{F}_c(t) \tag{2.47}$$

in which

$$B'_{cC} = [B'_{crk}] = [W_r(x_k, y_k)],\ r = 1,2,\ldots,n;\ k = 1,2,\ldots,N \tag{2.48a}$$

$$B'_{cR} = [B'_{crk}] = [W_r(x_k, y_k)],\ r = n+1, n+2,\ldots,\infty;\ k = 1,2,\ldots,N \tag{2.48b}$$

are referred to as modal participation matrices (ref. 4). Substitution of Eqs. (2.47) into Eqs. (2.45) results in the uncoupled modal equations

$$[I + C_C]\ddot{\underset{\sim}{q}}_C(t) + \Lambda_C \underset{\sim}{q}_C(t) = B'_{cC}\underset{\sim}{F}_c(t) + \underset{\sim}{f}_{dC}(t) \tag{2.49a}$$

$$[I + C_R]\ddot{\underset{\sim}{q}}_R(t) + \Lambda_R \underset{\sim}{q}_R(t) = B'_{cR}\underset{\sim}{F}_c(t) + \underset{\sim}{f}_{dR}(t) \tag{2.49b}$$

2.8 The Performance Measure for Optimal Control

The control objective is to minimize the acoustic radiation pressure $p(R,\theta,\phi)$ in a far-field region in the fluid medium (Figure 2.1). This objective is to be achieved using actuator forces $f_c(P,t)$ acting at points $P(x,y)$ on the plate surface. In state feedback control, the actuator forces depend on state variables (displacements and velocities). In optimal control design by the linear quadratic regulator (LQR) theory, the relation between the control forces and the state is linear, with the transformation matrix representing the control gain matrix. According to the LQR theory, the control law is optimal in the sense that it minimizes the system performance measure

$$\begin{aligned} J = {} & \frac{1}{2}\int_{\Omega}[h_1(P)w^2(P,t_f) + h_2(P)\dot{w}^2(P,t_f)]d\Omega + \\ & \frac{1}{2}\int_{t_i}^{t_f}\left\{\int_{\Omega}[w(P,t)\mathcal{L}w(P,t) + m(P)\dot{w}^2(P,t) + a(P)f_c^2(P,t)]d\Omega + \right. \\ & \left. \int_{\Omega_f} b(R,\theta,\phi)p^2(R,\theta,\phi)d\Omega_f\right\} dt \end{aligned} \tag{2.50}$$

where $h_1(P)$, $h_2(P)$, $a(P)$ and $b(R,\theta,\phi)$ are weighting functions, Ω_f is the far-field region in the fluid over which the pressure is to be minimized and t_i and t_f are the initial and final times, respectively. This performance measure places penalties on the state of the vibrating plate, the amount of control force applied, the far-field pressure, as well as the state at the final time. The weighting functions determine the amount of penalty placed on each of these factors, so that various control objectives can be achieved by changing these functions.

2.9 Optimal State Feedback Control

Control system design is carried out using the state-space representation of the system. To this end, we include only the controlled modal displacements and velocities in the state vector, or

$$\underset{\sim}{x}(t) = [\underset{\sim}{q}_C^T(t) \;\; \dot{\underset{\sim}{q}}_C^T(t)]^T \tag{2.51}$$

Then, adjoining the identity $\dot{\underset{\sim}{q}}_C(t) \equiv \dot{\underset{\sim}{q}}_C(t)$, Eq. (2.49a) can be written in the standard state form

$$\dot{\underset{\sim}{x}}(t) = A\,\underset{\sim}{x}(t) + B\,\underset{\sim}{F}_c(t) + D\,\underset{\sim}{f}_{dC} \tag{2.52}$$

where

$$A = \begin{bmatrix} 0 & I \\ -(I+C_C)^{-1}\Lambda_C & 0 \end{bmatrix},\quad B = \begin{bmatrix} 0 \\ (I+C_C)^{-1}B'_{cC} \end{bmatrix},\quad D = \begin{bmatrix} 0 \\ (I+C_C)^{-1} \end{bmatrix} \tag{2.53}$$

The performance measure given by Eq. (2.50) is expressed in terms of continuous functions. Before optimal controls can be determined, it is necessary to discretize the performance measure in a manner similar to the discretization of the plate equations. For control purposes, we are only considering the controlled modes, so that the expansion theorem can be modified so as to read

$$w(P,t) \cong \sum_{i=1}^{n} W_i(P)\, q_i(t) = \underset{\sim}{W}_C^T(P)\underset{\sim}{q}_C(t) \tag{2.54}$$

in which $\underset{\sim}{W}_C(P) = [W_1(P)\ W_2(P) \ldots W_n(P)]^T$ is an n-dimensional vector of the *in vacuo* eigenfunctions corresponding to the n controlled modes. Using Eq. (2.54), the first integrand in Eq. (2.50) can be approximated by

$$\int_\Omega [h_1(P)w^2(P,t_f) + h_2(P)\dot{w}^2(P,t_f)]d\Omega \cong \underset{\sim}{x}^T(t_f)H\underset{\sim}{x}(t_f) \tag{2.55}$$

where H is a weighting matrix defined as

$$H = \text{block-diag}\left[\int_\Omega h_1(P)\underset{\sim}{W}_C(P)\underset{\sim}{W}_C^T(P)d\Omega \quad \int_\Omega h_2(P)\underset{\sim}{W}_C(P)\underset{\sim}{W}_C^T(P)d\Omega\right] \tag{2.56}$$

Similarly,

$$\int_\Omega [w(P,t)\mathcal{L}w(P,t) + m(P)\dot{w}^2(P,t)]d\Omega \cong \underset{\sim}{x}^T(t)Q^*\underset{\sim}{x}(t) \tag{2.57}$$

where Q* is a weighting matrix defined as

$$Q^* = \begin{bmatrix} \Lambda_C & 0 \\ 0 & I \end{bmatrix} \tag{2.58}$$

Moreover, replacing x_k, y_k by P_k in Eq. (2.46), we can write

$$\int_{\Omega} a(P) f_c^2(P,t) d\Omega \cong \underset{\sim}{F}_c^T(t) R^* \underset{\sim}{F}_c(t) \tag{2.59}$$

where R^* is a weighting matrix given by

$$R^* = \mathrm{diag}\,[a(P_k)], \quad k=1,2,\ldots N \tag{2.60}$$

The only term remaining for discretization in Eq. (2.50) involves the far-field radiation pressure. Substituting Eq. (2.54) into Eq. (2.20), we obtain

$$p(R,\theta,\phi,t) = \underset{\sim}{P}^T(R,\theta,\phi) \underset{\sim}{\ddot{q}}_C(t) \tag{2.61}$$

where

$$\underset{\sim}{P}(R,\theta,\phi) \cong \frac{\rho e^{ikR}}{2\pi R} \int_{-b}^{b} \int_{-a}^{a} \underset{\sim}{W}_C(x,y) e^{-ik \sin\theta\, (x \cos\phi + y \sin\phi)}\, dx dy \tag{2.62}$$

Then, letting

$$\bar{P} = \int_{\Omega_f} b(R,\theta,\phi) \underset{\sim}{P}(R,\theta,\phi) \underset{\sim}{P}^T(R,\theta,\phi) d\Omega_f \tag{2.63}$$

we can write

$$\int_{\Omega_f} b(R,\theta,\phi) p^2(R,\theta,\phi) d\Omega_f \cong \underset{\sim}{\dot{x}}^T(t) P^* \underset{\sim}{\dot{x}}(t) \tag{2.64}$$

where P^* is a weighting matrix defined as

$$P^* = \begin{bmatrix} 0 & 0 \\ 0 & \overline{P} \end{bmatrix} \tag{2.65}$$

It should be noted that evaluation of the matrix $\overline{P}$ cannot be carried out in closed form. Indeed, as with the pressure at the plate surface, the integration in Eq. (2.63) must be carried out numerically. Inserting Eqs. (2.55), (2.57), (2.59) and (2.64) into Eq. (2.50), we obtain the discretized performance measure

$$J = \frac{1}{2}\underset{\sim}{x}^T(t_f)H\underset{\sim}{x}(t_f) + \frac{1}{2}\int_{t_i}^{t_f}(\underset{\sim}{x}^T Q^* \underset{\sim}{x} + \underset{\sim}{F}_c^T R^* \underset{\sim}{F}_c + \dot{\underset{\sim}{x}}^T P^* \dot{\underset{\sim}{x}})dt \tag{2.66}$$

