
by

Premkumar Muthedath

Thesis submitted to the Faculty of the
Virginia Polytechnic Institute and State University
in partial fulfillment of the requirements for the degree of
Master of Science
in
Aerospace Engineering

APPROVED:

______________________________
Dr. Stergios Liapis, Chairman

______________________________
Dr. Wayne L. Neu

______________________________
Dr. Dean T. Mook

May, 1992

Blacksburg, Virginia
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1992
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c. 2

by

Premkumar Muthedath

Dr. Stergios Liapis, Chairman

Aerospace Engineering

(ABSTRACT)

Nonlinear free-surface flows generated by the motion of a surface-piercing body in an ideal fluid are studied. A numerical scheme employing a mixed Eulerian-Lagrangian approach and involving time stepping is used to simulate the flow. At each time step, the boundary value problem is solved using the Complex Boundary Element Method. The numerical performance of the method is studied by considering cases where the exact solution is known. Computational results for the impulsive wavemaker problem and the wedge entry problem for wedges of half-angles up to 15 degrees are presented. The obtained results are found to be in good agreement with existing analytical and numerical solutions.
Acknowledgements

I would like to thank Dr. Liapis for his constant guidance and help during the course of this work. Working with him has been a fruitful experience for me and I am sure that it will benefit me for the rest of my life.

I am very grateful to my friend Evangelos Hytcopolos for his valuable advice and help concerning almost all aspects of this thesis. Special thanks go to Marc Kreider for helping me with the figures. I would also like to thank all my office mates at the Pack building for their company and encouragement.

I am grateful to Dr. Schetz for providing financial support during my studies at Virginia Tech. I wish to also thank Dr. W. L. Neu and Dr. D. T. Mook for being on my committee.

Last but not least, I would like to thank my family back in India for their constant encouragement and support.
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Introduction

The study of nonlinear ship motions is of greater importance today than ever before due to the increasing activities that are being undertaken in that environment. Today the sea is considered an important source of raw materials in addition to a medium of transportation. This new outlook has brought in the need for having permanent stations out in the sea, their locations being determined not by sea conditions but by the availability of the desired resource.

Typically, the investment required for setting up an offshore station is considerable, and therefore, its safety and reliability are of primary concern. To ensure that these objectives are adequately met, designers devote considerable effort during design stages to studying the response of the structure to a myriad of sea conditions, paying close attention to the worst possible scenario. An important aspect of such investigations is the estimation of the hydrodynamic forces experienced by the structure at sea. Accurate knowledge of these a priori is essential to assessing the reliability of the structure.
As a first step towards the study of such problems we consider in this thesis the impulsive wavemaker problem and the wedge entry problem. The latter is of considerable importance in studies concerned with the phenomenon of bow slamming which can lead to severe structural damage of ships in rough seas.

Theoretical studies of the problem assume the fluid to be ideal and the fluid motion to be irrotational at all times. Considerable simplification can result under these assumptions without sacrificing the accuracy required for engineering applications (Dommermuth et al. [4]). Nonetheless, great difficulty is encountered and further simplifying approximations have to be made to obtain a solution.

The principal difficulty encountered in analytical investigations of free-surface flows is that of satisfying the nonlinear constant-pressure condition on the free surface, whose location is unknown and must be found as part of the solution. To circumvent this problem, one assumes that the free-surface elevations are small, which allows the boundary condition to be satisfied at the undisturbed position of the free surface. This results in what are called linear theories. For the simple harmonic wavemaker, a steady state linear solution has been given by Havelock [8]. A transient solution for the same problem has been obtained by Cointe [2]. Their range of validity is known to be restricted to waves of small steepness (Ursell et al. [26]).

For the wedge entry problem, a linear closed form solution has been obtained by Mackie [17]. The free-surface boundary conditions are linearized. The body boundary condition is treated exactly. The solution is expected to hold only for very slender wedges (Dobrovol'skaya [3], Greenhow [7]); thus, the practical utility of these solutions is restricted.
The presence of a surface-piercing body introduces another difficulty related to the proper description of the flow in the vicinity of the intersection points between the body and the free surface; a singularity is present at the intersection point due to the confluence of boundary conditions.

The presence of the singularity was first pointed out by Kravtchenko [13] in the steady state linear solution of a simple harmonic wavemaker. For the transient wavemaker problem, Cointe [2] has demonstrated that the boundary conditions at the intersection point are disjoint. Recent work by Cointe [2], Roberts [23], Joo et al. [12] has produced a linear, bounded solution for the wavemaker problem when the wavemaker acceleration is bounded and typically less than that due to gravity. However, a bounded solution for impulsive motions has still not been found.

For the wedge entry problem, Hughes [10] has pointed out that singularities exist at the spray tip, spray root, and wedge vertex. However, at present no analysis exists that indicates the order or nature of the singularities.

The difficulty relating to the presence of the singularity is yet to be resolved. The presence of the singularity in the small time solution of the fully nonlinear impulsive wavemaker problem obtained by Peregrine [22] clearly rules out linearization as its origin. The complexity of the analysis involved has so far prevented the consideration of nonpotential effects such as surface tension and viscosity.

Fortunately, for the problems considered here, solutions taking into account the full nonlinear free-surface effects are available. A solution for small times valid everywhere except at the intersection point has been provided by Peregrine [22] for the impulsive wavemaker problem.
For the problem of constant-velocity wedge entry with general angle and fully nonlinear free-surface conditions, with gravity neglected, a solution can be obtained using similarity variables. Such a solution has been provided by Dobrovol'skaya [3] and Hughes [10] using the Wagner function. As verified by Hughes [10], the assumption of self-similarity is expected to hold everywhere except in the jet region where gravity effects are appreciable.

However, it must be borne in mind that these solutions or the techniques used to obtain them cannot be extended to other types of motions. Thus, an analytical solution that takes into account the full nonlinear effects for a simple harmonic wavemaker motion or an oscillating wedge does not exist.

Given these restrictions, a numerical approach would be the most desirable tool to study nonlinear free-surface problems. In light of this, we here, use a numerical scheme employing a mixed Eulerian-Lagrangian approach and involving time stepping to study the problem. The problem is solved as an initial-boundary value problem. The boundary value problem at each time step is solved by the Complex Boundary Element Method. The assumptions involved are that of an inviscid, incompressible, homogenous fluid. The scheme is based on the Cauchy's integral theorem method of Vinje and Brevig [27], with Lin's [14] modification for the treatment of the singularity.

Since at each time step the Complex Boundary Element Method is used to solve for the unknown values, it is important to study the performance of this method. In Chapter 2 we undertake such a study for some simple problems where the exact solutions are known. A comparative study is also made with two different formulations of this method and the results are presented.
Numerical results for the impulsive wavemaker motion and the two-dimensional, symmetrical, constant-speed entry of wedges of half-angles $\leq 15^\circ$, into initially calm water are presented in Chapter 3. The results are compared with existing analytical and numerical solutions.
Solution of Potential Problems Using Complex Boundary Element Method

Numerical solutions of time-dependent problems governed by the Laplace equation require the solution of a boundary value problem at each time step for the time marching to be continued. The problems we are interested in, such as the wedge entry problem, fall typically into such a category. The accuracy and stability of such numerical schemes are strongly dependent on the accuracy of the solution to the boundary value problem. Valuable information regarding the accuracy of the method used to solve the boundary value problem can be obtained by considering simple problems for which exact analytical solutions are available. The method employed here for the solution of the boundary value problem is the Complex Boundary Element Method. In this Chapter, we undertake a study to evaluate the performance of the Complex Boundary Element Method by using it to solve some simple potential problems.

Problems governed by the Laplace equation or the Poisson equation are called potential problems. The solutions of such problems are scalar functions of position and
exhibit many unique features. The feature that will concern us most here is the relationship that exists between the values of the function within the domain and its values on the boundary. For potential functions, the functional values within a domain are determined by its values on the boundary. From here on, we shall confine ourselves to problems governed by the 2-dimensional Laplace equation.

Analytical solutions to various problems governed by the Laplace equation that arise in Physics and Engineering are limited and very often a numerical solution is sought. It is then imperative that a computationally efficient and sufficiently flexible numerical method be found. The numerical approach that is most popular for the solution of such problems is called the Boundary Element Method. This method exploits a unique feature of potential functions, noted in the preceding paragraph, to reformulate the problem as a boundary integral equation involving only the known and unknown functional values on the boundary. Once the unknown functional values on the boundary are determined, the problem can be considered solved. An important advantage of this formulation is that it reduces the problem dimension by one. From a numerical standpoint, this would mean just the discretization of the boundary rather than the whole domain - a tremendous reduction in computational effort.

The traditional approach is to use Green's second identity to derive the integral equation. But in two dimensions, an integral equation can be derived using Cauchy's integral theorem. Such a formulation, making use of the properties of complex analytic functions, has been developed by Vinje and Brevig [27]. This approach has been noted by Hromadka and Lai [9] to be superior, in terms of simplicity and computational effort, to the traditional approach.
The method comes under the category of what are called Amalgamative Techniques, described by Hromadka and Lai [9], and essentially consists of two steps. The first step involves the reformulation of the problem as a boundary integral equation. This is done by exploiting the well known fact that the solution of the Laplace equation in two dimensions can be represented by a complex analytic function. The unknown function, in its domain of analyticity, should satisfy Cauchy's integral theorem. This reformulation, as mentioned previously, reduces the problem dimension by one, and is, therefore, computationally very advantageous.

The second step is concerned with the numerical solution of the integral equation obtained in the first step. This is done by approximating the boundary by panels or elements. Over each panel, the complex potential is assumed to vary in some fashion while being piece-wise continuous. These approximations reduce the integral equation to a system of linear algebraic equations. Solution of the linear system provides the unknown part of the complex potential on the boundary. The functional values within the domain can be calculated using Cauchy's integral formula.

Any numerical approach for the solution of differential equations incurs two types of errors, namely, the round-off error due to the finite digit arithmetic used by the computer, and the discretization error due to the approximation of the solution and problem geometry. In most cases, the solution accuracy is dictated by the discretization error. This is determined by the accuracy with which the numerical scheme approximates the problem. For the Boundary Element Method, the discretization error is dictated by the accuracy of approximation of the boundary integrals. An advantage of using complex analytic functions is that these boundary integrals become path independent and, therefore, independent of panel shape approximations. At present, no global error analysis
is available for Boundary Element Methods, although local error analysis relating the order of the local error introduced for a given approximation of the solution and geometry can be found in Romate [24]. In general, the accuracy of the numerical solution is affected by the panel length, types of boundary conditions, boundary shape, and the behavior of the solution itself.

Two variations of the boundary integral equation are examined here. In the conventional method, the kernel singularity is allowed to approach the boundary of the domain from outside, giving rise to what is called a singular boundary integral equation. An alternative is to move the kernel singularity away from the boundary and outside the domain, resulting in a desingularized boundary integral equation. This is known to give, in some cases, more accurate results for a given truncation or discretization. Schultz and Hong [25], in their study of 2-dimensional potential problems, found considerable improvement in accuracy for circular contours when the kernel is desingularized. Similar improvement in accuracy for a given discretization is reported by Cao et al. [1], in their study of 3-dimensional gravity waves generated by a moving submerged disturbance.

In general, the boundary is composed of portions where either the real or the imaginary part of the complex potential is known. The problem has a unique solution only when the specified boundary conditions are of the mixed type.

