AN APPLICATION OF SIGNATURE OF SMOOTH MANIFOLDS

by

Jesse H. Taylor

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APPROVED:

Peter E. Haskell

Peter E. Haskell, Chairman

Robert A. McCoy

Robert A. McCoy

Robert F. Olin

Robert F. Olin

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(ABSTRACT)

We prove that no disjoint union of any number of copies of even dimensional complex projective space can bound a smooth oriented compact manifold with boundary. We prove this by defining and computing certain algebraic invariants for smooth oriented manifolds. A non-diffeomorphic relationship is established between boundary manifolds and complex projective space by contrasting invariants computed for these spaces.

CONTENTS

INTRODUCTION

The purpose of this paper is to prove that no disjoint union of any number of copies of $\mathbb{CP}^{2k}$ (with the same orientation) can serve as the boundary of a smooth compact oriented $(4k+1)$-dimensional manifold. We prove this by defining and computing certain algebraic invariants for smooth oriented manifolds. In particular, we compute the invariants for $\bigcup \mathbb{CP}^{2k}$. We show that, for a boundary, these invariants never take the values computed for $\bigcup \mathbb{CP}^{2k}$. Our style of argument is typical of ways in which the algebraic invariants reveal the topology of the manifold: contrasting values indicate a non-diffeomorphic relationship between smooth manifolds. In our exposition, we define a particular invariant, de Rham cohomology, as a quotient linear space of closed and exact differential forms. Two of the computable properties we examine are dimension and signature, the latter having its expression in terms of a bilinear pairing on cohomology. Our exposition uses both with deeper structure yielding more general results.

We reveal the flow of the exposition with the following summary. The theme of Chapter 1 is that we can do analysis on smooth manifolds because they are essentially subsets of $\mathbb{R}^n$ pieced together in a smooth way. Thus, many of the definitions comprising the first half of Chapter 1 are extensions of multivariable calculus definitions to smooth manifolds. Among the most important are the definitions of C^∞ maps on manifolds and diffeomorphisms.

Chapter 1 concludes with a discussion of differential forms. We first define the concept of a differential form on $\mathbb{R}^n$. Because smooth manifolds are locally Euclidean, we define a differential form on a smooth manifold as a collection of differential forms on certain subsets of $\mathbb{R}^n$ with an "overlap condition." This "overlap condition" is structured so that the forms agree smoothly on different subsets of $\mathbb{R}^n$ in a way consistent with the C^∞-compatibility of subsets of the manifold.

Because volume is not intrinsic to the manifold, we cannot define the integral in a way directly modeled on the definition of Riemann integrals. Differential forms are intrinsic to smooth manifolds and are the objects we will integrate. We discuss integration of n-forms on smooth oriented n-dimensional manifolds. One purpose of our discussion of integration is to generalize Stokes' Theorem to smooth manifolds with boundary. With such a generalization, we have an important link between analysis and topology. To illustrate this link, we use a variation of Stokes' Theorem (Green's theorem in the plane) to motivate our study of de Rham cohomology. As remarked previously, we define cohomology with differential forms. The existence of a non-exact closed form on $\mathbb{R}^2 \setminus \{(0,0)\}$ shows that the presence of the puncture leads to nontrivial cohomology. We emphasize the significance of this because cohomological properties can reveal the topology of a given manifold. In particular, the Poincaré lemmas are important examples.

The primary objective of Chapter 3 is to develop the tools needed for simplifying the computations of cohomology. The Poincaré lemmas are fundamental, and also very useful is the Mayer-Vietoris sequence. This sequence is stated in two forms: in terms of the algebra of differential forms and in terms of de Rham cohomology. Often we can write a manifold as the union of two open sets U and V such that the cohomology groups of U, V, and $U \cap V$ are known. The Mayer-Vietoris sequence relates these cohomology groups with the cohomology groups of $U \cup V$ by an exact sequence, often simplifying the calculations of the cohomology of $U \cup V$. Employing the Mayer-Vietoris sequence in this exposition reduces calculation of cohomology to an application of the dimension theorem from linear algebra. Following the examples illustrating our use of the Mayer-Vietoris sequence, we introduce Poincaré duality as a cohomological property of smooth oriented manifolds. We use Poincaré duality to develop the concept of signature of a manifold. Computation of $signature \bigcup \mathbb{CP}^{2k}$ is necessary in the proof of our result.

We prove our result in Chapter 4. Before proving our result for any number of copies of $\mathbb{CP}^{2k}$, we prove that no odd number of copies of $\mathbb{CP}^{2k}$ can serve as a boundary by comparing the dimensions of cohomology. The purpose in separating this from our main result is to illustrate how little information is needed to conclude certain manifolds fail to bound smooth compact oriented $(4k+1)$-dimensional manifolds.

The proof for any number of copies of $\mathbb{CP}^{2k}$ proceeds by showing that if M is the boundary of an oriented smooth compact $(4k+1)$-dimensional manifold with boundary, then $H^{2k}(M)$ contains a subspace one half the dimension of $H^{2k}(M)$ on which the bilinear form $([\alpha],[\beta]) \mapsto \int_M \alpha \wedge \beta$ is identically zero. The calculation of this bilinear form on $H^{2k}(M)$ shows that if N is the union of any number of copies of $\mathbb{CP}$ with the same orientation, then $dim(H^{2k}(N)) > 0$ but $H^{2k}(N)$ contains no subspace of positive dimension on which the bilinear form is identically zero.

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DIFFERENTIABLE MANIFOLDS AND DIFFERENTIAL FORMS

This chapter defines topological manifolds and gives a precise definition of differentiable manifolds with and without boundary. We show that real projective space ($\mathbb{RP}^n$) and complex projective space ($\mathbb{CP}^n$) are smooth manifolds, as properties of the latter are the focus of this paper. Subsequently, we present differential forms and the exterior derivative d as their properties are important in the development of the integral and de Rham theory.

DIFFERENTIABLE MANIFOLDS.

DEFINITION 1.1. *A n-dimensional manifold M is a second countable Hausdorff space having the property that for each $m \in M$, there is an open neighborhood U_m of m and a homeomorphism ϕ_m mapping U_m onto an n-ball in $\mathbb{R}^n$. The pair (U_m, ϕ_m) is called a coordinate chart or a coordinate neighborhood.*

EXAMPLE 1.2: If M is any open subset of $\mathbb{R}^n$ with the subspace topology, then M is an n-dimensional manifold. M is certainly Hausdorff and second countable; that M is locally Euclidean follows from M being open.

DEFINITION 1.3. *Let $U, V \subset \mathbb{R}^n$. A map $F\colon U \to V$ is a diffeomorphism provided F is a homeomorphism and both F and F^{-1} are of class C^∞.*

DEFINITION 1.4. *Two coordinate neighborhoods (U, ϕ) and (V, ψ) are said to be C^∞-related or C^∞-compatible if $\phi \circ \psi^{-1}$ and $\psi \circ \phi^{-1}$ are diffeomorphisms of the open subsets $\psi(U \cap V)$ and $\phi(U \cap V)$ of $\mathbb{R}^n$.*

DEFINITION 1.5. *A differentiable or smooth (C^∞) structure on a topological manifold M is a family $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}$ of coordinate charts (neighborhoods) satisfying the following:*

(1) the U_α cover M,

(2) for any α, β the neighborhoods (U_α, ϕ_α) and (U_β, ϕ_β) are C^∞-compatible,

(3) any coordinate neighborhood (V, ψ) compatible with every $(U_\alpha, \phi_\alpha) \in \mathcal{U}$ is itself in $\mathcal{U}$. A C^∞ manifold is a topological manifold together with a C^∞ structure.

REMARK 1.6: Any covering $\{(U_\alpha, \phi_\alpha)\}$ of a smooth manifold M by C^∞-compatible coordinate neighborhoods is contained in a unique C^∞-structure on M.

DEFINITION 1.7. *Let M be a C^∞ manifold of dimension n with differentiable structure $\{(U_\alpha, \phi_\alpha)\}$. Suppose $x_1, \ldots, x_n$ represent the standard coordinates on $\mathbb{R}^n$. Write $\phi_\alpha = (u_1, \ldots, u_n)$ where $u_i = x_i \circ \phi_\alpha$ defines a coordinate system on U_α. We say that a function f on U_α is differentiable on U_α if $f \circ \phi_\alpha^{-1}$ is a differentiable (or C^∞) function on $\mathbb{R}^n$. If M and N are smooth manifolds and $f \colon M \to N$ is a map, then f is differentiable (or C^∞) if for every point $m \in M$, there are coordinate neighborhoods (U, ϕ) and (V, ψ) such that the following hold:*

(1) $m \in U$ and $f(m) \in V$,

(2) $f(U) \subset V$, and

(3) $\psi \circ f \circ \phi^{-1}$ is differentiable (or C^∞) on $\mathbb{R}^n$.

DEFINITION 1.8. *Suppose M and N are smooth manifolds. A map $F \colon M \to N$ is a diffeomorphism if F is a homeomorphism and both F and F^{-1} are C^∞ as in Definition 1.7. Two smooth manifolds are said to be diffeomorphic if there is a diffeomorphism mapping one to the other.*

DEFINITION 1.9. *A C^∞ manifold with boundary is a second countable Hausdorff space M with a differentiable structure $\mathcal{U}$ in the following sense: $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}$ consists of a family of open subsets U_α of M each with a homeomorphism ϕ_α onto an open subset of $\mathbb{H}^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_n \geq 0\}$ with the subspace topology such that:*

(1) the U_α cover M,

(2) if (U_α, ϕ_α) and (U_β, ϕ_β) are elements of $\mathcal{U}$, then $\phi_\beta \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ \phi_\beta^{-1}$ are diffeomorphisms of $\phi_\alpha(U \cap V)$ and $\phi_\beta(U \cap V)$, open subsets of $\mathbb{H}^n$,

(3) $\mathcal{U}$ is maximal with respect to properties 1 and 2.

REMARK 1.10: Recall that a map F is differentiable on an arbitrary subset $A \subset \mathbb{R}^n$ if and only if F is differentiable on some open subset U containing A. Hence the definition of a diffeomorphism extends to subsets of $\mathbb{H}^n$. We say two open subsets $U, V \subset \mathbb{H}^n$ are diffeomorphic if there is a one-to-one map F taking U onto V such that F and F^{-1} are C^∞. Notice that diffeomorphisms take boundary points to boundary points. For if U is an open subset of $\mathbb{H}^n$ with $U \cap \partial\mathbb{H}^n = \emptyset$, then the inverse function theorem implies $F(U)$ is an open subset of $\mathbb{R}^n$ because no open subset of $\mathbb{H}^n$ containing a point of $\partial\mathbb{H}^n$ can be open in $\mathbb{R}^n$. Hence $F(U) \cap \partial\mathbb{H}^n = \emptyset$.

REMARK 1.11: Recall that if X is a topological space and $\sim$ is an equivalence relation on X, then $X/\sim = \{[x] | x \in X\}$ can be given a topology. For a subset $A \subset X$, define $[A] = \{[a] | a \in A\}$. Let $\pi \colon X \to X/\sim$ be the projection given by $\pi(x) = [x]$ for all $x \in X$. We define the quotient topology on $X/\sim$ by defining a set $U \subset X/\sim$ to be open if $\pi^{-1}(U)$ is open in X. The projection π is then continuous by definition. We call $X/\sim$ with this topology the quotient space of X relative to $\sim$.

We say an equivalence relation $\sim$ is open whenever a subset $A \subset X$ is open implies $[A] \subset X/\sim$ is open. When $\sim$ is open, properties of $X/\sim$ can be revealed by examining X, namely,

(1) $\sim$ is open on X if and only if π is an open map. When $\sim$ is open and X is second countable, then $X/\sim$ is second countable.

(2) Let $\sim$ be an open equivalence relation on X. Then $R = \{(x, y) | x \sim y\}$ is a closed subset of $X \times X$ if and only if the quotient space $X/\sim$ is Hausdorff.

EXAMPLE 1.12: In order to verify that $\mathbb{RP}^n$ is a smooth n-dimensional manifold, we show that it is a topological manifold and that the coordinate neighborhoods are C^∞ compatible. $\mathbb{RP}^n$ is defined to be the space $(\mathbb{R}^{n+1} \setminus \{\vec{0}\})/\sim$ with the quotient topology where $x \sim y$ if there is nonzero $t \in \mathbb{R}$ such that $y = tx$. Denote $(\mathbb{R}^{n+1} \setminus \{\vec{0}\})$ by X. We visualize elements of $X/\sim$ as lines through the origin.

