Probability-Based Stability Analysis of a Laminated Composite Plate
Under Combined In-Plane Loads

by

Theofanis D. Rantis

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APPROVED:

S. Thangiytham, Chairman

R. A. Heller

E. R. Johnson

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Committee Chairman: Dr. S. Thangjitham

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(ABSTRACT)

The probabilistic stability of a laminated composite plate is investigated. Three different models are considered in this study, namely, the classical laminated plate theory, a first-order shear deformation theory, and a higher-order shear deformation theory. The probabilistic characteristics, such as the probability density and cumulative distribution functions for the resistance to buckling of the plate are obtained by employing the first-order second-moment method of reliability analysis. Uncertainties associated with material mechanical properties and fiber orientations of individual layers are modeled as statistically independent random variables. Numerical results are presented for rectangular simply-supported laminates, showing the effects of thickness ratio, stacking sequence, and number of layers on the probabilistic stability of the plates.
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Vita
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Chapter 1
Introduction

1.1 Motivation

The stability of plate structures has always been a subject of great interest in the area of structural design. The use of plates in structural applications has been considerable, and many different approaches to the analysis of plate response to various loading and boundary conditions have been proposed in the literature.

Conventional homogeneous, isotropic materials, such as steel or aluminum have long been the materials of choice for the construction of engineering structures. Recently, however, a new class of materials has come to the forefront of these applications, especially where a high stiffness-to-weight ratio is critical. This advanced material involves the macroscopic combination of two or more different materials into one, composite material. One application of this new material is the laminated, fiber-reinforced composite plate. The individual layers, or laminae, consist of high-strength fibers bonded to a matrix material in such a way that the resulting material is stronger and stiffer than either material standing alone. These individual laminae are then bonded together into a laminate. The mechanical properties of this laminate can then be tailored to the requirements of a specific application by altering the fiber orientations of its constitutive laminae.

There are many different criteria which can be defined as failure in a plate structure. One important failure mode is buckling. In some cases, the plate may be designed specifically to buckle and carry a reduced load in the post-buckling regime, while in others, buckling is considered failure. In either situation, a reliable prediction of the load or combination of loads at which the plate will buckle is critical.
Many factors can affect the predicted load carrying capacity of a plate. Some involve errors or oversimplifications made in the plate mechanics models, and others involve a basic uncertainty about the physical makeup of the plate itself. As is the case for "conventional" engineering materials, there is some statistical scatter in the mechanical properties of the constitutive materials of the plate. Also, for laminated fibrous composites, there is often some difference between the specified angular orientation of the fibers and what is actually present in the plate after fabrication. In general, the level of uncertainty is also higher for this class of composite materials, due to the greater number of design variables present.

There are two basic methods of laminated plate analysis, the exact elasticity solution or its approximation, laminated plate theory. Both theories assume that the individual laminae are linearly elastic, and that there is perfect bonding between layers. Elasticity solutions, though mathematically rigorous, are not possible for many complex boundary conditions. As a means of circumventing this difficulty, a simplified method of analysis is introduced in which the plate is considered to be in a state of plane stress. This is the approach taken by the laminated plate models. In many of these models, the mechanical properties of each individual lamina are combined into overall plate stiffness constants, and hence can be known as equivalent single-layer models. These models can be either stress- or displacement-based. Stress-based models use the state of stress within the plate as basic variables, while those which are displacement-based use the displacement field to model plate behavior. This study considers only displacement-based plate models.

The classical plate (Kirchoff-Love) theory (CPT) (Timoshenko and Woinowsky-Krieger, 1961) is based on the assumption that normals to the plate mid-plane before deformation remain normal to the mid-plane after deformation. This implies that the transverse shear strain is neglected, and is a valid assumption for thin plates in many applications, however, its usefulness is limited. In cases where the transverse shear strain is important, the stiffness of the plate is
overestimated. This is the case for many laminated composite plates, due to their high in-plane modulus to transverse shear modulus ratio.

There are several plate mechanics models which take into account the finite transverse shear-deformation which is prominent in these materials. Two representative models are considered in this study, a first-order shear-deformation theory (FSDT) proposed by Mindlin (1951), and a higher-order shear-deformation theory (HSDT) proposed by Reddy (1984).

In the FSDT, transverse shear strains are defined by a displacement field that is linear with respect to the thickness coordinate. In order to reconcile the energy stored in transverse shear predicted by the model with reality, a shear correction factor must be used. Higher-order models assume displacement fields which are non-linear with respect to the thickness coordinate. The HSDT model considered in this study assumes that the transverse displacement can be represented by a cubic polynomial, and no shear correction factor is necessary. In general, the higher order models more accurately predict plate behavior, because the assumed displacement field more closely represents the actual displacements in the plate.

1.2 Literature Review

The term “plate” defines a solid which is bounded by two parallel planes, the distance between which is small when compared to their lateral dimensions. An excellent overview of composite plate buckling was presented by Leissa (1987), in which some considerations and some general results are given. In order to fully comprehend plate stability theory, the different models governing plate behavior must be understood. There have been several reviews of the various plate bending models applicable to composite plates (Bert, 1984; Librescu and Reddy, 1989).

There are many two-dimensional plate theories in existence, and the simplest of these is the classical plate theory (CPT). This model employs the Kirchoff-Love assumption that normals to the mid-plane before deformation remain normal to the mid-plane after deformation (Jones,
This implies that the transverse shear strains are neglected, or the plate is infinitely rigid in the transverse direction. For thin plates, this assumption is valid because these strains are vanishingly small, however as the thickness of the plate increases, their effect becomes significant. In these situations, the plate stiffness can be overestimated, leading to an underestimation of deflections and a consequent overestimation of natural vibration frequencies and buckling loads, as shown by Khdeir (1989). The same study also shows the dramatic effect that a change in boundary conditions can have on the accuracy of the CPT critical load predictions. Geier and Rohwer (1989), however, maintain that CPT offers accurate, simple solutions for certain types of thin plates. Though its general use is quite limited, it can still be used for these specific cases with some confidence.

There have been a number of attempts at including transverse shear strain in plate models, and they are generally known as the shear deformation models. The effect of shear-deformation on the bending and stability of composite plates is examined in (Bert and Chen, 1978; Noor and Mathers, 1976; Whitney, 1987). One of the more widely used shear deformation models is the first-order shear deformation theory (FSDT) proposed by Mindlin (1951). This model assumes a constant transverse shear strain through-the-thickness. Because this is an approximation to the actual strain state, a shear correction factor is required in order to maintain the correct transverse shear energy magnitude. As a result, there are five independent displacement functions in this model. Rotational degrees of freedom allow cross-sections normal to the plate before deformation to rotate, with the constraint that they remain straight. The Mindlin model was extended to the layered anisotropic case by Yang, Norris, and Stavsky (1966), and Whitney and Pagano (1970).

Utilization of the FSDT model leads to a dramatic improvement in the accuracy of the solutions to plate response. Noor (1975) shows that FSDT results show good agreement with the three-dimensional elasticity solutions, however, they depend strongly on the selection of an appropriate shear correction factor. The study also shows that the transverse normal strain which
is taken into account in some first-order models, has a negligible effect. Turvey (1987) points out that the FSDT has two basic weaknesses, uncertainties associated with the selection of shear correction factors, and the inaccurate stress distributions predicted as compared to elasticity solutions.

A great number of higher-order shear deformation models have been proposed. One of these was proposed by Lo, Christensen, and Wu (1977). This is a third-order model, implying that the displacement in the plate thickness direction is allowed to vary cubically. The number of displacement variables in this model is relatively large; even transverse normal strain is taken into account. The complexity of this model makes boundary conditions difficult to interpret and impose. It also does not satisfy the stress-free surface boundary condition and requires the selection of an appropriate shear correction factor. Other higher-order models were proposed by Green and Naghdi (1982), Nelsen and Lorch (1974), and Librescu and Khdeir (1988).

Levinson (1980) and Murthy (1981) presented higher-order models based upon expanded displacements in powers of the thickness coordinate, and required that the transverse shear stresses be zero on the bounding surfaces of the plate. There are three independent displacement variables for both models, the same as for the FSDT. The principal differences between the models are that the latter uses average displacements through-the-thickness, and also develops the model for laminated plates. Both models, however, are based upon the equilibrium equations of a first-order shear deformation theory, making them variationally inconsistent.

Reddy (1984) introduces a refined higher-order plate model in which the governing equations are derived from the principle of virtual work, making them variationally consistent. This model makes use of the same displacement field as Levinson (1980) and Murthy (1981), allowing for a parabolic variation of transverse shear strain in the thickness direction. The model also satisfies the stress free surface boundary condition, and requires no shear correction factor. The number of independent displacement variables in this model is five, the same as for FSDT.
The exact solutions for plates exist for only a few, specific cases. The Navier solution technique is among the most popular methods of obtaining closed-form results. It utilizes a Fourier series expansion of the displacement field, and is therefore limited to simply supported rectangular plates of a specific stacking sequence. All of the previously mentioned models have been used to examine plate stability. The Levy-type solution procedure is used to find the exact buckling loads for various laminates, symmetric and anti-symmetric angle-ply, in (Khdeir, Reddy, and Librescu, 1987; Khdeir, 1988; Khdeir, 1989). As mentioned earlier, the necessity for the proper choice of shear correction factor for the FSDT is critical. It was shown by Noor (1975) that, when proper correction factors are chosen, the FSDT buckling loads are in good agreement with elasticity solutions. Reddy and Phan (1985) present stability results for composite plates based upon the HSDT. Harris (1975) examines the effect of biaxial loading on plate stability. It is found that a tensile perpendicular load has the effect of stabilizing the plate, while the converse is true for a compressive load.

Considerable work has been done in the area of optimization of composite laminates for maximum load-carrying capacity. Due to the unique ability of composite materials to be tailored to specific mechanical properties, the angular orientations of the individual laminae may be altered in such a way that a particular desired buckling capacity may be achieved. Muc (1988) found optimal layups for various types of laminate, including the anti-symmetric angle-ply. He concluded that the optimal angle for this type of plate is $\theta = 45$. It was also demonstrated that any bending-twisting coupling reduces the load carrying capacity of the plate. Muc's analysis was extended to cover hybrid composites by Adali and Duffy (1990). It was found that, in some cases, hybrids can have higher critical buckling loads than non-hybrids. Duffy and Adali (1990) also examine the effect of optimizing with respect to layer thickness. Other papers dealing with optimization are presented by Chao, Kob, and Sun (1975) and Hirano (1979).
1.3 Objectives

This study is primarily concerned with the application of the first-order second-moment technique of reliability analysis to examine the probabilistic stability of a laminated composite plate under combined in-plane loads. The effects introduced by various uncertainties, such as fiber orientation, material properties, and geometric parameters of the layers are investigated. The effects of these random variables upon the overall stability of the plate, as predicted by the three different plate models given above, are examined.
Chapter 2
Classical and Shear Deformable Plate Models

Plate bending models considered in this study are displacement based. These models are based upon an assumed displacement field of the general form

\[ u(x, y, z) = u_0(x, y) + z\psi(x, y) + z^2\xi(x, y) + z^3\zeta(x, y) + \ldots \] (2.1a)

\[ v(x, y, z) = v_0(x, y) + z\psi_y(x, y) + z^2\xi_y(x, y) + z^3\zeta_y(x, y) + \ldots \] (2.1b)

\[ w(x, y, z) = w_0(x, y) \] (2.1c)

where \( u, v, \) and \( w \) are displacements of an arbitrary point on the plate in the \( x-, y-, \) and \( z- \) directions, respectively, while \( u_0, v_0, \) and \( w_0 \) denote the corresponding displacements at the plate mid-plane. The functions \( \psi_x \) and \( \psi_y \) represent rotations of the normals to the mid-plane about the \( y- \) and \( x- \) axes, respectively. The remaining unknown functions \( \xi, \zeta, \xi_y, \) and \( \zeta_y \) can be determined by the application of the appropriate boundary conditions, i.e. the stress-free condition on the plate bounding surfaces.