Depending on the prescribed conditions, the singular integral equation can be manipulated to give a Fredholm integral equation of the second kind, which is known to have good numerical properties. On the other hand, desingularizing the kernel always results in a Fredholm integral equation of the first kind, which is undesirable from a numerical standpoint.
In this Chapter, we investigate the performance of both the singular and desingularized formulations when the complex potential is assumed to vary linearly over an element. In all the cases studied, the exact solution was available and used to specify the boundary conditions. Schultz and Hong [25] have conducted a similar study but have mainly confined their investigations to circular and elliptic contours. The problems investigated were of the Neumann type. To remove the non-uniqueness associated with the Neumann problem, they resorted to an overdetermined formulation. Since the physical problems that will occupy us in the next Chapter involve mixed type boundary conditions, it is appropriate to consider such boundary conditions for this study.

The problems considered here, therefore, involve mixed type boundary conditions, and computations are performed for circular and rectangular contours. The effect of boundary shape, curvature, change in boundary conditions, desingularization, and proximity of singularities on the solution accuracy are investigated. Also discussed are the strategies for placing the kernel singularity in the desingularized case and the numerical treatment of mixed type boundary conditions.

2.1. Mathematical Formulation

Consider the simply connected region $R$ bounded by the closed curve $C$, shown in Figure 1. Let $\beta(z)$ be the complex potential which is analytic in $R \cup C$, given by

$$\beta(z) = \phi(x, y) + i\psi(x, y)$$  \[2.1\]
where \( z = x + iy \) is the complex coordinate and \( i = \sqrt{-1} \). One can regard \( \phi \) and \( \psi \) as the velocity potential and stream function, respectively, for an incompressible, irrotational, inviscid fluid flow. The contour \( C \) consists of portions where \( \phi \) is known, denoted by \( C_\phi \), and where \( \psi \) is known, denoted by \( C_\psi \). The problem then is to find the unknown part of the complex potential on the boundary \( C \). Since \( \beta(z) \) is analytic in \( R \cup C \), it satisfies the Cauchy integral theorem

\[
\oint_C \frac{\beta(z)}{z - z_k} \, dz = 0 \tag{2.2}
\]

where \( z_k \) is outside \( R \cup C \), and the contour \( C \) is traversed such that the region \( R \) always lies to its left.

### 2.1.1. Singular Method

In the singular method, the kernel singularity \( z_k \) is allowed to approach the boundary \( C \) from outside, in a limiting sense; see Figure 2. In the limit, equation (2.2) becomes

\[
i\alpha_k \beta(z_k) + \oint_C \frac{\beta(z)}{z - z_k} \, dz = 0 \tag{2.3}
\]

where \( \alpha_k \) is the angle by which an infinitely small vector \( \overrightarrow{z_k z} \) rotates, when the point \( z \), remaining on the left of \( C \) and revolving around \( z_k \), passes from the part of the contour to the left of \( z_k \) to the part of the contour on the right of \( z_k \), as the contour \( C \) is traversed anticlockwise. The range of \( \alpha_k \) is \(-2\pi \leq \alpha_k \leq 0\). For smooth contours, \( \alpha_k \) is math-
ematically equal to $-\pi$. The integral in equation [2.3] represents the Cauchy principal value. The details of the derivation of equation [2.3] are given in Appendix 1.

Now, one can take either the real or the imaginary part of equation [2.3] and solve for the unknowns. Vinje and Brevig [27] have shown that taking the real part of the equation when $z_s$ is on $C_\phi$ and the imaginary part when $z_s$ is on $C_\psi$, yields a Fredholm integral equation of the second kind, which has good numerical properties. The details are provided in Appendix 1. Doing so, one obtains

$$\begin{align*}
-\alpha_k \psi(z_k) + \text{Re} \left\{ \int_C \frac{\beta(z)}{z - z_k} \, dz \right\} &= 0 \\
\text{[2.4]} \end{align*}$$

when $z_s$ is on $C_\phi$, and

$$\begin{align*}
-\alpha_k \phi(z_k) + \text{Re} \left\{ i \int_C \frac{\beta(z)}{z - z_k} \, dz \right\} &= 0 \\
\text{[2.5]} \end{align*}$$

when $z_s$ is on $C_\psi$.

2.1.2. Desingularized Method

In this method, the kernel singularity $z_s$ is placed outside $R \cup C$. Therefore, equation [2.2] remains unchanged; see Figure 3. The integral equation is of Fredholm's first kind and is expected to have numerical problems. Nevertheless, when $z_s$ is not too far from the boundary $C$, the behavior of the integral equation [2.2] can be expected to be very similar to that of equation [2.3]. This implies that one can make the same choice
as that for the singular method of taking the real or the imaginary part of the integral equation, depending on whether $z_s$ is close to $C_\phi$ or $C_\psi$. This translates to

$$Re \left\{ \oint_C \frac{\beta(z)}{z - z_k} \, dz \right\} = 0$$  \hspace{1cm} [2.6]

when $z_s$ is close to $C_\phi$, and

$$Re \left\{ i \oint_C \frac{\beta(z)}{z - z_k} \, dz \right\} = 0$$  \hspace{1cm} [2.7]

when $z_s$ is close to $C_\psi$.

### 2.2. Numerical Solution

The numerical solution procedure, presented in sections 2.2.1, 2.2.2, and 2.2.3, is applicable to both the singular and desingularized methods. The two formulations differ only in the placement of the control point $z_s$. As will be seen, this difference manifests itself only in the calculation of the influence coefficients. Sections 2.2.4 and 2.2.5 are devoted to the detailed discussion of this aspect.
2.2.1. Discretization of the Integral Equation

The boundary $C$ is divided into elements or panels by locating nodes on it as shown in Figures 4 and 5. Between nodes or over a panel, the complex potential $\beta$ is assumed to vary linearly in $z$. This can be expressed as

$$\beta(z) = \left\{ \frac{z - z_j}{z_j-1 - z_j} \right\} \beta_{j-1} + \left\{ \frac{z - z_{j-1}}{z_j - z_{j-1}} \right\} \beta_j$$ \hspace{0.5cm} [2.8]

when $z$ lies between $z_{j-1}$ and $z_j$, and

$$\beta(z) = \left\{ \frac{z - z_j}{z_{j+1} - z_j} \right\} \beta_{j+1} + \left\{ \frac{z - z_{j+1}}{z_j - z_{j+1}} \right\} \beta_j$$ \hspace{0.5cm} [2.9]

when $z$ lies between $z_j$ and $z_{j+1}$. It follows from equations [2.8] and [2.9] that a global description of $\beta$ will assume the form

$$\beta(z) = \sum_{j=1}^{N} \Lambda_j \beta_j$$ \hspace{0.5cm} [2.10]

where $N$ is the total number of nodes on the boundary, and $\Lambda_j$ are the linear basis functions, given by

$$\Lambda_j(z) = \frac{z - z_{j-1}}{z_j - z_{j-1}} \quad \text{for } z \text{ between } z_j \text{ and } z_{j-1}$$
$$\Lambda_j(z) = \frac{z - z_{j+1}}{z_j - z_{j+1}} \quad \text{for } z \text{ between } z_j \text{ and } z_{j+1}$$ \hspace{0.5cm} [2.11]
$$\Lambda_j(z) = 0 \quad \text{elsewhere}$$
Substituting [2.10] into [2.2], and noting that \( \Lambda_i(z) \) is nonzero only for \( z \) between \( z_{j-1} \) and \( z_{j+1} \), yields the numerical model of the integral equation

\[
\int_C \frac{\beta(z)}{z - z_k} \, dz = \sum_{j=1}^{N} \Gamma_{k,j} \, \beta_j \tag{2.12}
\]

where

\[
\Gamma_{k,j} = \left\{ \begin{array}{c}
\frac{z_k - z_j}{z_j - z_{j-1}} \ln \left( \frac{z_j - z_k}{z_j - z_{j-1}} \right) + \\
\frac{z_j - z_{j+1}}{z_{j+1} - z_j} \ln \left( \frac{z_j - z_k}{z_{j+1} - z_j} \right)
\end{array} \right. \tag{2.13}
\]

are the influence coefficients, and, \( z_j \) and \( z_k \), are the nodal and control point locations, respectively. The point \( z_s \) is also called the collocation point. The numerical solution is found by forcing the numerical model of the integral equation to vanish at some chosen control points \( z_s \). This is known as the collocation method. By choosing \( N \) control points \( z_s \), one can generate a \( N \times N \) matrix equation for the \( N \) unknowns on the boundary. This can be expressed as

\[
\sum_{j=1}^{N} \Gamma_{k,j} \, \beta_j = 0 \quad \text{for} \quad k = 1, 2, \ldots, N \tag{2.14}
\]

### 2.2.2. Application of Boundary Conditions

One can take, either the real or the imaginary part of [2.14], depending on the location of \( z_s \) as discussed in sections 2.1.1 and 2.1.2, giving
\[ \text{Re} \left\{ \sum_{j=1}^{N} \Gamma_{k,j} \beta_j \right\} = 0 \quad \text{for } z_k \text{ on or close to } C_{\phi} \]

\[ \text{Re} \left\{ i \sum_{j=1}^{N} \Gamma_{k,j} \beta_j \right\} = 0 \quad \text{for } z_k \text{ on or close to } C_{\psi} \]

However, special attention has to be paid to the nodal points which lie at the intersection of \( C_{\phi} \) where \( \phi \) is known, and \( C_{\psi} \) where \( \psi \) is known. As shown in Figures 4 and 5, \( C_{\phi} \) extends from node 1 to node \( N1 \), and \( C_{\psi} \) extends from node \( N1 \) to node 1 through node \( N \). This makes the nodes, 1 and \( N1 \), common to both \( C_{\phi} \) and \( C_{\psi} \), and the complex potential \( \beta(z) \) is completely determined there. It is then readily evident that the number of unknowns is \( N - 2 \) rather than \( N \).

In the development of the linear system, both \( \phi \) and \( \psi \) are specified at nodes 1 and \( N1 \), and no equations corresponding to the placement of the control point \( z_k \) on or close to these two nodes are considered. This approach is known to be an effective treatment for nodal points that have to satisfy two different boundary conditions. (e.g., see Lin [14], Yim [29])

2.2.3. Development and Solution of the Linear System

Expanding equations [2.15] and [2.16], and keeping all the unknown terms on the left hand side and moving all the known terms to the right, one obtains
for $k = 2, 3, \ldots, N1 - 1$, corresponding to $z_k$ on or close to $C_0$

\begin{align*}
- \sum_{j=2}^{N1-1} Im(\Gamma_{k,j})X_j + \sum_{j=N1+1}^{N} Re(\Gamma_{k,j})X_j \\
= - \sum_{j=2}^{N1-1} Re(\Gamma_{k,j})\phi_j + \sum_{j=N1+1}^{N} Im(\Gamma_{k,j})\psi_j \\
- Re \{\Gamma_{k,1} (\phi_1 + i\psi_1) + \Gamma_{k,N1} (\phi_{N1} + \psi_{N1})\}
\end{align*}

[2.17]  

and, for $k = N1 + 1, N1 + 2, \ldots, N$, corresponding to $z_k$ on or close to $C_0$

\begin{align*}
- \sum_{j=2}^{N1-1} Re(\Gamma_{k,j})X_j + \sum_{j=N1+1}^{N} Im(\Gamma_{k,j})X_j \\
= \sum_{j=2}^{N1-1} Im(\Gamma_{k,j})\phi_j + \sum_{j=N1+1}^{N} Re(\Gamma_{k,j})\psi_j \\
- Im \{\Gamma_{k,1} (\phi_1 + i\psi_1) + \Gamma_{k,N1} (\phi_{N1} + \psi_{N1})\}
\end{align*}

[2.18]  

where $X_j$ is the unknown part of the complex potential at the $j^{th}$ nodal point on the boundary $C$.

