CONDITIONAL DISTRIBUTIONS ARISING FROM VARIATION OF PARAMETERS IN NON-LINEAR RESPONSE FUNCTIONS

by

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I INTRODUCTION

Many research workers in the field of biology have analyzed data on growth, e.g., weight as a function of time, under the assumption that a whole group of similar individual organisms can be represented by one mathematical model having constant parameters and an additive random residual term which accounts for all deviations from the model. Almost all of the methods of analysis and the curve fitting procedures that have been devised have the same basic underlying assumption, that this residual term has a zero mean and a constant variance throughout the duration of the experiment.

In this paper we would like to propose the idea that the growth of an individual organism follows a given mathematical model very closely and that different individuals follow different members of the same parametric family of models. This idea implies that the variation between individuals observed at the same time arises primarily from the variation of the parameters in the model.

It is the purpose of this paper to investigate the consequences of random variation of parameters on several models, the probability distributions to be either independent uniform distributions or independent gamma distributions. Included also is an example which seems to justify the philosophy of the approach considered.

II. REVIEW OF LITERATURE

Several typical examples of the extensive literature on growth curve methodology are given in this review of literature.

An example of curve fitting was given in 1930 by H. S. Will [7] who realized that use of the fitted curves could conveniently be divided into two main classes depending upon the intent of the investigator. He said that if the purpose of fitting is to interpolate for unobserved time points between two observed ordinates, virtually any method will suffice since the fitting of the observed ordinates can be made exact. If the object is to secure representation of all the observations of the series by means of a single fitted function, the method must be such that a close fit is obtained. He then proceeded to give a method, known as the method of differences, for estimating the parameters of an assumed model.

W. L. Stevens [6] in 1951 considered the model

$$y_i = \alpha + \beta\rho^{x_i} + \varepsilon_i$$

where

$$0 < \rho < 1$$

for problems where the dependent variable y approaches a limit asymptotically. He discussed the importance of this model showing several transformations whereby he could produce the Mitscherlich curve, the logistic curve, and similar functions. His proposed method of fitting was least squares in which the normal equations are non-linear in the parameter estimates. The Newton-Raphson iterative procedure was to be used to solve them. The computational technique was simplified by a table of inverses of the matrix of coefficients in the Newton-Raphson procedure which turns out to be a function only of r, the estimate of ρ, and the equally spaced x values.

H. O. Hartley (1948) [3] considered the difference equation

$$(y_{i+1} - y_i) = -\tfrac{1}{2}b(y_{i+1} + y_i) + a$$

the solution of which is

$$y_i = \hat{y}(1 - fe^{kx_i})$$

where $\hat{y} = a/b$, the limiting response, and k, the exponential curvature, is given by $-k = 2\tanh^{-1}\tfrac{1}{2}b$. He then proceeded to estimate a and b from the difference equation by the methods of least squares. This technique avoids the non-linearity problems of the model considered.

Another approach avoiding non-linear estimation was proposed by G. E. P. Box [1] in 1950. As in Hartley's technique the differences of the response variable were obtained. These differences would then correspond to average growth rates during successive periods of time. He considered these successive periods as a classification variable "periods," the effect of any treatments on mean rate to be measured by the variation in the period averages, and the "shape" of the rate curve by the interaction of these treatments with "periods." The justification lies in the fact that in models such as those of Hartley and Stevens the taking of differences results in a very simple covariance pattern for the errors.

Rao [5] in 1958 proposed reducing a series of observations to a minimum number of quantities that would summarily contain the information provided by the data. He suggested that although the rate of growth is generally a complicated function of time, it is sometimes possible to make a transformation such that growth is uniform with respect to the chosen time metameter. An adequate representation is then often available in terms of the initial value and a redefined uniform rate.

Rao's method is to split up the transformed time axis into segments and let y_i be the increase in weight in the g_i th time period (transformed), then form

$$b = \frac{\Sigma y_i g_i}{\Sigma g_i^2}$$

which is an estimated rate of growth (average rate of growth).

Rao then suggested that, if the problem is one of comparing growth under different conditions, we need to test whether the mean value of b is the same in all groups by analysis of variance with respect to the single variable b, eliminating the initial value y_0 by analysis of covariance if necessary.