We show $\mathbb{RP}^n$ is second countable by appealing to property (1) above. Suppose $U \subset X$ is open. We show π is an open map. Now $\pi(U) = \bigcup_{x \in U}[x]$ is open in $X/\sim$ if and only if $\pi^{-1}(\bigcup_{x\in U}[x])$ is open. But

$$\pi^{-1}(\bigcup_{x\in U}[x]) = \bigcup_{x\in U} \pi^{-1}([x]) = \bigcup_{\substack{t\in\mathbb{R}\\ t\neq 0}}(\{tx | x \in U\})$$

which is open. So π is an open map and $\mathbb{RP}^n$ is second countable.

To show $\mathbb{RP}^n$ is Hausdorff, we use property (2) of Remark 1.11. Let

$$R = \{(x, y) | x \sim y\} \subset X \times X.$$

Define $f\colon X \times X \to \mathbb{R}$ by

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i \neq j} |x_i y_j - x_j y_i|^2.$$

Clearly f is continuous and $f(x, y) = 0$ if and only if $x \sim y$. Then $R = f^{-1}(0)$, hence R is a closed set.

Now we need to construct coordinate neighborhoods on $\mathbb{RP}^n$. Let $\tilde{U}_i$ be the open set $\{x \in X | x = (x_1, \ldots, x_{n+1})$ and $x_i \neq 0\}$. Then $U_i = \pi(\tilde{U}_i)$ is open in $X/\sim$. Define

$$\phi_i([x]) = \left(\frac{x_1}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right)$$

for any $x \in [x]$. Notice that $\phi_i\colon U_i \to \mathbb{R}^n$ is continuous, one-to-one, and onto. We have ϕ_i^{-1} continuous because $\phi_i^{-1}(x_1, \ldots, x_n) = [(x_1, \ldots, x_{i-1}, 1, x_i, \ldots, x_n)]$ which implies

ϕ_i^{-1} is the composition of π with a C^∞ map $\psi\colon \mathbb{R}^n \to \mathbb{R}^{n+1}$ defined by $\psi(x_1,\ldots,x_n) = (x_1,\ldots,x_{i-1},1,x_i,\ldots,x_n)$. Hence ϕ_i is a homeomorphism.

Now suppose $U_i \cap U_j \neq \emptyset$. Then

$$\begin{aligned}\phi_i \circ \phi_j^{-1}(x_1,\ldots,x_n) &= \phi_i([(x_1,\ldots,x_{j-1},1,x_j,\ldots,x_n)]) \\ &= \left(\frac{x_1}{x_i},\ldots,\frac{x_{i-1}}{x_i},\frac{x_{i+1}}{x_i},\ldots,\frac{x_{j-1}}{x_i},\frac{1}{x_i},\frac{x_j}{x_i},\ldots,\frac{x_n}{x_i}\right).\end{aligned}$$

Since $\phi_j(U_i \cap U_j) \neq \emptyset$, we have $x_i \neq 0$. The same results hold with the i and j in opposite places, so $\phi_i \circ \phi_j^{-1}$ is a diffeomorphism. We have verified that $\mathbb{RP}^n$ is an n-dimensional differentiable manifold.

Remark 1.13: We can define $\mathbb{RP}^n$ as $S^n/\sim_1$ with the quotient topology, where $x \sim_1 y$ if and only if $x = \pm y$, because antipodal points on S^n determine a line through the origin.

Example 1.14: $\mathbb{CP}^n$ is defined to be $(\mathbb{C}^{n+1} \setminus \{\vec{0}\})/\sim$ with the quotient topology, where $z \sim w$ if and only if $w = az$ for some $a \in \mathbb{C} \setminus \{0\}$. The proof that $\mathbb{CP}^n$ is a $2n$-dimensional smooth manifold amounts to replacing $\mathbb{R}$ with $\mathbb{C}$ in Example 1.12.

Remark 1.15: Just as in Remark 1.13, we can give an alternative definition for $\mathbb{CP}^n$. Since $\mathbb{C}^{n+1} \setminus \{\vec{0}\} \cong \mathbb{R}^{2n+2} \setminus \{\vec{0}\}$, we write $\mathbb{CP}^n$ as $S^{2n+1}/\sim_1$ with the quotient topology, where $z \sim_1 w$ if and only if there is $a \in S^1$ such that $w = az$.

Differential Forms.

Let $x_1,\ldots,x_n$ be the linear coordinates on $\mathbb{R}^n$. Define Ω^* to be the algebra over $\mathbb{R}$ generated by $dx_1,\ldots,dx_n$ with the relations

$$\begin{cases} dx_i\,dx_i = 0 \\ dx_i dx_j = -dx_j dx_i, i \neq j. \end{cases}$$

Then Ω^* is a vector space over $\mathbb{R}$ with basis

$$1, dx_i, dx_i dx_j (i<j), dx_i dx_j dx_k (i<j<k), \ldots, dx_1 \ldots dx_n.$$

We define the C^∞ differential forms on $\mathbb{R}^n$ as the elements of

$$\Omega^*(\mathbb{R}^n) = \{C^\infty \text{ functions on } \mathbb{R}^n\} \underset{\mathbb{R}}{\otimes} \Omega^*.$$

The collection of C^∞ q-forms on $\mathbb{R}^n$, denoted $\Omega^q(\mathbb{R}^n)$, consists of all C^∞ differential forms of the form $\sum f_I dx_I$ where I is an ordered set of q distinct indices. We say the *degree* of a C^∞ differential form ω on $\mathbb{R}^n$, $deg(\omega)$, equals q if and only if $\omega \in \Omega^q(\mathbb{R}^n)$. The algebra $\Omega^*(\mathbb{R}^n) = \oplus_{q=0}^n \Omega^q(\mathbb{R}^n)$.

NOTATION: Suppose $I = \{i_1, \ldots, i_k\}$ is an ordered collection of k distinct indices. We define $f_I = f_{i_1 \ldots i_k}$ and $dx_I = dx_{i_1} \ldots dx_{i_k}$. If $J = \{j_1, \ldots, j_m\}$ is another such collection, then $f_I f_J$ is usual multiplication with the product denoted by $f_{IJ} = f_{i_1, \ldots, i_k, j_1, \ldots, j_m}$ and $dx_I dx_J = dx_{i_1} \ldots dx_{i_k} dx_{j_1} \ldots dx_{j_m}$.

The differential operator $d\colon \Omega^q(\mathbb{R}^n) \to \Omega^{q+1}(\mathbb{R}^n)$ is defined by

$$\begin{cases} \text{if } f \in \Omega^0(\mathbb{R}^n), \text{ then } df = \sum_i \frac{\partial f}{\partial x_i} dx_i \\ \text{if } \omega = \sum_{I \in \Lambda} f_I dx_I, \text{ then } d\omega = \sum_{I \in \Lambda} df_I dx_I. \end{cases}$$

A short calculation verifies d is well defined.

REMARK 1.16: On $\mathbb{R}^3$, $\Omega^0(\mathbb{R}^3)$ and $\Omega^3(\mathbb{R}^3)$ have one generator and $\Omega^1(\mathbb{R}^3)$ and $\Omega^2(\mathbb{R}^3)$ have three generators as a module over $C^\infty(\mathbb{R}^3)$. We make the following identifications:

$$C^\infty(\mathbb{R}^3) \leftrightarrow \text{0-forms} \leftrightarrow \text{3-forms}$$

$$f \leftrightarrow f \leftrightarrow f\, dxdydz$$

and

$$\text{vector fields} \leftrightarrow \text{1-forms} \leftrightarrow \text{2-forms}$$

$$(f, g, h) \leftrightarrow fdx + gdy + hdz \leftrightarrow fdydz - gdxdz + hdxdy.$$

If $f \in C^\infty(\mathbb{R}^3)$, then f is a 0-form and

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

Hence $d|_{\text{0-forms}} = \text{gradient}$.

Suppose (f_1, f_2, f_3) is a vector field. By the identication above, we write the vector field as $f_1dx + f_2dy + f_3dz$ and thus

$$\begin{aligned}
&d(f_1dx + f_2dy + f_3dz)\\
&\qquad = \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dxdy + \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) dydz + \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right) dxdz\\
&\qquad \leftrightarrow \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}, \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}, \frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right).
\end{aligned}$$

This is the definition of curl in $\mathbb{R}^3$, hence $d|_{\textbf{1-forms}} = \text{curl}$.

Finally, if (f_1, f_2, f_3) is a vector field, we can identify it with $f_1dydz - f_2dxdz + f_3dxdy$ and apply d as follows:

$$d(f_1dydz - f_2dxdz + f_3dxdy) = \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\right) dxdydz.$$

We see that $d|_{\textbf{2-forms}} = \text{divergence}$.

DEFINITION 1.17. *The wedge product* $(\wedge)$ *of two differential forms is defined as follows: if* $\alpha = \sum f_I dx_I$ *and* $\beta = \sum g_J dx_J$, *then*

$$\alpha \wedge \beta = \sum f_I g_J dx_I dx_J.$$

REMARK 1.18: We list the following facts without proof because each can be verified by short calculations.

(1) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{deg(\alpha)} \alpha \wedge d\beta.$

(2) $d \circ d = d^2 = 0.$

Also notice in $\mathbb{R}^n$, Definition 1.17 equates $dx_1 \dots dx_n$ with $dx_1 \wedge \dots \wedge dx_n$ provided the order of the dx_i's is preserved. Moreover, for any permutation σ, we have

$$dx_1 \dots dx_n = sign(\sigma) dx_{\sigma(1)} \wedge \dots \wedge dx_{\sigma(n)}$$

where

$$sign(\sigma) = \begin{cases} +1, & \text{if } \sigma \text{ is even} \\ -1, & \text{if } \sigma \text{ is odd.} \end{cases}$$

Let $x_1, \ldots, x_m$ be the standard coordinates on $\mathbb{R}^m$ and $y_1, \ldots, y_n$ be the standard coordinates on $\mathbb{R}^n$. A smooth map $f\colon \mathbb{R}^m \to \mathbb{R}^n$ induces a *pullback* map on C^∞ functions, namely $f^*\colon \Omega^0(\mathbb{R}^n) \to \Omega^0(\mathbb{R}^m)$ defined by $f^*(g) = g \circ f$. In order to extend this pullback to all forms, we write $f^*\colon \Omega^*(\mathbb{R}^n) \to \Omega^*(\mathbb{R}^m)$ defined by

$$f^*(\sum g_I dy_I) = \sum (g_I \circ f) df_I$$

where $f_i = y_i \circ f$ is the i^{th} component of f and

$$df_I = df_{i_1} \wedge \cdots \wedge df_{i_k} = \sum_j \frac{\partial f_{i_1}}{\partial x_j} dx_j \wedge \cdots \wedge \sum_j \frac{\partial f_{i_k}}{\partial x_j} dx_j.$$

Notice

$$\begin{aligned}(f^* dy_i)(f^* dy_i) &= d(y_i \circ f) d(y_i \circ f) \\ &= f^*(dy_i dy_i) \\ &= f^*(0) = 0\end{aligned}$$

and

$$\begin{aligned}(f^* dy_i)(f^* dy_j) &= d(y_i \circ f) d(y_j \circ f) \\ &= f^*(dy_i dy_j) \\ &= f^*(-dy_j dy_i) \\ &= -d(y_j \circ f) d(y_i \circ f) \\ &= -(f^* dy_j)(f^* dy_i).\end{aligned}$$

Hence f^* respects the relations

$$\begin{cases} dx_i\, dx_i = 0 \\ dx_i dx_j = -dx_j dx_i, i \neq j \end{cases}$$

and is well defined.