Equations [2.17] and [2.18] can be written in the matrix form $[A][X] = [b]$, where the elements of the matrices are given by

for $k = 2, 3, 4, \ldots, N1 - 1$

$A_{k,j} = -Im\{\Gamma_{k,j}\}$ for $2 \leq j \leq N1 - 1$

$A_{k,j} = Re\{\Gamma_{k,j}\}$ for $N1 + 1 \leq j \leq N$

\begin{align*}
b_k = & - \sum_{j=2}^{N1-1} Re(\Gamma_{k,j})\phi_j + \sum_{j=N1+1}^{N} Im(\Gamma_{k,j})\psi_j \\
& - Re \{\Gamma_{k,1} (\phi_1 + i\psi_1) + \Gamma_{k,N1} (\phi_{N1} + \psi_{N1})\}
\end{align*}

[2.19]
and, for \( k = N1 + 1, N1 + 2, \ldots, N \)

\[
A_{k,j} = -Re\{ \Gamma_{k,j} \} \quad \text{for} \quad 2 \leq j \leq N1 - 1
\]

\[
A_{k,j} = -Im\{ \Gamma_{k,j} \} \quad \text{for} \quad N1 + 1 \leq j \leq N
\]

\[
b_k = \sum_{j=2}^{N1-1} Im(\Gamma_{k,j}) \phi_j + \sum_{j=N1+1}^{N} Re(\Gamma_{k,j}) \psi_j
\]

\[
- \quad Im \{\Gamma_{k,1} (\phi_1 + i\psi_1) + \Gamma_{k,N1} (\phi_{N1} + \psi_{N1})\}
\]

[2.20]

Two things are to be noted from equations [2.19] and [2.20]. First, although the formulation makes use of complex variables, the linear system obtained contains only real variables. In other words, all the elements of the [A] matrix and the [b] vector are real, and so are the unknown values \( X \). The matrix [A], in general, is fully populated and does not exhibit any kind of symmetry.

Second, we observe from equations [2.17] and [2.18] that at the intersection nodes 1 and \( N1 \), both \( \phi \) and \( \psi \) are specified, and the terms corresponding to them are moved to the right hand side. Further, no equations corresponding to the placement of the control point \( z_* \) on or close to these nodes appear in the linear system. This explains the absence of the 1\textsuperscript{st} and \( N1\textsuperscript{st} \) rows and columns in the matrix [A], given by equations [2.19] and [2.20].

Before one proceeds to solve the matrix equation for the \( N - 2 \) unknowns, a simple rearrangement is required, so that a continuous variation of rows and columns from 1 to \( N - 2 \) is obtained. The linear system is solved using L-U decomposition with subsequent iterative refinement.
2.2.4. Singular Method

In this method, the control point $z_a$ is allowed to approach each of the nodes on the boundary with the exception of the nodes marked 1 and N1; see Figure 4. Special cases are to be considered when $k$ equals $j - 1$, $j$, and $j + 1$, to evaluate the influence coefficients $\Gamma_{\kappa\nu}$, given by equation [2.13]. These are obtained from equation [2.13] using a limit process and are given by

$$
\Gamma_{j-1,j} = \left\{ \frac{z_{j-1} - z_{j+1}}{z_j - z_{j+1}} \right\} \ln \frac{z_{j+1} - z_{j-1}}{z_j - z_{j-1}} \quad \text{when } k = j - 1 \tag{2.21}
$$

$$
\Gamma_{j,j} = \ln \frac{z_{j+1} - z_j}{z_j - z_{j-1}} \quad \text{when } k = j \tag{2.22}
$$

$$
\Gamma_{j+1,j} = \left\{ \frac{z_{j+1} - z_{j-1}}{z_j - z_{j-1}} \right\} \ln \frac{z_j - z_{j+1}}{z_{j-1} - z_{j+1}} \quad \text{when } k = j + 1 \tag{2.23}
$$

The rest of the influence coefficients are calculated using equation [2.13].

In general, for smooth contours, the real part of $\Gamma_{\kappa\nu}$ is nearly zero and its imaginary part is nearly equal to $-\pi$, although for flat contours with equispaced nodes, the real part of $\Gamma_{\kappa\nu}$ is exactly zero and the imaginary part is $-\pi$. The imaginary parts of $\Gamma_{j-1,j}$ and $\Gamma_{j+1,j}$ are nearly zero, while their real parts are not small. The other influence coefficients have typically small real and imaginary parts.

It is very important to bear in mind that for the singular case, the numerical model given by equation [2.14], should approximate equation [2.3]. A close scrutiny should reveal that the term $\Gamma_{\kappa\nu}\beta_j$ in equation [2.14] represents the term $i\alpha_j\beta(z_a)$. This implies that
the imaginary part of $\Gamma_{ij}$ should represent $\alpha_i$. In numerical calculations, due to the branch cut used by the computer for evaluating the logarithmic function, the imaginary part of $\Gamma_{ij}$ does not represent the correct interval of $\alpha_i$, which is $[-2\pi,0]$ and can lead to erroneous results. Modification, therefore, is required so that the imaginary part of $\Gamma_{ij}$ lies in the interval $[-2\pi,0]$.

The numerical solution is obtained using equations [2.15] and [2.16] with the calculation of influence coefficients done for the special cases using equations [2.21], [2.22], and [2.23]. Subsequent steps towards setting up and solving the linear system are described in the previous sections. It should be noted that equations [2.15] and [2.16], when used for the singular case, should model equations [2.4] and [2.5].

An interesting observation can be made about the coefficient matrix $[A]$, obtained using equations [2.19] and [2.20]. We observe that, because of the properties of the influence coefficients $\Gamma_{j-1,i}$, $\Gamma_{j,i}$ and $\Gamma_{j+1,i}$, equations [2.17] and [2.18] will always give a diagonally dominant and well conditioned coefficient matrix $[A]$. This obviously avoids numerical problems related to ill-conditioning and it is due to the fact that the integral equations modeled by equations [2.15] and [2.16] are of Fredholm's second kind.

2.2.5. Desingularized Method

In this method, the control point $z_k$ is placed at $N-2$ locations outside the boundary $C$ to obtain the linear system; see Figure 5. The influence coefficients are calculated using equation [2.13], and no special cases are required as the control point $z_k$ never coincides with the nodal points. The numerical solution is obtained using
equations [2.15] and [2.16]. Subsequent steps leading to the development of the linear system are described in the previous sections.

The resulting system model is an integral equation of Fredholm's first kind, given by equations [2.6] and [2.7]. No general conclusions can be drawn about the influence coefficients $\Gamma_{k,j}$, although when $z_s$ is close to the boundary, they should approach the values for the singular case. The condition number of the $[A]$ matrix is strongly dependent on the values of the influence coefficients, which in turn depend on the control point locations.

The location of the control points must be carefully chosen to avoid problems related to ill-conditioning. Schultz and Hong [25] indicate that the panel length is an important parameter in deciding the location of the control point $z_s$. They propose that the control point $z_s$ be placed outside the domain, at a distance $L_d$, given by

$$L_d = f |z_{j+1} - z_j|$$  \hspace{1cm} [2.24]

where $|z_{j+1} - z_j|$ is the local node spacing and $f$ is an adjustable parameter to be discussed below.

The strategy adopted here for an arbitrary contour is as follows. The control point $z_s$ is placed outside the domain from the corresponding nodal point $z_j$ at a distance given by equation [2.24]. The direction of the vector $\overline{z_jz_s}$ is along the outward normal to the panel connecting the nodes $z_s$ and $z_{j+1}$; see Figure 5. Because of the treatment of the intersection nodes, 1 and $N1$, discussed in section 2.2.2, the control points associated with them are not required.
2.3. Results

Computations are carried out for circular and rectangular contours. Several complex potentials are considered and the effect of various factors such as the boundary conditions, contour geometry, presence of singularities, on the solution accuracy are examined. All computations were carried out in double precision. In what follows, the root mean square error $E_{\text{rms}}$ of the solution obtained on the boundary, $\phi$ on $C_\phi$ and $\psi$ on $C_\psi$, is calculated using the definition

$$E_{\text{rms}} = \left[ \sum_j \frac{(\text{Exact} - \text{Computed})^2}{N - 2} \right]^{1/2} \quad [2.25]$$

2.3.1. Circular Contour

A circular contour with center at the origin and unit radius was chosen for this study; see Figures 6 and 7. The complex potentials considered were $\beta(z) = \sin(z)$ and $\beta(z) = e^z$. Uniform node spacing is used, i.e., $z_i = \exp \left( \frac{2\pi ij}{N} \right)$. Here, $i = \sqrt{-1}$. For the desingularized case, the control point $z_k$ is located radially outside from the $k^{th}$ node, at a distance given by equation [2.24].
2.3.1.1. Effect of Desingularization

To determine the effect of desingularization on the solution accuracy, we look at the variation of $E_{rm}$ as $f$ is varied, while keeping the number of nodes $N$ constant. The results for the two complex potentials considered are shown in Figures 8 and 9. The results are provided for $N = 50, 100$ and $150$. For comparison purpose, the $E_{rm}$ for the singular method are provided.

Typically, for a given $N$, as $f$ is increased, the $E_{rm}$ first decreases and then reaches a minimum. Further increase in $f$ results in deterioration of solution accuracy. The $f$ value at which the minimum $E_{rm}$ occurs seems to be highly dependent on the panel length. As $N$ is increased, the minimum shifts to smaller $f$ values. Similar behaviour of the desingularized solution has been noted by Cao et al. [1] in their study of three dimensional problems. Nevertheless, if one chooses the value of 0.5 for $f$, a substantial improvement in accuracy can be attained for a broad range of panel lengths.

As stated before, desingularization results in a Fredholm integral equation of the first kind. The linear system is then poorly conditioned and not diagonally dominant. This can result in poor accuracy of the computed solution. The linear system was solved using L-U decomposition with subsequent iterative refinement. Figure 10 shows the variation of the condition number of the system when $f$ is varied. The condition number increases rapidly with $f$. Concomitant with this increase is the loss of accuracy. The condition number for a given value of $f$ is higher for larger values of $N$. This explains the broader range of values of $f$ available for smaller $N$; see Figures 8 and 9.
2.3.1.2. Variation of Solution Accuracy with Panel Length

Figures 11 and 12 show the variation of $E_{rm}$ for the singular and desingularized cases when the complex potentials are given by, $\beta(z) = \sin(z)$ and $\beta(z) = e^z$, respectively.

The behaviour of the desingularized solution for small $f$, as expected, is similar to the singular case, although improved accuracy is obtained. For larger values of $f$, the behaviour is markedly different, showing improvement in accuracy for small values of $N$, and deterioration in accuracy as $N$ is increased. This is the same behaviour discussed in the previous section.

In Tables 1 and 2, the $E_{rm}$ of the solution obtained using the singular method are given as a function of $N$. The convergence rate is calculated as follows. Let $E_1$ and $E_2$ be the RMS errors of the solution when $N_1$ and $N_2$ nodes are used, respectively. Then the convergence rate $n$ is defined as

$$n = \frac{\log_{10}(\frac{E_2}{E_1})}{\log_{10}(\frac{N_1}{N_2})} \tag{2.26}$$

To calculate the convergence rate, we use the $E_{rm}$ corresponding to $N_1 = 80$ and $N_2 = 160$. 
Table 1. Solution Convergence. Circle, $\beta(x) = \sin(x)$.

