AN APPLICATION OF RELAXATION METHODS

TO

TRANSONIC NOZZLE FLOW

by

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<td>( \hat{c} )</td>
<td>( c/c_0 )</td>
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<td>Total length of constant-area section and nozzle</td>
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<td>( P_b )</td>
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Subscripts

a  Actual
e  Refers to nozzle exit
id  Ideal
i  Mesh point index in the axial direction
I  Largest axial index
j  Mesh point index in the radial direction
J  Largest radial index
o  Stagnation conditions

\[ y \quad \text{Lateral coordinate in the physical plane (for planar flow)} \]
\[ \gamma \quad \text{Ratio of specific heats} \]
\[ \Delta x \quad \text{Mesh spacing in axial direction (physical plane)} \]
\[ \Delta r \quad \text{Mesh spacing in radial direction (physical plane)} \]
\[ \Delta \xi \quad \text{Mesh spacing in axial direction (computational plane)} \]
\[ \Delta \psi \quad \text{Mesh spacing in radial direction (computational plane)} \]
\[ \theta \quad \text{Flow angle} \]
\[ \theta_w \quad \text{Nozzle wall angle} \]
\[ \xi \quad \text{Axial coordinate in the computational plane} \]
\[ \rho \quad \text{Density} \]
\[ \phi \quad \text{Velocity potential function} \]
\[ \phi' \quad \text{Non-dimensionalized potential function} \]
\[ \tilde{\phi} \quad \text{Column vector of unknown } \phi \text{'s} \]
\[ \psi \quad \text{Radial coordinate in the computational plane} \]
\[ \psi \quad \text{Steam function} \]
\[ \omega \quad \text{Relaxation factor} \]
r  Radial derivative in the physical plane
th  Refers to nozzle throat
W  Refers to nozzle wall
x  Axial derivative in the physical plane
ξ  Axial derivative in the computational plane
ψ  Radial derivative in the computational plane

Superscripts
*
  Refers to sonic conditions
-
  Average
~
  Matrix or vector
1. INTRODUCTION

1.1 Problem Description

An exhaust nozzle is a critical component of any turbojet propulsion system. Its purpose is to create thrust by accelerating exhaust gases to a high velocity. In the exhaust nozzles found on today's turbojet aircraft, velocities reach and exceed the speed of sound. Accurate analytical methods capable of handling these transonic flows are required for the design and development of new nozzle configurations. This is underscored by studies of supersonic transport aircraft (1) which have shown that errors of one percent in such operating characteristics as the velocity coefficient result in errors in the predicted specific fuel consumption of over two percent and errors in the operating cost predictions of over three percent.

The desired nozzle performance data could be obtained analytically by the solution of the Navier-Stokes equations. Although some progress is today being made in their solution, the mathematical complexity of these equations preclude their use in this study. Instead, the simpler Euler's equations which assume inviscid flow were used. Adiabatic flow was also assumed, giving isentropic flow. These assumptions were justified by the simplicity of Euler's equations, the ability to include viscous effects by means of boundary layer analysis once Euler's equations are solved, and the fact that low Mach number shocks ($M \leq 1.3$) cause negligible entropy changes.
Even Euler's equations, when further simplified by the assumption of steady flow of a perfect gas, are presently without a closed-form, analytical (as opposed to numerical) solution. The equations contain second order partial derivatives and are only quasi-linear, or linear in the highest order derivatives. An additional difficulty arises from a change in equation type with the local Mach number, from elliptic in subsonic flow to hyperbolic in supersonic flow. Any solution method, therefore, must be capable of handling non-linear, partial differential equations of mixed type.

1.2 The Need for Numerical Relaxation

In 150 years of transonic nozzle flow investigation, observers have commented that as many solution methods have appeared as there are problems and investigators in the field. These methods, however, can be divided into two types, indirect and direct (1). In the indirect methods a boundary geometry is calculated from a specified velocity or pressure distribution, while in the direct methods a geometry is specified and the characteristics of the resulting flow field are calculated. Only direct methods will be considered here because in most exhaust nozzle problems the boundary geometry is specified and the resulting performance is desired. Direct methods can with a few exceptions be divided into series expansion, time-dependent, and relaxation methods. Of most current interest are the time-dependent and relaxation methods which require the use of digital computers.

The series expansion methods make use of polynomial representations of the dependent variables. The coefficients of these polynomials are
obtained from the specified nozzle geometry. Most of these methods use the small perturbation form of the transonic flow equation, and thus the results are restricted to nozzles with large radii of curvature. Furthermore, the accuracy of the solution is reduced in regions away from the nozzle throat. Even methods which use the full governing equations, such as Oswatitsch and Rothstein (2), are limited in terms of the throat radius of curvature. In addition, convergent nozzles cannot be handled due to the required specification of boundary geometry (1).

In time-dependent methods, the difficulty introduced by the equation being of a mixed type is overcome by the introduction of unsteady, time-dependent terms which change the governing equation to hyperbolic type independent of the local Mach number. The steady, mixed-flow, boundary (elliptic) and initial value (hyperbolic) problem is then solved as the asymptotic limit of an unsteady, initial value problem. Unlike series expansion methods, time-dependent solutions have been obtained for convergent nozzles, are accurate for very small radii of curvature, and have shown promise in calculating shock and viscous flows and flows with inlet non-uniformities (1). Disadvantages are excessive computation time (1) and inaccuracy in the subsonic region (3).