The common practice in optimal control by the LQR theory is to express the performance measure in terms of the state vector and not its time derivative. The derivative can be eliminated from Eq. (2.66) if we realize that the state equations, Eqs. (2.52), give us a relationship between the state and its derivative. Substituting Eqs. (2.52) into Eq. (2.66) results in the performance measure

$$J = \frac{1}{2}\underset{\sim}{x}^T(t_f)H\underset{\sim}{x}(t_f) + \frac{1}{2}\int_{t_i}^{t_f}(\underset{\sim}{x}^T Q\underset{\sim}{x} + \underset{\sim}{F}_c^T R\underset{\sim}{F}_c + \underset{\sim}{f}_{dc}^T S\underset{\sim}{f}_{dc}$$
$$+ \underset{\sim}{x}^T T\underset{\sim}{F}_c + \underset{\sim}{x}^T U\underset{\sim}{f}_{dc} + \underset{\sim}{F}_c^T V\underset{\sim}{f}_{dc})dt \tag{2.67}$$

where

$$Q = Q^* + A^T P^* A, \quad R = R^* + B^T P^* B, \quad S = D^T P^* D \qquad (2.68a,b,c)$$

$$T = 2A^T P^* B, \quad U = 2A^T P^* D, \quad V = 2B^T P^* D \qquad (2.68d,e,f)$$

The optimal control problem can now be stated as the problem of finding an admissible control force $\underset{\sim}{F}_c(t)$ which minimizes the performance measure, Eq. (2.67), subject to the state constraint given by Eq. (2.52). We propose to use Pontryagin's minimum principle as a method of deriving the optimal control law (ref. 4). The Hamiltonian for this system is given by

$$\mathcal{H} = \frac{1}{2}\underset{\sim}{x}^T Q\underset{\sim}{x} + \frac{1}{2}\underset{\sim}{F}_c^T R\underset{\sim}{F}_c + \frac{1}{2}\underset{\sim}{f}_{dc}^T S\underset{\sim}{f}_{dc} + \underset{\sim}{x}^T T\underset{\sim}{F}_c + \underset{\sim}{x}^T U\underset{\sim}{f}_{dc}$$

$$+ \underset{\sim}{F}_c^T V\underset{\sim}{f}_{dc} + \underset{\sim}{p}^T(A\underset{\sim}{x} + B\underset{\sim}{F}_c + D\underset{\sim}{f}_{dc}) \tag{2.69}$$

where $\underset{\sim}{p}$ is known as the costate vector. The process of minimizing the Hamiltonian results in a set of necessary conditions for an optimal solution (ref. 4). In this case, these necessary conditions are given by

$$\dot{\underset{\sim}{x}} = \frac{\partial \mathcal{H}}{\partial \underset{\sim}{p}} = A\underset{\sim}{x} + B\underset{\sim}{F}_c + D\underset{\sim}{f}_{dc} \tag{2.70a}$$

$$\dot{\underset{\sim}{p}} = -\frac{\partial \mathcal{H}}{\partial \underset{\sim}{x}} = -\left(Q\underset{\sim}{x} + T\underset{\sim}{F}_c + U\underset{\sim}{f}_{dc} + A^T\underset{\sim}{p}\right) \tag{2.70b}$$

$$\frac{\partial \mathcal{H}}{\partial \underset{\sim}{F}_c} = T^T\underset{\sim}{x} + R\underset{\sim}{F}_c + V\underset{\sim}{f}_{dc} + B^T\underset{\sim}{p} = \underset{\sim}{0} \tag{2.70c}$$

Equations (2.70a) and (2.70b) give us the state and costate equations respectively, while equation (2.70c) allows us to solve for the control law

$$\underset{\sim}{F}_c = -R^{-1}[T^T\underset{\sim}{x} + V\underset{\sim}{f}_{dc} + B^T\underset{\sim}{p}] \tag{2.71}$$

The control law, Eq. (2.71), depends on the costate vector as well as the state vector. Since a state feedback law is desired, we can eliminate the dependence on the costate by assuming that

$$\underset{\sim}{p} = K(t)\underset{\sim}{x} + \underset{\sim}{v} \tag{2.72}$$

where K is matrix and $\underset{\sim}{v}$ a vector, both still to be determined. It should be noted at this point that, in searching for a state feedback solution, we are assuming the entire state is measurable and available for feedback. Combining Eqs. (2.70a), (2.70b), (2.71) and (2.72), we obtain two expressions for the derivative of the costate vector

$$\dot{\underset{\sim}{p}} = \dot{K}\underset{\sim}{x} + K\dot{\underset{\sim}{x}} + \underset{\sim}{v} = (\dot{K} + KA - KBR^{-1}T^T - KBR^{-1}B^TK)\underset{\sim}{x} + K(D - BR^{-1}V)\underset{\sim}{f}_{dc}$$
$$- KBR^{-1}B^T\underset{\sim}{v} + \dot{\underset{\sim}{v}} \tag{2.73a}$$

and

$$\dot{\underset{\sim}{p}} = -(Q - TR^{-1}T^T + TR^{-1}K - A^TK)\underset{\sim}{x} + (TR^{-1}V - U)\underset{\sim}{f}_{dc} + (TR^{-1}B^T - A)\underset{\sim}{v} \tag{2.73b}$$

Equations (2.73a) and (2.73b) can be rendered equivalent by choosing K and $\underset{\sim}{v}$ so as to satisfy

$$\dot{K} = -Q + TR^{-1}T^T + KBR^{-1}B^TK - K(A - BR^{-1}T^T) - (A^T - TR^{-1}B^T)K, \quad K(t_f) = H \tag{2.74}$$

and

$$\dot{\underset{\sim}{v}} = (TR^{-1}B^T - A^T + KBR^{-1}B^T)\underset{\sim}{v} + [\,TR^{-1}V - U - K(D - BR^{-1}V)]\underset{\sim}{f}_{dC}, \quad \underset{\sim}{v}(t_f) = \underset{\sim}{0} \tag{2.75}$$

Equation (2.74) is known as the matrix differential Riccati equation. The boundary condition involves the final time t_f, so that the equation must be integrated backward in time to solve for the n x n symmetric matrix K(t). If the system is controllable, H=0, and A,B,Q,R and T are time-invariant, the solution K(t) tends to a constant matrix as t_f increases (ref. 4). Therefore, if the control time t_f is reasonably large, the matrix differential Riccati equation, Eq. (2.74), reduces to the matrix algebraic Riccati equation, which is obtained by letting $\dot{K} = 0$ in Eq. (2.74); its solution can be determined by means of Potter's algorithm (ref. 4).

To determine the costate, Eq. (2.72), and thus complete the control law, Eq. (2.71), we must solve Eq. (2.75) for the vector $\underset{\sim}{v}$. In the case in which the matrix K(t) depends on time, Eq. (2.75) represents a time-varying system, and its solution can be obtained using discrete-time techniques (ref. 4). When K is constant, Eq. (2.75) represents a time-invariant system and it solution has the form (ref. 4)

$$\underset{\sim}{v}(t) = \int_{t_f}^{t} \Phi_v(t-\tau)[TR^{-1}V - U - K(D - BR^{-1}V)]\underset{\sim}{f}_{dc}(\tau)d\tau \qquad (2.76)$$

where

$$\Phi_v(t) = \exp(TR^{-1}B^T - A^T + KBR^{-1}B^T)t \qquad (2.77)$$

is the transition matrix for system (2.75).

Inserting Eq. (2.72) into Eq. (2.71), we obtain finally the state-feedback control law

$$\underset{\sim}{F}_c = -R^{-1}[(T^T + B^T K)\underset{\sim}{x} + V\underset{\sim}{f}_{dc} + B^T \underset{\sim}{v}] \tag{2.78}$$

which gives the control force to be applied to the plate surface as a function of current measurements of the state vector $\underset{\sim}{x}(t)$, the persistent disturbance, $\underset{\sim}{f}_{dc}(t)$ and the just derived vector $\underset{\sim}{v}(t)$.