Suppose $U, V \subset \mathbb{R}^2$ are open, (x, y) and (u, v) are coordinates on U and V, and $f \colon U \to V$ is a smooth map with $f_1 = u \circ f$ and $f_2 = v \circ f$. Let $\omega = g(u, v)\, du \wedge dv$ be a smooth 2-form on V. Then

$$\begin{aligned} f^*\omega &= f^*(g\, du \wedge dv) \\ &= (g \circ f) d(u \circ f) \wedge d(v \circ f) \\ &= (g \circ f) df_1 \wedge df_2 \\ &= (g \circ f) \left(\frac{\partial f_1}{\partial x} dx + \frac{\partial f_1}{\partial y} dy \right) \left(\frac{\partial f_2}{\partial x} dx + \frac{\partial f_2}{\partial y} dy \right) \\ &= (g \circ f) \left(\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x} \right) dx \wedge dy, \end{aligned}$$

and $\left(\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x} \right)$ is the determinant of the Jacobian of f calculated in the given coordinates. An analogous calculation proves the following proposition.

PROPOSITION 1.19. *Suppose $f \colon \mathbb{R}^n \to \mathbb{R}^n$ is a smooth map on open subsets of Euclidean space. Let $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ be two coordinate systems on $\mathbb{R}^n$ such that $y_i = y_i \circ f$ (the i^{th} component function). If $\omega = g dy_1 \wedge \cdots \wedge dy_n$ is a smooth n-form defined on the image of f, then*

$$f^*\omega = (g \circ f) det(J_f) dx_1 \wedge \cdots \wedge dx_n.$$

LEMMA 1.20. *If $f \colon U \to V$ is a smooth map between open subsets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ and ω, ν are smooth forms on V, then*

(1) $f^*(\omega + \nu) = f^*\omega + f^*\nu$

(2) $f^*(\omega \wedge \nu) = (f^*\omega) \wedge (f^*\nu)$

(3) $d(f^*\omega) = f^*(d\omega)$

(4) $(\psi \circ \phi)^*\omega = \phi^* \circ \psi^*\omega$ *for any smooth maps* $\phi \colon U \to V$, $\psi \colon V \to W$.

PROOF: Properties (1), (2), and (4) follow by direct calculation. We prove (3). Suppose $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are the coordinates on U and V.

If $\omega = g\, dy_{i_1} \wedge \cdots \wedge dy_{i_q}$ is a q-form on V, then

$$\begin{aligned}
f^*(d\omega) &= f^*\left(\sum_j \frac{\partial g}{\partial y_j} dy_j \wedge dy_{i_1} \wedge \cdots \wedge dy_{i_q}\right) \\
&= \sum_j \left(\frac{\partial g}{\partial y_j} \circ f\right) d(y_j \circ f) \wedge d(y_{i_1} \circ f) \wedge \cdots \wedge d(y_{i_q} \circ f) \\
&= \sum_j \left(\frac{\partial g}{\partial y_j} \circ f\right) df_j \wedge df_{i_1} \wedge \cdots \wedge df_{i_q} \\
&= d(g \circ f) df_{i_1} \wedge \cdots \wedge df_{i_q} \\
&= d(f^*\omega). \quad \square
\end{aligned}$$

DEFINITION 1.21. *A differential form on a smooth manifold M with C^∞ structure $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}$ is a collection of forms*

$$\{\omega_{U_\alpha} \in \Omega^*(\phi_\alpha(U_\alpha)) | (U_\alpha, \phi_\alpha) \in \mathcal{U}\}$$

such that if $U_\alpha \cap U_\beta \neq \emptyset$ then

$$(\phi_\alpha \circ \phi_\beta^{-1})^* \omega_{U_\alpha}|_{\phi_\alpha(U_\alpha \cap U_\beta)} = \omega_{U_\beta}|_{\phi_\beta(U_\alpha \cap U_\beta)}.$$

REMARK 1.22: It is sufficient to define a differential form on a smooth manifold M by defining it on some subcovering $\mathcal{U}' \subset \mathcal{U}$. C^∞-compatibility determines the form uniquely.

A differential form ω on M is actually a collection of differential forms defined on the images of the coordinate maps with the "overlap condition" described above. For a C^∞ map between manifolds $f\colon M \to N$, we define $f^*\colon \Omega^*(N) \to \Omega^*(M)$ locally; i.e. if (U, ϕ) and (V, ψ) are as in Definition 1.7, then

$$\mathcal{F}_{U,V} = \psi \circ f \circ \phi^{-1}\colon \phi(U) \to \psi(V)$$

is a C^∞ map between open subsets of $\mathbb{R}^n$. Hence $\mathcal{F}_{U,V}^*$ pulls forms on $\psi(V)$ back to forms on $\phi(U)$. If ω is a differential form on N, then ω is a collection of forms $\{\omega_{V_\alpha}\}$

and $f^*\omega$ is expressed by the collection $\{\mathcal{F}^*_{U_\alpha,V_\alpha}\omega_{V_\alpha}\}$ wherever $\mathcal{F}_{U_\alpha,V_\alpha}$ is defined. $f^*\omega$ is a well defined differential form by virtue of Lemma 1.20.

Notice that, for $\omega \in \Omega^*(N)$ and $m \in M$, we want to define $f^*\omega$ at m. So for an arbitrary $m \in M$, we have $f(m) = n$ for some $n \in N$. Then for any coordinate neighborhood (V, ψ) containing the point n, there is a coordinate neighborhhod (W, ϕ) with $m \in W$ and $f(W) \subset V$. Clearly (W, ϕ) is C^∞-compatible with every coordinate neighborhood in the C^∞-structure on M. If we consider each $m \in M$, then the collection $\{(W_m, \phi_m)\}_{m \in M}$ is a covering of M by C^∞-compatible neighborhoods. Thus if ω is a differential form on N, then $f^*\omega$ is uniquely determined by Remark 1.22.

d is defined on smooth manifolds similarly. We simply replace $\mathbb{R}^n$ in the definition of d with the appropriate subset of $\mathbb{R}^n$. For if ω, defined by $\{\omega_{U_\alpha}\}$, is a differential form on a smooth manifold M, then $d\omega$ is the collection $\{d\omega_{U_\alpha}\}$. Because d commutes with pullbacks under smooth maps, we notice that the overlap condition for forms on manifolds is preserved. d and f^* are well defined operations on smooth manifolds, and because these operations are expressed on subsets of $\mathbb{R}^n$, Lemma 1.20 applies directly and hence holds for smooth forms on differentiable manifolds.

INTEGRATION ON MANIFOLDS AND DE RHAM COHOMOLOGY

The differentiable structure of an arbitrary smooth manifold does not determine volume of its subsets. Since volume is not intrinsic to the manifold, in reasoning based solely on differential topology, we cannot systematically assign volume to subsets of the manifold. Hence we cannot define the integral of a function over an arbitrary differentiable manifold in a way directly modeled on the definition of the Riemann integral. Instead, we must utilize the transformation properties of differential forms combined with the structure of differentiable manifolds to generalize the concept of an integral of a function over $\mathbb{R}^n$ to the integral of a differential form over a differentiable manifold without boundary. Once this is achieved, we present the details extending our definitions (without modification) to manifolds with boundary and generalize the classical Stokes' Theorem to differentiable manifolds.

We motivate the definition of the q^{th} de Rham cohomology for a differentiable manifold by providing a counterexample to the implication *closed* $\Rightarrow$ *exact*. This counterexample is based in vector calculus; we identify a vector field with a 1-form and use the observation that the integral of a conservative vector field is independent of path. Moreover, we comment on how such analysis can reveal topological properties of a space. Finally, we discuss the properties of de Rham cohomology with respect to maps, which we will use in the following chapters.

ORIENTED MANIFOLDS.

Let M be a smooth n-dimensional manifold with C^∞-structure $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}$. Recall that $\phi_\alpha \colon U_\alpha \to \mathbb{R}^n$ is given by $\phi_\alpha(u) = (x_1^\alpha(u), \ldots, x_n^\alpha(u))$ where $x_i^\alpha(u)$ is the i^{th} coordinate of $\phi_\alpha(u)$ in $\phi_\alpha(U_\alpha)$. The coordinate functions together with their order form a local oriented coordinate system on $\phi_\alpha(U_\alpha)$. If (U, ϕ) and (V, ψ) are elements of $\mathcal{U}$ with $U \cap V \neq \emptyset$, then $\phi \circ \psi^{-1}$ is a diffeomorphism between $\psi(U \cap V)$ and $\phi(U \cap V)$. We say (U, ϕ) and (V, ψ) are compatibly oriented if the Jacobian determinant of $\phi \circ \psi^{-1}$ is everywhere positive. That is, if $\phi(u) = (x_1(u), \ldots, x_n(u))$ and $\psi(v) = (y_1(v), \ldots, y_n(v))$ are the local coordinate systems on $\phi(U)$ and $\psi(V)$, then $det(J_{\phi \circ \psi^{-1}})$ calculated in local coordinates is everywhere positive.

DEFINITION 2.1. *Let M be a smooth manifold with C^∞ structure $\mathcal{U}$. A collection $\mathcal{U}' \subset \mathcal{U}$ of compatibly oriented coordinate neighborhoods is called an oriented cover of M provided the domains of coordinate neighborhoods in $\mathcal{U}'$ cover M. A particular choice of oriented covering is said to define an orientation for M, and M is then called oriented. Once an orientation is chosen for M, we confine our analysis to coordinate neighborhoods in the chosen oriented cover. M is said to be orientable if it can be oriented.*

REMARK 2.2: If U, $V \subset \mathbb{R}^n$ are open with coordinates $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, and if $\phi\colon U \to V$ is a diffeomorphism, then we say ϕ is orientation preserving if, when calculated in the chosen local coordinates, $det(J_\phi) > 0$ and orientation reversing when $det(J_\phi) < 0$. When a smooth connected manifold M is orientable, we always have two choices of orientation. We say two oriented coverings of M define the same orientation if their union is oriented. We define the equivalence class $[M]$ to be the collection of all oriented covers of M defining the same orientation. We denote by $-[M]$ the collection of all remaining oriented covers of M. These are the only two possible orientations for M because we have coordinate transformations with either positive or negative Jacobian determinant. $-[M]$ is the opposite orientation to $[M]$. For if (U, ϕ) is associated with some oriented covering in $[M]$ and (V, ψ) is associated with some oriented covering in $-[M]$ and $U \cap V \neq \emptyset$, then $\psi \circ \phi^{-1}$ has negative Jacobian determinant, and hence is an orientation reversing diffeomorphism.

Also, $-[M]$ is a nonempty collection because for any oriented covering $\{(U, \phi)\}$ in $[M]$, we find that $\{(U, \sigma \circ \phi)\}$ is an oriented covering in $-[M]$ when σ reverses the ordering of any chosen pair of coordinates.

In $\mathbb{R}^n$, we will denote the standard orientation by $[x_1, \ldots, x_n]$. We define

$$-[x_1, \ldots, x_n] = [x_2, x_1, x_3, \ldots, x_n].$$

These equivalence classes are defined just as for M. For example, if $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are two coordinate systems on $\mathbb{R}^n$ such that the change of coordinates $y_i = T_i(x_1, \ldots, x_n)$ are given by components of a diffeomorphism T, then $[x_1, \ldots, x_n] = [y_1, \ldots, y_n]$ if and only if $det(J_T) > 0$.

DEFINITION 2.3. *A partition of unity on a manifold is a collection of non-negative C^∞ functions $\{\rho_\alpha\}_{\alpha \in I}$ such that*

(1) every point has a neighborhood in which only finitely many ρ_α are not identically zero, and so pointwise $\sum \rho_\alpha$ is a finite sum

(2) $\sum_{\alpha \in I} \rho_\alpha = 1$.

THEOREM 2.4. *For any open cover $\{U_\alpha\}_{\alpha \in I}$ of a smooth manifold M, there is a partition of unity $\{\rho_\alpha\}_{\alpha \in I}$ such that the support of ρ_α is contained in U_α. In this case, we say $\{\rho_\alpha\}$ is a partition of unity subordinate to the open cover $\{U_\alpha\}$.*

PROOF: See [B].

PROPOSITION 2.5. *A C^∞ n-dimensional manifold M is orientable if and only if it has a global nowhere vanishing smooth n-form.*

PROOF: ($\Leftarrow$) Let ω be a smooth nonvanishing n-form on M. Let $\{(U_\alpha, \phi_\alpha)\}$ be a cover of M by C^∞ compatible coordinate neighborhoods. We will use the "overlap condition" in Definition 1.21 to alter the coordinate functions ϕ_α so that the compositions $\phi_\alpha \circ \phi_\beta$ (where defined) are orientation preserving diffeomorphisms.