Leech and Healy [4] (1959) suggest a method of analysis of experiments on growth rate when experiments are of short duration or treatment effects are small. They assume under these conditions that the average difference between treated and control groups increases linearly with time. They proceed to fit orthogonal polynomials to the treated group and to the control group. Differences between these groups are represented by another polynomial with the coefficients being the differences of the respective coefficients of the fitted curves. Assuming all the coefficients of terms of higher degree than one to be zero, they have a linear equation with slope λ. They estimate this λ and make their comparisons, then proceed to generalize the method to include more than one treatment and higher degree polynomials.

Wishart [8] in 1938 analyzed the data on an experiment by Woodman et al (1936) on nutrition studies with the bacon pig. Here he classified each individual growth curve as to litter, sex, and treatment, and proceeded to fit a second degree polynomial in time by the least squares method. Thus the numerous observations were replaced by the coefficients of the linear and quadratic terms of the polynomial. From this point, the analysis consisted of comparing the means of these coefficients under differing experimental conditions. A large portion of the differences in growth curves was concentrated in the linear growth rate rather than the quadratic term, thus simplifying the analysis.

These papers typify the work in this field. It should be noted that most were concerned with fitting problems involving non-linear exponential models with an additive error term at least implicit in the development.

The Wishart paper was really the only one in which the idea of an individual being associated with a single curve from a parametric family of similar curves was brought out and then in a model which seems to have collapsed to a single linear coefficient.

III. MODELS AND THEIR MOMENTS

3.1. Description of Models.

A perusal of the literature indicates that most models assumed on a priori grounds for the description of growth as a function of time are non-linear and of exponential nature with finite asymptotes as t increases. Polynomial models are generally used principally to approximate the growth phenomenon over a short period of time, or for statistical convenience, or for a combination of these reasons.

The models considered here were chosen for biological interpretability and proper asymptotic behavior and this resulted in all of them being of exponential form. No attempt was made to exhaust a class, but, we feel, the more typical forms are represented.

The models considered subsequently are then:

1. The Two Parameter Model is

$$Y_i(t) = \beta_{1i}(1 - e^{-\gamma_{1i}t}),$$

where $Y_i(t)$ is the response (e.g., growth) of the i th individual at time t, β_{1i} is the final or asymptotic "size" to which $Y_i(t)$ tends as t increases indefinitely, and γ_{1i} is the "growth factor" of the i th individual. This model may be used as a terminal curve to fit the final stages in the growth pattern using a predetermined size as an origin (which is to be subtracted from the observations to obtain $Y_i(t)$).

2. <u>The Three Parameter Model (A)</u> is

$$Y_i(t) = \beta_{0i} + \beta_{1i}(1 - e^{-\gamma_{1i}t}) ,$$

where β_{0i} is the initial size of the i th individual, β_{1i} the final size, and γ_{1i} the growth factor for the i th individual. This is similar to the two parameter model allowing, however, for a predetermined time origin to be used, resulting in a random initial size.

3. <u>The Three Parameter Model (B) or Logistic</u> is

$$Y_i(t) = \frac{\beta_i}{1 + \alpha_i e^{-\gamma_i t}} ,$$

where β_i is the final size, α_i is that multiple of the initial size that gives the total growth of the i th individual and γ_i is the growth factor. This has been used by a great number of investigators and often quite adequately graduates growth data.

4. <u>The Four Parameter Model</u> is

$$Y_i(t) = \beta_{0i}e^{-\gamma_{0i}t} + \beta_{1i}(1 - e^{-\gamma_{1i}t}) ,$$

where β_{0i}, β_{1i}, and γ_{1i} are defined as previously with γ_{0i}

representing an initial retarding factor for the i th individual that diminishes in importance as time passes.

A sample of n individuals observed at time t, then, owes its dispersion to the n different values of each of the parameters in the model. Thus the distribution of the observations arises not from an additive residual term, but from the distributions of the parameters present in the model. This distribution of the observations changes as time changes from the distribution of the initial size at time $t = 0$ to the distribution of the final or asymptotic size as t increases indefinitely. At any time t we have, then, a conditional distribution, conditional on time.

The next logical step is the assumption of suitable probability distributions for the parameters involved, and the investigation of the conditional distributions of $Y(t)$ resulting therefrom. It will be appreciated that obtaining the explicit probability distribution for $Y(t)$ results in virtually intractable integrations. Moments, however, of this variable are much more amenable mathematically and they are obtained in the following section.