<table>
<thead>
<tr>
<th>No of Nodes $N$</th>
<th>$-\log_{10}E_{rms}$</th>
<th>Convergence Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>2.103202</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>2.701260</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>3.272808</td>
<td></td>
</tr>
<tr>
<td>160</td>
<td>3.842676</td>
<td>1.89306</td>
</tr>
</tbody>
</table>

Table 2. Solution Convergence. Circle, $\beta(x) = e^x$.

<table>
<thead>
<tr>
<th>No of Nodes $N$</th>
<th>$-\log_{10}E_{rms}$</th>
<th>Convergence Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>1.955420</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>2.505050</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>3.044770</td>
<td>.</td>
</tr>
<tr>
<td>160</td>
<td>3.579547</td>
<td>1.776</td>
</tr>
</tbody>
</table>

The results for the desingularized method are not provided. This is done simply because of the behaviour of the solution explained in the preceding paragraphs, due to which the notion of convergence rate becomes invalid.

2.3.1.3. Effect of Boundary Conditions

Figures 13 and 14 show the variation of the relative error along the contour. For the singular method, an increase in error is seen at the nodes which are close to the intersection nodes or nodes common to both $C_s$ and $C_e$. The minimum value of the error occurs at nodal points midway from the intersection points. The increased error, although small, at nodal points close to the intersection points is clearly due to the change in boundary conditions. The results of the desingularized computations are also provided. For small values of $f$, the solution exhibits the same behavior as that obtained with the singular method, but the error at nodal points close to the intersection points
is greatly reduced. For higher values of $f$, the error due to boundary conditions is reduced to the values found at the nodal points midway from the intersection points. This can be seen from the results provided for $f = 0.5$. This reduction in error due to boundary conditions, explains the increased accuracy obtained with this method.

### 2.3.2. Rectangular Contour

A rectangular contour of dimensions $2 \times 1$ is chosen for this study. Uniform node spacing is used. The contour is shown in Figures 15 and 16. The contour is traversed anticlockwise, starting from the upper corner on the right hand side. The portion of the contour between nodes 1 and N1 is taken as $C_\phi$, and $\phi$ is specified there. The rest of the boundary, between nodes N1 and 1 belong to $C\psi$, and $\psi$ is specified there. This type of contour occurs in the wavemaker and wedge entry problems discussed in the next chapter. It is therefore of special interest to us.

#### 2.3.2.1. Effect of Desingularization

Figures 17 and 18 show the effect of desingularization for the complex potentials $\beta(z) = \sin(z)$ and $\beta(z) = e^z$. The control point $z_k$ is placed at a distance given by equation [2.24] from the corresponding nodal point, in the direction of the outward normal to the panel connecting the nodes $z_k$ and $z_{k+1}$. The desingularized solution does not show any improvement in accuracy for the broad range of $f$ values considered. For $f \geq 3$, the solution accuracy deteriorates rapidly. This is due to the problems related to ill-conditioning as discussed for the case of circular contours.
Schultz and Hong [25] have shown that, for equispaced nodes along flat panels, the dominant truncation error for the singular method vanishes. Thus, for such contours, the singular method provides the best results. The lack of improvement using the desingularized approach, therefore, is not surprising. It should be stressed here that the above discussion is valid only when one can speak of a dominant truncation error. This would require that the singularities of the complex potential be far away from the boundary. The effect of singularities present close to the boundary is examined in section 2.3.2.4.

2.3.2.2. Variation of Solution Accuracy with Panel Length

The variation of $E_{rm}$ with panel length for the complex potentials, $\beta(z) = \sin(z)$ and $\beta(z) = e^z$ are shown in Figures 19 and 20. As expected, the singular formulation provides better accuracy, although no significant loss of accuracy results by desingularization. The convergence of the solution obtained using the singular formulation for the two complex potentials is provided in Tables 3 and 4. The convergence rate is calculated using equation [2.26] using the $E_{rm}$ corresponding to $N_1 = 96$ and $N_2 = 180$. The convergence rate of the solution is equal to 2.0.

<table>
<thead>
<tr>
<th>No of Nodes $N$</th>
<th>$-\log_{10}E_{rm}$</th>
<th>Convergence Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>24</td>
<td>2.174228</td>
<td></td>
</tr>
<tr>
<td>48</td>
<td>2.807459</td>
<td></td>
</tr>
<tr>
<td>96</td>
<td>3.430998</td>
<td></td>
</tr>
<tr>
<td>180</td>
<td>3.980186</td>
<td>2.011</td>
</tr>
</tbody>
</table>

Table 3. Solution Convergence. Rectangle, $\beta(z) = \sin(z)$.
2.3.2.3. Effect of Corners

The rectangular contour studied here possesses corners. A large increase in error is known to result in the vicinity of such sharp curvatures or discontinuities, even when the solution there is well behaved. This has been shown by Schultz and Hong [25] to be due to the large increase in the truncation error, incurred in the approximation of the boundary integrals.

Figure 21 shows the variation of the relative error along $C_r$ for the complex potential, $\beta(z) = e^z$. The effect of the two corners is clearly evident, where the peaks occur. The dominant error in this case is due to the geometrical discontinuities, which is unlike the circular contour where the dominant error was due to the change in boundary conditions. The effect is very localized and the solution far away is not significantly affected. As can been seen, desingularization is not helpful in reducing this error.

To reduce this error, Schultz and Hong [25] propose a modification of the complex potential so that the dominant error at the corner vanishes. While this may lead to increased accuracy, it is clearly not practical, since in most physical problems of interest, the complex potential is unknown a priori. Another alternative is to bevel the corners.
But in many cases, the corner node is important and beveling will lead to erroneous results.

The dominant error can be reduced if the panel length is reduced. Figure 22 shows the results of a computation using 180 nodes on the boundary. Although the nature of error distribution remains the same, the peak error is reduced by 50%. The convergence rate of the solution does not seem to be affected by the presence of corners. The results for the desingularized calculations are also provided. As can be seen, no improvement is observed.

2.3.2.4. Effect of Singularities

The presence of singularities of the complex potential near the contour will lead to poor accuracy of the computed solution. This is due to the fact that, close to a singularity, the variation of the complex potential is very rapid. Therefore, the assumption of a linear variation will clearly lead to a loss of accuracy.

Figure 23 shows the results of the calculation for the complex potential, $\beta(z) = \log(z - 0.1(1 + i))$. The total number of nodes used was $N = 120$. A singularity is present outside the domain close to node 39. As expected, a large error is introduced on the portion of the boundary close to the singularity. The solution on other parts of the contour is not significantly affected, which indicates that the effect is very localized.

Also shown in the same Figure are the results obtained using the desingularized method. As the solution is well behaved locally for nodal points far away from the
singularity, it is sufficient to desingularize those portions close to the singularity. The portion of the boundary between nodes 28 and 41 is desingularized with \( f = 0.5 \). Considerable improvement in accuracy is obtained, with the peak error reduced by about 63%. No improvement in solution accuracy is observed at nodal points far away from the singularity.

Figure 24 shows the results using 180 nodes. The singularity is now close to node 58. The peak error for the singular method, in comparison with the results for \( N = 120 \), is reduced by 60%. The calculation using the desingularized method is done with \( f = 0.5 \). The portion between nodes 42 and 61 is desingularized. Desingularization is still very effective; the peak error obtained is 50% less than that for the singular method. The reduction in peak error for the desingularized method due to increase in the number of nodes is about 44%.

We have also carried out calculations for the complex potential, \( \beta(z) = \frac{1.0}{z - 0.1(1.0 + i)} \). This represents a stronger singularity than the one considered before. The results for the case of 180 nodes on the boundary are shown in Figure 25. The singularity is close to node 58. The singular method shows large errors close to the singularity. Desingularization leads to a great reduction in error in this case. The peak error is reduced by about 86%.

The effect of increasing the number of nodes is shown in Figure 26. The calculations were done with 240 nodes. In comparison to the results obtained with 120 nodes, the peak value of the error for the singular method is reduced by 45%. A similar reduction is obtained for the desingularized method. Even with \( N = 240 \), the accuracy near the singularity is poor for the singular method. On the other hand, we see that
desingularization can lead to greatly improved results. For this case, the maximum error is reduced by 87%.
Numerical Study of Nonlinear Free-Surface Flows

A numerical method involving time stepping and employing a mixed Eulerian-Lagrangian approach is used to study the nonlinear free-surface motions caused by the impulsive motion of a surface piercing body. The problem is solved as an initial-boundary-value problem. The boundary value problem at each time step is solved using the Complex Boundary Element Method. The time evolution of the flow is obtained by integrating the exact free-surface boundary conditions. The impulsive wavemaker problem, and the problem of a two-dimensional, symmetrical, constant-speed entry of wedges of various angles are considered.

The idea of employing a mixed Eulerian-Lagrangian approach to simulate free-surface flows was first proposed by Longuet-Higgins and Cokelet [16]. Their scheme combines the best features of the Eulerian and Lagrangian approaches. Under the assumption of irrotational flow, the free surface is composed of the same fluid particles. Therefore, a Lagrangian description permits the prediction of the free surface position at any instant in time, which is an important unknown in the Eulerian formulation. The particles on the free surface are followed by integrating the free-surface conditions. To do this,
however, the normal component of the velocity $\frac{\partial \phi}{\partial n}$ is required. An Eulerian description here is most desirable as it provides a linear governing equation, namely the Laplace equation. The normal velocity $\frac{\partial \phi}{\partial n}$ is determined by solving a boundary integral equation. The computational advantage of the scheme lies in the fact that all the information required for time stepping is on the boundary. The scheme was successfully used by them to simulate breaking waves in infinite water depth.

Vinje and Brevig [27], using the concept of a complex potential, while maintaining the key features of the above mentioned scheme, were able to simulate breaking waves in finite water depth. The governing integral equations for the boundary value problem were derived from Cauchy's integral theorem. The solution required the assumption of spatial periodicity. However, their method had numerical problems when a surface-piercing body was present.

At the intersection of the body and the free surface, due to the confluence of boundary conditions, a singularity is present (Lin [14], Cointe [2], Roberts [23], Yim [29]). The complex potential is not analytic there. The solution obtained from the boundary value problem, using the assumption of an analytic complex potential, therefore, cannot satisfy the boundary conditions. Modification in regard to the treatment of the intersection point was proposed by Lin [14].

According to Lin [14], the unknowns at the intersection point, namely, its location and the complex potential, are calculated from the body and free-surface boundary conditions. Thus at the intersection point, all quantities can be determined without solving the boundary value problem. This treatment was used by Lin [14] to study breaking waves generated by the motion of a simple harmonic wavemaker in a tank. It was subsequently used by Dommermuth et al. [4] to simulate deep-water plunging waves.
The numerical scheme used here is one that uses the method of Vinje and Brevig [27], with Lin's [14] modification for the treatment of the intersection point between the body and the free surface. Numerical results for the impulsive wavemaker motion and the two-dimensional, symmetrical, constant-speed entry of wedges of half-angles $\leq 15^\circ$ into initially calm water are presented. The results are compared with existing analytical and numerical solutions.

3.1. Mathematical Problem

The problem geometry and the coordinate axes system for the wavemaker and the wedge entry problems are shown in Figures 27 and 28, respectively. The fluid is contained in a rectangular tank and is assumed to be initially at rest. We present here a general mathematical formulation that describes the flow caused by the motion of a surface-piercing body.

