Nonlinear Finite Element Analysis of a Laminated Composite Plate
with Nonuniform Transient Thermal Loading

by

Thomas Harris Fronk

Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of
Master of Science

APPROVED:

Dr. J. R. Mahan, Chairman

Dr. C. E. Knight
Dr. J. B. Kosmatka

March 25, 1988
Blacksburg, Virginia
LD
5655
V855
1988
F165
C.2
Nonlinear Finite Element Analysis of a Laminated Composite Plate
with Nonuniform Transient Thermal Loading

by

Thomas Harris Fronk

Dr. J. R. Mahan, Chairman

(ABSTRACT)

Metal plates are being replaced by lighter but equally strong laminated composite plates in order to improve efficiency and increase performance of aerospace vehicles. But because of the complex construction of laminated plates they are very difficult to analyze. Conventional thin plate theories prove to be inadequate in predicting laminated composite plate behavior. Therefore, a finite element model which incorporates a first-order shear-deformation theory and nonlinear von Karman strains is described. The model is shown to accurately predict deflections in laminated composite plates due to nonuniform transient heat fluxes and transverse mechanical loads.
Acknowledgements

I would like to thank my adviser Dr. J.R. Mahan for his constant help and advice. Also I appreciate the advice and time given to me by the members of my committee, Dr. C.E. Knight, and Dr. J.B. Kosmatka. A special thanks goes to NASA Langley and Dr. John S. Mixson for their funding of this project under contract NAS1-18471, Task 3. I am indebted to my parents, brothers, and sisters for their example and encouragement. And most of all I thank my wife Monica and two sons Alexander and Aaron. Without their support it would not have been possible.
# Table of Contents

1. Introduction ............................................................................................................. 1

2. Laminated Plate Analysis .......................................................................................... 6
   2.1. Laminate Definition ............................................................................................. 6
   2.2. Classical Lamination Theory (CLT) ................................................................. 7
   2.3. Shear-Deformation Theory (SDT) ..................................................................... 9

3. Lamina Stresses, Strains, and Displacements ......................................................... 12
   3.1. Lamina Stress-Strain Relations ......................................................................... 12
   3.2. Strain-Displacement Relations ......................................................................... 20

4. Governing Equations for Laminated Plates ............................................................ 24
   4.1. Laminate Forces and Moments ......................................................................... 24
   4.2. Thermal Forces and Moments ......................................................................... 27
   4.3. Equations of Motion ......................................................................................... 33

5. Finite Element Model ............................................................................................... 37
5.1. Variational Formulation ............................................. 37
5.2. Interpolation and Approximate Functions .......................... 45
5.3. Gauss-Legendre Quadrature and Reduced Integration ............ 54
5.4. The Transient Solution (Newmark Method) ....................... 58
5.5. Newton-Raphson Method .......................................... 60

6. Verification of Model and Results .................................... 63
   6.1. Static Mechanical Load ......................................... 64
   6.2. Linear Temperature Variation Through an Isotropic Plate ....... 66
   6.3. Transient Mechanical Load ..................................... 67
   6.4. Transient Thermal Load ........................................ 68

7. Conclusions and Recommendations .................................... 70

References .......................................................................... 88

Appendix A. COMMEC2 PROGRAM LISTING ................................. 92

Vita ................................................................................ 109
List of Illustrations

Fig. 1. Typical Laminated Composite Plate. ........................................... 71

Fig. 2. Plate Element with Seven Degrees of Freedom, three Translations and Four Rotations. ...................................................... 72

Fig. 3. (a) Undeformed Laminated Composite Plate (b) Deformed Laminated Plate with CLT Displacement Field (c) Deformed Laminated Plate with First-Order SDT Displacement Field. ........................................... 73

Fig. 4. (a) Forces on the Laminated Plate (b) Moments on the Laminated Plate. .......................................................... 74

Fig. 5. Laminate Coordinate System and Nomenclature. ......................... 75

Fig. 6. Typical Infinitesimal Cube of Laminated Composite Plate. ............. 76

Fig. 7. Master Element for (a) 4-Node Rectangular (b) 8-Node Rectangular, and (c) 9-Node Rectangular Elements. ............................................. 77

Fig. 8. Finite Element Mesh used in the Plate Analysis, Four 9-Node Quadrilateral Elements. .................................................. 78

Fig. 9. Geometry and Boundary Conditions of the First Quadrant of a Simply-Supported Square Plate. ................................. 79

Fig. 10. Normalized Center Deflection of a Simply-Supported Isotropic Plate Under a Uniformly-Distributed Transverse Load. ......................... 80

Fig. 11. Geometry and Boundary Conditions of the First Quadrant of a Clamped Symmetric Cross-Ply [0/90/90/0] Laminated Composite Plate. ................. 81

Fig. 12. Center Deflection of a Clamped Symmetric Cross-Ply [0/90/90/0] Laminated Subjected to a Uniform Transverse Load (a = 4.72 cm (12 in.), h = 0.038 cm (0.096 in.)). ......................................................... 82

Fig. 13. Normalized Deflection of a Simply-Supported Isotropic Plate Subjected to a Linear Temperature Distribution. ................................. 83
Fig. 14. Evolution of Center Deflection with Time for a Simply-Supported Isotropic Square Plate Under a Uniform Load of 100 kPa (a = 25.0 cm, h = 5.0 cm). 84

Fig. 15. Evolution of Center Deflection with Time for a [0/90/0] Simply-Supported Laminated Composite Plate Under a Uniform Load of 100 kPa (a = 25.0 cm, h = 5.0 cm). .............................................. 85

Fig. 16. Geometry and Boundary Conditions of the First Quadrant of a Simply-Supported Cross-Ply [0/90/90/0] Laminated Composite Plate (a = 2.0 cm, h = 0.5 cm). .............................................. 86

Fig. 17. Evolution of Center Deflection with Time for a Simply-Supported [0/90/90/0] Laminated Composite Plate Subjected to a Nonuniform Heat Flux (a = 2.0 cm, h = 0.5 cm). .............................................. 87
The need for a strong yet light-weight structural material has evolved with the evolution of aerospace vehicles. Many missiles, aircraft, and even the space shuttle now use parts made of composite materials which have large strength-to-weight and stiffness-to-weight ratios. One of the most popular forms of composite material is the laminated plate. Several layers of unidirectional aligned fibers bound with resin, called *laminae*, are bonded together to form a solid plate, called a *laminate*. The individual laminae are oriented in certain directions giving the laminate the necessary strength and stiffness in the directions of the anticipated loads. A typical laminated plate is shown in Fig. 1. A laminated composite plate is an *orthotropic, nonhomogenous* material. Orthotropic means the material properties are directional dependent with three orthogonal planes of material property symmetry and nonhomogenous means the material properties are location dependent. Because of their orthotropy and nonhomogenity, many complications and complexities arise in the analysis and description of laminated composites that do not exist with conventional isotropic, homogenous materials. These include coupling between bending and extension, bending and twisting, and shear and extension.
The National Aeronautics and Space Administration (NASA) is currently studying the interaction of thermal stress and sonic fatigue in laminated composite structures of the proposed U.S. - U.K. advanced short take-off and vertical landing (ASTOVL) aircraft [1]. If the aft fuselage and thrust redirection and lifting surfaces of the ASTOVL can be made of laminated composites, a substantial weight savings will be gained. However, the engine exhaust causes high levels of acoustic radiation and heat flux to impinge on these structures. Although composite materials are said to have an almost infinite fatigue life [2], the extent of damage caused by the coupling of acoustically-induced vibration with thermally-induced stress is still unknown. Given the complex nature of composite materials, it is unlikely that their behavior can be predicted only on the basis of experience with isotropic materials. A finite element model of a laminated composite plate, developed in this thesis, is shown to accurately and economically predict the deflections caused by transient thermally-induced loads. This is the necessary first step in the development of a finite element model capable of describing the thermoacoustic response of fiber-reinforced composite materials.

The science of analyzing and designing laminated composite components has received significant attention only in the last twenty to thirty years. In the 1950's and 60's an increased interest arose in developing elasticity theories for plates that are nonhomogeneous and anisotropic. Stavsky and Reissner [3] established the elastostatic bending and stretching theory for nonhomogenous plates based on the Euler-Bernoulli hypothesis and the classical Kirchoff hypothesis. This evolved into what is now called the classical plate theory (CPT). Classical lamination theory (CLT) is an extension of CPT to laminated composite plates. Classical lamination theory is generally accepted and widely used to predict laminated composite plate response and in designing components made of laminated composites. Classical lamination theory is relatively easy to use because it employs several simplifying assumptions discussed in Chapter 2.
However, many researchers have recognized the inadequacies and limitations of classical lamination theory. In 1969, Pagano published a series of papers [4,5,6] in which the deficiencies of classical lamination theory are exposed and explained. One of the most limiting consequences of the CLT assumptions is that accurate predictions are obtained only when the plate has a high span-to-depth ratio and deflections are small. Therefore, CLT is really only a thin plate approximation of laminated composite plates.

The varying material properties of laminated plates make it impossible to set a specific thickness where CLT yields valid results. Hence, many have attempted to account for thickness effects by including shear-deformation terms. The Mindlin plate theory [7] includes transverse shear effects and, therefore, has served as the basis for developing theories which account for deformation caused by shear. Stravsky [8] was the first to propose shear deformation theory as a solution to analyze laminated isotropic plates. Later, Yang, Norris, and Stavsky [9] generalized the shear deformation theory to include laminated anisotropic heterogenous plates. Whitney and Pagano [10] applied the theory of Yang, Norris, and Stavsky (YNS) to bending of antisymmetric cross-ply and angle-ply plate strips under sinusoidal load distribution and free vibration of antisymmetric angle-ply strips and reported good results. Reissner [11], Lo, Christensen and Wu [12], and Librescu [13] have all developed higher-order shear-deformation theories to predict the response of laminated composites. Each one of these has included shear deformation effects and consequently led to improved accuracy of their predictions over that afforded by the traditional CLT results (e.g. Dong, Pister, and Taylor [14], and Bert and Mayberry [15]). Of great importance to the research of this thesis is the work done by Reddy and Nsu [16]. They used the YNS theory to evaluate the effects of shear deformation and anisotropy on the thermal bending of layered composite plates. Recently, Mei and Prasad [17] investigated the influence of large deflection and transverse shear
on rectangular symmetric laminated composite plates due to acoustic loads. They found that for thick plates the effects due to shear deformation are considerable.

The finite element method has been used extensively to analyze isotropic plates, but less effort has gone into the investigation of laminated composite plates. Thick laminated plates have been analyzed by Pryor and Barker [18], and Barker, Lin, and Dara [19] using the conventional displacement finite element method. A plate element with seven degrees of freedom per node (three displacements, two rotations, and two shear rotations as shown in Fig. 2) was used. Implementing an element with 80 degrees of freedom, Noor and Mathers [20] developed a finite element model based on Reissner's plate theory. They were able to study the effects of shear deformation on the accuracy of several models. While these models are very accurate they are also computationally very demanding. Models of plate elements with five degrees of freedom (three displacements, two rotations) have been formulated by Yang, Norris, and Stravsky [9] and Reddy [21,22]. They show that in most cases the use of only five degrees of freedom proves sufficiently accurate. Reddy [23] has improved the accuracy of the predictions of the finite element model incorporating the method of reduced integration on the shear deformation terms as proposed by Zienkiewicz, Taylor, and Too [24].

The problem of thermally-induced stress in nonhomogenous materials was first addressed by Pell [25], Sharma [26], and Nowacki [27]. Later Sugano [28] studied the effects of transient thermal stresses in transverse isotropic cylinders. The thermal stress problem was expanded to include orthotropic laminated plates by Tauchert and Akoz [29]. However, this was only a steady-state solution. Most recently Wang, and Chou [30] present a solution of transient thermal stresses in thermally and elastically orthotropic two-dimensional materials based on a displacement-potential approach.

The previous work cited above has provided the basis for the nonlinear transient finite element model developed in this thesis. The model is nonlinear in the sense that
the in-plane strain-displacement relationships include the square of the transverse de-
flexion; i.e. the von Karman strains are implemented. The model is capable of accu-
rately and economically predicting the dynamic behavior of the laminated composite
components of the type proposed for use in the ASTOVL aircraft, and is capable of ex-
amining the effects of imposed three-dimensional, transient temperature fields while ac-
counting for the nonlinear shear deformation terms with the use of reduced integration.
The temperature field is restricted to the semicoupled case in which the heat generated
by the kinetic behavior of the plate is assumed to be negligible. This assumption has
been extensively validated by Jia [31]. To the author's knowledge, the finite element
model in this thesis is unique in its accuracy, versatility and low computational demand.
The goals of the work described in this thesis are, first, to formulate the finite element
model just described and, second, to demonstrate its capabilities by considering the
transient, three-dimensional thermoelastic response of a laminated plate.
2. Laminated Plate Analysis

The purpose of this chapter is to describe the components of a laminated composite plate and to develop the theories used in laminated composite plate analysis. One of the most prevalent lamination theories, classical lamination theory (CLT), is described and shown to be inadequate for the application envisioned in this thesis. Other theories which assume higher-order displacement fields and include shear deformation effects are discussed. Finally, a first-order theory which does not rely on Kirchoff's assumption is shown to be the most suitable for the application of this thesis.

2.1. Laminate Definition

Consider a composite laminated plate of thickness \( h \) and width \( a \) as shown in Fig. 1. Throughout this thesis the laminate orientation will be referred to a global \( x, y, z \) system of Cartesian coordinates. The \( x-y \) plane of the coordinate system is the midplane of the plate and the \( z \)-axis is normal to the midplane. The top of the plate, therefore,
will be located at \( z = + \frac{h}{2} \) and the bottom at \( z = - \frac{h}{2} \). The principal material axes of an individual lamina are aligned parallel to the direction of the unidirectional fibers in each case. The angle between the laminate \( x \)-axis and the principal material axis is referred to as the angle of rotation and is denoted by \( \theta \) in Fig. 1. The angle of rotation is positive in the clockwise direction.

Each layer is assumed to be orthotropic and homogeneous. Orthotropic means the lamina has three planes of geometric and material property symmetry. The orthotropic assumption is made with little or no error. There are many methods and theories for mathematically deriving orthotropic material properties of a heterogenous lamina. However, these methods are tedious and rarely more accurate than empirical characterizations of the material properties [3]. Therefore, for this analysis no micromechanical calculations are performed. The lamina properties are assumed to be known and homogenous throughout each lamina.

2.2. Classical Lamination Theory (CLT)

Classical lamination theory (CLT) is the prevalent analysis and design tool of laminated composite plates because of its relative simplicity compared to three-dimensional theory. The assumptions made in classical lamination theory are listed below along with the direct implication of each assumption.

1. **Assumption:** Plane sections originally plane and perpendicular to the middle surface remain plane and perpendicular to the middle surface after extension and bending.

   **Implication:** The deflection of the plate is associated only with the bending strains. The shear strains \( \varepsilon_{xy} \) and \( \varepsilon_{xz} \) are neglected.
2. **Assumption:** The displacements of the middle surface are small compared with the thickness of the plate.

**Implication:** The slope of the deflected surface is small compared to the in-plane strains and the square of the slope is negligible; i.e., linear strain-displacement relations are valid, and the influence of in-plane forces upon the transverse deflection may be ignored.

3. **Assumption:** The thickness of the plate is small compared with other dimensions of the plate.

**Implication:** The plate is in a state of plane stress. \( \sigma_z = \tau_{xz} = \tau_{yz} = 0. \)

4. **Assumption:** The laminate consists of perfectly bonded laminae and the bonds are infinitesimally thin as well as nonshear-deformable.

**Implication:** There is no possibility of delamination since laminae cannot slip with respect to each other.

5. **Assumption:** The plate is constructed of linearly-elastic material with properties that are independent of temperature.

**Implication:** Hooke's law is applicable.

Although these assumptions simplify the analysis and design of composite materials, the implications often cause erroneous results. Pagano and his associates in a series of papers [4,5,6] show that CLT predictions are reasonable only in the case of relatively thin plates. This is because of:

1. the neglect of shear deformation terms, implied by the Kirchoff hypothesis (normals remain normal)
2. the assumption of linear in-plane displacements through the thickness

3. the presence of only two boundary conditions per edge in the bending theory

4. the assumption of a state of plane stress which eliminates the possibility of interlaminar stress calculations.

Research has been done to eliminate some of the restricting assumptions of CLT and still maintain the computational simplicity of plate theory. Shear-deformation theories (SDT) are a result of this research.

2.3. Shear-Deformation Theory (SDT)

The Kirchoff restriction is unnecessary in shear-deformation theories and, therefore, they are a compromise between the accuracy of three-dimensional theory and the simplicity of plate theory. Shear-deformation theories can be grouped into either stress-based or displacement-based theories. Reissner [32] is responsible for the first stress-based SDT and Basset [33] is one of the first to develop a shear deformation theory based on displacements. The theory developed in this thesis is also a displacement-based theory. Basset assumes that displacements can be expanded in power series of the z-coordinate. For example, the displacement in the x-coordinate is written

\[ u(x, y, z) = u^0(x, y) + \sum_{n=1}^{N} z^n \psi_x^n(x, y), \]  

(2.1)

where \( u^0 \) is the middle surface displacement in the x-direction and \( \psi_x^n \) means
\[ \psi^{(n)}(x, y) = \frac{\partial^n u}{\partial z^n} \bigg|_{z=0}, \quad n = 1, 2, \ldots \] (2.2)

A similar expression can be written for the displacement in the y-coordinate. The quantities \( \psi \) and \( \psi_y \) have a physical significance when \( n = 1 \) as the slope of the deflected surface. A special case of Basset's displacement field is the first-order shear-deformation theory. The first-order SDT assumes the following displacement field

\[
\begin{align*}
    u(x, y, z) &= u^\circ(x, y) + z\psi_x(x, y), \\
    v(x, y, z) &= v^\circ(x, y) + z\psi_y(x, y), \\
    w(x, y, z) &= w^\circ(x, y).
\end{align*}
\] (2.3)

The displacement fields assumed by the first-order SDT and CLT are similar in that they both assume a linear through-the-thickness displacement field. However, in the first-order SDT transverse displacements are allowed to rotate and are not necessarily perpendicular to the middle surface after bending or extension. Figure 3 shows a graphical comparison of the displacement field assumed by the first-order SDT and the displacement field assumed in CLT because of the Kirchoff-Love hypothesis. Figure 3(a) shows an undeformed laminate, and the deformed laminate with the displacement field assumed by CLT is shown in Fig. 3(b). Notice in Fig. 3(b) that the solid line depicting the displacement field is still normal to the middle surface. The solid line shown in Fig. 3(c) is a typical first-order SDT displacement field and is not necessarily normal to the middle surface.

2. Laminated Plate Analysis
Another observation about the first-order SDT is that the shear strains are assumed uniform through the thickness of the plate. In general this assumption is incorrect and for this reason Mindlin [7] introduced a shear correction factor, $k^2$. The shear correction factor is evaluated by comparing solutions with an exact elasticity solution. For the materials most commonly used in laminated composite plates $k^2 = \frac{5}{6}$. This factor is multiplied by the shear moduli $G_{23}$ and $G_{31}$.

In 1969 Yang, Norris, and Stravsky [9] assumed the displacement field of the first-order SDT in order to investigate elastic wave propagation in heterogenous plates and found good agreement with the exact solution. Later Gol'deneizer [34], Schmidt [35], Whitney and Pagano [10], Librescu [13], and Lo, Christensen, and Wu [12] independently included higher order terms in Basset's power series assumption to describe the displacement fields. Reddy and his associates [22] have refined a higher-order theory that is extremely accurate and yet somewhat less cumbersome and computationally less demanding than the three-dimensional elastic theory. However, even this refined higher-order theory is substantially more cumbersome than the first-order SDT. In chapter 3 the nonlinear deflections are introduced. By including the nonlinear deflections it becomes even more computationally cumbersome to include the higher-order shear deformation theories. Therefore, it is assumed in the present work that the first-order theory is sufficiently accurate and the displacement field of Eq. (2.3) is used as the basis for defining the strains. Thus only five degrees of freedom are considered: $u$, $v$, $w$, $\psi_x$, and $\psi_y$. 