The resulting displacement field is given as

\[ u = u_0 + z \left[ \psi_x - \delta_A \frac{4}{3} \left( \frac{z}{h} \right)^2 \left( \psi_x + \frac{\partial w}{\partial x} \right) \right] \] (2.2a)

\[ v = v_0 + z \left[ \psi_y - \delta_A \frac{4}{3} \left( \frac{z}{h} \right)^2 \left( \psi_y + \frac{\partial w}{\partial y} \right) \right] \] (2.2b)

\[ w = w_0 \] (2.2c)

where \( h \) is the plate thickness, as shown in Fig. 2.1. The parameter \( \delta_A \) is set to \( \delta_A = 1 \) for the refined shear deformation model introduced by Reddy (1984). When \( \delta_A = 0 \), the displacement field is reduced to that of the first-order model.
Fig. 2.1 – Plate configuration and coordinate system.
The classical plate model displacement field can be obtained from this by further setting

\[ \psi_x = -\frac{\partial w}{\partial x}, \quad \psi_y = -\frac{\partial w}{\partial y} \] (2.3)

Consequently, the displacement fields of both the HSDT and FSDT are governed by five independent functions, while that of the CPT is reduced to three. All three displacement fields are illustrated in Fig. 2.2.

The plate equilibrium equations will be derived for three different plate bending models in this chapter. From these equations, the critical buckling load can be arrived at through an analytical approach (under the appropriate conditions), or a more general solution can be found by using a numerical approximation to the solution of the differential equations.

2.1 Strain-Displacement Relations

The first step in developing the constitutive equations which govern the behavior of a laminated composite plate is the derivation of the constitutive equations for a single lamina of that plate. The lamina is composed of a matrix material, reinforced with unidirectional fibers. A local “material” coordinate system is defined such that the 1-direction is along the fiber, the 2-direction is normal to the fiber direction and in the plane of the lamina, and the 3-direction is out-of-plane. This coordinate system is illustrated in Fig. 2.3.

The strain measures considered in this study are based upon the linear strain relations, and are related to the gradients of displacement as follows

\[ \epsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \epsilon_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right), \quad \epsilon_{13} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) \]

\[ \epsilon_{21} = \epsilon_{12}, \quad \epsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \epsilon_{23} = \frac{1}{2} \left( \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) \]

\[ \epsilon_{31} = \epsilon_{13}, \quad \epsilon_{32} = \epsilon_{23}, \quad \epsilon_{33} = \frac{\partial u_3}{\partial x_3} \] (2.4)

where the variables \( u_1, u_2, \) and \( u_3 \) are defined as displacements in the 1-, 2-, and 3-directions, respectively, and \( x_1, x_2, \) and \( x_3 \) describe the location of a point in the 1-2-3 material coordinate system.

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Fig. 2.2 – Displacement fields for (a) CPT, (b) FSDT, (c) HSDT.
Fig. 2.3 – Material coordinate systems: a) 1-2-3 lamina system, and b) x-y-z laminate system.
A substitution of the displacement field given in Eq. 2.2 into Eq. 2.4 gives the strain-displacement relations

\[ \epsilon_{11} = \varepsilon_{11}^0 + z\kappa_{11}^0 + z^2\kappa_{11}^2 \]

\[ \epsilon_{22} = \varepsilon_{22}^0 + z\kappa_{22}^0 + z^2\kappa_{22}^2 \]

\[ \epsilon_{33} = 0 \]

\[ \gamma_{12} = \gamma_{12}^0 + z\kappa_{12}^0 + z^2\kappa_{12}^2 \]  \hspace{1cm} (2.5)

\[ \gamma_{13} = \gamma_{13}^0 + z\kappa_{13}^0 + z^2\kappa_{13}^2 \]

where

\[ \varepsilon_{11}^0 = \frac{\partial u_1}{\partial x_1}, \quad \kappa_{11}^0 = \frac{\partial \psi_1}{\partial x_1}, \quad \kappa_{11}^2 = -\delta_A \frac{4}{3h^2} \left( \frac{\partial^2 u_3}{\partial x_1^2} + \frac{\partial^2 u_3}{\partial x_2^2} \right) \]

\[ \varepsilon_{22}^0 = \frac{\partial u_2}{\partial x_2}, \quad \kappa_{22}^0 = \frac{\partial \psi_2}{\partial x_2}, \quad \kappa_{22}^2 = -\delta_A \frac{4}{3h^2} \left( \frac{\partial^2 u_3}{\partial x_2^2} + \frac{\partial^2 u_3}{\partial x_1^2} \right) \]

\[ \gamma_{12}^0 = \psi_2 + \frac{\partial u_3}{\partial x_2}, \quad \gamma_{12}^0 = \frac{\partial \psi_2}{\partial x_2} + \frac{\partial \psi_1}{\partial x_1} \quad \kappa_{12}^2 = -\delta_A \frac{4}{h^2} \left( \psi_1 + \frac{\partial u_3}{\partial x_1} \right) \]

\[ \gamma_{13}^0 = \psi_1 + \frac{\partial u_3}{\partial x_1}, \quad \gamma_{13}^0 = \frac{\partial \psi_1}{\partial x_1} + \frac{\partial \psi_2}{\partial x_2} \quad \kappa_{13}^2 = -\delta_A \frac{4}{h^2} \left( \psi_2 + \frac{\partial u_3}{\partial x_2} \right) \]

In the preceding equations, the superscript \( \varepsilon^0 \) indicates a strain occurring at the 1-2 plane, where \( x_3 = 0 \). The corresponding relations for the first-order shear deformation model can be recovered by setting \( \delta_A = 0 \). Similarly, the terms for the classical deformation model can be found by the substitution of eqn. (2.3) into the first-order model.

### 2.2 Constitutive Equations

The stress associated with a fully anisotropic elastic material can be defined by the following equation relating the stresses, \( \sigma_{ij} \), to the strains, \( \epsilon_{ij} \),

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{22} & C_{23} & C_{24} & C_{25} & C_{26} & C_{27} \\
C_{33} & C_{34} & C_{35} & C_{36} & C_{37} & C_{38} \\
\text{symm.} & C_{44} & C_{45} & C_{46} & C_{47} & C_{48} \\
& C_{55} & C_{56} & C_{57} & C_{58} & C_{59} \\
& & C_{66} & & & C_{68}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{11} \\
\epsilon_{22} \\
\epsilon_{33} \\
\epsilon_{12}
\end{bmatrix}
\]

\hspace{1cm} (2.7)

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where \( C_{ij} \) are the constants defining the stiffness matrix in the principal (1-2-3) material directions. In the case of a transversely isotropic material, the number of independent material constants defining \( C_{ij} \) is reduced from twenty-one to five, and the constitutive equation may be expressed more simply, depending on the plate model used. The following sections will develop the equilibrium equations for the plate bending models studied.

By restricting the materials studied to those which are transversely isotropic, i.e. exhibiting planes of material symmetry parallel to the 1-2 plane, the constitutive equations for the \( k^{th} \) layer can be written as

\[
\begin{align*}
\begin{bmatrix}
\sigma_{11}^{(k)} \\
\sigma_{22}^{(k)} \\
\sigma_{12}^{(k)}
\end{bmatrix}
& =
\begin{bmatrix}
\bar{Q}_{11}^{(k)} & \bar{Q}_{12}^{(k)} & 0 \\
\bar{Q}_{22}^{(k)} & 0 & 0 \\
\bar{Q}_{66}^{(k)} & &
\end{bmatrix}
\begin{bmatrix}
\epsilon_{11}^{(k)} \\
\epsilon_{22}^{(k)} \\
\epsilon_{12}^{(k)}
\end{bmatrix} \\
\begin{bmatrix}
\sigma_{23}^{(k)} \\
\sigma_{13}^{(k)}
\end{bmatrix}
& =
\begin{bmatrix}
\bar{Q}_{44}^{(k)} & 0 & 0 \\
0 & \bar{Q}_{55}^{(k)} & 0 \\
0 & 0 & \bar{Q}_{66}^{(k)}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{23}^{(k)} \\
\epsilon_{13}^{(k)}
\end{bmatrix}
\end{align*}
\]  

(2.8)

The constants \( \bar{Q}_{ij} \) are the plane-stress reduced elastic stiffness coefficients in the \( k^{th} \) layer, and are defined in terms of the basic material mechanical constants as

\[
\begin{align*}
\bar{Q}_{11} & = \frac{E_1}{1 - \nu_{12}\nu_{21}}, & \bar{Q}_{12} & = \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}}, & \bar{Q}_{22} & = \frac{E_2}{1 - \nu_{12}\nu_{21}} \\
\bar{Q}_{44} & = G_{23}, & \bar{Q}_{55} & = G_{13}, & \bar{Q}_{66} & = G_{12}
\end{align*}
\]  

(2.9)

Upon transformation into the general plate coordinate system, the lamina stress-strain relation can be expressed as

\[
\begin{align*}
\begin{bmatrix}
\sigma_x^{(k)} \\
\sigma_y^{(k)} \\
\sigma_{xy}^{(k)}
\end{bmatrix}
& =
\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{22} & \bar{Q}_{26} & \bar{Q}_{26} \\
\bar{Q}_{66} & \bar{Q}_{66} & \bar{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_x^{(k)} \\
\epsilon_y^{(k)} \\
\epsilon_{xy}^{(k)}
\end{bmatrix} \\
\begin{bmatrix}
\sigma_{yz}^{(k)} \\
\sigma_{xz}^{(k)}
\end{bmatrix}
& =
\begin{bmatrix}
\bar{Q}_{44} & \bar{Q}_{45} & \bar{Q}_{45} \\
\bar{Q}_{45} & \bar{Q}_{55} & \bar{Q}_{55}
\end{bmatrix}
\begin{bmatrix}
\epsilon_{yz}^{(k)} \\
\epsilon_{xz}^{(k)}
\end{bmatrix}
\end{align*}
\]  

(2.10)

where \( Q_{ij} \) are the transformed material stiffness coefficients for the \( k^{th} \) layer, and are given by (Jones, 1975)

\[
\begin{align*}
Q_{11} & = \bar{Q}_{11} \cos^4 \theta + 2 (\bar{Q}_{12} + 2 \bar{Q}_{66}) \sin^2 \theta \cos^2 \theta + \bar{Q}_{22} \sin^4 \theta \\
Q_{12} & = (\bar{Q}_{11} + \bar{Q}_{22} - 4 \bar{Q}_{66}) \sin^2 \theta \cos^2 \theta + \bar{Q}_{12} (\sin^4 \theta + \cos^4 \theta)
\end{align*}
\]

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\[ Q_{22} = \bar{Q}_{11} \sin^4 \theta + 2(\bar{Q}_{12} + \bar{Q}_{56}) \sin^2 \theta \cos^2 \theta + \bar{Q}_{22} \cos^4 \theta \]

\[ Q_{15} = (\bar{Q}_{11} - \bar{Q}_{12} - 2\bar{Q}_{66}) \sin \theta \cos^3 \theta + (\bar{Q}_{11} - \bar{Q}_{22} + 2\bar{Q}_{66}) \sin^3 \theta \cos \theta \]

\[ Q_{56} = (\bar{Q}_{11} - \bar{Q}_{12} - 2\bar{Q}_{66}) \sin^3 \theta \cos \theta + (\bar{Q}_{12} - \bar{Q}_{22} + 2\bar{Q}_{66}) \sin \theta \cos^3 \theta \]

\[ Q_{66} = (\bar{Q}_{11} + \bar{Q}_{22} - 2\bar{Q}_{66}) \sin^2 \theta \cos^2 \theta + \bar{Q}_{66}(\sin^4 \theta + \cos^4 \theta) \]

\[ Q_{44} = \bar{Q}_{44} \cos^2 \theta + \bar{Q}_{55} \sin^2 \theta \]

\[ Q_{45} = (\bar{Q}_{55} + \bar{Q}_{44}) \cos \theta \sin \theta \]

\[ Q_{55} = \bar{Q}_{55} \cos^2 \theta + \bar{Q}_{44} \sin^2 \theta \] (2.11)

As shown in Fig. 2.3, the variable \( \theta \) is defined as the angular orientation of the fiber with respect to the general plate x-y-z coordinate system.