Relaxation methods encompass the iterative techniques of Richardson, Liebman (or Gauss and Seidel) and Southwell (as applied cyclically for machine computation) (4). Stripped of modifications which increase the rate of convergence, relaxation consists simply of obtaining successive approximations to the solution of the finite-difference form of the governing equations. The method as originally developed was applied
to the solution of elliptic, boundary value problems, but Murman and Cole (5) determined that the use of backward differencing in forming derivative approximations in hyperbolic (supersonic) regions would enable relaxation to handle mixed-type problems. According to Murman, relaxation methods when thus modified are capable not only of handling transonic flows, but of doing so at five to ten times the computational speed of time-dependent methods with no loss in accuracy (6). A study by Brown (1) comparing times for the two methods in terms of computation time per point, normalized with respect to the same computer, shows relaxation to be an order of magnitude faster than time-dependent solutions. The computational efficiency of relaxation methods has as yet only been demonstrated for external flow problems such as airfoil calculations; however, similar economy is expected for nozzle flow problems. Because of this the relaxation method was chosen for use in this investigation.
2. LITERATURE REVIEW

The papers on numerical relaxation related to transonic flows deal almost exclusively with external flow, specifically flow about airfoils and bodies of revolution such as cone-cylinder combinations. In a later section the extension of these methods to the problem of transonic nozzle flow will be discussed.

2.1 Solution of Transonic Small Perturbation Equation

In his first paper with Cole (5) and in a later individual work (6), Murman described the use of relaxation with a type-dependent, finite-difference technique to solve the transonic small perturbation equation which was written as

\[ [K - (\gamma + 1)\phi_x] \phi_{xx} + \phi_{yy} = 0 \]

where \( K \) is the transonic similarity parameter. The type-dependent finite-difference technique had the purpose of maintaining the domain of dependence of the differential equation in its finite-difference representation by relating the finite-difference expressions to the local Mach number.

In the subsonic (elliptic) region, a centered difference expression was used for the derivatives in the flow direction (\( \phi_x \) and \( \phi_{xx} \)) consistent with the equation's domain of dependence which extends infinitely far in all directions. In the supersonic (hyperbolic) region, however, an upwind difference expression was used for these derivatives since
the domain of dependence extends only upstream, bounded by right and left running characteristics. (This bounded domain of dependence in supersonic flow is represented physically as the well-known zones of silence and action.) This retarded differencing, with either an explicit formulation obeying the Courant-Friedrichs-Lewy criterion or an implicit formulation, is necessary for stability in the supersonic region. The centered difference is consistent with the domain of dependence of the radial derivative ($\partial_r$) in both regions, since a column of points is solved simultaneously.

Murman and Cole's centered differences were of second order accuracy and used two mesh points for the first derivative and three points for the second. Both first order accurate (two and three point) and second order accurate (four point) backward differences were tried, with the conclusion that the first order form gave more reliable results. The algebraic equations obtained by the substitution of these finite-difference expressions were solved column by column in the downstream direction by successive line relaxation using a tridiagonal algorithm (7).

The small perturbation method of Murman and Cole was modified to handle sonic mesh points by Murman and Krupp (8), to insure that exactly sonic points resulted in hyperbolic equations. The method was applied successfully to numerous external flow problems involving airfoils and bodies of revolution by these investigators (9), and by Bailey (10), Bailey and Ballhaus (11), Ballhaus and Bailey (12), and Caughey (13). Bailey and Steger (14) used a combination of the potential
function and primitive velocity variable forms of the small disturbance equation. All results agreed well with available theoretical and experimental data.

Yoshihara (15) discussed the small disturbance procedure of Murman and Cole in a general review of transonic computational methods. He noted the possibility of a matching problem between flow types along the sonic line in cases where the centered difference indicated a supersonic point, but the backward difference resulted in a subsonic velocity at the point. Yoshihara added that Krupp (16) arbitrarily used a centered finite-difference in such cases, and that while such a treatment has no physical or mathematical basis, it would have significant consequences only when a shock occurred near the sonic line.

Yoshihara also pointed out the stabilizing effect of line relaxation by columns for transonic flow. According to Yoshihara, point relaxation converges for nearly incompressible flows because the low propagation of disturbances causes only small residuals at points surrounding the point being relaxed. In high subsonic flows, however, propagation of disturbances in the radial direction is greatly increased. Point relaxation of these flows would result in large residuals forming at points above and below the point being relaxed, inhibiting the convergence of a solution. Relaying all points in a given column simultaneously avoids this problem and aids convergence.

Yoshihara felt that solutions of the transonic small perturbation equation were incapable of properly capturing shocks. In any event
he concluded that since for two-dimensional flows the use of the full governing equations requires essentially the same computation time, little was gained by using the small perturbation approximation in the first place.

### 2.2 Solution of Euler's Equations

The second group of procedures reviewed are those which use the type-dependent procedure of Murman and Cole to solve Euler's equations rather than the transonic small perturbation equation. Garabedian and Korn (17) used the potential function form of Euler's equations, which for planar, two-dimensional flow is

\[
(c^2 - u^2)\phi_{xx} - 2uv \phi_{xy} + (c^2 - v^2)\phi_{yy} = 0
\]

where

\[
\phi_x \equiv u \\
\phi_y \equiv v
\]

and \(c\), the local speed of sound, is given by the first law of thermodynamics as

\[
c^2 = c_o^2 - \frac{\gamma - 1}{2} v^2.
\]

The potential function is used because, according to Steger and Lomax (18), the governing equation in terms of a derived variable (the potential function, \(\phi\), or stream function, \(\psi\)) converges to solution more rapidly than the equation in terms of primitive velocity variables. Furthermore, the potential function is used in preference to the stream function because the density-stream function relationship is non-unique,
as noted by Steger and Lomax (18) and Colehour (19).