2. Laminated Plate Analysis
3. Lamina Stresses, Strains, and Displacements

Stress-strain relations for an orthotropic lamina are reviewed in this chapter. Starting with the generalized Hooke’s law for an elastic solid, the stress-strain relations are reduced to the three-dimensional orthotropic form using contracted notation. The stiffness matrix is transformed so that the on-axis stresses and strains can be found for a lamina with arbitrary orientation of the principal material direction. The stiffness matrix is further reduced to exclude $\sigma_z$ and $\varepsilon_z$. Finally, the nonlinear strain-displacement relations are derived.

3.1. Lamina Stress-Strain Relations

The generalized Hooke’s law for an elastic solid in indicial notation is expressed

$$\sigma_{ij} = C_{ijkl} \varepsilon_{kl},$$

(3.1)

where $\sigma_{ij}$ is the stress tensor, $C_{ijkl}$ is the stiffness tensor, and $\varepsilon_{kl}$ is the strain tensor. It is assumed in this relationship that there are no residual stresses from previous deforma-
mations. With this assumption the generalized Hooke's law is the most general form of a linear relationship between the stress and strain components. In this general form C_{ijkl} represents 81 independent material constants. From equilibrium conditions it can be shown that, if there are no body moments, the σ_{ij} tensor is symmetric (σ_{ij} = σ_{ji}). And by definition of the strain tensor from the Eulerian viewpoint (see Eq. (3.18)), it can be concluded that the strain tensor ε_{kl} is symmetric. Since σ_{ij} and ε_{ij} are symmetric, it follows that C_{ijkl} is symmetric. The independent constants are now reduced to 36 and so contracted notation can be employed to simplify the expression for Hooke's law to

\[ \sigma_i = C_{ij} \varepsilon_j. \] (3.2)

Contracted notation is a simplification of indicial notation that reduces the number of indices by one-half. The indices are replaced as follows:

\[ 11 \rightarrow 1, \quad 22 \rightarrow 2, \quad 33 \rightarrow 3, \]
\[ 23 \rightarrow 4, \quad 31 \rightarrow 5, \quad 12 \rightarrow 6. \]

For example, \( \sigma_{11} \rightarrow \sigma_1, \quad \varepsilon_{23} \rightarrow \varepsilon_4, \quad \text{and} \quad C_{3132} \rightarrow C_{56}. \) The number of independent elastic constants is further reduced to 21 by recognizing that there exists a quadratic strain energy function \( W \) such that

\[ W = \frac{1}{2} C_{ij} \varepsilon_i \varepsilon_j. \] (3.3)

Therefore \( C_{ij} \) is symmetric.

In Chapter 2 the lamina was assumed to be orthotropic. This implies that there are three planes of geometric and material symmetry. By systematically equating terms from symmetric coordinate systems it can be shown that the number of independent material
constants can be reduced to nine and that the generalized Hooke's law in matrix form for an individual lamina can now be written [3]

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6 \\
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\
C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & C_{66} \\
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6 \\
\end{bmatrix}
\]  
(3.4)

By inverting the stiffness matrix \([C_{ij}]\), the compliance matrix \([S_{ij}]\), can be found along with the strain-stress relationship

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6 \\
\end{bmatrix} = \begin{bmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\
S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & S_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & S_{55} & 0 \\
0 & 0 & 0 & 0 & 0 & S_{66} \\
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\sigma_4 \\
\sigma_5 \\
\sigma_6 \\
\end{bmatrix}
\]  
(3.5)

From the strain-stress relationship, the compliance terms can be logically deduced in terms of engineering constants [3]:

3. Lamina Stresses, Strains, and Displacements
where $E_i$ are the Young's moduli in the $i$-direction, $v_{ij}$ are the Poisson's ratios for strain in the $j$ direction resulting from stress in the $i$ direction, and $G_{ij}$ are the shear moduli in the $i$-$j$ plane. Recall from Chapter 2 that the shear moduli $G_{23}$ and $G_{13}$ can now be reduced by the shear correction factor, $k^2$. Although twelve material constants appear in Eqs. (3.6), there are only nine independent constants because

$$
\frac{v_{ij}}{E_i} = \frac{v_{ji}}{E_j}, \quad i = 1, 2, 3, \quad j = 1, 2, 3.
$$

The stress-strain relations of Eqs. (3.4) are true only when the stresses and strains are in the principal material direction of the lamina (direction in which the fibers are aligned). When the various principal axes of the laminae are in different directions it becomes difficult to analyze the laminate. This necessitates a method of transforming
the stress-strain relations from one coordinate system to another. Consider a lamina whose principal material direction is at some angle $\theta$ with respect to the laminate $x$-axis. In tensor notation the stresses in the $x$-$y$ coordinate system $(\sigma_{ij})$, are transformed to the 1-2 coordinate system $(\sigma'_{ij})$, by [36]

$$\sigma'_{ij} = a_{ii} a_{mj} \sigma_{lm},$$

where

$$[a_{ij}] = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}.$$

This yields

$$\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\tau_{23} \\
\tau_{31} \\
\tau_{12}
\end{bmatrix} = \begin{bmatrix}
c^2 & c^2 & 0 & 0 & 0 & 2cs \\
s^2 & s^2 & 0 & 0 & 0 & -2cs \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & c & -s & 0 \\
0 & 0 & 0 & s & c & 0 \\
-cs & cs & 0 & 0 & 0 & c^2 - s^2
\end{bmatrix} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{yx} \\
\tau_{zx} \\
\tau_{xy}
\end{bmatrix},$$

where now

$$c = \cos \theta \text{ and } s = \sin \theta.$$

In this thesis the transformation matrix of Eq. (3.9) is represented by the symbol $[T_1]$. Tensorial strains are transformed in the same way; however, tensorial strains and engi-
neering strains are not equal. Tensorial shear strains are one-half of engineering strains. Thus,

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\gamma_{23} \\
\gamma_{31} \\
\gamma_{12}
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\varepsilon_4 \\
\varepsilon_5 \\
\varepsilon_6
\end{bmatrix},
\tag{3.10}
\]

where the conversion matrix is commonly called the Reuter matrix \([R]\). Engineering strains can now be transformed from the \(x-y\) coordinate system \(\{\varepsilon\}_x\) to the \(1-2\) coordinate system \(\{\varepsilon\}_1\) by

\[
\{\varepsilon\}_1 = [R][T_2][R]^{-1}\{\varepsilon\}_x,
\]

or, carrying out the indicated matrix multiplications,

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\gamma_{23} \\
\gamma_{31} \\
\gamma_{12}
\end{bmatrix} =
\begin{bmatrix}
c^2 & c^2 & 0 & 0 & 0 & cs \\
s^2 & s^2 & 0 & 0 & 0 & -cs \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & c & -c & 0 \\
0 & 0 & 0 & s & s & 0 \\
-2cs & 2cs & 0 & 0 & 0 & c^2 - s^2
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{yz} \\
\gamma_{zx} \\
\gamma_{xy}
\end{bmatrix},
\tag{3.11}
\]

This transformation matrix is represented by the symbol \([T_2]\). Expressing the on-axis stresses and strains \((\{\sigma\}_1\) and \(\{\varepsilon\}_1\)) of Eqs. (3.4) in terms of the off-axis stresses and strains (Eqs. (3.9) and (3.11)) yields

3. Lamina Stresses, Strains, and Displacements
Rearranging Eqs. (3.12) yields

\[
\{\sigma\}_x = [T_1]^{-1}C[T_2]\{\varepsilon\}_x,
\]

or

\[
\{\sigma\}_x = [\bar{C}]\{\varepsilon\}_x,
\]

where

\[
[\bar{C}] = [T_1]^{-1}C[T_2].
\]

Expanding Eqs. (3.13), the off-axis stress-strain relationship for the given lamina is

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{yz} \\
\tau_{zx} \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
\bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & \bar{C}_{16} \\
\bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & \bar{C}_{26} \\
\bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & \bar{C}_{36} \\
0 & 0 & 0 & \bar{C}_{44} & \bar{C}_{45} & 0 \\
0 & 0 & 0 & \bar{C}_{45} & \bar{C}_{55} & 0 \\
\bar{C}_{16} & \bar{C}_{26} & \bar{C}_{36} & 0 & 0 & \bar{C}_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{yz} \\
\gamma_{zx} \\
\gamma_{xy}
\end{bmatrix},
\]

where

\[
\bar{C}_{11} = c^4C_{11} + 2c^2s^2(C_{12} + 2C_{66}) + c^4C_{22},
\]

\[
\bar{C}_{12} = c^2s^2(C_{11} + C_{22} - 4C_{66}) + (s^4 + c^4)C_{12},
\]

\[
\bar{C}_{13} = c^2C_{13} + s^2C_{23}.
\]
\[ \bar{C}_{16} = c^4(C_{11} - C_{12} - 2C_{66}) + s^2(C_{12} - C_{22} + 2C_{66}) \] ,

\[ \bar{C}_{22} = s^4C_{11} + 2c^2s^2(C_{12} + 2C_{66}) + c^4C_{22} \] ,

\[ \bar{C}_{23} = c^2C_{13} + s^2C_{23} \] ,

\[ \bar{C}_{26} = cs[c^2(C_{11} - C_{12} - 2C_{66}) + c^2(C_{12} - C_{22} + 2C_{66})] \] ,

\[ \bar{C}_{33} = C_{33} \] ,

\[ \bar{C}_{36} = cs(C_{13} - C_{23}) \] ,

\[ \bar{C}_{44} = c^2C_{44} + s^2C_{55} \] ,

\[ \bar{C}_{45} = cs(C_{55} - C_{44}) \] ,

\[ \bar{C}_{55} = s^2C_{44} + c^2C_{55} \] ,

and

\[ \bar{C}_{66} = c^2s^2(C_{11} - 2C_{12} + C_{22}) + C_{66}(c^2 - s^2)^2. \]

The transformed stiffness matrix is reduced by recognizing that \( \sigma_z = 0 \) from the plane stress assumption. When zero is substituted for \( \sigma_z \) in Eqs. (3.14) there results

\[ \sigma_z = 0 = \varepsilon_x \bar{C}_{13} + \varepsilon_y \bar{C}_{23} + \varepsilon_z \bar{C}_{33} + \gamma_{yz} \bar{C}_{36}. \] \hspace{1cm} (3.15)

Solving Eq. (3.15) for \( \varepsilon_z \),

\[ \varepsilon_z = -\frac{1}{\bar{C}_{33}} [\varepsilon_x \bar{C}_{13} + \varepsilon_y \bar{C}_{23} + \gamma_{yz} \bar{C}_{36}] . \] \hspace{1cm} (3.16)
By eliminating $\sigma_z$ and substituting Eq. (3.16) for $\varepsilon_z$, the reduced stiffness matrix and stress-strain relationship become

$$
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{xz}
\end{bmatrix} = 
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} & 0 & 0 \\
Q_{12} & Q_{22} & Q_{23} & 0 & 0 \\
Q_{13} & Q_{23} & Q_{33} & 0 & 0 \\
0 & 0 & 0 & Q_{44} & Q_{45} \\
0 & 0 & 0 & Q_{45} & Q_{55}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy} \\
\gamma_{yz} \\
\gamma_{xz}
\end{bmatrix}
$$

(3.17)

where $[Q_{ij}]$ is the transformed reduced stiffness matrix. In terms of the transformed stiffness matrix its elements are

$$
Q_{11} = \bar{C}_{11} - \frac{\bar{C}_{13}^2}{\bar{C}_{33}}, \quad Q_{12} = \bar{C}_{12} - \frac{\bar{C}_{13} \bar{C}_{23}}{\bar{C}_{33}}, \quad Q_{13} = \bar{C}_{13} - \frac{\bar{C}_{13} \bar{C}_{36}}{\bar{C}_{33}},
$$

$$
Q_{22} = \bar{C}_{22} - \frac{\bar{C}_{23}^2}{\bar{C}_{33}}, \quad Q_{23} = \bar{C}_{23} - \frac{\bar{C}_{26} \bar{C}_{26}}{\bar{C}_{33}}, \quad Q_{33} = \bar{C}_{66} - \frac{\bar{C}_{36}^2}{\bar{C}_{33}},
$$

$$
Q_{44} = \bar{C}_{44}, \quad Q_{45} = \bar{C}_{45}, \quad Q_{55} = \bar{C}_{55}.
$$

3.2. Strain-Displacement Relations

In the Eulerian viewpoint the strains are described in indicial notation as [36]

$$
\varepsilon_{ij} = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right]; \quad i = 1,2,3, \quad j = 1,2,3.
$$

(3.18)
where \( u, v, \) and \( w \) are the displacements in the \( x, y, \) and \( z \) directions and are referred to as \( u, v, \) and \( w \), respectively, in the analysis that follows; and the quantities \( x, y, \) and \( z \) refer to the orthogonal coordinate directions redesignated \( x, y, \) and \( z \), respectively, in the analysis that follows. After substituting the first-order SDT displacement field into the Eulerian strain description, the strains become

\[
\varepsilon_x = \frac{\partial u^o}{\partial x} + \frac{1}{2} \left[ (\frac{\partial u^o}{\partial x})^2 + (\frac{\partial v^o}{\partial x})^2 + (\frac{\partial w^o}{\partial x})^2 \right] \\
+ z \left( \frac{\partial \psi_x}{\partial x} + \frac{z^2}{2} \left[ (\frac{\partial \psi_x}{\partial x})^2 + (\frac{\partial \psi_y}{\partial x})^2 \right] \right),
\]

(3.19)

\[
\varepsilon_y = \frac{\partial v^o}{\partial y} + \frac{1}{2} \left[ (\frac{\partial u^o}{\partial y})^2 + (\frac{\partial v^o}{\partial y})^2 + (\frac{\partial w^o}{\partial y})^2 \right] \\
+ z \left( \frac{\partial \psi_y}{\partial y} + \frac{z^2}{2} \left[ (\frac{\partial \psi_x}{\partial y})^2 + (\frac{\partial \psi_y}{\partial y})^2 \right] \right),
\]

(3.20)

\[
\varepsilon_z = \frac{\partial w^o}{\partial z} + \frac{1}{2} \left[ (\frac{\partial u^o}{\partial z})^2 + (\frac{\partial v^o}{\partial z})^2 + (\frac{\partial w^o}{\partial z})^2 \right] + \frac{1}{2} \left[ \psi_x^2 + \psi_y^2 \right],
\]

(3.21)

\[
\varepsilon_{xy} = \frac{1}{2} \left[ \frac{\partial u^o}{\partial y} + \frac{\partial v^o}{\partial x} + \frac{\partial u^o}{\partial x} + \frac{\partial u^o}{\partial y} + \frac{\partial v^o}{\partial y} + \frac{\partial w^o}{\partial x} \frac{\partial w^o}{\partial y} \right] \\
+ z \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} + z^2 \left( \frac{\partial \psi_x}{\partial x} \frac{\partial \psi_y}{\partial y} + \frac{\partial \psi_x}{\partial x} \frac{\partial \psi_y}{\partial y} \right) \right),
\]

(3.22)

\[
\varepsilon_{yz} = \frac{1}{2} \left[ \frac{\partial v^o}{\partial z} + \frac{\partial w^o}{\partial y} + \frac{\partial u^o}{\partial z} + \frac{\partial u^o}{\partial y} \frac{\partial v^o}{\partial z} + \frac{\partial v^o}{\partial z} + \frac{\partial w^o}{\partial z} \frac{\partial w^o}{\partial y} \right] \\
+ \psi_y + z(\psi_x \frac{\partial \psi_x}{\partial y} + \psi_y \frac{\partial \psi_y}{\partial y} ),
\]

(3.23)
Many of the higher-order terms in the strain expressions are negligible in comparison with the first-order terms. Using von Karman’s criterion only the squares and products of \( \partial w/\partial x \) and \( \partial w/\partial y \) are retained. Removing all other higher-order terms and replacing tensorial strains with engineering strains, the strain-displacement relations become

\[
\varepsilon_x = \frac{1}{2} \left[ \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \frac{\partial u}{\partial x} \frac{\partial w}{\partial z} + \frac{\partial v}{\partial x} \frac{\partial w}{\partial z} + \frac{\partial w}{\partial z} \frac{\partial w}{\partial z} \right] + \psi_x + z(\psi_x \frac{\partial \psi_x}{\partial x} + \psi_y \frac{\partial \psi_y}{\partial y}),
\]

(3.24)

For convenience in notation let

\[
\gamma_{xy} = \frac{\partial w}{\partial y} + \psi_x
\]

(3.28)

\[
\gamma_{yx} = \frac{\partial w}{\partial y} + \psi_y,
\]

(3.29)

\[
\gamma_{zx} = \frac{\partial w}{\partial z} + \psi_x
\]

(3.30)
\[ \varepsilon_x^0 = \frac{\partial u^0}{\partial x} + \frac{1}{2} \left( \frac{\partial w^0}{\partial x} \right)^2, \quad \varepsilon_y^0 = \frac{\partial v^0}{\partial y} + \frac{1}{2} \left( \frac{\partial w^0}{\partial y} \right)^2, \quad \varepsilon_z^0 = 0, \]

\[ \gamma_{yz}^0 = \frac{\partial w^0}{\partial y} + \psi_y, \quad \gamma_{zx}^0 = \frac{\partial w^0}{\partial x} + \psi_x, \quad \gamma_{xy}^0 = \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} + \frac{\partial w^0}{\partial x} \frac{\partial w^0}{\partial y}, \]

\[ \kappa_x = \frac{\partial \psi_x}{\partial x}, \quad \kappa_y = \frac{\partial \psi_y}{\partial y}, \quad \text{and} \quad \kappa_{xy} = \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}. \]

The total strains can then be written

\[ \varepsilon_x = \varepsilon_x^0 + z \kappa_x, \quad (3.31) \]

\[ \varepsilon_y = \varepsilon_y^0 + z \kappa_y, \quad (3.32) \]

\[ \varepsilon_z = \varepsilon_z^0, \quad (3.33) \]

\[ \epsilon_{yz} = \epsilon_{yz}^0, \quad (3.34) \]

\[ \epsilon_{zx} = \epsilon_{zx}^0, \quad (3.35) \]

and \[ \epsilon_{xy} = \epsilon_{xy}^0 + z \kappa_{xy}. \quad (3.36) \]
4. Governing Equations for Laminated Plates

In Chapter 3 the stress-strain relationships for an arbitrary lamina, Eq. (3.14), were found. In this chapter those relationships are extended to an arbitrary laminate resulting in force-strain relationships. Thermal forces are also included in the force-strain relationship. Finally, the equations of motion are expanded into five equilibrium equations which fully describe the five degrees of freedom of the laminate: \( u, v, w, \psi_x, \) and \( \psi_y. \)

4.1. Laminate Forces and Moments

The resultant forces and moments per unit width acting on a laminate can be found by integrating the laminae stress-strain relationship of Eq. (3.14) over the laminate thickness, yielding

\[
(N_x, N_y) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y)dz ,
\]

(4.1)
\[(N_{xy}, N_{yz}, N_{xz}) = \int_{-h/2}^{h/2} (\tau_{xy}, \tau_{yz}, \tau_{xz}) dz , \quad (4.2)\]

and

\[(M_x, M_y, M_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy}) z \, dz , \quad (4.3)\]

where \(\tau_{xy} = \tau_{yx}, \quad \tau_{xz} = \tau_{zx}, \quad \tau_{yz} = \tau_{zy}, \quad N_{xy} = N_{yx}, \quad \text{and} \quad M_{xy} = M_{yx}.\) These forces and moments are labeled on the plate in Fig. 4. Figure 5 depicts a laminate of \(N\) laminae and gives the lamina numbering system that is used in the analysis. The z-axis is orthogonal to the midplane, positive being up. The laminae are numbered sequentially from 1 to \(N\) starting with the top lamina. The laminae interfaces are numbered corresponding to lamina directly below the interface with the laminate top surface being numbered 1 and the laminate bottom surface numbered \(N+1.\) The z-coordinate of the midplane of the \(k^{th}\) lamina is denoted by \(Z_k.\) Equations (4.1), (4.2) and (4.3) can now be rewritten using the notation of Fig. 5:

\[\left(N_x, N_y \right) = \sum_{k=1}^{N} (\sigma_x, \sigma_y) dz , \quad (4.4)\]

\[\left(N_{xy}, N_{yz}, N_{xz} \right) = \sum_{k=1}^{N} (\tau_{xy}, \tau_{yz}, \tau_{xz}) dz , \quad (4.5)\]

and

4. Governing Equations for Laminated Plates
\[(M_x, M_y, M_{xy}) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} (\sigma_x, \sigma_y, \tau_{xy}) \cdot dz . \] \hspace{1cm} (4.6)\]

The off-axis stresses can now be replaced with the equivalent stiffness matrix and middle surface strains, Eqs. (3.31) through (3.36). Neither the middle surface strains nor the stiffness matrix are dependent on \( z \) so they can be brought outside the integral. After these changes are made Eqs. (4.4), (4.5), and (4.6) can now be written in matrix form as

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
N_{yz} \\
N_{xz} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} & A_{16} & 0 & 0 & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & 0 & 0 & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & 0 & 0 & B_{16} & B_{26} & B_{36} \\
0 & 0 & 0 & A_{44} & A_{45} & 0 & 0 & 0 \\
0 & 0 & 0 & A_{45} & A_{55} & 0 & 0 & 0 \\
B_{11} & B_{12} & B_{16} & 0 & 0 & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & 0 & 0 & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & 0 & 0 & D_{16} & D_{26} & D_{36}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x^o \\
\varepsilon_y^o \\
\gamma_{xy}^o \\
\gamma_{yz}^o \\
\gamma_{xz}^o \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}, \hspace{1cm} (4.7)
\]

where

\[
A_{ij} = \sum_{k=1}^{N} Q_{ij}(z_k - z_{k+1}), \hspace{1cm} (4.8)
\]

\[
B_{ij} = \frac{1}{2} \sum_{k=1}^{N} Q_{ij}(z_k^2 - z_{k+1}^2), \hspace{1cm} (4.9)
\]

and

4. Governing Equations for Laminated Plates
4.2. Thermal Forces and Moments

The three-dimensional thermal strains are induced in the laminate when the temperature of the laminate changes from its stress-free state. The stress-free state occurs at a temperature very close to that at which the composite plate is cured. If the cure temperature and coefficients of thermal expansion are known, the thermally-induced strains can be found from

$$\epsilon_i = \alpha_i \Delta T, \quad i = 1, 2, 3,$$

where $\alpha_i$ are the coefficients of thermal expansion in the $i^{th}$ material direction and $\Delta T$ is the change in temperature from the stress-free state to the present temperature.