### 2.3 Equilibrium Equations

Resultant forces and moments acting on a laminate can be obtained by the integration of the stresses in each layer through the laminate thickness. These resultants are defined as follows

\[ (N_x, N_y, N_{xy}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_x, \sigma_y, \sigma_{xy}) \, dz \] (2.11a)

\[ (M_x, M_y, M_{xy}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_x, \sigma_y, \sigma_{xy}) z \, dz \] (2.11b)

\[ (Q_y, Q_x) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{yx}, \sigma_{xz}) \, dz \] (2.11c)

The cubic variation of \( u \) and \( v \) through the laminate thickness introduces the additional force resultants

\[ (P_x, P_y, P_{xy}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_x, \sigma_y, \sigma_{xy}) z^3 \, dz \] (2.12a)
\[(R_y, R_x) = \int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{yz}, \sigma_{xz}) z^2 \, dz \quad (2.12b)\]

Now, recalling that the strains, \(\epsilon_i\), are defined by terms which are independent of \(z\), Eq. 2.10 can be rewritten as

\[
\begin{pmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy} \\
P_x \\
P_y \\
P_{xy}
\end{pmatrix} =
\begin{pmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & E_{11} & E_{12} & E_{16} \\
A_{22} & A_{26} & B_{22} & B_{26} & E_{22} & E_{26} \\
A_{66} & B_{16} & B_{26} & E_{16} & E_{26} & E_{66} \\
D_{12} & D_{16} & F_{12} & F_{16} & K_x^0 & K_x^0 \\
D_{22} & D_{26} & F_{22} & F_{26} & K_x^0 & K_x^0 \\
D_{66} & F_{16} & F_{26} & F_{66} & K_x^0 & K_x^0 \\
H_{11} & H_{12} & H_{16} & H_{26} & K_y^0 & K_y^0 \\
H_{22} & H_{26} & F_{66} & K_y^0 & K_y^0
\end{pmatrix}
\begin{pmatrix}
\epsilon_x^0 \\
\epsilon_y^0 \\
\epsilon_{xy}^0 \\
\gamma_{yy}^0 \\
\gamma_{xx}^0 \\
\kappa_{xx}^0 \\
\kappa_{yy}^0 \\
\kappa_{xy}^0 \\
\kappa_{xy}^0
\end{pmatrix}
\quad (2.13a)
\]

\[
\begin{pmatrix}
Q_y \\
Q_z \\
R_y \\
R_x
\end{pmatrix} =
\begin{pmatrix}
A_{44} & A_{45} & D_{44} & D_{45} \\
A_{55} & D_{45} & D_{55} \\
F_{44} & F_{45} & \kappa_x^0 \\
F_{55} & \kappa_x^0
\end{pmatrix}
\begin{pmatrix}
\gamma_{yy}^0 \\
\gamma_{xx}^0 \\
\kappa_{xx}^0 \\
\kappa_{yy}^0
\end{pmatrix}
\quad (2.13b)
\]

where the stiffness terms are defined as

\[(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}(1, z, z^2, z^3, z^4, z^5) \, dz \quad i, j = 1, 2, 6 \quad (2.14)\]

\[(A_{ij}, D_{ij}, F_{ij}) = \int_{-\frac{h}{2}}^{\frac{h}{2}} Q_{ij}(1, z^2, z^4) \, dz \quad i, j = 4, 5 \]

The equilibrium equations for the HDST can be found by the application of Hamilton's principle, which finds the functional describing the minimum energy (potential and kinetic) of the system. When the static case is considered, the principle can be stated in the form

\[
\int_{-\frac{h}{2}}^{\frac{h}{2}} \int_{\Omega} \left( \sigma_{x} \delta \epsilon_{xz} + \sigma_{y} \delta \epsilon_{yz} + \sigma_{xy} \delta \epsilon_{xy} + \sigma_{yz} \delta \epsilon_{yx} + \sigma_{xz} \delta \epsilon_{xz} \right) \, dx \, dy \, dz
\]

\[
+ \int_{\Omega} \left[ \left( N_{xi} \frac{\partial w}{\partial x} + N_{yi} \frac{\partial w}{\partial y} \right) \frac{\partial \delta w}{\partial x} + \left( N_{xi} \frac{\partial w}{\partial x} + N_{yi} \frac{\partial w}{\partial y} \right) \frac{\partial \delta w}{\partial y} \right] \, dx \, dy = 0 \quad (2.15)
\]

where \(N_{xi}^i, N_{yi}^i,\) and \(N_{xy}^i\) denote the initial in-plane applied loads, and \(\Omega\) is the domain of the plate.

Classical and Shear Deformable Plate Models
Using (2.11) and (2.12) in (2.15), the following equation is obtained

\[
\int_{\Omega} \left\{ N_z \left( \frac{\partial \delta u}{\partial x} + M_x \frac{\partial \delta \psi_x}{\partial x} + P_x \left[ -\frac{4}{3h^2} \left( \frac{\partial \delta \psi_x}{\partial x} + \frac{\partial^2 \delta w}{\partial x^2} \right) \right] \right. \\
+ N_y \frac{\partial \delta v}{\partial y} + M_y \frac{\partial \delta \psi_y}{\partial y} + P_y \left[ -\frac{4}{3h^2} \left( \frac{\partial \delta \psi_y}{\partial y} + \frac{\partial^2 \delta w}{\partial y^2} \right) \right] \right. \\
+ N_{xy} \left( \frac{\partial \delta u}{\partial y} + \frac{\partial \delta v}{\partial x} \right) + M_{xy} \left( \frac{\partial \delta \psi_x}{\partial y} + \frac{\partial \delta \psi_y}{\partial x} \right) \\
+ P_{xy} \left[ -\frac{4}{3h^2} \left( \frac{\partial \delta \psi_x}{\partial y} + \frac{\partial \delta \psi_y}{\partial x} + 2 \frac{\partial^2 \delta w}{\partial x \partial y} \right) \right] + Q_x \left( \delta \psi_x + \frac{\partial \delta w}{\partial x} \right) + Q_y \left( \delta \psi_y + \frac{\partial \delta w}{\partial y} \right) \\
+ R_y \left[ -\frac{4}{h^2} \left( \delta \psi_x + \frac{\partial \delta w}{\partial x} \right) \right] + R_x \left[ -\frac{4}{h^2} \left( \delta \psi_y + \frac{\partial \delta w}{\partial y} \right) \right] \\
+ \left( N_z \frac{\partial w}{\partial x} + N_{xz} \frac{\partial w}{\partial y} \right) \frac{\partial \delta w}{\partial x} + \left( N_y \frac{\partial w}{\partial x} + N_{yz} \frac{\partial w}{\partial y} \right) \frac{\partial \delta w}{\partial y} \right\} \, dx \, dy = 0
\]

(2.16)

Integrating (2.16) by parts and collecting coefficients of \( \delta u, \delta v, \delta w, \delta \psi_x, \) and \( \delta \psi_y, \) the following equilibrium equations are obtained

\[
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 
\]

(2.17a)

\[
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 
\]

(2.17b)

\[
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \frac{\partial}{\partial x} \left( N_{z} \frac{\partial w}{\partial x} + N_{xy} \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left( N_{xz} \frac{\partial w}{\partial x} + N_{yz} \frac{\partial w}{\partial y} \right) \]

\[
- \frac{4}{h^2} \left( \frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y} \right) + \frac{4}{3h^2} \left( \frac{\partial^3 P_x}{\partial x^3} + 2 \frac{\partial^2 P_{xy}}{\partial x \partial y} + \frac{\partial^2 P_y}{\partial y^2} \right) = 0
\]

(2.17c)

\[
\frac{\partial M_x}{\partial y} + \frac{\partial M_{xy}}{\partial x} - Q_x + \frac{4}{h^2} R_x - \frac{4}{3h^2} \left( \frac{\partial P_x}{\partial x} + \frac{\partial P_{xy}}{\partial y} \right) = 0
\]

(2.17d)

\[
\frac{\partial M_{xy}}{\partial y} + \frac{\partial M_y}{\partial x} - Q_y + \frac{4}{h^2} R_y - \frac{4}{3h^2} \left( \frac{\partial P_{xy}}{\partial x} + \frac{\partial P_y}{\partial y} \right) = 0
\]

(2.17e)

Similarly, the equilibrium equations for the FSDT can be shown to be

\[
\frac{\partial N_z}{\partial x} + \frac{\partial N_{zy}}{\partial y} = 0 
\]

(2.18a)

\[
\frac{\partial N_{zy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 
\]

(2.18b)

\[
\frac{\partial Q_z}{\partial x} + \frac{\partial Q_z}{\partial y} = 0 
\]

(2.18c)

\[
\frac{\partial M_x}{\partial y} + \frac{\partial M_{zy}}{\partial x} - Q_z = 0 
\]

(2.18d)
\[
\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \quad (2.18c)
\]

and the CPT is reduced to three governing equations

\[
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (2.19a)
\]

\[
\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \quad (2.19b)
\]

\[
\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = 0. \quad (2.19c)
\]

The differences in the plate behavior when different displacement fields are allowed can be seen in the increasing complexity of the higher order models. Clearly, the more complex set of equations offer the most flexibility in describing the displacement field.
Chapter 3
Navier Solution Procedure

A solution to the equilibrium equations must be found in order to find the critical buckling load of a plate. One method of finding the analytical solution of this problem for the classical, first-order, and higher-order models is the Navier solution method. This technique employs the double Fourier series representation of the displacement field; but due to the nature of the boundary conditions which must be satisfied, solutions for only a limited number of cases may be obtained.

3.1 The Navier Solution of the Classical Plate Theory

Due to the necessity that the physical boundary conditions of the problem be satisfied, only plates which have specific lamination schemes and boundary constraints can be solved by the Navier method. This is due to the inseparability of the governing differential equations. In this study, one simply-supported case in particular is examined. When the CPT is used, constraints applied to the edges of a rectangular plate of dimension $a \times b$ (Fig. 2.1) are as follows

$$u_0(0, y) = u_0(a, y) = v_0(x, 0) = v_0(x, b) = 0 \quad (3.1a)$$

$$w_0(x, 0) = w_0(x, b) = w_0(0, y) = w_0(a, y) = 0 \quad (3.1b)$$

$$N_{xy}(0, y) = N_{xy}(a, y) = N_{xy}(x, 0) = N_{xy}(x, b) = 0 \quad (3.1c)$$

$$M_x(0, y) = M_x(a, y) = M_y(x, 0) = M_y(x, b) = 0 \quad (3.1d)$$

In order to satisfy the boundary conditions for the simply supported plate, Eqs. 3.1, the displacement components are assumed to be of the form

$$u_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \quad (3.2a)$$
\[ v_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \]  
\[ w_0 = \sum_{m>1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \]  

(3.2b)  

(3.2c)

where \( m \) and \( n \) are integers indicating the number of half sine waves present in the buckling mode, and the constants \( U_{mn} \), \( V_{mn} \), and \( W_{mn} \) are the corresponding modal amplitudes.

A substitution of the displacement field given in Eq. 3.2 into the governing equations Eq. 2.19, gives a matrix eigenvalue problem for each pair of \( m \) and \( n \), such that

\[
\begin{bmatrix}
    K_{11} & K_{12} & K_{13} \\
    K_{21} & K_{22} & K_{23} \\
    \text{symm} & \text{symm} & K_{33}
\end{bmatrix}
\begin{bmatrix}
    U_{mn} \\
    V_{mn} \\
    W_{mn}
\end{bmatrix} = \lambda_{mn}
\begin{bmatrix}
    0 & 0 & 0 \\
    0 & 0 & 0 \\
    \text{symm} & G_{33}
\end{bmatrix}
\begin{bmatrix}
    U_{mn} \\
    V_{mn} \\
    W_{mn}
\end{bmatrix}
\]

(3.3)

where \( \lambda_{mn} \) is the eigenvalue and the constants \( U_{mn} \), \( V_{mn} \), and \( W_{mn} \) describe the associated eigenvector. The coefficients of the stiffness matrix, \([K]\), and stability matrix, \([G]\), are given in Appendix A. In terms of \( \lambda \) for the \((m, n)\) mode, the corresponding critical buckling load of the plate is given as

\[
(P_{cr})_{mn} = \frac{\pi^4 h^4}{a^2} E_2 \lambda_{mn}
\]

(3.4)

where \( E_2 \) is the elastic modulus of the plate constitutive layers in the in-plane direction normal to the fiber axis. The critical buckling load of the plate, \( P_{cr} \) is then the smallest of \((P_{cr})_{mn}\).