Garabedian and Korn (17) used a type-dependent difference for the second derivatives in the flow direction ($\phi_{xx}$ and $\phi_{xy}$), but used a centered difference for the axial first derivative ($\phi_x$), regardless of the local Mach number. Unlike Murman and Cole, a four and six point retarded difference was used for supersonic $\phi_{xx}$ and $\phi_{xy}$ respectively, and a damping coefficient was introduced to control the amount of artificial viscosity. The purpose of the increased number of points and the damping coefficient was to obtain a stable supersonic procedure of second order accuracy. The geometry considered was that of a shockless airfoil for which wind tunnel data was available. The calculated values agreed well with this data, and Garabedian and Korn concluded that boundary layer effects were not large enough to make the inviscid equation unrealistic.

Colehour (19) mentioned the importance of having the proper finite-difference forms for a stable solution, and similar to Garabedian and Korn used type dependent differences only for the axial second derivatives ($\phi_{xx}$ and $\phi_{xy}$). These differences, using three and four mesh points respectively, were first order accurate and did not employ the damping coefficient of Garabedian and Korn. Colehour also performed a coordinate transformation in order to obtain a curvilinear, orthogonal coordinate system which was always aligned with the local flow direction. This was done to insure stability by maintaining the proper domains of dependence of the finite-difference equations. Colehour applied his method to airfoils, axisymmetric bodies, and
turbine engine inlets. He found good agreement with predicted airfoil results, except downstream of shocks due to shock-boundary layer interactions. For the inlet calculations he noted deviation from predicted results caused by boundary layer blockage. Colehour concluded that the lack of viscous effects significantly influenced his calculations, and recommended the addition of a boundary layer calculation.

Similar to Colehour, Jameson (20) and South and Jameson (21) used centered differences for all first derivatives and retarded differences for the axial and mixed supersonic second derivatives. Jameson's supersonic derivatives are identical to Colehour's, except that they contain both old and new potential function values. To guarantee stability, South and Jameson used a rotating difference scheme in the supersonic region to align the coordinate system with the flow direction. These investigators reported good agreement between calculated pressure distributions and wind tunnel data for a variety of lifting airfoils and axisymmetric bodies in regions where the entropy change was not large. A large discrepancy was noted in a comparison with a time-dependent numerical solution which did not assume irrotational flow.

The supersonic differences of Steger and Lomax (22) included a retarded, three point, axial first derivative along with four and six point models for the second derivatives. All of these were of second order accuracy. Steger and Lomax considered airfoils and bodies for which experimental data and theoretical hodograph and time-dependent solutions were available. Pressure distributions were generally in good agreement. Shock location differences were attributed to
boundary layer interaction.

2.3 A Comment on Relaxation Factors and Convergence

The relaxation factor, $\omega$, when used in an expression such as

$$\phi_{i,j}^{n+1} = \phi_{i,j}^n + \omega(\phi_{i,j}^{n+1} - \phi_{i,j}^n)$$

can perform the critical functions of increasing the convergence rate or insuring computational stability, if chosen correctly. Yet at this time there are no rigorous methods for selecting this parameter. According to Murman (6), who recommended values between 0.5 and 1.95 for subsonic flow and a value less than 1.0 for supersonic flow to insure stability, the optimum $\omega$ must be found by numerical experimentation. Steger and Lomax (22) state that theoretical guidelines for an optimum $\omega$ will come only after future study of the eigenvalue spectrums of the finite-difference flow equations. Meanwhile, they and Bailey (10) recommend user adjustment of the relaxation factor during computer runs by means of "interactive graphics" with a cathode ray tube display.

The proper selection of convergence criteria is even more poorly established. Many authors do not discuss this at all. Some, such as Murman and Krupp (8), and Bailey and Ballhaus (11), required, for example, that airfoil surface pressures or coefficients of lift and drag change by less than 0.02% during the course of ten iterations. Others, such as Colehour, examined the change in potential function as the solution progressed, and assumed convergence when the maximum change in $\phi$ from one iteration to the next was less than $10^{-5}$. Any
stricter criteria, he said, was wasteful and would not noticeably improve the accuracy of the resulting solution.
3. **METHOD DESCRIPTION**

This section describes the application of the relaxation methods to the problem of transonic nozzle flow. The governing equations are stated, a coordinate transformation used to simplify computation is described, and the finite-difference representation of the governing equations is discussed. The treatment of the boundary conditions is described, and the relaxation process used to solve the resulting algebraic equations is outlined.

3.1 **Governing Equations**

The potential function, \( \phi \), is introduced so that the derivatives of \( \phi \) in the physical plane represent the velocity components, or

\[
\frac{\partial \phi}{\partial x} = u
\]

\[
\frac{\partial \phi}{\partial r} = v .
\]

The equation of steady, axisymmetric, isentropic flow can then be written as

\[
(c^2 - \phi_x^2)\phi_{xx} + (c^2 - \phi_r^2)\phi_{rr} - 2\phi_{x}\phi_{r}\phi_{xr} + \frac{c^2}{r} \frac{\partial \phi}{\partial r} = 0
\]

(1)

where \( c \) is the local speed of sound, given in terms of the stagnation sound velocity, \( c_0 \), by

\[
c^2 = c_0^2 - \frac{\gamma - 1}{2} (u^2 + v^2) .
\]
In this investigation, the velocity components are non-dimensionalized by the speed of sound at stagnation conditions, resulting in a redefined potential function, $\phi'$, given by

$$\frac{\partial \phi'}{\partial x} = \frac{u}{c_0}$$

and

$$\frac{\partial \phi'}{\partial r} = \frac{v}{c_0}.$$

The sound speed term in Eq. 1 then becomes

$$c^2 = \left(\frac{c}{c_0}\right)^2 = 1 - \frac{1}{\gamma - 1} \left(\phi_x^2 + \phi_r^2\right)$$

and the potential function form of Euler's equations is

$$(c^2 - \phi_x^2)\phi_{xx} + (c^2 - \phi_r^2)\phi_{rr} - 2\phi_x\phi_r\phi_{x} + \frac{c^2}{r} \phi_r = 0,$$

where the primes introduced in the redefined potential function have been dropped for simplicity.