The remainder of this chapter consists in obtaining the r th moments of models 1, 2, 3, and 4 assuming independent uniform distributions for the parameters with population means equal to β, β_0, β_1, α, γ, γ_0, and γ_1 respectively with subsequently defined ranges, also are obtained r th moments of models 1, 2, and 4 assuming independent gamma distributions with the same population means and suitably defined coefficients of variation.

3.2. Mathematical Development.

Model 1

$$Y = \beta_1(1 - e^{-\gamma_1 t}) .$$

Let us call the parameters β_1 and γ_1 the random variables x_1 and z_1, say, so that the model becomes

$$Y = x_1(1 - e^{-z_1 t}) ,$$

where

$$f(x_1) = \frac{1}{2\rho} \quad \text{for } \beta_1 - \rho \leq x_1 \leq \beta_1 + \rho$$

and

$$g(z_1) = \frac{1}{2\alpha} \quad \text{for } \gamma_1 - \alpha \leq z_1 \leq \gamma_1 + \alpha .$$

Then

$$E(Y^r) = E\left\{[x_1(1 - e^{-z_1 t})]^r\right\}$$

$$= \int_{\gamma_1-\alpha}^{\gamma_1+\alpha} \int_{\beta_1-\rho}^{\beta_1+\rho} \frac{1}{2\rho}\,\frac{1}{2\alpha}[x_1(1 - e^{-z_1 t})]^r \, dx_1 \, dz_1 .$$

Thus

$$E(Y^r) = \frac{[(\beta_1+\rho)^{r+1} - (\beta_1-\rho)^{r+1}]}{4\rho\alpha t(r+1)} \sum_{i=0}^{r} \binom{r}{i}(-1)^{r-i} \frac{1}{r-i}$$

$$\cdot [e^{-(r-i)(\gamma_1-\alpha)t} - e^{-(r-i)(\gamma_1+\alpha)t}] .$$

Now let:

$$f(x_1) = \frac{1}{\Gamma(\frac{1}{\lambda_1^2})} \frac{1}{\beta_1\lambda_1^2} \left(\frac{x_1}{\beta_1\lambda_1^2}\right)^{1/\lambda_1^2 - 1} e^{-\frac{x_1}{\beta_1\lambda_1^2}}$$

$$\text{for } 0 \leq x < \infty ,$$

where β_1 is the mean and λ_1 the coefficient of variation, and

$$g(z_1) = \frac{1}{\Gamma(\frac{1}{\lambda_2^2})} \frac{1}{\gamma_1\lambda_2^2} \left(\frac{z_1}{\gamma_1\lambda_2^2}\right)^{\frac{1}{\lambda_2^2} - 1} e^{-\frac{z_1}{\gamma_1\lambda_2^2}}$$

where γ_1 is the mean and λ_2 the coefficient of variation.

Under this assumption the r th moment of Y is

$$E(Y^r) = E\left\{[x_1(1 - e^{-z_1 t})]^r\right\}$$

$$= E\left\{x_1^r \sum_{i=0}^{r} \binom{r}{i}(-1)^{r-i} e^{-(r-i)z_1 t}\right\} .$$

This results in

$$E(Y^r) = \frac{(\beta_1\lambda_1^2)^r \Gamma(\frac{1}{\lambda_1^2} + r)}{\Gamma(\frac{1}{\lambda_1^2})} \sum_{i=0}^{r} \binom{r}{i}(-1)^{r-i} \frac{1}{[(r-i)t\gamma_1\lambda_2^2+1]^{1/\lambda_2^2}} .$$

<u>Model 2</u>.

$$y = \beta_0 + \beta_1(1 - e^{-\gamma_1 t})$$

becomes

$$y = x_0 + x_1(1 - e^{-z_1 t})$$

where

$$f(x_0) = \frac{1}{2\rho}\,, \qquad \text{for } \beta_0-\rho \le x_0 \le \beta_0+\rho$$

$$g(x_1) = \frac{1}{2\rho}\,, \qquad \text{for } \beta_1-\rho \le x_1 \le \beta_1+\rho$$

$$h(z_1) = \frac{1}{2\alpha}\,, \qquad \text{for } \gamma_1-\rho \le z_1 \le \gamma_1+\rho \quad .$$