The fluid in the region is assumed to be homogenous, incompressible and inviscid. Since the fluid is initially undisturbed, it follows from the inviscid assumption that the flow will remain irrotational at all times, everywhere. The flow can be described by using the velocity potential $\phi (x, y; t)$ defined as

$$\bar{u} = \nabla \phi \quad \text{in the fluid domain } R \quad [3.1]$$

where $\bar{u}$ denotes the velocity vector at the spatial location $(x, y)$ at time $t$. The description is Eulerian and so, $x$, $y$, and $t$ are independent variables. Since the flow is incompressible, we have
\[ \nabla \cdot \vec{u} = 0 \] in the fluid domain \( R \)  \[ {\text{[3.2]}} \]

Using equation \([3.1]\) in equation \([3.2]\), we obtain the governing equation for the flow

\[ \nabla^2 \phi = 0 \] in the fluid domain \( R \)  \[ {\text{[3.3]}} \]

which is the Laplace equation. Once the boundary and initial conditions are specified, the description of the problem is complete. The following boundary conditions are required to be satisfied.

On the free surface \( y = \eta(x, t) \), the kinematic and dynamic boundary conditions are to be satisfied. These are expressed in the sense of following the marked-particles on the free surface. These aspects are discussed below.

Kinematic free-surface boundary condition:

Suppose at some initial instant in time \( t = t_0 \), the free surface is defined by the curve \( y = \eta(x, t_0) \) and the velocity potential \( \phi(x, y; t_0) \) and its derivatives are known on it. Here, \( x, y, \) and \( t \) are the Eulerian variables. Let \( x_0(t), y_0(t) \) denote the Lagrangian coordinates of a particle on the free surface. Then the equation of motion for this particle is

\[ \frac{dx_0}{dt} = \frac{\partial \phi(x_0, y_0, t)}{\partial x} \]  \[ {\text{[3.4]}} \]

\[ \frac{dy_0}{dt} = \frac{\partial \phi(x_0, y_0, t)}{\partial y} \]  \[ {\text{[3.5]}} \]

Numerical Study of Nonlinear Free-Surface Flows
Integration of equations [3.4] and [3.5] in time would yield the position of the same particle, a short time \( dt \) later. Now, the free surface is always composed of the same particles. Therefore, the new location obtained for the marked-particle would lie on the curve defining the free surface at time \( t_0 + dt \). Thus, by considering a number of such marked free-surface particles, one can generate the new free-surface contour at time \( t_0 + dt \), whose equation is given by \( y = \eta(x, t_0 + dt) \). We note that in the Lagrangian approach, the kinematic free-surface boundary condition simply reduces to the equation of motion of marked-particles on the free surface; see equations [3.4] and [3.5].

Dynamic free-surface boundary condition:

This states that the pressure on the free surface is constant. This condition can be expressed using the Bernoulli equation. Thus, at a point \( x_0, y_0 \) occupied by a generic marked-particle on the free surface, we must have

\[
\frac{\partial \phi}{\partial t} = -gy_0 - \frac{|\nabla \phi|^2}{2} - \frac{p}{\rho} \tag{3.6}
\]

Now, the material derivative or the derivative following the motion is defined as

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \nabla \phi \cdot \nabla \tag{3.7}
\]

Using the definition of the derivative \( \frac{D}{Dt} \) given in equation [3.7], one can write

\[
\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + |\nabla \phi|^2 \tag{3.8}
\]

Eliminating \( \frac{\partial \phi}{\partial t} \) from equations [3.6] and [3.8], the dynamic free-surface boundary condition can be written as
\[
\frac{D\phi}{Dt} = -g\nu_0 + \frac{|\nabla \phi|^2}{2} - \frac{p}{\rho} \tag{3.9}
\]

In equations [3.6] and [3.9], \( g \) is the acceleration due to gravity, \( p \) is the pressure on the free surface, and \( \rho \) is the density of the fluid.

The velocity potential \( \phi(x, y; t) \) is expressed in the Eulerian description. Equation [3.9] expresses the rate of change of \( \phi \) following the motion of the free surface from one position to another. The field points \( x, y \) at which \( \phi \) is to be determined on the free surface are the Lagrangian coordinates of the marked-particles. Then, if one integrates equation [3.9] in time, one would obtain the value of \( \phi \) at time \( t_0 + dt \) on the new free-surface contour \( y = \eta(x, t_0 + dt) \), at \((x, y)\) locations of the marked-particles. Clearly, the obtained values of \( \phi \) on the new free-surface position would ensure that the dynamic free-surface boundary condition is satisfied.

Thus, by making use of the Lagrangian description, one can follow the marked-particles on the free surface and their Eulerian velocity potential \( \phi \) in time. This enables one to determine the position of the free surface at every instant in time. Also, by following the motion, the velocity potential \( \phi \) can be determined at field points \((x, y)\) occupied by the marked-particles on the free surface.

An Eulerian description for the velocity potential \( \phi \) has two major advantages. First, it enables one to work with a linear governing equation, namely, the Laplace equation; see equation [3.3]. Thus, one can make use of powerful properties of potential theory. Second, it permits the use of the Bernoulli equation to express the constant-pressure condition.
The other boundary conditions to be satisfied are the following.

No penetration condition on the body surface:

\[(\nabla \phi - \vec{V}) \cdot \vec{n} = 0\]  \[\text{[3.10]}\]

where \(\vec{V}\) is the velocity of the body, and \(\vec{n}\) is the unit outward normal to the body surface.

No penetration condition on the tank bottom:

\[\frac{\partial \phi}{\partial y} \bigg|_{y = -h} = 0\]  \[\text{[3.11]}\]

where \(h\) is the depth of the tank.

No penetration condition on the vertical wall downstream:

\[\frac{\partial \phi}{\partial x} \bigg|_{x = L} = 0\]  \[\text{[3.12]}\]

where \(L\) is the length of the tank.

For the wedge problem, we have the symmetry condition:

\[\frac{\partial \phi}{\partial x} = 0 \quad \text{on the axis of symmetry}\]  \[\text{[3.13]}\]

The initial conditions to be satisfied by the fluid everywhere in \(R\) are:
\[ \phi = \phi_t = 0 \text{ at } t = 0 \] \hspace{1cm} [3.14]

3.2. Nondimensional Parameters

One can nondimensionalize the variables involved by choosing length and time scales. The length and time scales are chosen to be \( h \) and \( (h/g)^{1/2} \), respectively. All other variables are nondimensionalized by the proper combination of \( g \) and \( h \). Here, \( h \) is the depth of the tank, and \( g \) is the acceleration due to gravity.

3.3. Method of Complex Analytic Function

3.3.1. Outline of the Method

Under the assumptions made in section 3.1, the flow can be described by a complex potential \( \beta(z; t) \) which is analytic in the fluid domain \( R \). This can be expressed as

\[ \beta(z; t) = \phi(x, y; t) + i\psi(x, y; t) \] \hspace{1cm} [3.15]

where \( z = x + iy \) is the complex coordinate and \( i = \sqrt{-1} \). Here, \( \phi \) and \( \psi \) are the velocity potential and stream function, respectively, for an incompressible, irrotational, inviscid fluid flow. These are defined at any point in the flow field as
\[
\frac{\partial \phi}{\partial x} = u \\
\frac{\partial \phi}{\partial y} = v \\
\frac{\partial \psi}{\partial y} = u \\
\frac{\partial \psi}{\partial x} = -v
\]  

[3.16]

where \( u \) and \( v \) are the velocities in the \( x \) and \( y \) directions, respectively at that point.

Since \( \beta \) is analytic in the fluid domain, it satisfies Cauchy's integral theorem. This implies

\[
\oint_C \frac{\beta(z)}{z - z_k} \, dz = 0
\]  

[3.17]

where \( C \) is a closed contour consisting of the free surface, the body surface, the portion between the body and the tank bottom, the tank bottom and the vertical rigid wall downstream; see Figures 27 and 28. The point \( z_k \) is outside \( R \cup C \), and the contour \( C \) is traversed such that the region \( R \) always lies to its left.

The contour \( C \) consists of portions where \( \phi \) is known, denoted by \( C_\phi \), and where \( \psi \) is known, denoted by \( C_\psi \). The free surface belongs to \( C_\psi \). On it, \( \phi \) can be calculated by integrating equations [3.4], [3.5], and [3.9], at all times, except at \( t = 0 \), when both \( \phi \) and the elevation of the free surface are zero.

The rest of the boundary is taken as \( C_\psi \) since the stream function \( \psi \) can be calculated from the normal velocity. On the tank bottom, downstream vertical wall, and on the
axis of symmetry, the stream function is taken to be zero. On the body surface, one can 
calculate analytically the stream function from the prescribed body motion.

Depending on the position of the kernel singularity $z_s$, one can derive a pair of integral 
equations relating the known and unknown values of the complex potential on the 
boundary. These aspects were described in detail in Sections 2.1.1 and 2.1.2 of Chapter 
2. These equations completely determine $\beta(z)$ on the boundary at any instant in time; 
see equations [2.4] and [2.5], and, [2.6] and [2.7].

The time evolution of the flow is determined by following the free-surface particles. 
The free-surface boundary conditions given by equations [3.4], [3.5] and [3.9] can be 
written as

$$\frac{dx_0}{dt} = u$$  \hspace{1cm} [3.18]

$$\frac{dy_0}{dt} = v$$  \hspace{1cm} [3.19]

$$\frac{D\beta}{Dt} = -y_0 + \frac{1}{2} w\bar{w} - \frac{p}{\rho}$$  \hspace{1cm} [3.20]

where $w$ is the complex velocity, and $\bar{w}$ is its complex conjugate. $x_0$ and $y_0$ denote the 
Lagrangian coordinates of a generic marked-particle on the free surface. The complex 
velocity $w$ is given by

$$w = u - iv = \frac{\partial \beta}{\partial z}$$  \hspace{1cm} [3.21]

The pressure $p$ on the free surface is taken to be zero.
The position of the body can be calculated analytically as its motion is prescribed. The other parts of the boundary are stationary. Therefore, no integration is required.

3.3.2. Numerical Solution

It is clear from the discussion in Section 3.3.1 that the solution of the problem essentially consists of two steps. The first step involves the solution of the boundary value problem to determine the unknown part of the complex potential $\beta$ on the boundary. The second step is concerned with the time stepping. These aspects will be discussed in detail in the following sections.

3.3.2.1. Solution of the Boundary Value Problem

From equations [3.18], [3.19], and [3.20] we note that in order to follow the evolution of the free surface, the velocities of the marked-particles on the free surface must be known. To calculate the the velocity field, the complex potential $\beta$ must be completely determined on the boundary; see equation [3.21]. This requires the solution of an integral equation for $\beta$.

To solve the integral equation for the complex potential, the Complex Boundary Element Method described in Section 2.2 of Chapter 2 is used. The only aspect that warrants further discussion regarding the solution of the boundary value problem is the numerical treatment of the intersection points.
As mentioned previously, a singularity is present at the intersection point between the body and free surface. Proper numerical treatment is required there to obtain realistic profiles. This can happen only if the relevant boundary conditions are satisfied. The treatment proposed by Lin [14] to accommodate the singularity is explained below.

As shown in Figures 27 and 28, node 1 represents the intersection of the free surface and the vertical wall downstream; node N1 represents the intersection of the free surface and the body surface. At both these nodes the stream function is known as it is prescribed on the respective \( C_\psi \) boundaries. The velocity potential \( \phi \) at these nodes can be calculated from the free-surface boundary conditions.

Thus at the intersection points, both the velocity potential \( \phi \) and the stream function \( \psi \) are known from the boundary conditions. Clearly, the complex potential is completely determined there and no effort is made to calculate it from the boundary integral equation. As can be seen, this method of calculating the complex potential ensures that at the intersection points, both the free-surface and the body boundary conditions are satisfied.