### 3.2 Coordinate Transformation

In order to simplify the numerical application of Eq. 2 at the wall boundary of a given nozzle, a coordinate transformation from the physical plane to a computational plane was performed (see Fig. 1). The transformation results in a rectangular computational grid, avoiding the problem of irregular finite-difference molecules at the nozzle wall. Transformation equations of the form

$$\xi = \xi(x)$$

$$\psi = \psi(x,r)$$
Fig. 1. Coordinate Transformation.
were used. Derivatives of the potential function with respect to the transformed variables were obtained through the chain rule, resulting in

\[
\begin{align*}
\phi_x &= \phi_\xi \xi_x + \phi_\psi \psi_x \\
\phi_r &= \phi_\psi \psi_r \\
\phi_{xx} &= \phi_\xi \xi_\xi x_x^2 + \phi_\psi \psi_\xi x_x + \phi_\psi \psi_x^2 \\
\phi_{rr} &= \phi_\psi \psi_r^2 \\
\phi_{xr} &= \phi_\psi \psi_\xi \xi_r + \phi_\psi \psi_r \psi_r.
\end{align*}
\]

Using these relationships, Euler's equation can be written in the quasi-linear form

\[
A \phi_\xi + B \phi_\psi + C \phi_\psi = D,
\]

where \(A, B, C,\) and \(D\) are functions containing the first derivatives of \(\phi\) in the computational plane, \(\phi_\xi\) and \(\phi_\psi\), and the transformation derivatives, \(\xi_x, \psi_x,\) and \(\psi_r\). In the work reported here, the specific transformation equations used were

\[
\begin{align*}
\xi &= \frac{x}{L} \\
\psi &= \frac{r}{R(x)}
\end{align*}
\]

where \(L\) is the total nozzle length and \(R\) is the nozzle wall radius which is a function of the axial coordinate. The particular transformation derivatives used were then

\[
\begin{align*}
\xi_x &= \frac{1}{L} \\
\psi_x &= -\frac{r}{R^2 \frac{dR}{dx}}
\end{align*}
\]
and

$$\psi_r = \frac{1}{R}.$$  

3.3 Finite-Difference Approximation

The transformed governing equation was solved numerically using finite-difference expressions to approximate the derivatives in the computational plane. The differences in the flow direction were type dependent, that is, were altered depending on the local Mach number.

In the subsonic region, centered differences of second order accuracy were used which results in

$$\phi_{\xi} \bigg|_{i,j} = \frac{\phi_{i+1,j} - \phi_{i-1,j}}{2\Delta \xi}$$

and

$$\phi_{\xi \xi} \bigg|_{i,j} = \frac{\phi_{i+1,j+1} - 2\phi_{i,j} + \phi_{i-1,j+1}}{(\Delta \xi)^2}.$$ 

In the supersonic region, backward finite-differences of first order accuracy were used for axial derivatives. The relationships used were

$$\phi_{\xi} \bigg|_{i,j} = \frac{\phi_{i,j} - \phi_{i-1,j}}{\Delta \xi}$$

and

$$\phi_{\xi \xi} \bigg|_{i,j} = \frac{\phi_{i+1,j} - 2\phi_{i-1,j} + \phi_{i-2,j}}{(\Delta \xi)^2}.$$
In both the subsonic and supersonic regions, the radial derivatives of $\phi$ were approximated by centered differences of second order accuracy, given by

$$
\frac{\phi_i^{j+1} - \phi_i^{j-1}}{2\Delta\psi}, \quad J_1, J_2
$$

These finite-difference relationships can be visualized as computational molecules, which are shown in Fig. 2.

### 3.4 Boundary Conditions

The finite-differences of the previous section were used to evaluate derivatives at interior points, that is, points lying one or more mesh spaces within the boundary of the computational grid. As with exact solutions of differential equations, finite-difference calculations require specification of the boundary conditions to obtain the desired solution. For the problem of transonic nozzle flow, conditions at the wall, inlet, and centerline are required; the hyperbolic nature of the equation in the supersonic region precludes the need for an exit boundary condition.

At the centerline, the radial velocity component vanishes. Thus

$$
v = \phi \frac{c}{r_o} = \phi \psi \frac{c}{r_o} = 0
$$

or

$$
\psi = 0.
$$
\[
\phi_{\xi} \big|_{i,j} = \frac{\phi_{6} - \phi_{4}}{2\Delta\xi}
\]
\[
\phi_{\psi} \big|_{i,j} = \frac{\phi_{2} - \phi_{8}}{2\Delta\psi}
\]
\[
\phi_{\xi \xi} \big|_{i,j} = \frac{\phi_{6} - 2\phi_{5} + \phi_{4}}{(\Delta\xi)^2}
\]
\[
\phi_{\psi \psi} \big|_{i,j} = \frac{\phi_{2} - 2\phi_{6} + \phi_{8}}{(\Delta\psi)^2}
\]
\[
\phi_{\xi \psi} \big|_{i,j} = \frac{\phi_{3} - \phi_{9} - \phi_{1} + \phi_{7}}{4\Delta\xi\Delta\psi}
\]