In this case

$$E(Y^r) = E\left\{[x_0 + x_1(1 - e^{-z_1 t})]^r\right\}$$

$$= \frac{1}{8\rho^2 t\alpha} \sum_{i=0}^{r} \binom{r}{i} \frac{1}{(i+1)(r-i+1)} [(\beta_0+\rho)^{i+1} - (\beta_0-\rho)^{i+1}]$$

$$\cdot[(\beta_1+\rho)^{r-i+1} - (\beta_1-\rho)^{r-i+1}]$$

$$\cdot \sum_{j=0}^{r-i} \binom{r-i}{j} (-1)^j \frac{1}{j} [e^{-j(\gamma_1-\alpha)} - e^{-j(\gamma_1+\alpha)}] \ .$$

For the independent Gamma distributions we have:

$$E(Y^r) = \sum_{i=0}^{r} \binom{r}{i} \left[\frac{(\beta_0\lambda_0^2)^i \Gamma(\frac{1}{\lambda_0^2} + i)}{\Gamma(\frac{1}{\lambda_0^2})} \right]$$

$$\cdot \left[\frac{(\beta_1\lambda_1^2)^{r-i} \Gamma(\frac{1}{\lambda_1^2} + r - i)}{\Gamma(\frac{1}{\lambda_1^2})} \right]$$

$$\cdot \sum_{j=0}^{r-i} \binom{r-i}{j} (-1)^j \frac{1}{[jt\gamma_1\lambda_2^2 + 1]^{1/\lambda_2^2}}$$

for model 2.

<u>Model 3</u>.

$$Y = \frac{\beta}{1 + \alpha e^{-\gamma t}}$$

becomes

$$Y = \frac{x}{1 + we^{-zt}}$$

where

$$f(x) = \frac{1}{2\rho} \quad , \quad \text{for } \beta-\rho \leq x \leq \beta+\rho$$

$$g(w) = \frac{1}{2\lambda_1} \quad , \quad \text{for } \alpha-\lambda_1 \leq w \leq \alpha+\lambda_1$$

$$h(z) = \frac{1}{2\lambda_2} \quad , \quad \text{for } \gamma-\lambda_2 \leq z \leq \gamma+\lambda_2 \ .$$

With these assumptions

$$E(Y) = \frac{\beta}{4\lambda_1\lambda_2 t} \quad (A_2+B_2)\ln(A_2+B_2) - (A_2+B_1)\ln(A_2+B_1)$$
$$- (A_1+B_2)\ln(A_1+B_2) + (A_1+B_1)\ln(A_1+B_1)$$

where

$$A_1 = e^{(\gamma-\lambda_2)t}, \quad A_2 = e^{(\gamma+\lambda_2)t}$$

$$B_1 = \alpha - \lambda_1, \qquad B_2 = \alpha + \lambda_1$$

$$E(Y^2) = \frac{3\beta^2 + \rho^2}{12\lambda_1\lambda_2 t} \quad B_2 \ln(\frac{A_2 + B_2}{A_1 + B_2}) - B_1 \ln(\frac{A_2 + B_1}{A_1 + B_1})$$

and for $r > 2$

$$E(Y^r) = K \left\{ \sum_{i=0}^{r-3} \binom{r-1}{i} (-1)^{r-i-2} \frac{B_1^{r-i-1}}{r-i-2} \left[\frac{1}{(A_2+B_1)^{r-i-2}} - \frac{1}{(A_1+B_1)^{r-i-2}} \right] - \sum_{i=0}^{r-3} \binom{r-1}{i} (-1)^{r-i-2} \frac{B_2^{r-i-1}}{r-i-2} \right.$$
$$\cdot \left[\frac{1}{(A_2+B_2)^{r-i-2}} - \frac{1}{(A_1+B_2)^{r-i-2}} \right]$$
$$\left. -(r-1)B_1 \left[\ln(\frac{A_2+B_1}{A_1+B_1}) \right] + (r-1)B_2 \left[\ln(\frac{A_2+B_2}{A_1+B_2}) \right] \right\},$$

where

$$K = \frac{1}{8\rho\lambda_1\lambda_2(r^2-1)t}[(\beta+\rho)^{r+1} - (\beta-\rho)^{r+1}]$$

and A_1, A_2, B_1, B_2 are as defined above.