For each U_α, we write $\omega_{U_\alpha} = f_\alpha\, dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha$ in oriented local coordinates for some smooth nonvanishing f_α. We may assume $f_\alpha > 0$ for all α because if f_α is negative, we replace ϕ_α with $\sigma \circ \phi_\alpha$ where σ negates a single component of ϕ_α. The "overlap condition" implies that $\phi_\alpha \circ \phi_\beta^{-1}$ is an orientation preserving diffeomorphism. To see this, suppose $U_\alpha \cap U_\beta$ is nonempty. Then

$$\omega_{U_\alpha} = f_\alpha\, dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha$$

and

$$\omega_{U_\beta} = f_\beta\, dx_1^\beta \wedge \cdots \wedge dx_n^\beta$$

where f_α and f_β are positive functions on U_α and U_β. Then

$$(\phi_\alpha \circ \phi_\beta^{-1})^* \omega_\alpha = \omega_\beta$$

by the "overlap condition." But

$$\begin{aligned}(\phi_\alpha \circ \phi_\beta^{-1})^* \omega_\alpha &= f_\alpha \circ \phi_\alpha \circ \phi_\beta^{-1} det(J_{\phi_\alpha \circ \phi_\beta^{-1}}) dx_1^\beta \wedge \cdots \wedge dx_n^\beta \\ &= f_\beta dx_1^\beta \wedge \cdots \wedge dx_n^\beta;\end{aligned}$$

hence $det(J_{\phi_\alpha \circ \phi_\beta^{-1}}) > 0$ and M has an oriented cover.

($\Rightarrow$) Let $\{(U_\alpha, \phi_\alpha = (x_1^\alpha, \ldots, x_n^\alpha))\}$ be an oriented cover of M. We will use a subordinate partition of unity $\{f_\alpha\}$ to construct a nowhere vanishing smooth n-form on M.

For each α, let $\omega_\alpha = dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha$. Because $\{(U_\alpha, \phi_\alpha)\}$ is an oriented cover,

$$(\phi_\alpha \circ \phi_\beta^{-1})^* \omega_\alpha = g \omega_\beta$$

for some positive C^∞ function g on $\phi_\beta(U_\beta)$ where defined. Let $\{f_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. Set $\omega = \sum_\alpha f_\alpha \omega_\alpha$. Recall that not all f_α can vanish at a point, hence ω is smooth and nonvanishing. $\square$

REMARK 2.6: An n-form of the type mentioned in Proposition 2.5 is called a *volume form* on M and is denoted by Ω. The space of smooth n-forms on M is one dimensional over $C^\infty(M)$, so any smooth n-form ω on M is of the form $f\Omega$, $f \in C^\infty(M)$.

We can define the orientation of M by considering equivalence classes of smooth nowhere vanishing n-forms on M. We say two n-forms ω and ω' are equivalent if there is a smooth positive function f such that $\omega = f\omega'$. Hence, on a connected orientable manifold M the nowhere vanishing smooth n-forms fall into two equivalence classes. Either class determines an orientation for M, written $[M]$.

Orientations defined by smooth n-forms are equivalent to orientations defined by choosing oriented covers. The proof of Proposition 2.5 reveals the relationship between oriented covers and smooth nowhere vanishing n-forms. For example, the standard orientation on $\mathbb{R}^n$ is given by $[dx_1 \wedge \cdots \wedge dx_n]$ and is equivalent to $[x_1, \ldots, x_n]$. If M and N are oriented manifolds oriented by smooth n-forms Ω_M and Ω_N, then we say a diffeomorphism $F\colon M \to N$ is orientation preserving if $F^*\Omega_N = f\Omega_M$ for some positive C^∞ function f.

INTEGRATION ON MANIFOLDS.

For the purposes of integration, will consider not only smooth n-forms, but n-forms of the type $g\Omega$, where g is compactly supported, almost continuous, and bounded (i.e. integrable). Such an n-form will be called an *integrable* n-form.

The data we need to integrate over an n-dimensional manifold M (or $\mathbb{R}^n$) are a chosen orientation for M and an integrable n-form.

DEFINITION 2.7. *Let $x_1, \ldots, x_n$ be the chosen oriented orthonormal coordinates on $\mathbb{R}^n$. Let ω be a compactly supported integrable n-form on $\mathbb{R}^n$ and let U be an open subset containing support(ω). ω can be expressed uniquely on U as $f\, dx_1 \wedge \cdots \wedge dx_n$*

for some integrable function f. Then

$$\begin{aligned}\int_{\mathbb{R}^n} \omega &= \int_U \omega \\ &= \int_U f\, dx_1 \wedge \cdots \wedge dx_n \\ &= \int_{\mathbb{R}^n} f\, dx_1 \cdots dx_n. \quad \textit{(Riemann)}\end{aligned}$$

Notice that we express ω in local coordinates so that the order of the differentials corresponds to the order of the coordinates. We can always express ω in local coordinates, for if $x_2, x_1, \ldots, x_n$ were our chosen oriented coordinates, then our expression of ω in local coordinates would change to $-f\, dx_2 \wedge dx_1 \wedge \cdots \wedge dx_n$ and we would define $\int_U \omega = -\int_{\mathbb{R}^n} f\, dx_2 dx_1 \cdots dx_n$.

Recall that an n-form ω on a smooth manifold M is a collection of n-forms ω_U defined on $\phi(U)$ where (U, ϕ) is in the smooth structure on M with the property that if $(U, \phi), (V, \psi)$ are coordinate neighborhoods on M and $U \cap V \neq \emptyset$, then

$$(\phi \circ \psi^{-1})^* \omega_U|_{\phi(U \cap V)} = \omega_V|_{\psi(U \cap V)}.$$

This suggests that the integral of an n-form on a manifold reduces to the integral of an n-form over subsets of $\mathbb{R}^n$. This is indeed the case. We require that M is oriented so that our definition is independent of the representative oriented cover chosen from $[M]$ (Proposition 2.12). Also notice that by choosing an orientation for M, we are specifying coordinate systems on subsets of $\mathbb{R}^n$ so that the coordinate transformations (where defined) have positive Jacobian determinant. Hence, by choosing an orientation for M, we are merely specifying coordinates on subsets of $\mathbb{R}^n$ in a special way. We define the integral of an n-form over a smooth oriented manifold as follows.

DEFINITION 2.8. *Suppose $[M]$ is the orientation for M. Let ω be an n-form with compact support contained in the domain U of some coordinate neighborhood (U, ϕ),* *meaning support*$(\omega) \subset \phi(U)$. *We suppose further that* $(y_1, \ldots, y_n)$ *represents* ϕ *componentwise (so that* $y_1, \ldots, y_n$ *are the oriented local orthonormal coordinates on* $\phi(U)$*). Then we can write* $\omega = \omega_U = g\, dy_1 \wedge \cdots \wedge dy_n$ *in local coordinates. We define*

$$\begin{aligned} \int_M \omega &= \int_U \omega \\ &= \int_{\phi(U)} \omega_U \\ &= \int_{\mathbb{R}^n} \omega_U \\ &= \int_{\mathbb{R}^n} g\, dy_1 \wedge \cdots \wedge dy_n \\ &= \int_{\mathbb{R}^n} g\, dy_1 \cdots dy_n. \quad \textit{(Riemann)}. \end{aligned}$$

REMARK 2.9: Recall from calculus the change of variables formula for Riemann integrals. If $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are two coordinate systems on $\mathbb{R}^n$ and $G\colon \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism such that $y_i = y_i \circ G$, then

$$\int_{\mathbb{R}^n} f(y_1, \ldots, y_n) dy_1 \cdots dy_n = \int_{\mathbb{R}^n} f \circ G\, |J_G| dx_1 \cdots dx_n.$$

This formula resembles the expression for pullbacks of forms (Proposition 1.19), and hence is often useful when studying how $\int \omega$ transforms under diffeomorphisms.

PROPOSITION 2.10. *Reversing the orientation of* M *will reverse the sign of* $\int_M \omega$, *i.e.*

$$\int_{-M} \omega = - \int_M \omega$$

when $-M$ *denotes* M *with the opposite orientation.*

PROOF: Let ω be a compactly supported integrable n-form contained in the domain of $(U, \phi = (x_1, \ldots, x_n))$ associated with some oriented cover in $[M]$. Furthermore, suppose $(U', \psi = (y_1, \ldots, y_n))$ is some coordinate neighborhood associated with an oriented cover in $-[M]$ with $U = U'$ and $det(J_{\phi \circ \psi^{-1}}) < 0$. By definition,

$$(\phi \circ \psi^{-1})^* \omega_U = \omega_{U'}.$$

Then

$$\int_{-M} \omega = \int_{\psi(U')} \omega_{U'} = \int_{\psi(U')} (\phi \circ \psi^{-1})^* \omega_U.$$

So $\omega_U = f\, dx_1 \wedge \cdots \wedge dx_n$ in local coordinates and

$$\int_{\psi(U')} \omega_{U'} = \int_{\psi(U')} (f \circ \phi \circ \psi^{-1}) det(J_{\phi \circ \psi^{-1}}) dy_1 \wedge \cdots \wedge dy_n.$$

But our change of variables formula for integrals implies

$$\int_{\phi(U)} \omega_U = \int_{\psi(U')} (f \circ \phi \circ \psi^{-1}) |det(J_{\phi \circ \psi^{-1}})| dy_1 \wedge \cdots \wedge dy_n.$$

But $det(J_{\phi \circ \psi^{-1}}) < 0$, hence

$$\int_{\psi(U')} \omega_{U'} = -\int_{\phi(U)} \omega_U = -\int_M \omega. \quad \square$$

For an arbitrary integrable n-form ω on M with compact support, we cover $support(\omega)$ with the domains of coordinate neighborhoods $(U_1, \phi_1), \ldots, (U_m, \phi_m)$. Then $\mathcal{C} = \{U_1, \ldots, U_m, M \setminus support(\omega)\}$ forms an open cover of M. We choose a partition of unity $\{\rho_i\}$ subordinate to $\mathcal{C}$ so that $\omega = \sum_i \rho_i \omega$. In this setting, we make the following definition.

DEFINITION 2.11.

$$\int_M \omega = \sum_i \int_M \rho_i \omega$$

Each term in the sum above is defined by the case where $support(\rho_i \omega)$ is contained in the domain of a coordinate neighborhood.

PROPOSITION 2.12. *The definition of $\int_M \omega$ is independent of the choice of coordinate neighborhoods $\{(U_i, \phi_i)\}_{i=1}^m$ and the partition of unity $\{\rho_i\}$.*

PROOF: We suppose $\{(U_i, \phi_i)\}$ and $\{(V_j, \psi_j)\}$ are oriented covers of M such that their union is oriented. Suppose $\{V_j\}_{j=1}^n$ is a covering of *support*(ω) and $\{\sigma_j\}$ is a partition of unity subordinate to $\{V_j\}$. Then

$$\sum_i \int_{U_i} \rho_i \omega = \sum_{i,j} \int_{U_i} \rho_i \sigma_j \omega$$

since $\sum_j \sigma_j = 1$. But $\rho_i \sigma_j \omega$ has compact support in $U_i \cap V_j$, so

$$\int_{U_i} \rho_i \sigma_j \omega = \int_{V_j} \rho_i \sigma_j \omega$$

and

$$\sum_i \int_{U_i} \rho_i \omega = \sum_{i,j} \rho_i \sigma_j \omega = \sum_j \int_{V_j} \sigma_j \omega. \quad \square$$

PROPOSITION 2.13. *Let M and N be oriented differentiable manifolds. If $F\colon M \to N$ is a diffeomorphism and ω is an integrable n-form on N, then*

$$\int_M F^*\omega = \begin{cases} \int_N \omega, & \text{if } F \text{ preserves orientation} \\ -\int_N \omega, & \text{if } F \text{ reverses orientation.} \end{cases}$$

PROOF: Suppose $\omega = f dx_1 \ldots dx_n$ is an integrable n-form on N with support contained in the domain of some coordinate neighborhood (U, ϕ). Since F is a diffeomorphism, $F^*\omega$ has support in the domain of a coordinate neighborhood (V, ψ) on M. Recall from Chapter 1 that

$$F^*\omega = (f \circ \mathcal{F}_{V,U}) det(J_{\mathcal{F}_{V,U}}) dy_1 \wedge \cdots \wedge dy_n.$$