III. Optimal Observer Design

3.1 Introduction

The control law developed in Chapter 2 is a function of all of the modal state variables, i.e., all of the modal displacements and velocities. The control law was formulated under the basic assumption that the complete state is available for measurement. For a complex system such as a vibrating plate, measurement of all these variables would require a very large number of sensors located on the plate surface, and is not likely to be feasible. We propose to overcome this difficulty by designing a state estimator, also known as an observer, which uses the measurement of a limited number of the state variables to generate an estimate of the entire state. This estimated state can then be used in place of the actual state in the feedback control law. The idea of using an observer to estimate the state variables for a deterministic system was developed by Luenberger (ref. 10). He considered a system in the general state form

$$\dot{\underset{\sim}{x}}(t) = A\underset{\sim}{x}(t) + B\underset{\sim}{u}(t) \tag{3.1a}$$

$$\underset{\sim}{y}(t) = C\underset{\sim}{x}(t) \tag{3.1b}$$

where $\underset{\sim}{y}(t)$ is the output vector from the system sensors expressed as a linear combination of the state variables. An observer for the system is given by

$$\dot{\hat{\underset{\sim}{x}}}(t) = A\hat{\underset{\sim}{x}}(t) + B\underset{\sim}{u}(t) + L(t)[\,\underset{\sim}{y}(t) - C\hat{\underset{\sim}{x}}(t)] \tag{3.2}$$

where $\hat{\underset{\sim}{x}}(t)$ is the estimated state vector and L(t) is a matrix of observer gains. A block diagram of a Luenberger observer is shown in Figure 3.1. This

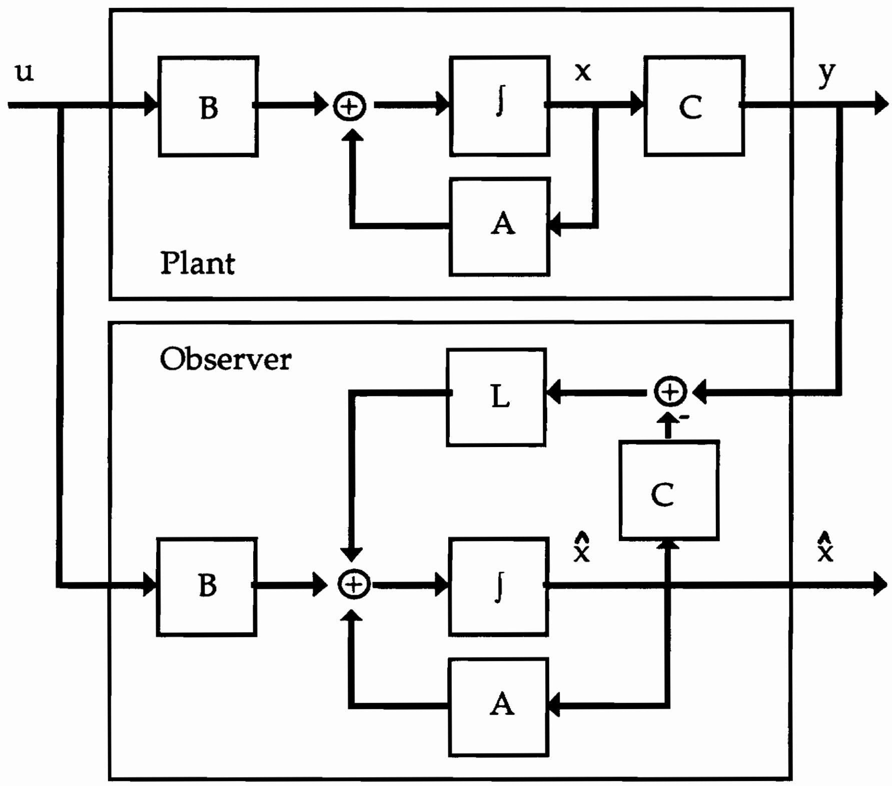

Figure 3.1 Block Diagram of Luenberger Observer

type of observer is referred to as a full-order observer because the entire state is included in the estimate. The observer is a model of the original system containing identical dynamics with an extra driving term proportional to the difference between the output of the system and the output of the observer. When the estimate of the state coincides with the actual state, then the forcing term drops out of Eq. (3.2) and the estimate is locked onto the state due to the identical dynamics. Luenberger has shown (ref. 10) that the observer gain matrix L(t) can be chosen so that the observer is asymptotically stable and that the observation error, the difference between the actual and estimated state vectors, is reduced to zero as $t \rightarrow \infty$ independently of the observer initial conditions.

Observer design involves choosing the gain matrix so that the observation error is asymptotically reduced to zero. It is desirable to choose the observer gains so that the dynamic characteristics of the observer are faster than those of the original system. Faster convergence is achieved by placing the poles deeper in the left-half complex plane. This gives the observer a chance to converge to a good estimate of the state before the estimate is fed back into the control law. However, if the poles of the observer are placed too far to the left in the complex plane, then the observer tends to magnify small amounts of noise introduced through the measurements of the system outputs (ref. 8). Clearly, a suitable choice of observer gains must be a compromise between these two effects.

A Luenberger observer is suitable for estimation of unknown state variables in the deterministic case, i.e., when the noise to signal ratio is negligible. However, in practical applications measurements always contain a degree of uncertainty due to noise, such as background electrical activity and current fluctuations. Uncertainties and noise are inherent in all sensors and actuators. Kalman and Bucy (ref. 11) solved the problem of estimating unknown state variables while at the same time filtering out the effects of sensor and actuator noise on the system. The resulting observer, taking into account the stochastic nature of practical problems, is known as a Kalman-Bucy filter.

Kalman and Bucy have also solved the problem of observer gain selection in an optimal fashion, optimal in the sense that the mean square observation error is minimized. In fact, Kalman and Bucy have shown in reference 11 that the solution of the optimal observer problem is equivalent, or *dual,* to the solution of the linear quadratic regulator problem in optimal control theory. This equivalence between the two solutions is known as the *Duality Principle* (ref. 11) and it permits the use of the well-defined methodology of optimal control theory in the optimal selection of observer gains and initial conditions.

In this chapter, the design of an observer for the estimation of modal displacements and velocities of a plate vibrating in fluid is described. As a logical first step, the design is carried out assuming a deterministic system representation, thus resulting in a Luenberger observer. Then, after introducing some of the statistical nomenclature of stochastic system theory, the design is reworked including the effects of sensor and actuator noise in the system equations. As the final stage of the design, optimal observer gains and initial conditions are calculated and the closed-loop system equations are developed.

3.2 Deterministic State Estimator Design

Before designing an observer for the state space system given in Eq. (2.52), it is necessary to derive a relationship between the sensor measurements and the state vector. In general, the measured output is a linear combination of the state and this relationship can be written in the form

$$\underset{\sim}{y}(t) = C_o \underset{\sim}{x}(t) \tag{3.3}$$

where $\underset{\sim}{y}(t)$ is a q-dimenisonal vector of sensor outputs and C_o is a q x 2n matrix. We propose to take displacement measurements using point sensors located at points (x_i, y_i) $(i=1,2,\ldots,q)$ on the plate surface. A general plate configuration using thirty-seven sensors (q = 37) is shown in Figure 3.2. The

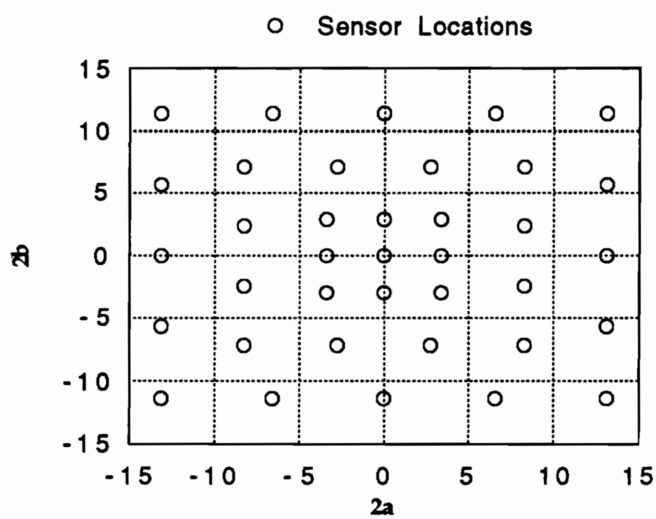

Figure 3.2 Plate Model with 37 Sensors

components of the output vector are

$$y_i(t) = w(x_i, y_i, t) = \sum_{r=1}^{n} W_r(x_i, y_i)\, q_r(t), \qquad i=1,2,\ldots,q \tag{3.4}$$

where we have substituted the approximate modal expansion, Eq. (2.54) with $P = (x_i, y_i)$. In matrix notation, we then write

$$\underset{\sim}{y}(t) = [C' \vdots\ 0]\underset{\sim}{x}(t) = C_o\underset{\sim}{x}(t) \tag{3.5}$$

where

$$[C'] = [W_r(x_i,y_i)], \quad i=1,2,\ldots,q, \quad r=1,2,\ldots,n \tag{3.6}$$

is a q x n submatrix. The general state space representation of the vibrating plate system is finally given by

$$\dot{\underset{\sim}{x}}(t) = A\ \underset{\sim}{x}(t) + B\ \underset{\sim}{F}_c(t) + D\ \underset{\sim}{f}_{dc} \tag{3.7a}$$

$$\underset{\sim}{y}(t) = C_o\underset{\sim}{x}(t) \tag{3.7b}$$

with A, B and D given by Eqs. (2.53).