<u>Model 4</u>.

$$Y = \beta_0 e^{-\gamma_0 t} + \beta_1(1 - e^{-\gamma_1 t})$$

becomes

$$Y = x_0 e^{-z_0 t} + x_1(1 - e^{-z_1 t})$$

where

$$f(x_0) = \frac{1}{2\rho} \quad \text{for } \beta_0-\rho \leq x_0 \leq \beta_0+\rho$$

$$g(x_1) = \frac{1}{2\rho} \quad \text{for } \beta_1-\rho \leq x_1 \leq \beta_1+\rho$$

$$h(z_0) = \frac{1}{2\alpha} \quad \text{for } \gamma_0-\alpha \leq z_0 \leq \gamma_0+\alpha$$

$$k(z_1) = \frac{1}{2\alpha} \quad \text{for } \gamma_1-\alpha \leq z_1 \leq \gamma_1+\alpha \ .$$

$$E(Y^r) = \frac{1}{16\rho^2\alpha^2t^2} \sum_{i=0}^{r} \binom{r}{i} \frac{1}{i(i+1)(r-i+1)} \{[A_2^{i+1}B_2^{r-i+1}C_1^i$$

$$-A_2^{i+1}B_1^{r-i+1}C_1^i - A_2^{i+1}B_2^{r-i+1}C_2^i + A_2^{i+1}B_1^{r-i+1}C_2^i$$

$$-A_1^{i+1}B_2^{r-i+1}C_1^i + A_1^{i+1}B_1^{r-i+1}C_1^i + A_1^{i+1}B_2^{r-i+1}C_2^i$$

$$-A_1^{i+1}B_1^{r-i+1}C_2^i][\sum_{j=0}^{r-i}\binom{r-i}{j}(-1)^{r-i-j}(e^{-(r-i-j)t(\gamma_1-\alpha)}$$

$$-e^{-(r-i-j)t(\gamma_1+\alpha)})]\}$$

where

$$A_1 = \beta_0 - \rho, \qquad A_2 = \beta_0 + \rho$$

$$B_1 = \beta_1 - \rho, \qquad B_2 = \beta_1 + \rho$$

$$C_1 = e^{-(\gamma_0-\alpha)t}, \quad C_2 = e^{-(\gamma_0+\alpha)t} .$$

Using independent gamma distributions for the parameters with different coefficients of variation for each we have

$$E(Y^r) = \sum_{i=0}^{r} \binom{r}{i} \frac{(\beta_0\lambda_0^2)^i \Gamma(\frac{1}{\lambda_0^2} + i)}{\Gamma(\frac{1}{\lambda_0^2})} \cdot \frac{(\beta_1\lambda_1^2)^{r-i}\Gamma(\frac{1}{\lambda_1^2} + r - i)}{\Gamma(\frac{1}{\lambda_1^2})}$$

$$\cdot \frac{1}{(i\gamma_0\lambda_2^2 t+1)^{1/\lambda_2^2}} \sum_{j=0}^{r-i} \binom{r-i}{j} (-1)^j$$

$$\cdot \frac{1}{[(j)t\gamma_1\lambda_3^2+1]^{1/\lambda_3^2}} .$$

IV. A GROWTH CURVE EXPERIMENT

The mere proposal of a new probability model is not sufficient scientifically to warrant its adoption. Evidence to indicate some improvement over past approaches must be assembled to check on the validity of the assumptions and theoretical predictions. Dr. Paul B. Siegel of the Poultry Department of the Virginia Polytechnic Institute volunteered to conduct such an experiment, for if the major outlines of the basic idea could be shown to be valid, there follow significant implications to the field of quantitative inheritance.

The experiment was set up with the chicken as the organism under study. The chickens were selected from a broad genetic base, giving an opportunity to check on the assumption of consistence of the mathematical form of the growth curves. They were all fed the same low energy ration, which contained no additives, and were not inoculated so that "normal" growth patterns would be exposed if present. The chickens were carefully weighed twice a day for the first two weeks (early in the morning and late in the evening), to eliminate as far as possible variations due to food ingestion and defecation. Following this period the weighing was performed once a day since this type of variation then became insignificant as compared to total body weight.

The scale used was carefully calibrated and was not disturbed so that no extraneous variation from this source would influence the results. The chickens were placed in individual pens to eliminate social stratification effects and were not moved except to larger pens as the experiment progressed.

A graph of the data for the first 120 days of the experiment of the thirteen chickens which survived and appeared free of disease is given in Figure I. A perusal will show that additive deviations from the assumed smooth growth functions is of such a minor nature that they are not distinguishable on the scale used. The assumption that variability among individuals observed at the same time arises not from additive sources but from true individuality of growth curves seems to be well justified. The assumption of common parametric form for the individual growth curves also appears reasonable.