3.2 The Navier Solution of the Shear Deformation Theories

Additional unknown functions are introduced when shear-deformation models are employed, hence more comprehensive boundary conditions must be applied. The FSDT introduces the following additional constraint

\[
\psi_z(x, 0) = \psi_z(x, b) = \psi_y(0, y) = \psi_y(a, y) = 0
\]

(3.5)

while the HSDT model additionally requires that (Reddy)[35]

\[
P_x(0, y) = P_x(a, y) = P_y(x, 0) = P_y(x, b) = 0
\]

(3.6)

Navier Solution Procedure
In order to satisfy the boundary conditions for the simply supported plate, Eqs. 3.1, the displacement field is assumed to be of the form

\[
\begin{align*}
{u}_0 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn} \sin \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \\
{v}_0 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} V_{mn} \sin \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \\
{w}_0 &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} W_{mn} \sin \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \\
\psi_x &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} X_{mn} \sin \frac{m \pi x}{a} \cos \frac{n \pi y}{b} \\
\psi_y &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} Y_{mn} \sin \frac{m \pi x}{a} \cos \frac{n \pi y}{b}
\end{align*}
\]  
(3.7)

where \(m\) and \(n\) are integers indicating the number of half sine waves present in the buckling mode, and the constants \(U_{mn}, V_{mn}, W_{mn}, X_{mn},\) and \(Y_{mn}\), are the corresponding modal amplitudes.

As for the CPT, a substitution of the displacement field, Eqs. 3.7 into the governing equations Eq. 2.17, gives a matrix eigenvalue problem for each pair of \(m\) and \(n\), such that

\[
\begin{bmatrix}
K_{11} & K_{12} & K_{13} & K_{14} & K_{15} \\
K_{22} & K_{23} & K_{24} & K_{25} \\
K_{33} & K_{34} & K_{35} & & \\
\text{symm} & K_{44} & K_{45} & & \\
\text{symm} & & & K_{55}
\end{bmatrix}
\begin{bmatrix}
U_{mn} \\
V_{mn} \\
W_{mn} \\
X_{mn} \\
Y_{mn}
\end{bmatrix}
= \lambda_{mn}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
G_{33} & 0 & 0 & & \\
\text{symm} & 0 & 0 & & \\
\text{symm} & & & 0
\end{bmatrix}
\begin{bmatrix}
U_{mn} \\
V_{mn} \\
W_{mn} \\
X_{mn} \\
Y_{mn}
\end{bmatrix}
\]  
(3.8)

where \(\lambda_{mn}\) is the eigenvalue and the constants \(U_{mn}, V_{mn}, W_{mn}, X_{mn}\), and \(Y_{mn}\) describe the associated eigenvector. The coefficients of the stiffness matrix, \([K]\), and stability matrix, \([G]\), for the FSDT and HSDT are given in Appendix A. The critical buckling load of the plate is then given by Eq. 3.4.
Chapter 4
Probabilistic Analysis

4.1 Background

Traditional structural design relies primarily upon deterministic analysis. Appropriate dimensions, material properties, and loading conditions are assumed completely determined, and an analysis is performed which predicts the response of the structure. In the past, uncertainties were not ignored, but it was believed that a realistic upper limit to any load and lower limit to strength could be established. The structure could then be designed to survive this “worst case” scenario. Thus, safety factors were generally established by the judgment and experience of those who worked on a particular design. These safety factors generally failed to produce designs with uniform probabilities of failure, because of variabilities specific to individual designs.

In recent years, a more rigorous approach to reliability has been introduced. Economic factors have proved to be a driving force in the development of this design concept. Many structures in the past were greatly overdesigned because of a lack of detailed knowledge in probability theory. This leads to high cost without a significant improvement in reliability. The development of techniques which accurately reflected the probability of structural failure have enabled the designer to reduce material costs while maintaining a suitably high level of safety.

4.2 Probability-Based Methods

Many methods of reliability analysis exist. A basic precept of all models is the separation of the structural performance into two distinct regions, “failure” and “non-failure.” In practice, the complexity of systems makes the true reliability, \( p_f \), of the system which accounts for all failure modes, virtually impossible to determine. Instead, reliability methods determine a reliability, \( p_f \).
which considers only a specific set of failure modes. This $p_r$ is then used as a measure of $p_r'$.

The probability of failure, $p_f$, of a system is related to $p_r$ by the relation

$$p_f = 1 - p_r.$$  \hspace{1cm} (4.1)

A real system can be modeled with differing degrees of accuracy, therefore a system of classification has been established (Madsen, Krenk, and Lind, 1986). This classification is based upon the extent of information about the problem that is used or provided.

Reliability methods which employ only one “characteristic” value (the mean) of each random variable are known as Level I methods. Methods which employ two values of each uncertain parameter (usually mean and variance), along with a measure of the correlation between the parameters (covariance), are called Level II methods. Methods which use probability of failure as a measure, and therefore require a knowledge of the joint distribution of all uncertain parameters are called Level III methods. Finally, a reliability method which considers factors outside of the specific structural problem itself, i.e. consequences of failure, cost of fabrication, frequency of repair, etc. is classified as a Level IV method. These methods are appropriate when the structure is of major importance, for example, a highway bridge.

Although this study is concerned with a situation where only one parameter (the plate buckling resistance) is random, while the loading is assumed deterministic, insight into the problem can be gained by looking at the classic supply vs. demand problem, as shown by Ang and Tang (1984). When the safety of a structure is considered, it must be insured that the strength (supply) of the structure is sufficient to withstand the maximum applied loading (demand). A realistic determination of available supply and maximum demand is, of course, difficult to arrive at. Some degree of estimation and prediction is necessary, and therefore, the supply and demand are modeled as the random variables

$$X = \text{supply}$$

$$Y = \text{demand}$$  \hspace{1cm} (4.2)
with distributions shown in Fig. 4.1. It is the objective of reliability analysis to insure that \( X > Y \) throughout the design life of the structure. Because \( X \) and \( Y \) are random variables, it is not possible to absolutely assure that \( Y \) will never be greater than \( X \). A measure of the degree of certainty that \( X > Y \) is the reliability, \( P(X > Y) \). It follows, from eqn. (4.1), that the probability of failure, \( p_f \), is \( P(X < Y) \).

If the probability distributions of \( X \) and \( Y \) are known, \( p_f \) may be expressed as

\[
p_f = P(X < Y) = \sum_y P(X < Y|Y = y) P(Y = y)
\]  

(4.3)

When \( X \) and \( Y \) are continuous and statistically independent, (4.3) becomes

\[
p_f = \int_{-\infty}^{\infty} F_X(y) f_Y(y) \, dy
\]  

(4.4)

where \( F_X(y) \) is the cumulative distribution function of \( X \) evaluated at \( y \), and \( f_Y(y) \) is the probability density function of \( y \) evaluated at \( y \).

Demand exceeds supply in the region where the two curves in Fig. 4.1 overlap. Upon close examination of the curves, it is apparent that the region of overlap depends on the relative distance between \( f_X(x) \) and \( f_Y(y) \), and on their degree of dispersion. Their relative position can be quantitatively measured by the ratio of the mean demand to the mean supply, \( \mu_X / \mu_Y \). This quantity is known as the central safety factor. The dispersion may be measured in terms of the coefficients of variation, \( \delta_X \) and \( \delta_Y \).

The problem of supply and demand may also be recast in terms of the state or performance function (Ang and Tang, 1984),

\[
g = X - Y
\]  

(4.5)

where failure is defined as the condition \( g \leq 0 \). The probability of failure is given by

\[
p_f = \int_{-\infty}^{g} f_g(g) \, dg = F_g(g)
\]  

(4.6)
Fig. 4.1 – Probability density functions for supply and demand.
and is illustrated in Fig. 4.2.

The application of classical probability theory (Freudenthal, 1956) to reliability assessment requires a full description of the joint statistical distribution of the random (design) variables, \( (X_1, X_2, ..., X_n) \), as well as the definition of the limit-state equation

\[
g(X_1, X_2, ..., X_n) = 0.
\]  

The performance function, \( g \), determines the state of the system. The limiting performance requirement can be defined as the condition when \( g = 0 \), the "limit-state" of the system. If viewed geometrically, the limit-state equation, \( g(X) \) is an \( n \)-dimensional surface which may be called the "failure surface.”

The probability of failure, \( p_f \), of the system in then defined to be

\[
p_f = \int \int ... \int f_{X_1, X_2, ..., X_n} (x_1, x_2, ..., x_n) \, dx_1, \, dx_2, \, ..., \, dx_n
\]  

where \( f_{X_1, X_2, ..., X_n} \) is the joint probability density function for \( X_1, X_2, ..., X_n \) and the integration is performed over the region where \( g < 0 \).

In practice, the data required to determine the joint probability density function is usually inadequate. Furthermore, even if the joint density function is known, it is usually impractical to evaluate (4.8). Consequently, alternatives based upon simplifications of the classical theory have been developed. The first-order second-moment (FOSM) method which is a Level II method, estimates the probability of failure by using only the first two moments – the means and covariances, of the design variables, is frequently employed (Ang and Tang, 1984 and Shinozouka, 1983). The formulation is based upon the assumption that design variables are statistically independent random variables. It involves the linearization of the function \( g \) at the most probable failure point \( (X_1, X_2, ..., X_n) = (x_1^*, x_2^*, ..., x_n^*) \) on the failure surface \( g = 0 \).

As the failure surface, \( g(X) = 0 \), moves further or closer to the origin, the safe region \( g(X) > 0 \), increases or decreases. The location on the failure surface of the point with minimum
Fig. 4.2 – Probability density function for performance function, $g$. 

Probabilistic Analysis
distance to the origin is known as the most probable failure point. This distance may be used as an approximate measure of the system reliability.

In order gain a good understanding of the meaning of second-moment reliability, a problem involving one random variable will be examined (Hasofer and Lind, 1974). Suppose that a structure having a deterministic resistance, $X$, is subjected to a random load, $Y$. Failure is defined to occur when $Y > X$. It is the function of the reliability analysis method to determine the probability of failure, $P(Y > X)$. Generally, if this probability is less than some small value, $\epsilon$, the design is viewed as acceptable. Because often, sufficient information on the tail of the distribution of $S$ is not available, the failure criterion $P(Y > X) < \epsilon$ is replaced by a criterion involving the mean and standard deviation of $Y$. This criterion can be stated as

$$X > \mu_Y + \beta \sigma_Y \quad (4.9)$$

where $\mu_Y$ is the mean value of $Y$, $\sigma_Y$ is its standard deviation, and $\beta$ is a "reliability coefficient." An illustration of this condition is given in Fig. 4.3. The largest value of $\beta$ which satisfies (4.9) is called the safety index of the design. The rationale for the use of (4.9) is that it can be expected that most of the probability of the design variable, $Y$, will be concentrated within a few standard deviations of the mean.

One may also use a "reduced load",

$$Y' = \frac{Y - \mu_Y}{\sigma_Y} \quad (4.10)$$

In the space of $Y'$, there will be a corresponding new safe region, $G(Y')$, and the failure criterion will be defined such that the interval $(-\beta, +\beta)$ will be entirely within $G(Y')$. More simply stated, if the failure region is denoted as $G^*(Y')$, the distance from the point $\mu_Y$ to the region $G^*(Y')$, when $S$ is measured in units of standard deviation, must be greater than $\beta$ (Hasofer and Lind, 1974).