An observer for system (3.7) must contain identical dynamics and forcing functions with an additional forcing term proportional to the deviation between the actual and estimated output vectors. In state space notation, the observer can be written as

$$\dot{\hat{\underset{\sim}{x}}}(t) = A\ \hat{\underset{\sim}{x}}(t) + B\ \underset{\sim}{F}_c(t) + D\ \underset{\sim}{f}_{dc} + L[\underset{\sim}{y}(t) - \hat{\underset{\sim}{y}}(t)] \tag{3.8}$$

where

$$\hat{\underset{\sim}{y}}(t) = C_o\hat{\underset{\sim}{x}}(t) \tag{3.9}$$

is the estimate of the system output vector as reconstructed by the observer. It is possible to include the forcing term $\underset{\sim}{f}_{dc}$ in the observer equation, as the disturbance force is a known harmonic function. In practice, the disturbance force is generally measured for feedback in the control law, Eq. (2.78), and thus it can also be used as an input to the state estimator. In the case where the disturbing force is unknown or unmeasurable, it is possible to include the force in the state vector and generate an estimate of the disturbance along with an estimate of the original state (refs. 12 and 13).

Observation error is defined as the difference between the actual state and the estimated state. Subtracting Eq. (3.8) from Eq. (3.7a) results in

$$\dot{\underset{\sim}{e}}(t) = [A - LC_o]\underset{\sim}{e}(t) \tag{3.10}$$

in which

$$\underset{\sim}{e}(t) = \{\underset{\sim}{x}(t) - \hat{\underset{\sim}{x}}(t)\} \tag{3.11}$$

is the error vector. Equation (3.10) represents a first-order system of linear homogeneous differential equations. The stability of the solution vector $\underset{\sim}{e}(t)$ depends solely on the eigenvalues of the coefficient matrix A-LC_o, which are known as the observer poles. If the observer poles are located in the left-half of the complex plane, then the system (3.10) is asymptotically stable and the error vector approaches zero asymptotically as $t \to \infty$. This guarantees that the state estimate approaches the actual state asymptotically.

The location of the observer poles in the complex plane depends on the choice of the observer gain matrix L in Eq. (3.10). If the pair (A,C_o) is completely observable, then all of the observer poles can be placed arbitrarily in the complex plane within the restriction that complex poles must occur in pairs of complex conjugates. Complete observability indicates that any initial state $\underset{\sim}{x}(t_o)$ can be determined from the output $\underset{\sim}{y}(t)$, $t_o \leq t < t_f$. Mathematically, this condition implies that the nq x n observability matrix

$$O_o = \begin{bmatrix} C_o \\ C_oA \\ \vdots \\ C_oA^{n-1} \end{bmatrix} \tag{3.12}$$

has rank n. In general, complete observability can be achieved with a suitable choice of sensor locations.

While the choice of observer gains is relatively arbitrary for the deterministic system (3.7), the question remains as to how the gains affect the closed-loop system response when the state estimate is used in the feedback control law. If we define a matrix of feedback control gains as

$$G = R^{-1}[T^T + B^T K] \tag{3.13}$$

then the feedback control law, Eq. (2.78), can be written as

$$\underset{\sim}{F}_c = -G\hat{\underset{\sim}{x}} - R^{-1}V\underset{\sim}{f}_{dc} - R^{-1}B^T\underset{\sim}{v} \tag{3.14}$$

where the state estimate has been substituted for the actual state. Substituting Eq. (3.14) into Eq. (3.7a) and combining the result with Eq. (3.10), we obtain the closed-loop system

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A\text{-}BG & BG \\ 0 & A\text{-}LC_o \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} D\text{-}R^{-1}V \\ 0 \end{bmatrix} [f_{dc}] - \begin{bmatrix} R^{-1}B^T \\ 0 \end{bmatrix} [v] \tag{3.15}$$

The response of the closed-loop system is governed by the eigenvalues of the closed-loop coefficient matrix

$$A_c = \begin{bmatrix} A\text{-}BG & BG \\ 0 & A\text{-}LC_o \end{bmatrix} \tag{3.16}$$

This matrix is in block-triangular form and the characteristic equation is given by

$$\det[sI - A_c] = \det[sI - (A - BG)] \cdot \det[sI - (A - LC_o)] \tag{3.17}$$

As a result, the set of eigenvalues of A_c is the union of the two sets of eigenvalues of the submatrices A-BG and A-LC_o. Therefore, the original system eigenvalues are not affected by the choice of the observer gains. This separation of system and observer poles is known as the *deterministic separation principle* (ref. 4). It implies that the observer gains can be chosen independently of the feedback control gains of the original system.

3.3 Stochastic State Estimator Design

In practical applications, noise and uncertainties included in sensor measurements can adversely affect the convergence of a Luenberger observer. It is therefore natural to consider a stochastic model of the system, i.e., one that includes sensor and actuator noise. With noise accounted for in the model, an observer can be designed which not only filters out the noise effects but also allows for optimal selection of observer gains. Such a stochastic state estimator is known as a Kalman-Bucy Filter (ref. 8).

Stochastic systems are described in terms of random variables whose values at a given time cannot be predicted. Due to the random nature of the sensor and actuator noise, the noise properties must be described in terms of statistical quantities. Pertinent statistical quantities will be defined as we proceed with the derivation of the filter equations. For a detailed discussion of stochastic systems and definitions, the reader is urged to consult refs. 4 and 8.

The effects of sensor and actuator noise are included in the system model by rewriting the state equations as

$$\dot{\underset{\sim}{x}}(t) = A\,\underset{\sim}{x}(t) + B\,\underset{\sim}{F}_c(t) + D\,\underset{\sim}{f}_{dc} + \underset{\sim}{w}_1(t) \tag{3.18a}$$

$$\underset{\sim}{y}(t) = C_o\underset{\sim}{x}(t) + \underset{\sim}{w}_2(t) \tag{3.18b}$$

where $\underset{\sim}{x}(t)$ is now a stochastic or random vector. The additional forcing vector $\underset{\sim}{w}_1(t)$ in Eq. (3.18a) is a random state excitation vector. It can be regarded as including the effects of actuator noise and/or any random external disturbances applied to the plate. Similarly, the random vector $\underset{\sim}{w}_2(t)$, added to the system output, represents the effects of sensor or observation noise in the output. These random forcing vectors are assumed to be white noise processes, the significance of which is discussed shortly.