The broad outlines of the proposal being thus justified, a method was then devised to investigate at least to a first approximation the more specific developments of the theory.

We selected the logistic model as the simplest form offering some hope of approximately graduating the data for the time period from 0 to 120 days. Then, using the data for the time points 0, 60, and 120 days we calculated the

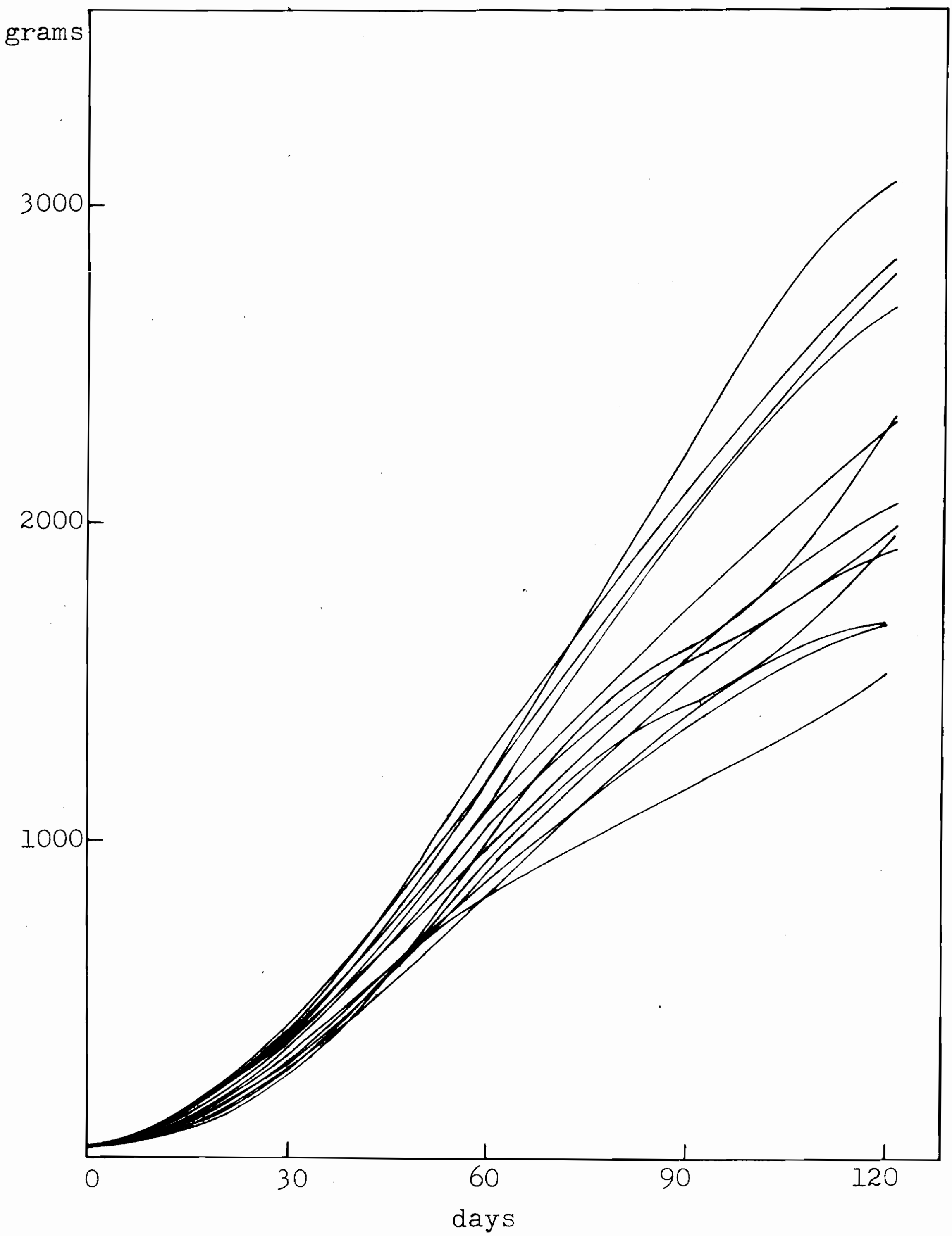

FIGURE I. Growth Curves for the Chicken Data in Table 1 individual logistic curves for the thirteen chickens which would pass through these three points. After obtaining the thirteen parameter values for each of three parameters in the logistic curve we then estimated the parameters, i.e., β, α, γ, λ_1, λ_2, and ρ, assumed in the theoretical development by the method of moments. Using these estimates in the formulae given in Chapter III estimates of E(Y) and V(Y) were obtained for the time periods 0, 30, 60, 90, and 120 days. These are, then, to be compared to the observed means and variances obtained from the original data for these time points. While this procedure is admittedly only a very rough check on the proposal, it should shed some light on the feasibility of persuing it further at some later date.