The problem can now be extended to the case of two random variables. Now, assume that the resistance, $X$, from the previous problem is considered to be random. As before, reduced
Fig. 4.3 – Safety index for one variable.
variables can be introduced, \( X' = (X - \mu_X) / \sigma_X \) and \( Y' = (Y - \mu_Y) / \sigma_Y \). The plane of \( X' \) and \( Y' \) is now divided into a safe region, \( G(X', Y') \), and a failure region, \( G^*(X', Y') \). It is now required that the circle of radius \( \beta \) centered at the origin, lie entirely within \( G(X', Y') \), as illustrated in Fig. 4.4. Again, the reason for this is that it can be expected that most of the joint probability of \( X \) and \( Y \) will be concentrated within the circle, and will be associated with safe values of \( X \) and \( Y \).

The minimum distance to the origin can be determined as follows (Shinozouka, 1983). The distance from a point \((X'_1, X'_2, ..., X'_n)\), where \( X'_i = (X_i - \mu_{X_i}) / \sigma_{X_i} \), on the failure surface \( g(X) = 0 \) to the origin is given by

\[
D = \sqrt{X'_1^2 + X'_2^2 + \ldots + X'_n^2} = \sqrt{(X' \cdot X')}
\]  

(4.11)

The point on the failure surface, \((x'_1^*, x'_2^*, ..., x'_n^*)\), which has the minimum distance to the origin can be found by the minimization of the function \( D \) subject to the constraint \( g(X) = 0 \). In order to accomplish this, the method of Lagrange's multiplier can be used as follows. Let

\[
L = D + \lambda g(X)
\]  

(4.12)

or

\[
L = \sqrt{(X' \cdot X')} + \lambda g(X)
\]  

(4.13)

In minimizing \( L \), the following set of \( n + 1 \) equations with \( n + 1 \) unknowns is arrived at

\[
\frac{\partial L}{\partial X'_i} = \frac{X'_i}{\sqrt{X'_1^2 + X'_2^2 + \ldots + X'_n^2}} + \lambda \frac{\partial g}{\partial X'_i} = 0 \quad i = 1, 2, ..., n
\]  

(4.14)

and

\[
\frac{\partial L}{\partial \lambda} = g(X_1, X_2, ..., X_n) = 0
\]  

(4.15)

The solution of these equations will yield the most probable failure point, \((x'_1^*, x'_2^*, ..., x'_n^*)\).

Probabilistic Analysis
Fig. 4.4 – Safety index for two variables.
Following the approach presented by Ang and Tang (1984), the gradient vector

\[ G = \left( \frac{\partial g}{\partial X_1}, \frac{\partial g}{\partial X_2}, \ldots, \frac{\partial g}{\partial X_n} \right) \]  

(4.16)

where

\[ \frac{\partial g}{\partial X_i} = \frac{dX_i}{dX_i} = \sigma_X \frac{dX_i}{dX_i} \]  

(4.17)

Equations (4.14) can then be rewritten as

\[ X = -\lambda DG \]  

(4.18)

From this, it can be shown that

\[ \lambda = (G^tG)^{-1/2} \]  

(4.19)

and by further substituting into (4.14) and (4.15), the minimum distance from the failure surface to the origin, \( D_{min} = \beta \), is

\[ \beta = \frac{-G^tX^*}{(G^tG)^{1/2}} \]  

(4.20)

where \( G^* \) is the gradient vector at the most probable failure point. Equation (4.20) can be rewritten in scalar form as

\[ \beta = \frac{-\sum_i^r \frac{g_{x_i}}{s_{x_i}} \mu_{x_i}}{\sqrt{\sum_i^r \left( \frac{g_{x_i}}{s_{x_i}} \right)^2 \mu_{x_i}}} = \frac{m_g}{s_g} \]  

(4.21)

where \( m_g \) and \( s_g \) are the first order approximations of the mean and variance of the function \( g \) as evaluated at the most probable failure point.

When the function \( g \) is linear in the design variables \( (X_1, X_2, \ldots, X_n) \), and these variables are statistically independent and normally distributed, \( \beta \) relates to the probability of failure, \( p_f \), by

\[ \beta = \Phi^{-1}(1 - p_f), \quad p_f = \Phi(-\beta) \]  

(4.22)
where $\Phi(.)$ is the standard normal distribution function.

In the case where the probability distribution of the design variables is not normal, the relation given in (4.22) is not valid. For this situation, an equivalent normal transformation technique can be applied. This is accomplished by the approximation of the actual distribution of the design variable $X_i$ by a normal distribution at the location corresponding to the most probable failure point on the failure surface.

An equivalent normal distribution for a non-normally distributed variable can be obtained so that both the cumulative probability as well as the probability density of the equivalent normal distribution are equal to that of the corresponding non-normal distribution at the point $x_i^*$ on the failure surface (Ang and Tang, 1984).

Equating the cumulative probabilities at the failure point, $x_i^*$, gives

$$
\Phi \left( \frac{x_i^* - \mu_{X_i}^N}{\sigma_{X_i}^N} \right) = F_{X_i^*}(x_i^*)
$$

(4.23)

where $\mu_{X_i}^N$ and $\sigma_{X_i}^N$ are the mean and standard deviation of the equivalent normal distribution, and $F_{X_i^*}(x_i^*)$ is the original cumulative distribution function (CDF) of $X_i$ evaluated at $x_i^*$. This then yields

$$
\mu_{X_i}^N = x_i^* - \sigma_{X_i}^N \Phi^{-1}[F_{X_i^*}(x_i^*)]
$$

(4.24)

Equating the probability density at $x_i^*$ implies that

$$
\frac{1}{\sigma_{X_i}^N} \phi \left( \frac{x_i^* - \mu_{X_i}^N}{\sigma_{X_i}^N} \right) = f_{X_i}(x_i^*)
$$

(4.25)

where $\phi(.)$ indicates the probability density function (PDF) of the standard normal distribution. From this, the following relation can be obtained

$$
\sigma_{X_i}^N = \frac{1}{f_{X_i}(x_i^*)} \phi^{-1}[F_{X_i}(x_i^*)]
$$

(4.26)

The value of the safety index, $\beta$, may now be calculated in a similar fashion as with normal variables, by using the equivalent normal mean and standard deviation.
For correlated design variables, the application of the FOSM method requires the original reduced variables to be transformed to a set of uncorrelated reduced variables, \((Z_1, Z_2, ..., Z_n)\). This is accomplished by the method of orthogonal transformation (Ang and Tang, 1984)

\[
Z = [T]^T Y
\]  

(4.27)

where \(Y\) and \(Z\) are vectors of the reduced correlated and uncorrelated variables. The transformation matrix \([T]^T\) contains the normalized eigenvectors of the correlation matrix \([C]\) of the original variables.
Chapter 5
Results and Conclusions

5.0 Introduction

In this study, the thickness of each layer, $h_i$, $i = 1, 2, ... N$, and the plate lateral dimensions, $a$ and $b$ are assumed deterministic. The mechanical properties and fiber angle of each layer are considered random variables with a specified mean, standard deviation, and probability density function. Depending on the plate model considered in the analysis, the total number of random variables for each transversely isotropic layer varies. For the classical plate model, there are four independent linear elastic mechanical constants ($E_1, E_2, G_{12}, v_{12}$) and the fiber orientation angle, $\theta$. For first- and higher-order shear deformation models, however, an additional independent mechanical constant, $G_{23}$, is required. Thus, for an $N$-layer laminated composite plate, the total number of random variables, $n$, is $5N$ for the CPT and $6N$ for both the FSDT and HSDT.

To eliminate the need for material specific information concerning the mechanical and geometric properties of the laminae, the following nondimensional parameters are introduced into the analysis,

$$E_i \rightarrow \frac{E_i}{E_2}, \quad G_{ij} \rightarrow \frac{G_{ij}}{E_2}, \quad i, j = 1, 2, 3$$

(5.1)

where the over-bar indicates the mean value.

For comparative purposes, the critical buckling load of the composite plate, $P_{cr}$, will be normalized with respect to the mean critical buckling load of the classical plate theory, $P_{cr}^{CPT}$, obtained by using the mean values of mechanical properties and fiber orientation of the constitutive layers. The buckling resistance of the plate is denoted by nondimensional parameter

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where

\[ R = \frac{P_{cr}}{\bar{P}_{cr}} \]  \hspace{1cm} (5.2)

The probability of failure of the composite plate, \( p_f \), due to buckling instability is defined as the probability that the resisting buckling load of the plate, \( R \), is less than or equal to the applied load, \( s \). That is

\[ p_f = \text{Prob}[R \leq s] \]  \hspace{1cm} (5.3)

where \( s \) is a nondimensional load parameter obtained by dividing the actual applied load by \( \bar{P}_{cr} \).

As a result of uncertainties associated with material mechanical properties and fiber angle orientation for each layer, the plate critical buckling resistance, \( R \), is a random variable with a cumulative probability distribution function (CDF), \( F_R(r) \), given as

\[ F_R(r) = \text{Prob}[R \leq r] \]  \hspace{1cm} (5.4)

and the corresponding probability density function (PDF), \( f_R(r) \), is obtained by

\[ f_R(r) = \frac{dF_R(r)}{dr} \]  \hspace{1cm} (5.5)

For the problem at hand, the state function \( g \) is defined as

\[ g(X_1, X_2, \ldots X_n) = R(X_1, X_2, \ldots X_n) - r \]  \hspace{1cm} (5.6)

where \( X_i \) are the random variables.

In applying the FOSM method, the value of the most probable buckling resistance (strength) of the plate must be evaluated. This can be accomplished by considering the eigenvalue problem of the plate, Eqn. 3.3 and 3.8, using the most probable values of the mechanical properties and fiber angle of the constitutive layers (most probable failure point). However, due to the highly

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nonlinear nature of the state function $g$, these most probable values of the design variables are not known a priori. Consequently, the most probable value of $R$ is obtained through an iterative approach introduced by Rackwitz and Fiessler (1978). Computer implementation of this approach involves the calculation of the buckling load of the plate several hundred times in order to determine the probability of failure of a plate subject to a particular loading situation. The actual number of critical loads which must be computed is dependent upon the rate of convergence of the algorithm for a particular state function, $g(x)$.

In this investigation, the following (nondimensionalized) material properties for Gr/Ep composites are used.

\begin{align*}
E_1 &\equiv (40.0, 4.0), \\
E_2 &\equiv (1.0, 0.1), \\
\nu_{12} &\equiv (0.25, 0.025), \\
G_{12} &\equiv (0.6, 0.06), \\
G_{23} &\equiv (0.5, 0.05), \\
\theta &\equiv (\bar{\theta}, 5^\circ),
\end{align*} \tag{5.7}

where the first and second values in the parentheses are the mean and standard deviation of the corresponding variable, respectively. For the purposes of this study, a coefficient of variation of 10% is imposed upon the material properties examined, as this is deemed to be a reasonable approximation of the variability found in many engineering materials.

5.1 Uniaxial Loading of a Square Plate

Numerical results will be presented for the case of a square $N$-layer composite plate ($a = b$) subjected to a unidirectional, in-plane loading along the $x$-axis ($\mu = 0$). All random variables will be considered statistically independent and normally distributed unless otherwise stated. A shear correction factor of $5/6$ is used in the FSDT.

The probability of failure of the plate is evaluated for the first five (lowest energy) buckling modes. The resulting probabilities of failure are, however, found to be insignificant when compared to that for the critical mode, as shown in Fig. 5.1.1. Consequently, the following probabilistic stability analysis of the composite plate will be based on the assumption that the plate will buckle only in the first, critical mode. This assumption is valid for cases in which

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the lowest energy mode is sufficiently distinct from higher modes. It should be noted that the critical mode may not have the lowest mode number; it is defined as the buckling mode with the lowest value of $P_{cr}$.