The lamina thermally-induced stresses can now be found in the principal material directions from Hooke's law as [3]

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}_T = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \Delta T \end{bmatrix}. \quad (4.12)$$
As before, we are more concerned with stresses and strains on the laminate coordinate axes than along the principal material directions. Therefore, the coefficients of thermal expansion must be transformed to the on-axis directions by $[T_2]^{-1}$:

$$
\begin{bmatrix}
\alpha_x \\
\alpha_y \\
\alpha_z \\
\alpha_{xy}
\end{bmatrix} =
\begin{bmatrix}
c^2 & s^2 & 0 & 0 & 0 & -cs \\
s^2 & c^2 & 0 & 0 & 0 & cs \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & c & s & 0 \\
2cs & -2cs & 0 & 0 & 0 & c^2 - s^2
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\alpha_2 \\
\alpha_3 \\
0
\end{bmatrix},
$$

or, written out,

$$\alpha_x = c^2 \alpha_1 + s^2 \alpha_2, \quad (4.14)$$

$$\alpha_y = s^2 \alpha_1 + c^2 \alpha_2, \quad (4.15)$$

$$\alpha_z = \alpha_3, \quad (4.16)$$

and

$$\alpha_{xy} = 2cs (\alpha_1 - \alpha_2). \quad (4.17)$$

Upon substituting the on-axis thermally-induced strains into Eq. (4.12), the corresponding stresses can be expressed.
As was done in Section 3.1, $\sigma_z$ is assumed to be zero and therefore, can be eliminated from Eqs. (4.18). Now, $\alpha_z$ can be rewritten in terms of $\alpha_x$, $\alpha_y$, and $\alpha_{xy}$ as

$$\alpha_z = -\frac{1}{C_{33}} [\alpha_x C_{13} + \alpha_y C_{23} + \alpha_{xy} C_{36}].$$

Substituting Eq. (4.19) into Eq. (4.18) yields

$$\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{xz}
\end{bmatrix}^T =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} & 0 & 0 \\
Q_{12} & Q_{22} & Q_{23} & 0 & 0 \\
Q_{13} & Q_{23} & Q_{33} & 0 & 0 \\
0 & 0 & 0 & Q_{44} & Q_{45} \\
0 & 0 & 0 & Q_{45} & Q_{55}
\end{bmatrix}
\begin{bmatrix}
\alpha_x \\
\alpha_y \\
\alpha_{xy} \\
\Delta T \\
\Delta T
\end{bmatrix},$$

where $[Q]$ is the same stiffness matrix as in Eq. (3.17). At this point the strain caused by mechanical loads can be added to the thermally-induced strains to give the total strain. The purely mechanically induced stresses can now be expressed as
\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy} \\
\tau_{yz} \\
\tau_{xz}
\end{bmatrix}_M = 
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} & 0 & 0 \\
Q_{12} & Q_{22} & Q_{23} & 0 & 0 \\
Q_{13} & Q_{23} & Q_{33} & 0 & 0 \\
0 & 0 & 0 & Q_{44} & Q_{45} \\
0 & 0 & 0 & Q_{45} & Q_{55}
\end{bmatrix} \begin{bmatrix}
\bar{\varepsilon}_x - \alpha_x \Delta T \\
\bar{\varepsilon}_y - \alpha_y \Delta T \\
\bar{\gamma}_{xy} - \alpha_{xy} \Delta T \\
\bar{\gamma}_{yz} \\
\bar{\gamma}_{xz}
\end{bmatrix}, \tag{4.21}
\]

where \(\bar{\varepsilon}_x, \bar{\varepsilon}_y, \bar{\gamma}_{xy}, \bar{\gamma}_{xz}\), and \(\bar{\gamma}_{xz}\) now represent the total strains, mechanical plus free thermal.

With the use of the expressions developed at the beginning of this chapter, Eqs. (4.4) through (4.10), Eqs. (4.21) can be integrated over the laminate thickness to yield an expression for the laminate mechanical forces and moments:

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
N_{yz} \\
N_{xz} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix}_T = 
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & 0 & 0 & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & 0 & 0 & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & 0 & 0 & B_{16} & B_{26} & B_{36} \\
0 & 0 & 0 & A_{44} & A_{45} & 0 & 0 & 0 \\
0 & 0 & 0 & A_{45} & A_{55} & 0 & 0 & 0 \\
B_{11} & B_{12} & B_{16} & 0 & 0 & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & 0 & 0 & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & 0 & 0 & D_{16} & D_{26} & D_{36}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x^o \\
\varepsilon_y^o \\
\gamma_{xy}^o \\
\gamma_{yz}^o \\
\gamma_{xz}^o \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}, \tag{4.22}
\]

where the superscript \(T\) on the right-hand side indicates thermal forces and moments. The total forces and moments can be expressed by adding the thermal forces and moments to both sides of Eqs. (4.22) to give

4. Governing Equations for Laminated Plates 30
\[
\begin{bmatrix}
\bar{N}_x \\
\bar{N}_y \\
\bar{N}_{xy} \\
\bar{N}_{yz} \\
\bar{N}_{xz} \\
\bar{M}_x \\
\bar{M}_y \\
\bar{M}_{xy} \\
\end{bmatrix} = \begin{bmatrix}
A_{11} & A_{12} & A_{16} & 0 & 0 & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & 0 & 0 & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & 0 & 0 & B_{16} & B_{26} & B_{36} \\
0 & 0 & 0 & A_{44} & A_{45} & 0 & 0 & 0 \\
0 & 0 & 0 & A_{45} & A_{55} & 0 & 0 & 0 \\
B_{11} & B_{12} & B_{16} & 0 & 0 & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & 0 & 0 & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & 0 & 0 & D_{16} & D_{26} & D_{36}
\end{bmatrix} \begin{bmatrix}
\bar{\varepsilon}_x^o \\
\bar{\varepsilon}_y^o \\
\bar{\gamma}_{xy}^o \\
\bar{\gamma}_{yz}^o \\
\bar{\gamma}_{xz}^o \\
\bar{\kappa}_x \\
\bar{\kappa}_y \\
\bar{\kappa}_{xy}
\end{bmatrix},
\tag{4.23}
\]

where

\[
\bar{N}_x = N_x^T + N_x, \quad \bar{N}_y = N_y^T + N_y, \\
\bar{N}_{xy} = N_{xy}^T + N_{xy}, \quad \bar{N}_{yz} = N_{yz}, \\
\bar{N}_{xz} = N_{xz}, \quad \bar{M}_x = M_x^T + M_x, \\
\bar{M}_y = M_y, \text{ and } \bar{M}_{xy} = M_{xy}^T + M_{xy}.
\]

In the paragraphs which follow, expressions for the thermally induced forces and moments \((N_x^T, N_y^T, N_{xy}^T, M_x^T, M_y^T, \text{ and } M_{xy}^T)\) are derived on the basis of a piecewise-linear temperature distribution in the z-direction. A typical graphite lamina thickness is on the order of 0.15 mm (.006 in.) and so it seems reasonable to assume that the temperature distribution through the lamina thickness is linear and, therefore, piecewise linear through the laminate thickness. The temperature field can then be described as

\[
T(x,y,z) = T_k(x,y) + \frac{1}{t_k} \left( T_{k+1}(x,y) - T_k(x,y) \right) (z_k - z),
\tag{4.24}
\]

4. Governing Equations for Laminated Plates
where \( T_k \) is the temperature at the lamina interface above the \( k \)th lamina, \( T_{k+1} \) is the temperature at the lamina interface above the \( k+1 \) lamina, \( z_t \) is the z-coordinate of the lamina interface above the \( k \)th lamina, and \( t_e \) is the thickness of the \( k \)th lamina. If \( T_o \) refers to the stress-free temperature then

\[
\Delta T = T(x,y,z) - T_o .
\]  

(4.25)

The thermally-induced forces can now be found in the same manner as the mechanical forces were found in Chapter 3. The thermal forces and moments are therefore written

\[
\begin{bmatrix}
N_x^T \\
N_y^T \\
N_{xy}^T
\end{bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{12} & Q_{22} & Q_{23} \\
Q_{13} & Q_{23} & Q_{33}
\end{bmatrix} \begin{bmatrix}
\alpha_x \\
\alpha_y \\
\alpha_{xy}
\end{bmatrix} \Delta T \, dz
\]  

(4.26)

and

\[
\begin{bmatrix}
M_x^T \\
M_y^T \\
M_{xy}^T
\end{bmatrix} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{12} & Q_{22} & Q_{23} \\
Q_{13} & Q_{23} & Q_{33}
\end{bmatrix} \begin{bmatrix}
\alpha_x \\
\alpha_y \\
\alpha_{xy}
\end{bmatrix} \Delta T \, z \, dz .
\]  

(4.27)

The stiffness matrix \([Q_{ij}]\) and the coefficients of thermal expansion \(\alpha_x\), \(\alpha_y\), and \(\alpha_{xy}\) do not depend on \(z\) and can be taken outside of the integral. The terms remaining in the integrand are \(z\) and \(\Delta T\). Replacing \(\Delta T\) with Eqs. (4.24) through (4.25) and integrating over the lamina thickness leads to

\[
\int_{z_k}^{z_{k+1}} \Delta T \, dz = t_k \left[ (T_k + T_{k+1})/2 - T_o \right] .
\]  

(4.28)
and

\[
\int_{z_k}^{z_{k+1}} \Delta T \, dz = t_k \left[ T_k \left( \frac{z_k}{2} + \frac{t_k}{12} \right) + T_{k+1} \left( \frac{z_k}{2} - \frac{t_k}{12} \right) - T_0 z_k \right],
\]

(4.29)

where \( z_k \) is the \( z \)-coordinate of the middle of the \( k \)th lamina. The thermal forces and moments can now be written as

\[
\begin{bmatrix}
N_{x}^T \\
N_{y}^T \\
N_{xy}^T
\end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{12} & Q_{22} & Q_{23} \\
Q_{13} & Q_{23} & Q_{33}
\end{bmatrix} \begin{bmatrix}
\alpha_x \\
\alpha_y \\
\alpha_{xy}
\end{bmatrix}_k
t_k \left[ (T_k + T_{k+1})/2 - T_0 \right]
\]

(4.30)

and

\[
\begin{bmatrix}
M_{x}^T \\
M_{y}^T \\
M_{xy}^T
\end{bmatrix} = \sum_{k=1}^{N} \begin{bmatrix}
Q_{11} & Q_{12} & Q_{13} \\
Q_{12} & Q_{22} & Q_{23} \\
Q_{13} & Q_{23} & Q_{33}
\end{bmatrix} \begin{bmatrix}
\alpha_x \\
\alpha_y \\
\alpha_{xy}
\end{bmatrix}_k
t_k [T_k \left( \frac{z_k}{2} + \frac{t_k}{12} \right) + T_{k+1} \left( \frac{z_k}{2} - \frac{t_k}{12} \right) - T_0 z_k]
\]

(4.31)

4.3. Equations of Motion

Up to this point the strains have been defined in terms of the five degrees of freedom: \( u, v, w, \psi_x, \) and \( \psi_y \). The stresses have been found from Hooke’s law in terms of the strains, and the laminate forces and moments have been defined in terms of the mechanical and thermal strains. In order to completely describe the dynamic behavior of
the three-dimensional laminate, the equations of equilibrium must also be satisfied. Consider the infinitesimal cube with dimensions $dx$, $dy$, and $dz$ shown in Fig. 6. All of the nonzero stresses considered in this thesis are assumed to be acting on the cube. The net force in the $x$- and $y$-directions can be written

$$ F_x = \frac{\partial \sigma_x}{\partial x} \ dxdydz + \frac{\partial \tau_{xy}}{\partial y} \ dydxdz \quad (4.32) $$

and

$$ F_y = \frac{\partial \sigma_y}{\partial y} \ dydxdz + \frac{\partial \tau_{xy}}{\partial x} \ dxdydz. \quad (4.33) $$

Because of the plate geometry, the net force in the $z$-direction also contains in-plane stress terms which cause nonzero $z$-directed forces when the plate becomes nonplanar [37]. The net force in the $z$-direction is therefore

$$ F_z = \frac{\partial \tau_{yz}}{\partial y} \ dydxdz + \frac{\partial \tau_{xz}}{\partial x} \ dxdydz + \frac{\partial}{\partial x} (\sigma_x \frac{\partial w}{\partial x} + \tau_{xy} \frac{\partial w}{\partial y}) dxdydz + \frac{\partial}{\partial y} (\sigma_y \frac{\partial w}{\partial y} + \tau_{xy} \frac{\partial w}{\partial x}) dxdydz. \quad (4.34) $$

The equations of equilibrium can now be found by substituting the net forces into Newton’s second law of motion. By assuming that only symmetric laminates will be analyzed we can ignore rotary inertia terms and therefore the acceleration terms in the three directions are

$$ \ddot{u} + z\ddot{\psi}_x. $$
\( \ddot{v} + z \ddot{\psi}_y, \)

and

\( \ddot{w}, \)

and the mass is denoted by \( \rho dxdydz \), where \( \rho \) is the mass density. Newton's second law then becomes

\[
\rho (\dddot{u} + z \dddot{\psi}_x) dxdydz = \left[ \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} \right] dxdydz,
\]

\[
\rho (\dddot{v} + z \dddot{\psi}_y) dxdydz = \left[ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} \right] dxdydz,
\]

and

\[
\rho \dddot{w} dxdydz = \left[ \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial}{\partial x} \left( \sigma_x \frac{\partial w}{\partial x} + \tau_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left( \tau_{xy} \frac{\partial w}{\partial x} + \sigma_y \frac{\partial w}{\partial y} \right) \right] dxdydz.
\]

Similarly, moment equilibrium equations can also be derived. The moment equilibrium equations based on the net moments about the x- and y-axes are

\[
\rho z (\ddot{v} + z \ddot{\psi}_y) dxdydz = (z \frac{\partial \tau_{xy}}{\partial x} + z \frac{\partial \sigma_y}{\partial y} - \tau_{yz}) dxdydz \quad (4.35)
\]

and

\[
\rho z (\ddot{u} + z \ddot{\psi}_x) dxdydz = (z \frac{\partial \sigma_x}{\partial x} + z \frac{\partial \tau_{xy}}{\partial y} - \tau_{xz}) dxdydz. \quad (4.36)
\]
Equations (4.32) through (4.36) can now be divided through by \( \text{d}x \text{d}y \text{d}z \). Recalling Eqs. (4.4) through (4.6), the equilibrium equations can be integrated through the laminate thickness yielding

\[
\rho h \frac{\partial^2 u}{\partial t^2} = \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y}, \quad (4.37)
\]

\[
\rho h \frac{\partial^2 v}{\partial t^2} = \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y}, \quad (4.38)
\]

\[
\rho h \frac{\partial^2 w}{\partial t^2} = \frac{\partial N_{xz}}{\partial x} + \frac{\partial N_{yz}}{\partial y} + \frac{\partial}{\partial x} (N_x \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y}) + \frac{\partial}{\partial y} (N_{xy} \frac{\partial w}{\partial x} + N_y \frac{\partial w}{\partial y}), \quad (4.39)
\]

\[
\frac{\rho h^3}{12} \frac{\partial^2 \psi_x}{\partial t^2} = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - N_{xz}, \quad (4.40)
\]

and

\[
\frac{\rho h^3}{12} \frac{\partial^2 \psi_y}{\partial t^2} = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - N_{yz}. \quad (4.41)
\]
5. Finite Element Model

The salient features of the finite element model are explained in this chapter. First, the equations of dynamic equilibrium are expanded and recast into a variational formulation. Interpolation functions are derived and the equilibrium equations are rewritten in matrix form. The Newmark method is employed to integrate the equations with respect to time. The nonlinearity of the dynamic equilibrium equations is treated using the Newton-Raphson method. Gauss-Legendre quadrature is used to evaluate the integrals. Finally, reduced integration is explained with respect to the shear terms of the dynamic equilibrium equations.