The data for the weights at 0, 30, 60, 90 and 120 days, then, are given in Table I with their means and variances.

The formulae for estimating the parameters in the logistic using a unit of 60 days are:

$$u_i = \frac{Y_{0i}(Y_{2i} - Y_{1i})}{Y_{2i}(Y_{1i} - Y_{0i})}$$

where

$$u_i = e^{-\gamma_i}$$

Y_{0i} = weight of i th individual at 0 days

Y_{1i} = weight of i th individual at 60 days

Y_{2i} = weight of i th individual at 120 days,

	Time in Days				
	0	30	60	90	120
	33	370	1198	2223	3080
	36	387	1248	2113	2833
	31	390	1190	2032	2807
	35	355	1157	2028	2690
	29	273	1063	1622	2348
Weight	34	374	1143	1783	2340
in	35	372	1003	1632	2074
Grams	34	358	1049	1592	1999
	32	286	943	1452	1973
	31	302	940	1559	1922
	33	260	846	1442	1702
	32	299	904	1402	1690
	36	335	824	1208	1537
Mean	33.1538	335.4615	1039.0769	1699.0769	2230.3846
Variance	4.4744	2085.4359	19932.5775	97497.7441	245491.9233

TABLE I. Data from Siegel's Experiment

$$\alpha_i = \frac{Y_{1i} - Y_{0i}}{Y_{0i} - Y_{1i}u_i} ,$$

and

$$\beta_i = Y_{0i}(1 + \alpha_i).$$

These are given in Table II with their means and variances.

The comparison of the means and variances of the data with the predicted means and variances from the formulae are given in Table III.

Within the qualifications and limits of this check, we feel the probability model proposed shows real promise of offering new insight into the nature of biological variation in this field of endeavor.

	γ	α	β
	4.057	94.99	3168
	4.097	79.42	2895
	4.173	91.52	2868
	4.030	77.75	2756
	4.177	81.52	2393
	4.155	68.99	2380
	3.981	59.49	2117
	4.140	58.67	2029
	3.999	61.94	2014
	4.050	62.17	1958
	3.892	51.69	1739
	4.071	52.62	1716
	3.854	42.51	1566
mean	4.052	67.94	2277
variance	.010361	256.10	262150
range	.17630	27.718	886.82

TABLE II. Estimates of Logistic Parameters, Their Means, Variances and Ranges

Time in Days	0	30	60	90	120
Mean from Data	33.154	335.46	1039.1	1699.1	2230.4
Moment Method Mean	34.999	240.07	1061.4	1987.2	2230.2
Variance from Data	4.4744	2085.4	19933	97498	245492
Moment Method Variance	141.71	6231.7	80292	133460	251300

TABLE III. Comparison of Theoretical and Observed Means and Variances for Selected Time Points

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VITA

Max Henry Myers was born in Lynchburg, Virginia on July 2, 1936. He was graduated from E. C. Glass High School in June 1954.

In September of 1954, he entered Bridgewater College. At Bridgewater College he took part in Glee Club, Acapella Choir, Chapel Choir, and was feature editor of the college newspaper and later became associate editor. He became a member of Pi Delta Epsilon, national journalism fraternity, and later was elected president of the local chapter. He was graduated cum laude from Bridgewater College in June 1958 with the B.A. Degree in Mathematics.

He entered the Graduate School of Virginia Polytechnic Institute holding a fellowship in statistics from the National Institutes of Health.

He was married in August 1959.

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ABSTRACT

This paper proposed the idea that the growth of an individual organism follows a mathematical model very closely and that different individuals follow different members of the same parametric family of models. This idea implies that the variation observed between individuals measured at the same time arises not from an additive term as has been previously supposed, but primarily from variation of the parameters of the model. A graph of data from an experiment on chickens is included which points up this individuality and the increased variation resulting from the passage of time.

The four models considered were growth curves employing two, three, and four parameters with biological interpretations existing for the parameters.

The parameters were allowed to follow independent uniform distributions and independent gamma distributions.