To describe a white noise process accurately, we first give several definitions. For a general stochastic vector $\underset{\sim}{v}(t)$, the *mean value* is defined as

$$\bar{\underset{\sim}{v}}(t) = E\{\underset{\sim}{v}(t)\} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \underset{\sim}{v}(t) p(v_1, v_2, \ldots, v_n) dv_1 dv_2 \ldots dv_n \qquad (3.19)$$

in which E{ } is known as the mean value, or expected value, and $p(v_1, v_2, \ldots, v_n)$ is the *probability density* function. The *covariance matrix* is given by

$$C_v(t_1, t_2) = E\{[\underset{\sim}{v}(t_1) - \bar{\underset{\sim}{v}}(t_1)][\underset{\sim}{v}(t_2) - \bar{\underset{\sim}{v}}(t_2)]^T\} \qquad (3.20)$$

and the *correlation matrix*, or *second-order joint moment matrix*, has the form

$$R_v(t_1, t_2) = E\{\underset{\sim}{v}(t_1) \underset{\sim}{v}^T(t_2)\} \qquad (3.21)$$

The special covariance matrix obtained by letting $t_1 = t_2 = t$ is known as the *variance matrix* of the stochastic process. Similarly, for two random vectors, the *cross-covariance matrix* is defined as

$$C_{vz}(t_1, t_2) = E\{[\underset{\sim}{v}(t_1) - \bar{\underset{\sim}{v}}(t_1)][\underset{\sim}{z}(t_2) - \bar{\underset{\sim}{z}}(t_2)]^T\} \qquad (3.22)$$

and the *cross-correlation matrix* by

$$R_{vz}(t_1,t_2) = E\{\underset{\sim}{v}(t_1)\underset{\sim}{z}^T(t_2)\} \tag{3.23}$$

Now, consider a zero-mean stochastic process in which $\underset{\sim}{v}(t_1)$ and $\underset{\sim}{v}(t_2)$ are uncorrelated even for values of t_1 and t_2 that are close to each other. In this case, the correlation matrix is nonzero only for $t_1=t_2$ and can be written in simplified form as

$$R_v(t_1,t_2) = V(t_1)\delta(t_2 - t_1) \tag{3.24}$$

where $V(t_1)$ is known as the *intensity* of the process at time t_1 and $\delta(t_2\text{-}t_1)$ is the Dirac delta function. When the intensity of the process is constant, the process is said to be *wide-sense stationary* and the correlation matrix reduces to a function of $\tau = t_2 - t_1$ only. The Fourier transform of the correlation matrix is known as the *power spectral density matrix*. Taking the Fourier transform of Eq. (3.24), with constant intensity $V(t_1) = V$, we conclude that

$$S_v(\omega) = \int_{-\infty}^{\infty} R_v(\tau)e^{-i\omega\tau}d\tau = \int_{-\infty}^{\infty} V\delta(\tau)e^{-i\omega\tau}d\tau = V \tag{3.25}$$

or the power spectral density is constant over all frequencies. A random process $\underset{\sim}{v}(t)$ with a constant power spectral density matrix is called *white noise* by analogy with white light, which has a flat spectrum over the visible range (ref. 4).

Considering once again the vibrating plate of Eqs. (3.18), we assume that $\underset{\sim}{w}_1(t)$ and $\underset{\sim}{w}_2(t)$ are white noise processes with intensities $V_1(t)$ and $V_2(t)$, respectively. The correlation matrices have the form

$$E\{\underset{\sim}{w}_1(t_1)\underset{\sim}{w}_1^T(t_2)\} = V_1(t_1)\delta(t_2 - t_1) \tag{3.26a}$$

$$E\{\underset{\sim}{w}_2(t_1)\underset{\sim}{w}_2^T(t_2)\} = V_2(t_1)\delta(t_2 - t_1) \tag{3.26b}$$

Moreover, assuming that the two processes are uncorrelated, the cross-correlation is simply

$$E\{\underset{\sim}{w}_1(t_1)\underset{\sim}{w}_2^T(t_2)\} = E\{\underset{\sim}{w}_2(t_1)\underset{\sim}{w}_1^T(t_2)\} = 0 \tag{3.27}$$

Since the state vector is now a stochastic variable we have the initial condition

$$E\{\underset{\sim}{x}(t_o)\} = \bar{\underset{\sim}{x}}_o \tag{3.28}$$

with an initial variance matrix given by

$$E\{[\underset{\sim}{x}(t_o) - \bar{\underset{\sim}{x}}_o][\underset{\sim}{x}(t_o) - \bar{\underset{\sim}{x}}_o]^T\} = Q_o \tag{3.29}$$

Furthermore, the initial state vector is assumed to be uncorrelated with the state excitation noise and the observation noise, or

$$E\{\underset{\sim}{x}(t_o)\underset{\sim}{w}_i^T(t)\} = 0, \quad i = 1,2 \tag{3.30}$$

We consider the case in which all of the components of the output vector are corrupted by noise. This property is characterized mathematically by a sensor noise intensity matrix which is positive definite, $V_2(t) > 0$, and the resulting problem is known as the nonsingular state estimation problem. As in the deterministic case, we assume a full-order observer for system (3.18) having the form

$$\dot{\hat{\underset{\sim}{x}}}(t) = A\,\hat{\underset{\sim}{x}}(t) + B\,\underset{\sim}{F}_c(t) + D\,\underset{\sim}{f}_{dc} + L(t)[\underset{\sim}{y}(t) - C_o\hat{\underset{\sim}{x}}(t)] \tag{3.31}$$

where $L(t)$ is a matrix of observer gains. The optimal observer problem is that of determining the optimal gain matrix, $L^*(t)$, and the optimal observer initial state $\hat{\underset{\sim}{x}}(t_o)$ so that the observation error is minimized. To this end, we seek to minimize the quadratic cost function

$$J = E\{\underset{\sim}{e}^T(t)U(t)\underset{\sim}{e}(t)\} \tag{3.32}$$

where $\underset{\sim}{e}(t)$ is the observation error vector defined as

$$\underset{\sim}{e}(t) = \{\underset{\sim}{x}(t) - \hat{\underset{\sim}{x}}(t)\} \tag{3.33}$$

and $U(t)$ is a positive definite weighting matrix. Subtracting Eq. (3.31) from Eq. (3.18a) and using Eq. (3.18b), we obtain

$$\dot{\underset{\sim}{e}}(t) = [A - L(t)C_o]\underset{\sim}{e}(t) + \underset{\sim}{w}_1(t) - L(t)\underset{\sim}{w}_2(t) \tag{3.34}$$

which is the differential equation governing the observation error and is subject to the initial condition

$$\underset{\sim}{e}(t_o) = \{\underset{\sim}{x}(t_o) - \hat{\underset{\sim}{x}}(t_o)\} \tag{3.35}$$

To proceed with the minimization of the performance measure, it is helpful to expand Eq. (3.32) in terms of the error vector mean value and variance matrix. The mean value of the error vector is defined as

$$\bar{\underset{\sim}{e}}(t) = E\{\underset{\sim}{e}(t)\} \tag{3.36}$$

and the corresponding variance matrix is

$$\tilde{Q}(t) = E\{[\underset{\sim}{e}(t) - \bar{\underset{\sim}{e}}(t)][\underset{\sim}{e}(t) - \bar{\underset{\sim}{e}}(t)]^T\} \tag{3.37}$$

Using Eqs. (3.36) and (3.37), we can express the second-order joint moment matrix of the error vector as

$$E\{\underset{\sim}{e}(t)\underset{\sim}{e}^T(t)\} = \tilde{Q}(t) + \bar{\underset{\sim}{e}}(t)\bar{\underset{\sim}{e}}^T(t) \tag{3.38}$$

Now, Eq. (3.36) can be used along with Theorem 1.50 of ref. 8 to express the performance measure as

$$J = E\{\underset{\sim}{e}^T(t)U(t)\underset{\sim}{e}(t)\} = \bar{\underset{\sim}{e}}^T(t)U(t)\bar{\underset{\sim}{e}}(t) + tr[\tilde{Q}(t)U(t)] \tag{3.39}$$

In this form, the performance measure is broken into two independent terms that can be minimized separately. It can be shown (ref. 8) that the mean value of the error vector must satisfy

$$\dot{\bar{\underset{\sim}{e}}}(t) = [A - L(t)C_o]\bar{\underset{\sim}{e}}(t) \tag{3.40}$$

which is a homogeneous first-order differential equation. For a positive definite weighting matrix U(t), the first term in Eq. (3.39) is clearly minimized by letting $\bar{\underset{\sim}{e}}(t_o) = \underset{\sim}{0}$ since this choice implies $\bar{\underset{\sim}{e}}(t) = \underset{\sim}{0}$ for all time. We can make $\bar{\underset{\sim}{e}}(t_o) = \underset{\sim}{0}$ by choosing

$$\hat{\underset{\sim}{x}}(t_o) = \bar{\underset{\sim}{x}}_o \tag{3.41}$$

as the initial condition of the observer.