5.1. Variational Formulation

The equations of dynamic equilibrium, Eqs. (4.37) through (4.41) are more easily treated under the following substitutions:

\[ N_x = \bar{N}_x - N_x^T, \quad M_x = \bar{M}_x - M_x^T, \]
When the total forces and moments are expressed in terms of the corresponding strains and stiffness's, Eqs. (4.21), the equations of dynamic equilibrium, Eqs. (4.37) through (4.41), become

\[
\rho \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left[ A_{11} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + A_{12} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right] \\
+ A_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + B_{11} \frac{\partial \psi_x}{\partial x} + B_{12} \frac{\partial \psi_y}{\partial y} + B_{16} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \\
- \frac{\partial}{\partial y} \left[ A_{16} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + A_{26} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) + A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \right) + B_{16} \frac{\partial \psi_x}{\partial x} + B_{26} \frac{\partial \psi_y}{\partial y} + B_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] = - \frac{\partial N_x^T}{\partial x} - \frac{\partial N_{xy}^T}{\partial y}, \quad (5.1)
\]

\[
\rho \frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x} \left[ A_{16} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + A_{26} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right] \\
+ A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + B_{16} \frac{\partial \psi_x}{\partial x} + B_{26} \frac{\partial \psi_y}{\partial y} + B_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \\
- \frac{\partial}{\partial y} \left[ A_{12} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + A_{22} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) + A_{26} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \right) + B_{12} \frac{\partial \psi_x}{\partial x} + B_{22} \frac{\partial \psi_y}{\partial y} + B_{26} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] = - \frac{\partial N_y^T}{\partial x} - \frac{\partial N_{yx}^T}{\partial y}, \quad (5.2)
\]
\[
\frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + B_{12} \frac{\partial \psi_x}{\partial x} + B_{22} \frac{\partial \psi_y}{\partial y} + B_{26} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) = -\frac{\partial N_{xy}^T}{\partial x} - \frac{\partial N_y^T}{\partial y}, \tag{5.2}
\]

\[
\rho \frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x} \left[ \frac{\partial w}{\partial x} \left[ A_{11} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + A_{12} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right]
\right.
\]

\[
+ A_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + B_{11} \frac{\partial \psi_x}{\partial x} + B_{12} \frac{\partial \psi_y}{\partial y} + B_{16} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right]
\]

\[
+ \frac{\partial w}{\partial y} \left[ A_{16} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + A_{25} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right]
\]

\[
+ A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + B_{15} \frac{\partial \psi_x}{\partial x} + B_{26} \frac{\partial \psi_x}{\partial y} + B_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right]
\]

\[
- \frac{\partial}{\partial y} \left[ \frac{\partial w}{\partial x} \left[ A_{16} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + A_{26} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right]
\right.
\]

\[
+ A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + B_{16} \frac{\partial \psi_x}{\partial x} + B_{26} \frac{\partial \psi_x}{\partial y} + B_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right]
\]

\[
+ \frac{\partial w}{\partial y} \left[ A_{12} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + A_{22} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right]
\]

\[
+ A_{26} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + B_{12} \frac{\partial \psi_x}{\partial x} + B_{22} \frac{\partial \psi_x}{\partial y} + B_{26} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right]
\]

5. Finite Element Model
\[- \frac{\partial}{\partial x} \left[ A_{55} \left( \psi_x + \frac{\partial w}{\partial x} \right) + A_{45} \left( \psi_y + \frac{\partial w}{\partial y} \right) \right] - \frac{\partial}{\partial y} \left[ A_{45} \left( \psi_x + \frac{\partial w}{\partial x} \right) + A_{44} \left( \psi_y + \frac{\partial w}{\partial y} \right) \right] = p + \frac{\partial}{\partial x} \left[ N_{x}^T \frac{\partial w}{\partial x} + N_{y}^T \frac{\partial w}{\partial y} \right] + \frac{\partial}{\partial y} \left[ N_{x}^T \frac{\partial w}{\partial x} + N_{y}^T \frac{\partial w}{\partial y} \right], \quad (5.3)\]

\[
\frac{\rho h^3}{12} \frac{\partial^2 \psi_x}{\partial t^2} - \frac{\partial}{\partial x} \left[ B_{11} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + B_{12} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right] + B_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + D_{11} \frac{\partial \psi_x}{\partial x} + D_{12} \frac{\partial \psi_y}{\partial y} + D_{16} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \]

\[- \frac{\partial}{\partial y} \left[ B_{16} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + B_{26} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right] + B_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + D_{16} \frac{\partial \psi_x}{\partial x} + D_{26} \frac{\partial \psi_y}{\partial y} + D_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \]

\[+ A_{55} \left[ \psi_x + \frac{\partial w}{\partial x} \right] + A_{45} \left[ \psi_y + \frac{\partial w}{\partial y} \right] = - \frac{\partial M_{x}^T}{\partial x} - \frac{\partial M_{xy}^T}{\partial y}, \quad (5.4)\]

and

\[
\frac{\rho h^3}{12} \frac{\partial^2 \psi_y}{\partial t^2} - \frac{\partial}{\partial x} \left[ B_{16} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + B_{26} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right] + B_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + D_{16} \frac{\partial \psi_x}{\partial x} + D_{26} \frac{\partial \psi_y}{\partial y} + D_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \]

5. Finite Element Model
Variational formulation in the context of this thesis means the recasting of differential equations, Eqs. (5.1) through (5.5), into an equivalent integral form by distributing the differentiation between a test, \( \lambda \), and the dependent variables. The reason for this manipulation is to minimize the order of differentiation of the dependent variables and to specify the secondary variables of the problem. For example, Eq. (5.1) is first multiplied by a test function, \( \lambda \), and then integrated over the element area, \( \Omega \), yielding

\[
\int_{\Omega} \left[ \rho \frac{\partial^2 u}{\partial t^2} - \lambda \frac{\partial}{\partial x} \left[ A_{11} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + A_{12} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right] \\
+ A_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + B_{11} \frac{\partial \psi_x}{\partial x} + B_{12} \frac{\partial \psi_y}{\partial y} + B_{16} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] \\
- \lambda \frac{\partial}{\partial y} \left[ A_{16} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + A_{26} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) \right] \\
+ A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + B_{10} \frac{\partial \psi_x}{\partial x} + \right]
\]
\[ B_{36} \frac{\partial \psi_y}{\partial y} + B_{66} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \] 
\[ + \frac{\partial N_x^T}{\partial x} + \frac{\partial N_{xy}^T}{\partial y} \] 
\[ \int_{\Omega} dx dy = 0. \]

Wherever there are second derivatives in Eq. (5.6) the differentiation can be equally distributed between $\lambda$ and the dependent variable by noting that for any two differentiable functions $f$ and $g$

\[ \int_{\Omega} g \frac{\partial f}{\partial x} dx dy = - \int_{\Omega} f \frac{\partial g}{\partial x} dx dy + \int_{\Omega} \frac{\partial}{\partial x} (fg) dx dy \]

and

\[ \int_{\Omega} g \frac{\partial f}{\partial y} dx dy = - \int_{\Omega} f \frac{\partial g}{\partial y} dx dy + \int_{\Omega} \frac{\partial}{\partial y} (fg) dx dy. \]

But by the divergence theorem

\[ \int_{\Omega} \frac{\partial}{\partial x} (fg) dx dy = \int_{\Gamma} (fg) n_x ds \]

and

\[ \int_{\Omega} \frac{\partial}{\partial y} (fg) dx dy = \int_{\Gamma} (fg) n_y ds. \]

Therefore,
\[ \int_{\Omega} g \frac{\partial f}{\partial x} \, dxdy = - \int_{\Omega} f \frac{\partial g}{\partial x} \, dxdy + \int_{\Gamma} (fg)n_{x} \, ds \]

and

\[ \int_{\Omega} g \frac{\partial f}{\partial y} \, dxdy = - \int_{\Omega} f \frac{\partial g}{\partial y} \, dxdy + \int_{\Gamma} (fg)n_{y} \, ds , \]

where \( n_{x} \) and \( n_{y} \) are the \( x \) and \( y \) components, respectively, of the unit normal to the element boundary, \( \Gamma \), and \( ds \) is an infinitesimal arc length along the boundary. With these considerations Eq. (5.6) becomes

\[ \int_{\Omega} \left[ \lambda \rho h \frac{\partial^{2} u}{\partial t^{2}} + \frac{\partial \lambda}{\partial x} \left[ A_{11} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^{2} \right) + A_{12} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^{2} \right) \right] 
+ A_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + B_{11} \frac{\partial \psi_{x}}{\partial x} + B_{12} \frac{\partial \psi_{y}}{\partial y} + B_{16} \left( \frac{\partial \psi_{x}}{\partial y} + \frac{\partial \psi_{y}}{\partial x} \right) \right] 
+ \frac{\partial \lambda}{\partial y} \left[ A_{16} \left( \frac{\partial u}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^{2} \right) + A_{26} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^{2} \right) + 
+ A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + B_{16} \frac{\partial \psi_{x}}{\partial x} + B_{26} \frac{\partial \psi_{y}}{\partial y} + B_{66} \left( \frac{\partial \psi_{x}}{\partial y} + \frac{\partial \psi_{y}}{\partial x} \right) \right] \right] \, dxdy = 

\[ \int_{\Gamma} \lambda \left[ A_{11} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^{2} \right) + A_{12} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^{2} \right) \right] \]

(5.7)
The primary variables are defined as the dependent variables; \( u, v, w, \psi_x, \) and \( \psi_y \). The secondary variables are defined from the variational form as the coefficients of \( \lambda \) in the boundary integrals. In Eq. (5.7) the secondary variables are

\[
\left[ A_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + B_{16} \frac{\partial \psi_x}{\partial x} + B_{12} \frac{\partial \psi_y}{\partial y} + B_{16} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] n_x + \\
\lambda \left[ A_{16} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + A_{26} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) + A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right] n_y \right] ds + \int_\Omega \left[ N_x^T \frac{\partial \lambda}{\partial x} + N_{xy}^T \frac{\partial \lambda}{\partial y} \right] dx dy.
\]

The primary variables are defined as the dependent variables; \( u, v, w, \psi_x, \) and \( \psi_y \). The secondary variables are defined from the variational form as the coefficients of \( \lambda \) in the boundary integrals. In Eq. (5.7) the secondary variables are

\[
\left[ A_{11} \left( \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \right) + A_{12} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) + A_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right] n_x + \\
+ B_{11} \frac{\partial \psi_x}{\partial x} + B_{12} \frac{\partial \psi_y}{\partial y} + B_{16} \left( \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x} \right) \right] n_x
\]

and

\[
\left[ A_{16} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) + A_{26} \left( \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2 \right) + A_{66} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right) \right] n_y,
\]

which are recognized from Eqs. (4.7) as \( (N_x + N_{xy}) n_x \) and \( (N_{xy} + N_y) n_y \). The variational form of Eq. (5.1) can finally be written as.
The variational forms of Eqs. (5.2) through (5.5) can be derived in a similar manner.

5.2. Interpolation and Approximate Functions

In the finite element method the region being examined is discretized into small finite areas called elements. The element boundaries are established by discrete points called nodes. Values of the dependent variables \(u, v, w, \psi_x, \psi_y\) are directly obtained only at the nodes. The values between the nodes are found by interpolation. In the Ritz method [37] the dependent variables for a single element are approximated by
\[ u = \sum_{i=1}^{P} u^{(i)} \phi_i, \quad (5.11) \]

\[ v = \sum_{i=1}^{P} v^{(i)} \phi_i, \quad (5.12) \]

\[ w = \sum_{i=1}^{P} w^{(i)} \phi_i, \quad (5.12) \]

\[ \psi_x = \sum_{i=1}^{P} \psi_x^{(i)} \phi_i, \quad (5.14) \]

\[ \psi_y = \sum_{i=1}^{P} \psi_y^{(i)} \phi_i, \quad (5.15) \]

where \( u^{(i)}, v^{(i)}, w^{(i)}, \psi_x^{(i)}, \) and \( \psi_y^{(i)} \) are the nodal values at the point \((x_i, y_i)\), the \( \phi_i \) are the interpolation functions, and the symbol \( P \) represents the number of nodes the element contains. Equations (5.11) through (5.15) are referred to as approximate functions. The interpolation functions of Eqs. (5.11) through (5.15) are not unique but must obey to the following criteria:
1. The interpolation functions must be differentiable to the same order as the dependent variables.

2. Any set of interpolation functions \{\phi_i\} must be linearly independent.

3. The set of interpolation functions must be complete; i.e., if the interpolation functions contain a term of the \(n\)th degree, they must also contain terms of the \(n - 1\)st, \(n - 2\)nd, \ldots, and \(0\)th degree.

4. Interpolation functions are nonzero only on the element to which they are assigned.

5. An interpolation function \(\phi_i\) is equal to unity at the coordinate \(x_i, y_i\) and equal to zero at any other nodal location.

6. At any point \(x_i, y_i\) on an element the value of the interpolation functions at that point must sum to unity.

If these criteria are satisfied the element is said to be conforming which means that:

1. There are no interelement gaps or overlaps and no sudden changes in slope between elements.

2. Rigid body modes are present; i.e. when the body is in rigid motion the element must exhibit zero strain.

3. The elemental interpolation functions are complete.
Figure 7 shows three master rectangular elements in local Cartesian coordinates which are all conforming elements. The following three sets of interpolation functions correspond to these three elements.

**Linear Rectangular Element (4 Nodes)**

\[ \phi_1 = \frac{1}{4} (1 - \zeta)(1 - \eta) , \]
\[ \phi_2 = \frac{1}{4} (1 + \zeta)(1 - \eta) , \]
\[ \phi_3 = \frac{1}{4} (1 - \zeta)(1 + \eta) , \]
and
\[ \phi_4 = \frac{1}{4} (1 + \zeta)(1 + \eta) . \]

**Quadratic Rectangular Element (8 Nodes)**

\[ \phi_1 = \frac{1}{4} (\zeta^2 - \zeta)(\eta^2 - \eta) , \]
\[ \phi_2 = \frac{1}{4} (\zeta^2 + \zeta)(\eta^2 - \eta) , \]
\[ \phi_3 = \frac{1}{4} (\zeta^2 + \zeta)(\eta^2 + \eta) , \]
\[ \phi_4 = \frac{1}{4} (\zeta^2 - \zeta)(\eta^2 + \eta) , \]
\[ \phi_5 = \frac{1}{2} (1 - \zeta^2)(\eta^2 - \eta) . \]
\[ \phi_6 = \frac{1}{2} (\zeta + \zeta^2)(1 - \eta^2) , \]

\[ \phi_7 = \frac{1}{2} (1 - \zeta^2)(\eta^2 + \eta) , \]

and

\[ \phi_8 = \frac{1}{2} (\zeta^2 - \zeta)(1 - \eta^2) . \]

Quadratic Rectangular Element (9 Nodes)

These are the same as the interpolation functions for the 8-noded element with the addition of

\[ \phi_9 = (1 - \zeta^2)(1 - \eta^2) . \]

The interpolation functions therefore depend on the element type. The computer code COMMEC2 contained in the appendix is capable of utilizing all of these elements however only results of the analysis using the 9-noded quadratic rectangular element are reported in this thesis.

The test function \( \lambda \) must be capable of satisfying the boundary conditions but other than that it is arbitrary. It is convenient and usual to let the test function assume the same form as the interpolation functions. After this substitution is made along with the substitution of the approximate functions, the variational formulation can be written in matrix form as
\[
\begin{bmatrix}
M_{ij}^{11} & 0 & 0 & 0 & 0 \\
0 & M_{ij}^{22} & 0 & 0 & 0 \\
0 & 0 & M_{ij}^{33} & 0 & 0 \\
0 & 0 & 0 & M_{ij}^{44} & 0 \\
0 & 0 & 0 & 0 & M_{ij}^{55}
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_i \\
\ddot{v}_i \\
\ddot{w}_i \\
\ddot{\psi}_{xi} \\
\ddot{\psi}_{yi}
\end{bmatrix}
\]

\[
\begin{bmatrix}
K_{ij}^{11} & K_{ij}^{12} & K_{ij}^{13} & K_{ij}^{14} & K_{ij}^{15} \\
K_{ij}^{21} & K_{ij}^{22} & K_{ij}^{23} & K_{ij}^{24} & K_{ij}^{25} \\
K_{ij}^{31} & K_{ij}^{32} & K_{ij}^{33} & K_{ij}^{34} & K_{ij}^{35} \\
K_{ij}^{41} & K_{ij}^{42} & K_{ij}^{43} & K_{ij}^{44} & K_{ij}^{45} \\
K_{ij}^{51} & K_{ij}^{52} & K_{ij}^{53} & K_{ij}^{54} & K_{ij}^{55}
\end{bmatrix}
\begin{bmatrix}
u_i \\
v_i \\
w_i \\
\psi_{xi} \\
\psi_{yi}
\end{bmatrix}
= \begin{bmatrix} F_i^1 \\ F_i^2 \\ F_i^3 \\ F_i^4 \\ F_i^5 \end{bmatrix},
\]

(5.16)

where

\[M_{ij}^{11} = \int_{\Omega} \rho \phi_i \phi_j \, dx \, dy,\]

\[M_{ij}^{22} = M_{ij}^{11},\]

\[M_{ij}^{33} = M_{ij}^{11},\]

\[M_{ij}^{44} = \int_{\Omega} \frac{\rho h^3}{12} \phi_i \phi_j \, dx \, dy,\]

\[M_{ij}^{55} = M_{ij}^{44}.\]
\[ K_{ij}^{11} = \int_{\Omega} (A_{11}\phi_{i,x}\phi_{j,x} + A_{16}(\phi_{i,x}\phi_{j,y} + \phi_{i,y}\phi_{j,x}) + A_{66}\phi_{i,y}\phi_{j,x})dxdy , \]

\[ K_{ij}^{12} = \int_{\Omega} (A_{12}\phi_{i,x}\phi_{j,y} + A_{16}\phi_{i,x}\phi_{j,x} + A_{26}\phi_{i,y}\phi_{j,y} + A_{66}\phi_{i,y}\phi_{j,x})dxdy , \]

\[ K_{ij}^{13} = \frac{1}{2} \int_{\Omega} (A_{11}\phi_{i,x}\phi_{j,x}w_{x} + A_{12}\phi_{i,x}\phi_{j,y}w_{y} + A_{26}\phi_{i,y}\phi_{j,x} + A_{66}\phi_{i,y}\phi_{j,y}w_{x,y} + \]

\[ A_{66}\phi_{i,y}(\phi_{j,x}w_{y} + \phi_{j,y}w_{x}) + A_{16}(\phi_{i,x}(\phi_{j,y}w_{x} + \phi_{j,x}w_{y}) + \phi_{i,y}\phi_{j,x}w_{x,y})dxdy , \]

\[ K_{ij}^{14} = \int_{\Omega} (B_{11}\phi_{i,x}\phi_{j,x} + B_{16}(\phi_{i,x}\phi_{j,y} + \phi_{i,y}\phi_{j,x}) + B_{66}\phi_{i,y}\phi_{j,y})dxdy , \]

\[ K_{ij}^{15} = \int_{\Omega} (B_{12}\phi_{i,x}\phi_{j,y} + B_{16}\phi_{i,x}\phi_{j,x} + B_{26}\phi_{i,y}\phi_{j,y} + B_{66}\phi_{i,y}\phi_{j,x})dxdy , \]

\[ K_{ij}^{21} = K_{ji}^{12} , \]

\[ K_{ij}^{22} = \int_{\Omega} (A_{66}\phi_{i,x}\phi_{j,x} + A_{26}(\phi_{i,x}\phi_{j,y} + \phi_{i,y}\phi_{j,x}) + A_{22}\phi_{i,y}\phi_{j,y})dxdy , \]

\[ K_{ij}^{23} = \frac{1}{2} \int_{\Omega} (A_{16}\phi_{i,x}\phi_{j,y}w_{x} + A_{22}\phi_{i,y}\phi_{j,y}w_{y} + A_{12}\phi_{i,y}\phi_{j,x}w_{x} + \]

\[ A_{66}\phi_{i,x}(\phi_{j,y}w_{x} + \phi_{j,x}w_{y}) + A_{26}(\phi_{i,y}(\phi_{j,y}w_{x} + \phi_{j,x}w_{y}) + \phi_{i,x}\phi_{j,y}w_{x,y})dxdy , \]

5. Finite Element Model
\[ K_{ij}^{24} = \int_{\Omega} (B_{66} \phi_{i,x} \phi_{j,y} + B_{16} \phi_{i,x} \phi_{j,x} + B_{26} \phi_{i,y} \phi_{j,y} + B_{12} \phi_{i,y} \phi_{j,x}) \, dx \, dy, \]

\[ K_{ij}^{25} = \int_{\Omega} (B_{66} \phi_{i,x} \phi_{j,x} + B_{26} (\phi_{i,x} \phi_{j,y} + \phi_{i,y} \phi_{j,x}) + B_{22} \phi_{i,y} \phi_{j,y}) \, dx \, dy, \]

\[ K_{ij}^{31} = 2 K_{ji}^{13}, \]

\[ K_{ij}^{32} = 2 K_{ji}^{23}, \]

\[ K_{ij}^{33} = \frac{1}{2} \int_{\Omega} \left( A_{11} \phi_{i,x} \phi_{j,x}(w_x)^2 + A_{22} \phi_{i,y} \phi_{j,y}(w_y)^2 + A_{16} (\phi_{i,y} \phi_{i,x} w_x)^2 + \phi_{j,x} \phi_{i,y} w_x w_y \right) \, dx \, dy, \]

\[ -2 \phi_{j,x} \phi_{i,x} w_x w_y + A_{26} (\phi_{i,x} \phi_{j,y} w_y)^2 + \phi_{j,x} \phi_{i,y} w_x w_y \right) \, dx \, dy, \]

\[ A_{12} (\phi_{j,y} \phi_{i,x} w_x w_y + \phi_{j,x} \phi_{i,y} w_x w_y) + A_{66} (\phi_{i,y} \phi_{i,x} w_x w_y + \phi_{i,x} \phi_{i,y} w_x w_y) \, dx \, dy, \]

\[ K_{ij}^{24} = \int_{\Omega} (B_{11} \phi_{i,x} \phi_{j,x} w_x + B_{25} \phi_{i,y} \phi_{j,y} w_y + B_{12} \phi_{i,y} \phi_{j,x} w_y + B_{66} \phi_{i,y} (\phi_{i,y} w_x + \phi_{i,x} w_y) + \]

\[ B_{16} (\phi_{i,x} (\phi_{i,y} w_x + \phi_{j,x} w_y) + \phi_{j,x} \phi_{i,y} w_x) \, dx \, dy, \]

\[ K_{ij}^{25} = \int_{\Omega} (B_{16} \phi_{i,x} \phi_{j,x} w_x + B_{22} \phi_{i,y} \phi_{j,y} w_y + B_{12} \phi_{i,x} \phi_{j,y} w_x + \]

\[ B_{66} \phi_{i,x} (\phi_{i,y} w_x + \phi_{j,x} w_y) + B_{26} (\phi_{i,y} (\phi_{i,y} w_x + \phi_{j,x} w_y) + \phi_{i,x} \phi_{j,y} w_x) \, dx \, dy, \]