This is exactly the treatment discussed in Section 2.2.2. Subsequent development of the linear system and its solution procedure are provided in Section 2.2.3. For the sake of clarity, we state here that the final linear system of equations to be solved is given by equations [2.19] and [2.20].
3.3.2.1. Time Stepping

After the complex potential $\beta$ is completely determined on the boundary, the free-surface boundary conditions given by equations [3.18], [3.19] and [3.20] are integrated to step forward in time.

For numerical purposes, equations [3.18], [3.19], and [3.20] are treated as a system of ordinary differential equations. As there are $N_1$ nodes on the free surface, the system consists of $3 \times N_1$ ordinary, nonlinear, coupled, first order differential equations. They are integrated in time using Hamming's fourth order predictor-corrector method with a fourth order Runge-Kutta method as a starter. Since the Runge-Kutta algorithm requires four functional evaluations, the boundary value problem has to be solved four times every time step. The Hamming's algorithm requires two functional evaluations, which implies that the boundary value problem has to be solved twice at each time step.

The position and velocity potential at the intersection nodes 1 and $N_1$ at each time step are found by integrating the free-surface boundary conditions given by equations [3.18], [3.19], and [3.20]. The pressure at the intersection points is taken to be zero as they belong to the free surface.

To integrate equations [3.18], [3.19], and [3.20], one requires the fluid particle velocities. These are computed using a second order differential scheme provided by Vinje and Brevig [27]. For completeness, this is provided in Appendix 2. Table 5 summarizes the sequence of steps involved in the simulation.
Table 5. Brief Summary of the Simulation Algorithm

1. Input initial coordinate locations, law of motion of the body, problem geometry, analytical expression for the stream function on the body surface, stream function values on the solid boundaries, initial conditions, time step size, number of time steps.
2. Update the position and velocity potential of the marked free-surface particles by integrating the free-surface boundary conditions. Update also the body position and the stream function values on it.
3. Solve the boundary value problem to determine the stream function on the free surface and the velocity potential on the rest of the boundary.
4. Output the free-surface particle locations.
5. Repeat steps 2 to 4 for the required number of time steps.

3.3.3. Pressure Calculation

The primary motivation for the study of the wedge entry problem is the estimation of the pressure experienced by a body during slamming motions. The pressure is calculated using the Bernoulli equation. To do so, one requires \( \frac{\partial \phi}{\partial t} \) on the wedge since

\[
- \frac{p}{\rho} = \frac{\partial \phi}{\partial t} + \frac{1}{2} \ w \bar{w} + y \tag{3.22}
\]

This is found by solving a boundary value problem.

Since \( \frac{\partial B(z; t)}{\partial t} \) is analytic in the domain \( R \), one can write

\[
\oint_{c} \frac{\partial B(z; t) |\partial t}{z - z_k} \ dz = 0 \tag{3.23}
\]

On the free surface, \( \frac{\partial \phi}{\partial t} \) is known since

\[
\frac{\partial \phi}{\partial t} = - \frac{1}{2} \ w \bar{w} - y \tag{3.24}
\]
Vinje and Brevig [28] have derived an analytical expression for calculating \( \frac{\partial \psi}{\partial t} \) on the surface of an arbitrary, two-dimensional body undergoing a combination of translational and rotational motions. For the case of a wedge moving with a constant speed \( V \) in the negative y-direction, this yields

\[
\frac{\partial \psi}{\partial t} = Vu
\]

[3.25]

Here \( u \) is the velocity of the fluid particle. On the axis of symmetry and on the solid boundaries

\[
\frac{\partial \psi}{\partial t} = 0
\]

[3.26]

Thus, we have a boundary value problem to be solved to determine the unknowns, namely, \( \frac{\partial \phi}{\partial t} \) on the body surface, axis of symmetry, tank bottom, and the far vertical wall, and \( \frac{\partial \psi}{\partial t} \) on the free surface. The problem is solved numerically and the procedure is exactly as outlined for the complex potential \( \beta \). It is to be noted here that the influence coefficients \( \Gamma_{\alpha} \) required for the numerical solution are the ones used for the solution of boundary value problem for \( \beta \). Once \( \frac{\partial \phi}{\partial t} \) on the body surface is determined, the pressure can be calculated using equation [3.22].

3.4. Impulsive Wavemaker Problem

The numerical simulation of the free-surface flow due to the impulsive motion of a wavemaker in a rectangular tank containing the fluid is considered; see Figure 27. A
wavemaker starts to move with a constant horizontal velocity \( U \) from a state of rest. At \( t = 0^+ \) the velocity is a step function and the acceleration is infinite.

For this problem, an analytical solution valid for small times has been obtained by Peregrine [22]. According to this solution, the velocity potential \( \phi \), at any point \((x, y)\) in the domain at \( t = 0^+ \) is given by

\[
\phi_o = \sum_{n=0}^{\infty} \frac{2U}{h k_n^2} \sin k_n y \exp(-k_n x) \tag{3.27}
\]

where

\[
k_n = \frac{(2n + 1) \pi}{2h} \tag{3.28}
\]

and the free surface elevation \( \eta \) at a distance \( x \) measured from the instantaneous position of the wavemaker at time \( t \) is given by

\[
\eta = -\frac{2U t}{\pi} \log \left( \tanh \frac{\pi x}{4h} \right) \tag{3.29}
\]

In equations (3.27), (3.28), and (3.29), \( U \) is the constant horizontal velocity of the wavemaker and \( h \) is the depth of the tank. The solution was obtained using a perturbation expansion in time. The details of the derivation of the solution can be found in [14]. The free-surface profile predicted by equation (3.29) has been found by Lin [14] to agree well with experimental results everywhere except at the intersection point where the analytical solution predicts a contact point at infinity.
For the numerical study, a tank of nondimensional dimensions $5 \times 1$ was chosen. The nondimensional velocity of the wavemaker is 1.0. The simulation was carried out for small nondimensional time; from 0 to 0.2. No reflections were observed from the far boundary for this duration of simulation. The stream function on the wavemaker is given by

$$\psi = U (\nu + 1)$$

[3.30]

The panel length and the time step for this nonlinear problem have to be found from numerical experiments. A relation between the time step and the panel length based on a linear von Neumann stability analysis for the fourth order Runge-Kutta algorithm is provided by Dommermuth et al. [4]. It was found that this can only be used as a guide as the time step required to obtain a reasonably accurate solution was much smaller.

Figure 29 shows the time evolution of the free surface obtained using 120 nodes. The computation was done using the singular formulation. The initial node spacing was uniform. The time step size for this run was 0.01. For subsequent times, the panel lengths on both sides of the four corner nodes were adjusted to be equal. The results are compared with the analytical solution given by equation [3.29]. As can be seen, good agreement is obtained. The contact point obtained from the numerical results, however, is strictly finite. This is indicated by the highest symbol in each curve. For small times, slight disagreement is observed in the calculated position of the free-surface particles in the region close to the wavemaker.

The contour here has corners, which as seen in Chapter 2, can introduce errors. The convergence of the solution was, therefore, tested by doubling the number of nodes $N$ and halving the time step. Figure 30 shows the results obtained using a singular for-
mulation with \( N = 240 \) and a time step size of 0.005. A uniform initial node spacing was used. This results in a higher intersection point and the removal of the slight disagreement observed in the jet region for the previous run. The rest of the free-surface profile is unaffected.

For this problem, the location of the contact point is really arbitrary as a denser grid near the intersection will lead to a higher intersection point. The rest of the free-surface profile is unaffected. Thus, in that sense it can be said that the numerical solution does mimic the singular behavior predicted by the analytical result.

Figure 31 shows the distribution of the velocity potential on the wavemaker at \( t = 0^+ \) obtained using \( N = 240 \). The result is compared with the analytical solution given by equation [3.27]. Excellent agreement is obtained.

No problems related to stability were encountered in simulating this problem inspite of the fact that a singularity is predicted at the intersection point. The time stepping could be continued as long as problems related to reflections from the far boundary were not encountered.

Computations were also performed using the desingularized formulation. No improvement in the solution was observed for smaller number of nodes when \( f \leq 0.5 \). For larger values of \( f \) the accuracy of the solution deteriorated. Figure 32 shows the results of a computation using \( f = 0.5 \) with \( N = 120 \). The time step size was 0.01. The calculated intersection point is higher than that for the corresponding singular case; see Figure 29. Although it may be argued that this is more accurate, for problems where a stronger singularity is present, this is clearly not desirable as the large velocities at the
intersection point will ruin the numerical scheme. In view of this, the desingularized formulation was not pursued further.

3.4. Wedge Entry Problem

The problem of a two-dimensional, symmetrical, constant-speed entry of a wedge is considered. The wedge is assumed to enter at the center of the free surface of calm water contained in a rectangular tank. Due to the symmetry of the problem, only one half of the geometry is considered; see Figure 28.

For the numerical simulation, a rectangular tank of nondimensional dimensions $1 \times 1$ is chosen. The wedge vertex at time $t = 0$ is at the nondimensional depth of $-0.005$ and is at rest. The undisturbed free-surface level at time $t = 0$ coincides with the $x$-axis. At $t = 0^+$ the wedge begins to move impulsively with a constant speed $V$ in the negative $y$-direction. The stream function on the wedge is given by

$$\psi = Vx \quad [3.31]$$

For this problem, singularities are expected at the spray tip, spray root, and wedge vertex (Hughes [10]). The presence of the singularities can lead to numerical instability and, therefore, ruin the time stepping scheme. The panel length and the time step must be carefully chosen from numerical experiments to obtain an accurate solution.

As an exact analytical solution to this problem is not available, the panel length and the time step were determined on the basis of checks made on mass conservation; that
is, the area under the wave above the $t = 0$ free surface must be equal to the area surrounded by the instantaneous position of the wedge surface at time $t > 0$, the wedge surface at time $t = 0$, the free surface at $t = 0$, and the $y$-axis. It was found that the panel lengths on both sides of the intersection point must bear a certain ratio to obtain results that satisfactorily conserve mass.

Computations were carried out for a wedge of half-angle 9° using a total of 185 nodes on the boundary with $N1 = 100$, $N5 = 104$, $N8 = 140$, $N11 = 165$. The nondimensional speed of the wedge was 1.0. To obtain good resolution near the intersection point, the four nodes on the free surface closest to the intersection point $N1$ were placed at a nodal spacing of 0.004. Although a closer spacing is desirable, it was found that a very fine resolution of the jet flow leads to very high values for the calculated velocity. This gives rise to stability problems, which arrest the possibility of simulating the flow for the desired duration.

At subsequent times, the number of nodes on the wedge surface was varied. The three nodes on the wedge closest to the intersection node $N1$ were placed at a nodal spacing equal to the size of the panel on the free surface containing the node $N1$. For the rest of the wedge surface, a uniform node spacing equal to one-half of the size of the panel on the wedge surface containing the node $N1$ was used. It was found that placing nodes too close to the wedge vertex leads to numerical instability.

Figure 33 shows the time evolution of the free surface for a wedge of half-angle 9°. The simulation was carried out for 60 time steps, the size of the nondimensional time step being 0.0025. For this duration, no reflections from the far boundary were observed. For this calculation, the mass was conserved to within 1% of the mass of the fluid displaced by the wedge. As the wedge begins to move, a thin film of fluid is formed.
on the wedge surface. As time increases, this jet begins to moves up along the wedge surface. An interesting aspect of the flow is the movement of the marked free-surface particles. As time proceeds, the contact point and the node closest to it on the free surface tend to move away from the rest of the nodes. This may be due to the higher velocities calculated in the jet region. There is no substantial change in nodal spacing for the rest of the nodes on the free surface. No stability problems were encountered for this span of time and the obtained solution did not require any explicit smoothing.