**Centered**

\[
\phi_{\xi} \big|_{i,j} = \frac{\phi_{6} - \phi_{5}}{\Delta\xi}
\]
\[
\phi_{\psi} \big|_{i,j} = \frac{\phi_{3} - \phi_{9}}{2\Delta\psi}
\]
\[
\phi_{\xi \xi} \big|_{i,j} = \frac{\phi_{6} - 2\phi_{5} + \phi_{4}}{(\Delta\xi)^2}
\]
\[
\phi_{\psi \psi} \big|_{i,j} = \frac{\phi_{3} - 2\phi_{6} + \phi_{9}}{(\Delta\psi)^2}
\]
\[
\phi_{\xi \psi} \big|_{i,j} = \frac{\phi_{3} - \phi_{9} - \phi_{1} + \phi_{7}}{4\Delta\xi\Delta\psi}
\]

**Backward**

Fig. 2. Finite-Difference Molecules.
At the wall, the tangency condition requires that

\[ \frac{v}{u} = \frac{\frac{\phi_r}{\phi_x}}{\tan \theta} = \frac{dR(x)}{dx} \cdot \frac{\partial}{\partial x} = w \]

Expressing \( \phi_r \) and \( \phi_x \) in terms of derivatives in the computational plane allows the evaluation of \( \phi \psi \) as a function of the wall slope, transform derivatives, and \( \phi \xi \) or

\[ \phi \psi = \frac{(dR/dx) \xi_x \phi \xi}{[\psi - (dR/dx) \psi_x]} \cdot \]

Since the radial derivatives of \( \phi \) on the wall and centerline can now be calculated, the values of \( \phi \) on the wall and centerline can be obtained by means of the one-sided finite-difference expressions

\[ \phi_{1,1} = \phi_{1,2} - \bar{\phi}_\psi \Delta \psi \]

and

\[ \phi_{1,J} = \phi_{1,J-1} + \bar{\phi}_\psi \Delta \psi \cdot \]

where \( \bar{\phi}_\psi \) is the average of the values of \( \phi \psi \) at the boundary and at the adjacent interior point for the centerline and wall, respectively.

The values of \( \phi \) at the inlet of the constant area duct upstream of the nozzle (see Fig. 1) were obtained by linear extrapolation from the \( \phi \) values at the two adjacent interior points. The expression used was

\[ \phi_{1,j} = 2\phi_{2,j} - \phi_{3,j} \cdot \]
3.5 Relaxation Process

In the solution of the potential function form of Euler's equations A, B, C, and D were held constant, resulting in a linearization of the governing equation. If only a single column in the matrix of unknown \( \phi \) values is considered, Eq. 3 can be rewritten in matrix form as

\[
\tilde{A} \tilde{\phi} = \tilde{f}
\]

where \( \tilde{A} \) is the coefficient matrix which contains A, B, and C, \( \tilde{\phi} \) is a vector of the unknown \( \phi \)'s, or

\[
\tilde{\phi} = \begin{bmatrix}
\phi_1,2 \\
\vdots \\
\vdots \\
\phi_{i,j-1}
\end{bmatrix}
\]

and \( \tilde{f} \) contains D and all products of A, B, or C and the \( \phi \) values upstream or downstream of the column under consideration (see Appendix).

The coefficient matrix \( \tilde{A} \) is tridiagonal, and the matrix equation can be easily solved for \( \tilde{\phi} \) by the Thomas algorithm (7). However, due to the first derivatives of \( \phi \) appearing in A, B, and C, values of \( \phi \) in the column under consideration also appear in \( \tilde{A} \) and \( \tilde{f} \). The Thomas algorithm must therefore be applied repeatedly to the linearized equation, updating \( \tilde{A} \) and \( \tilde{f} \) after each application, as in

\[
\tilde{A}(\tilde{\phi}_i^r) \rho_i^{n+1} = \tilde{f}(\tilde{\phi}_i^n),
\]

where \( n \) is the iteration number. This column iteration procedure was applied column by column to solve for interior \( \phi \) values, proceeding downstream from the column adjacent to the inlet of the constant area section to the column forming the nozzle exit plane. New \( \phi \) values were
used as soon as available, making the procedure one of successive line relaxation (SLR). After new \( \phi \) values were obtained in the nozzle interior, the boundary conditions were applied to obtain updated boundary \( \phi \) values.

The application of this procedure requires the calculation of the Mach number at each point to determine the appropriate finite-difference type. When these calculations (using centered differences) indicated that any column contained even one supersonic point, the change to backward differences was made for the entire column. After new \( \phi \) values in each column had been obtained, the \( \phi \) array was over-relaxed in the subsonic region to speed convergence and under-relaxed in the supersonic region to insure stability. The equation used for this was

\[
\phi_{i,j}^{n+1} = \phi_{i,j}^n + \omega(\phi_{i,j}^{n+1} - \phi_{i,j}^n)
\]

where \( \omega \), the relaxation factor, was calculated according to Frankel (4) in the subsonic region. This resulted in values between 1.5 and 2.0 depending on the size of the computational mesh. In the supersonic region the relaxation factor was set equal to 0.9.