The second term in Eq. (3.39) does not depend on $\bar{\underset{\sim}{e}}(t)$ and therefore it can be minimized independently. It is shown in ref. 8 that the variance matrix of the error must satisfy the matrix differential equation

$$\dot{\tilde{Q}}(t) = [A - L(t)C_o]\tilde{Q}(t) + \tilde{Q}(t)[A - L(t)C_o]^T + V_1(t) + L(t)V_2(t)L^T(t) \tag{3.42}$$

with the corresponding initial condition

$$\tilde{Q}(t_o) = Q_o \tag{3.43}$$

Furthermore, two transformations involving time reversals (ref. 8) lead to the selection of the optimal observer gain matrix as

$$L^*(t) = Q(t)C_o^T V_2^{-1}(t) \tag{3.44}$$

where Q(t) is the n x n variance matrix of $\underset{\sim}{e}(t)$ which satisfies the nonlinear matrix differential Riccati equation

$$\dot{Q}(t) = AQ(t) + Q(t)A^T + V_1(t) - Q(t)C_o^T V_2^{-1}(t)C_oQ(t), \quad Q(t_o) = Q_o \tag{3.45}$$

The mean-square weighted error for the optimal observer is then given by

$$E\{\underset{\sim}{e}^T(t)U(t)\underset{\sim}{e}(t)\} = \mathrm{tr}[Q(t)U(t)] \tag{3.46}$$

It was pointed out earlier that the optimal observer problem and the optimal regulator problem are dual in nature. This duality is manifested in the fact that both solutions depend on a solution of an n^{th} order matrix differential equation of the Riccati type. In the regulator problem the differential equation must be integrated backward in time from a final condition. However, in the observer problem, equation (3.45) can be integrated forward in time from the given initial condition. Thus the solution of the matrix equation for the optimal observer is reduced in complexity compared to that of the optimal regulator. In both cases, the solution of a nonlinear n x n differential matrix equation can be reduced to the solution of a linear 2n x 2n differential matrix equation.

If $\underset{\sim}{w}_1(t)$ and $\underset{\sim}{w}_2(t)$ are white noise processes with constant intensities V_1 and V_2, respectively, then the Riccati matrix Q(t) approaches a constant matrix as time increases, independently of the value of Q_o. For this steady-state case, the differential Riccati equation reduces to the algebraic Riccati equation given by

$$AQ + QA^T + V_1 - QC_o^T V_2^{-1} C_o Q = 0 \tag{3.47}$$

and the resulting steady-state optimal observer gain matrix has the form

$$L^* = QC_o^T V_2^{-1} \tag{3.48}$$

The steady-state observer is optimal in the sense that the mean-square weighted error $E\{\underset{\sim}{e}^T(t)U\underset{\sim}{e}(t)\}$ is minimized for the constant weighting matrix U.

In Section 3.2 we demonstrated the *Separation Principle* which states that for a deterministic observer the closed loop-characteristics of the plant are independent of the choice of observer gains. This principle can also be extended to the case of the optimal stochastic observer. To this end, we write the closed-loop system in terms of the plant state and the observation error. Substituting the feedback control law, Eq. (3.14), into Eq. (3.18a) and combining the result with Eq. (3.34), we obtain the closed-loop equations in matrix form

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A\text{-}BG & BG \\ 0 & A\text{-}LC_o \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} D\text{-}R^{-1}V \\ 0 \end{bmatrix} [f_{dc}] - \begin{bmatrix} R^{-1}B^T \\ 0 \end{bmatrix} [v] + \begin{bmatrix} I & 0 \\ I & -L \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{3.49}$$

As in the deterministic case, the closed-loop coefficient matrix

$$A_c = \begin{bmatrix} A\text{-}BG & BG \\ 0 & A\text{-}LC_o \end{bmatrix} \tag{3.50}$$

is in block-triangular form. We therefore conclude that the *Separation Principle* holds for the optimal stochastic observer and the optimal gain matrix $L^*(t)$ does not effect the closed-loop characteristics of the feedback system.

IV. Numerical Integration Techniques

4.1 Introduction

Before we implement the control system/observer design developed in the previous chapters, it is necessary to calculate the modal acoustic pressures exerted on the plate surface. Indeed, the coefficient matrix of the state space system depends explicitly on the fluid-loading matrix. The modal surface pressures are given by Eq. (2.36) in the form

$$f_{pmn}(t) = -\frac{i\rho}{4\pi^2}\sum_{r,s=1}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\tilde{W}_{rs}(\gamma_x,\gamma_y)\tilde{W}^*_{mn}(\gamma_x,\gamma_y)}{(k^2-\gamma_x^2-\gamma_y^2)^{1/2}}d\gamma_x d\gamma_y\right]\ddot{q}_{rs}(t) \tag{4.1}$$

and their relationship to the fluid-loading matrix is given by Eq. (2.39), or

$$\begin{aligned} f_{pr}(t) &= -\frac{i\rho}{4\pi^2}\sum_{s=1}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\tilde{W}_{s}(\gamma_x,\gamma_y)\tilde{W}^*_{r}(\gamma_x,\gamma_y)}{(k^2-\gamma_x^2-\gamma_y^2)^{1/2}}d\gamma_x d\gamma_y\right]\ddot{q}_{s}(t) \\ &= -\sum_{s=1}^{\infty} c_{rs}\ddot{q}_s(t), \quad r = 1,2,\ldots,\infty \end{aligned} \tag{4.2}$$

which has the matrix form

$$\underset{\sim}{f}_p(t) = -[C]\underset{\sim}{\ddot{q}}(t) \tag{4.3}$$

The difficulty in calculating the modal pressures lies in the calculation of the double integral in Eq. (4.1). First of all, there is a square root singularity at $\gamma_x^2+\gamma_y^2 = k^2$ and, secondly, the limits of integration are infinite, thus creating an infinite number of singularities in the integrand. There is no closed-form solution for the double integral. However, Chang and Leehey (ref. 14) have developed a computer code that can be used to calculate the integral numerically using a combination of Romberg and Tschebychev quadratures. The computer code is presented in detail in (ref. 15). The purpose of this chapter is to nondimensionalize Eq. (4.1) and manipulate it into a form that can be readily calculated by means of the algorithm of Chang and Leehey.

4.2 Modal Surface Pressures

As a first step in evaluating the integral in Eq. (4.1), we need to determine the nature of the Fourier transforms $\tilde{W}_{rs}(\gamma_x,\gamma_y)$ and $\tilde{W}^*_{mn}(\gamma_x,\gamma_y)$, given by

$$\tilde{W}_{rs}(\gamma_x,\gamma_y) = \int_{-b}^{b}\int_{-a}^{a} W_{rs}(x,y)e^{-i[\gamma_x x + \gamma_y y]}dxdy, \quad r,s=1,2,... \tag{4.4}$$

and

$$\tilde{W}^*_{mn}(\gamma_x,\gamma_y) = \int_{-b}^{b}\int_{-a}^{a} W_{mn}(x,y)e^{i[\gamma_x x + \gamma_y y]}dxdy, \quad m,n=1,2,... \tag{4.5}$$

respectively. To evaluate the above expressions, it is helpful to translate the coordinate system to one of the corners of the plate. The new coordinates are shown in Figure 4.1.

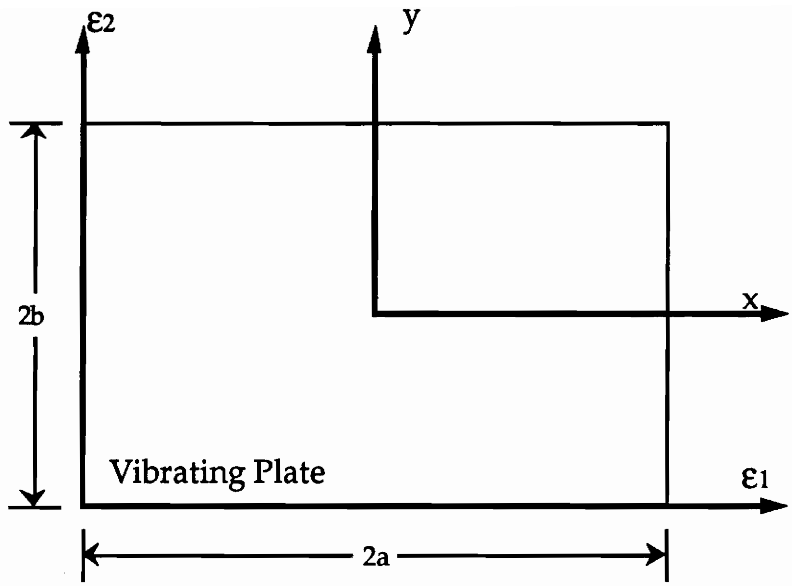

Figure 4.1 Coordinate System Transformation

This coordinate translation can be written as

$$\varepsilon_1 = x + a, \qquad \varepsilon_2 = y + b \tag{4.6a,b}$$

Substitution of Eqs. (4.6) into the mode shapes given in Eq. (2.27) results in

$$W_{rs}(\varepsilon_1,\varepsilon_2) = \frac{1}{\sqrt{m\ ab}} \sin k_r\varepsilon_1 \sin k_s\varepsilon_2 \tag{4.7}$$