A numerical solution based on the assumption of self-similarity was obtained by Dobrovol'skaya [3]. The solution neglects gravity. As verified by Hughes [10], this approximation is expected to hold everywhere except in the jet region as long as the neglected term $gt\nu$ is small. Dobrovol'skaya's [3] solution was obtained using the Wagner function.

Figure 34 shows the free-surface profiles plotted in terms of the similarity variables, $x/(vt)$ and $y/(vt)$, for a wedge of half-angle $9^{\circ}$. The free-surface profiles at time steps 30 and 60 are plotted. Except for the location of the contact point, the two profiles are indistinguishable indicating that the solution is self-similar. Also shown in the same Figure is Dobrovol'skaya's [3] result. Good agreement is observed. However, the calculated contact point differs from Dobrovol'skaya's value. It must be realized that in the present scheme, the position of the contact point is really arbitrary. A denser grid on the free surface close to the wedge will result in a higher intersection point.

Figure 35 shows the distribution of the dynamic pressure on the wedge surface for a wedge of half-angle $9^{\circ}$. The method used for calculating the pressure on the wedge surface is outlined in Section 3.3.3. The quantity plotted is the nondimensional pressure $2p/\rho$ obtained from equation [3.22] with the hydrostatic component subtracted out.
As the flow is self-similar, one of the pressure profiles is chosen here for comparison with Dobrovol’skaya’s [3] result. The maximum pressure occurs at the wedge vertex. The pressure decreases as one moves up along the wedge. Fairly good agreement with Dobrovol’skaya’s result is found everywhere except in the jet region where small negative pressures is observed.

To verify the convergence of the solution, a run was made with a time step of 0.00125. Figure 36 shows the time evolution of the free surface. Comparing with Figure 33, we find that the free-surface profiles are unchanged. Also, no difference in the pressure distribution was observed. As the contour has corners, one can expect large errors there. It was found that increasing the number of nodes on the solid boundaries did not affect the solution, indicating that the panel length used for the solid boundaries is adequate.

Computations were carried out for a wedge of half-angle 15° moving with a constant nondimensional speed equal to 1.0. The values of $N1$, $N5$, $N8$, and $N11$ are the same as that for the 9° case. The size of the two panels on the free surface closest to the node $N1$ was taken as 0.008.

As in the previous case, the number of nodes on the wedge was varied. The size of the three panels on the wedge surface closest to the intersection point was taken to be three-fifths of the size of the panel on the free surface containing the intersection point $N1$. For the rest of the wedge surface, a uniform nodal spacing equal to one-half of the size of the panel on the wedge surface containing the node $N1$ was used.

Figure 37 shows the time evolution of the free surface for a wedge of half-angle 15°. The simulation was carried out for 60 time steps with a time step of 0.0025. For this calculation the mass was conserved to within 3% of the mass of the fluid displaced by
the wedge. Although no problems related to stability were encountered, for later time steps the free-surface profile near the wedge surface showed problems concerning smoothness.

To investigate this, a run was made with a time step of 0.00125. The results are shown in Figure 38. This results in a slightly higher contact point for the later time steps and yields a smooth free-surface profile. The mass was conserved to within 2%. The rest of the free-surface profile is unaffected. No difference was found in the calculated pressure on the wedge. In view of this, it was deemed adequate to carry out further calculations with a time step of 0.0025.

The obtained solution was tested for self-similarity. Figure 39 shows the plot of the free surface profiles in terms of the similarity variables. The profiles at time steps 30 and 55 are provided. The solution, as can be seen, is self-similar.

The computed pressure distribution on the wedge is shown in Figure 35. The behavior of the pressure distribution on the wedge surface is similar to that for the 9° case, although for this case, higher pressures are experienced by the wedge. The results are compared with the values obtained by Greenhow [7]. Greenhow's results are obtained from a computational scheme which locates the intersection point as the point at which the pressure vanishes. Good agreement is obtained everywhere except in the jet region where discrepancies are noted. This is clearly due to the treatment of the intersection point. The results obtained also show negative pressures on the wetted portion of the wedge close to the contact point which may indicate separation.

We have tried to simulate the wedge entry problem for a wedge of half-angle 30°. Severe difficulties related to numerical instability were encountered. The principal diffi-
culty is that a sufficient resolution of the flow close to the intersection point required to obtain satisfactory mass conservation leads to very large velocities for the intersection point. This eventually leads to the breakdown of the scheme within a few time steps. This problem is present regardless of the time step. The evolution of the flow, therefore, could not be simulated for the desired duration of time.

On the other hand, sparse nodal spacing near the intersection point led to very poor mass conservation. Large negative pressures were seen on the wedge. The situation is found to get worse for wedges of larger angles.

3.5. Discussion

It is clear from the discussion in the previous Section that the severe difficulties encountered by the present numerical scheme in simulating the flow for larger wedge angles is related to the proper description of the flow in the jet region. The problem of instability is present even for smaller wedge angles, as a very fine resolution near the intersection point does lead to breakdown of the scheme. However, time stepping can be achieved with sufficient resolution of the flow to obtain a solution that satisfactorily conserves mass.

Exactly what causes this instability for larger wedge angles is unknown. Unfortunately, very little is known about the behavior of the flow in the jet region from an analytical point of view. Some results regarding the shape of the free surface and the contact angle between the wedge and the free surface are provided by Mackie [19]. But
verification of these results is not possible with the present scheme as it would require a very fine resolution of the flow near the contact point. The solution obtained by Dobrovol'skaya [3] using the Wagner function predicts a strictly finite value for the velocity of the intersection point. The bounded solution is obtained by imposing a constraint on the contact angle. The solution, however, is not expected to provide an accurate description of the flow in the jet region. The presence of singularities at the spray tip, spray root, and wedge vertex was first pointed out by Hughes [10]. But he does not provide any information about its nature.

In the present numerical scheme, the intersection point is clearly not a regular point. This has been demonstrated by Yim [29]. He pointed out that Lin's [14] technique for the treatment of the intersection points is the only way to obtain realistic free-surface profiles. The presence of a singularity really should only affect the flow locally. But in a time stepping scheme, the presence of a singularity contaminates the solution everywhere, which eliminates the possibility of obtaining an accurate solution far from the intersection point.

Clearly, the intersection point in nature does not have an infinite velocity. Experimental observations by Lin [14] and Greenhow [7] indicate that as the wedge begins to move, a thin film of fluid quickly moves along the wedge, which subsequently peels off the wedge surface. The negative pressures seen on the wedge may thus be due to retaining the intersection point which actually does separate.

A numerical scheme that mimics this phenomenon was proposed by Greenhow [7]. His scheme was able to simulate wedges of half-angles up to 30°. No negative pressures were seen on the wedge. However, for higher wedge angles, the same problem associated
with numerical instability is encountered, which indicates that the appearance of negative pressures is only part of the problem.

Some insight into the nature of the flow near the juncture point can be obtained from the closely related wavemaker problem for which considerably more analytical results are available. Investigations of the transient wavemaker motions by Cointe [2] and Roberts [23] have revealed the existence of a singularity at the intersection point in the analytical solutions. Roberts [23] pointed out that the description of the flow in the region close to the wavemaker requires the use of an inner variable. According to him, in the inner region, the space and time variables are coupled and, therefore, cannot be separated. The solution obtained by Peregrine [22], using a perturbation expansion in time, is thus really an outer solution. By introducing an inner variable to describe the flow in the inner region, Roberts was able to find bounded solutions for wavemaker motions when the acceleration is less than \( g \). However, for an impulsive wavemaker, where the initial acceleration is infinite, a bounded solution could not be found.

The existence of an inner region has also been pointed out by Joo et al. [12]. Although their analysis takes into account surface tension, no bounded solution at the intersection point could be obtained for large accelerations. It must be pointed out that these analysis were performed using a linear theory.

The present scheme essentially separates spatial and time variables, which may be the reason for the numerical instability. However, more investigations have to be undertaken to understand the behavior of the flow near the juncture point before any conclusions can be reached.
Conclusions

In this thesis, we have undertaken a numerical study of time-dependant nonlinear free-surface flows caused by the impulsive motion of a surface-piercing body. The numerical scheme is based on the Cauchy-theorem method of Vinje and Brevig [27] with Lin's [14] modification for the treatment of the intersection points.

The numerical performance of the method used to solve the boundary value problem at each time step, namely, the Complex Boundary Element Method, is studied for the singular and the desingularized formulations by considering some simple complex potentials. It is found that changes in boundary conditions, presence of corners, presence of singularities close to the boundary, result in a loss of accuracy of the obtained solution. However, these effects are highly localized and do not affect the solution accuracy elsewhere.

The accuracy of the solution obtained using the desingularized formulation seems to be highly dependent on the desingularization distance. It is found that the desingularization distance required to obtain an accurate solution is strongly dependent
on the panel length. To avoid problems related to ill-conditioning, a value of 0.5 for the desingularization parameter \( f \) is found to be adequate.

The performance of the desingularized formulation is also dependent on the problem geometry and on the solution characteristics. For circular contours, desingularization results in substantial improvement in accuracy - by an order of magnitude. However, for rectangular contours, improvement in accuracy is observed only when a singularity is present. This may be useful in the study of problems which are not time-dependant and where a singularity is known to exist outside the domain but close to the boundary.

The numerical scheme with Lin's [14] treatment of the intersection point has been successful in simulating the flow due to the motion of an impulsive wavemaker. The obtained results agree extremely well with analytical predictions. Although the calculated value at the intersection point is finite, it is found that the numerical scheme does mimic the singular behaviour predicted by the analytical solution.

The calculations performed using the desingularized formulation resulted in a higher intersection point than that obtained using the singular formulation. It is, therefore, more accurate in the sense that it mimics more closely the singular behaviour at the intersection. However, for problems where a stronger singularity is present, a singular formulation is preferable as a desingularized formulation will result in higher velocities at the intersection point and, therefore, earlier breakdown of the numerical scheme.

The problem of wedge entry has been simulated for wedges of half-angles \( \leq 15^\circ \). It is found that the flow is indeed self-similar as assumed in earlier theories. The calculated free-surface profiles and pressure distributions are in satisfactory agreement, everywhere except in the jet region, with existing solutions. Small negative pressures are seen on the
wetted portion of the wedge close to the juncture point; the magnitude of it is found to increase with the wedge angle.

For wedges of larger half-angles severe problems related to numerical instability were faced. Sufficient resolution of the flow near the juncture point required to conserve mass, results in very high velocities in the jet. This eventually leads to the breakdown of the scheme within a few time steps.

In order to simulate slamming motion of ship-shaped bodies, it is necessary to consider motions of wedges of large included angles. The lack of understanding of the behavior of the flow near the juncture point, especially the nature of the singularities present, represents the biggest obstacle to making any modifications in the present scheme to carry out such studies. Future work must, therefore, be directed towards understanding and modeling the flow in the jet region that would eventually permit numerical simulation of wedges of large angles.
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\[ C = C_\phi \cup C_\psi \]
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$C = C_\phi \cup C_\psi$
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Desingularized Method: $f=0.5$

Desingularized Method: $f=1.0$

Desingularized Method: $f=2.0$
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Figure 14. Variation of Relative Error in Computed Velocity Potential: Circular contour, $\beta(z) = e^z$. Total number of nodes $N = 60$. 