Hence

$$\int_{\mathbb{R}^n} \mathcal{F}_{V,U}^* \omega = \int_{\mathbb{R}^n} (f \circ \mathcal{F}_{V,U}) det(J_{\mathcal{F}_{V,U}}) dy_1 \wedge \cdots \wedge dy_n.$$

But our change of variables formula for integrals implies

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} (f \circ \mathcal{F}_{V,U})|det(J_{\mathcal{F}_{V,U}})|dy_1 \wedge \cdots \wedge dy_n.$$

Hence

$$\int_M F^*\omega = \pm \int_N \omega,$$

depending on whether or not F preserves or reverses orientation. If F preserves orientation, then $det(J_{\mathcal{F}_{V,U}})$ is positive, and if F reverses orientation, $det(J_{\mathcal{F}_{V,U}})$ is negative. □

In the case that M is an oriented differentiable manifold with boundary, M determines the differentiable structure on ∂M. Coordinate neighborhoods on ∂M are given by $(\tilde{U}, \tilde{\phi})$ where $\tilde{U} = U \cap \partial M$ and $\tilde{\phi} = \phi|_{U \cap \partial M}$. Hence if $[M]$ is determined by the oriented cover $\{(U_\alpha, \phi_\alpha)\}$, then $\tilde{\phi}_\alpha \circ \tilde{\phi}_\beta^{-1}$ (where defined) is the restriction of an orientation preserving diffeomorphism, and thus is itself an orientation preserving diffeomorphism. So $\{(\tilde{U}_\alpha, \tilde{\phi}_\alpha)\}$ forms an oriented cover of ∂M and defines the orientation *induced* by $[M]$, denoted $[\partial M]$. For proof of this, consult [B]. Notice further that the definition of the integral remains unchanged with this added structure.

STOKES' THEOREM. *Let M be an oriented n-dimensional differentiable manifold with boundary ∂M given the orientation induced by M. If ω is a compactly supported $(n-1)$-form on M then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

PROOF: Proofs of Stokes' Theorem can be found in many introductory texts on differentiable manifolds. In particular, refer to [B],[BoT], or [GP]. □

REMARK 2.14: Clearly, if M is a surface in $\mathbb{R}^3$ bounded by curves, then the version of Stokes' Theorem above is a restatement of the classical version. Moreover, notice that if M is an interval, then Stokes' Theorem is the Fundamental Theorem of Calculus.

Furthermore, if M is a region in $\mathbb{R}^2$ ($\mathbb{R}^3$) bounded by simple closed smooth curves (surfaces), then Stokes' Theorem gives the results of Green's Theorem in the plane (Gauss' Theorem). We explain the case where M is an open connected subset of $\mathbb{R}^2$ bounded by simple closed curves.

Suppose $\omega = fdx + gdy$ where f and g are C^1 functions on some open set containing M. Then

$$d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy.$$

Stokes' Theorem implies

$$\int_M \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = \int_{\partial M} f dx + g dy.$$

The integral on the left is an ordinary Riemann integral over $M \subset \mathbb{R}^2$ and the integral on the right can be expressed as

$$\sum_i \int_{C_i} f dx + g dy$$

where $C_1 + \cdots + C_n$ comprises the boundary of M, oriented appropriately. Hence

$$\iint_M \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx dy = \sum_i \int_{C_i} f dx + g dy,$$

which is the classical statement of Green's Theorem.

DE RHAM COHOMOLOGY.

DEFINITION 2.15. *A differential form ω on a manifold M is called closed if $d\omega = 0$ and ω is called exact if ω is in the image of d.*

REMARK 2.16: Every exact form is closed because $d(d\omega)) = 0$ for all forms ω. It is not the case that every closed form is exact, as is illustrated by the following example.

EXAMPLE 2.17: Let ω be the 1-form defined on $\mathbb{R}^2 \setminus \{(0,0)\}$ by

$$\left(\frac{-y}{x^2 + y^2}\right) dx + \left(\frac{x}{x^2 + y^2}\right) dy.$$

Calculation shows that $d\omega = 0$, hence ω is a closed form. If ω were exact, then it could be written as $d\zeta$ for some function ζ on $\mathbb{R}^2 \setminus \{(0,0)\}$. Recall from Chapter 1 that a differential 1-form can be identified with a vector field. Also recall that a vector field is called *conservative* if it is the gradient of some function (called a potential function). We have the result that the integral of a vector field is independent of path if and only if the vector field is conservative. We use this fact to show ω is not exact. Suppose C_1 is a semicircle connecting $(1,0)$ to $(-1,0)$ oriented counterclockwise and C_2 is another semicircle connecting these points oriented clockwise, i.e. $C_1 - C_2$ is the unit circle oriented counterclockwise. Notice that $(0,\infty) \times [0,\pi]$ and C_1 are smooth orientable manifolds and

$$F\colon (0,\infty) \times [0,\pi] \to \{(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\} | y \geq 0\}$$

defined by $F(r,\theta) = (r\cos\theta, r\sin\theta)$ is an orientation preserving diffeomorphism. Hence $F|_{\{1\}\times[0,\pi]}$ is an orientation preserving diffeomorphism onto C_1. Then

$$\int_{\{1\}\times[0,\pi]} F^*\omega = \int_{C_1} \omega$$

by Proposition 2.7. A straightforward calculation reveals

$$\begin{aligned}(F|_{\{1\}\times[0,\pi]})^*\omega &= \frac{-\sin\theta}{\cos^2\theta + \sin^2\theta}(1)(-\sin\theta)d\theta + \frac{\cos\theta}{\cos^2\theta + \sin^2\theta}(1)(\cos\theta)d\theta \\ &= d\theta\end{aligned}$$

Hence $\int_{C_1} \omega = \int_0^\pi d\theta = \pi$.

Similar analysis allows us to conclude

$$\int_{C_2} \omega = -\int_\pi^{2\pi} d\theta = -\pi.$$

Clearly these integrals are not independent of path, so $\omega \neq d\zeta$ for any 0-form ζ. Hence ω is not exact. Notice that $\int_{S^1} \omega \neq 0$ because S^1 encircles the puncture. Hence

S^1 is not the boundary of a compact manifold on which ω is defined. If, instead, we integrated ω over any smooth closed path γ not encircling the puncture, then Green's Theorem would imply $\int_\gamma \omega = 0$ because γ is the boundary of some compact region in $\mathbb{R}^2$.

If we change our domain from $\mathbb{R}^2 \setminus \{(0,0)\}$ to $\mathbb{R}^2$, the Poincaré lemma, stated in Chapter 3, will show that every closed differential p-form on $\mathbb{R}^2$ is exact for $p \geq 1$.

DEFINITION 2.18. *For a smooth manifold M, the vector space*

$$H^q(M) = \frac{\{\textit{closed differential } q\textit{-forms on } M\}}{\{\textit{exact differential } q\textit{-forms on } M\}}$$

called the q^{th} de Rham cohomology of M. Two forms α and β are in the same cohomology class, i.e. $[\alpha] = [\beta]$, if and only if their difference is exact. $[0]$ is the class of exact q-forms on M.

DEFINITION 2.19. *The q^{th} de Rham cohomology of M with compact supports, denoted $H^q_c(M)$, is defined just as $H^q(M)$ is, except all forms are compactly supported.*

REMARK 2.20: Notice that if a differentiable manifold M can be expressed as the disjoint union $U \uplus V$ of two open sets U and V, then $H^*(U \uplus V) = H^*(U) \oplus H^*(V)$.

REMARK 2.21: Example 2.17 exhibits a closed 1-form ω on $\mathbb{R}^2 \setminus \{(0,0)\}$ which is not exact and contrasts this with the fact that every closed 1-form on $\mathbb{R}^2$ is exact. Thus $[0]$ is the only cohomology class in $H^1(\mathbb{R}^2)$, hence $H^1(\mathbb{R}^2)$ is zero dimensional. When $\mathbb{R}^2 \setminus \{(0,0)\}$ is our space, the example shows $[\omega] \neq [0]$. Hence the presence of a hole leads to nontrivial cohomology. The q^{th} de Rham cohomology measures the size of the space of nonexact closed q-forms on a manifold and can reveal topological properties of the manifold. This motivates the next theorem.

REMARK 2.22: Suppose $f\colon M \to N$ is a C^∞ map between smooth manifolds. Because f^* and d commute on forms (Lemma 1.20), we find that

$$f^*(\{\text{closed } q\text{-forms on } N\}) \subset \{\text{closed } q\text{-forms on } M\}$$

and

$$f^*(\{\text{exact } q\text{-forms on } N\}) \subset \{\text{exact } q\text{-forms on } M\}.$$

Hence f^* induces a linear map on cohomology, also denoted f^*. More precisely, we have

DEFINITION 2.23. *A C^∞ map $f\colon M \to N$ between smooth manifolds induces a linear map on cohomology, $f^*\colon H^*(N) \to H^*(M)$, defined by $f^*[\omega] = [f^*\omega]$.*

THEOREM 2.24. *Let M and N be differentiable n-dimensional manifolds. If $f\colon M \to N$ is a diffeomorphism, then $H^q(M) \cong H^q(N)$.*

PROOF: Suppose $f\colon M \to N$ is a diffeomorphism. Since both f and f^{-1} are smooth, $f^*\colon H^q(N) \to H^q(M)$ and $f^{-1^*}\colon H^q(M) \to H^q(N)$ are linear maps. We see that $f^{*-1} = f^{-1^*}$ because

$$f^{-1^*} \circ f^* = (f \circ f^{-1})^* = id^* \qquad \text{and}$$
$$f^* \circ f^{-1^*} = (f^{-1} \circ f)^* = id^*.$$

Hence f^* is invertible and is thus a vector space isomorphism. So $H^q(M) \cong H^q(N)$.
□

DEFINITION 2.25. *A homotopy between two maps f and g from M to N is a C^∞ map $F\colon M \times [0,1] \to N$ such that*

$$\begin{cases} F(x,t) = f(x) & \text{for } t = 1, \\ F(x,t) = g(x) & \text{for } t = 0. \end{cases}$$

M and N are said to have the same homotopy type in the C^∞ sense if there are C^∞ maps $f\colon M \to N$ and $g\colon N \to M$ such that $g \circ f$ and $f \circ g$ are C^∞ homotopic to the identity on M and N respectively.

DEFINITION 2.26. *A deformation retraction of X onto A is a C^∞ map $F\colon X\times[0,1]\to X$ such that*

$$\begin{cases} F(x,t)=x & \text{for all } x\in X \text{ and } t=0,\\ F(x,t)\in A & \text{for } x\in X \text{ and } t=1,\\ F(a,t)=a & \text{for } a\in A \text{ and } t\in[0,1] \end{cases}$$

If such an F exists, then A is called a deformation retract of X.

REMARK 2.27: Suppose F is a deformation retraction of X onto A. Let $i\colon A\to X$ be the inclusion and let $r\colon X\to A$ be defined by $r(x)=F(x,1)$. Then $i\circ r$ is the identity on A and F is a homotopy between $r\circ i$ and the identity on X. Hence a deformation retraction is a special case of homotopy equivalence. We can visualize a deformation retraction as a shrinking of a space X to a subspace A such that A remains fixed throughout the shrinking process.

The underlying idea in the following two propositions is that homotopic maps induce equal maps on cohomology. We state them without proof. For proof of these results, see [BoT].

PROPOSITION 2.28. *Two manifolds with the same homotopy type have the same de Rham cohomology.*

PROPOSITION 2.29. *If A is a deformation retract of M, then A and M have the same de Rham cohomology.*

REMARK 2.30: Proposition 2.29 is a consequence of Proposition 2.28 because deformation retractions are special cases of homotopy equivalence. Since these results are stated without proof, we give the following examples to illustrate the way in which these results will be used in subsequent chapters.

EXAMPLE 2.31: $H^*(S^{n-1})=H^*(\mathbb{R}^n\setminus\{\vec{0}\})$.