Plots comparing the cumulative distribution function (CDF), $F_R$, and the associated probability density function (PDF), $f_R$, for the resistance to buckling, $r$, of a composite plate obtained via three different plate mechanics models are shown in the following figures for various thickness ratios, $a/h$, and stacking sequences. Figures 5.1.2 – 5.1.4 illustrate the effect of $a/h$ on the probabilistic buckling resistance of a composite plate with a $[+45/-45]$ stacking sequence. Consistent with the results obtained through a deterministic analysis (Phan and Reddy, 1987), the mean critical buckling loads calculated by the FSDT and HSDT are found to be smaller than that obtained through the CPT. In terms of the CDF and PDF, it can be seen that the distributions of the resistance to buckling of the plate obtained by the shear-deformable models are generally less dispersive. This can be observed from the steeper slope of the CDF and the higher peak of the PDF. As $a/h$ increases, the influence of the transverse shear deformation decreases and the probability distributions for the critical buckling load obtained by the three plate models converge. It should be noted that both the CDF and PDF for the CPT are not affected by the change in the $a/h$ ratio. This is due to the CPT’s inability to model transverse shear deformation.

One factor which has a significant effect on the distribution of the critical loads for the composite plates examined is the number of random variables in a particular model. Again, it should be noted that there are 5 or 6 random variables per layer (depending on the plate model used), and the material properties and fiber orientation in each layer are allowed to vary independently. As the number of random variables (number of layers) is increased, the dispersion of the buckling loads decreases. This is illustrated in a comparison of Figs. 5.1.3, 5.1.5, and 5.1.6, where a $[+45/-45]$ laminate is examined in three different ways, each with an increasing number of random variables. This is because, as the number of random variables

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Fig. 5.1.1 – Comparison of mode contributions to cumulative distribution and probability density functions for the resistance to buckling of a [+45/-45] composite plate.
Fig. 5.1.2 – Cumulative distribution and probability density functions for the resistance to buckling of a [+45/-45] composite plate.
Fig. 5.1.3 – CDF (a) and PDF (b) for the resistance to buckling of a [+45/-45] composite plate with $a/h = 50$, $\rho = 1$, and $\mu = 0$. 

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Fig. 5.1.4 – CDF (a) and PDF (b) for the resistance to buckling of a $[+45/-45]$ composite plate with $a/h = 100$, $\rho = 1$, and $\mu = 0$. 

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becomes large, the influence of one individual variable on the overall strength of the plate becomes less pronounced. In the extreme, the distribution of $P_{cr}$ reduces to the deterministic load as the number of random variables approaches infinity.

A similar trend is also found in Figs. 5.1.7 – 5.1.10 for plates with a similar stacking sequence but a larger number of layers. Increasing the number of layers is shown to produce less variation in the distribution of the plate buckling resistance. This may be caused by the fact that the global mechanical properties of the plate are now more homogeneous, and the plate, therefore becomes less sensitive to the change in the mechanical properties of each individual layer.

Probability functions for the critical buckling load of a composite plate with a $\{+30/-30\}$ stacking sequence are shown in Figs. 5.1.11 and 5.1.12. The overall statistical variation of the resistance in this case is found to be almost identical to that of the previous case of the $\{+45/-45\}$ laminate.

In Figs. 5.1.13 – 5.1.16, it is interesting to note that probability distribution of the critical buckling loads for a plate with a $\{0/90/90/0\}$ stacking sequence has a much larger variation than that for a plate with a $\{90/0/0/90\}$ stacking sequence. This is caused by fibers in layers with $\theta = 0^\circ$ at the outermost layers having a greater contribution to the load carrying capacity of the plate. The plate is therefore highly sensitive to any variation in the mechanical properties and fiber orientation angle of these layers.

Because the mean values of mechanical properties of the laminae are all located more than five standard deviations away from zero, assuming either normal or lognormal random variables will have an insignificant effect on the probability distribution of the critical buckling load of the plate.

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Fig. 5.1.5 – CDF (a) and PDF (b) for the resistance to buckling of a 
$[+45_2/-45_2]_T$ composite plate
with $a/h = 50$, $\rho = 1$, and $\mu = 0$. 

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Fig. 5.1.6 – CDF (a) and PDF (b) for the resistance to buckling of a \([+45_3/-45_3]_T\) composite plate with \(a/h = 50\), \(\rho = 1\), and \(\mu = 0\).
Fig. 5.1.7 – CDF (a) and PDF (b) for the resistance to buckling of a \([±45/−45]_{2T}\) composite plate with \(a/h = 20\), \(\rho = 1\), and \(\mu = 0\).
Fig. 5.1.8 – CDF (a) and PDF (b) for the resistance to buckling of a $[+45/-45]_{2T}$ composite plate with $a/h = 50$, $\rho = 1$, and $\mu = 0$. 

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Fig. 5.19 – CDF (a) and PDF (b) for the resistance to buckling of a \([+45/-45]_{3T}\) composite plate with \(a/h = 20, \rho = 1,\) and \(\mu = 0\).
Fig. 5.1.10 – CDF (a) and PDF (b) for the resistance to buckling of a $[+45/-45]_3T$ composite plate with $a/h = 50$, $p = 1$, and $\mu = 0$. 

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Fig. 5.1.11 – CDF (a) and PDF (b) for the resistance to buckling of a \([+30/-30]_T\) composite plate with \(a/h = 20\), \(\rho = 1\), and \(\mu = 0\).
Fig. 5.1.12 - CDF (a) and PDF (b) for the resistance to buckling of a $[+30/-30]_T$ composite plate with $a/h = 50$, $\rho = 1$, and $\mu = 0$. 

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Fig. 5.1.13 – CDF (a) and PDF (b) for the resistance to buckling of a $[0/90]_S$ composite plate with $a/h = 20$, $\rho = 1$, and $\mu = 0$. 

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Fig. 5.1.14 - CDF (a) and PDF (b) for the resistance to buckling of a [0/90]s composite plate with a/h = 50, ρ = 1, and μ = 0.
Fig. 5.1.15 – CDF (a) and PDF (b) for the resistance to buckling of a $[90/0]_S$ composite plate with $a/h = 20$, $p = 1$, and $\mu = 0$. 

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Fig. 5.1.16 – CDF (a) and PDF (b) for the resistance to buckling of a $[90/0]_S$ composite plate with $a/h = 50$, $p = 1$, and $\mu = 0$.
5.2 Biaxial Loading of a Square Plate

In practical applications, plates are not commonly subjected to purely uniaxial loading. Some degree of biaxiality in the loading is often present. In cases where this type of loading is dominant, different physical phenomena can play important roles in the stability of the plate. In this section, the effect of an applied load with varying degrees of biaxiality on the probabilistic stability of laminated composite plates is examined. It is important to note that all critical loads in this section have been normalized to the uniaxial critical load as predicted by the CPT model for that particular plate.

Initially, the case of a square plate ($\rho = a/b = 1$) with a $[45/-45]$ stacking sequence is considered, where the lamina fiber orientations are indicated by $[\pm \theta/ -\theta]$ and $\theta$ is measured in degrees. Predictions for the CPT, FSDT, and HSDT models are given in Figs. 5.2.1 – 5.2.3 for $\mu = 0, 0.25, 0.5, 0.75, 1$. It can be readily seen that the directionality of the applied loading has a strong effect on the magnitude of the mean critical buckling load. Yet, for the range of $\mu$ examined, the directionality of the loading has no effect on the critical modeshape (the buckling mode associated with the critical buckling load). This result can be explained by the symmetry of the problem. The plate is square ($\rho = 1$) and the fibers are initially oriented so that they will offer the same stiffness to loads applied in either the 1- or 2- directions, thus the plate exhibits the same sensitivity to material and geometric (fiber orientation) property variations in either direction. It is also important to note that the dispersion of the critical loads becomes much smaller as the degree of biaxiality is increased. This implies that as $\mu$ is increased, the critical buckling load becomes less sensitive to variations in material parameters in the plate. The same result is obtained when the $[+45/-45]_{1T}$ layup is examined with the HSDT model (Fig. 5.2.4).

It would be reasonable to expect the same result for the $[90/0]_S$ layup because of the same symmetries in the laminate, however a quite different result is arrived at. The PDF’s are clearly broken out into two distinct groups with different dispersive characteristics: one for $\mu = 0,$
Fig. 5.2.1 – CDF (a) and PDF (b) for the resistance to buckling of a \([+45/-45]_T\) composite plate with \(a/h = 50\), \(\rho = 1\), and \(\mu = 0, 1/4, 1/2, 3/4, 1\).
Fig. 5.2.2 – CDF (a) and PDF (b) for the resistance to buckling of a $[+45/-45]_T$ composite plate with $a/h = 50$, $\rho = 1$, and $\mu = 0, 1/4, 1/2, 3/4, 1$. 

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Fig. 5.23 – CDF (a) and PDF (b) for the resistance to buckling of a $[+45/-45]_T$ composite plate with $a/h = 50$, $\rho = 1$, and $\mu = 0, 1/4$. 

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Fig. 5.2.4 – CDF (a) and PDF (b) for the resistance to buckling of a $[+45/-45]_{2T}$ composite plate with $a/h = 50$, $\rho = 1$, and $\mu = 1/2, 3/4, 1$. 

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1/4, and another for \( \mu \geq 0.5 \) (Fig. 5.2.5). Initially, this seems inconsistent with previous results, but upon closer inspection, it can be seen that the reason for this disparity is in the critical modeshapes. For the \( \mu < 0.5 \), the critical modeshape is \((m, n) = (2, 1)\), while for the higher degrees of biaxiality, \((m, n) = (1, 1)\) becomes the lowest energy one. Within each buckling mode, however, the trend towards decreasing sensitivity of the critical load with increasing biaxiality is again displayed.

The \([0/90]_s\) layup has the same critical modeshape \((m, n) = (1, 1)\) for \( \mu = 0, 0.25, 0.5, 0.75, \) and 1, and therefore the respective CDF's and PDF's display the same dispersive characteristics. As seen in the previous cases, increasing biaxiality results in decreased dispersion of critical buckling loads. Graphical illustrations of these phenomena are given in Fig. 5.2.6.

5.3 Generally Rectangular Plates with Biaxial Loading

A more broad class of laminated plate includes the generally rectangular geometry. In this section, the generally rectangular plate is studied under both uniaxial and biaxial loading conditions.

One parameter which has a strong influence on the plate rigidity, and hence on the resistance to buckling of the plate, is the inclusion of the transverse shear deformation degree of freedom. As the degree of allowed shear deformation is increased, the resistance to buckling is diminished. Results presented in this section are normalized with respect to the mean critical buckling load of the CPT, \( P_{cr}^{CPT} \). The buckling resistance of the plate is again denoted by \( R \), where

\[
R = \frac{P_{cr}}{P_{cr}^{CPT}} \tag{5.3.1}
\]

The first case examined is, again, the baseline \([+45/-45]_T\) laminate. A comparison of plates with three different aspect ratios, \((\rho = 1, 2, \text{ and } 4)\) subjected to uniaxial loading is given in Figs. 5.3.1 – 5.3.3 for each of the plate deformation models.

The introduction of statistical parameters such as the mean \((\mu_X)\), standard deviation \((\sigma_X)\),

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Fig. 5.2.5 – CDF (a) and PDF (b) for the resistance to buckling of a [90/0]_s composite plate with $a/h = 50, \rho = 1$, and $\mu = 0, 1/4, 1/2, 3/4, 1$. 

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Fig. 5.2.6 – CDF (a) and PDF (b) for the resistance to buckling of a $[0/90]_S$ composite plate with $a/h = 50$, $\rho = 1$, and $\mu = 0, 1/4, 1/2, 3/4, 1$.
Fig. 5.3.1 - CDF (a) and PDF (b) for the resistance to buckling of a $[+45/-45]_T$ composite plate using CPT with $\rho = 1, 2, 4$ ($a/h = 50, \mu = 0$).
Fig. 5.3.2 – CDF (a) and PDF (b) for the resistance to buckling of a \([+45/-45]_T\) composite plate using FSDT with \(\rho = 1, 2, 4\) \((a/h = 50, \mu = 0)\).
Fig. 5.3.3 – CDF (a) and PDF (b) for the resistance to buckling of a \([+45/-45]_T\) composite plate using HSDT with \(\rho = 1, 2, 4\) (\(a/h = 50, \mu = 0\)).
and skewness, can aid in the understanding of the statistical information presented in the probability distribution functions (Ang and Tang, 1975). These statistical parameters can be put into the following nondimensional forms to aid in comparisons of different plate responses.