where we have defined

$$k_r = \frac{r\pi}{2a}, \qquad k_s = \frac{s\pi}{2b} \tag{4.8}$$

as modal wavenumbers. Introducing Eq. (4.7) into Eq. (4.4), we obtain

$$\tilde{W}_{rs}(\gamma_x,\gamma_y) = \frac{1}{\sqrt{mab}}\int_0^{2b}\int_0^{2a} \sin k_r\varepsilon_1 \sin k_s\varepsilon_2\, e^{-i[\gamma_x\varepsilon_1+\gamma_y\varepsilon_2]} d\varepsilon_1 d\varepsilon_2$$

$$= \frac{1}{\sqrt{mab}}\int_0^{2a} \sin k_r\varepsilon_1\, e^{-i\gamma_x\varepsilon_1} d\varepsilon_1 \int_0^{2b} \sin k_s\varepsilon_2\, e^{-i\gamma_y\varepsilon_2} d\varepsilon_2$$

$$= \frac{1}{\sqrt{mab}}\; \frac{k_r\, k_s[1-(-1)^r e^{-2i\gamma_x a}][1-(-1)^s e^{-2i\gamma_y b}]}{(k_r^2-\gamma_x^2)\,(k_s^2-\gamma_y^2)}, \quad r,s = 1,2,... \qquad (4.9)$$

Similarly, inserting Eq. (4.7) with r and s replaced by m and n, respectively, into Eq. (4.5), we obtain

$$\tilde{W}^*_{mn}(\gamma_x,\gamma_y) = \frac{1}{\sqrt{mab}}\; \frac{k_m\, k_n[1-(-1)^m e^{2i\gamma_x a}][1-(-1)^n e^{2i\gamma_y b}]}{(k_m^2-\gamma_x^2)\,(k_n^2-\gamma_y^2)} \qquad (4.10)$$

Then, combining Eqs. (4.10), (4.11) and (4.1), we can write an expression for the modal surface pressures

$$f_{pmn}(t) = -A\sum_{r,s=1}^{\infty}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{k_m k_n k_r k_s\,[1+(-1)^{m+r} - (-1)^m e^{2i\gamma_x a} - (-1)^r e^{-2i\gamma_x a}]}{(k_m^2-\gamma_x^2)\,(k_r^2-\gamma_x^2)}\right.$$

$$\left. \times\frac{[1+(-1)^{n+s} - (-1)^n e^{2i\gamma_y b} - (-1)^s e^{-2i\gamma_y b}]}{(k_n^2-\gamma_y^2)\,(k_s^2-\gamma_y^2)\,(k^2-\gamma_x^2-\gamma_y^2)^{1/2}} d\gamma_x d\gamma_y\right] \ddot{q}_{rs}(t) \qquad (4.11)$$

where

$$A = \frac{i\rho}{4\pi^2 mab} \qquad (4.12)$$

The exponential terms in Eq. (4.11) can be rewritten in terms of trigonometric functions using Euler's Formula, $e^{i\theta} = \cos\theta + i\sin\theta$, with the goal of reducing the complexity of the integrand. In particular, we can write

$$(-1)^n e^{2ib\gamma_y} + (-1)^s e^{-2ib\gamma_y} = [(-1)^n + (-1)^s]\cos 2b\gamma_y + [(-1)^n - (-1)^s]\sin 2b\gamma_y \qquad (4.13a)$$

and

$$(-1)^m e^{2ia\gamma_x} + (-1)^r e^{-2ia\gamma_x} = [(-1)^m + (-1)^r]\cos 2a\gamma_x + [(-1)^m - (-1)^r]\sin 2a\gamma_x \qquad (4.13b)$$

Now, consider several cases involving different values of the mode numbers, r, s, m and n. If n+s is an odd number, then Eq. (4.13a) reduces to an odd function of γ_y. Since all other terms in Eq. (4.11) are even functions of γ_y, the integrand reduces to an odd function of γ_y, for which

$$\int_{-\infty}^{\infty} f_{odd}(\gamma_y)d\gamma_y = 0, \quad n+s = \text{odd} \qquad (4.14)$$

Similarly, if m+r is an odd number, then Eq. (4.13b) reduces to an odd function of γ_x which in turn results in an integrand which is an odd function of γ_x. As in Eq. (4.14) above, we then have

$$\int_{-\infty}^{\infty} f_{odd}(\gamma_x)d\gamma_x = 0, \quad m+r = \text{odd} \qquad (4.15)$$

Hence, the integral in Eq. (4.11) is different from zero only for even values of n+s and m+r. The resulting expression for the modal surface pressures takes the form

$$f_{pmn}(t) = \sum_{r,s=1}^{\infty} J_{mnrs}\, \ddot{q}_{rs}(t) \tag{4.16}$$

where

$$J_{mnrs} = \frac{-2i\rho k_m k_n k_r k_s}{\pi^2 m ab} \int_0^{\infty}\int_0^{\infty} \frac{[1-(-1)^n \cos 2b\gamma_y]}{(k_n^2-\gamma_y^2)(k_s^2-\gamma_y^2)(k_m^2-\gamma_x^2)}$$

$$\times \frac{[1-(-1)^m \cos 2a\gamma_x]}{(k_r^2-\gamma_x^2)(k^2-\gamma_x^2-\gamma_y^2)^{1/2}} d\gamma_x d\gamma_y \tag{4.17a}$$

when both n+s and m+r are even and

$$J_{mnrs} = 0 \tag{4.17b}$$

otherwise.

4.3 Nondimensional Form of the Modal Surface Pressures

To reduce the total number of independent variables involved, and thus further reduce the complexity of the numerical integration, we non-dimensionalize the integrand in the expression for the modal surface pressures, Eq. (4.17a). The Fourier transform parameters and the wavenumbers are nondimensionalized using the plate length 2a as a characteristic length, resulting in the nondimensional quantities,

$$K = 2ak, \quad G_x = 2a\gamma_x, \quad G_y = 2a\gamma_y = 2bR\gamma_y \tag{4.18a,b,c}$$

$$K_m = 2ak_m = m\pi, \quad K_n = 2ak_n = n\pi R \tag{4.18d,e}$$

$$K_r = 2ak_r = r\pi, \quad K_s = 2ak_s = s\pi R \tag{4.18f,g}$$

where R = a/b is the plate aspect ratio. Introducing Eqs. (4.18) into Eq. (4.17a), we obtain

$$J_{mnrs} = \frac{-8i\rho\pi^2 mnrsR^3 a}{m} \int_0^\infty \int_0^\infty \frac{I_{mr}\, I_{ns}}{(K^2 - G_x^2 - G_y^2)^{1/2}} dG_x dG_y \tag{4.19}$$

where

$$I_{mr} = \frac{1-(-1)^m \cos G_x}{(K_m^2 - G_x^2)\,(K_r^2 - G_x^2)}, \quad I_{ns} = \frac{1-(-1)^n \cos(G_y/R)}{(K_n^2 - G_y^2)\,(K_s^2 - G_y^2)} \tag{4.20a,b}$$

The nondimensional integrand in Eq. (4.19) depends on only the modal wavenumbers, m, n, r and s, and the plate aspect ratio, R.

4.4 Discussion of Numerical Integration

The double integral in Eq. (4.19) is still very complicated. There are several difficulties encountered in evaluating such an integral, including:

1) the integrand has a square root singularity for $G_x^2 + G_y^2 = K^2$. This represents an infinite number of singular points since $G_x^2 + G_y^2 = K^2$ ($0 \le G_x \le \infty$, $0 \le G_y \le \infty$) traces out a quarter-circle in the G_x,G_y plane,

2) the integral is an improper one, with the upper limits extending to $+\infty$ and

3) the integrand is a very complex function with four indeterminate points that must be evaluated using L'Hospital's rule.