We show that S^{n-1} is a deformation retract of $\mathbb{R}^n\setminus\{\vec{0}\}$ and apply Proposition

2.28. Suppose $X = \mathbb{R}^n \setminus \{\vec{0}\}$. Define $F\colon X \times [0,1] \to X$ by

$$F(\vec{x}, t) = (1-t)\vec{x} + \frac{t\vec{x}}{||\vec{x}||}.$$

Clearly F satisfies the conditions of Definition 2.19, so indeed S^{n-1} is a deformation retract of $\mathbb{R}^n \setminus \{\vec{0}\}$. Hence $H^*(S^{n-1}) = H^*(\mathbb{R}^n \setminus \{\vec{0}\})$.

Example 2.32: Let

$$U = \{[(z_1, \ldots, z_{n+1})] \in \mathbb{CP}^n | z_{n+1} = 0\}$$

and

$$V = \{[(z_1, \ldots, z_{n+1})] \in \mathbb{CP}^n | z_i \neq 0 \text{ for at least one } i \in \{1, \ldots, n\}\},$$

then $H^*(U) = H^*(V)$. Again, we show that U is a deformation retract of V and apply Proposition 2.28.

Define $F\colon V \times [0,1] \to V$ by $F(\vec{z}, t) = (z_1, \ldots, z_n, (1-t)z_{n+1})$. Then F satisfies the conditions of Definition 2.26, so U is a deformation retract of V. Notice that U can be identified with $\mathbb{CP}^{n-1}$ because the last coordinate is zero, hence $\mathbb{CP}^{n-1}$ is a deformation retract of V. We will refer to this result in the next chapter when we compute the cohomology of $\mathbb{CP}^n$.

EXTRA STRUCTURE OF THE DE RHAM COHOMOLOGY OF ORIENTED MANIFOLDS

This chapter discusses the Poincaré lemma, Mayer-Vietoris sequences, Poincaré duality, and the signature of a bilinear form. Proofs of these results are omitted and may be found in [BoT]. We emphasize by example techniques using these results to compute cohomology and to use cohomology to establish non-existence results.

THE POINCARÉ LEMMA AND THE MAYER-VIETORIS SEQUENCE.

POINCARÉ LEMMA. $H^*(\mathbb{R}^n) = H^*(point) = \begin{cases} \mathbb{R}, & \textit{in dimension } 0 \\ 0, & \textit{otherwise.} \end{cases}$

For compactly supported cohomology, we have the following variation of the Poincaré lemma:

POINCARÉ LEMMA FOR COMPACT SUPPORTS. $H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & \textit{in dimension } n \\ 0, & \textit{otherwise.} \end{cases}$

To compute the cohomology of an arbitrary manifold M, it may be the case that we can express M as the union of two open sets U and V such that the cohomology groups of U, V, and $U \cap V$ are known. The Mayer-Vietoris sequence relates the cohomology of M to the cohomology of these subspaces. In some cases the cohomology of M can be deduced using only the familiar dimension theorem in linear algebra.

DEFINITION 3.1. *A sequence* $\cdots \xrightarrow{f} \Omega^*(M) \xrightarrow{g} \Omega^*(N) \longrightarrow \cdots$ *is exact at* $\Omega^*(M)$ *provided image*(f) = *kernel*(g). *A sequence is called exact if the sequence is exact everywhere.*

For $M = U \cup V$, U and V open, define the Mayer-Vietoris sequence to be

$$0 \longrightarrow \Omega^*(M) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{\delta} \Omega^*(U \cap V) \longrightarrow 0$$

where $\delta(\omega, \tau) = \tau|_{U \cap V} - \omega|_{U \cap V}$. An argument with partitions of unity verifies the exactness of the Mayer-Vietoris sequence [BoT].

The Mayer-Vietoris sequence $0 \longrightarrow \Omega^*(M) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \longrightarrow \Omega^*(U \cap V) \longrightarrow 0$ induces a long exact sequence in cohomology,

$$\begin{tikzcd}
H^{q+2}(M) \arrow[r] & \cdots & \\
H^{q+1}(U \cap V) \arrow[u] & H^{q+1}(U) \oplus H^{q+1}(V) \arrow[l] & H^{q+1}(M) \arrow[l] \\
H^q(M) \arrow[r] & H^q(U) \oplus H^q(V) \arrow[r] & H^q(U \cap V) \arrow[u] \\
H^{q-1}(U \cap V) \arrow[u] & \cdots \arrow[l] &
\end{tikzcd}$$

also called the Mayer-Vietoris sequence.

EXAMPLE 3.2:

$$H^i(S^n) = \begin{cases} \mathbb{R}, & i = 0, n \\ 0, & \text{otherwise.} \end{cases}$$

This result follows from induction on n, however the cohomologies are computed directly for $n = 0, 1$, and 2 to illustrate the technique. Since S^0 is a point, the Poincaré lemma implies $H^0(S^0) = \mathbb{R}$ and $H^i(S^0) = 0$ for all $i \geq 1$. In the case of $n = 1$, we cover S^1 by two open neighborhoods U and V such that U contains the upper half circle, V contains the lower half circle, and $U \cap V$ is the union of two disjoint arcs. Because $U \cap V$ is equivalent to the union of two disjoint open intervals in $\mathbb{R}$, $H^0(U \cap V) = \mathbb{R} \oplus \mathbb{R}$. Let $U \Cup V$ denote the disjoint union of U and V. Because $U \Cup V$ is also equivalent to the union of two disjoint open intervals, $H^0(U \Cup V) = \mathbb{R} \oplus \mathbb{R}$. We write the incomplete Mayer-Vietoris sequence below as a visual aid:

$$\begin{tikzcd}
 & S^1 & U \Cup V & U \cap V \\
H^2 & 0 \arrow[r] & 0 \arrow[r] & 0 \\
H^1 & ? \arrow[r] & 0 \arrow[r] & 0 \arrow[ull] \\
H^0 & ? \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "\delta"] & \mathbb{R} \oplus \mathbb{R} \arrow[ull]
\end{tikzcd}$$

Because δ is the difference map, i.e. $(\alpha, \beta) \mapsto \beta - \alpha$, $image(\delta) = \mathbb{R}$ and $kernel(\delta) = \mathbb{R}$, hence $H^i(S^1) = \mathbb{R}$ for $i = 0, 1$.

For $n = 2$, let U and V be open neighborhoods of the upper and lower hemispheres. Then $U \Cup V$ is equivalent to the union of two disjoint open disks in the plane and S^1 is a deformation retract of $U \cap V$.

$$\begin{tikzcd}
 & S^2 & U \Cup V & U \cap V \\
H^3 & 0 \arrow[r] & 0 \arrow[r] & 0 \\
H^2 & ? \arrow[r] & 0 \arrow[r] & 0 \arrow[llu] \\
H^1 & ? \arrow[r] & 0 \arrow[r, "\delta'"] & \mathbb{R} \arrow[llu] \\
H^0 & ? \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "\delta"] & \mathbb{R} \arrow[llu]
\end{tikzcd}$$

Because $0 = rank(\delta') + nullity(\delta')$, we have $H^2(S^2) = \mathbb{R}$. The difference map δ satisfies $rank(\delta) = 1$. This result and the observation that $nullity(\delta') = 0$ imply that $H^1(S^2) = 0$ and $H^0(S^2) = \mathbb{R}$.

For larger n, we can identify $U \Cup V$ with two disjoint open disks in $\mathbb{R}^n$ and $U \cap V$ with S^{n-1} and proceed by induction.

DEFINITION 3.3. *$\mathbb{CP}^n$ is defined to be the set of all complex lines through $(0, \ldots, 0)$ in $\mathbb{C}^{n+1}$. Each line is determined by a linear equation $a_1 y_1 + \cdots + a_{n+1} y_{n+1} = 0$ unique up to nonzero constant multiples. More precisely, we have*

$$\mathbb{CP}^n = \{(z_1, \ldots, z_{n+1}) | z_i \neq 0 \textit{ for at least one } i\} \, / \sim$$

where $(z_1, \ldots, z_{n+1}) \sim (z'_1, \ldots, z'_{n+1})$ if and only if there is $a \in \mathbb{C} \setminus \{0\}$ such that $z_i = a z'_i$ for all i.

REMARK 3.4: The calculation of $H^*(S^k)$ has a role in the calculation of $H^*(\mathbb{CP}^n)$ after making the following identifications. $\{(z_1, \ldots, z_{n+1}) | z_{n+1} \neq 0\}$ is identified with $\mathbb{C}^n$

by noticing

$$(z_1, \dots, z_n, z_{n+1}) \sim \left(\frac{z_1}{z_{n+1}}, \dots, \frac{z_n}{z_{n+1}}, 1\right) \leftrightarrow \left(\frac{z_1}{z_{n+1}}, \dots, \frac{z_n}{z_{n+1}}\right) \in \mathbb{C}^n$$

and $\{(z_1, \dots, z_n, 0) \in \mathbb{CP}^n\} \leftrightarrow \mathbb{CP}^{n-1}$.

EXAMPLE 3.5:

$$H^k(\mathbb{CP}^n) = \begin{cases} \mathbb{R}, & 0 \le k \le 2n, \ k \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

We begin as in the previous example by computing $H^*(\mathbb{CP}^n)$ directly for $n = 1$ and 2 and proving the result by induction on n. Notice that $\mathbb{CP}^1$ can be identified with S^2. So $H^k(\mathbb{CP}^1) = \mathbb{R}$ for $k = 0,\ 2$. For $n = 2$, we let $U = \{p \in \mathbb{CP}^2 | z_3 \neq 0\}$ and $V' = \{p \in \mathbb{CP}^2 | z_3 = 0\}$. By the remark above, we identify U with $\mathbb{C}^2$ and V' with $\mathbb{CP}^1$. Since V' is compact, we choose $V = \{p \in \mathbb{CP}^1 | z_1 \neq 0 \text{ or } z_2 \neq 0\}$ as our open neighborhood of V. We know from Example 2.32 that V' is a deformation retract of V, so they have the same cohomology. Moreover,

$$U \cap V \leftrightarrow \{(z_1, z_2, 1) | z_1 \neq 0 \text{ or } z_2 \neq 0\} \leftrightarrow \mathbb{C}^2 \setminus \{(0,0)\} \cong S^3.$$

The calculation of $H^*(\mathbb{CP}^2)$ now reduces to dimension counting.

$$\begin{tikzcd}
& \mathbb{CP}^2 & U \Cup V & U \cap V \\
H^4 & \mathbb{R} \arrow[r] & 0 \oplus 0 \arrow[r] & 0 \\
H^3 & 0 \arrow[r] & 0 \arrow[r] & \mathbb{R} \arrow[ull] \\
H^2 & \mathbb{R} \arrow[r] & 0 \oplus \mathbb{R} \arrow[r] & 0 \arrow[ull] \\
H^1 & 0 \arrow[r] & 0 \arrow[r, "\delta'"] & 0 \arrow[ull] \\
H^0 & \mathbb{R} \arrow[r] & \mathbb{R} \oplus \mathbb{R} \arrow[r, "\delta"] & \mathbb{R} \arrow[ull]
\end{tikzcd}$$

Now suppose

$$H^k(\mathbb{CP}^n) = \begin{cases} \mathbb{R}, & 0 \le k \le 2n,\ k \text{ even} \\ 0, & \text{otherwise.} \end{cases}$$

Let $U = \{p \in \mathbb{CP}^{n+1} | z_{n+2} \neq 0\}$ and $V = \{p \in \mathbb{CP}^{n+1} | z_i \neq 0 \text{ for some } i = 1, \dots, n+1\}$. Now V retracts to $\mathbb{CP}^n$ and $U \cap V$ retracts to S^{2n+1} because

$$U \cap V = \{(z_1, \dots, z_{n+1}, 1) | z_i \neq 0 \text{ for some } i = 1, \dots, n+1\} \cong \mathbb{C}^{n+1} \setminus \{(0, \dots, 0)\}.$$

For the k^{th} step in the Mayer-Vietoris sequence, $0 < k < 2n+2$, k even, we have

$$\begin{tikzcd}
H^k(U \cap V) & H^k(U) \oplus H^k(V) \arrow[l] & H^k(\mathbb{CP}^{n+1}) \arrow[l] \\
H^{k-1}(\mathbb{CP}^{n+1}) \arrow[r] & H^{k-1}(U) \oplus H^{k-1}(V) \arrow[r] & H^{k-1}(U \cap V) \arrow[u] \\
H^{k-2}(U \cap V) \arrow[u] & \cdots \arrow[l] &
\end{tikzcd}$$

But S^{2n+1} is a deformation retract of $U \cap V$, so

$$H^k(U \cap V) = H^{k-1}(U \cap V) = H^{k-2}(U \cap V) = 0.$$

Also $U \cong \mathbb{C}^{n+1}$, so $H^k(U) = H^{k-1}(U) = 0$. Since $V \cong \mathbb{CP}^n$, $k > 0$ even implies $H^k(V) = \mathbb{R}$ and $H^{k-1}(V) = 0$. So we rewrite the sequence as

$$0 \to H^{k-1}(\mathbb{CP}^{n+1}) \to 0 \to 0 \to H^k(\mathbb{CP}^{n+1}) \to \mathbb{R} \to 0.$$

We count dimensions and conclude that $H^{k-1}(\mathbb{CP}^{n+1}) = 0$ and $H^k(\mathbb{CP}^{n+1}) = \mathbb{R}$.