5. Finite Element Model
\begin{align*}
K_{ij}^{41} &= K_{ji}^{14}, \\
K_{ij}^{42} &= K_{ji}^{24}, \\
K_{ij}^{43} &= K_{ji}^{34}/2, \\
K_{ij}^{44} &= \int_{\Omega} (D_{11} \phi_{i,x} \phi_{j,x} + D_{16} (\phi_{i,x} \phi_{j,y} + \phi_{i,y} \phi_{j,x}) + D_{66} \phi_{i,y} \phi_{j,y}) dxdy, \\
K_{ij}^{45} &= \int_{\Omega} (D_{12} \phi_{i,x} \phi_{j,y} + D_{16} \phi_{i,x} \phi_{j,x} + D_{26} \phi_{i,y} \phi_{j,y} + D_{66} \phi_{i,y} \phi_{j,x}) dxdy, \\
K_{ij}^{51} &= K_{ji}^{15}, \\
K_{ij}^{52} &= K_{ji}^{25}, \\
K_{ij}^{53} &= K_{ji}^{35}/2, \\
K_{ij}^{54} &= K_{ji}^{45}, \\
K_{ij}^{55} &= \int_{\Omega} (D_{26} \phi_{i,x} \phi_{j,y} + D_{66} \phi_{i,x} \phi_{j,x} + D_{22} \phi_{i,y} \phi_{j,y} + D_{26} \phi_{i,y} \phi_{j,x}) dxdy, \\
\Gamma_{i}^l &= -\sum_{j=1}^{P} \int_{\Omega} \phi_{i} [N_{x}^{T} \phi_{j,x} + N_{xy}^{T} \phi_{j,y}] dxdy,
\end{align*}

5. Finite Element Model
\[ F_i^2 = - \sum_{j=1}^{p} \int_{\Omega} \phi_i \left[ N_{y}^{T} \phi_{j,x} + N_{xy}^{T} \phi_{j,x} \right] dx dy, \]

\[ F_i^3 = - \sum_{j=1}^{p} \int_{\Omega} \phi_i \left[ N_{x}^{T} \phi_{j,x} w_{x} + N_{x}^{T} (\phi_{j,x} w_{x} + \phi_{j,y} w_{y}) + N_{y}^{T} \phi_{j,y} w_{y} \right] dx dy, \]

\[ F_i^4 = - \sum_{j=1}^{p} \int_{\Omega} \phi_i \left[ M_{x}^{T} \phi_{j,x} + M_{xy}^{T} \phi_{j,y} \right] dx dy, \]

and

\[ F_i^5 = - \sum_{j=1}^{p} \int_{\Omega} \phi_i \left[ M_{y}^{T} \phi_{j,y} + M_{xy}^{T} \phi_{j,x} \right] dx dy. \]

The matrices and vectors of Eq. (5.16) are referred to as the mass matrix \([M]\), the direct stiffness matrix \([K]\), the force vector \([F]\) and the displacement vector \([U]\), so that Eq. (5.16) can be written

\[ [M] \{\ddot{U}\} + [K] \{U\} = \{F\}. \]  \(5.17\)

### 5.3. Gauss-Legendre Quadrature and Reduced Integration

In order to incorporate the approximate and interpolation functions into the variational form, all the functions and areas must be described in the same coordinate sys-
The transformation from the local coordinate system to the global coordinate system is done with the Jacobian matrix. For example,

\[
\begin{bmatrix}
\frac{\partial \phi_i}{\partial \zeta} \\
\frac{\partial \phi_i}{\partial \eta}
\end{bmatrix} = [J] \begin{bmatrix}
\frac{\partial \phi_i}{\partial x} \\
\frac{\partial \phi_i}{\partial y}
\end{bmatrix},
\]

and

\[
\begin{bmatrix}
dx \\
dy
\end{bmatrix} = [J]^T \begin{bmatrix}
d\zeta \\
d\eta
\end{bmatrix},
\]

where

\[
[J] = \begin{bmatrix}
\frac{dx}{d\zeta} & \frac{dy}{d\zeta} \\
\frac{dx}{d\eta} & \frac{dy}{d\eta}
\end{bmatrix}.
\]

The elements used in this thesis are isoparametric, meaning that the x- and y-coordinates are approximated using the same nodes and interpolation functions as were used to approximate the dependent variables. Thus

\[
x = \sum_{i=1}^{P} x_i \phi_i
\]

5. Finite Element Model
For computational convenience the Jacobian matrix can be rewritten as

\[
[J] = \begin{bmatrix}
\frac{\partial \phi_1}{\partial \zeta} & \frac{\partial \phi_2}{\partial \zeta} & \cdots & \frac{\partial \phi_n}{\partial \zeta} \\
\frac{\partial \phi_1}{\partial \eta} & \frac{\partial \phi_2}{\partial \eta} & \cdots & \frac{\partial \phi_n}{\partial \eta} \\
\vdots & \vdots & & \vdots \\
x_1 & y_1 & & x_n & y_n
\end{bmatrix}
\]

A wide variety of numerical integration methods is available to evaluate the integrands of the previous equations. Gauss-Legendre quadrature is one of the most widely used and has proven to be reliable. For any function \( G(x) \) the integral can be found by

\[
\int_a^b \int_c^d G(x,y) \, dx \, dy = \int_{-1}^1 \int_{-1}^1 G(\zeta, \eta) |J| d\eta d\zeta = \sum_{i=1}^{GP} \sum_{j=1}^{GP} G(\zeta_i, \eta_j) \omega_i \omega_j |J|,
\]

where \( \zeta \) and \( \eta \) are the so-called Gauss points in the local system (roots of the Legendre polynomial), \( \omega_i \) and \( \omega_j \) are weight factors, and \( |J| \) is the determinant of the Jacobian matrix. The integral of a polynomial of the \( n \)th degree can be found if the number of Gauss points is greater than or equal to \((n + 1)/2\) [38].

Although the integral can be evaluated using the Gauss-Legendre quadrature, the terms in Eq. (5.16) involving \( A_{44}, A_{45}, \) and \( A_{55} \) are still erroneous. This is because the problem under consideration is based on two-dimensional plate theory, but the trans-
verse shear terms are included. The result is that these terms are too stiff in the finite element model. In other words, the solution to the governing equations is dominated by the presence of the shear deformation terms regardless of their value. Zienkiewicz, Taylor, and Too [24] addressed this problem and developed a solution, called reduced integration, which is very simple to employ. The first step in reduced integration is to separate the transverse shear terms from the tangent stiffness matrix. These terms are

\[ G_{ij}^{33} = \int_{\Omega} \left[ A_{55} \phi_{i,x} \phi_{j,x} + A_{44} \phi_{i,y} \phi_{j,y} + A_{45} (\phi_{i,y} \phi_{j,x} + \phi_{i,x} \phi_{j,y}) \right] dxdy , \]

\[ G_{ij}^{34} = \int_{\Omega} \left[ A_{55} \phi_{i,x} \phi_{j} + A_{45} \phi_{i,y} \phi_{j} \right] dxdy , \]

\[ G_{ij}^{35} = \int_{\Omega} \left[ A_{45} \phi_{i,x} \phi_{j} + A_{44} \phi_{i,y} \phi_{j} \right] dxdy , \]

\[ G_{ij}^{43} = \int_{\Omega} \left[ A_{55} \phi_{i,x} \phi_{j} + A_{45} \phi_{j,y} \phi_{i} \right] dxdy , \]

\[ G_{ij}^{44} = \int_{\Omega} \left[ A_{55} \phi_{j} \phi_{i} \right] dxdy , \]

\[ G_{ij}^{45} = \int_{\Omega} \left[ A_{45} \phi_{j} \phi_{i} \right] dxdy , \]
Instead of using \((n + 1)/2\) Gauss points, as is done when finding the exact solution, the transverse shear terms are integrated using one less Gauss point. This relaxes the stiffness and gives a better weighting to those terms.

5.4. The Transient Solution (Newmark Method)

There are many choices of numerical solutions to the transient problem represented by Eq. (5.12). However, the Newmark method is one of the most reliable and is unconditionally stable. In the Newmark method the essential boundary conditions and their derivatives with respect to time at the \(n + 1\)\(^{\mathrm{st}}\) time step are given by

\[
\{U\}^{n+1} = \{U\}^n + \{\dot{U}\}^n \Delta t + 0.25[\{\ddot{U}\}^n + \{\dot{U}\}^{n+1}](\Delta t)^2 \tag{5.19}
\]

and
\[
\{\dot{U}\}^{n+1} = \{\dot{U}\}^n + 0.5[\{\ddot{U}\}^n + \{\ddot{U}\}^{n+1}]\Delta t ,
\]  
(5.20)

where now \(U\) represents any one of the dependent variables. Solving for \(\{\ddot{U}\}^{n+1}\),

\[
\{\ddot{U}\}^{n+1} = \[\{U\}^{n+1} - \{U\}^n\] \frac{4}{(\Delta t)^2} - \frac{4}{\Delta t}\{\dot{U}\}^n - \{\ddot{U}\}^n .
\]  
(5.21)

Now multiplying both sides of Eq. (5.21) by the mass matrix \([M]^{n+1}\) yields

\[
[M]^{n+1}\{\ddot{U}\}^{n+1} = [M]^{n+1}[\{U\}^{n+1} - \{U\}^n] \frac{4}{(\Delta t)^2} - \frac{4}{\Delta t}[M]^{n+1}\{\dot{U}\}^n - [M]^{n+1}\{\ddot{U}\}^n .
\]  
(5.22)

However,

\[
[M]^{n+1}\{\ddot{U}\}^{n+1} = \{F\}^{n+1} - [K]^{n+1}\{U\}^{n+1} ,
\]  
(5.23)

so that Eq. (5.22) can now be written

\[
[[K] + a_0[M]]^{n+1}\{U\}^{n+1} = \{F\}^{n+1} + \left[a_0\{U\}^n + a_1\{\dot{U}\}^n + \{\ddot{U}\}^n\right][M]^{n+1} ,
\]

where \(a_0 = \frac{4}{(\Delta t)^2}\) and \(a_1 = \frac{4}{\Delta t}\). For simplification in notation let

\[
[K] = ([K] + a_0[M])^{n+1}
\]

and

\[
\{F\} = \{F\}^{n+1} + \left[a_0\{U\}^n + a_1\{\dot{U}\}^n + \{\ddot{U}\}^n\right][M]^{n+1} .
\]

Then

\[
[K]\{U\}^{n+1} = \{F\}^{n+1} .
\]
5.5. Newton-Raphson Method

The Newton-Raphson method of iteration is a very fast and accurate scheme for solving nonlinear problems. The theory is as follows. Consider a residual vector \( \{R\} \) such that at the \( r \)th iteration

\[
\{R\}_r^t = [\overline{K}]^r \{U\}_r^t - \{F\}.
\]

(5.24)

When the solution finally converges the residual matrix ideally would be zero. The residual matrix on the iteration after the \( r \)th iteration will be

\[
\{R\}_r^{t+1} = \{R\}_r^t + \frac{\partial \{R\}_r^t}{\partial \{U\}_r^t} \left[ U_r^{t+1} - U_r^t \right].
\]

(5.25)

If \( \{R\}_r^{t+1} \) is set to zero, Eq. (5.25) can be written as

\[
\frac{\partial \{R\}_r^t}{\partial \{U\}_r^t} \left[ U_r^{t+1} - U_r^t \right] = - \{R\}_r^t,
\]

(5.26)

where

\[
[K]^T = \frac{\partial \{R\}_r^t}{\partial \{U\}_r^t} = \frac{\partial}{\partial \{U\}_r^t} ([K]^r \{U\}_r^t)
\]

(5.27)

and is commonly called the tangent stiffness matrix. Unlike the direct stiffness matrix \([K]\), the tangent stiffness matrix, \([K]^T\) is symmetric. The individual terms of the tangent stiffness matrix can be written in terms of the direct stiffness matrix and mass matrix:

\[
K_{ij}^{11} = K_{ij}^{D11} + M_{ij}^{11}, \quad K_{ij}^{12} = K_{ij}^{D12}, \quad K_{ij}^{13} = 2K_{ij}^{D13},
\]
\[ K_{ij}^{T14} = K_{ij}^{D14}, \quad K_{ij}^{T15} = K_{ij}^{D15}, \quad K_{ij}^{T22} = K_{ij}^{D22} + M_{ij}^{22}, \]
\[ K_{ij}^{T23} = 2K_{ij}^{D23}, \quad K_{ij}^{T24} = K_{ij}^{D24}, \quad K_{ij}^{T25} = K_{ij}^{D25}, \]
\[ K_{ij}^{T33} = K_{ij}^{D33} + M_{ij}^{33} + H_{ij}, \quad K_{ij}^{T34} = K_{ij}^{D34}, \quad K_{ij}^{T35} = K_{ij}^{D35}, \]
\[ K_{ij}^{T44} = K_{ij}^{D44} + M_{ij}^{44}, \quad K_{ij}^{T45} = K_{ij}^{D45}, \]

and

\[ K_{ij}^{T55} = K_{ij}^{D55} + M_{ij}^{55}, \]

where

\[ H_{ij} = \int_{\Omega} \left[ A_{11} \left[ \left( \frac{\partial w}{\partial x} \right)^2 \frac{\partial \phi_i}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial \phi_i}{\partial x} \right] + A_{12} \left[ \frac{\partial u}{\partial x} \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} + \right. \right. \]
\[ + \frac{\partial v}{\partial y} \frac{\partial \phi_i}{\partial x} + \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} + \]
\[ + \frac{1}{2} \frac{\partial w}{\partial y} \frac{\partial \phi_i}{\partial y} + \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2 \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \right] + A_{16} \left[ \frac{\partial u}{\partial y} \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial y} + \right. \right. \]
\[ + \frac{\partial u}{\partial x} \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial y} + \frac{\partial u}{\partial x} \frac{\partial \phi_i}{\partial y} + \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial v}{\partial y} \frac{\partial \phi_i}{\partial x} + \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \phi_i}{\partial x} + \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \]
\[ + \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} \right] + A_{26} \left[ \frac{\partial u}{\partial y} \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} + \right. \right. \]
\[ + \left( \frac{\partial w}{\partial x} \right)^2 \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} + \left( \frac{\partial w}{\partial x} \right)^2 \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \right] + A_{26} \left[ \frac{\partial u}{\partial y} \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} + \right. \right. \]

5. Finite Element Model
Equation (5.26) can now be written as

\[
[K^T](U^{t+1} - U^t) = \{F\} - [K]^f(U)^f. \tag{5.28}
\]

Given the initial displacements, velocities, and accelerations, Eq. (5.28) can be numerically solved for any other displacements, velocities, or accelerations in time.
6. Verification of Model and Results

The results predicted by the finite element model are presented and compared to known results of plate problems. In order to verify the finite element model, discrete boundary conditions are applied to plates of various geometries and the known results are compared to those predicted by the model. Each condition verifies an aspect of the model and demonstrates its capabilities. The conditions are:

• Simply-supported isotropic plate subjected to a uniformly-distributed transverse mechanical load at steady state.

• Clamped composite plate subjected to a uniformly-distributed transverse mechanical load at steady state.

• Simply-supported isotropic plate subjected to a linear temperature variation through the thickness of the plate (steady state).

• Simply-supported isotropic plate subjected to a sudden uniformly-distributed transverse mechanical load (transient).
• Simply-supported laminated composite plate subjected to a sudden uniformly-distributed transverse mechanical load (transient).

• Simply-supported composite plate subjected to a sudden heat flux on the upper surface (transient).

Each case is analyzed with a finite element model of four quadrilateral 9-node elements as shown in Fig. 8.

6.1. Static Mechanical Load

Numerical results are presented in this section for a simply-supported isotropic plate under uniformly distributed static loads and for a clamped laminated plate under the same loading.

The geometry and boundary conditions of the rectangular isotropic plate are shown inFig. 9. Because of the biaxial symmetry only the first quadrant of the plate must be modeled. The isotropic material properties are $E = 21.0$ GPa, and $v = 0.3$.

An analytical solution for a simply supported rectangular isotropic plate with uniform transverse loading is available for comparison. However, the analytical theory is based on a linear relationship between the deflection in the z-direction and the transverse load and therefore it is limited to the case of a thin plate with small deflections. For the case of a square plate with $v = 0.3$ the analytical theory gives a maximum deflection at the center of the plate of

$$w_{\text{max}} = 0.0443 \frac{P_0 a^4}{E h^3}.$$
Figure 10 shows the normalized maximum deflections versus the plate width-to-depth (a/h) ratio of three theories. The continuous line represents the thin plate theory [39] and the dashed line represents the first-order SDT finite element solution of this thesis. The results of a higher-order shear deformation theory of Reddy [22] are indicated with the triangles. The deflections in Fig. 10 are normalized as

\[ \bar{w} = \frac{wEh^3}{p_r a^4}. \]

The finite element solutions agree with the linear theory until the width-to-depth ratio becomes less than about 10. At this point thin plate theory breaks down because it assumes that the in-plane loads are negligible. However, as the width-to-depth ratio becomes small more of the load is being carried in-plane and less transversely. Because the finite element solutions have incorporated this coupling between the deflections in the u-, y-, and z-directions they diverge from the linear solution. The two finite element solutions agree within 0.1 percent difference.

The geometry and boundary conditions of the laminated composite plate are shown in Fig. 11. The plate is symmetric about the x-y plane with a 0° and a 90° layer of equal thickness above and below the midplane; i.e. a [0/90/90/0] layup. It is clamped on all edges and subjected to uniformly-distributed transverse load. The orthotropic properties of the laminae are

\[
E_{11} = 12.605 \text{ GPa}, \quad E_{22} = 12.628 \text{ GPa}, \quad E_{33} = 12.628 \text{ GPa},
\]

\[
v_{12} = 0.23949,
\]

\[
G_{12} = 2.155 \text{ GPa}, \quad G_{31} = 2.155 \text{ GPa}, \quad \text{and} \ G_{23} = 2.155 \text{ GPa}.
\]

6. Verification of Model and Results
The finite element model results are compared to the linear solution of Zaghloul and Kennedy [40] in Fig. 12. Experimental results are also available for this case [40] as well as results from a higher-order finite element model [22]. The dot-dashed line in Fig. 12 represents the linear solution, the dashed line is the experimental data, and the solid line is the SDT finite element solution from this thesis. The triangles represent the values from the higher-order finite element model [22]. Again the nonlinear and linear solutions converge for small loads. But now it can also be seen that the experimental solution agrees very well with the finite element solutions. The small discrepancies that occur between the two finite element solutions and the experimental results may be due to a number of things, including errors in modeling experimental boundary conditions. Another important result is the agreement between the higher-order finite element model and the first-order model of this thesis. The deflections differ by less than 0.01 percent.

6.2. Linear Temperature Variation Through an Isotropic Plate

The nonlinearity of the plate deflection due to a thermal load is demonstrated using a simply-supported isotropic plate. The boundary conditions and geometry of the plate are shown in Fig. 9 (a = 25.4 cm), and the material properties are $E = 68.9$ GPa, $v = 0.3$, and $\alpha = 0.000124$ m/m/°C. The midplane of the plate is kept at the stress-free temperature and the temperature varies linearly through the thickness of the plate and is uniform on any x-y plane. Figure 13. shows the nondimensional deflections versus the width-to-depth ratio (a/h). The deflection in nondimensional form is

$$\bar{w} = \frac{10.4wh}{\alpha a^2 \Delta T},$$
where \( \Delta T \) is the temperature difference between the top and bottom surfaces, \( \alpha \) is the coefficient of thermal expansion, and \( w \) is the actual deflection. The solid line in Fig. 13 depicts the thin-plate small-deflection solution [41]. When the temperature variation is small producing small deflections the finite element predictions agree with the thin-plate small-deflection theory. The dashed line in Fig. 13 shows the nondimensional finite element solution when the temperature difference between the top surface and bottom surface is 500°C. Again the effects of the nonlinear strains are shown in Fig. 13. As the width-to-depth ratio becomes small the nonlinear and linear solutions converge.