A nondimensional measure of the dispersion or spread in a distribution is known as the Coefficient of Variation (COV, $\delta_X$), and can be expressed as

$$\delta_X = \frac{\sigma_X}{\mu_X}$$  \hspace{1cm} (5.3.2)

A nondimensional measure of the skewness can be found by dividing the third central moment (Ang and Tang, 1975) by the cube of the standard deviation as follows

$$\theta = \frac{\int_{-\infty}^{\infty} (x - \mu_X)^3 f_X(x) dx}{\sigma_X^3}$$  \hspace{1cm} (5.3.3)

This parameter is then known as the skewness coefficient. A positive value of $\theta$ indicates a probability distribution which is skewed to the values less the mean, $\mu_X$, and conversely a negative skewness coefficient describes a distribution which is skewed to the right of the mean.

When the statistical parameters for the $[+45/-45]_T$ laminate given in Table 5.3.1 are examined, the general trends of the data become more straightforward to analyze. The data for the uniaxial loading ($\mu = 0$) indicates that as the complexity of the plate bending model is increased, there is a minor decrease in the COV, and hence the dispersion of $P_{cr}$. This indicates that variations in material properties of the laminate have a slightly smaller influence on the stability of the plate when shear deformation is included in the bending models.

The CPT shows no increase in the COV with increasing $\rho$, while both the shear deformation models again show a small decrease in their dispersive properties with increasing $\rho$. All distributions for the uniaxial loading case are symmetric about $\mu_X$.

When the loading is made uniformly biaxial ($\mu = 1$), a dramatic effect is found upon the dispersive characteristics of the shear deformation model, as shown in Fig. 5.3.4. As mentioned
Table 5.3.1 – Statistical parameters for a $[+45/-45]_T$ plate with $\mu = 0$ and $a/h = 50$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Model</th>
<th>$\mu_X$</th>
<th>$\delta_X$, %</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>CPT</td>
<td>1.00</td>
<td>5.66</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>0.99</td>
<td>5.61</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>0.99</td>
<td>5.62</td>
<td>0.00</td>
</tr>
<tr>
<td>2</td>
<td>CPT</td>
<td>1.00</td>
<td>5.66</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>0.96</td>
<td>5.46</td>
<td>-0.01</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>0.97</td>
<td>5.50</td>
<td>-0.01</td>
</tr>
<tr>
<td>4</td>
<td>CPT</td>
<td>1.00</td>
<td>5.66</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>0.86</td>
<td>5.32</td>
<td>-0.06</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>0.89</td>
<td>5.10</td>
<td>-0.03</td>
</tr>
</tbody>
</table>
in the previous section, an increase in µ decreases the effective stiffness of the plate. Compressive loading in the transverse direction reduces the plate stiffness in the normal direction, and therefore the plate is dramatically more sensitive to variations in its constitutive properties. This is numerically illustrated in Table 5.3.2. The high degree of skewness for ρ=2, 4 should be noted.

The next case examined is that of the [45/−45]_{2T} laminate. Again, due to the anti-symmetry of the laminate, the same dispersive characteristics are seen for the CPT predictions in the uniaxial loading case, as shown in Fig. 5.3.5. When the HSDT model is used for the same plate configuration, the results diverge in much the same manner as for the [45/−45]_{T} plate (Fig. 5.3.6). When the biaxial loading case is considered, the same trends are apparent as were described earlier (Figs. 5.3.7, 5.3.8). There is an increase in the dispersion for the plates with higher aspect ratio. In general, Tables 5.3.3 and 5.3.4 show that the 4 lamina is generally less dispersive than the [45/−45]_{T} configuration. This is caused by the larger number of random variables considered in the analysis.

Now the case of the [0/90]_{s} plate is examined. Again, the relative magnitudes of the critical loads are seen to be the same, however there is a marked increase in the dispersion seen for plates with higher ρ. This is because, as ρ increases, more of the load is being supported by the outer, 0° plies, as they make the greatest contribution to the plate stiffness. Hence, the plate is much more sensitive to variations in parameters which control the stiffness of those layers. This increases the dispersion in the predicted buckling loads, as is shown in Figs. 5.3.9 – 5.3.11 and Tables 5.3.5 and 5.3.6. For the case of biaxial loading, there is again the increase in dispersion, along with the shift to lower relative critical loads (Fig. 5.3.11 – 5.3.12). It should also be noted that the buckling mode associated with changes as ρ is increased, and therefore the dispersion is decreased then increased with still higher aspect ratios.

Cumulative and probability density functions for the [90/0]_{s} laminate using the CPT and HSDT are presented in Figs 5.3.13 and 5.3.14, and results are presented in a tabular form in

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Table 5.3.2 – Statistical parameters for a \([+45/\ -45]^T\) plate with \(\mu = 1\) and \(a/h = 50\).

\[
\begin{array}{|c|c|c|c|c|}
\hline
\rho & \text{Model} & \mu_X & \delta_X, \% & \theta \\
\hline
1 & \text{CPT} & 1.00 & 5.66 & 0.00 \\
   & \text{FSDT} & 0.99 & 5.61 & 0.00 \\
   & \text{HSDT} & 0.99 & 5.62 & 0.00 \\
\hline
2 & \text{CPT} & 1.00 & 9.35 & 0.16 \\
   & \text{FSDT} & 0.98 & 9.23 & 0.14 \\
   & \text{HSDT} & 0.98 & 9.25 & 0.14 \\
\hline
4 & \text{CPT} & 1.00 & 14.57 & 0.34 \\
   & \text{FSDT} & 0.95 & 14.10 & 0.32 \\
   & \text{HSDT} & 0.96 & 14.21 & 0.32 \\
\hline
\end{array}
\]
Fig. 5.3.4 – CDF (a) and PDF (b) for the resistance to buckling of a $[+45/-45]_T$ composite plate using HSDT with $\rho = 1, 2, 4$ ($a/h = 50, \mu = 1$).

Results and Conclusions
Fig. 5.3.5 – CDF (a) and PDF (b) for the resistance to buckling of a \([+45/-45]_{2T}\) composite plate using CPT with \(\rho = 1, 2, 4\) \((a/h = 50, \mu = 0)\).
Fig. 5.3.6 – CDF (a) and PDF (b) for the resistance to buckling of a [+45/-45]_{2T} composite plate using HSDT with $\rho = 1, 2, 4$ ($a/h = 50, \mu = 0$).
Table 5.3.3 – Statistical parameters for a $[±45/-45]_{2T}$ plate with $\mu = 0$ and $a/h = 50$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Model</th>
<th>$\mu_X$</th>
<th>$\delta_X$ (%)</th>
<th>$\theta$</th>
</tr>
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<tbody>
<tr>
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<td>CPT</td>
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<td>4.80</td>
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</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>0.97</td>
<td>4.68</td>
<td>-0.01</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>0.97</td>
<td>4.68</td>
<td>-0.01</td>
</tr>
<tr>
<td>2</td>
<td>CPT</td>
<td>1.00</td>
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<td>4.39</td>
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<td></td>
<td>HSDT</td>
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<td>-0.02</td>
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<td>4.81</td>
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<td>FSDT</td>
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<td>HSDT</td>
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<td>4.78</td>
<td>-0.01</td>
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Fig. 5.3.7 – CDF (a) and PDF (b) for the resistance to buckling of a [+45/-45]_{2T}
composite plate using CPT with $\rho = 1, 2, 4$ ($a/h = 50$, $\mu = 1$).

Results and Conclusions
Fig. 5.3.8 – CDF (a) and PDF (b) for the resistance to buckling of a \([+45/-45]_{2T}\) composite plate using HSDT with \(\rho = 1, 2, 4\) (\(a/h = 50, \mu = 1\)).

Results and Conclusions 76
Table 5.3.4 – Statistical parameters for a $[+45/-45]_{2T}$ plate with $\mu = 1$ and $a/h = 50$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Model</th>
<th>$\mu_X$</th>
<th>$\delta_X, %$</th>
<th>$\theta$</th>
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</thead>
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</tr>
<tr>
<td></td>
<td>FSDT</td>
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<td>4.68</td>
<td>-0.01</td>
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<td>HSDT</td>
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</tr>
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<td></td>
<td>HSDT</td>
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<td>12.20</td>
<td>0.26</td>
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Fig. 5.3.9 – CDF (a) and PDF (b) for the resistance to buckling of a $[0/90]_S$ composite plate using CPT with $\rho = 1, 2, 4$ ($a/h = 50$, $\mu = 0$).
Fig. 5.3.10 – CDF (a) and PDF (b) for the resistance to buckling of a $[0/90]_S$ composite plate using HSDT with $\rho = 1, 2, 4$ ($a/h = 50, \mu = 0$).
Table 5.3.5 – Statistical parameters for a $[0/90]_S$ plate with $\mu = 0$ and $a/h = 50$.

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Model</th>
<th>$\mu_X$</th>
<th>$\delta_X$, %</th>
<th>$\theta$</th>
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<tr>
<td></td>
<td>FSDT</td>
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<td>5.56</td>
<td>-0.01</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>0.97</td>
<td>5.56</td>
<td>-0.01</td>
</tr>
<tr>
<td>2</td>
<td>CPT</td>
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<td>4.49</td>
<td>0.00</td>
</tr>
<tr>
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<td>FSDT</td>
<td>0.98</td>
<td>4.41</td>
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</tr>
<tr>
<td></td>
<td>HSDT</td>
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<td>4.39</td>
<td>0.00</td>
</tr>
<tr>
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<td>CPT</td>
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</tr>
<tr>
<td></td>
<td>FSDT</td>
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<td></td>
<td>HSDT</td>
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<td>3.87</td>
<td>-0.03</td>
</tr>
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</table>
Fig. 5.3.11 – CDF (a) and PDF (b) for the resistance to buckling of a $[0/90]_S$ composite plate using CPT with $\rho = 1, 2, 4$ ($a/h = 50, \mu = 1$).

Results and Conclusions
Fig. 5.3.12 – CDF (a) and PDF (b) for the resistance to buckling of a $[0/90]_S$ composite plate using HSDT with $\rho = 1, 2, 4$ ($a/h = 50, \mu = 1$).
Table 5.3.6 – Statistical parameters for a [0/90]_s plate with μ = 1 and a/h = 50.

<table>
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<tr>
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<th>Model</th>
<th>μ_X</th>
<th>δ_X, %</th>
<th>θ</th>
</tr>
</thead>
<tbody>
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<td>5.72</td>
<td>0.00</td>
</tr>
<tr>
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<td>5.56</td>
<td>-0.01</td>
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<td></td>
<td>HSDT</td>
<td>0.97</td>
<td>5.56</td>
<td>-0.01</td>
</tr>
<tr>
<td>2</td>
<td>CPT</td>
<td>1.00</td>
<td>4.49</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>0.98</td>
<td>4.41</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>0.97</td>
<td>4.39</td>
<td>-0.01</td>
</tr>
<tr>
<td>4</td>
<td>CPT</td>
<td>1.00</td>
<td>5.81</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>FSDT</td>
<td>0.93</td>
<td>5.45</td>
<td>-0.02</td>
</tr>
<tr>
<td></td>
<td>HSDT</td>
<td>0.92</td>
<td>5.35</td>
<td>-0.02</td>
</tr>
</tbody>
</table>
Table 5.3.7. The $[90/0]_S$ laminate shows a generally smaller $\delta_X$ for the uniaxial loading case than the $[90/0]_S$. This could be explained in that the outer, $90^\circ$ plies have less of an impact on the plate stiffness in the loading direction, and the plate is therefore less sensitive to variations in material parameters of those layers.