Since the singularity in Eq. (4.19) traces out a circle in the G_x, G_y plane, it appears that a suitable transformation of coordinates may reduce the complexity of the problem. Indeed, transformation to a polar coordinate system in the G_x, G_y plane reduces the singularity to a single point along the radial coordinate. To this end, we introduce the polar coordinate transformations

$$\theta = \tan^{-1}\frac{G_y}{G_x}, \quad X = \frac{\sqrt{G_x^2 + G_y^2}}{K}, \quad Y = \frac{K}{\sqrt{G_x^2 + G_y^2}} \qquad (4.21a,b,c)$$

Using this transformation permits the real and imaginary parts of J_{mnrs} to be separated. The equations in polar coordinates are

$$J_{mnrs} = \Gamma_{mnrs} + i\Theta_{mnrs} \qquad (4.22)$$

where

$$\Gamma_{mnrs} = \frac{-8\rho\pi^2 mnrsR^3 aK}{m}\int_0^1\int_0^\pi \frac{I_{mr}\, I_{ns}}{Y^2(1-Y^2)^{1/2}}\,d\theta dY \qquad (4.23a)$$

$$\Theta_{mnrs} = \frac{8\rho\pi^2 mnrsR^3 aK}{m}\int_0^1\int_0^\pi \frac{I_{mr}\, I_{ns}}{(1-X^2)^{1/2}}\,X d\theta dX \qquad (4.23b)$$

Integrals of the form in Eqs. (4.23) were derived for the case of fluid flow over the surface of a rectangular plate in refs. 14 and 15. Equations (4.23) can be recovered from the corresponding equations in these references by substitution of zero Mach number corresponding to a no flow condition over the plate surface.

4.5 Romberg Quadrature Over the Angular Coordinate

The singularities in Eqs. (4.23) occur only in the integration over the radial coordinate. We propose to handle the numerical integration over the angular coordinate using Romberg quadrature, which uses the trapezoidal rule over subintervals, bisected repeatedly, until the difference between two successive extrapolated estimates falls within a specified error tolerance (ref. 15). It has been proven that a Romberg quadrature converges rapidly to the true value of the integral (ref. 16). Chang and Leehey (ref. 15) have developed a FORTRAN subroutine to perform calculation of integrals using a modified version of the Romberg quadrature with faster convergence than the standard version. A complete listing of the FORTRAN code is given in Appendix II of reference 15.

4.6 Tschebychev Quadrature Over the Radial Coordinate

The singularity encountered along the radial coordinate in Eq. (4.23) can be elegantly handled numerically using a form of the standard Tschebychev quadrature, or

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{n} \sum_{j=1}^{n} f(a_j) + E' \tag{4.25}$$

where

$$E' = \frac{2\pi}{2^{2n}\,(2n)!} f^{(2n)}(\xi), \quad a_j = \cos\frac{(2j-1)\pi}{2n}, \quad j = 1,2,\ldots,n \tag{4.26}$$

and $\xi \in [-1,1]$. The order of the quadrature is n, which is also the number of abscissas used for evaluation of the integrand. To calculate the integrals in Eqs. (4.23) it is necessary to shift the limits from [-1, 1] to [0, 1]. Chang and Leehey (ref. 15) have developed a FORTRAN subroutine to calculate integrals of the type given in Eqs. (4.23) using a modified version of the Tschebychev quadrature given above. A complete listing of the FORTRAN code is included in Appendix II of reference 15. For a detailed discussion of both Romberg and Tschebychev quadratures used in numerical integration, the reader is urged to consult reference 16.

V. Summary and Suggestions for Further Work

5.1 Summary

The problem of suppressing the far-field acoustic pressure radiated by a structure vibrating in fluid has received much attention recently due to its many applications in the fields of aeroacoustics and hydroacoustics. When a vibrating structure is submerged in fluid, the fluid acts as a coupling medium, causing a transfer of energy between the modes of vibration of the structure. Of the vibrating modes, some radiate acoustic pressure more than others.

Suppression of the far-field acoustic radiation pressure is possible using either active or passive controls, where active control offers many advantages over passive control. For the case of a vibrating structure submerged in fluid, only certain modes contribute significantly to the far-field acoustic radiation pressure. Only through the use of active controls is it possible to tailor feedback control forces so as to automatically concentrate on the modes contributing the most to the far-field pressure.

Meirovitch (ref. 1) has developed a general modern control theory for the optimal control of the far-field acoustic pressure radiating from a structure submerged in fluid. The theory involves simultaneous solution of the boundary-value problem for the structure, the interaction between structure and fluid and the control design. The selection of modes to be controlled is made in an optimal fashion by including a penalty on the far-field acoustic pressure in the performance measure, in addition to the standard penalties on the state and on the control effort. The distributed parameter problem is governed by partial differential equations for which control design is not within the state of the art. Therefore, the problem is discretized in space by using the *in vacuo* modes of the structure as a series of admissible functions. The resulting state feedback control law determines the control forces as a function of all the modal state variables. It is developed under the assumption that all state variables are directly measurable and available for feedback.

This thesis extends the general control theory developed by Meirovitch to the control of the far-field acoustic pressure radiating from a vibrating plate submerged in fluid for the case in which not all state variables are available from sensor data. The model consists of a simply supported isotropic plate bounded on one side by air and on the other by fluid. External excitation is applied to the dry side of the plate in the form of a harmonic persistent disturbing force and control forces. To estimate the complete state for feedback, the control system incorporates a full-order optimal stochastic observer, or Kalman-Bucy filter. The observer is optimal in the sense that the mean-square estimation error is minimized.

5.2 Suggestions for Future Work

To assess the performance of the closed-loop system incorporating feedback control and state estimation, it is necessary to carry out numerical simulation of the system equations, Eqs. (3.49). Before simulation can be carried out, the numerical integrations discussed in Chapter 4 must be incorporated into a routine for the calculation of the fluid pressure at the plate surface. Similarly, a routine must be developed for the calculation of the far-field pressure through numerical integration of Rayleigh's formula, Eq. (2.63). Once these routines are developed, numerical simulation of the closed-loop equations is likely to be straight-forward, albeit very intensive computationally.

REFERENCES

1. Meirovitch, L., "A Theory for the Optimal Control of the Far-Field Acoustic Pressure Radiating from Submerged Structures," Journal of the Acoustical Society of America, (to be published).

2. Firth, D., "Acoustic Vibrations of Structures in Liquids," Shock and Vibration Digest, Vol. 9, No. 9, 1977, pp. 3-7.

3. Junger, M. C. and Feit, D., Sound, Structures and Their Interaction, 2nd ed., MIT Press, Cambridge, MA, 1982.

4. Meirovitch, L., Dynamics and Control of Structures, Wiley-Interscience, New York, 1990.

5. Warnaka, G. E., "Active Attenuation of Noise - the State of the Art," Noise Control Engineering, Vol. 18, No. 3, 1982, pp. 100-110.

6. Luenberger, D. G., "An Introduction to Observers," IEEE Transactions on Automatic Control, Vol. AC-16, No. 6, 1971, pp. 596-602.

7. Chen, C. T., Linear System Theory and Design, Holt, Rinehart and Winston, New York, 1984.

8. Kwakernaak, H. and Sivan, R., Linear Optimal Control Systems, Wiley-Interscience, New York, 1972.

9. Meirovitch, L., Analytical Methods in Vibrations, The Macmillan Co., New York, 1967.

10. Luenberger, D. G., "Observers for Multivariable Systems," IEEE Transactions on Automatic Control, Vol. AC-11, No. 2, 1966, pp. 190-197.

11. Kalman, R. E. and Bucy, R. S., "New Results in Linear Filtering and Prediction Theory," Journal of Basic Engineering, Vol. 83, 1961, pp. 95-108.

12. Johnson, C. D., "Accommodation of Disturbances in Optimal Control Problems," International Journal on Control, Vol. 15, No. 2, 1971, pp. 209-231.

13. Kwatny, H. G. and Kalnitsky, K. C., "On Alternative Methodologies for the Design of Robust Linear Multivariable Regulators," IEEE Transactions on Automatic Control, Vol. AC-23, No. 5, 1978, pp. 930-933.

14. Chang, Y. M. and Leehey, P., "Acoustic Impedance of Rectangular Panels," Journal of Sound and Vibration, Vol. 64, No. 2, 1979, pp. 243-256.

15. Chang, Y. M., "The Mean Flow Effect on the Acoustic Impedance of a Rectangular Panel," MIT Acoustics & Vibration Laboratory Report No. 82464-1, 1977.

16. Ralston, A., A First Course in Numerical Analysis, New York, McGraw-Hill, 1965.

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