The convergence criterion used for the column iteration was that the maximum change in \( \phi \) from one iteration to the next be less than \( 1 \times 10^{-4} \). The number of nozzle iterations required for a final solution was determined by examination of the behavior of the Mach number at the nozzle inlet, throat, and exit. This will be discussed in greater detail in the following section.
After the final solution was obtained, the Crocco number, Mach number, and flow angle at each point were calculated. In addition, the discharge coefficient* was found at each axial station from

\[ C_D = \frac{m_a}{m_{id}} = \frac{2 \pi \int_0^R \left( \frac{\rho}{\rho_o} \right) \left( \frac{u}{c_o} \right) r \, dr}{\pi R_{th}^2 \left( \frac{\rho^*}{\rho_o} \right) \left( \frac{u^*}{c_o} \right)} \]

where

\[ \frac{\rho}{\rho_o} = \frac{1 - \gamma - 1}{2} \left( \frac{u}{c_o} \right)^2 + \left( \frac{\gamma}{c_o} \right)^2 \]

and the velocity coefficient** was calculated at the exit by

\[ C_V = \frac{T_a}{T_{id}} = \frac{2 \pi \int_0^R \left[ \left( \frac{\rho}{\rho_o} \right) \left( \frac{u}{c_o} \right)^2 + 1/\gamma \left( \frac{p}{p_o} - \frac{p_b}{p_o} \right) \right] r \, dr}{\left( \frac{u}{c_o} \right)_e C_{D_{th}} \left( \frac{\rho^*}{\rho_o} \right) \left( \frac{u^*}{c_o} \right) \pi R_{th}^2} \]

where \( p/p_o \) is defined in terms of \( \rho/\rho_o \) by the isentropic relationship and \( p_b/p_o \) is the pressure ratio corresponding to \( A_e/A_{th} \). Finally, interpolation was performed to obtain the desired lines of constant Mach number.

* The discharge coefficient represents the ratio of actual mass flow to the mass flow which would result if the flow in the nozzle were uniform and one-dimensional.

** The velocity coefficient represents the ratio of the actual thrust to the thrust of an ideal, one-dimensional nozzle with the same mass flow as the actual nozzle and operating at perfectly expanded conditions.
4. RESULTS AND DISCUSSION

To determine the accuracy of the relaxation technique when applied to the solution of transonic nozzle-flow problems, the procedure described in the preceding section was coded for an IBM 370/158 digital computer using FORTRAN IV (see Appendix for flow chart). The test case chosen was an axisymmetric, hyperbolic nozzle with a radius of curvature of five inches and a throat radius of one inch (see Fig. 3). The results of this calculation were compared with a theoretical solution obtained by Hopkins and Hill (23) using a series expansion method. A constant-area section upstream of the nozzle inlet was found necessary to provide compatibility with the inlet boundary treatment which was used.

The results presented here were obtained with a grid of 25 points in the axial direction and 11 points in the radial direction. A solution was also obtained for a 25 x 21 grid, but these results differed negligibly from the 25 x 11 grid and will therefore not be reported here. The present solution is compared with the results of Hopkins and Hill in Fig. 4, which shows lines of constant Mach number. The results shown were obtained with 300 nozzle iterations. This was determined to be the final solution by examination of the curves in Fig. 5 in which the Mach number at various locations in the nozzle is plotted as a function of the number of iterations. Inspection of this figure shows that no significant change in the Mach number occurs after the 300th iteration. The calculated points of constant Mach number in Fig. 4
Fig. 3. Test-Case Nozzle Geometry.

Constant Area Section

Hyperbolic Nozzle

\[ R = [1 + 0.2 (x-2)^2]^{\frac{1}{2}} \]
Fig. 4. Lines of Constant Mach Number.
Fig. 5. Change of Mach Number At Various Nozzle Locations With Number of Iterations (Δ = Wall, 0 = Centerline).
show excellent agreement with the series expansion method in the entire region shown.

The discharge coefficient calculated at various axial stations is plotted in Fig. 6. The discharge coefficient at the throat was 0.999, and became as high as 1.06 times this at the nozzle exit (not shown). Since the discharge coefficient should be constant (at some value less than one due to the two-dimensional nature of the flow) and not a function of axial position, this figure indicates an error in the calculation and therefore is of some concern. There are two possible explanations for this unexpected behavior. First, there is the possibility that the $\phi$ values are in error due to the fact that the calculations were stopped at the 300th iteration. At this iteration the maximum change in $\phi$ which is occurring is of the order of $10^{-4}$. An approximate order-of-magnitude estimate of the effect of these changes reveals that an error on the order of $10^{-3}$ in the discharge coefficient should be expected. This explains the variation of the discharge coefficient in the vicinity of the throat. In the supersonic region, however, the error is much greater. Here it is possible that a portion of the error is due to the first-order accuracy of the finite-difference expressions used where the flow is supersonic. This error is of the order of $(\Delta \xi)^2$ or approximately 0.002. If this error accumulated as calculations progressed downstream, the variation of discharge coefficient shown in Fig. 6 could result.

The results of two previous and unsuccessful forms of the relaxation program are worth noting. One form calculated the boundary $\phi$
Fig. 6. Discharge Coefficient.
values at the wall and centerline from the radial first derivative of $\phi$ at the adjacent, interior points only, rather than by averaging the derivatives as described in section 3.4. This effectively ignored the known radial derivatives on the wall and centerline, and the results shown in Fig. 7 underline the importance of imposing the proper boundary conditions in obtaining an accurate solution. The second unsuccessful program used type-dependent axial first derivatives in the column iteration procedure, but calculated the axial first derivative by centered differences everywhere for the final output after 300 iterations. These results, shown in Fig. 7, are quite different from the output obtained with type-dependent first derivatives (Fig. 4). The finite-difference solution is thus dependent on the finite-difference expressions used in its calculation, and these expressions cannot be changed even when stability is no longer a factor.