For the last step of the sequence,

$$\begin{tikzcd}
H^{2n+2}(S^{2n+1}) & H^{2n+2}(U) \oplus H^{2n+2}(V) \arrow[l] & H^{2n+2}(\mathbb{CP}^{n+1}) \arrow[l] \\
H^{2n+1}(\mathbb{CP}^{n+1}) \arrow[r] & H^{2n+1}(U) \oplus H^{2n+1}(V) \arrow[r] & H^{2n+1}(S^{2n+1}) \arrow[u] \\
H^{2n}(S^{2n+1}) \arrow[u] & H^{2n}(U) \oplus H^{2n}(V) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

we rewrite as follows after substitution:

$$0 \oplus \mathbb{R} \to 0 \to H^{2n+1}(\mathbb{CP}^{n+1}) \to 0 \to \mathbb{R} \to H^{2n+2}(\mathbb{CP}^{n+1}) \to 0 \to 0.$$

We apply the dimension theorem to get $H^{2n+1}(\mathbb{CP}^{n+1}) = 0$ and $H^{2n+2}(\mathbb{CP}^{n+1}) = \mathbb{R}$. $\square$

REMARK 3.6: Non-existence results can be established by locating differences in the de Rham cohomology of two manifolds. For example, we can conclude S^4 is not diffeomorphic to $\mathbb{CP}^2$ because $H^2(\mathbb{CP}^2) = \mathbb{R}$ and $H^2(S^4) = 0$.

POINCARÉ DUALITY AND SIGNATURE.

DEFINITION. *An open cover $\{U_\alpha\}$ of a manifold is said to be a good cover if all non-empty intersections $U_{\alpha_1} \cap \cdots \cap U_{\alpha_m}$ are diffeomorphic to $\mathbb{R}^n$.*

REMARK 3.7: A compact manifold has a finite good cover. Furthermore, so does the interior of a compact manifold with boundary.

PROPOSITION. *If M has a finite good cover, then $H^*(M)$ and $H_c^*(M)$ are finite dimensional.*

PROOF: See [BoT].

Let M be an oriented n-dimensional manifold without boundary having a finite good cover. We can consider the pairing

$$\int : H^q(M) \times H_c^{n-q}(M) \to \mathbb{R}$$

defined by the integral of the wedge product of two forms representing cohomology classes. Notice that the integral is defined since one of the forms is compactly supported. Poincaré duality states that this pairing is nondegenerate (since $H^q(M)$ and $H_c^{n-q}(M)$ are finite dimensional). Therefore we can interpret this pairing as an isomorphism $H^q(M) \to H_c^{n-q}(M)^*$. Similarly, $H_c^{n-q}(M) \cong H^q(M)^*$. Notice $H^q(M) = H_c^q(M)$ when M is compact.

DEFINITION 3.8. *A bilinear form on a real vector space is a map $\Phi\colon V \times V \to \mathbb{R}$ that is linear in each variable separately. The form is called symmetric if $\Phi(u,v) = \Phi(v,u)$ for all $u, v \in V$ and is called nonsingular if any $v \in V$ satisfying $\Phi(u,v) = 0$ for all $u \in V$ must equal the zero vector.*

PROPOSITION 3.9. *Let M be a smooth oriented compact $4k$-dimensional manifold without boundary. Then $\Phi\colon H^{2k}(M) \times H^{2k}(M) \to \mathbb{R}$ defined by*

$$\Phi([\alpha],[\beta]) = \int_M \alpha \wedge \beta$$

defines a well defined nondegenerate symmetric bilinear form.

PROOF: The integral is independent of the representative of the cohomology class chosen by Stokes' Theorem, so Φ is well defined on cohomology. Nondegeneracy follows from Poincaré duality. Φ is symmetric because $\alpha \wedge \beta = \beta \wedge \alpha$.

REMARK 3.10: A basis can be chosen for V so that Φ is represented by a diagonal matrix A. The signature of Φ is defined as the number of positive diagonal entries in A minus the number of negative diagonal entries in A and is denoted by $signature(\Phi)$. We define the signature of a compact oriented $4k$ dimensional manifold M to be the signature of Φ as defined in Proposition 3.9.

The following is a result we will need in the next chapter.

LEMMA 3.11. *The signature of $\mathbb{CP}^{2k}$ is either 1 or -1.*

PROOF: By Example 3.5, we have $dim(H^{2k}(\mathbb{CP}^{2k})) = 1$. Let α be any nonzero $2k$ form on $\mathbb{CP}^{2k}$. Since $\mathbb{CP}^{2k}$ is compact and the pairing given by the Poincaré duality is nondegenerate, there is another $2k$-form β on $\mathbb{CP}^{2k}$ such that $\int_{\mathbb{CP}^{2k}} \alpha \wedge \beta \neq 0$. But $H^{2k}(\mathbb{CP}^{2k})$ is one-dimensional, so $\alpha = k\beta$ for some nonzero k. Hence $\int_{\mathbb{CP}^{2k}} \alpha \wedge \alpha \neq 0$ for any nonzero $2k$-form $\alpha \in H^{2k}(\mathbb{CP}^{2k})$. Since $dim(H^{2k}(\mathbb{CP}^{2k})) = 1$, Φ is represented by a 1×1 nonzero matrix, hence $signature(\Phi) = signature(\mathbb{CP}^{2k}) = \pm 1$, depending on the choice of orientation. $\square$

FAILURE OF $\mathbb{CP}^{2k}$ TO BOUND A SMOOTH ORIENTED COMPACT MANIFOLD

This chapter shows that neither $\mathbb{CP}^{2k}$ nor the disjoint union of two or more copies of $\mathbb{CP}^{2k}$ (with the same orientation) can serve as the boundary of an oriented smooth compact manifold with boundary. The proof proceeds by showing that if M is the boundary of an oriented smooth compact $(4k+1)$-dimensional manifold with boundary, then $H^{2k}(M)$ contains a subspace one half the dimension of $H^{2k}(M)$ on which the bilinear form $([\alpha],[\beta]) \mapsto \int_M \alpha \wedge \beta$ is identically zero. The calculation of this bilinear form on $H^{2k}(M)$ (Lemma 3.11) shows that if N is the union of any number of copies of $\mathbb{CP}^{2k}$ with the same orientation, then $dim(H^{2k}(N)) > 0$ but $H^{2k}(N)$ contains no subspace of positive dimension on which the bilinear form is identically zero. In fact, for a boundary M as above, our reasoning shows that $dim(H^{2k}(M))$ is even. Thus for certain manifolds N, e.g. the union of an odd number of copies of $\mathbb{CP}^{2k}$, very little information about $H^{2k}(N)$ is needed to conclude that N is not the boundary of an oriented smooth compact $(4k+1)$-dimensional manifold with boundary.

DEFINITIONS.

Let W be an oriented smooth compact $(4k+1)$-dimensional manifold with boundary M. Assume that W contains a neighborhood of M diffeomorphic to $(-R,0] \times M$. Let $\mathcal{W}$ be the noncompact manifold formed by attaching along $\{0\} \times M$ a manifold of the form $[0,R) \times M$ so that $\mathcal{W}$ contains an open set diffeomorphic to $(-R,R) \times M$. We fix one such diffeomorphism throughout the remainder of this chapter. The results stated for $(-R,R) \times M$ are thus assumed for the preimage of $(-R,R) \times M$ without explication. Let $j_r \colon M \to \{r\} \times M$ and notice $j_r^* \colon H^*(\mathcal{W}) \to H^*(M)$ is independent of r since j_{r_1} is homotopic to j_{r_2} for all $r_1, r_2 \in (-R,R)$.

Since W is a deformation retract of $\mathcal{W}$ it follows that $H^*(W) \cong H^*(\mathcal{W})$. Abusing notation slightly, define the inclusions $j\colon M \to W$ and $j\colon M \to \mathcal{W}$ identifying M with $\{0\} \times M$ and define $j^*\colon H^*(W) \to H^*(M)$ and $j^*\colon H^*(\mathcal{W}) \to H^*(M)$.

Define $\phi\colon (-R, R) \to \mathbb{R}$ to be a smooth, symmetric about 0, compactly supported, nonnegative bump function satisfying $\int_{-R}^{R} \phi(r)dr = 1$. ϕ extends to $(-R, R) \times M$ by $\phi(r, m) = \phi(r)$. Define functions $\mu, \nu\colon (-R, R) \to \mathbb{R}$ by

$$\mu(r) = \int_{-R}^{r} \phi(s)ds$$
$$\nu(r) = \mu(r) - 1.$$

Let $\pi\colon (-R, R) \times M \to M$ be the projection on M.

REMARK 4.1: Both functions can be extended to $(-R, R) \times M$ by $\mu(r, m) = \mu(r)$ and $\nu(r, m) = \nu(r)$. For some $\epsilon > 0$, both functions are constant in the set $(-R, -R+\epsilon) \times M$ and extend constantly to all of $\mathcal{W}$. The extended ν has compact support on $\mathcal{W}$.

LEMMA 4.2. *Let α be a closed i-form on M. Define $g\colon \Omega^i(M) \to \Omega_c^{i+1}(\mathcal{W})$ by $g(\alpha) = \pi^*\alpha \wedge \phi(r)dr$. Then $g(\alpha)$ is a closed $i+1$-form on $\mathcal{W}$ and the resulting map (also called g)*

$$g\colon H^i(M) \to H_c^{i+1}(\mathcal{W})$$

is well defined on cohomology.

PROOF: Let $\alpha \in \Omega^i(M)$ be closed. Since α is closed and $d(\phi(r)dr) = 0$, we have

$$\begin{aligned} d(g(\alpha)) &= d\pi^*\alpha \wedge \phi(r)dr + (-1)^i \pi^*\alpha \wedge d\phi(r)dr \\ &= \pi^*(d\alpha)) \wedge \phi(r)dr \\ &= 0. \end{aligned}$$

Hence $g(\alpha)$ is closed. To see that g is well defined on cohomology, suppose $[\alpha] = [\beta]$. Then $\alpha - \beta$ is exact and hence equal to $d\gamma$ for some γ. Now

$$\begin{aligned} g(\alpha) - g(\beta) &= (\pi^*\alpha - \pi^*\beta) \wedge \phi(r)dr \\ &= \pi^* d\gamma \wedge \phi(r)dr. \end{aligned}$$

In order for the map to be well defined on cohomology, we need $\pi^*d\gamma \wedge \phi(r)dr = d(\eta)$ for some compactly supported η. Clearly $d\nu = \phi dr$. ν is a compactly supported form and so is $d\pi^*\gamma \wedge \nu$. Because

$$d(d\pi^*\gamma \wedge \nu) = \pm d\pi^*\gamma \wedge \phi(r)dr,$$

take $\eta = d\pi^*\gamma \wedge \nu$ with the appropriate sign. Hence $[g(\alpha)] = [g(\beta)]$ and g is well defined on cohomology. □

PROPOSITION 4.3. *$H^i(\mathcal{W}) \xrightarrow{j^*} H^i(M) \xrightarrow{g} H_c^{i+1}(\mathcal{W})$ is exact in the middle.*

PROOF: We show that $image(j^*) = kernel(g)$. Let ζ be a closed i-form on $\mathcal{W}$ such that $j^*(\zeta) = \xi$ on M. Hence ξ is a closed form on M. By the lemma above, $g(\xi) = \pi^*\xi \wedge \phi(r)dr = \pi^*j^*\zeta \wedge \phi(r)dr$ is also a closed form on $\mathcal{W}$. But $\nu\zeta$ is a compactly supported form on $\mathcal{W}$ and

$$\begin{aligned} d(\nu\zeta) = d((\mu(r) - 1)\zeta) &= d\left(\left(\int_{-R}^{r} \phi(s)ds - 1\right) \wedge \zeta\right) \\ &= \phi(r)dr \wedge \zeta - \left(\int_{-R}^{r} \phi(s)ds - 1\right) \wedge d\zeta. \end{aligned}$$

But ζ is closed, hence $d\zeta = 0$ and $d(\nu\zeta) = \phi(r)dr \wedge \zeta$. Since $j \circ \pi$ is homotopic to $Id_{M\times(-R,R)}$, we have

$$[\pi^*j^*\zeta] = [Id^*\zeta] = [\zeta] \text{ in } H^*(M \times (-R, R)).$$

Thus $\pi^*j^*\zeta - \zeta$ is an exact form. So there is β on $M \times (-R, R)$ with $d\beta = \pi^*j^*\zeta - \zeta$ and $\phi(r)dr \wedge \beta$ is compactly supported. Now then

$$d(\phi(r)dr \wedge \beta) = \phi(r)dr \wedge (\pi^*j^*\zeta - \zeta)$$

implies $\phi(r)dr \wedge \pi^* j^* \zeta = g(j^*\zeta)$ represents the same cohomology class as $\phi(r)dr \wedge \zeta$. Also $\phi(r)dr \wedge \zeta = d(\nu\zeta)$, so $g(\xi)$ represents the same cohomology class as zero. By Lemma 4.2, we have $g(\xi) = 0$ and $image(j^*) \subset kernel(g)$.