### 6.3. Transient Mechanical Load

The ability to accurately predict deflections as a function of time due to a uniformly-distributed transverse load is demonstrated for a simply-supported isotropic square plate and laminated composite square plate. The isotropic material properties are \( E = 2.1 \text{ GPa}, \, \nu = 0.3, \) and \( \rho = 800.0 \text{ kg/m}^3 \). The plate is 5.0 cm thick and 25.0 cm wide and suddenly subjected to a 100 kPa uniform transverse load. The deflections through half of a cycle are shown in Fig. 14. The solid line represents the solution using the present first-order SDT model and the triangles represent the results using another first-order SDT finite element model [42]. There is no appreciable difference in the two finite element solutions.

Figure 15 shows the center deflection versus time of a \([0/90/0]_s\) laminated composite plate under the same 100 kPa load. The laminated plate is 5.0 cm thick and 25.0 cm square and the lamina material properties are

\[
E_{11} = 52.1 \text{ GPa}, \quad E_{22} = 2.1 \text{ GPa}, \quad E_{33} = 2.1 \text{ GPa},
\]
\[ v_{12} = 0.3, \]

\[ G_{12} = 1.05 \text{ GPa}, \quad G_{31} = 1.05 \text{ GPa}, \quad G_{23} = 1.05 \text{ GPa}. \]

It can be seen in Fig. 15 that the SDT finite element solution from the literature [42] again agrees well with the results obtained using the first-order SDT model of this thesis.

### 6.4. Transient Thermal Load

The first-order SDT finite element model is shown to accurately predict the time-dependent deflections caused by a nonuniform heat flux. The predictions are compared to the solution of a three-dimensional finite element model [31]. Since the finite element model of this thesis does not have the capability to perform heat conduction calculations, the temperature field calculated by the three-dimensional model is read into the model at time steps of \( \Delta t = 0.075 \text{ seconds} \). The comparison is made on a simply-supported square \([0/90/90/0]\) laminated composite plate of thickness 0.005 m and 0.020 m width, as shown in Fig. 16. The lamina material properties of the plate are

\[ E_{11} = 132.0 \text{ GPa}, \quad E_{22} = 10.8 \text{ GPa}, \quad E_{33} = 10.8 \text{ GPa}, \]

\[ v_{12} = 0.24, \]

\[ G_{12} = 5.7 \text{ GPa}, \quad G_{31} = 5.7 \text{ GPa}, \quad G_{23} = 3.6 \text{ GPa}. \]

The plate is subjected to a nonuniform heat flux on the upper surface described by

\[ q = 30(1 - 1250(x^2 + y^2)) \text{ kW/m}^2. \]
Figure 17 shows the center deflections of the plate caused by the time-dependant thermal load. The solid line shows the first-order SDT finite element solution and the dashed line represents the three-dimensional finite element solution. Because the inertia effects are extremely small the plate response is dominated by the thermal load. The two finite element solutions differ by about 5.0 percent. There are probably two reasons for this difference:

1. The differences that arise in simulating the boundary conditions of a two-dimensional plate and a three-dimensional plate.

2. The three-dimensional finite element theory permits strains in the z-direction; in other words the plate can become thinner. The two-dimensional theory does not permit transverse strains.
7. Conclusions and Recommendations

The first-order SDT finite element model predicts the deflections of the isotropic and laminated composite plates very accurately under steady-state and transient thermal and mechanical loads. In all cases presented in this thesis the first-order SDT solution agrees with the other two-dimensional finite element solutions and is very close to the three-dimensional finite element solution for transient thermal problems.

The convergence of the model is very fast using the Newton-Raphson method. No more than 3 iterations were ever needed; thus, CPU time requirements are very small. The finite element model should be improved by including the capability to perform heat conduction calculations. Also, to correctly predict the dynamic response of nonsymmetric laminates, rotary inertia terms must also be included. In order to fully predict laminate behavior a postprocessor must be added in which the laminate stresses are calculated. With these modifications the model will be a very valuable tool in predicting laminated composite plate behavior.
Fig. 1. Typical Laminated Composite Plate.
Fig. 2. Plate Element with Seven Degrees of Freedom, Three Translations, and Four Rotations.
Fig. 3 (a) Undeformed Laminated Plate, (b) Deformed Laminated Plate with CLT Displacement Field, (c) Deformed Laminated Plate with First-Order SDT Displacement Field.
Fig. 4. (a) Forces and (b) Moments on the Laminated Plate.
Fig. 5. Laminate Coordinate System and Nomenclature.
Fig. 6. Typical Infinitesimal Cube of Laminated Composite Plate.
Fig. 7. Master Elements for (a) 4-Node Rectangular (b) 8-Node Rectangular, and (c) 9-Node Rectangular Elements.
Fig. 8. Finite Element Mesh used in the Plate Analysis, Four 9-Node Quadrilateral Elements.
Fig. 9. Geometry and Boundary Conditions of the First Quadrant of a Simply-Supported Square Plate.
Fig. 10. Normalized Center Deflection of a Simply-Supported Isotropic Plate Under a Uniformly-Distributed Transverse Load.
Fig. 11. Geometry and Boundary Conditions of the First Quadrant of a Clamped Symmetric Cross-Ply [0/90/90/0] Laminated Composite Plate.
Fig. 12. Center Deflection of a Clamped Symmetric Cross-Ply [0/90/90/0] Laminate Subjected to a Uniform Transverse Load (a = 4.72 cm (12. in.), h = 0.038cm (0.096 in.)).
Fig. 13. Normalized Deflection of a Simply-Supported Isotropic Plate Subjected to a Linear Temperature Distribution.
Fig. 14. Evolution of Center Deflection with Time for a Simply-Supported Isotropic Square Plate Under a Uniform Load of 100 kPa ($a = 25.0$ cm, $h = 5.0$ cm).
Fig. 15. Evolution of Center Deflection with Time for a [0/90/0] Simply-Supported Laminated Composite Plate Under a Uniform Load of 100 kPa (a = 25.0 cm, h = 5.0 cm).
Fig. 16. Geometry and Boundary Conditions of the First Quadrant of a Simply-Supported Cross-Ply \([0/90/90/0]\) Laminated Composite Plate \((a = 2.0 \text{ cm}, h = 0.5 \text{ cm})\).
Fig. 17. Evolution of Center Deflection with Time for a Simply-Supported [0/90/90/0] Laminated Composite Plate Subjected to a Nonuniform Heat Flux ($a = 2.0$ cm, $h = 0.5$ cm).
References


Appendix A. COMMEC2 PROGRAM LISTING

IMPLICIT REAL*8(A-H,O-Z)
DIMENSION IBDY(85),YBDYC85,XC85),Y85),TITLE(70)
DIMENSION IBFC1),YBF1),GSTIF125,65),ELXY(9,2),UI125),GR(125)
DIMENSION TNY(25),TMX(25),TYX(25),TXK(25),TYM(25),TXY(25)
DIMENSION TEMS25),T5,T5,TH5),DX5),DY5),LMT5)
DIMENSION E1(2),E2(2),E3(2),P12(2),P31(2),P23(2)
DIMENSION G12(2),G31(2),G23(2),ALP12(2),ALP21(2),ALP12(2),RHO(2)
DIMENSION C16),Z(20),C21),SH(20,3),QB(20,6)
DIMENSION ALPX20),ALPY20),ALPXY20)
DIMENSION TEMS2),T2),TH2),DX2),DY2),LMT2)
DIMENSION El2),E22),E32),P122),P312),P232)
DIMENSION G122),G312),G232),ALP122),ALP212),ALP122),RHO22)
DIMENSION COMMON/NOD(4,9)
COMMON/STFL/EK(45,45),REV(45),A(6),B(6),D(6),AS(3)
COMMON/STF2/SAO,SA1,SA2,SA3,SA*11,X11,112
COMMON/STF3/EUL(45),EUL(45),EUL(45),EUL(45)
COMMON/STF4/TNX(9),TYN(9),TXY(9),TYX(9),TMY(9)
DATA NRMAX,NCHAX125,65/
READ(S,*)IPE,NDF,ITHEL
IFCNPE.EQ.4)THEN
  IELEM=2
ELSE
  IELEM=3
ENDIF
NN=NPE*NDF
IF(IHESH.EQ.1)GOTO 500
READ(S,*)NEH,NNH
DO 510 N=1,NEH
  READ(S,*)(NODtN,I),I=1,NPE)
  READ(S,*)(X(I),Y(I),I=1,NNH)
  GOTO 520
510 READ(S,*)NX,NY
  READ(S,*)(DX(I),I=1,NX)
  READ(S,*)(DY(I),I=1,NY)
  CALL MESH(IPE,NPE,NX,Y,NH,NEM,DX,DOY)
520 CONTINUE
READ(S,*)TMAX,TOLER,SEED
READ(S,*)HT,NL,SK,TEO
DO 5 T=1,HT
  READ(S,*)EI1(T),E12(I),EI3(I),PI2(I),P31(I),P23(I)
  READ(S,*)G12(I),G31(I),G23(I),ALP1(I),ALP2(I),ALP12(I),RHO(I)
  T=0.
  DO 7 T=1,NL
    READ(S,*)TH(I),T(I),LMT(I)
  7 TT=TT+I
READ(S,*)NLOAD,PINIT,PDELT
C
C TRANSIENT INPUT
C
READ(S,*)ITIM
IF(ITIM.EQ.1)THEN
  READ(S,*)HT,TMAX
  DELT=TMAX/HT
C
C
C
SAO=4.DO/(DELT**2)
SA1=SAO*DELT
SA2=1.DO
SA3=DELT/2.DO
SA4=SA3
ELSE
SAO=0.DO
SA1=0.DO
SA2=0.DO
SA3=0.DO
SA4=0.DO
ENDIF
C
WRITE INPUT DATA TO OUTPUT FILE
WRITE(7,3)TITLE
WRITE(7,6399)
WRITE(7,938)NPE
WRITE(7,940)NEM
WRITE(7,942)NMM
WRITE(7,6399)
WRITE(7,8999)
DO 8998 I=1,NEM
WRITE(7,2376)I,(NOD(I,J),J=1,NPE)
WRITE(7,6399)
WRITE(7,2999)
DO 3998 I=1,NEM
WRITE(7,2876)I,(X(I),Y(I))
WRITE(7,6399)
WRITE(7,8210)MT
WRITE(7,8211)NL
WRITE(7,8212)TEO
WRITE(7,8213)SK
WRITE(7,8214)TOLER
WRITE(7,8215)ISEED
WRITE(7,8216)PINIT
WRITE(7,3416)PDAM
WRITE(7,6399)
DO 5164 I=1,MT
WRITE(7,9922)MT
WRITE(7,3522)
WRITE(7,3523)E1(I),E2(I),E3(I),P12(I),P31(I),P23(I)
WRITE(7,6382)
WRITE(7,6384)G12(I),G31(I),G23(I),ALP1(I),ALP2(I),ALP12(I)
WRITE(7,3416)RHO1(I)
WRITE(7,6399)
WRITE(7,7446)TH(I),T(I),LMT(I)
WRITE(7,6399)
WRITE(7,7444)
DO 7445 I=1,NL
WRITE(7,7446)TH(I),T(I),LMT(I)
WRITE(7,6399)
WRITE(7,7444)
7444 FORMAT( ,ANGLE',,THICKNESS',,MAT. TYPE')
7446 FORMAT( ,F5.2,5X,F7.4,9X,I2)
8210 FORMAT( ,NO. OF MATERIAL TYPES =',I1)
8211 FORMAT( ,NO. OF LAYERS IN PLATE =',I2)
8212 FORMAT( ,STRESS-FREE TEMPERATURE =','F8.4)
8213 FORMAT( ,SHEAR CORRECTION FACTOR =','F8.6)
8214 FORMAT( ,TOLERANCE ON LS ERROR =','F8.4)
8215 FORMAT( ,INITIAL DISPLACEMENTS =','F8.6)
8216 FORMAT( ,INITIAL LOAD IN THE Z-DIRECTION=',F13.6)
3416 FORMAT( ,DELTA LOAD IN THE Z-DIRECTION=','F13.6)
3 FORMAT( ,70A1)
938 FORMAT( ,NO. OF NODES PER ELEMENT= ',I2)
940 FORMAT( ,NO. OF ELEMENTS IN MESH= ',I2)
942 FORMAT( ,NO. OF NODES IN MESH= ',I2)
2876 FORMAT( ',I2,3X,2(F7.4,2X))
2376 FORMAT( ',3X,I2,7X,(I2,2X))
8999 FORMAT( ,ELEMENT', 'NODES')
2999 FORMAT( ',NODE', 'X', 'Y')
6399 FORMAT( ',/'}
INPUT NONZERO BOUNDARY FORCES AND DISPLACEMENTS

READ(5,*,NBDY,NBF)
IF(NBDY.EQ.0)THEN
IBDY(1)=0
VBDY(1)=0.00
ELSE
READ(5,*,IBDY(I),I=1,NBDY)
READ(S,*,VBDY(I),I=1,NBDY)
ENDIF

IF(NBF.EQ.0)THEN
IBF(1)=0
VBF(1)=0.00
ELSE
DO 30 I=1,NBF
READ(S,*,IBF(I))
DO 33 I=1,NBF
READ(S,*,VBF(I))
WRITE(7,35)(VBF(I),I=1,NBF)
FORMAT(' ',SECONDARY KNOTS=',4(F10.4,2X),//')
ENDIF

CALCULATION OF MATERIAL PROPERTIES

DO 27 I=1,HT
P32=E3(I)*PZ3(I)/EZ(I)
P21=E2(I)*PIZ(I)/El(I)
P13=El(I)*P31(I)/E3(I)
COEl=(1.0-Pl2(I))*P21-P23(I)*P3Z-P31(I)*P13-Z.DO*PZ1*P32*P13)/
& (El(I)*E2(I)*E3(I))
C(I,1)=1.00-P12(I)*P21-P23(I)*P3Z-P31(I)*P13-Z.DO*PZ1*P32*P13)/
& (El(I)*E2(I)*E3(I))
C(I,2)=P12(I)*P21+P23(I)*P3Z+P31(I)*P13-Z.DO*PZ1*P32*P13)/
& (El(I)*E2(I)*E3(I))
C(I,3)=P12(I)*P21+P23(I)*P3Z+P31(I)*P13-Z.DO*PZ1*P32*P13)/
& (El(I)*E2(I)*E3(I))
C(I,4)=P21+P23(I)*P3Z+P31(I)*P13-Z.DO*PZ1*P32*P13)/
& (El(I)*E2(I)*E3(I))
C(I,5)=P21+P23(I)*P3Z+P31(I)*P13-Z.DO*PZ1*P32*P13)/
& (El(I)*E2(I)*E3(I))
C(I,6)=P12(I)*P21+P23(I)*P3Z+P31(I)*P13-Z.DO*PZ1*P32*P13)/
& (El(I)*E2(I)*E3(I))
27

TRANSFORMED REDUCED STIFFNESS MATRIX (Q8)

Z(I)=TT/2.00
DO 15 I=1,NL
II=I+1
Z(II)=Z(I)+T(II)/2.00
TTH=TH(I)+3.141592653589793D0/180.00
CS=COS(TTH)
SN=SIN(TTH)
CS4=CS*CS
CS2=CS*CS
SN4=SN*SN
SN2=SN*SN
K=LHT(I)
CC1=C(K,2)+2.00*G12(K)
CC2=C(K,1)+C(K,4)-4.00*G12(K)
CC3=C(K,1)-C(K,2)-2.00*G12(K)
CC4=C(K,1)-2.00*C(K,2)+C(K,4)
CC5=(C(K,2)-C(K,4)+2.D0*G12(K))
CC11=CS0*(C(K,1)+2.D0*G2*SN2*CC1+SN4*MC(K,4))
CC12=CS2*SN2*CC2*(SN4+CS4)*MC(K,2)
CC13=CS0*(C(K,3)+SN4*(C(K,5))
CC16=SN4*(CS2*CS+SN2*CC5)
CCB22=SN4*(C(K,1)+2.D0*CS2*SN2*MC(K,4)+CS4*MC(K,4))
CCB26=SN2*MC(K,5)+CS2*MC(K,5)
C25=SN2*MC(K,5)+CS2*MC(K,5)
C23=CS0*(C(K,3)-C(K,5))
C2B6=SN2*MC(K,5)+CS2*MC(K,5)
C236=CS0*(C(K,6))
C266=CS0*(C(K,6))
C312=CS0*(C(K,1)+2.D0*CS2*SN2*CC1+SN4*CIK,4)
C311=CS2*SN2*CC2+(SN4+CS41*C(K,2)
C313=CS2*CtK,3)+SN2*CIK,5)
C316=SN*CS*(SN2*CC3+CS2*CCS)
C366=CS2*SN2*CC4+G12(KJ*(CSZ-SNZ)**Z)
C5=CS2*G23(K)+SNZ*G31CK)
C5=CS2*G23(K)+SNZ*G31CK)
C5=SN2*G23(K)+CS2*G31(K)
C5=CS2*G23(K)+SNZ*G31(K)
C511=CS0*(C(K,1)+2.D0*CS2*SN2*CC1+CS4*C(K,4))
C512=SN4*C(K,5)+2.DO*CS2*SN2*CC1+SN4*CIK,4)
C513=CS2*SN2*CC2+(SN4+CS41*C(K,2)
C516=SN2*MC(K,5)+CS2*MC(K,5)
C522=SN4*C(K,1)+2.DO*CS2*SN2*MC(K,4)+CS4*MC(K,4)
C523=SN2*MC(K,5)+CS2*MC(K,5)
C526=SN*CS*(SN2*CC3+CS2*CCS)
C533=CtK,6)
C536=CS*SN*CCIK,3)-CIK,5)
C566=CS2*SN2*CC4+G12(KJ*(CSZ-SNZ)**Z)
C6=CS2*G23(K)+SNZ*G31CK)
C6=CS2*G23(K)+SNZ*G31(K)
C6=CS2*G23(K)+SNZ*G31(K)
C611=CS0*(C(K,1)+2.D0*CS2*SN2*CC1+CS4*C(K,4))
C612=SN4*C(K,5)+2.DO*CS2*SN2*CC1+SN4*CIK,4)
C613=CS2*SN2*CC2+(SN4+CS41*C(K,2)
C616=SN2*MC(K,5)+CS2*MC(K,5)
C622=SN4*C(K,1)+2.DO*CS2*SN2*MC(K,4)+CS4*MC(K,4)
C623=SN2*MC(K,5)+CS2*MC(K,5)
C626=SN*CS*(SN2*CC3+CS2*CCS)
C633=CtK,6)
C636=CS*SN*CCIK,3)-CIK,5)
4572 FORMAT(' '10X,|',2X,3(D13.6,2X),'|')
4573 FORMAT(' '5X,'Ai=$',1X,'|',17X,2(D13.6,2X),'|'), i,j=1,2,6')
4574 FORMAT(' '10X,|',132X,D13.6,2X),'|')
5573 FORMAT(' ',5X,'Bi=$',1X,'|',17X,2(D13.6,2X),'|'), i,j=1,2,6')
7573 FORMAT(' ',5X,'Di=$',1X,'|',17X,2(D13.6,2X),'|'), i,j=1,2,6')
7571 FORMAT(' ',1X,'A44,A45=$',1X,'|',2X,2(D13.6,2X),'|')
7572 FORMAT(' ',5X,'AS=$',1X,'|',17X,D13.6,2X,'|')

WRITE(7,3076)RH

3076 FORMAT(' ','AVERAGE DENSITY=',013.6)
C
C...------------------------------------------------------------------
C
C COMPUTATION OF HALF BAND WIDTH
C
C...------------------------------------------------------------------
C
3100 NHBM=0
DO 3110 N=1,NEBM
DO 3110 I=1,NPE
DO 3110 J=1,NPE
N=((ABS(NOMOD(N,I)-MOD(N,J)))+1)*NDF
3110 IF (NHBM.LT.NHBM)NHBM=N
WRITE(7,3400)NHBM,NHBW
3400 FORMAT(' ',HALF BAND WIDTH=',12',' FULL BAND WIDTH=',13)
NEG=NHBM*NDF
DO 288 I=1,NEQ
GF0(I)=0.00
GF1(I)=0.00
GF2(I)=0.00
288 UI(I)=SEED
DO 3458 I=1,NBDY
UIBDY(I)=VBDY(I)
3458 VBDY(I)=0.00
READ5,*ITEMP
C
C BEGIN TIME DO LOOP
C
C...------------------------------------------------------------------
C
C TIME=DELT
IF(ITIM.EQ.0)GOTO 179
C NT1=NT+1
DO 1999 NT1=1,NT
IF(ITIM.GT.80)GOTO 1001
C
C TEMPERATURE INDUCED FORCES
C
C...------------------------------------------------------------------
C
179 DO 5820 ILOAD=1,NLOAD
Q=QINIT+PDDEL*(ILOAD-1)
IF(ITIM.EQ.11)THEN