Although successful runs were made with grids of $25 \times 11$ and $25 \times 21$ points, runs of a $49 \times 21$ grid failed because the column iterations in the throat region did not converge. A slight improvement resulted when centered differences were used to reassign the flow type in columns which were indicated as being subsonic with backward differences. Also, using under-relaxation in the column iterations themselves seemed to improve convergence. A recent run which delayed the use of backward differences until the physical throat was reached (instead of one or two columns upstream of the throat) has allowed 25 complete nozzle iterations with no apparent difficulties. At this time, all that can be said is that the problem may be caused by the way in which the
Fig. 7. Effect of Boundary Treatment and Finite-Difference Form On Relaxation Solutions.
change of finite-difference expressions from centered to backward is handled.

The CPU time required for 300 iterations of the 25 x 11 mesh was 199 seconds, corresponding to a time per point of 0.72 seconds. Normalizing this time with respect to an IBM 370/165 computer according to Peitsch (24) gives a time per point of 0.24 seconds. This time is less than that required by the time-dependent solutions of internal flow tabulated by Brown (1) by an approximate factor of four and less than the time required by error-minimization-type internal flow solutions by a factor of five. Although this present method is not as fast as the majority of external flow relaxation solutions, it must be kept in mind that no attempt has been made to optimize the program logic from the standpoint of computational efficiency. Variation of the subsonic relaxation factor from 1.4 to 1.9 and the supersonic factor from 0.9 to 1.0 indicates that an optimum subsonic factor is given by Frankel's method and that little improvement is obtained by an increase of the supersonic factor. It is not unlikely, however, that refinements in the program logic will produce significant improvements in the computational time.
5. CONCLUSIONS

The relaxation method reported here shows excellent agreement with the series solution of Hopkins and Hill. In the throat region, the calculated discharge coefficients are reasonable and within the expected error. The computational time required is less than that of time-dependent and error-minimization-type internal flow solutions. The relaxation program in its present form is an accurate and competitive computational tool for transonic nozzle-flow problems, and deserves further development.

Further work is needed to devise a procedure for handling the finite-difference scheme in the throat so that column convergence is not jeopardized. Also program refinements should be sought to decrease the computational time and thus further increase the computational-speed advantage which the present method has over other nozzle analysis methods.
6. APPENDIX

6.1 Derivation Of The Column Matrix Form Of The Governing Equation

The potential function form of Euler's equations is given in Section 3.1 as

\[(c^2 - \phi_x^2) \phi_{xx} + (c^2 - \phi_r^2) \phi_{rr} - 2\phi_x \phi_r \phi_{xr} + \frac{c^2}{r} \phi_r = 0 \]  \hfill (2)

When the transformation relations for the derivatives of \( \phi \) obtained in Section 3.2 are introduced, the transformed governing equation can be written as

\[(c^2 - [\phi_{\xi_x}^2 + \phi_{\psi_x}^2]) (\phi_{\xi_x}^2 - \phi_{\psi_x}^2) + (c^2 - [\phi_{\psi_r}^2]) (\phi_{\psi_r}^2)
\]

\[-2(\phi_{\xi_x} \phi_{\psi_x} + \phi_{\psi_x} \phi_{\xi_x}) (\phi_{\psi_r} \phi_{\psi_r} + \phi_{\psi_r} \phi_{\psi_r})
\]

\[+ \frac{c^2}{r} (\phi_{\psi_r} \phi_{\psi_r}) = 0. \]

This equation is written in Section 3.2 as

\[A \phi_{\xi_x} + B \phi_{\psi_x} + C \phi_{\psi_r} = D \]  \hfill (3)

where

\[A = (c^2 - [\phi_{\xi_x}^2 + \phi_{\psi_x}^2]) \phi_{\xi_x}^2 \]

\[B = (c^2 - [\phi_{\xi_x}^2 + \phi_{\psi_x}^2]) \phi_{\psi_x}^2
\]

\[-2(\phi_{\xi_x} \phi_{\psi_x} + \phi_{\psi_x} \phi_{\xi_x}) (\phi_{\psi_r} \phi_{\psi_r} + \phi_{\psi_r} \phi_{\psi_r}) \]

\[C = (c^2 - [\phi_{\psi_r}^2]) \phi_{\psi_r}^2
\]

\[+ (c^2 - [\phi_{\psi_r}^2]) \phi_{\psi_r}^2 \]
Because of its non-linear character, expressing this equation in finite-difference form would result in an equation not suited to solution by the Thomas algorithm. Thus the coefficients A, B, and C are taken to be constants which are updated after each application of the Thomas algorithm. Replacing the second derivatives in Eq. 3 by their finite-difference approximations (for subsonic flow) gives

\[ A\left(\phi_{6} - 2\phi_{5} + \phi_{4}\right)/\left(\Delta \xi\right)^{2} \]
\[ + B\left(\phi_{3} - \phi_{9} - \phi_{1} + \phi_{7}\right)/\left(4\Delta \xi\Delta \psi\right) \]
\[ + C\left(\phi_{2} - 2\phi_{5} + \phi_{8}\right)/\left(\Delta \psi\right)^{2} = D \]

where the nomenclature introduced in Fig. 2 has been used. Since it is desired to solve the governing equation column by column and \(\phi_{2}, \phi_{5},\) and \(\phi_{8}\) all lie in the column under consideration, the equation above is written as

\[ \frac{C}{\left(\Delta \psi\right)^{2}} \phi_{8} - \left[ \frac{2A}{\left(\Delta \xi\right)^{2}} + \frac{2C}{\left(\Delta \psi\right)^{2}} \right] \phi_{5} + \frac{C}{\left(\Delta \psi\right)^{2}} \phi_{2} = \]
\[ - \frac{A}{\left(\Delta \xi\right)^{2}} \left(\phi_{6} + \phi_{4}\right) - \frac{B}{4\Delta \xi\Delta \psi} \left(\phi_{3} - \phi_{9} - \phi_{1} + \phi_{7}\right) + D \]