Singular Method

Desingularized Method: $f = 0.1$

: $f = 0.5$
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Desingularized Method: $N = 60$
Singular Method: $N = 60$
Figure 19. Variation of RMS error with N: Rectangular contour, \( \beta(x) = \sin(x) \).
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**Singular Method**  
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Figure 38. Development of the flow for a Wedge of Half-Angle 15 degrees: The results are for \( N = 185 \) and time step size of 0.00125. The symbols in each curve represent the marked free surface particles.
Figure 39. Self-Similar Flow for a Wedge of Half-Angle 15 degrees: The computed results are for $N = 185$ and time step size of 0.0025.
References


Appendix 1

The Fredholm Integral Equation

Consider the complex potential $\beta(z) = \phi(z) + i\psi(z)$ analytic in the region $R \cup C$; see Figure 40. Then Cauchy's integral theorem yields

$$\oint_C \frac{\beta(z)}{z - z_o} \, dz = 0 \tag{A1.1}$$

where $z_o$ is outside $R \cup C$, and the contour $C$ is traversed such that the region $R$ always lies to its left. We let $z_o$ approach the point $z_k$ on the boundary $C$ from outside. In the limit we obtain

$$\oint_C \frac{\beta(z)}{z - z_o} \, dz = \oint_C \frac{\beta(z)}{z - z_k} \, dz + \oint_{C_1} \frac{\beta(z)}{z - z_k} \, dz \tag{A1.2}$$

$$= 0$$
where $C_\epsilon$ is the part of $C$ consisting of a circular arc of radius $\epsilon$ centered at the point $z_k$; see Figure 41. The first integral on the right hand side in equation [A1.2] represents the Cauchy principal value.

Along $C_\epsilon$, we have

$$z = z_k + \epsilon e^{i\alpha}$$ \hspace{1cm} [A1.3]

$$dz = \epsilon ie^{i\alpha}d\alpha = (z - z_k)id\alpha$$ \hspace{1cm} [A1.4]

A Taylor expansion of the complex potential around $z = z_k$ yields

$$\beta(z) = \beta(z_k) + \frac{d\beta(z_k)}{dz} \epsilon e^{i\alpha} + O(\epsilon^2) + ...$$ \hspace{1cm} [A1.5]

Using results [A1.3], [A1.4], and [A1.5], the second integral on the right hand side of equation [A1.2] becomes

$$\int_{C_\epsilon} \frac{\beta(z)}{z - z_k} \, dz = i\beta(z_k)\alpha_k$$ \hspace{1cm} [A1.6]

$$+ \epsilon e \frac{d\beta(z_k)}{dz} (e^{i\alpha_k} - 1) + O(\epsilon^2) + ...$$

where the integration is carried out from $\alpha = 0$ to $\alpha = \alpha_k$. In the limit when $\epsilon \to 0$, we obtain

$$\lim_{\epsilon \to 0} \int_{C_\epsilon} \frac{\beta(z)}{z - z_k} \, dz = i\alpha_k\beta(z_k)$$ \hspace{1cm} [A1.7]

$\alpha_k$ is equal to $-\pi$ when $z_k$ is on a smooth curve.
Substituting equation [A1.7] into [A1.2] and replacing $dz$ by $e^{i\theta} \, ds$ where $ds$ is a line element of the contour $C$, yields

$$\oint_C \frac{\beta(z)}{z - z_k} \, dz = \oint_C \frac{\beta(z)}{z - z_k} \, e^{i\theta} \, ds + i\alpha_k \beta(z_k) \quad [A1.8]$$

One could put either the real or the imaginary part of the integral equation [A1.8] equal to zero. However, for numerical purposes a Fredholm integral equation of the second kind is desired. This is obtained as follows.

If $\psi$ is the unknown part of the complex potential $\beta$ at $z_k$, i.e. $z_k$ belongs to $C*$, we take the real part of equation [A1.8] to be zero. Thus

$$Re \left\{ \oint_C \frac{\beta(z)}{z - z_k} \, dz \right\} = -\alpha_k \psi(z_k)$$

$$+ \oint_C \phi(s) Re \left( \frac{e^{i\theta}}{z - z_k} \right) ds$$

$$- \oint_C \psi(s) Im \left( \frac{e^{i\theta}}{z - z_k} \right) ds$$

which can be written as

$$Re \left\{ \oint_C \frac{\beta(z)}{z - z_k} \, dz \right\} = -\alpha_k \psi(z_k) + \underline{h}(z_k)$$

$$+ \int_C \psi(s) \underline{g}(z_k, s) \, ds = 0 \quad [A1.10]$$
which is an inhomogenous Fredholm integral equation of the second kind.

If \( \phi \) is the unknown part of the complex potential \( \beta \) at \( z \), i.e. \( z \) belongs to \( C_p \), we take the imaginary part of equation [A1.8] to be zero. Thus

\[
\text{Re}\left\{ i \oint_C \frac{\beta(z)}{z - z_k} \, dz \right\} = \alpha_k \phi(z_k) \\
- \oint_C \phi(s) \text{Im}\left( \frac{e^{i\theta}}{z - z_k} \right) \, ds \\
- \oint_C \psi(s) \text{Re}\left( \frac{e^{i\theta}}{z - z_k} \right) \, ds
\]

which can be written as

\[
\text{Re}\left\{ i \oint_C \frac{\beta(z)}{z - z_k} \, dz \right\} = \alpha_k \phi(z_k) + l(z_k) \\
+ \oint_C \phi(s)m(z_k, s) \, ds = 0
\]

which is a inhomogenous Fredholm integral equation of the second kind.
Figure 40. Domain and Bounding Surfaces: The region where the complex potential is analytic is shown as \( R \).
Figure 41. Deformation of the Contour near the Kernel Singularity: The point $z_\varepsilon$ is let to approach the point $z_k$ on the contour $C$ from outside.
Appendix 2

Calculation of the Velocities on the Boundary

To determine the velocities at the nodal points on the boundary, a second order scheme is used. The velocity at the nodal point $z_j$ is given by

$$u_j + iv_j = \left( \frac{d\beta}{dz} \right)_j$$  \hspace{1cm} [A2.1]

where the bar denotes complex conjugate and $i = \sqrt{-1}$. We write the derivative of the complex potential at $z_j$ as (see Figure 42)

$$\left( \frac{d\beta}{dz} \right)_j = a_{j-1} \beta_{j-1} + a_j \beta_j + a_{j+1} \beta_{j+1} - R$$  \hspace{1cm} [A2.2]

Here $R$ is an error term. Expanding $\beta_{j-1}$ and $\beta_{j+1}$ in a Taylor series about the point $z_j$ gives
$\beta_{j-1} = \sum_{p=0}^{\infty} \frac{\beta_j^{(p)}}{p!} (z_{j-1} - z_j)^p \quad [A2.3]$ 

$\beta_{j+1} = \sum_{p=0}^{\infty} \frac{\beta_j^{(p)}}{p!} (z_{j+1} - z_j)^p \quad [A2.4]$ 

Substituting equations [A2.3] and [A2.4] into equation [A2.2], we obtain

$$
\left( \frac{d^2 \beta}{dz^2} \right)_j = (a_{j-1} + a_j + a_{j+1}) \beta_j \\
+ (a_{j-1} (z_{j-1} - z_j) + a_{j+1} (z_{j+1} - z_j)) \left( \frac{d \beta}{dz} \right)_j \\
+ \frac{1}{2} (a_{j-1} (z_{j-1} - z_j)^2 + a_{j+1} (z_{j+1} - z_j)^2) \left( \frac{d^2 \beta}{dz^2} \right)_j \\
+ \sum_{p=3}^{\infty} \frac{\beta_j^{(p)}}{p!} \left( (a_{j-1} (z_{j-1} - z_j)^p + a_{j+1} (z_{j+1} - z_j)^p) \right)
$$

which gives

$$a_{j-1} + a_j + a_{j+1} = 0 \quad [A2.6]$$

$$a_{j-1} (z_{j-1} - z_j) + a_{j+1} (z_{j+1} - z_j) = 1 \quad [A2.7]$$

$$a_{j-1} (z_{j-1} - z_j)^2 + a_{j+1} (z_{j+1} - z_j)^2 = 0 \quad [A2.8]$$

$$R = - \sum_{p=3}^{\infty} \frac{\beta_j^{(p)}}{p!} \left( (a_{j-1} (z_{j-1} - z_j)^p + a_{j+1} (z_{j+1} - z_j)^p) \right) \quad [A2.9]$$
The solution of the linear system given by equations [A2.6], [A2.7], and [A2.8] yields

\[
a_{j-1} = \frac{(z_{j+1} - z_j)}{(z_{j+1} - z_j)(z_{j-1} - z_j) - (z_{j-1} - z_j)^2} \quad [A2.10]
\]

\[
a_{j+1} = \frac{(z_{j-1} - z_j)}{(z_{j-1} - z_j)(z_{j+1} - z_j) - (z_{j+1} - z_j)^2} \quad [A2.11]
\]

\[
a_j = - a_{j-1} - a_{j+1} \quad [A2.12]
\]

For calculating the tangential velocity \( v_t \) of the fluid particle at the node \( N \), which represents the intersection of the free surface with the moving body (see Figures 27 and 32), we write

\[
\left( \frac{d\phi}{ds} \right)_{N1} = a_{N1} \phi_{N1} + a_{N1+1} \phi_{N1+1} + a_{N1+2} \phi_{N1+2} - R \quad [A2.13]
\]

where \( s \) represents the tangential coordinate along the body surface and \( (v_t)_N = \left( \frac{d\phi}{ds} \right)_{N1} \).

The coefficients \( a_{N1} \), \( a_{N1+1} \), and \( a_{N1+2} \) are obtained by expanding \( \phi_{N1+1} \) and \( \phi_{N1+2} \) in a Taylor series about \( s_{N1} \) and using these expressions in equation [A2.13]. The coefficients are

\[
a_{N1+1} = \frac{(s_{N1+2} - s_{N1})}{(s_{N1+1} - s_{N1})(s_{N1+2} - s_{N1}) - (s_{N1+1} - s_{N1})^2} \quad [A2.14]
\]

\[
a_{N1+2} = \frac{(s_{N1+1} - s_{N1})}{(s_{N1+2} - s_{N1})(s_{N1+1} - s_{N1}) - (s_{N1+2} - s_{N1})^2} \quad [A2.15]
\]
\[ a_{N1} = \frac{a_{N1+1} - a_{N1+2}}{} \]  \[ A2.16 \]

The normal velocity is calculated using the body boundary condition. Once the normal and tangential velocities are calculated at \( N1 \), a suitable transformation is used to obtain the \( x \) and \( y \) components of the fluid particle velocity at node \( N1 \).

A similar procedure is used to calculate the velocity at node \( I \) which represents the intersection of the free surface with the far vertical wall. The tangential coordinate here is along the \( y - \) axis.
Figure 42. Variation of Complex Potential on the Boundary: The complex velocity at node $z_j$ is calculated using $\beta_{j-1}$, $\beta_j$, and $\beta_{j+1}$. 
Premkumar Muthedath was born in 1965 in India. He received his B. Tech degree in Naval Architecture from Indian Institute of Technology, Kharagpur, India, in June 1989. In the same year, he was enrolled as a graduate student in the Department of Aerospace and Ocean Engineering at Virginia Tech. Mr. Muthedath is intending to continue his graduate studies in the Department of Environmental Sciences and Engineering at University of North Carolina at Chapel Hill.