96
"NNN=NL+1

C READ(5,*)((TEM(I,J),I=1,NNN),J=1,NNN)
C DO 9333 I=1,NNN
9333 READ(5,*)((TEM(I,J),J=1,NNH),I=1,NNN)
C DO 9444 J=1,NNM
C9444 WRITE(7,8368)((TEM(I,J),I=1,NNN),J=1,NNH)
C8368 FORMAT( ,2(13.6,3X))
C DO 39 J=1,NNH
C39 CONTINUE
C ELSE
DO 1139 J=1,NNM
C1139 TNX(J)=0.00
TNY(J)=0.00
TNX(J)=0.00
TMX(J)=0.00
TMY(J)=0.00
DO 39 I=1,NNL
II=I+1
C TEMT1=TEM(I,J)+(ZBI(I)/2.00*(TEM(I,J)+TEM(I,J))-TEO)
Q TEMT2=TEM(I,J)+(ZBI(I)/2.00*(TEM(I,J)-TEO))
A TEMT1=TEM(I,J)+(ZBI(I)/2.00*(TEM(I,J)-TEO))
B TEMT2=TEM(I,J)+(ZBI(I)/2.00*(TEM(I,J)-TEO))
C TNX(J)=QB(I,1)*ALPX(I)+QB(I,2)*ALPX(I)+QB(I,3)*ALPX(I)+QB(I,4)*ALPX(I)+QB(I,5)*ALPX(I)+QB(I,6)*ALPX(I)+QB(I,7)*ALPX(I)
D TMX(J)=QB(I,1)*ALPX(I)+QB(I,2)*ALPX(I)+QB(I,3)*ALPX(I)+QB(I,4)*ALPX(I)+QB(I,5)*ALPX(I)+QB(I,6)*ALPX(I)+QB(I,7)*ALPX(I)
E TNY(J)=QB(I,1)*ALPX(I)+QB(I,2)*ALPX(I)+QB(I,3)*ALPX(I)+QB(I,4)*ALPX(I)+QB(I,5)*ALPX(I)+QB(I,6)*ALPX(I)+QB(I,7)*ALPX(I)
F TMY(J)=QB(I,1)*ALPX(I)+QB(I,2)*ALPX(I)+QB(I,3)*ALPX(I)+QB(I,4)*ALPX(I)+QB(I,5)*ALPX(I)+QB(I,6)*ALPX(I)+QB(I,7)*ALPX(I)
G +TNX(J)
H +TMX(J)
I +TNY(J)
J +TMY(J)
K CONTINUE
ELSE
DO 1139 J=1,NNM
C1139 TNX(J)=0.00
TNY(J)=0.00
TNX(J)=0.00
TMX(J)=0.00
TMY(J)=0.00
DO 39 I=1,NNL
II=I+1
"
DO 90 I=1,NPE
TTNX(I)=TNX(NOD(N,I))
TTNY(I)=TNY(NOD(N,I))
TTNX(I)=TNY(NOD(N,I))
TTNY(I)=TNX(NOD(N,I))
90 ELXY(I,1)=X(NOD(N,I))
CALL STIFFIELEM,NN,NPE,NDF,ElXY,QJ

ASSEMBLE ELEMENT STIFFNESS MATRICES TO GET GLOBAL STIFFNESS
DO 280 I=1,NPE
NR=(NOD(N,I)-1)*NDF
DO 280 II=1,NDF
NR=NR+1
L=(I-1)*NDF+II
GR(NR)=GR(NR)+R(L)
280 CONTINUE
100 CONTINUE

C
C ASSEMBLE MATRIX EQUATIONS TO GET GLOBAL STIFFNESS
C
C-----------------------------------------------------------------------
C ASSEMBLED MATRIX EQUATIONS ARE NOW READY FOR IMPLEMENTATION OF
C THE BOUNDARY CONDITIONS ON PRIMARY AND SECONDARY VARIABLES
C-----------------------------------------------------------------------
C
110 CALL BNDY(NRMAX,NCMAX,NEQ,NHBF,GSTIF,GR,NBDY,IBDY,VBDY)
IRES=0
CALL SOLVE (NRMAX,NCMAX,NEQ,NHBF,GSTIF,GR,IRES)
C-----------------------------------------------------------------------
C CALL SUBROUTINE 'SOLVE' TO SOLVE THE SYSTEM OF EQUATIONS FOR THE
C PRIMARY DEGREES OF FREEDOM (THE SOLUTION IS RETURNED IN ARRAY GR
C-----------------------------------------------------------------------
C
C LEAST-SQUARES ERROR MEASUREMENT
C
ERR1=0.DO
ERR2=0.0
DO 43 I=1,NEQ
WRITE(7,768)I,GR(I),ERR1
C768 FORMAT(' ',I3,2(013.6,2X))
ERR1=ERR1+GR(I)**2
43 ERR2=ERR2+GR(I)**2
DO 44 I=1,NEQ
UII=UI(I)+GR(I)
44 IF(44.EQ.0)GOTO 1000
ERROR=SQRT(ERR1/ERR2)
DO 44 I=1,NEQ
UII=UI(I)+GR(I)
44 IF(44.LT.TOLER)GOTO 1000
999 CONTINUE
WRITE(7,998)
998 FORMAT(' ', 'SORY, IT DID NOT CONVERGE!')
1000 WRITE(7,999)
WRITE(7,612)
WRITE(7,6399)
WRITE(7,994)I,TIME
994 FORMAT(' ', 'FOR Q=',F12.3,5X,'AT TIME=',F12.9)
WRITE(7,993)IITER
C

933 FORMAT( 'IT CONVERGED AFTER ', I2, ' ITERATIONS' ) 

WRITE(7,6458) 

612 FORMAT( 'SOLUTION' ) 

6458 FORMAT( ',NODE',5X,'U',11X,'V',11X,'W',10X,'PHIX',9X,'PHIV') 

DO 3349 I=1,NN 

II=I+NDF-I 

3349 WRITE(7,610)I,(UIJ),J=II,III 

610 FORMAT( 5(D15.4,2X)) 

5820 CONTINUE 

IF (ITIM.EQ.0) GOTO 1001 

TIME=TIME+DELT 

DO 180 I=1,NEQ 

GF0(I)=SAO*(UIJ-GF0(I)) 

GF0(I)=GF0(I)+SA3*GF2(I) 

180 GF0(I)=U(I) 

1999 CONTINUE 

1001 CONTINUE 

STOP 

END 

SUBROUTINE STIFFCELL(NEQ,NN,NDF,ELXY,Q) 

IMPLICIT REAL*8(A-H,O-Z) 

COMMON/STF1/ELKT(45,45),R(45),A(6),B(6),D(6),AS(3) 

COMMON/STF2/SAO,SAl,SAZ,SA3,SA4,XIN1,XIN2 

COMMON/STF3/ELXX(45),ELXY(45),ELXZ(45) 

COMMON/SHP/SGSF(2,9),SGF(9) 

DIMENSION GAUS(4,4),MT(4,4),ELXY(9,2),TU(45),ELF(9,1) 

DIMENSION ELF(9,1),ELF(9,1),TU(45),TU(45),TU(45) 

DIMENSION TK1(9,9),TK2(9,9),TK3(9,9),TK4(9,9),TK5(9,9) 

DIMENSION TK1(9,9),TK2(9,9),TK3(9,9),TK4(9,9),TK5(9,9) 

DIMENSION TK1(9,9),TK2(9,9),TK3(9,9),TK4(9,9),TK5(9,9) 

DIMENSION TK1(9,9),TK2(9,9),TK3(9,9),TK4(9,9),TK5(9,9) 

DIMENSION TK1(9,9),TK2(9,9),TK3(9,9),TK4(9,9),TK5(9,9) 

DATA GAUS/4*0.0000,-.577350270,-.77459667D0,.33998104D0, 

DATA MT/4*0.0000,2*1.0000,2*0.0000,55555555D0,88888888D0, 

DATA NN/55555555D0,0.0000,3478548500,2*6521451500,3478548500/ 

NGP=IELEM 

C REARRANGE U 

NC1=1 

NC2=2 

NC3=3 

NC4=4 

NC5=5 

DO 892 I=1,NPE 

I2=I+NPE 

I4=I+4*NPE 

TU(I)=ELU(I) 

TU(I2)=ELU(I2) 

TU(I4)=ELU(I4) 

TU(I5)=ELU(I5)
$\text{TU2(I)} = \text{ELU2(NC1)}$

$\text{TU2(I2)} = \text{ELU2(NC2)}$

$\text{TU2(I3)} = \text{ELU2(NC3)}$

$\text{TU2(I4)} = \text{ELU2(NC4)}$

$\text{TU2(I5)} = \text{ELU2(NC5)}$

\begin{align*}
\text{NC1} & = \text{NC1} + \text{NOF} \\
\text{NC2} & = \text{NC2} + \text{NOF} \\
\text{NC3} & = \text{NC3} + \text{NOF} \\
\text{NC4} & = \text{NC4} + \text{NOF} \\
\text{NC5} & = \text{NC5} + \text{NOF}
\end{align*}

\text{INITIALIZE MATRICES}

\begin{align*}
\text{DO 415 } I &= 1, \text{NRT} \\
R(I) &= 0.00 \\
\text{DO 415 } J &= 1, \text{NRT} \\
\text{ELKT(I,J)} &= 0.00
\end{align*}

\begin{align*}
\text{DO 416 } I &= 1, \text{NPE} \\
\text{ELFI(I)} &= 0.00 \\
\text{ELF2(I)} &= 0.00 \\
\text{ELF3(I)} &= 0.00 \\
\text{ELF4(I)} &= 0.00 \\
\text{ELFS(I)} &= 0.00 \\
\text{DO 416 } J &= 1, \text{NPE} \\
\text{GHA(I,J)} &= 0.00 \\
\text{TK1U(I,J)} &= 0.00 \\
\text{TK12(I,J)} &= 0.00 \\
\text{TK13(I,J)} &= 0.00 \\
\text{TK14(I,J)} &= 0.00 \\
\text{TK15(I,J)} &= 0.00 \\
\text{TK2U(I,J)} &= 0.00 \\
\text{TK22(I,J)} &= 0.00 \\
\text{TK23(I,J)} &= 0.00 \\
\text{TK24(I,J)} &= 0.00 \\
\text{TK25(I,J)} &= 0.00 \\
\text{TK31(I,J)} &= 0.00 \\
\text{TK32(I,J)} &= 0.00 \\
\text{TK33(I,J)} &= 0.00 \\
\text{TK34(I,J)} &= 0.00 \\
\text{TK35(I,J)} &= 0.00 \\
\text{TK41(I,J)} &= 0.00 \\
\text{TK42(I,J)} &= 0.00 \\
\text{TK43(I,J)} &= 0.00 \\
\text{TK44(I,J)} &= 0.00 \\
\text{TK45(I,J)} &= 0.00 \\
\text{TK51(I,J)} &= 0.00 \\
\text{TK52(I,J)} &= 0.00 \\
\text{TK53(I,J)} &= 0.00 \\
\text{TK54(I,J)} &= 0.00 \\
\text{TK55(I,J)} &= 0.00 \\
\text{XX33(I,J)} &= 0.00 \\
\text{GI3(I,J)} &= 0.00 \\
\text{GI4(I,J)} &= 0.00 \\
\text{GI5(I,J)} &= 0.00 \\
\text{GI6(I,J)} &= 0.00 \\
\text{GI7(I,J)} &= 0.00 \\
\text{GI8(I,J)} &= 0.00 \\
\text{GI9(I,J)} &= 0.00
\end{align*}

\begin{align*}
\text{416} \\
\text{THIS LOOP REPRESENTS THE SUMMATION OF THE GAUSS POINTS} \\
\text{CONTRIBUTION TO KT & Q.}
\end{align*}

\begin{align*}
\text{DO 400 } IGP &= 1, \text{NGP} \\
\text{XI} &= \text{GAUSS(IGP,NGP)} \\
\text{DO 400 } JGP &= 1, \text{NGP} \\
\text{ETA} &= \text{GAUSS(JGP,NGP)}
\end{align*}
"GAUSS" contains the Gauss points in an array with each column containing the xi for 1, 2, 3, and 4 Gauss points respectively.

CALL SHAPE(XI, ETA, NPE, ELXY,IELM,DET)

"SHAPE" calculates the Jacobian (G.J), the local interpolation functions (SF), and their derivatives (DSF), and puts the local derivatives into global terms.

GDSF=DSF/G.J

DO 150 I=I,NPE
II=I+NPE
III=II+NPE
14=III+NPE

TNX=TNX+TTNX(I)*SF(I)
TNY=TNY+TTNY(I)*SF(I)
TNXY=TNXY+TTNXY(I)*SF(I)
TMX=TMX+TTMX(I)*SF(I)
THY=THY+TTHY(I)*SF(I)
T11XY=T11XY+TT11XY(I)*SF(I)

WRITE(7,615)TTNX(I),TNX,TTNY(I),TNY
WRITE(7,616)TTMX(I),TMX,TTHY(I),THY
WRITE(7,617)TT11XY(I),T11XY,TTNXY(I),TNXY

DFXX=DFXX+GDSF(I,I)*TU(I)
DFXY=DFXY+GDSF(I,2)*TU(I)
DFYY=DFYY+GDSF(2,2)*TU(I)

DFNX=DFNX+GDSF(I,I)*TU(II)
DFNY=DFNY+GDSF(I,2)*TU(II)

DFNX=DFNX+DFXY+GDSF(1,1)*TU(III)
DFNY=DFNY+DFXY+GDSF(1,2)*TU(III)

DNX=DNX+GDSF(1,1)*TU(IV)
DNY=DNY+GDSF(1,2)*TU(IV)

DVX=DVX+GDSF(2,1)*TU(IV)
DVY=DVY+GDSF(2,2)*TU(IV)

DX=DX+GDSF(1,1)*TU(V)
DY=DY+GDSF(1,2)*TU(V)

150 DUX=DUX+GDSF(2,1)*TU(V)

S=TNX*DNX+TNXY*DNY
S=TNXY*DNX+TNY*DNY

CON01=A(1)*HT(IGP,NGP)*HT(JGP,NGP)*DET
CON02=A(2)*HT(IGP,NGP)*HT(JGP,NGP)*DET
CON03=A(3)*HT(IGP,NGP)*HT(JGP,NGP)*DET
CON04=A(4)*HT(IGP,NGP)*HT(JGP,NGP)*DET
CON05=A(5)*HT(IGP,NGP)*HT(JGP,NGP)*DET
CON06=A(6)*HT(IGP,NGP)*HT(JGP,NGP)*DET
& DXK=DXK+(XJY+XJY)*CONA22*YY*(DVY+DVY*DVY) COM07630
& DX3I=DX3I*(DVXY+DVX+DVX)*CONB11*XX*DVY+DVX*DVX*DVX COM07640
& XJY=DFX*XX*CONB16*DFX*DFY*XJY COM07650
& CONB12=YF*DFXY+DFXY+DFXY+DFXY+DFXY+DFXY*XJY COM07660
& CONB12=YY*DFY COM07670
& CONGDF(1,J)*TNK*GDSF(1,I)+TNK*GDSF(2,J) COM07680
& CONGDF(2,J)*TNK*GDSF(1,I)+TNK*GDSF(2,J) COM07690
159 CONTINUE COM07700
ELF1(I)=ELF1(I)+(TNK*GDSF(1,I)+TNK*GDSF(2,I))*CONST COM07710
ELF2(I)=ELF2(I)+(TNK*GDSF(1,I)+TNK*GDSF(2,I))*CONST COM07720
ELF3(I)=ELF3(I)+(TNK*GDSF(1,I)+TNK*GDSF(2,I))*CONST COM07730
ELF4(I)=ELF4(I)+(TNK*GDSF(1,I)+TNK*GDSF(2,I))*CONST COM07740
ELF5(I)=ELF5(I)+(GDSF(1,I)+GDSF(2,I))*CONST COM07750
160 CONTINUE COM07760
400 CONTINUE COM07770
BEGIN REDUCED INTERGRATION COM07780
C COM07790
C BEGIN REDUCED INTERGRATION COM07800
C NGP=NGP-1 COM07810
DO 401 IGP=1,NGP COM07820
ETA=GAUSS(IGP,NGP) COM07830
CALL SHAPE(XI,ETA,NPE,ELXY,IELEH,DET) COM07840
CONA44=AS(1)*HT(I,NGP)*HT(I,NGP)*DET COM07850
CONA45=AS(2)*HT(I,NGP)*HT(I,NGP)*DET COM07860
CONA55=AS(3)*HT(I,NGP)*HT(I,NGP)*DET COM07870
DO 402 I=1,NPE COM07880
DO 402 J=1,NPE COM07890
XX=GDSF(1,J)*GOSF(1,I) COM07900
YY=GOSF(2,J)*GDSF(2,I) COM07910
G33(I,J)=G33(I,J)+CONA55*XX+CONA44*YY+CONA45*(GOSF(2,I)*GOSF(1,J)*GOSF(2,J)) COM07920
G34(I,J)=G34(I,J)+CONA55*YY+CONA45*XX+CONA44*Y*GDSF(1,J) COM07930
G35(I,J)=G35(I,J)+CONA45*XX+CONA55*GDSF(1,J)*GOSF(2,I) COM07940
G43(I,J)=G43(I,J)+CONA45*YY+CONA55*GDSF(1,I)*GOSF(2,J) COM07950
G44(I,J)=G44(I,J)+CONA45*YY+CONA55*GDSF(1,J)*GOSF(2,J) COM07960
G45(I,J)=G45(I,J)+CONA55*YY+CONA45*GDSF(1,J)*GOSF(2,J) COM07970
G53(I,J)=G53(I,J)+CONA45*YY+CONA55*GDSF(1,J)*GOSF(2,J) COM07980
G54(I,J)=G54(I,J)+CONA45*XX+CONA55*GDSF(1,J)*GOSF(2,J) COM07990
G55(I,J)=G55(I,J)+CONA45*YY+CONA55*GDSF(1,J)*GOSF(2,J) COM08000
402 CONTINUE COM08010
ASSEMBLE ELEMENT KT & R MATRICES COM08020
ASSEMBLE ELEMENT KT & R MATRICES COM08030
II=1 COM08040
DO 170 I=1,NPE COM08050
J=I+1,NPE COM08060
II=II+1 COM08070
DO 170 J=1,NPE COM08080
J2=J+2,NPE COM08090
J3=J+3,NPE COM08100
J4=J+4,NPE COM08110
J5=J+5,NPE COM08120
GM1=GMASS(I,J)*XIN1*SAO COM08130
GM2=GMASS(I,J)*XIN2*SAO COM08140
ELKT(I,J)=TK11(I,J)+GM1 COM08150
ELKT(I,J)=TK12(I,J)+GM2 COM08160
ELKT(I,J)=TK13(I,J)+GM1 COM08170
ELKT(I,J)=TK14(I,J)+GM2 COM08180
ELKT(I,J)=TK15(I,J)+GM1 COM08190
ELKT(I,J)=TK16(I,J)+GM2 COM08200
170 CONTINUE COM08210
ASSEMBLE ELEMENT KT & R MATRICES COM08220
ASSEMBLE ELEMENT KT & R MATRICES COM08230
II=1 COM08240
DO 180 J=1,NPE COM08250
J2=J+1,NPE COM08260
J3=J+2,NPE COM08270
J4=J+3,NPE COM08280
JM=J+M,NPE COM08290
GM1=GMASS(I,J)*XIN1*SAO COM08300
GM2=GMASS(I,J)*XIN2*SAO COM08310
ELKT(I,J)=TK11(I,J)+GM1 COM08320
ELKT(I,J)=TK12(I,J)+GM2 COM08330
ELKT(I,J)=TK13(I,J)+GM1 COM08340
ELKT(I,J)=TK14(I,J)+GM2 COM08350
ELKT(I,J)=TK15(I,J)+GM1 COM08360
ELKT(I,J)=TK16(I,J)+GM2 COM08370
180 CONTINUE COM08380
ASSEMBLE ELEMENT KT & R MATRICES COM08390
ASSEMBLE ELEMENT KT & R MATRICES COM08400
II=1 COM08410
DO 190 J=1,NPE COM08420
J2=J+1,NPE COM08430
J3=J+2,NPE COM08440
J4=J+3,NPE COM08450
J5=J+4,NPE COM08460
GM1=GMASS(I,J)*XIN1*SAO COM08470
GM2=GMASS(I,J)*XIN2*SAO COM08480
ELKT(I,J)=TK11(I,J)+GM1 COM08490
ELKT(I,J)=TK12(I,J)+GM2 COM08500
ELKT(I,J)=TK13(I,J)+GM1 COM08510
ELKT(I,J)=TK14(I,J)+GM2 COM08520
ELKT(I,J)=TK15(I,J)+GM1 COM08530
ELKT(I,J)=TK16(I,J)+GM2 COM08540
190 CONTINUE COM08550
ASSEMBLE ELEMENT KT & R MATRICES COM08560
ASSEMBLE ELEMENT KT & R MATRICES COM08570
II=1 COM08580
DO 200 J=1,NPE COM08590
J2=J+1,NPE COM08600
J3=J+2,NPE COM08610
J4=J+3,NPE COM08620
J5=J+4,NPE COM08630
GM1=GMASS(I,J)*XIN1*SAO COM08640
GM2=GMASS(I,J)*XIN2*SAO COM08650
ELKT(I,J)=TK11(I,J)+GM1 COM08660
ELKT(I,J)=TK12(I,J)+GM2 COM08670
ELKT(I,J)=TK13(I,J)+GM1 COM08680
ELKT(I,J)=TK14(I,J)+GM2 COM08690
ELKT(I,J)=TK15(I,J)+GM1 COM08700
ELKT(I,J)=TK16(I,J)+GM2 COM08710
200 CONTINUE COM08720
ASSEMBLE ELEMENT KT & R MATRICES COM08730
ASSEMBLE ELEMENT KT & R MATRICES COM08740
SUBROUTINE SHAPE(XI, ETA, NPE, ELXY, IELEH, OET)
IMPLICIT REAL*8(A-H,O-Z)