Similar consideration of a point lying in the supersonic region of the flow field gives, in the nomenclature of Fig. 2,
\[ A(\psi_6 - 2\psi_5 + \psi_4)/(\Delta \xi)^2 \]
\[ + B (\psi_3 - \psi_9 - \psi_1 + \psi_7)(4\Delta \xi \Delta \psi) \]
\[ + C (\psi_3 - 2\psi_6 + \psi_9)/(\Delta \psi)^2 = D \]

or
\[
\begin{bmatrix}
-\frac{B}{4\Delta \xi \Delta \psi} + \frac{C}{(\Delta \psi)^2} \\
\frac{B}{4\Delta \xi \Delta \psi} + \frac{C}{(\Delta \psi)^2}
\end{bmatrix}
\phi_9 + \begin{bmatrix}
-\frac{A}{(\Delta \xi)^2} - \frac{2C}{(\Delta \psi)^2} \\
-\frac{A}{(\Delta \xi)^2}\frac{2\psi_5 + \psi_4}{-2\psi_5 + \psi_4}
\end{bmatrix}
\phi_6
\]
\[
+ \begin{bmatrix}
-\frac{B}{4\Delta \xi \Delta \psi} (-\phi_1 + \phi_7) + D
\end{bmatrix}
\]

Since it is assumed in the above equations that the central point in the computational molecule lies in the interior region of the flow field, consideration must next be given to points lying on the boundary of the interior region (at \( j = 2 \) or \( J-1 \)). At the nozzle wall \( \psi_2 \) or \( \phi_3 \) (depending on whether the flow is subsonic or supersonic) is assumed to be known, and at the nozzle centerline \( \psi_8 \) or \( \psi_9 \) is assumed to be known. This means that the number of unknowns in the two equations written at the interior boundaries is reduced from three to two. Therefore writing these equations for all interior points in a given column gives rise to a set of simultaneous equations with a tridiagonal coefficient matrix when written in the form
\[ \tilde{A} \tilde{\phi} = \tilde{f} \]
of Section 3.2.
6.2 Flow Chart of Relaxation Program
START

INITIAL SOLUTION

BEGIN NOZZLE SCAN

IS NOZZLE SCAN COMPLETE AND TYPE SUPersonic?

NO

CENTERED \( \phi_e \) AND \( \phi_\psi \)

YES

BACKWARD \( \phi_e \)

CENTERED \( \phi_\psi \)

CALCULATE \( \phi_x \), \( \phi_r \), AND \( M \);
ASSIGN FLOW TYPE USING CENTERED DIFFERENCES

NEXT PAGE
IS NOZZLE SCAN COMPLETE?

PRINT INITIAL SOLUTION ON FIRST ITERATION

LOAD ARRAYS OF COLUMN MATRIX EQUATION ACCORDING TO TYPE

CALCULATE ALL COLUMN FIRST DERIVATIVES ACCORDING TO TYPE

HAS COLUMN ITERATION CONVERGED?

TO PAGE 40

FROM PAGE 40

NEXT PAGE
UPDATE COLUMN FIRST DERIVATIVES

ENTER FROM PREVIOUS PAGE

FROM PAGE 39

2

UPDATE COLUMN BOUNDARY \( \phi \)'S

TO PAGE 39

3

IS NOZZLE SCAN COMPLETE?

TO PAGE 39

NO

OVER/UNDER-RELAX ENTIRE NOZZLE

TO PAGE 38

4

PRINT OUTPUT?

YES

CALCULATE \( M, \theta, c_p, \) AND \( c_v \)

NEXT PAGE

FROM PAGE 39

5

TO PAGE 39

6

TO PAGE 41
7. REFERENCES


8. VITA

The author was born on July 10, 1950, in Alexandria, Virginia. He attended school in Annandale, Virginia; Djakarta, Indonesia; and Manila, Republic of the Philippines. He graduated from Annandale High School in June of 1968.

In September of 1968 he entered Virginia Polytechnic Institute and State University. While there he participated in the Cooperative Education Program, working for the Ford Motor Company and the Central Intelligence Agency. He obtained a Bachelor of Science degree in Mechanical Engineering in June of 1973, and began graduate study in September of that year toward a Master of Science degree in Mechanical Engineering.

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Kevin E. Walsh
AN APPLICATION OF RELAXATION METHODS

TO

TRANSONIC NOZZLE FLOW

By

Kevin Eugene Walsh

(ABSTRACT)

An application of relaxation techniques to the solution of transonic flow in a converging-diverging nozzle is presented. The assumptions of steady, isentropic flow of a perfect gas were made. Successive line relaxation, similar to that of Garabedian and Korn, was employed in a transformed computational plane. The potential function at interior points was obtained column by column through repeated application of the Thomas algorithm. Once the interior-point calculations were complete the values of the potential function on the boundaries were obtained by extrapolation using the wall-tangency and centerline-symmetry conditions where appropriate. A final solution was considered obtained when negligible changes were observed in the Mach number at all flow-field points from one iteration to the next.

To determine the accuracy of the method, the flow through a hyperbolic converging-diverging nozzle was calculated and the results were compared with an existing solution obtained by a series expansion method. The calculated lines of constant Mach number were in excellent agreement with the series expansion solution. The time required for this solution
was faster than time-dependent and error-minimization-type solutions by more than a factor of four even though no attempt was made to optimize the computational efficiency of the program logic.