Now suppose ξ is a closed i-form on M with $[g(\xi)] = [0]$, i.e. with $\pi^*\xi \wedge \phi(r)dr$ exact (represents the same cohomology class as 0). So there is a compactly supported β on $\mathcal{W}$ with $d\beta = \pi^*\xi \wedge \phi(r)dr$. Then

$$\begin{aligned}
d(\beta - (-1)^i \mu(r)\pi^*\xi) &= d\beta - (-1)^i d(\mu(r)\pi^*\xi) \\
&= d\beta - (-1)^i d\left(\int_{-R}^{r} \phi(s)ds \wedge \pi^*\xi\right) \\
&= d\beta - (-1)^i \phi(r)dr \wedge \pi^*\xi - \mu(r)\pi^* d\xi \\
&= d\beta - (-1)^i \phi(r)dr \wedge \pi^*\xi \\
&= \pi^*\xi \wedge \phi(r)dr - (-1)^i \phi(r)dr \wedge \pi^*\xi \\
&= \pi^*\xi \wedge \phi(r)dr - \pi^*\xi \wedge \phi(r)dr \\
&= 0.
\end{aligned}$$

So $\beta - (-1)^i \mu(r)\pi^*\xi$ is closed. Also for any $r_0 \in (-R, R)$

$$\begin{aligned}
j_{r_0}^*(\beta - (-1)^i \mu(r)\pi^*\xi) &= j_{r_0}^*\beta - (-1)^i j^* \mu(r)\pi^*\xi \\
&= j_{r_0}^*\beta - (-1)^i \mu(r_0) j^*\pi^*\xi.
\end{aligned}$$

By construction of μ and because β has compact support, it is possible to choose r_0 so that $j_{r_0}^*\beta = 0$ and $\mu(r_0) = 1$. Recall that for any r_0, $j_{r_0}^* = j^*$ as maps on cohomology.

Hence

$$\begin{aligned} j^*[\beta - (-1)^i \mu(r)\pi^*\xi] &= j^*_{r_0}[\beta - (-1)^i \mu(r)\pi^*\xi] \\ &= [j^*_{r_0}(\beta - (-1)^i \mu(r)\pi^*\xi)] \\ &= [(-1)^i j^*_{r_0}\pi^*\xi] \\ &= (-1)^i j^*[\pi^*\xi] \\ &= (-1)^i[\xi]. \end{aligned}$$

Thus $(-1)^i[\xi]$, and hence $[\xi]$ itself is in $image(j^*)$. □

LEMMA 4.4. *If α is a closed $2k$-form on M and γ a closed $2k$-form on $\mathcal{W}$, then*

$$\int_M j^*(\gamma) \wedge \alpha = \int_{\mathcal{W}} \gamma \wedge g(\alpha).$$

PROOF: Let $I = (-R, R)$. Since $\gamma \wedge g(\alpha)$ vanishes off $I \times M$, we write

$$\begin{aligned} \int_{\mathcal{W}} \gamma \wedge g(\alpha) &= \int_{\mathcal{W}} \gamma \wedge \phi(r) dr \wedge \pi^*\alpha \\ (*) \qquad &= \int_{I\times M} \gamma(r,m) \wedge \phi(r) dr \wedge \pi^*\alpha(m) \end{aligned}$$

where (r, m) indicates dependency on both I and M. But $(*)$ is equal to

$$\begin{aligned} &\int_I \phi(r) \int_M \gamma(r,m) \wedge \alpha(m) dr \quad \text{by Fubini's theorem} \\ &= \int_I \phi(r) \int_M j_r^*(\gamma) \wedge \alpha dr \\ &= \int_I \phi(r) \left(\int_M j^*(\gamma) \wedge \alpha \right) dr \quad \text{since } j_r^* \text{ is independent of } r \\ &= \int_M j^*\gamma \wedge \alpha. \quad \square \end{aligned}$$

REMARK 4.5: We can conclude that for H^{2k}, g is the dual map to j^* under the identification of $H^{2k}(M)$ with $H^{2k}(M)^*$ and of $H_c^{2k+1}(\mathcal{W})$ with $H^{2k}(\mathcal{W})^*$. To check this, define

$$G\colon H^{2k}(M) \to H^{2k}(\mathcal{W})^* \text{ by } G(\alpha)(\gamma) = \int_{\mathcal{W}} \gamma \wedge g(\alpha) \text{ and}$$

$$J\colon H^{2k}(\mathcal{W}) \to H^{2k}(M)^* \text{ by } J(\gamma)(\alpha) = \int_M j^*(\gamma) \wedge \alpha$$

By Lemma 4.4, equality of the integrals establishes G and J as dual maps.

PROPOSITION 4.6. *If S and T are dual linear maps of vector spaces, then*

$$dim(image(T)) = dim(image(S)).$$

PROOF: Suppose V and W are vector spaces, $T\colon V \to W$, $S\colon W^* \to V^*$ is the dual map of T, and $dim(V) = n$ and $dim(W) = m$. Suppose $dim(image(T)) = r$. Then

$$dim((image(T))^0) = dim(W) - dim(image(T)) = m - r$$

and $ker(S) = (image(T))^0$, where V^0 denotes the annihilator of V. Hence $dim(ker(S)) = m - r$. Finally, we have $dim(image(S)) = dim(W^*) - dim(ker(S)) = dim(W^*) - dim(W^*) + dim(image(T))$. $\square$

THEOREM 4.7. *The disjoint union of any odd number of copies of $\mathbb{CP}^{2k}$ is not the boundary of an oriented compact smooth manifold W.*

PROOF: By Proposition 4.3,

$$H^i(\mathcal{W}) \xrightarrow{j^*} H^i(M) \xrightarrow{g} H_c^{i+1}(\mathcal{W})$$

is an exact sequence. Hence $dim(image(j^*)) = dim(ker(g))$. By Proposition 4.6, we know that $dim(image(j^*) = dim(image(g))$. Then

$$\begin{aligned} dim(H^{2k}(M) &= dim(image(g)) + dim(ker(g)) \\ &= dim(image(j^*)) + dim(ker(g)) \\ &= dim(image(j^*)) + dim(image(j^*)) \\ &= 2dim(image(j^*)). \end{aligned}$$

In particular, $dim(H^{2k}(M))$ is even. We know from Example 8.4 that $dim(H^{2k}(\mathbb{CP}^{2k})) = 1$, and since cohomology is additive on disjoint unions, $dim(H^{2k}(\bigcup_{n=1}^{Nodd} \mathbb{CP}^{2k}) = N$. Since diffeomorphic manifolds have the same cohomology, $\mathbb{CP}^{2k}$ is not diffeomorphic to M. Hence we conclude that $\mathbb{CP}^{2k}$ is not the boundary of W. □

REMARK 4.8: From the proof above, $image(j^*)$ is a subspace of $H^{2k}(M)$ with half the dimension. For α and β closed $2k$ forms on W, Stokes' Theorem implies

$$\int_M j^*\alpha \wedge j^*\beta = \int_W d(\alpha \wedge \beta) = 0.$$

Hence the subspace $image(j^*)$ pairs to 0 with itself. In this setting, an argument in linear algebra shows that $signature(M) = 0$ [Vi, Lemma 5.39]. With this fact, calculation of $signature(\mathbb{CP}^{2k})$ (Lemma 3.11) shows $\mathbb{CP}^{2k}$ cannot be the boundary of a smooth compact oriented manifold. Similarly, the disjoint union of two or more copies of $\mathbb{CP}^{2k}$ with the same orientation cannot bound a smooth compact oriented manifold. The following theorem is a restatement of this fact. In our setting we can avoid the more general argument in linear algebra and give a proof based on the observation that no nonzero element of $H^{2k}(\mathbb{CP}^{2k} \cup \cdots \cup \mathbb{CP}^{2k})$ pairs to zero with itself under $\int \cdot \wedge \cdot$.

THEOREM 4.9. *The disjoint union of two or more copies of $\mathbb{CP}^{2k}$ with the same orientation is not the boundary of a smooth compact oriented manifold W.*

PROOF: Suppose ∂W is the disjoint union of n copies of $\mathbb{CP}^{2k}$ with the same orientation, i.e. $\partial W = \mathbb{CP}^{2k} \cup \cdots \cup \mathbb{CP}^{2k}$. By the proof of Theorem 4.7, we know that $H^{2k}(\mathbb{CP}^{2k} \cup \cdots \cup \mathbb{CP}^{2k})$ is $2m$-dimensional and contains an m-dimensional subspace which pairs to zero with itself under $\int_{\partial W} \cdot \wedge \cdot$. Since $signature(\mathbb{CP}^{2k}) = \pm 1$ and $dim(H^{2k}(\mathbb{CP}^{2k})) = 1$, any nonzero $2k$ form ω we choose as a basis element of $H^{2k}(\mathbb{CP}^{2k})$ will have the property $\int_M \omega \wedge \omega \neq 0$. From the i^{th} copy of $\mathbb{CP}^{2k}$ choose a nonzero $2k$

form ω_i. All copies of $\mathbb{CP}^{2k}$ have the same orientation, so without loss of generality we can assume $sign\left(\int_{\partial W} \omega_i \wedge \omega_i\right)$ is constant for all i. Let

$$\alpha = c_1\omega_1 + \cdots + c_n\omega_n \in H^{2k}(\mathbb{CP}^{2k} \cup \cdots \cup \mathbb{CP}^{2k}).$$

Then we have

$$\int_M \alpha \wedge \alpha = \sum_{i=1}^{n} \int_{\mathbb{CP}^{2k}} c_i^2 \omega_i \wedge \omega_i \neq 0.$$

So $H^{2k}(\mathbb{CP}^{2k} \cup \cdots \cup \mathbb{CP}^{2k})$ does not contain any subspace which pairs to zero with itself, thus contradicting our assumption that $\mathbb{CP}^{2k} \cup \cdots \cup \mathbb{CP}^{2k}$ bounds a smooth compact oriented manifold. □

REMARK 4.10: The method of proof of Theorem 4.9 can be modified to show that $\mathbb{CP}^{2k}$ cannot bound a smooth oriented compact manifold by taking $n = 1$. Theorem 4.9 establishes the conclusion of Theorem 4.7, but these results are listed separately because the less general technique of proof of Theorem 4.7 requires less information about $H^{2k}(M)$.

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Jesse H. Taylor

ADDRESS:
600 East Washington Street
Apartment 9
Blacksburg, Virginia 24060

EDUCATION:
Bachelor of Science, Mathematics, May, 1992
Bachelor of Science, Statistics, December, 1992