DIMENSION DSF(2,*), SF(*), GJ(2,2), GJINV(2,2), XSIGN(9,2), NP(*)
DATA XSIGN/1.0DO, 2*1.0DO, -1.0DO, 0.0DO, 1.0DO, 2*0.0DO, -1.0DO, 0.0DO, 1.0DO, 2*0.0DO/
DATA NP/1, 2, 3, 4, 5, 6, 7, 8, 9/
IF(NPE.EQ.9)GOTO 500
IF(NPE.EQ.5)GOTO 100
LINEAR INTERPOLATION FUNCTIONS
DO 410 I=1,4
XP=XSIGN(I,1)
VP=XSIGN(I,2)
XLO=I.0DO+XI*XP
ETAO=l.DO+ETA*VP
SFI(I)=.ZS00*XLO*ETAO
OSF(I,I)=.2S00*XP*ETAO
OSF(2,I)=.2S00*VP*XLO
410 CONTINUE
IF(NPE.EQ.5)GOTO 600
IF(NPE.EQ.9)GOTO 500
LINEAR INTERPOLATION FUNCTIONS
DO 410 I=1,4
XP=XSIGN(I,1)
VP=XSIGN(I,2)
XLO=I.0DO+XI*XP
ETAO=l.DO+ETA*VP
SFI(I)=.ZS00*XLO*ETAO
OSF(I,I)=.2S00*XP*ETAO
OSF(2,I)=.2S00*VP*XLO
410 CONTINUE
RETURN
END

SUBROUTINE SHAPE(XI, ETA, NPE, ELXY, IELEH, OET)
IMPLICIT REAL*8(A-H,O-Z)
COMMON/SHP/GDSF(2,*), SF(*)
DIMENSION DSF(2,*), SF(*), GJ(2,2), GJINV(2,2), XSIGN(9,2), NP(*)
DATA XSIGN/1.0DO, 2*1.0DO, -1.0DO, 0.0DO, 1.0DO, 2*0.0DO, -1.0DO, 0.0DO, 1.0DO, 2*0.0DO/
DATA NP/1, 2, 3, 4, 5, 6, 7, 8, 9/
IF(NPE.EQ.9)GOTO 500
IF(NPE.EQ.5)GOTO 100
LINEAR INTERPOLATION FUNCTIONS
DO 410 I=1,4
XP=XSIGN(I,1)
VP=XSIGN(I,2)
XLO=I.0DO+XI*XP
ETAO=l.DO+ETA*VP
SFI(I)=.ZS00*XLO*ETAO
OSF(I,I)=.2S00*XP*ETAO
OSF(2,I)=.2S00*VP*XLO
410 CONTINUE
IF(NPE.EQ.5)GOTO 600
IF(NPE.EQ.9)GOTO 500
LINEAR INTERPOLATION FUNCTIONS
DO 410 I=1,4
XP=XSIGN(I,1)
VP=XSIGN(I,2)
XLO=I.0DO+XI*XP
ETAO=l.DO+ETA*VP
SFI(I)=.ZS00*XLO*ETAO
OSF(I,I)=.2S00*XP*ETAO
OSF(2,I)=.2S00*VP*XLO
410 CONTINUE
RETURN
END

SUBROUTINE SHAPE(XI, ETA, NPE, ELXY, IELEH, OET)
IMPLICIT REAL*8(A-H,O-Z)
COMMON/SHP/GDSF(2,*), SF(*)
DIMENSION DSF(2,*), SF(*), GJ(2,2), GJINV(2,2), XSIGN(9,2), NP(*)
DATA XSIGN/1.0DO, 2*1.0DO, -1.0DO, 0.0DO, 1.0DO, 2*0.0DO, -1.0DO, 0.0DO, 1.0DO, 2*0.0DO/
DATA NP/1, 2, 3, 4, 5, 6, 7, 8, 9/
IF(NPE.EQ.9)GOTO 500
IF(NPE.EQ.5)GOTO 100
LINEAR INTERPOLATION FUNCTIONS
DO 410 I=1,4
XP=XSIGN(I,1)
VP=XSIGN(I,2)
XLO=I.0DO+XI*XP
ETAO=l.DO+ETA*VP
SFI(I)=.ZS00*XLO*ETAO
OSF(I,I)=.2S00*XP*ETAO
OSF(2,I)=.2S00*VP*XLO
410 CONTINUE
IF(NPE.EQ.5)GOTO 600
IF(NPE.EQ.9)GOTO 500
LINEAR INTERPOLATION FUNCTIONS
DO 410 I=1,4
XP=XSIGN(I,1)
VP=XSIGN(I,2)
XLO=I.0DO+XI*XP
ETAO=l.DO+ETA*VP
SFI(I)=.ZS00*XLO*ETAO
OSF(I,I)=.2S00*XP*ETAO
OSF(2,I)=.2S00*VP*XLO
410 CONTINUE
RETURN
END

SUBROUTINE SHAPE(XI, ETA, NPE, ELXY, IELEH, OET)
IMPLICIT REAL*8(A-H,O-Z)
COMMON/SHP/GDSF(2,*), SF(*)
DIMENSION DSF(2,*), SF(*), GJ(2,2), GJINV(2,2), XSIGN(9,2), NP(*)
DATA XSIGN/1.0DO, 2*1.0DO, -1.0DO, 0.0DO, 1.0DO, 2*0.0DO, -1.0DO, 0.0DO, 1.0DO, 2*0.0DO/
DATA NP/1, 2, 3, 4, 5, 6, 7, 8, 9/
IF(NPE.EQ.9)GOTO 500
IF(NPE.EQ.5)GOTO 100
LINEAR INTERPOLATION FUNCTIONS
DO 410 I=1,4
XP=XSIGN(I,1)
VP=XSIGN(I,2)
XLO=I.0DO+XI*XP
ETAO=l.DO+ETA*VP
SFI(I)=.ZS00*XLO*ETAO
OSF(I,I)=.2S00*XP*ETAO
OSF(2,I)=.2S00*VP*XLO
410 CONTINUE
IF(NPE.EQ.5)GOTO 600
IF(NPE.EQ.9)GOTO 500
LINEAR INTERPOLATION FUNCTIONS
DO 410 I=1,4
XP=XSIGN(I,1)
VP=XSIGN(I,2)
XLO=I.0DO+XI*XP
ETAO=l.DO+ETA*VP
SFI(I)=.ZS00*XLO*ETAO
OSF(I,I)=.2S00*XP*ETAO
OSF(2,I)=.2S00*VP*XLO
410 CONTINUE
RETURN
END
GOTO 100

C QUADRATIC INTERPOLATION FUNCTIONS (EIGHT-NODE ELEMENTS)

800 DO 810 I=1,8
NI=NP(I)
XP=XSIGN(NI,1)
YP=XSIGN(NI,2)
XI0=1.00+XI*XP
ETAO=1.00+ETA*YP
XI1=1.00-XI*XI
ETA1=1.00-ETA*ETA
IF(I.GT.4)GOTO 8Z0
SF(NI)=.2500*XI0*ETAO*(XI*XP+ETA*YP-1.00)
DSF(1,NI)=.2500*XP*ETAO*(1.DO+Z.DO*XI)*XI
DSF(2,NI)=.2500*YP*XI0*(1.DO+Z.DO*ETA)*ETA
GOTO 810
8Z0 IF(I.GT.6)GOTO 830
SF(NI)=.5000*XI0*ETAO
DSF(1,NI)=-XI*ETAO
DSF(2,NI)=.5000*XP*XI1
GOTO 810
830 CONTINUE
GOTO 100

C QUADRATIC INTERPOLATION FUNCTIONS (NINE-NODE ELEMENTS)

900 DO 910 I=1,9
NI=NP(I)
XP=XSIGN(NI,1)
YP=XSIGN(NI,2)
XI0=1.00+XI*XP
ETAO=1.00+ETA*YP
XI1=1.00-XI*XI
ETA1=1.00-ETA*ETA
XIZ=XP*XI
ETAZ=YP*ETA
IF(I.GT.4)GOTO 9Z0
SF(NI)=.5000*XI0*ETAO*XIZ*ETAZ
DSF(1,NI)=.2500*XP*ETAO*ETAZ*(1.DO+Z.DO*XIZ)
DSF(2,NI)=.2500*YP*XIO*XIZ*(1.DO+Z.DO*ETAZ)
GOTO 910
9Z0 IF(I.GT.6)GOTO 930
SF(NI)=.5000*XI0*ETAO
DSF(1,NI)=-XI*ETAO
DSF(2,NI)=.5000*YP*XI1
GOTO 910
930 IF(I.GT.8)GOTO 940
SF(NI)=.5000*ETA1*XIO
DSF(1,NI)=.5000*XP*ETA1
DSF(2,NI)=-ETA*XIO
GOTO 910
940 CONTINUE
GOTO 100

100 DO 560 I=1,2
DO 560 J=1,2
GJ(I,J)=0.00
DO 560 K=1,NPE

560 GJ(I,J)=GJ(I,J)+DSF(I,K)*WELXY(K,J)
DET=GJ(1,1)*GJ(2,2)-GJ(1,2)*GJ(2,1)
GJINV(1,1)=GJ(2,2)/DET
GJINV(1,2)=-GJ(1,2)/DET
GJINV(2,1)=GJ(1,1)/DET
GJINV(2,2)=GJ(1,1)/DET
DO 570 I=1,2
DO 570 J=1,NPE

105
GDSF(I,J) = 0.00
DO 570 K = 1, 2
570 GDSF(I,J) = GDSF(I,J) + GJINV(I,K) * DSF(K,J)
RETURN
END

SUBROUTINE BNDY(NRHAX, NCHAX, NEq, NHBH, S, SL, NBDY, IBDY, VBDY)

This program imposes the prescribed boundary conditions on the banded symmetric matrix and modifies the right-hand-side (SL). S is the coefficient matrix (stiffness matrix), SL is the right-hand-side column (force) vector.

IMPLICIT REAL*8(A-H, O-Z)
DIMENSION SINRHAX, NCHAX, SLINRHAX, IBDY(NBDY), VBDY(NBDY)
COMMON/IO/IN, ITT

WRITE(7, 980) S(1, 1), S(1, 2), S(1, 3), S(1, 4)
WRITE(7, 980) S(2, 1), S(2, 2), S(2, 3), S(2, 4)
WRITE(7, 980) S(3, 1), S(3, 2), S(3, 3), S(3, 4)
WRITE(7, 980) S(4, 1), S(4, 2), S(4, 3), S(4, 4)
WRITE(7, 980) S(5, 1), S(5, 2), S(5, 3), S(5, 4)
WRITE(7, 980) S(6, 1), S(6, 2), S(6, 3), S(6, 4)
WRITE(7, 980) S(7, 1), S(7, 2), S(7, 3), S(7, 4)

C FORMAT( ' ,4(D13.6,ZX))
980 FORMAT( ' ,4(D13.6,ZX))

DO 30 NB = 1, NBDY
IE = IBDY(NB)
SVAL = VBDY(NB)
IT = NHBH - 1
I = IE - IT
DO 10 II = 1, IT
I = I + 1
IF (I .LT. 1) GO TO 10
J = IE - I + 1
SL(I) = SL(I) - S(I, J) * SVAL
S(I, J) = 0.0
10 CONTINUE
SL(IE, 1) = 1.0
SL(IE) = SVAL
I = IE
DO 20 II = 2, NHBH
I = I + 1
IF (I .GT. NEq) GO TO 20
SL(I) = SL(I) - S(IE, II) * SVAL
S(IE, II) = 0.0
20 CONTINUE

30 CONTINUE
RETURN
END

SUBROUTINE SOLVE(NRHM, NCM, NEQNS, NBH, BAND, RHS, IRES)

This program solves a banded symmetric system of equations. The banded matrix is input through BAND(NEQNS,NBH), and RHS is the right hand-side (force) vector) of the equation. NEQNS is the no. of equations and NBH is the half band width. In resolving, IRES .GT. 0, LHS elimination is skipped.

IMPLICIT REAL*8(A-H, O-Z)
DIMENSION BAND(NRHM, NCM), RHS(NRHM)
COMMON/IO/IN, ITT
MEGS = NEQNS - 1
IF (IRES .GT. 0) GO TO 40
DO 30 NPIV = 1, MEQS
NPIVOT = NPIV + 1
LSTSUB = NPIV + NBH - 1
IF (LSTSUB .GT. MEQS) LSTSUB = MEQS
DO 20 NROW = NPIVOT, LSTSUB
NCOL = NROW - NPIV + 1
20 CONTINUE
FACTOR = BAND(NPIV, NCOL) / BAND(NPIV, 1)
DO 10 NCOL = NROM, LSTSUB
   ICOL = NCOL - NROM + 1
   JCOL = NCOL - NPIV + 1
10       BAND(NROH, ICOL) = BAND(NROH, ICOL) - FACTOR * BAND(NPIV, JCOL)
            RHS(NROH) = RHS(NROH) - FACTOR * RHS(NPIV)
GO TO 70
40       DO 60 NPIV = 1, NEQNS
               NPIVOT = NPIV + 1
            LSTSUB = NPIV + NBH - 1
            IF (LSTSUB > NEQNS) LSTSUB = NEQNS
               DO 50 NROH = NPIVOT, LSTSUB
                     NCOL = NROH - NPIV + 1
                     FACTOR = BAND(NPIV, NCOL) / BAND(NPIV, 1)
                     RHS(NROH) = RHS(NROH) - FACTOR * RHS(NPIV)
50            CONTINUE
60       CONTINUE
C     BACK SUBSTITUTION
70       DO 90 IJK = 2, NEQNS
               NPIV = NEQNS - IJK + 2
            RHS(NPIV) = RHS(NPIV) / BAND(1, 1)
            LSTSUB = NPIV + NBH - 1
            IF (LSTSUB > LT. 1) LSTSUB = NEQNS
               NPIVOT = NPIV - 1
               DO 80 JKI = LSTSUB, NPIVOT
                     NROH = NPIVOT - JKI + LSTSUB
                     NCOL = NPIV - NROH + 1
                     FACTOR = BAND(NROH, NCOL) / BAND(NPIV, 1)
                     RHS(NROH) = RHS(NROH) - FACTOR * RHS(NPIV)
80            CONTINUE
90       CONTINUE
C     SUBROUTINE MESH(IELEM, NX, NY, NEP, X, Y, XM, YM, DX, DY)
      IMPlicit REAL*8(A-H,O-Z)
      COMMON/MEX/NOD(1, NPE)
      DIMENSION X(9), Y(9), DX(5), DY(5)
      IELEM = IEL - 1
C     QUADRILATERAL ELEMENTS
      NEX = NX + 1
      NEY = NY + 1
      NOX = NX * IEL
      NYX = NY * IEL
      NDX = NOX + 1
      NDX = NOX + 1
      XM = XM
      YX = YX
      IF (NPE.EQ.9) XR = RX + 1
      KO = 0
      IF (NPE.EQ.9) KO = 1
      NODI(1, 1) = 1
      NODI(1, 2) = IELEM + 1
      NODI(1, 3) = NOX + IELEM - 1 + NEX + IEL + 1
      IF (NPE.EQ.9) NODI(1, 3) = 4 * NX + 5
      NODI(1, 4) = NODI(1, 3) - IEL
      IF (NPE.EQ.9) GOTO 100
      NODI(1, 5) = 2
      NODI(1, 6) = NOX(IPE - 6)
      NODI(1, 7) = NODI(1, 3) - 1
      NODI(1, 8) = NOX + 1
      IF (NPE.EQ.9) NODI(1, 9) = NOX + 2
200     IF (NX.EQ.1) GOTO 230
      M = 1
      DO 220 N = 2, NY
            L = (N - 1) * NX + 1
      210     NODI(L, I) = NODI(M, I) + NOX(IELEM - 1) * NEX + KO * NX
      220     M = L
      230     IF (NX.EQ.1) GOTO 270
      DO 260 NI = 2, NX
      DO 240 I = 1, NPE
K1=IEL
IF (I.EQ.6.OR.I.EQ.8) K1=1+K0
240 NOD(NI,I)=NOD(NI-1,I)+K1
M=NI
DO 260 NJ=2,NI
L=(NI-1)*NX+NI
DO 250 J=1,NPE
250 NOD(N,J)=NOD(M,J)+NXX1+(IEL-1)*NEX1+K0*NX
260 M=L
270 YC=0.0
    IF (NPE.EQ.9) GOTO 310
    DO 300 NI=1,NY1
      I=(NXX1+(IEL-1)*NEX1+NI-1)+1
      J=(NI-1)*IE1+1
      X(I)=0.0
      Y(I)=YC
      DO 280 NJ=1,NI
        I=I+1
        X(I)=X(I-1)+DX(NI)
        Y(I)=YC
        IF (NI.GT.NY.OR.I.EQ.1) GOTO 300
        J=J+1
        YC=YC+DY(J-1)
    280 Y(I)=YC
    IF (NI.GT.NY.OR.I.EQ.1) GOTO 300
    J=J+1
    YC=YC+DY(J)
    RETURN
300 YC=YC+DY(J)
RETURN
310 N=NI+1
    DO 320 NJ=1,NYY1
      I=NXX1*(NI-1)
      X(I)=0.0
      DO 320 NJ=1,NYY1
        I=I+1
        X(I)=X(I)+DX(K)+DX(K+1)
      320 Y(I)=YC
      RETURN
330 YC=YC+DY(NY)
RETURN
END
Vita

Thomas Harris Fronk was born February 17, 1961 in Tremonton, Utah, the fifth of six children. He was educated in Tremonton until his graduation from Bear River High School in June 1979. He served a lay mission in South America for the Church of Jesus Christ of Latter-Day Saints from 1980 to 1982. He received his B.S. degree in mechanical engineering from Utah State University in June 1985 while working for Lockheed Missile and Space Co. his senior year. From June 1985 to September 1986 he was employed in the composite structures design group at Morton Thiokol Inc.. He began his graduate studies at Virginia Polytechnic and State University in September 1986. He married Monica Layne Smith in June 1984 and they are the parents of two boys ages four months and two and one-half years.