5.4 Conclusions

The conclusions of this work can be briefly summarized. In general, as shear deformation is included in the plate governing equations, the influence of variations in material properties and fiber angle orientation on the resistance to buckling is diminished. As the degree of biaxiality of the loading is increased, the mean value of the buckling load is decreased, but the dispersion of the critical loads is not significantly influenced provided that the critical mode is not altered. And finally, as the plate aspect ratio is increased, dispersion of the critical load decreases for the uniaxial case, but is dramatically increased for the case of biaxial loading. Also a large skew in the resultant critical loads is in evidence for the $[+45/-45]_{nT}$ laminates. This skewness is due to the fact that, for the biaxial loading case, perturbations in material parameters (especially fiber orientation) are more likely to decrease the critical load than to increase it, because the initial fiber angles are in the orientation which offers the highest resistance to buckling.
Fig. 5.3.13 – CDF (a) and PDF (b) for the resistance to buckling of a \([90/0]_s\) composite plate using CPT with \(\rho = 1, 2, 4\) (\(a/h = 50, \mu = 0\)).
Fig. 5.3.14 – CDF (a) and PDF (b) for the resistance to buckling of a [90/0]_S composite plate using HSDT with \( \rho = 1, 2, 4 \) \((a/h = 50, \mu = 0)\).
Table 5.3.7 – Statistical parameters for a \([90/0]_S\) plate with \(\mu = 0\) and \(a/h = 50\).

\[
\begin{array}{|c|c|c|c|c|}
\hline
\rho & \text{Model} & \mu_X & \delta_X, \% & \theta \\
\hline
1 & CPT & 1.00 & 4.49 & 0.00 \\
    & FSDT & 0.98 & 4.41 & 0.00 \\
    & HSDT & 0.97 & 4.39 & -0.01 \\
\hline
2 & CPT & 1.00 & 4.35 & 0.00 \\
    & FSDT & 0.93 & 4.02 & -0.02 \\
    & HSDT & 0.92 & 4.00 & -0.02 \\
\hline
4 & CPT & 1.00 & 4.35 & 0.00 \\
    & FSDT & 0.77 & 3.47 & -0.03 \\
    & HSDT & 0.75 & 3.41 & -0.02 \\
\hline
\end{array}
\]
References


References


References

Appendix A

Coefficients of Stiffness Matrices for the Navier Solution

The elements of the stiffness and stability matrices in Eqn. 3.3 (CPT) and Eqn. 3.8 (FSDT, HSDT) are given by

CPT:

\[
K_{11} = \left(\frac{a}{\pi}\right)^2 \frac{1}{h^3} \left[ A_{11}m^2 + A_{66}\left(\frac{a}{b}\right)^2 n^2 \right]
\]

\[
K_{12} = \left(\frac{a}{\pi}\right)^2 \frac{1}{h^3} \left(\frac{a}{b}\right)mn \left[ A_{12} + A_{66} \right]
\]

\[
K_{13} = \left(\frac{a}{\pi}\right) \frac{1}{h^2} \left[ -3B_{16}\left(\frac{a}{b}\right)m^2n - B_{26}\left(\frac{a}{b}\right)^3 n^3 \right]
\]

\[
K_{22} = \left(\frac{a}{\pi}\right)^2 \frac{1}{h^3} \left[ A_{66}m^2 + A_{22}\left(\frac{a}{b}\right)^2 n^2 \right]
\]

\[
K_{23} = \left(\frac{a}{\pi}\right) \frac{1}{h^2} \left[ -B_{16}m^3 - 3B_{26}\left(\frac{a}{b}\right)^2 mn^2 \right]
\]

\[
K_{33} = \left(\frac{a}{\pi}\right)^2 \frac{N_z}{Eh^4} \left[ m^2 + \mu\left(\frac{a}{b}\right)^2 n^2 \right] + \frac{1}{h} \left[ D_{11}m^4 + (2D_{12} + 4D_{66})\left(\frac{a}{b}\right)^2 m^2n^2 + D_{22}\left(\frac{a}{b}\right)^4 n^4 \right]
\]

FSDT:

\[
K_{11} = \left(\frac{a}{\pi}\right)^2 \frac{1}{h^3} \left[ A_{11}m^2 + A_{66}\left(\frac{a}{b}\right)^2 n^2 \right]
\]

\[
K_{12} = \left(\frac{a}{\pi}\right)^2 \frac{1}{h^3} \left(\frac{a}{b}\right)mn \left[ A_{12} + A_{66} \right]
\]

\[
K_{13} = 0
\]

\[
K_{14} = \left(\frac{a}{\pi}\right)^2 \frac{1}{h^3} \left[ 2B_{16}\left(\frac{a}{b}\right)mn \right]
\]

\[
K_{15} = \left(\frac{a}{\pi}\right) \frac{1}{h^2} \left[ B_{16}m^2 + B_{26}\left(\frac{a}{\pi}\right)^2 n^2 \right]
\]

\[
K_{22} = \left(\frac{a}{\pi}\right)^2 \frac{1}{h^3} \left[ A_{66}m^2 + A_{22}\left(\frac{a}{b}\right)^2 n^2 \right]
\]
$K_{23} = 0$

$K_{24} = \left(\frac{a}{\pi}\right)^2 \frac{1}{h^2} \left[ B_{16} m^2 + B_{26} \left(\frac{a}{b}\right)^2 n^2 \right]$

$K_{25} = \left(\frac{a}{\pi}\right)^2 \frac{1}{h^2} \left[ 2B_{26} \left(\frac{a}{b}\right) mn \right]$

$K_{33} = \left(\frac{a}{\pi}\right)^2 \frac{N_1}{Eh^4} \left[ m^2 + \mu \left(\frac{a}{b}\right)^2 n^2 \right] + \left(\frac{a}{\pi}\right)^2 \frac{1}{h^3} \left[ kA_{55} m^2 + kA_{44} \left(\frac{a}{\pi}\right)^2 n^2 \right]$

$K_{34} = \left(\frac{a}{\pi}\right)^3 \frac{1}{h^3} \left[ kA_{55} m \right]$

$K_{35} = \left(\frac{a}{\pi}\right)^3 \frac{1}{h^3} \left[ kA_{44} \left(\frac{a}{b}\right) n \right]$

$K_{41} = K_{14} \frac{1}{h}$

$K_{42} = K_{24} \frac{1}{h}$

$K_{43} = K_{34} \frac{1}{h}$

$K_{44} = \left(\frac{a}{\pi}\right)^2 \frac{1}{h^2} \left[ D_{11} m^2 + D_{66} \left(\frac{a}{b}\right)^2 n^2 \right] + \left(\frac{a}{\pi}\right)^4 \frac{1}{h^4} \left[ kA_{55} \right]$

$K_{45} = \left(\frac{a}{\pi}\right)^2 \frac{1}{h^2} \left[ \left(\frac{a}{b}\right) mn \left( D_{12} + D_{66} \right) \right]$

$K_{51} = K_{15} \frac{1}{h}$

$K_{52} = K_{25} \frac{1}{h}$

$K_{53} = K_{35} \frac{1}{h}$

$K_{54} = K_{45} \frac{1}{h}$

$K_{55} = \left(\frac{a}{\pi}\right)^2 \frac{1}{h^2} \left[ D_{66} m^2 + D_{22} \left(\frac{a}{b}\right)^2 n^2 \right] + \left(\frac{a}{\pi}\right)^4 \frac{1}{h^4} \left[ kA_{44} \right]$

HSDT:

$K_{11} = \left(\frac{a}{\pi}\right)^2 \frac{1}{h^3} \left[ A_{11} m^2 + A_{66} \left(\frac{a}{b}\right)^2 n^2 \right]$

$K_{12} = \left(\frac{a}{\pi}\right)^2 \frac{1}{h^3} \left(\frac{a}{b}\right) mn \left[ A_{12} + A_{66} \right]$

Appendix A
\[ K_{13} = \left( \frac{a}{\pi} \right)^2 \frac{1}{h^2} \left[ -4E_{16} \left( \frac{a}{b} \right) m^2 n - \frac{4}{3} E_{26} \left( \frac{a}{b} \right)^3 n^3 \right] \]

\[ K_{14} = \left( \frac{a}{\pi} \right)^2 \frac{1}{h^2} \left[ 2 \left( B_{16} - \frac{4}{3} E_{16} \right) \left( \frac{a}{b} \right) mn \right] \]

\[ K_{15} = \left( \frac{a}{\pi} \right)^2 \frac{1}{h^2} \left[ \left( B_{16} - \frac{4}{3} E_{16} \right) m^2 + \left( B_{26} - \frac{4}{3} E_{26} \right) \left( \frac{a}{b} \right)^2 n^2 \right] \]

\[ K_{21} = \left( \frac{a}{\pi} \right)^2 \frac{1}{h^3} \left[ A_{66} m^2 + A_{22} \left( \frac{a}{b} \right)^2 n^2 \right] \]

\[ K_{23} = \left( \frac{a}{\pi} \right)^2 \frac{1}{h^2} \left[ -4E_{26} \left( \frac{a}{b} \right)^2 mn^2 - \frac{4}{3} E_{16} m^3 \right] \]

\[ K_{24} = \left( \frac{a}{\pi} \right)^2 \frac{1}{h^2} \left[ \left( B_{16} - \frac{4}{3} E_{16} \right) m^2 + \left( B_{26} - \frac{4}{3} E_{26} \right) \left( \frac{a}{b} \right)^2 n^2 \right] \]

\[ K_{25} = \left( \frac{a}{\pi} \right)^2 \frac{1}{h^2} \left[ 2 \left( B_{26} - \frac{4}{3} E_{26} \right) \left( \frac{a}{b} \right) mn \right] \]

\[ K_{35} = \left( \frac{a}{\pi} \right)^2 \frac{N_f}{E h^4} \left[ m^2 + \mu \left( \frac{a}{b} \right)^2 n^2 \right] \]

\[ + \left( \frac{a}{\pi} \right)^2 \frac{1}{h^3} \left[ \left( A_{55} - 8D_{55} + 16F_{55} \right) m^2 - k \left( A_{44} - 8D_{44} + 16F_{44} \right) \left( \frac{a}{\pi} \right)^2 n^2 \right] \]

\[ + \frac{16}{9} \frac{1}{h} \left[ H_{11} m^4 + 2(H_{12} + 2H_{66}) + H_{22} \left( \frac{a}{b} \right)^4 n^4 \right] \]

\[ K_{34} = \left( \frac{a}{\pi} \right)^3 \frac{1}{h^3} \left[ (A_{55} - 8D_{55} + 16F_{55}) m \right] \]

\[ K_{35} = \left( \frac{a}{\pi} \right)^3 \frac{1}{h^3} \left[ kA_{44} \left( \frac{a}{b} \right) n \right] \]

\[ K_{41} = K_{14} \frac{1}{h} \]

\[ K_{42} = K_{24} \frac{1}{h} \]

\[ K_{43} = K_{34} \frac{1}{h} \]

\[ K_{44} = \left( \frac{a}{\pi} \right)^2 \frac{1}{h^2} \left[ D_{11} m^2 + D_{66} \left( \frac{a}{b} \right)^2 n^2 \right] + \left( \frac{a}{\pi} \right)^4 \frac{1}{h^4} \left[ kA_{55} \right] \]

\[ K_{45} = \left( \frac{a}{\pi} \right)^2 \frac{1}{h^2} \left[ (\frac{a}{b}) m n (D_{12} + D_{66}) \right] \]

\[ K_{51} = K_{15} \frac{1}{h} \]

\[ K_{52} = K_{25} \frac{1}{h} \]

\[ K_{53} = K_{35} \frac{1}{h} \]

\[ K_{54} = K_{45} \frac{1}{h} \]

\[ K_{55} = \left( \frac{a}{\pi} \right)^2 \frac{1}{h^2} \left[ D_{66} m^2 + D_{22} \left( \frac{a}{b} \right)^2 n^2 \right] + \left( \frac{a}{\pi} \right)^4 \frac{1}{h^4} \left[ kA_{44} \right] \]
Vita

Theofanis Demosthenes Rantis was born in Falls Church, Virginia on January 21, 1967. He graduated from Annandale High School, Annandale, Virginia in 1984 and entered Virginia Polytechnic Institute and State University. After graduating with a Bachelor of Science degree in Engineering Science and Mechanics in the spring of 1989, he returned to VPI to pursue a Master of Science degree in Engineering Mechanics. He took employment with the Carderock Division of the Naval Surface Warfare Center, David Taylor Model Basin in the fall of 1991, and is currently employed there as a Mechanical Engineer.